Cost Functions of Incomplete Markets*

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Abstract

Incomplete markets without arbitrage opportunities are characterized by the existence of multiple risk neutral probabilities. A cost function describes the minimum value necessary for superhedging any claim and a well-known property states that a cost function is necessarily the maximum of expectations with respect to a given family of probabilities. Hence cost functions must satisfy conditions obtained by some characterizations existing in the literature (*e.g.*, Huber (1981), Gilboa and Schmeidler (1989), Chateauneuf (1991)). However, these properties are not sufficient for the characterizations of cost functions.

Our main result gives a full characterization of cost functions of frictionless financial market without arbitrage opportunities in the two periods framework. We obtain some criteria that allows to know if a given function is actually a cost function. Also, we provide some results that clarify the understanding about the interdependence between the market structure and the functional form of the cost function. For example, we obtain that a cost function is a Choquet integral if and only if the subspace of attainable claims is a Riesz space in which case the corresponding capacity is a very particular concave one.

1 Introduction

Since the Arrow's *Role of Securities* seminal paper the theory of equilibrium for markets in which both spot commodities and securities are traded is the fundamental scope for the study of basic problems of economic theory such as

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equilibrium existence[1], asset pricing and soo on. The general equilibrium model assumes that the price of assets satisfies *equilibrium conditions* in a setting where many agents demand assets profiles in acordance with their preferences and their endowments. It provide the main elements for the study of financial market models given by the set of basic securities and the respective price system. A fundamental result says that for an economy with financial markets satisfying mild conditions, at equilibrium, financial markets must not offer arbitrage opportunities for any agent[2]. For a two period economy it implies the impossibility at equilibrium to realize positive net financial returns in the second period without spending at the initial period some amount of money on the asset market.

The *principle of no-arbitrage* can be viewed as the central principle of modern finance because it is the key for the determination of the *value* of the assets. As is well-known, no-arbitrage principle and the assumption of complete markets[3] enforce linear princing rule: the cost of replication of any asset is given by the mathematical expectation of his payoffs under the unique risk neutral probability obtained by no-arbitrage principle. On the other hand, market incompleteness says that not all securities can be replicated by feasible portfolios on the market. Equivalently, while in a complete market every asset can be hedged perfectly, in the incomplete market case it is possible to stay on the safe side for many cases only by *superhedging strategies*, *i.e.*, a portfolio strategy which generates payoffs across the states that is at least as large as the underlying contingent claim. A *sine qua non* condition for incomplete markets without arbitrage opportunities are the *existence of multiple risk neutral probabilities.*

A *cost function* describes the minimum value necessary for the replication or a superreplication of any contingent claim, and a corresponding strategy is referred to as a *minimum-cost superhedging strategy.* An essential fact for the determination of cost functions is that the standard linear approach fails for any non attainable claim[4]. In this sense, a very known result says that the set

[1] Arrow (1953) proposed this approuch for the presence of a complete securities markets and used the results from Arrow and Debreu (1954) as well as McKenzie (1954) for the existence of equilibrium. However, as is widely accept, incomplete markets is a more natural and intuitive hypothesis (Magill and Quinzii (1996) and Magill and Shafer (1991) are basics references for general equilibrium analysis of incomplete markets, where its is possible to find the list of main contributions for incomplete markets theory. Föllmer and Shied (2004) provided a treatment of basics results in incomplete markets following the lines of finance theory).

[2] See, for instance, Florenzano (1999), page 18.

[3] Recall that a financial market is complete if the trading of basic assets reproduce any financial payoff, otherwise we have the incompleteness of financial market.

[4] Some results show that this medothological problem is typical for some importants classes of assets, for example, a well known result from the work of Ross (1976) says that whenever the payoff of every *call* or *put option* can be replicated, the securities market must be complete. Also, Aliprantis and Tourky (2002) showed that if the number of securities is less than half the number of states of the world, then generically we have the absence of perfect replication of *any* option. Hence, the approach of finding the value of an option by reference to the prices of the primitive securities breaks down for any option. In another way, Baptista (2007) showed that (generically) if every *risk binary contingent claim* is non attainable then every option is non attainable.

of risk-neutral probabilities plays an important role for the determination of a cost function: in fact, the cost of any contingent claim can be determinaded by his maximum expected value with respect to all risk neutral probability. Hence cost functions satisfies conditions obtained by some characterizations existing in the literature (*e.g.*, Huber (1981), Gilboa and Schmeidler (1989), Chateauneuf (1991)). However, these properties are not sufficient for the characterizations of cost functions, for instance, as we will see in the Example 47 the epsilon-contaminated functions never can be a cost function.

The main result of this paper gives a full characterization of cost functions of frictionless financial market without arbitrage opportunities in the two periods framework. We obtain some criteria that allows to know if a given function is actually a cost function. Also, we provide some results that clarify the understanding about the interdependence between the market structure and the functional form of the cost function. For example, we obtain that a cost function is a Choquet integral if and only if the subspace of attanaible claims is a Riesz space in which case the corresponding capacity is a very particular concave one.

2 Preliminaries

Let $S = \{s_1, ..., s_n\}$ be a finite set of states of nature. At date one, one and only one state s will occur, and an asset $X \in \mathbb{R}^S$ bought at date $t = 0$ will deliver payoff $X(s)$ at date 1 if s occurs.

