Collective Bargaining and Walrasian Equilibrium.*

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Abstract

This paper contributes to the research agenda on non-cooperative foundations of Walrasian Equilibrium. A class of barganing games in which agents bargain over prices and maximum trading constraints is considered: It is proved that all the Stationary Subgame Perfect Equilibria of these games implement Walrasian allocations as the bargaining frictions vanish. The main novelty of the result is twofold: (1) it holds for any number of agents; (2) it is robust to different specifications of the bargaining process.

1 Introduction.

This paper contributes to the literature on non-cooperative foundations of the Walrasian Equilibrium. This research agenda dates back at least to the early works on market games by Shubik (1972), Shapley and Shubik (1977) and Postlewaite and Schmeidler (1978). More recently, the development of the theory of strategic bargaining, pioneered by Stahl (1972) and Rubinstein (1982), motivated the investigation of the foundations of the competitive equilibrium in the context of strategic bargaining games:[1] rather than assuming an abstract price mechanism or fictitious auctioneers that deliver the market equilibrium, this literature explicitely models the interaction among agents as a bargaining problem, and determines the conditions under which the competitive outcome

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[1] See Osborne and Rubinstein (1990) for a survey of this literature.

emerges as the equilibrium of the game. The main concern so far has been the idea that the competitive outcome should emerge in economies with a large number of agents: most of the works in this literature explored this question in economies with an infinite number of agents. The important task of extending the argument to finite economies has proved of difficult solution: only recently, Gale and Sabourian (2005) provided strategic bargaining foundations to the competitive hypothesis, in the context of a single good economy.[2]

To the best of my knowledge, only two contributions have studied strategic bargaining foundations for finite walrasian economies: Yildiz (2005) and Dàvila and Eeckhout (2007) consider pure exchange economies with two agents and an arbitrary number of goods. Analyzing different bargaining procedures, they both provide a particularly striking result: the equilibria of their games yield walrasian outcomes, as the two players become infinitely patient. In Yildiz (2005) it is shown that a bargaining procedure *à la Rubinstein*, in which the proposer offers an allocation, yields non-walrasian outcomes. In contrast, it is proved that, under certain conditions, a bargaining procedure in which proposals consist of price vectors implement competitive outcomes. Nonetheless, Dàvila and Eeckhout (2007) proved Yildiz's conditions to be generically violated in the space of economies, and obtain the competitive result adopting a different bargaining procedure, in which the two players alternatingly announce prices and a maximum trading constraint.[3] If the responder agrees, he can demand any trade consistent with the constraints he has agreed upon. These results point out the sensitivity of the competitive outcome to the specification of the bargaining process: a thouroughly neglected topic in this literature.

The next step that seems natural to undertake is to generalize the results of Dàvila and Eeckhout to economies with an arbitrary number of agents. This presents non trivial issues of modelling choice though. The reason is that in the two-agents economy of Dàvila and Eeckhout, the only possible pairwise meeting also coincides with the grand-coalition of the economy itself. It is not clear then what the natural generalization should be. The focus of this paper is on the properties of this particlar bargaining protocol. For this reason trade is assumed to occur in a centralized market, in a one-shot exchange.[4] A further question that arises naturally is whether these results are robust to different specifications of

[2] See Gale (2000) for a thourough account of this literature, and a discussion of the issues raised by the finite number of agents.

[3] The importance of maximum trading constraints for the case of axiomatic bargaining was analyzed by Binmore (1987) first.

[4] This issue and possible alternative specifications are discussed in section 5.

the bargaining protocol.[5]

The main contribution of this paper is precisely to generalize Dàvila and Eeckhout's results to economies with an arbitrary number of agents and to different bargaining processes. The class of bargaining games considered here encompasses all the bargaining procedures of alternating offers in which the proposer announces prices and maximum trading constraints, in which responses are sequential, trade occurs upon unanimous acceptance, and the continuation game in case of rejection does not depend on the actions previously taken by the players.[6] If an agreement is reached, the proposer acts as the residual claimant of a centralized market: responders simultaneously choose their demands, subject to the maximum trading constraints and the standard budget conditions, and the market is cleared by the proposer at the announced prices.

It is proved that, as the bargaining frictions vanish, the Stationary Subgame Perfect Equilibria of this class of games implement Walrasian allocations in economies with an arbitrary number of agents and commodities. To the best of my knowledge, this is the first work that provides strategic bargaining foundations in such general environments. Furthermore, the convergence result is robust to details of the bargaining process such as differences in players' discount factors and the process according to which the proposer is selected.

A remarkable aspect of the result is that it doesn't require a large economy, or an approximation of that such as a *replica* economy: the result holds for any number of agents. The limit only concerns the players' discount factors. This suggests that the details of the bargaining process may play a crucial role in determining the competitive outcome, independently on the number of agents in the economy, and that a careful analysis of alternative bargaining protocols may be of great use to the research agenda that seeks to provide strategic bargaining foundations to the competitive hypothesis.

The rest of the paper is organized as follows: section 2 introduces the economy, and the basic notation; section 3 contains the description of the class of bargaining games, and introduces further notation and definitions. Section 4 contains the analysis of the game and the main results of the paper. Section 5 discusses the related literature and some alternative specifications. Section 6 concludes. Proofs are relegated to the Appendix 1. In the Appendix 2 it is shown how the main result can be extended to bargaining processes in which traders respond simultaneously, applying to an equilibrium refinement reminiscent of Selten's

[5] Cf. Gale (2000) and references therein.

[6] In section 6 we also discuss how to extend the result to procedures in which responses are simultanenous.

(1975) trembling hand equilibrium.

2 The Economy

A pure exchange economy is defined as a tuple $\mathcal{E} = \left\langle I, r, (X_i, e_i, u_i)_{i\in I} \right\rangle$: $I = \{1, ..., n\}$ is the set of agents in the economy, indexed by $i \in I$; $r \in \mathbb{R}^C_{++}$ denotes the total endowments of the C commodities in the economy. For each agent i, $X_i \subseteq \mathbb{R}^C_+$ is i's consumption possibility set, assumed compact. We assume that $0 \in X_i$. Each agent is endowed with a bundle $e_i \in X_i$ such that $\forall i$, $0 << e_i << r$ and $\sum_{i\in I} e_i = r$.[7] Agents have utility functions $u_i : X_i \to \mathbb{R}$. We assume, without loss of generality, that $u_i(e_i) = 0$ for all i. Allocations are denoted by $x = (x_i)_{i\in I} \in \mathbb{R}^{nC}_+$, where for each i, $x_i = \left(x_i^1, ..., x_i^C\right) \in \mathbb{R}^C_+$ is the consumption bundle of agent i. An allocation $(x_i)_{i\in I}$ is *feasible* if $\sum_{i\in I} x_i = r$ and $x_i \in X_i$ for each i. X denotes the set of feasible allocations:

$$X = \left\{ x \in \mathbb{R}^{nC}_+ : x_i \in X_i \text{ for all } i, \text{ and } \sum_{i\in I} x_i = r \right\}$$

Prices are denoted by $p \in \mathbb{R}^C_{++}$. The set of Pareto Efficient Allocations is denoted by X^{PE}:

$$X^{PE} := \left\{ x \in X : \nexists x' \in X \text{ such that } \left(u_i\left(x_i'\right)\right)_{i\in I} > \left(u_i\left(x_i\right)\right)_{i\in I} \right\}.$$

Definition 1 *The set of* Walrasian Allocations *is the set of feasible allocations x for which there exists a price vector such that (p, x) is a Walrasian Equilibrium. This set is denoted by X^*. Formally, X^* is such that: $X^* \subseteq X$ and $\forall x \in X^*$, $\exists p \in \mathbb{R}^C_{++}$: for each $i \in I$,*

$$x_i \in \arg\max_{y_i\in X_i} u_i(y_i)$$
$$s.t.\ \ p(y_i - e_i) \le 0$$

Mantained Assumptions (A):

- **(A1):** for each $i \in I$, u_i is differentiably strictly increasing and differentiably strictly quasi-concave on X_i.
- **(A2):** for each $i \in I$, u_i is *strongly concave*, in the sense that

$$\det\left\{ 2D^2 u_i(x) + \left[\sum_{k=1}^{n} D_{ikj} u_i(x) \left[x_k - e_{i,k}\right] \right]_{ij} \right\}$$

  does not change sign.

[7] "<<" is strict for all components. "<" allows the equality for some component, but not all. "≤" means "<" or "=".

Assumption (A2) guarantees that the offer curves have no inflexion points. This condition is satisfied whenever the substitution effect dominates the income effect.

3 The Bargaining Game.

In this section the bargaining procedure in Dàvila and Eeckhout (2007, DE hereafter) is adapted to the case of an economy with an arbitrary number of agents, and it is generalized to a wide class of bargaining processes.

Let $\sigma = (\sigma_0, \sigma_1, ...)$ denote a temporally homogeneous Markov process realizing values in a (possibly infinite) compact measurable space S. Let π be a measurable function, such that $\forall s \in S$, $\pi(s)$ is a permutation on I: $\pi(s) = (\pi_1(s), ..., \pi_n(s))$ identifies the order in which agents move in state s. We refer to the agent $\pi_1(s) \equiv a(s)$ as the *auctioneer* in state s; the other agents are the *traders*. The selected *auctioneer* $a(s) \in I$ announces a price vector p, and a vector $q = (q_j)_{j \neq a(t)} \in \mathbb{R}^{C(n-1)}$, where q_j represents player j's maximum excess demand (hereafter, we will refer to q_j as *maximum trading constraints, MTC*). The remaining agents $j \in I \backslash \{a(s)\}$, play sequentially[8], $\pi_2(s)$ moving first, and so on, until $\pi_n(s)$: they may either accept (action "Y") or reject (action "N"). If everybody accepts, trade can take place in the centralized market at the price p announced by the *auctioneer*, subject to the traders' MTCs $(q_j)_{j \neq a(t)}$: traders simultaneously choose excess demands $(z_j)_{j \neq a(t)}$ s.t. $z_j \in B_j(p, q)$, where

$$B_j(p, q) := \left\{ z \in \mathbb{R}^C : z \leq q \text{ and } pz \leq 0 \right\}. \tag{1}$$

The aggregate excess demand $\sum_{j \neq a(t)} z_j$ is cleared by the auctioneer, acting as the residual claimant of the market. After trade has taken place, agents leave the market and consume the bundle of goods they own.