We assume that at date 0 agents can trade a finite number of assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with respective prices q_j. Also, we suppose that[5]

$$X_0 = S^* := (1, ..., 1) \in \mathbb{R}^S$$

is the *riskless bond* and for sake of simplicity we suppose that $q_0 = 1$. A portfolio of an agent is identified with a vector $\theta = (\theta_0, \theta_1, ...\theta_m) \in \mathbb{R}^{m+1}$, where θ_j denotes the quantities of assets X_j possessed by the agent.

An arbitrage opportunity is a portfolio strategy with no cost that yields a strictly positive profit in some contingences and exposes no loss risk. The existence of such an arbitrage opportunity may be view as a kind of market inefficiency. The following definition establishes the basic properties of prices for efficient financial markets:

Definition 1 *The market* $\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m)$ *is assumed to offer* no-

[5] For any $A \subset S$, we will denote by A^* the characteristic function of the *event* A:

$$\begin{aligned} A^* &: S \to \{0, 1\} \\ s &\in A^*(s) = 1 \text{ iff } s \in A. \end{aligned}$$

arbitrage opportunity *(NAO) if*[6] *for any portfolio* $\theta \in \mathbb{R}^{m+1}$,

$$\sum_{j=0}^{m} \theta_j X_j > 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j > 0,$$
$$\sum_{j=0}^{m} \theta_j X_j = 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j = 0.$$

Denote by $F := span\,(X_0, X_1, ..., X_m)$ the *subspace of income transfers*[7] or the *set of attainable claims.* Let 2^S be the field of all subsets of S and Δ the set of all probability measures on $(S, 2^S)$. A well known property says that[8]:

Remark 2 *The market $\mathcal{M} = (X_j, q_j;\ 0 \le j \le m)$ offers no arbitrage opportunity if and only if there exists a strictly positive probabilitity*[9] $P_0 \in \Delta$ *such that* $E_{P_0}(X_j) = q_j$, $0 \le j \le m$.

Since, in general, the probability measure P_0 above is not uniquely determined an importante definition follows as:

Definition 3 *The set*

$$\mathcal{Q} = \{P \in \Delta : E_P(X_j) = q_j,\, \forall j \in \{0, ..., m\}\},$$

is called the set of risk-neutral probabilities (or martingal measures).

Note that the set $\mathcal{Q}$ of all risk-neutral probabilities describes the family of all probability measures that agree about the value of all basic assets. Remark 2 is known as the *fundamental theorem of asset princing* and it says that $\mathcal{Q}$ is nonempty if and only if NAO is true.

As it is usual, we say that the market $\mathcal{M} = (X_j, q_j;\ 0 \le j \le m)$ is complete if every claim $Y \in \mathbb{R}^S$ is attanaible, *i.e.*, $F = \mathbb{R}^S$. Othewise, we say that the market $\mathcal{M}$ is incomplete. A basic fact says that, if the prices of basic securities satisfies the NAO property then completeness of financial market is equivalent to the equality $\mathcal{Q} = \{P_0\}$, where P_0 is the probability measure obtained in the Remark 2.

[6] We use the following notation: For $X \in \mathbb{R}^S$, $X > 0$ means that $X \ge 0$ (i.e., $X(s) \ge 0$ for any $s \in S$) and $X \ne 0$.

[7] Recall that a (linear) *subspace* $W \subset \mathbb{R}^S$ satisfies: for any $\omega_1, \omega_2 \in W$ and $\lambda \in \mathbb{R}$ it is true that $\lambda\omega_1 + \omega_2 \in W$. Moreover, given a set of vectors $\{\omega_1, \omega_2, ..., \omega_N\}$ it generates the following subspace:

$$span\,\{\omega_1, \omega_2, ..., \omega_N\} := \left\{\sum_{n=1}^{N} \alpha_n \omega_n : \alpha_n \in \mathbb{R},\, 1 \le n \le N\right\}$$

[8] A nice reference for the well know results used here is the chapter 1 of Föllmer and Schied (2004).

[9] Note that P_0 strictly positive means that $P_0(\{s\}) > 0$ for any $s \in S$. We are denoting $E_P(X)$ as the integral of the random variable X w.r.t. the the probabilitiy P.

The preceding discussion is concerning the fundamental tools for pricing and hedging any attanaible claim. However, in general, financial markets are incomplete by the lack of a some financial instrumentals. A possible strategy against a non attainable claim X is to consider a portfolio $\theta \in \mathbb{R}^{m+1}$ such that $\sum_{j=0}^{m} \theta_j X_j \geq X$: this portfolio is often called a *superhedging strategy* or *superreplication* of X[10]. Hence, we may view the cost of a non attainable claim as the lowest possible price of a superreplication of X. Summing up, we obtain the following definition:

Definition 4 *For any claim $X \in \mathbb{R}^S$, the cost of X is given by*

$$C(X) = inf \left\{ \sum_j \theta_j q_j : \sum_j \theta_j X_j \geq X \right\}.$$

Moreover, C will be called the cost function of the market $\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m)$.

Remark 5 *It is worth to noticing that under NAO assumption the cost of any attanaible claim X trivially writes:*

$$C(X) = \sum_{j=0}^{m} \theta_j q_j,$$

for any portfolio $\theta = (\theta_0, \theta_1, ...\theta_m)$ such that $X = \sum_{j=0}^{m} \theta_j X_j$. Moreover, the NAO condition says that C is a strictly positive functional on F, in fact, following the notation of the Remark 2, note that

$$C(X) = E_{P_0}(X), \text{ for any } X \in F.$$

The set of risk-neutral probabilities $\mathcal{Q}$ plays an important role for the determination of a cost function and it is a well-known property that can be enunciated as:

Remark 6 *For a market $\mathcal{M}$ offering no-arbitrage opportunity, the cost function satisfies, for any $X \in \mathbb{R}^S$:*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Our main goal is to give a full characterization of cost functions and to describe certain classes of cost functions related to some specific types of incompleteness of financial markets. From Remark 6 a cost function is necessarily the maximum of expectations with respect to a given family of probabilities for which characterizations exist in the literature (*e.g.*, Huber (1981), Gilboa and Schmeidler (1989) and Chateauneuf (1991)). Based on these characterizations, C must be:

[10] For instance, the existence of superhedging strategies for any non attanaible claim follows from the existence of the riskless bond.

1. subadditive, *i.e.*,

$$C(X+Y) \leq C(X) + C(Y), \forall X, Y \in \mathbb{R}^S;$$

2. Positively affinely homogeneous, *i.e.*,

$$C(\alpha X + kS^*) = \alpha C(X) + k, \forall X \in \mathbb{R}^S, \forall k \in \mathbb{R}, \forall \alpha \geq 0;$$

3. Monotone, *i.e.*,

$$X \geq Y \Rightarrow C(X) \geq C(Y), \forall X, Y \in \mathbb{R}^S.$$

Remark 7 *As is well-known, any function with these three properties is Lipschitz continuous on* $\mathbb{R}^S$ *with respect to the supnorm* $X \mapsto \|X\|_\infty := \max_{s \in S} |X(s)|$. i.e.,

$$|C(X) - C(Y)| \leq \|X - Y\|_\infty, \text{ for all } X, Y \in \mathbb{R}^S.$$

However, these conditions are not sufficient for the characterizations of cost functions as will be clear in the next sections[11]. Some natural questions are:

- How to recognise that a particular function is a cost function?
- How to derive the underlying market from a given cost function?
- Is possible to find some property with economic meaning in order to determine if some function is in fact a cost function?

Building on the well-known properties discussed in Remarks 2 and 6, we derive an equivalent definition of cost functions:

Definition 8 *The mapping* $C : X \in \mathbb{R}^S \to C(X) \in \mathbb{R}$ *is the cost function of a market of securities without arbitrage opportunities if:*

1) *There exist* $X_0,\ X_1, ...,\ X_m \in \mathbb{R}^S$ *with* $X_0 = S^*$ *and a strictly positive probability* P_0 *such that:*

$$E_{P_0}(X_j) = C(X_j),\ 0 \leq j \leq m;$$

2) *Denoting,* $\mathcal{Q} := \{P \in \Delta : E_P(X_j) = C(X_j),\ 0 \leq j \leq m\}$, *then* $\forall X \in \mathbb{R}^S$:

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X)$$

[11] As we will see, by the no-arbitrage principle, a cost function must be *strictly positive*: $X > 0 \Rightarrow C(X) > 0$. However, adding this condition to the classical conditions mentioned above we still have a set of necessary but not sufficient conditions.

3 Characterization of Cost Functions

We denote by $\mathcal{H}_0$ the family of all *subadditive, positively affinely homogeneus* and *monotone* functions $C : \mathbb{R}^S \to \mathbb{R}$. Also, we denote by $\mathcal{H}$ the family of all function in $\mathcal{H}_0$ that are *strictly positive*. The class $\mathcal{H}$ describes the candidates to be a cost function of a market with securities without arbitrage opportunity.

Remark 9 *We know that $C \in \mathcal{H}_0$ if and only if there exists a nonempty, closed and convex set set $\mathcal{K} \subset \Delta$ such that for any $X \in \mathbb{R}^S$,*

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

(See for instance Huber (1981)). Moreover, we note that $C \in \mathcal{H}$ if and only if there exists a strictly positive probability $P_0 \in \mathcal{K}$. In fact, if $C \in \mathcal{H}_0$ and it is strictly positive we have that (putting $S = \{s_1, ..., s_n\}$)

$$C\left(\{s_i\}^*\right) > 0, \forall i \in \{1, ..., n\}.$$

Hence, for every state $s_i \in S$ there exists a probability $P_i \in \mathcal{K}$ such that $E_{P_i}\left(\{s_i\}^*\right) > 0$, since $\mathcal{K}$ is convex we obtain that it is possible to find a strictly positive probability in $\mathcal{K}$. For the converse, we note that by assumption there exists a strictly positive probability $P_0 \in \mathcal{K}$, hence if $X > 0$

$$C(X) \geq E_{P_0}(X) \geq \max_{s \in S} P_0(\{s\}) X(s) > 0.$$

Given a function $C \in \mathcal{H}_0$, we define the set of *unambiguous assets* as,

$$F_C := \left\{Y \in \mathbb{R}^S : C(Y) + C(-Y) = 0\right\}.$$

The set F_C describes all well princing claim by the function C. In fact, taking C as the rule for the determination asset prices, the family of claims F_C describes the assets in which there is no pricing distinction between a selling position or a buying position. The set of probabilities that agree about the expected value of all unambiguous assets is given by,

$$Q_C := \{P \in \Delta : \ E_P(Y) = C(Y), \text{ for any } Y \in F_C\}.$$

A first elementary fact is:

Lemma 10 *Given a function $C \in \mathcal{H}_0$, the set unambiguous events F_C is a linear subspace.*

Proof. First, consider $Y \in F_C$ and $\lambda \in \mathbb{R}_+$, since C is positively homogeneous we have that $C(\lambda Y) = \lambda C(Y)$ and $C(\lambda(-Y)) = \lambda C(-Y)$, then $C(\lambda Y) + C(-\lambda Y) = 0$, *i.e.*, $\lambda Y \in F_C$. If $\lambda < 0$, by the definition $-Y \in F_C$ and then $(-\lambda)(-Y) \in F_C$, *i.e.*, $\lambda Y \in F_C$.

Now, if $Y, Z \in F_C$, since C is subadditive

$$\begin{aligned} C(Y+Z) &\leq C(Y) + C(Z), \text{ and} \\ C(-(Y+Z)) &\leq C(-Y) + C(-Z), \end{aligned}$$

hence, adding these two inequalities

$$0 = C(0) \leq C(Y+Z) + C(-(Y+Z)) \leq 0,$$

i.e., $Y + Z \in F_C$. ■

Another simple fact follows as:

Lemma 11 *Consider a market $\mathcal{M} = (X_j, q_j, 0 \leq j \leq m)$ without arbitrage oportunity, a claim $X \in F$ if and only if $P(X) = Q(X)$ for any $P, Q \in \mathcal{Q}$.*

Proof. That $X \in F$ implies that all risk measures agree is obvious, in fact, it is the law of one price.

In order to prove the reverse implication, assume that $X \notin F$ and $P(X) = Q(X)$ for any $P, Q \in \mathcal{Q}$, *i.e.*, the law of one price is true for some non-atanaible claim.

First, we note that:

$$C(X) = \min\{C(Y) : Y \geq X \text{ and } Y \in F\}.$$

In fact, by the NAO assumption there exists a strictly positive probabililty P_0 such that such that $C(Y) = E_{P_0}(Y)$ for any $Y \in F$. For any $n \in \{1, 2, ...\}$ consider the attanaible claim Y^n such that $E_{P_0}(Y^n) \leq C(X) + n^{-1}$. Hence, for any $s \in S$

$$Y^n(s) \leq P_0(\{s\})^{-1}\left(C(X) + n^{-1}\right) \leq (C(X) + 1) \max_{s \in S} P_0(\{s\})^{-1} := k$$

therefore $Y^n \leq kS^*$ for any $n \geq 1$. Clearly,

$$C(X) = \inf\{C(Y) : X \leq Y \ \leq kS^* \text{and } Y \in F\},$$

and since $\{Y \in F : X \leq Y \ \leq kS^*\}$ is compact and C is continuous (by Remark 7) we obtain that the *min* can be substituted to *inf* in the definition of C.

Hence, given $X \in \mathbb{R}^S \backslash F$ there exist $Y_0 \in F$ such that $Y_0 > X$ and

$$C(X) = E_{P_0}(Y_0).$$

Also, we have that $E_{P_0}(Y_0) > E_{P_0}(X)$. Now, as it is true that

$$C(X) = \sup_{P \in \mathcal{Q}} E_P(X),$$

and we suppose that $P(X) = Q(X)$ for any $P, Q \in \mathcal{Q}$, it turns that $E_{P_0}(X) = C(X)$, hence

$$C(X) = E_{P_0}(Y_0) > E_{P_0}(X) = C(X),$$

a contraction. ■

Lemma 11 says that a claim is attanaible if and only if it satisfies the law of one price. Hence, it is intuitive that if C is a cost function then the subspace of unambiguous assets is equal to the subspace of attanaible claims, in fact:

Lemma 12 *If $C : \mathbb{R}^S \to \mathbb{R}$ is a cost funtion of a market of securities with no-arbitrage oportunity then $F = F_C$.*

Proof. Since $E_P(X) = C(X)$ for any $X \in F$ and for any $P \in \mathcal{Q}$ clearly $F \subset F_C$.

Conversely, let $X \in F_C$, since for any $P \in \mathcal{Q}$,

$$E_P(X) \leq C(X) \text{ and } E_P(-X) \leq C(-X),$$

and

$$E_P(X) + E_P(-X) = 0 = C(X) + C(-X),$$

we obtain that $E_P(X) < C(X)$ is impossible for any $P \in \mathcal{Q}$, *i.e.*, for all $X \in F_C$ the mapping $P \mapsto \Phi_X(P) := E_P(X)$ is constant over $\mathcal{Q}$, and by Lemma 11 $X \in F$. ■

A first characterization of cost functions follows as:

Theorem 13 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*

(i) C is the cost function of a market of securities with no-arbitrage oportunity;

(ii) C is strictly positive positive linear form on F_C and

$$C(X) = \max_{P \in \mathcal{Q}_C} E_P(X).$$

Furthemore, under (i) and (ii) F_C is the set of attanaible claims and $\mathcal{Q}_C$ is the set of risk-neutral probabilities of the underlying market.

Proof. $(i) \Rightarrow (ii)$

By our assumption, there exists $X_0,\ X_1, ...,\ X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 on 2^Ssuch that $E_{P_0}(X_j) = C(X_j)$, $0 \leq j \leq m$. Moreover, $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : E_P(X_j) = C(X_j),\ 0 \leq j \leq m\}$.