For the game to be well-defined, it must be guaranteed that the auctioneer is indeed capable of clearing all the individual demands consistent with the individual budget constraints. For this purpose, it is assumed that $\forall t$, $\sum_{j \neq a(t)} q_j \leq e_{a(t)}$: the maximum quantities the auctioneer announces that can be traded must be cleareable by him.

If any player rejects, no trade occurs and the system moves to the next period according to the process σ.

Agents discount time: for each $i \in I$, let $\delta_i \in (0, 1]$ denote agent i's discount factor, and $\delta = (\delta_1, ..., \delta_n)$ denote the profile of discount factors.

[8] In Appendix 2 we consider a specific bargaining process in which the traders respond simultaneously: the main result is obtained for an equilibrium refinement considering trembles in the traders' responses.

The payoff in case of perpetual disagreement is assumed to be zero. If agreement occurs at period t, and agent i holds the bundle x_i after trade, he consumes it and derives a utility of $u_i(x_i)$. Player i's payoff for this outcome of the game is $\delta_i^t u_i(x_i)$. The definition of the set of histories and of players' strategies is straightforward but notationally cumbersome, therefore it is omitted. Strategy profiles are denoted by $f = (f_1, ..., f_n)$, f_i being i's strategy.

Notice that the class of games considered here encompasses all the bargaining procedures that use price-posting and maximum trading constraints in which trade occurs upon unanimous acceptance, responses are sequential, and the continuation game in case of rejection does not depend on the actions previously taken by the players. It includes, for example, deterministic processes of alternating offers, or a game in which at every period, each player is equally likely to occupy any position in the order of move, and so on. The important feature is that the transition probabilities only depend on the current state, not on the previous history or on the players' actions.

The following assumptions on the bargaining process will be used for the main result:

Mantained Assumptions (R):

- **(R1):** For each $i \in I$, $\exists s \in S : a(s) = i$ and
- **(R2):** From any state $s \in S$, all states $s' \in S$ are reached in finite time with probability one: $\forall s \in S$, $\forall s' \in S$,

$$\Pr(\{\exists m < \infty : \sigma_m = s'\} | \sigma_0 = s) = 1.$$

These assumptions guarantee that, from any initial condition, each agent is selected as the auctioneer in finite time with probability one.

4 Analysis.

Let's consider the utility possibility set of the economy, defined as

$$\mathcal{U} := \{v \in \mathbb{R}^n : \exists x \in X \text{ s.t. } u(x) = v\}$$

Each strategy profile f induces an *outcome* of the bargaining game, defined by a pair (τ^f, η^f), where τ^f is a *stopping time*, denoting the time at which agreement occurs, and η^f is a random variable that takes values in $\mathcal{U}$, denoting the utilities agents get from the consumption bundles they own at that period:[9] notice that because of the underlying stochastic

[9] Given a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t=1}^{\infty}$, a *stopping time* is a random variable τ that takes values $t = 1, 2, ...$ such that for each $t < \infty$, $\{\tau = t\} \in \mathcal{F}_t$.

process σ, (τ^f, η^f) are in general non-degenerate random variables, even if f is a profile of pure strategies. The dependence of the outcome on the strategy profile will be suppressed, and the outcome simply denoted by (τ, η), when its meaning is clear from the context.

For any pair (τ, η), and for each state s, let

$$\mathbb{E}[\delta^\tau \eta | \sigma_0 = s] = (\mathbb{E}[\delta_i^\tau \eta_i | \sigma_0 = s])_{i \in I}$$

denote the profile of expected utilities of the outcome (τ, η) when the state is s.

In this section we only focus on the *Stationary Subgame Perfect Equilibria* (SSP) of the game:

Definition 2 *A strategy profile f is a* Stationary Subgame Perfect Equilibrium (SSP) *if it is a Subgame Perfect Equilibrium and if, for each player i, the continuation of the strategy f_i after any partial history of lenght t is completely identified by the realized state σ_t.*

In words, players can condition their strategies only on the current state and the moves previously made in that period: they don't remember actions taken in previous periods, nor previous states. Player $a(s)$ always makes the same offer when selected as the auctioneer in state s, and for each s, $\pi_j(s)$'s response after a particular sequence of responses of agents $\pi_i(s)$, $i = 2, ..., j-1$ is always the same.

An outcome (τ, η) is *stationary* if there exists a measurable subset $S^* \subseteq S$ and a measurable function $\xi : S \to X$ such that: (i) $\sigma_t \notin S^*$ for all $t = 0, 1, ..., \tau - 1$; (ii) $\sigma_\tau \in S^*$; (iii) $\eta = u(\xi(\sigma_\tau))$. In words, a *stationary outcome* can be characterized by a pair (ξ, S^*) such that $S^* \subseteq S$ is the set of states in which agreement occurs, and the random variable ξ denotes the resulting allocation. Condition (iii) means precisely that the consumption utilities are the utilities of the allocations determined by ξ in the states in which agreement occurs. Using the latter condition, for any strategy profile f that induces a stationary outcome, we may define the value function of f at state s, $v^f(s) = \mathbb{E}[\delta^\tau \mu(\sigma_\tau) | \sigma_0 = s]$. Clearly, an SSP must induce a *stationary outcome.* Hence, the subsequent analysis will focus on *stationary outcomes* only.

The next definition introduces *SSPs with no delay*:

Definition 3 *An* SSP with immediate acceptance *(or* with no delay*) is an SSP in which agreement occurs in all states. Formally: f is an SSP with immediate acceptance if it induces a stationary outcome (ξ, S^*) s.t. $S^* \equiv S$. (This clearly entails that $\tau = 0$: no delay.)*

Notice that an *SSP with immediate acceptance* can be characterized by a tuple $(p^s, q^s)_{s\in S}$, where for each s, (p^s, q^s) is the offer made by $a(s)$ in state s. Traders $j \neq a(s)$ accept and choose consumption bundles $x_j(p^s, q^s)$ such that:

$$\begin{aligned} x_j(p^s, q^s) \in \arg\max_{x_j} u_j(x_j) \\ \text{s.t.} \quad p^s[x_j - e_j] \leq 0 \\ [x_j - e_j] \leq q_j^s \end{aligned} \tag{2}$$

This is obvious, from the definition of subgame perfection: once an agreement is reached, subgame perfection requires that each responder solves the optimization problem defined in (2). In an *SSP with immediate acceptance*, $a(s)$'s always offers the same (p^s, q^s) in state s. Furthermore, under the mantained assumptions (A1), $x_j(p^s, q^s)$ is uniquely determined for each j and s. Hence, an *SSP with immediate acceptance* can be completely characterized by a tuple $(p^s, q^s)_{s\in S}$, which in turn determines a tuple $(x^s)_{s\in S}$ of corresponding allocations.

Given this observation, in an *SSP with immediate acceptance*, at each state s the proposer $a(s)$ optimizes under the constraint that none of the traders $j \neq i$ has an incentive to deviate, that is $\forall s \in S$:

$$(p^s, q^s) \in \arg\max_{(p,q)} u_i\left(r - \sum_{j\neq a(s)} x_j(p^s, q^s)\right) \tag{3}$$

$$\text{s.t.:} \qquad \left\{ \begin{array}{l} u_j(x_j) \geq \delta_j \mathbb{E}\left[u_j\left(x_j^{\sigma_1}\right) | \sigma_0 = s\right] \\ \text{for } x_j(p^s, q^s) \text{ defined as in (2)} \end{array} \right\}_{j\neq a(s)}$$

The first constraint in (3) is the incentive compatibility necessary for the responders to actually accept the offer, rather than delaying the agreement and moving to the next period in state σ_1. The second constraint is simply the subgame perfect condition discussed above.

It is worth to point out that once an agreement is reached, players do not face a strategic situation anymore: they are simply left with the solution of the optimization problem in (2), and they behave as price takers. The agents' strategic behavior is confined to the responses to the offers. Once the bargaining process is over, agents do not behave strategically.[10]

Since, upon agreement, the responders are free to choose any consumption bundle consistent with the constraints in (2), in any *SSP with*

[10]Notice though that this is not an assumption: it is an immediate consequence of the structure of the game.

immediate acceptance $(p^s, q^s)_{s\in S}$ the induced allocations $(x^s)_{s\in S}$ must be such that, for each state s and agent $j \neq a(s)$,

$$Du_j\left(x_j^s\right)\left[x_j^s - e_j\right] \geq 0.$$

The inequality is strict if the maximum trading constraint q_j^s is binding in (2). Furthermore, since the proposer at s chooses the tuple $\left(q_j^s\right)_{j\neq a(s)}$, conditional on the responders accepting the offer, $a(s)$ can induce any allocation s.t. $Du_j\left(x_j^s\right)\left[x_j^s - e_j\right] \geq 0$: simply making the MTC tighter. Hence, an *SSP with immediate acceptance* can be characterized by allocation offers $\left(\left(x_j^s\right)_{j\in I}\right)_{s\in S}$, where $\left(x_j^s\right)_{j\in I}$ is the allocation offered by $a(s)$ at s, such that:[11]

$$\begin{aligned} &\forall s \in S: \\ &(x_l^s)_{l\in I} \in \arg\max_{(x_l)_{l\in I}\in X} u_{a(s)}\left(x_{a(s)}\right) \\ &\text{s.t.:} \quad \left\{\begin{array}{c} u_j(x_j) \geq \delta_j \mathbb{E}\left[u_j\left(x_j^{\sigma_1}\right)|\sigma_0 = s\right] \\ Du_j(x_j)\left[x_j - e_j\right] \geq 0 \end{array}\right\}_{j\neq a(s)} \end{aligned} \tag{4}$$

The rest of the analysis proceeds as follows: first, it is shown that if players are impatient (i.e. $\delta_i < 1$ for all $i \in I$), in all the SSPs of the bargaining game, agreement occurs with no delay (this is done in section 4.1); second, the attention is focused on the *SSP with immediate acceptance* for the case of infinitely patient players (i.e. $\delta = \mathbf{1}$), and it is shown that these equilibria induce Walrasian allocations (section 4.2). Finally (section 4.3), a continuity argument simply delivers the main result, summarized here:

Theorem 1 *Under the set of mantained assumptions (R) and (A), as* $\delta \to \mathbf{1}$*, the* SSP outcomes *converge to* Walrasian allocations*.*

4.1 Impatient players

In this section it is proved that in all the SSP of the game with impatient players, agreement occurs with no delay. The argument exploits techniques that are similar to those used by Merlo and Wilson (1995, MW hereafter), but it entails few important modifications. As in MW, the analysis is conducted in the space of utilities: *SSP payoffs* are characterized as the fixed points of a self-map in a space of measurable functions, representing the utility profiles induced by profiles of stationary strategies. The main departure from MW stems from the particular bargaining

[11] See Lemma A1 in Dàvila and Eeckhout (2007)

procedure considered here: in this setup, since agents still have room to choose their consumption bundles after an agreement is reached, the set of feasible utilities is an endogenous object, and therefore the SSP payoffs cannot be characterized as the fixed points of the same operator used in MW. Loosely speaking, MW's operator cannot be applied here because agents are not bargaining over final allocations or utilities. Rather, they *bargain over a procedure*: once agreement is reached, the actual allocation is chosen by the agents according to the procedure they have agreed upon. A second, minor departure from MW's analysis is that we allow for heterogeneous discounting, while they assume a common discount factor.