Now, note that no-arbitrage principle implies that C is a strictly positive linear form on F; actually, by Remark 2 there exists a strictly positive probability P_0 such that $\forall Y \in F$, $C(Y) = E_{P_0}(Y)$. By Lemma 12 we know that $F = F_C$, hence C is a strictly positive linear form on F_C.

Since $\mathcal{Q}_C$ and $\mathcal{Q}$ are nonempty, closed and convex set of probabilities, remains to show that $\mathcal{Q}_C = \mathcal{Q}$. If $P \in \mathcal{Q}$ we know that $C(Y) = E_P(Y)$ for any $Y \in F$, since $F = F_C$ we obtain that $P \in \mathcal{Q}_C$. Now, $P \in \mathcal{Q}_C$ says that $C(Y) = E_P(Y)$ for any $Y \in F_C$. Again, since $F = F_C$ entails that

$$F_C = span(X_0, ..., X_m),$$

in particular, $C(X_j) = E_P(X_j)$ for any $j \in \{0, 1, ..., m\}$, *i.e.*, $P \in \mathcal{Q}$.

$(ii) \Rightarrow (i)$

Since $S^* \in F_C$, let us consider $X_0,\ X_1, ..., X_m$, with $X_0 = S^*$, a basis of the linear subspace F_C. We intent to show that C is a cost function with respect to this family of securities $X_0,\ X_1, ..., X_m$.

By our assumption the restriction $C\mid_{F_C}$ of C on the linear subspace F_C of the Euclidian space $\mathbb{R}^S$ is a strictly positive linear form, hence it admits a strictly positive linear extension $\overline{C}\mid_{F_C}$ on $\mathbb{R}^S$ (see, for instance, Gale (1960)). Clearly, it is true that $\overline{C}\mid_{F_C}(S^*) = 1$, therefore there exists a striclty positive probability P_0 on $\left(S, 2^S\right)$ such that $E_{P_0}(X) = \overline{C}\mid_{F_C}(X)$, for any $X \in \mathbb{R}^S$; in particular, $E_{P_0}(X_j) = \overline{C}\mid_{F_C}(X_j) = C(X_j)$, $0 \leq j \leq m$. So, the condition 1) of the Definition 8 is satisfied. Recalling that $F = \mathit{span}(X_0, ..., X_m)$, by our construction F_C is the set of attanaible claims. The proof of (ii) implies (i) will be completed if we show prove that C satisfies the condition 2) of the Definition 8, or equally, that $\mathcal{Q}_C = \mathcal{Q}$, where $\mathcal{Q}$ is the set of risk neutral probabilities. By definition,

$$\mathcal{Q}_C := \left\{P \in \Delta :\ E_P(Y) = C(Y), \text{ for any } Y \in F_C\right\},$$

wich is nonempty because we saw that there exists a strictly positive probability $P_0 \in \mathcal{Q}_C$.

Since for any $j \in \{0, 1, ..., m\}$ the security X_j is unambiguous, we obtain that every probability $P \in \mathcal{Q}_C$ is a risk-neutral probability for the market $\mathcal{M} = (X_j, q_j := C(X_j), 0 \leq j \leq m)$[12]. Remains to prove that every risk-neutral probability belongs to $\mathcal{Q}_C$. In fact, let $P \in \mathcal{Q}$ and $Y \in F_C$, *i.e.*,

$$E_P(X_j) = C(X_j),\ 0 \leq j \leq m,$$

and there exists $\lambda_0, \lambda_1, ..., \lambda_m \in \mathbb{R}$ such that

$$Y = \sum_{j=0}^{m} \lambda_j X_j.$$

Since the restriction $C\mid_{F_C}$of C on the linear subspace F_C is a linear mapping,

$$\begin{aligned} E_P(Y) &= E_P\left(\sum_{j=0}^{m} \lambda_j X_j\right) = \sum_{j=0}^{m} \lambda_j E_P(X_j) = \\ \sum_{j=0}^{m} \lambda_j E_P(X_j) &= \sum_{j=0}^{m} \lambda_j C(X_j) = C\left(\sum_{j=0}^{m} \lambda_j X_j\right) = C(Y). \end{aligned}$$

henceforth,

$$E_P(Y) = C(Y)\ , \text{ for any } Y \in F_C,$$

this entails that $P \in \mathcal{Q}_C$, which completes the proof. ■

The examples below illustrate the usefulness of the criterion given by Theorem 13 .

[12] By the existence of the strictly positive probability P_0, the financial market $\mathcal{M}$ is a market of securities with no-arbitrage opportunity.