4.1.1 Immediate agreement in SSP

Proposition 1 *If $\delta << \mathbf{1}$, in any SSP of the game, agreement occurs with no delay*

(The proof of the proposition is left the appendix).

Sketch of Proof. As mentioned, the proof is conducted in the utility space $\mathcal{U}$, through a characterization of the SSP payoffs as fixed points of an operator defined on a space of value functions. From the previous analysis, we know that the consumption utilities induced by any SPE must be such that, for each i, $Du_i(x_i)[x_i - e_i] \geq 0$: hence, any SSP determines a *stationary outcome* (τ, η) such that $\eta \in \mathcal{U}^*$, where:

$$\mathcal{U}^* := \{v \in \mathbb{R}^n : \exists x \in X \text{ s.t. } u(x) = v \text{ and } Du(x)[x - e] \geq 0\}$$

Let $\mathcal{W}$ be the set of measurable functions $w : S \to \mathcal{U}^*$. For each agent i, define the function $\varphi_i^* : \mathbb{R}^n \to \mathbb{R}$ such that, for all $d \in \mathbb{R}^n$,

$$\varphi_i^*(d) := \begin{cases} 0 & \text{if } \nexists v \in \mathcal{U}^* : v_{-i} \geq d_{-i} \\ \max\{v_i : v \in \mathcal{U}^*, \text{ and } v_{-i} \geq d_{-i}\} & \text{otherwise} \end{cases}$$

Define the operator $E : \mathcal{W} \to \mathcal{W}$ such that for $w \in \mathcal{W}$,

$$E_i(w)(s) = \begin{cases} \max\{\varphi_i^*(\mathbb{E}[\delta w(\sigma_1)|\sigma_0 = s]) ; \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s]\} & \text{if } i = a(s) \\ \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s] & \text{otherwise} \end{cases}$$

The crucial step of the proof is contained in the following lemma:
Lemma 2: *w^* is an SSP payoff if and only if $E(w^*) = w^*$.*

This characterization of the SSP payoff implies that the SSP are efficient (constrained to $\mathcal{U}^*$). Assumptions (A1) and (A2) imply that as long as $\delta << \mathbf{1}$, any delay is inefficient. Hence, the constrained efficiency entailed by lemma 2 delivers the result.

To guarantee that proposition 1 is not vacuous, it is proved next that SSPs exist indeed. The proof exploits standard fixed point arguments, and it is left to the appendix:

Proposition 2 *There exists an SSP for the game with* $\delta << \mathbf{1}$.

4.2 Infinitely Patient Players.

In this section it is proved that the *SSP with immediate acceptance* in the game without discounting (i.e. $\delta = \mathbf{1}$) induce walrasian allocations.

With infinitely patient agents, under assumptions (R1) and (R2), the proposer's problem in (4) can be rewritten as:

$$\begin{aligned} &\forall s \in S: \\ &(x_l^s)_{l\in I} \in \arg\max_{(x_l)_{l\in I}} u_{a(s)}\left(x_{a(s)}\right) \\ \text{s.t.:}\quad &\left\{\begin{array}{c} Du_{ij}\left(x_j\right)\left[x_j - e_j\right] \geq 0 \\ u_j\left(x_j\right) \geq \max\left\{u_j\left(x_j^{s'}\right) : s' \in S\right\} \end{array}\right\}_{j\neq a(s)} \\ &r = \sum_{l\in I} x_l \end{aligned} \tag{5}$$

The reason is that under (R1) and (R2), from any state s, any state s' is reached in finite time with probability one. With infinitely patient players then the incentive compatibility constraint is that above because a player would reject as long as the utility he gets in state s is lower than what he would obtain in any other state.

Theorem 2 *Under the mantained assumptions (A1), (R1) and (R2), if* $\delta_i = 1$ *for every* $i \in I$*, the outcome* $(x_l^*)_{l\in I}$ *of an* ***SSP with immediate acceptance*** *is a Walrasian Allocation.*

The proof of the theorem is completed by the next three lemmata. Only the main lines of the argument are discussed here. The full proof is left to the appendix.

Lemma 2.1: *Under the mantained assumptions (A1), (R1) and (R2), if* $\delta = 1$*, in a SSP with immediate acceptance,* $(x_l^s)_{l\in I} = (x_l^*)_{l\in I}$ *for all* $s \in S$.

The argument of the proof here shows that for each s, x^s is an efficient allocation. Under the mantained assumptions, in an *SSP with immediate acceptance* it must be the case that the second constraint in (5) is binding, which implies that agents get the same utility in every state. The strict concavity of preferences, then implies that also the allocation is always the same.

Lemma 2.2: *At* $(x_l^*)_{l\in I}$, $Du_i(x_i^*)[x_i^* - e_i] = 0$ *for each* i.

In this lemma the efficiency of x^*, is used to prove that if $Du_i(x_i^*)[x_i^* - e_i] > 0$ for some i, the allocation x^* does not satisfy the constraints of problem (5) for agent i. Hence, for each player, the first constraint in (5) is also binding at x^*

Lemma 2.3: $(x_l^*)_{l\in I}$ *is a Walrasian allocation.*

From the previous lemma, it sufficies to set $p^* = Du_1(x_1^*)$, to have that, $\forall i$,

$$\begin{aligned} x_i^* &\in \arg\max_{x_i} u_i(x_i) \\ p^* x_i &= p^* e_i \end{aligned}$$

This concludes the proof of theorem 2.

It is proved next that, if agents are infinitely patient, every Walrasian allocation can be sustained as an outcome of an SSP with immediate acceptance: the proof is constructive, and it is left in the main body of the text.

Theorem 3 *Let* $\left(p^*, (x_l^*)_{l\in I}\right)$ *be a WE of the economy* $\mathcal{E}$. *Then, if* $\delta_i = 1$ *for all* $i \in I$, *there is an SSP with immediate acceptance of the game* $\Gamma(\mathcal{E})$ *with outcome* $(x_l^*)_{l\in I}$.

Proof. *Given the WE* $\left(p^*, (x_l^*)_{l\in I}\right)$, *consider the following strategy profile:* (p, q) *such that* $\forall i \in I$,

- *whenever* i *makes a proposal (i.e. for all* $s \in a^{-1}(i)$*) he offers* (p^i, q^i) *such that:*

  $p^i = p^*$; $\left(q_j^i\right)_{j\neq i}$ *are slack, i.e.* $q_j^i \geq z_j(p^*)$, *where*

$$\begin{aligned} z_j(p^*) &= \arg\max_{z_j\in\mathbb{R}^C} u_j(e_j + z_j) \\ s.t{:} &\quad p^* z_j \leq 0 \end{aligned}$$

- *whenever* i *is responding, he accepts any offer* (p', q') *such that* $u_i(x_i(p', q')) \geq u_i(x_i^*)$.

The outcome of this strategy profile, starting from any subgame in which some $j \in I$ *has to make an offer, is clearly* $(x_l^*)_{l\in I}$. *Now we check that it is an SSP indeed.*

If i *deviates, he may induce one of the following types of outcomes.*[12]

[12] Notice that with no discounting the o*ne-shot deviation principle* doesn't apply. Hence we consider all the possible deviations.

(1) Agreement is never reached: this outcome is clearly not preferred to $(x_l^*)_{l\in I}$ by i.

(2) At a later stage in the game, i offers (p', q') such that $\forall j \neq i$, $u_j(x_j(p', q')) \geq u_j(x_j^*)$ and it's accepted: Since $(x_l^*)_{l\in I}$ is efficient, and $u_j(x_j(p', q')) \geq u_j(x_j^*)$ for all $j \neq i$, it cannot be that $u_i\left(R - \sum_{j\neq i} x_j(p', q')\right) > u_i(x_i^*)$. So that this outcome cannot be preferred to $(x_l^*)_{l\in I}$ by i.

(3) At a later stage in the game, i accepts the offer (p^*, q^j) made by $j \neq i$, which yields the same outcome $(x_l^*)_{l\in I}$: therefore, deviations to this outcomes are not profitable either. ■

Clearly, Theorem 3 also proves the existence of *SSPs with immediate acceptance* when there is no discounting.

4.3 The Convergence result

From proposition 1, in any SSP an agreement is reached with no delay. Therefore, the SSP outcomes can be represented by measurable functions $y : S \to X$, assigning a feasible allocation to each state. Let Y be the set of such measurable functions. For any initial state s, $y(s)$ is the allocation induced by the acceptance of $a(s)$'s offer. Similarly to the above, the SSP allocations are fixed points of the operator $\rho : Y \to Y$, defined as:

$$\rho(y;\delta)(s) = \arg\max_{x\in X} u_{a(s)}\left(x_{a(s)}\right)$$
$$\text{s.t. for all } j \neq a(s), \quad \begin{array}{c} u_j(x_j) \geq \mathbb{E}\left[\delta_j u_j(y_j(\sigma_1)) | \sigma_0 = s\right] \\ Du_j(x_j)[x_j - e_j] \geq 0 \end{array}$$

With a slight abuse of notation, let's consider the operator ρ as a function of δ and define the correspondance $\Lambda : [0,1]^n \rightrightarrows X$ such that

$$\Lambda(\delta) = \{y \in Y : y \in \rho(y;\delta)\}$$

$\Lambda(\delta)$ is the set of fixed points of ρ, as a function of δ.

Proposition 3 *$\Lambda(\delta)$ is an u.h.c. correspondance*

(The proof is in the appendix)

Hence, as $\delta \to \mathbf{1}$, the outcomes of the the SSPs converge to *SSPs with immediate acceptance* with infinitely patient players, that we have proved (Theorem 2) to be Walrasian allocations. Hence, theorem 2 and propositions 1 and 3 together prove theorem 1.

5 On the related literature and some possible developments.

In perspective. Negishi (1989) distinguishes two major schools in the analysis of markets. On one hand, the french school, represented by the works of Cournot (1838) and Walras (1874), abstracts from the analysis of trading mechanisms, and models the behavior of the agents as determined by prices, both in and out of the equilibrium. A second school, associated with Jevons (1879) and Edgeworth (1881) instead studies the exchange activity focusing more explicitely on the *bargaining process* the economic agents are involved in. The difference between the Jevonsian and the Edgeworthian approaches is that in the former the trading mechanism is conceived as being based on pairwise interactions among agents, who exploit the gains from bilateral trades occurring in a *fully decentralized market*. In contrast, in the Edgeworthian view, groups of agents are allowed to interact in larger groups. Thus, although differing from the Walrasian tradition in the conceptualization of the trading mechanism, it shares with it somewhat of an idea of *centralized exchange*, rather in contrast to the Jevonsian view.

Negishi (1989) sees the *core equivalence theorem* by Debreu and Scarf (1961) as an important contribution to a research agenda that attempts to explore the connections between the Edgeworthian and Walrasian views. Other *cooperative foundations* of Walrasian Equilibrium have been studied in the literature: following Negishi, they can all be cast whitin the *Edgeworthian* tradition.

With a little delay respect to the cooperative foundations of the concept of Walrasian Equilibrium, also non-cooperative foundations have been studied: the first wave has been the literature on market games (see for instance Shubik (1972), Shapley and Shubik (1977) and Postlewaite and Schmeidler 1978). More recently, following the development of the theory of non-cooperative bargaining, models of economies with explicit strategic bargaining have studied the non-cooperative foundations of the competitive equilibrium. The classic works for *Walrasian Markets* (i.e. General Equilibrium economies*)* are by Gale (1986a,b), who studies the strategic foundations in an economy with *decentralized trade* and an *infinite number of agents*.[13] The case of finite number of agents instead has been studied by Rubinstein and Wolinsky (1990), Sabourian (2004) and Gale and Sabourian (2005), but only for the case of *Marshallian*

[13]McLennan and Sonnenschein (1991) and more recently Dagan, Serrano and Volij (2000) and Kunimoto and Serrano (2004) also studied walrasian economies with a non-atomic continuum of agents.

markets.[14]

All these contibutions that apply to an explicit model of strategic bargaining assume that trade occurs in *pairwise meetings*: again, following Negishi, we can cast these works whithin the *Jevonsian tradition.*[15]

As discussed in the introduction, this paper contributes to the literature on (strategic) bargaining foundations for the competitive equilibrium, providing a generalization to Dàvila and Eeckhout's (2007) bargaining procedure to economies with an arbitrary (finite) number of agents. It is not completely obvious though what the natural generalization should be: in the two-agents economy of Dàvila and Eekchout, the only possible pairwise meeting also coincides with the grand-coalition of the economy itself. The generalization of their work can be done in at least two different ways, reconductible to the *Edgeworthian* and the *Jevonsian* tradition, respectively.

An Edgeworthian Model. The present paper constitutes an *Edgeworthian* generalization of Dàvila and Eeckhout's (2007) model: agents strategically bargain over prices and maximum trading constraints, and if an agreement is reached, trade occurs in a *centralized* way: the non-cooperative foundation here concerns the terms at which trade occurs, not how these are affected by the possibility of trade occurring in a fully decentralized market (i.e. in a sequence of meetings). In this sense, it is quite in the spirit of Debreu and Scarf's *equivalence theorem*, just considering a model of strategic bargaining rather than applying to co-operative game theory: the focus is not on *how* trade occurs, but rather on *what* trade can occur, as the result of bargaining between strategic rational agents.

A Jevonsian Model. An alternative generalization of Dàvila and Eeckhout (2007), following the *Jevonsian tradition*, would be one in which agents are sequentially matched in pairs, and the exchange process consists of a sequence of bilateral trades. Such a setup is considered in a companion paper (Penta, 2007): in that model, agents are exogenously matched in pairs, whithin which a bargaining procedure similar to that analyzed here is used. At the moment, it has been proved that in large enough economies, if the initial allocation is close enough to the set of Pareto efficient allocations, then the walrasian equilibrium can be reached in a decentralized way, through a sequence of bilateral matchings in which agents bargain and trade. How close the endowments need to be to the Pareto set depends on the degree of substitutability of goods:

[14]That is economies with a single indivisible good, exchanged against a single divisible good (interpreted as non-fiat money).

[15]See also Kunimoto and Serrano (2001).

more substitutability allows to obtain the competitive outcome in a decentralized way for a larger set of initial conditions. The trade-off is clearly related to the possibility of strategically manipulate the terms of trade when either big trades are involved, or the marginal rates of substitution change a lot. Making trades smaller, or reducing the effect of trades on the marginal rates of substitution, reduces the extent to which the terms of trade can be manipulated. The solution of the problem for arbitrary initial endowment is a subject for future reearch.

In general, the *jevonsian* approach raises harder challenges: in the papers that focus on *marshallian markets* we have mentioned above,[16] agents' gains from trade can be exhausted in a single pairwise exchange. Hence, although decentralized, from the point of view of each agent there is only one relevant exchange: the real focus of the analysis is on prices. On the contrary, considering *walrasian economies*, the gains from trade cannot in general be exhausted in any given pairwise meeting: each agent in a decentralized economy has to go through a sequence of bilateral trades before the gains from trade are exhausted. For this reason, the environment in *Jevonsian models of general equilibrium* is significantly more complex than in *Jevonsian models of partial equilibrium* (i.e. in marshallian markets), and than in *Edgeworthian models of general equilibrium* (as in this paper): in the latter two, each agent's trade occurs in a one-shot exchange. The high non-stationarity of the environment in *Jevonsian models of general equilibrium* is the main difficulty one has to deal with in models with a finite number of agents.

Directed Search: an argument in favor of a centralized model of exchange. As mentioned above, one possible source of dissatisfaction with the model of this paper is that trade is centralized: most of the literature on the strategic bargaining foundations of general equilibrium instead has considered jevonsian models, assuming an exogenous matching function that matches agents in pairs (see Gale, 2000). As mentioned, Penta (2007) is an attempt to analyze this kind of setup.

One might instead give up an exogenous matching process altogether, and assume that agents may *direct* their search, and choose whom (or which group) to be matched with. For instance, consider a model in which agents sequentially choose a location: if nobody is there, they are the proposers in the local market formed by all the agents who choose to go to that same location, and a variant of the game above is played: if everybody agrees, trade occurs; in case of a rejection, the economy moves to the next period and a random process selects the order of

[16] Rubinstein and Wolinsky (1985, 1990), Gale (1987), Sabourian (2001), Gale and Sabourian (2005).

moves (hence the opportunities of choosing to be a proposer in a given location). Of course, if an agent is alone at a given location, he doesn't trade. It seems quite clear that in such a setup agents have an interest in coordinating on a given position:[17] the larger the market, the larger the gains from trade. Hence, for negligible transportation costs, a *directed-search* model would endogenously induce a centralized market-place, and together with a bargaining procedure such as the one analyzed in this paper, implement a Walrasian outcome.

6 Conclusions.

6.1 On the Competitive Result:

In this model agents bargain over prices and maximum trading constraints. If an agreement is reached, trade occurs in a *centralized* way. The results of Dàvila and Eeckhout for two-agents economies are generalized to economies with an arbitrary (finite) number of agents, and to different bargaining procedures.

It is proved that, as the bargaining frictions vanish, the Stationary Subgame Perfect Equilibria implement Walrasian allocations in economies with an arbitrary number of agents and commodities. To the best of my knowledge, this is the first work that provides strategic bargaining foundations in such environments.

A remarkable aspect of the result is that it doesn't require a large economy, or an approximation of that such as a *replica* economy: the result holds for any number of agents. The limit only concerns the players' discount factors, not the number of agents. This suggests that the details of the bargaining process may play a crucial role in determining the competitive outcome, independently on the number of agents in the economy.

The role that different bargaining procedures may play in providing strategic bargaining foundations of walrasian equilibrium is a thouroughly unexplored question: the existing literature in this research agenda has considered almost exclusively a specific bargaining procedure (namely, a *take-it or leave-it* exchange proposal. See for instance Gale, 2000 and references therein), and has focused mainly on the role played by the number of agents in the economy. The findings of this paper, and the sensitivity of the competitive result to different specifications of the bargaining process shown by the works of Yildiz (2005) and Dàvila and

[17] This is quite in contrast with standard models of *directed search* (see for example Burdett, Shi and Wright, 2001), in which coordination frictions arise due to an exogenous capacity constraint that induces agents not to coordinate on the same location to minimize the probability of being rationed.

Eeckhout (2007, DE hereafter), suggest that a careful analysis of alternative bargaining protocols may be of great use to this research agenda, and a promising direction for future research.

6.2 On the Robustness Result:

In this paper the robustness of the results obtained from our bargaining procedure is also addressed: the results hold for a class of games that encompasses all the bargaining procedures of alternating offers in which the proposer announces prices and maximum trading constraints, in which trade occurs upon unanimous acceptance, the continuation game in case of rejection does not depend on the actions previously taken by the players, and responses are sequential. In appendix 2 a setup in which *traders* respond simultaneousy is considered: it is shown that a refinement of the SSP (the SSP*) yields the same results as in the setup above. An SSP* considers trembles in the players' responses to an offer. This is done to rule out implausible equilibria in which agents reject offers they would like, only because someone else is rejecting the offer: if in a SSP player k is rejecting an offer at some history, all players $j \neq k$ are indifferent between rejecting and accepting that offer, because k's rejection makes j's actions at that history are all outcome-equivalent. For this reason, if players respond simultaneously, we may have for instance an equilibrium in which everybody rejects every offer: in that unilateral deviations wouldn't upset the outcome anyway. The consideration of "trembles" in the players' responses rules out this sort of equilibria based on players' coordinations on a rejection. For the sake of clarity, in the appendix it is considered only a specific case, in which the auctioneer process is deterministic. Along the lines of the main setup, the argument can be easily generalized to other processes in which agents respond simultaneously. The main message is that, in general, one can choose to model the bargaining game with responders moving simultaneously or sequentially. Whether the SSP* or the SSP has to be used as a solution concept consequently follows. The argument is reminiscent of that relying behind the refinements of the equilibria in the normal form of a dynamic game (see Selten 1975, or Van Damme, 1983).