Example 14 *Suppose that* $S = \{s_1, s_2, s_3\}$ *and that*

$$\begin{array}{rcl} C & : & \mathbb{R}^3 \to \mathbb{R} \\ X & \mapsto & C(X) = \max\{E_{P_1}(X), E_{P_2}(X)\}, \end{array}$$

where $P_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ *and* $P_2 = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. *Hence, denoting* $X(s_k) = x_k$, $k = 1, 2, 3:$

$$C(X) = \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3, \text{ if } x_1 \geq x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3, \text{ if } x_1 < x_2 \end{cases}.$$

Note that: (a) $C(S^*) = 1$; *(b)* $C(X) = -C(-X)$ *if and only if* $x_1 = x_2$, *and then* F_C *is a linear subspace; (c) on* F_C *we have that* $C(X) = \frac{3}{4}x_1 + \frac{1}{4}x_3 = \frac{3}{4}x_2 + \frac{1}{4}x_3$, *which implies that* C *is a strictly positive linear form on* F_C; *(d) Note that we may take* $\{X_0, X_1\} \equiv \{(1,1,1), (0,0,1)\}$ *as a basis of* F_C, *where* $C(X_0) = 1$, $C(X_1) = \frac{1}{4}$ *and*

$$\mathcal{Q}_C = \left\{ \left(p_1, p_2, \frac{1}{4}\right) : \ p_1, p_2 \geq 0 \text{ and } p_1 + p_2 = \frac{3}{4} \right\}.$$

Now, note that if $x_1 < x_2$

$$\max_{P \in \mathcal{Q}_C} E_P(X) = \frac{3}{4}x_2 + \frac{1}{4}x_3 < \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = C(X),$$

which allows us to conclude that C *is not a cost function of a market without arbitrage opportunity. An interesting fact is that this kind of function appears as a particular case of* insurance functionals *in Castagnoli, Maccheroni and Marinacci (2002). So, in this case, the insurance market admits some frictions (e.g., transactions costs).*

Example 15 *Again, consider the case with three states of nature and the function* $C : \mathbb{R}^3 \to \mathbb{R}$ *that satisfies:*

$$C(X) = \begin{cases} x_3, \text{ if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2), \text{ if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}$$

Note that: (a) $C(S^*) = 1$; *(b)* $C(X) = -C(-X)$ *if and only if* $x_1 + x_2 - 2x_3 = 0$, *hence* F_C *is a linear subspace; (c) on* F_C *we have that* $C(X) = x_3 = \frac{1}{2}(x_1 + x_2)$, *which implies that* C *is a strictly positive linear form on* F_C; *(d) Note that we may take* $\{X_0, X_1\} \equiv \{(1,1,1), (2,0,1)\}$ *as a basis of* F_C, *where* $C(X_0) = 1$, $C(X_1) = 1$ *and*

$$\mathcal{Q}_C = \left\{ (p, p, 1 - 2p) : \ 0 \leq p \leq \frac{1}{2} \right\}.$$

It turns out that:

$$\max_{0 \leq p \leq \frac{1}{2}} (px_1 + px_2 + (1 - 2p)x_3) = C(X),$$

hence C *is a cost function for the market* $\mathcal{M} = ((1,1,1), (2,0,1); \ 1, 1)$.

Based on the previous results, we may view incomplete markets financial models as an extreme approach to the interaction of agents in an uncertainty environments: an asset is marketed if and only if it is unambiguous. In this sense, agents can trade only those claims for which the expectation is giving by an exact measurement. Also, the risk neutral probabilities are those probabilities that agree on the set of well defined expectations.

3.1 Cost Functions and Wasteful Assets

Now, we introduce a fundamental notion for the characterization of cost functions. For motivation, suppose that C is a *potential* cost function and consider the case where there are two assets X and Y such that $Y > X$ and $C(X) = C(Y)$. If X and Y are available for an investor and he chooses X then he incurs into a *payoff wasteful* because at the same cost the payoff stream promised by Y is at least equal to payoff promised by X and for some contingence Y deliver a strict bigger payment. A reasonable investor´s behavior never is to choose claims that implies in a payoff wasteful unless he believe that the event $\{s \in S : Y(s) > X(s)\}$ is a miracle[13].

Definition 16 *Let $\mathbb{R}^S$ be the set of claims and $C : \mathbb{R}^S \to \mathbb{R}$ a function in $\mathcal{H}_0$. We say that a contigent claim X is wasteful asset if there exists a contingent claim $Y > X$ such that $C(Y) = C(X)$.*

A no-wasteful asset is a contingent claim with the property that if some payoff assigned to a state by the claim is replaced by a better payoff, then the resulting contingent claim is strictly more expensive than the original one. A investor behavior that seems reasonable is never consistence with a choice of some wasteful asset.

Given a function $C \in \mathcal{H}_0$, we denote by L_C the set of all no-wasteful assets[14], *i.e.*,

$$L_C := \left\{X \in \mathbb{R}^S : Y > X \;\Rightarrow C(Y) > C(X)\right\}.$$

Now we are able to derive an interesting result saying that a *potential cost function* is actually a cost function if and only if the respectives set of unambiguous assets and set of no-wasteful assets are the same.

Theorem 17 *C is a cost function of a market of securities with no-arbitrage opportunity if and only if $C \in \mathcal{H}_0$ and $L_C = F_C$.*

Proof. $(\Rightarrow)$ The fact that $C \in \mathcal{H}_0$ is immediate.

It remains to show that $L_C = F_C$. Suppose that $X \in F_C$, since C is a cost function we know that there exists a strictly positive probability P_0 such that $C(X) \geq E_{P_0}(X)$, $\forall X \in \mathbb{R}^S$ and $C(X) = E_{P_0}(X), \forall X \in F_C$. Hence, if $Y > X$ then $C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X)$.

[13] But such believes are not consistente with the setting where every simple bet $\{s\}^*$ has positive cost.

[14] In the context of decision theory under ambiguity, Lehrer (2007) provided a representation for preferences using a similar notion called *fat-free acts*.

Now, suppose that $X \in L_C$ then by definition $Y > X \Rightarrow C(Y) > C(X)$. Suppose that $X \notin F_C = F$ (Lemma 12), since

$$C(X) = \min\{C(Y) : Y \geq X \text{ and } Y \in F_C\} \overset{(X \notin F_C)}{=} \min\{C(Y) : Y > X \text{ and } Y \in F_C\}$$

there exists $Z \in F_C$ such $Z > X$ and $C(Z) = C(X)$, a contradiction.

($\Leftarrow$) Since $C \in \mathcal{H}$ we know that there exists a nonempty, closed and convex set $\mathcal{K} \subset \Delta$ such that for any $X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

By Theorem 13 and Remark 9 it is enough to show that C is strictly positive and $\mathcal{K} = \mathcal{Q}_C$.

Consider $X > 0$, since $0 \in F_C$ and $F_C = L_C$ we obtain that $C(X) > C(0) = 0$, therefore C is strictly positive, so $C \in \mathcal{H}$. The inclusion $\mathcal{K} \subset \mathcal{Q}_C$ is simple: Consider $P \in \mathcal{K}$, if $P \notin \mathcal{Q}_C$ then there exists $X \in F_C$ such that $E_P(X) < C(X) = -C(-X)$, hence $E_P(-X) > C(-X) = \max_{P \in \mathcal{K}} E_P(-X)$, a contradiction.

So we need to show that $\mathcal{Q}_C \subset \mathcal{K}$, or equally that $\mathcal{K} \subsetneq \mathcal{Q}_C$ is impossible. Assume that there exists $P_1 \in \mathcal{Q}_F$ such $P_1 \notin \mathcal{K}$. Then through the classical strictly separation theorem (see, for instance, Dunford and Schwartz (1958)) there exists a contingent claim X_0 such that

$$E_{P_1}(X_0) > \max_{P \in \mathcal{K}} E_P(X_0) = C(X_0).$$

If we prove that there exists $Y \in F_C,\ Y \geq X_0$ such that $C(X_0) = C(Y)$, this will entail a contradiction, since

$$E_{P_1}(X_0) > C(X_0) = C(Y) = E_{P_1}(Y) \geq E_{P_1}(X_0).$$

So it is enough to show that for any contingent claim X, setting

$$E_X := \left\{Y \in \mathbb{R}^S : Y \geq X \text{ and } C(Y) = C(X)\right\},$$

there exists $Y \in F_C \cap E_X$.

This result is obvious if $X \in F_C$, so let us assume that $X \notin F_C$. First, since $C \in \mathcal{H}$ by Remark 9 we know $\mathcal{K}$ contains at least a strictly positive probability P_0.

Let us now prove that E_X is bounded from above, otherwise there would a sequence $\{Y_k\}_{k \geq 1}$, $Y_k \in E_X$, $\forall k \geq 1$ and $s_0 \in S$ such that $\lim_k Y_k(s_0) = +\infty$. But

$$\begin{aligned} \lim_k C(Y_k) &\geq \lim_k E_{P_0}(Y_k) = \lim_k \sum_{s \in S} P_0(s) Y_k(s) \\ &\geq \sum_{s \neq s_0} P_0(s) X(s) + \lim_k P_0(s_0) Y_k(s_0) = \infty, \end{aligned}$$

contrading $C(Y_k) = C(X)$, $\forall k \geq 1$.

Let us now show that E_X has a maximal element for the partial preorder $\geq$ on $\mathbb{R}^S$. Thanks to Zorn´s lemma we just need to prove that every chain $(Y_\lambda)_{\lambda\in\Phi}$ in E_X has an upper bound. Define Y by

$$Y(s) := \sup_{\lambda\in\Phi} Y_\lambda(s), \forall s \in S,$$

E_X bounded from above implies that $Y \in \mathbb{R}^S$. It remains to check that $C(Y) = C(X)$, let $\varepsilon > 0$ be given, and let $s_i \in S$, hence there exists $\lambda_i \in \Phi$ such that $Y(s_i) \leq Y_{\lambda_i}(s_i) + \varepsilon$, since $(Y_\lambda)_{\lambda\in\Phi}$ is a chain there exists $n \geq 1$ and $\widetilde{\lambda} \in \{\lambda_1, ..., \lambda_n\}$ such that $Y_{\widetilde{\lambda}} \leq Y \leq Y_{\widetilde{\lambda}} + \varepsilon$, therefore $C\left(Y_{\widetilde{\lambda}}\right) \leq C(Y) \leq C\left(Y_{\widetilde{\lambda}}\right) + \varepsilon$, since $C\left(Y_{\widetilde{\lambda}}\right) = C(X)$ it turns out that $C(Y) = C(X)$. Let now Y_0 be a maximal element of E_X, the proof will be completed if we show that $Y_0 \in F_C$. From the hypothesis $F_C = L_C$, it is enough to show that $Y_0 \in L_C$. Let Y_1 be an arbitrary contingent claim such that $Y_1 > Y_0$, since Y_0 is a maximal element in E_X , it comes that $Y_1 \notin E_X$, but $Y_1 > X$, therefore $C(Y_1) > C(X) = C(Y_0)$, so $Y_0 \in L_C$ which completes the proof. ■