6.3 On the Stationarity restriction:

It is important to emphasize that in general the restriction to Stationary equilibria is a strong one. Other than the simplicity of the analysis, the general argument in favor of stationary strategies is that they entail relatively simpler behavior, and would therefore be chosen by somewhat boundedly rational agents. But this argument does not seem to be convincing in general games.[18] Recently, Sabourian (2004) and Gale and Sabourian (2005) have made precise the sense in which boundedly rational agents would play stationary strategies in the equilibria of their models, which allows them to overcome the difficulties arisen in Rubinstein and Wolinsky (1990) without assuming away the use of non-stationary strategies. The present paper didn't focus on these issues of complexity, and the stationarity of strategies is simply assumed. Chatterjee and Sabourian (2000) obtain stationarity of the behavior in multi-person bargaining games through the introduction of complexity costs. I conjecture that similar notions of bounded rationality can be used in the setup of this paper to justify the restriction to stationary strategies.

[18]Indeed, the very notion of *state space* can be problematic in general. Mailath and Samuelson (2006, ch.5) make this point very clear.

Appendix 1: Proofs.

Proof of Propositions 1 and 2.

Let's consider the utility possibility set of the economy, defined as

$$\mathcal{U} := \left\{v \in \mathbb{R}^I : \exists x \in X \text{ s.t. } u(x) = v\right\}$$

An outcome (τ, η) is *stationary* if there exists a measurable subset $S^\mu \subseteq S$ and a measurable function $\mu : S \to \mathbb{R}^n$ such that: (i) $\sigma_t \notin S^\mu$ for all $t = 0, 1, ..., \tau - 1$; (ii) $\sigma_\tau \in S^\mu$; (iii)$\eta = \mu(\sigma_\tau)$. Using the latter condition, for any stationary outcome, we may define, for all s, $v^\mu(s) = \mathbb{E}[\delta^\tau \mu(\sigma_\tau) | \sigma_0 = s]$.

Let $\mathcal{V}^n$ denote the set of bounded and measurable functions $v : S \to \mathbb{R}^n$.

Lemma 1.1 If (μ, S^μ) is a stationary outcome, then v^μ is the unique function in V^n such that:

$$\begin{aligned} v^\mu(s) &= \mu(s) \text{ for all } s \in S^\mu, \text{ and} \\ v^\mu(s) &= \mathbb{E}[\delta v(\sigma_1) | \sigma_0 = s] \text{ for all } s \in S \backslash S^\mu. \end{aligned}$$

Proof: *Given (μ, S^μ), define $V : V^n \to V^n$ s.t. $\forall v \in V^n$,*

$$V(v)(s) = \begin{cases} \mu(s) & \text{if } s \in S^\mu \\ \mathbb{E}[\delta f(\sigma_1) | \sigma_0 = s] & \text{otherwise} \end{cases}$$

The lemma is established if v^μ is the unique solution in V^n to $V(v) = v$.

Step 1: $V(\cdot)$ is a contraction.

Let $\|\cdot\|$ denote the supnorm on R^n, and $\|\cdot\|_\infty$ the supnorm on V^n. Let $v, v' \in V^n$. Then, if $s \in S^\mu$, $\|V(v)(s) - V(v')(s)\| = 0$, if $s \in S \backslash S^\mu$,

$$\begin{aligned} \|V(v)(s) - V(v')(s)\| &= \|\mathbb{E}[\delta[v(\sigma_1) - v'(\sigma_1)] | \sigma_0 = s]\| \\ &\leq \beta \|\mathbb{E}[v(\sigma_1) - v'(\sigma_1)] | \sigma_0 = s\| \\ &\leq \beta \|v - v'\|_\infty \end{aligned}$$

where $\beta := \max\{\delta_i : i \in I\}$. Hence, $\exists \beta \in (0,1) : \|V(v) - V(v')\|_\infty \leq \beta \|v - v'\|_\infty$. Since V^n is a complete metric space, Banach's theorem implies that $V(\cdot)$ has a unique fixed point. We now show that v^μ defined above is indeed a fixed point of $V(\cdot)$.

Step 2: $V(v^\mu) = v^\mu$.

Define the stopping time for agreement starting at period $t = 1$ as τ_1, such that: $\sigma_{\tau_1} \in S^\mu$ and $\sigma_t \notin S^\mu$ for $t = 1, ..., \tau_1 - 1$. Then, for any $s \in S$,

$$\begin{aligned} v^\mu(s) &= \mathbb{E}[\delta^\tau v(\sigma_\tau) | \sigma_0 = s] \\ &= \mathbb{E}\left[\delta^{\tau_1 - 1} v(\sigma_{\tau_1}) | \sigma_1 = s\right] \end{aligned}$$

If $s \in S^{\mu}$, $V(v^{\mu})(s) = v^{\mu}(s)$ *simply by definition. If* $\sigma_0 = s \in S \backslash S^{\mu}$, *then* $\tau = \tau_1$, *so:*

$$\begin{aligned} V(v^{\mu})(s) &= \mathbb{E}[\delta v(\sigma_1)|\sigma_0 = s] \\ &= \mathbb{E}\left[\delta \mathbb{E}\left[\delta^{\tau_1 - 1} v(\sigma_{\tau_1})|\sigma_1\right]|\sigma_0 = s\right] \\ &= \mathbb{E}\left[\delta \mathbb{E}\left[\delta^{\tau - 1} v(\sigma_{\tau})|\sigma_1\right]|\sigma_0 = s\right] \\ &= \mathbb{E}[\delta^{\tau} v(\sigma_{\tau})|\sigma_0 = s] \\ &= v^{\mu}(s) \end{aligned}$$

QED.

From section 4, we know that a necessary condition for an allocation x to be an SPE outcome is that, for each i, $Du_i(x_i)[x_i - e_i] \geq 0$. Hence, for the analysis of the SSP, we can restrict attention to the utility space

$$\mathcal{U}^* := \left\{v \in \mathbb{R}^I : \exists x \in X \text{ s.t. } u(x) = v \text{ and } Du(x)[x - e] \geq 0\right\}$$

Any SSP determines a *stationary outcome* (τ, η) such that $\eta \in \mathcal{U}^*$. Under the mantained assumptions (A1) and (A2), the set $\mathcal{U}^*$ is compact and convex, and $0 \in \mathcal{U}^*$. Hence, for any (τ, η) such that the image of η lies in $\mathcal{U}^*$, we have that $\mathbb{E}[\delta^{\tau}\eta|\sigma_0 = s] \in \mathcal{U}^*$ for all s.

Let $\mathcal{W}$ be the set of measurable functions $w : S \to \mathcal{U}^*$. For each agent i, define the function $\varphi_i^* : \mathbb{R}^n \to \mathbb{R}$ such that, for all $d \in \mathbb{R}^n$,

$$\varphi_i^*(d) := \begin{cases} 0 & \text{if } \nexists v \in \mathcal{U}^* : v_{-i} \geq d_{-i} \\ \max\{v_i : v \in \mathcal{U}^*, \text{ and } v_{-i} \geq d_{-i}\} & \text{otherwise} \end{cases}$$

Under the mantained assumptions for each i the fuction φ_i^* is well defined and continuous.

Define the operator $E : \mathcal{W} \to \mathcal{W}$ such that for $w \in \mathcal{W}$,

$$E_i(w)(s) = \begin{cases} \max\{\varphi_i^*(\mathbb{E}[\delta w(\sigma_1)|\sigma_0 = s]); \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s]\} & \text{if } i = a(s) \\ \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s] & \text{otherwise} \end{cases}$$

Clearly, E^* is a continuous map.

Lemma 1.2: v^* is an SSP payoff if and only if $E(v^*) = v^*$.

Proof: ($\Rightarrow$) *Let* v^* *be an SSP payoff. Fix an* $s \in S$ *and let* $i = a(s)$ *If agreement does not occur at* s, *then it must be* $v^*(s) = E[\delta v^*(\sigma_1)|\sigma_0 = s]$. *Now, consider an alternative proposal* $v \in U^*$ *at some* s. *If* $v_j < E[\delta v^*(\sigma_1)|\sigma_0 = s]$ *for some* j, *the proposal is rejected; if* $v_j \geq E\left[\delta_j v_j^*(\sigma_1)|\sigma_0 = s\right]$ *for all* j, *in a SSP proposal* v *would be accepted. Hence, a payoff maximizing proposer would get* $\varphi_i^*(\mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s])$ *from any proposal that is accepted. Since he can induce a rejection, for agreement to occur it must be the case that* $\varphi_i^*(\mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s]) \geq$

$E\left[\delta_i v_i\left(\sigma_1\right)|\sigma_0 = s\right]$. *In other words, if* v^* *is an SSP payoff, it is such that*

$$v^*\left(s\right) = \begin{cases} \mathbb{E}\left[\delta v^*\left(\sigma_1\right)|\sigma_0 = s\right] & \text{if } s \in S \backslash S^\mu \\ \Big(\max\left\{\varphi^*_{a(s)}\left(\mathbb{E}\left[\delta v^*\left(\sigma_1\right)|\sigma_0 = s\right]\right); \mathbb{E}\left[\delta_{a(s)} v_{a(s)}\left(\sigma_1\right)|\sigma_0 = s\right]\right\}, \\ \qquad\qquad \mathbb{E}\left[\delta_{-a(s)} v^*_{-a(s)}\left(\sigma_1\right)|\sigma_0 = s\right]\Big) & \text{if } s \in S^\mu \end{cases}$$

which clearly satisfies $E\left(v^*\right) = v^*$.

($\Leftarrow$) *it's obvious: from the OSDP and the stationarity of the players' strategy, any deviation that induces a rejection when an acceptance is due would simply yield the continuation payoff. From the definition of* E*, in a FP the continuation is never strictly greater than the value at any given* s*. Hence, a FP of* E *would be sustained as a SSP of the game.* **QED.**

Remark: By construction, if v^* is a fixed point of E, then $v^* \in bd\left(\mathcal{U}^*\right)$

Proposition 1: In any SSP, agreement occurs with no delay

Proof: *From the strict convexity of* U^**, if* $\delta << 1$*, for any outcome* (τ, η)*, we have that* $E\left[\delta^\tau \eta|\sigma_0 = s\right] \in U^*$ *for all* s*. Now, let* v^* *be an SSP payoff, and suppose that there exists a state* s *in which agreement is not reached. Then,* $v^*\left(s\right) = E\left[\delta^\tau \eta|\sigma_0 = s\right] \in int\left(\mathcal{U}^*\right)$*. But this is inconsistent with* v^* *being a fixed point of* E *(see remark 1).***QED.**

Lemma 1.3: $\langle \mathcal{W}, \|\cdot\|_\infty \rangle$ is a compact, convex, complete metric space.

Proof: *Convexity: let* $w, w' \in W$*,* $w \neq w'$*. For* $\alpha \in (0,1)$*, let* $w^\alpha\left(s\right) = \alpha w\left(s\right) + \left(1-\alpha\right) w'\left(s\right)$ *for all* s*. Since* U^* *is convex, clearly* $w^\alpha : S \to U^*$*. It's clearly measurable, hence* $w^\alpha \in W$*.*

Compactness: for any sequence $\left\{w^\nu\right\}_{\nu\in\mathbb{N}} \subseteq W$*, for each* $s \in S$*,* $\left\{w^\nu\left(s\right)\right\}_{\nu\in\mathbb{N}}$ *is a sequence in* U^**, hence with a limit* $\bar{w}\left(s\right) \in U^*$*. Hence,* $\left\{w^\nu\right\}_{\nu\in\mathbb{N}} \to \bar{w}$ *pointwise. Hence in the supnorm. Since it's a compact subest of a complete metric space, it's also complete.***QED.**

Proposition 2: there exists an SSP.

Proof: *Since* E *is a continuous self-map and* $\langle \mathcal{W}, \|\cdot\|_\infty \rangle$ *is a non-empty, compact, convex, subset of a linear metric space, the existence of a fixed point follows from Schauder conjecture, proved by Cauty (2001). (see also Ok (2007), p.626).***QED.**

Proof of Theorem 2.

Theorem 2: Under the mantained assumptions R, if $\delta_i = 1$ for every $i \in I$, the outcome $\left(x^*_l\right)_{l\in I}$ of an *SSP with immediate acceptance* is a Walrasian Allocation.

The proof of the theorem is completed by the next three lemmata.

Lemma 2.1: Under property 1, if $\delta = 1$, in a *SSP with immediate acceptance*, $(x_l^s)_{l\in I} = (x_l^*)_{l\in I}$ for all $s \in S$.

Proof: *Suppose not, i.e. for* $s \neq s'$, $(x_l^s)_{l\in I} \neq \left(x_l^{s'}\right)_{l\in I}$.

Notice that for each s*, the constraints* $u_j(x_j) \geq \max\left\{u_j\left(x_j^{s'}\right) : s' \in S\right\}$ *must be binding for every* j *in equilibrium, which clerly implies that in equilibrium, for each* j, $u_j\left(x_j^s\right) = u_j\left(x_j^{s'}\right)$ *for all* $s,s' \in S$.

From strict concavity of preferences, if $u_j\left(x_j^s\right) = u_j\left(x_j^{s'}\right)$ *and* $x_j^s \neq x_j^{s'}$*, it must be the case that* $(x_l^s)_{l\in I}$ *is inefficient, hence pairwise inefficient:* $\exists k, j$ *for whom there is a transfer* z *from* j *to* k *that would make both of them better off:*

$$u_k\left(x_k^s - z\right) > u_k\left(x_k^s\right)$$
$$u_j\left(x_j^s + z\right) > u_j\left(x_j^s\right)$$

Hence, if $k = a(s)$*, (i.e. he is making the offer* x^s*), provided that he would have a profitable deviation from the equilibrium offering* $\left(x_j^s + z\right)$ *to the trader* j*. This offer would be accepted, and* k *would be better off.*

If $k \neq a(s)$ *(i.e.* k *is one of the responders), he would have a profitable deviation by rejecting all the offers until he becomes the proposer, and then assigning* $x_{a(s)} = x_{a(s)}^s$*, and* $x_j = x_j^s + z$*, as above. Hence, if we are in an equilibrium,* $(x_l^s)_{l\in I}$ *cannot be inefficient. And if it's not inefficient, it must be that* $(x_l^s)_{l\in I} = (x_l^*)_{l\in I}$ *for all* $s \in S$. **QED.**

Lemma 2.2: at $(x_l^*)_{l\in I}$, $Du_i(x_i^*)[x_i^* - e_i] = 0$ for each i.

Proof: *Given lemma 2.1, in a SSP with immediate acceptance it must be the case that, for each* i*, and* $j \neq k \neq i \neq j$, $(x_k^*)_{k\neq i}$ *solves:*

$$(x_k^*)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(R - \sum_{k\neq i} x_k\right)$$
$$\text{s.t: } \left\{\begin{array}{c} Du_k(x_k)[x_k - e_k] \geq 0 \\ u_k(x_k) \geq u_k(x_i^*) \end{array}\right\}_{k\neq i}$$

Now, suppose $\exists j : Du_j\left(x_j^*\right)\left[x_j^* - e_j\right] > 0$*, then* $(x_k^*)_{k\neq i}$ *would also solve*

$$(x_k^*)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(R - \sum_{k\neq i} x_k\right)$$
$$\text{s.t.} \quad u_j(x_j) \geq u_j\left(x_j^*\right)$$
$$\left\{\begin{array}{c} Du_k(x_k)[x_k - e_k] \geq 0 \\ u_k(x_k) \geq u_k(x_i^*) \end{array}\right\}_{k\neq i,j}$$

Notice that

$$[x_i^* - e_i] = \left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right]$$

From efficiency, nder the mantained assumptions of the smoothe general equilibrium model, we know that $\forall k \neq j,\ \exists \gamma_{jk} > 0 : Du_j\left(x_j^*\right) = \gamma_{jk} Du_k\left(x_k^*\right)$. *Let* $\gamma_{ji} = \min\left\{\gamma_{jk} : k \neq j\right\}$

Adding up the constraints for j's optimization problem

$$\begin{aligned}
0 &\leq \sum_{k\neq j} Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] \\
&= Du_j\left(x_j^*\right)\left[\sum_{k\neq j} \frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right] + \sum_{k\neq i,j} \frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[e_j - x_j^*\right] + \sum_{k\neq i,k}\left(\frac{1}{\gamma_{jk}} - \frac{1}{\gamma_{ji}}\right)\left[x_k^* - e_k\right]\right] \\
&= \frac{1}{\gamma_{ji}} Du_j\left(x_j^*\right)\left[e_j - x_j^*\right] + \sum_{k\neq i,j}\left(1 - \frac{\gamma_{jk}}{\gamma_{ji}}\right) Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] < 0
\end{aligned}$$

*which yields the desired contradiction. (The absurd hypothesis implies that the first term is negative, while the terms in the summation are negative by construction. The contradiction follows immediately).***QED.**

Lemma 2.3: $\left(x_l^*\right)_{l\in I}$ is a Walrasian allocation.

Proof:*From the previous lemma, it sufficies to set* $p^* = Du_1\left(x_1^*\right)$, *to have that,* $\forall i$,

$$\begin{aligned}
x_i^* &\in \arg\max_{x_i} u_i\left(x_i\right) \\
p^* x_i &= p^* e_i
\end{aligned}$$

*Lemma 3 concludes the proof of theorem 1.***QED.**

Proof of Proposition 3.

Proposition 3: $\Lambda(\delta)$ is an u.h.c. correspondance

Proof: *Since (from Berge's Maximum Theorem)* ρ *is u.h.c. in* δ, *also* $\Lambda(\delta)$ *is u.h.c. (this follows from lemma A3 in DE).*

Appendix 2: simultaneous responses

In this section we consider a particular specification of the process selecting the auctioneer, in which $\forall t = 1, 2, \ldots\ \pi_{t(\text{mod}(n))}(t) = 1$. In words: agent 1 makes an offer in the first period, agent 2 in the second, and so on and so forth, with agent 1 offering again at periods $t = kn + 1$, for $k \in \mathbb{N}$.

A *stationary subgame perfect equilibrium (SSP) with immediate acceptance* is characterized by a tuple $(p^i, q^i)_{i \in I_n}$ such that in every subgame where i is called to make an offer, i offers (p^i, q^i) and $j \neq i$ accept and choose the excess demand

$$\begin{aligned} x_j\left(p^i, q^i\right) &\in \arg\max_{x_j} u_j\left(x_j\right) \\ \text{s.t.}\quad & p^i\left[x_j - e_j\right] \leq 0 \\ & \left[x_j - e_j\right] \leq q^i_j \end{aligned}$$

Similarly to above, an *SSP with immediate acceptance* can be characterized by allocations offers $\left(\left(x^i_j\right)_{j \in I_n}\right)_{i \in I_n}$, where $\left(x^i_j\right)_{j \in I_n}$ is the allocation offered by i when he is selected as the auctioneer, such that:[19]

$$\begin{aligned} & \forall i \in I_n : \\ \left(x^i_l\right)_{l \in I_n} &\in \arg\max_{(x_l)_{l \in I_n}} u_i\left(x_i\right) \\ \text{s.t.:}\quad & \left\{ \begin{array}{l} Du_{i+s}\left(x_{1+1}\right)\left[x_{i+s} - e_{i+s}\right] \geq 0 \\ u_{i+s}\left(x_{i+s}\right) \geq \delta^{s-1}_{i+s} u_{i+s}\left(x^{i+s}_{i+s}\right) \end{array} \right\}_{s \in \mathbb{Z}_n \backslash \{1(\text{mod}(n)\}} \\ & R = \sum_{l \in I_n} x_l \end{aligned}$$

Infinitely Patient Players

Theorem 4 *If $\delta_i = 1$ for every $i \in I_n$, the outcome $(x^*_l)_{l \in I_n}$ of an* ***SSP with immediate acceptance*** *is a Walrasian Allocation.*
Proof. *(The theorem follows from the next three lemmas).* ■

Notice that if $\delta = 1$ for every player, in a *SSP with immediate accep-*

[19] $\mathbb{Z}_n$ is the ring $\mathbb{N}_{\text{mod}(n)}$.

tance it must be the case that

$$\begin{aligned}
&\forall i \in I_n : \\
&(x_l^*)_{l\in I_n} \in \arg\max_{(x_l)_{l\in I_n}} u_i(x_i) \\
&\text{s.t.:} \qquad \left\{ \begin{array}{c} Du_{i+s}(x_{1+1})[x_{i+s} - e_{i+s}] \geq 0 \\ u_{i+s}(x_{i+s}) \geq u_{i+s}\left(x_{i+s}^{i+s}\right) \end{array} \right\}_{s\in\mathbb{Z}_n\setminus\{1(\text{mod}(n)\}} \\
&R = \sum_{l\in I_n} x_l
\end{aligned}$$

Lemma 1 *If $\delta = 1$ for every player, in a SSP with immediate acceptance, $(x_l^i)_{l\in I_n} = (x_l^*)_{l\in I_n}$ for all $i \in I_n$.*

Proof. *Suppose not, i.e. for $i \neq j$, $(x_l^i)_{l\in I_n} \neq \left(x_l^j\right)_{l\in I_n}$.*

Notice that for each i, the constraints $u_{i+s}(x_{i+s}) \geq u_{i+s}\left(x_{i+s}^{i+s}\right)$ must be binding for every s in equilibrium. Hence, for each $k \in I_n$, $u_k(x_k^i) = u_k\left(x_k^k\right)$.

From strict concavity of preferences, if $u_k(x_k^i) = u_k\left(x_k^j\right) = u_k\left(x_k^k\right)$ and $x_k^i \neq x_k^j$, it must be the case that $(x_l^i)_{l\in I_n}$ is inefficient: $\exists j, k$ for whom there is a transfer z from j to k that would make both of them better off:

$$\begin{aligned}
u_k\left(x_k^i - z\right) > u_k\left(x_k^i\right) \\
u_j\left(x_j^i + z\right) > u_j\left(x_j^i\right)
\end{aligned}$$

Hence, if $k = i$, (i.e. he is making the offer x^i), provided that he would have a profitable deviation from the equilibrium offering $\left(x_j^k + z\right)$ to j. It would be accepted, and k would be better off.

If $k \neq i$ (i.e. k is one of the responders), he would have a profitable deviation by rejecting all the offers until he becomes the proposer, and then assigning $x_i^k = x_i^i$, and $x_j^k = x_i^i + z$, as above. Hence, if we are in an equilibrium, $(x_l^i)_{l\in I_n}$ cannot be inefficient. And if it's not inefficient, it must be that $(x_l^i)_{l\in I_n} = (x_l^*)_{l\in I_n}$ for all $i \in I_n$. ■

Notice that in the proof of the previous lemma it's been argued that the resulting allocation is *efficient*.

Lemma 2 *at $(x_l^*)_{l\in I_n}$, $Du_i(x_i^*)[x_i^* - e_i] = 0$ for each i.*

Proof. *Given lemma1, in a SSP with immediate acceptance it must be the case that, for each i, and $j \neq k \neq i \neq j$, $(x_k^*)_{k\neq i}$ must solve:*

$$\begin{aligned}
&(x_k^*)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(R - \sum_{k\neq i} x_k\right) \\
&\textit{s.t:} \left\{ \begin{array}{c} Du_{i+s}(x_{1+1})[x_{i+s} - e_{i+s}] \geq 0 \\ u_{i+s}(x_{i+s}) \geq u_{i+s}\left(x_{i+s}^*\right) \end{array} \right\}_{s\in\mathbb{Z}_n\setminus\{1(\text{mod}(n)\}}
\end{aligned}$$

Now, suppose $\exists j : Du_j\left(x_j^*\right)\left[x_j^* - e_j\right] > 0$*, then* $(x_k^*)_{k\neq i}$ *would also solve*

$$(x_k^*)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(R - \sum_{k\neq i} x_k\right)$$
$$s.t. \qquad u_j\left(x_j\right) \geq u_j\left(x_j^*\right)$$
$$\left\{\begin{array}{c} Du_{i+s}\left(x_{1+1}\right)\left[x_{i+s} - e_{i+s}\right] \geq 0 \\ u_{i+s}\left(x_{i+s}\right) \geq u_{i+s}\left(x_{i+s}^*\right) \end{array}\right\}_{\substack{s\in\mathbb{Z}_n\setminus\{1(\mathrm{mod}(n)\},\\ s\neq j-i}}$$

Notice that

$$\left[x_i^* - e_i\right] = \left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right]$$

From efficiency, under the mantained assumption of the smoth general equilibrium model, we know that $\forall k \neq j$, $\exists\gamma_{jk} > 0 : Du_j\left(x_j^*\right) = \gamma_{jk} Du_k\left(x_k^*\right)$. *Let* $\gamma_{ji} = \min\left\{\gamma_{jk} : k \neq j\right\}$

Adding up the constraints for j's optimization problem

$$\begin{aligned}
0 &\leq \sum_{k\neq j} Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] \\
&= Du_j\left(x_j^*\right)\left[\sum_{k\neq j}\frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right] + \sum_{k\neq i,j}\frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[e_j - x_j^*\right] + \sum_{k\neq i,k}\left(\frac{1}{\gamma_{jk}} - \frac{1}{\gamma_{ji}}\right)\left[x_k^* - e_k\right]\right] \\
&= \frac{1}{\gamma_{ji}} Du_j\left(x_j^*\right)\left[e_j - x_j^*\right] + \sum_{k\neq i,j}\left(1 - \frac{\gamma_{jk}}{\gamma_{ji}}\right) Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] < 0
\end{aligned}$$

which yields the desired contradiction. (The absurd hypothesis implies that the first term is negative, while the terms in the summation are negative by construction. The contradiction follows). ■

Lemma 3 $(x_l^*)_{l\in I_n}$ *is a Walrasian allocation.*
Proof. *From the previous lemma, it sufficies to set* $p^* = Du_1\left(x_1^*\right)$*, to have that,* $\forall i$*,*

$$x_i^* \in \arg\max_{x_i} u_i\left(x_i\right)$$
$$p^* x_i = p^* e_i$$

■

Lemma 3 concludes the proof of the theorem.

We prove next that, if agents are infinitely patient, every Walrasian allocation can be sustained as an outcome of an SSP with immediate acceptance:

Theorem 5 *Let* $\left(p^*, (x_l^*)_{l\in I_n}\right)$ *be a WE of the economy* $\mathcal{E}$*. Then, if* $\delta_i = 1$ *for all* $i \in I_n$*, there is an SSP with immediate acceptance of the game* $\Gamma(\mathcal{E})$ *with outcome* $(x_l^*)_{l\in I_n}$*.*

Proof. *Given the WE* $\left(p^*, (x_l^*)_{l\in I_n}\right)$*, consider the following strategy profile:* (p, q) *such that* $\forall i \in I_n$*,*

- *when* i *makes a proposal he offers* (p^i, q^i) *such that:*

  $p^i = p^*$*;* $\left(q_j^i\right)_{j\neq i}$ *are slack, i.e.* $q_j^i \geq z_j(p^*)$*, where*

$$z_j(p^*) = \arg\max_{z_j\in\mathbb{R}^C} u_j(e_j + z_j)$$
$$s.t: \quad p^* z_j \leq 0$$

- *when* i *is responding, he accepts any offer* (p', q') *such that* $u_i(x_i(p', q')) \geq u_i(x_i^*)$*.*

The outcome of this strategy profile, starting from any subgame in which some $j \in I_n$ *has to make an offer, is clearly* $(x_l^*)_{l\in I_n}$*. Now we check that it is an SSP indeed.*

If i *deviates,*[20] *he may induce one of the following types of outcomes.*

(1) Agreement is never reached: this outcome is clearly not preferred to $(x_l^*)_{l\in I_n}$ *by* i*.*

(2) At a later stage in the game, i *offers* (p', q') *such that* $\forall j \neq i$*,* $u_j(x_j(p', q')) \geq u_j\left(x_j^*\right)$ *and it's accepted: Since* $(x_l^*)_{l\in I_n}$ *is efficient, and* $u_j(x_j(p', q')) \geq u_j\left(x_j^*\right)$ *for all* $j \neq i$*, it cannot be that* $u_i\left(R - \sum_{j\neq i} x_j(p', q')\right) > u_i(x_i^*)$*. So that this outcome cannot be preferred to* $(x_l^*)_{l\in I_n}$ *by* i*.*

(3) At a later stage in the game, i *accepts the offer* (p^*, q^j) *made by* $j \neq i$*, which yields the same outcome* $(x_l^*)_{l\in I_n}$*: therefore, deviations to this outcomes are not profitable either.* ■

[20] Notice that with no discounting the o*ne-shot deviation principle* doesn't apply. Hence we consider all the possible deviations.

Impatient Players

Definition 4 *Given the game* $\Gamma(\mathcal{E}) = \left\langle I_n, (F_i, v_i)_{i\in I_n}\right\rangle$*, define the strategy sets* $(F_i^*)_{i\in I_n}$*:*

$$\forall i \in I_n,\ \forall f_i = (\alpha_i, \rho_i) \in F_i^*,$$
$$\textit{for each } h \in H\backslash H_i,\ \forall r \in \{Y, N\},\ \rho(r|h) > 0$$

A strategy profile f^* *is an SSP* equilibrium* $\Gamma(\mathcal{E})$ *if and only if it is an SSP of* $\Gamma(\mathcal{E})$ *and there exists a sequence* $\{f^\nu\}_{\nu=1}^\infty$ *such that:*

1. $\forall \nu \in \mathbb{N}, f^\nu \in \times_{i\in I_n} F_i^*$.
2. $f^* = \lim_{\nu\to\infty} f^\nu$.
3. $\forall i, \forall \nu, f_i^* \in \arg\max_{f_i\in F_i} v_i\left(f_i, f_{-i}^\nu\right)$.

(We refer to the elements of sequence $\{f^\nu\}_{\nu=1}^\infty$ *as a "perturbations of* f^**").*

An SSP* considers trembles in the players' responses to an offer. This is done to rule out implausible equilibria in which agents reject offers they would like, only because somebody else is rejecting the offer. The consideration of "trembles" in the players' responses rules out these sort of equilibria based on players' coordinations on a rejection. As the next lemma shows, the consideration of SSP* restricts the agent responses to being somehow "sincere", i.e. disregarding issues of coordination with other agents rejecting at some history

Lemma 4 *Let* f^* *be an SSP* in which agreement occurs at period* $t > 1$*. Let* $\delta_j < 1$*. Then at* $(t-1)$ *player* j *accepts any offer* x' *such that*

$$u_j\left(x_j'\right) \geq \delta_j u_j\left(x_j^{a(t)}\right)$$

Proof. *Let* $V_j^{(t+s)}$ *denote* j*'s continuation value in the perturbed game from the beginning of period* $(t+s)$*. For* f^* *to be an SSP in which agreement occurs at* t*, it's necessarily the case that* $\forall j$*,* $u_j\left(x_j^{a(t)}\right) \geq \delta_j V_j^{(t+1)}$ *(otherwise it would be optimal to reject* $a(t)$*'s offer). For any perturbation of a SSP the probability that everybody else (i.e. all* $i \neq j, a^{(t)}$*) accept any given offer at any given period is positive. Then, if* j *accepts at* $(t-1)$ *and at* t*, agreement occurs at* $(t-1)$ *with probability* $\gamma_1 \in (0,1)$*, in period* t *with probability* $(1-\gamma_1)\gamma_2 \in (0,1)$*, and later with probability* $(1-\gamma_1)(1-\gamma_2)$*. If instead* j *rejects at* $(t-1)$*, an agreement* *occurs with probability γ_2 at t, and $(1-\gamma_2)$ later on. Then, the induced payoffs are:*

If j accepts
$$\gamma_1 u_j\left(x'_j\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_1)(1-\gamma_2)\delta_j^2 V_j$$
If j rejects
$$\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_2)\delta_j^2 V_j$$

Notice that, under $u_j\left(x_j^{a(t)}\right) \geq \delta_j V_j^{(t+1)}$ and $u_j\left(x'_j\right) \geq \delta_j\left(u_j x_j^{a(t)}\right)$ we obtain:

$$\begin{aligned}
&\gamma_1 u_j\left(x'_j\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_1)(1-\gamma_2)\delta_j^2 V_j \\
&= \gamma_1 u_j\left(x'_j\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_2)\delta_j^2 V_j + (\gamma_1\gamma_2-\gamma_1)\delta_j^2 V_j \\
&\geq \gamma_1 u_j\left(x_j^{a(t)}\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_2)\delta_j^2 V_j + (\gamma_1\gamma_2-\gamma_1)\delta_j^2 V_j \\
&> \gamma_1\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + \gamma_1(\gamma_2-1)\delta_j^2 V_j + (1-\gamma_2)\delta_j^2 V_j \\
&\geq \gamma_1\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_1)\gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + \gamma_1(\gamma_2-1)\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_2)\delta_j^2 V_j \\
&= \gamma_2\delta_j u_j\left(x_j^{a(t)}\right) + (1-\gamma_2)\delta_j^2 V_j
\end{aligned}$$

Which means that accepting is optimal. ■

Proposition 4 *If $\delta_i < 1$ for each $i \in I_n$, in any SSP* agreements occurs with no delay.*
Proof. *Let $(p^i, q^i)_{i\in I_n}$ be a candidate SSP* strategy profile, in which agreement occurs at period $t \geq 1$. Let J be the set of players that have rejected $a(t-1)$'s offer $\left(p^{a(t-1)}, q^{a(t-1)}\right)$ at period t.*

From the previous lemma, for $j \in J$ to be optimal to reject at t it must be the case that:

$$\forall j \in J:$$
$$\delta_j u_j\left(x_j^{a(t)}\right) > u_j\left(x_j^{a(t-1)}\right)$$

On the other hand, in a SSP they would have accepted any offer $\tilde{x}$ made by $a(t-1)$ s.t. $u_j(\tilde{x}) \geq \delta_j u_j\left(x_j^{a(t)}\right)$. Hence, for $a(t-1)$ not to make* *such an offer it must be the case that he himself rather preferred to have his offer rejected, and get* $x_{a(t-1)}^{a(t)}$ *next period. That is:*

$$\delta_{a(t_{-1})}u_{a(t-1)}\left(x_{a(t-1)}^{a(t)}\right) \geq u_{a(t-1)}\left(\tilde{x}_{a(t-1)}\right)$$
$$\text{for all } \tilde{x} : \forall j \neq a\left(t-1\right), \ u_j\left(\tilde{x}\right) \geq \delta_j u_j\left(x_j^{a(t)}\right)$$

In other words, $\delta_{a(t_{-1})}u_{a(t-1)}\left(x_{a(t-1)}^{a(t)}\right) \geq v^*$ *where*

$$v^* = \max_{(x_i)_{i\in I_n}} u_{a(t-1)}\left(x_{a(t-1)}\right)$$
$$s.t. \left\{\begin{array}{l} Du_k\left(x_k\right)\left[x_k - e_k\right] \geq 0 \\ u_k\left(x_k\right) \geq \delta_k u_k\left(x_k^{a(t)}\right) \end{array}\right\}_{k\neq a(t-1)}$$
$$R = \sum_{i\in I_n} x_i$$

Now, since $\delta_{a(t_{-1})} < 1$, $u_{a(t-1)}\left(x_{a(t-1)}^{a(t)}\right) > \delta_{a(t_{-1})}u_{a(t-1)}\left(x_{a(t-1)}^{a(t)}\right) \geq v^*$, *it must be the case that the allocation* $x^{a(t)}$ *(i.e. the one that would be offered in the next period) doesn't satisfy the constraints in the agent* $a\left(t-1\right)$*'s optimization problem above. That is, either* $\exists k : u_k\left(x_k^{a(t)}\right) < \delta_k u_k\left(x_k^{a(t)}\right)$ *(which is clearly impossible), or for some* k

$$Du_k\left(x_k^{a(t)}\right)\left[x_k^{a(t)} - e_k\right] < 0$$

Now, if such $k \neq a\left(t\right)$, *we have a contradiction to* $x^{a(t)}$ *actually solving the optimization for* $a\left(t\right)$, *that's still constrained to* $Du_k\left(x_k^{a(t)}\right)\left[x_k^{a(t)} - e_k\right] \geq 0$ *for all* $k \neq a\left(t\right)$. *If instead* $k = a\left(t\right)$ *and* $Du_k\left(x_k^{a(t)}\right)\left[x_k^{a(t)} - e_k\right] < 0$, *then* $x^{a(t)}$ *is not the solution to* $a\left(t\right)$*'s problem. Hence, in an SSP* there is no rejection on the equilibrium path.* ■

Proposition 5 *With impatient players, if for each* i, u_i *is* strongly concave,[21] *there exists an SSP* with immediate acceptance.*

[21] In the sense that

$$\det\left\{2D^2u_i\left(x\right) + \left[\sum_{k=1}^{n} D_{ikj}u_i\left(x\right)\left[x_k - e_{i,k}\right]\right]_{ij}\right\}$$

does not change sign. This guarantees that the offer curves has no inflexion points, and hece the constrained domain delimited by the offer curve is convex. This condition is satisfied whenever the substitution effect dominates the income effect.

Proof. *Consider the correspondance* $\Phi : X^n \rightrightarrows X^n$ *defined as*

$$\forall \tilde{x} = \left(\left(\tilde{x}_j^1 \right)_{j \in I_n}, ..., \left(\tilde{x}_j^n \right)_{j \in I_n} \right) \in X^n,$$

$$\Phi\left(\tilde{x}\right) \, : \, = \times_{i \in I_n} \arg\max u_i\left(x_i\right)$$

$$s.t. \qquad \left\{ \begin{array}{c} Du_{i+s}\left(x_{i+1}\right)\left[x_{i+s} - e_{i+s}\right] \geq 0 \\ u_{i+s}\left(x_{i+s}\right) \geq \delta_{i+s} u_{i+s}\left(\tilde{x}_{i+s}^{i+1}\right) \end{array} \right\}_{s \in \mathbb{Z}_n \setminus \{1 (\mathrm{mod}(n)\}}$$

Notice that a fixed point of Φ *is an SSP* with immediate acceptance. It's immediate to check that the conditions for Kakutani's Fixed Point Theorem are satisfied for the correspondance* Φ*, and hence an SSP* exists.* ■

Theorem 6 *As* $\delta \to 1$*, the SSP* allocation is a Walrasian Allocation.*
Proof. *Consider the correspondance* $\tilde{\Phi} : [0,1]^n \times X^n \rightrightarrows X^n$ *defined as*

$$\forall \left(\delta_j\right)_{j \in I_n}, \, \forall \tilde{x} = \left(\left(\tilde{x}_j^1 \right)_{j \in I_n}, ..., \left(\tilde{x}_j^1 \right)_{j \in I_n} \right) \in X^n,$$

$$\tilde{\Phi}\left(\tilde{x}; \left(\delta_j\right)_{j \in I_n}\right) = \Phi\left(\tilde{x}\right) := \times_{i \in I_n}$$

From Berge's Maximum Theorem, $\tilde{\Phi}$ *is clearly upper hemi-continuous in* $\left(\delta_j\right)_{j \in I_n}$*.Let* $\Gamma : [0,1]^n \rightrightarrows X^n$ *be such that*

$$\Gamma\left(\left(\delta_j\right)_{j \in I_n}\right) = \left\{ \tilde{x} \in X^n : \tilde{x} \in \Phi\left(\tilde{x}; \left(\delta_j\right)_{j \in I_n}\right) \right\}$$

Γ *is u.h.c. as well*[22]*, so that for* $\left(\delta_j\right)_{j \in I_n} \to \mathbf{1}$*, the outcome of the SSP* with impatient players converges to the outcome of the* SSP with immediate acceptance *with patient players, that is a walrasian allocation, from theorem 1.* ■

Hence, summing up:

Any SSP* outcome of the market game with impatient players converges to a walrasian allocation as the players become infinitely patient.

Comment:

The notion of SSP* was not needed in the main setup: if traders respond sequentially, the notion of SSP is enough to rule out this sort of equilibria. This claim is based on the following observation: in a SSP of the game in which traders respond sequentially, if an offer x' is rejected at $(t-1)$, there exist at least one trader j such that $u_j\left(x_j'\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$: if not, the last responder would accept in the subgame in which everybody has previously accepted. The previous responder, in that subtree,

[22] See lemma A3 in Dàvila and Eeckhout (2006).

would anticipate this and hence accept. By backward induction then, if $u_j\left(x'_j\right) > \delta_j u_j\left(x_j^{a(t)}\right)$ for eveybody, agreement would occur. Notice though that in a SSP it is not true that traders reject if and only if $u_j\left(x'_j\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$: it must be that $u_j\left(x'_j\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$ for the last player who rejects. All the players responding before him, are still indifferent among all their actions, and therefore may reject even if $u_j\left(x'_j\right) > \delta_j u_j\left(x_j^{a(t)}\right)$. Nonetheless, the fact that, upon rejection, there exists at least one trader j such that $u_j\left(x'_j\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$, is sufficient to prove that agreement occurs with no delay in a SSP of the game with sequential responses.In general, one can choose to model the bargaining game with responders moving simultaneously or sequentially. Whether the SSP* or the SSP has to be used a solution concept consequently follows.[23] The argument is reminiscent of that relying behind the refinements of the equilibria in the normal form of a dynamic game (see Selten (1975), or Van Damme (1983)).

[23]Notice that the SSP* in the simultaneous responses game is stronger than the SSP in the sequential responses game, in the sense that the former implies that every j rejects an offer x' at $(t-1)$ if and only if $u_j\left(x'_j\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$, while the latter implies only that, upon rejection, there exists at least one trader j such that $u_j\left(x'_j\right) \leq \delta_j u_j\left(x_j^{a(t)}\right)$.

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