An immediate useful corollary follows as,

Corollary 18 *C is a cost function of a market of securities with no-arbitrage opportunity if and only if $C \in \mathcal{H}$ and $L_C \subset F_C$.*

Proof. In fact, it is enough to show that for any $C \in \mathcal{H}_0$, $L_C = F_C$ iff C is strictly positive and $L_C \subset F_C$. Note that, as in the previous proof $F_C \subset L_C$ implies that, C is strictly positive. For the converse, by Remark 9 we know that there exists a strictly positive probability P_0 such that $C(X) \geq E_{P_0}(X)$ for any contingent claim X and $C(Y) = E_{P_0}(Y)$ for any unambiguous asset Y. Let $X \in F_C$ and consider a contigent claim $Y > X$. Hence,

$$C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X),$$

i.e., $C(Y) > C(X)$, so $F_C \subset L_C$. ■

Remark 19 *The fact that in Example 14, C is not a cost function, although it belongs to $\mathcal{H}$, can be easily shown by exhibiting some $X \in L_C$, which does not belong to F_C, in fact $X = (1, 0, 0)$ does the trick.*

3.2 Markets of $\{0, 1\}$-Securities

Arrow (1963) introduced the notion of contingent markets where agents can trade promisses concerning the future uncertainty realizations. A wide class of assets used is known as Arrow securities characterized by a promisse on a particular state of nature $s \in S$, *i.e.*, in a financial market the set of possibles Arrow securities is given by $\mathbb{A} := \left\{\{s\}^* : s \in S\right\}$[15]. Given an event A, the $\{0, 1\}$-security A^* is also often called a *bet on the event* A. For the classes of markets with only $\{0, 1\}$-securities the definition of cost function follows as:

[15] Of course, markets with only Arrow securities is a very particular case of markets with $\{0, 1\}$-securities.

Definition 20 *The mapping $C : X \in \mathbb{R}^S \to C(X) \in \mathbb{R}$ is the cost function of a market of $\{0,1\}$-securities without arbitrage opportunities if:*

1) There exist a list of events $B_0,\ B_1, ...,\ B_m \in 2^S$ with $B_0 = S$ and a striclty positive probability P_0 on 2^S such that

$$P_0\left(B_j\right) = C\left(B_j^*\right),\ \textit{for any } j \in \{0, 1, ..., m\};$$

2) Denoting, $\mathcal{Q} = \{P \in \Delta : P\left(B_j\right) = C\left(B_j^*\right),\ 0 \leq j \leq m\}$, for any claim $X \in \mathbb{R}^S$:

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Following the notation used in the previous discussion about cost function, given a subadditive, positively affinely homogeneous, monotone and normalized function $C : \mathbb{R}^S \to \mathbb{R}$ we induced the set function

$$\begin{array}{rcl} \mu_C & : & 2^S \to [0,1] \\ A & \mapsto & \mu_C\left(A\right) := C\left(A^*\right). \end{array}$$

Therefore, we define the set of unambiguous events by

$$\mathcal{E}_{\mu_C} = \left\{B \in 2^S : \mu_C(A) + \mu_C(A^c) = 1\right\},$$

which induce the following set of probabilities

$$\mathcal{Q}_{\mu_C} = \left\{P \in \Delta :\ P\left(B\right) = \mu_C(B), \forall B \in \mathcal{E}_{\mu_C}\right\}$$

and finally the linear subspace generated by $\mathcal{E}_{\mu_C}$:

$$F_{\mathcal{E}_{\mu_C}} := span\left\{B^* : B \in \mathcal{E}_{\mu_C}\right\}.$$

Lemma 21 *Let C be the cost funtion of a market without arbitrage opportunities and let $B \subset S$, then the two following assertions are equivalent:*

(ii) $B \in \mathcal{E}_{\mu_C}$, i.e.*, B is an unambiguous event*

(ii) $B^* \in F$, i.e.*, B^* is an attanaible claim.*

Proof. We recall that from Lemma 11, $X \in F$ iff $E_P\left(X\right) = E_Q\left(X\right)$, for any $P, Q \in \mathcal{Q}$.

$(i) \Rightarrow (ii)$

Let $B \in \mathcal{E}_{\mu_C}$, we need to show that $P\left(B\right) = Q\left(B\right)$, for any $P, Q \in \mathcal{Q}$. Assume that there exists $P_1, P_2 \in \mathcal{Q}$ such that $P_1\left(B\right) > P_2\left(B\right)$. Hence,

$$1 = C\left(B\right) + C\left(B^c\right) = \max_{P \in \mathcal{Q}} P\left(B\right) + \max_{P \in \mathcal{Q}} P\left(B^c\right) > P_2\left(B\right) + \max_{P \in \mathcal{Q}} P\left(B^c\right),$$

that is, $P_2\left(B^c\right) > \max_{P \in \mathcal{Q}} P\left(B^c\right)$, but $P_2 \in \mathcal{Q}$ hence the contraction $P_2\left(B^c\right) > P_2\left(B^c\right)$. Therefore, $B^* \in F$.

$(ii) \Rightarrow (i)$

Let $B^* \in F$, hence $P\left(B\right) = Q\left(B\right)$, for any $P, Q \in \mathcal{Q}$ and therefore $\mu_C(B^c) = P_0\left(B\right)$, but $S^* \in F$ implies also $S^* - B^* \in F$ and then $\mu_C(B^c) = P_0\left(B^c\right)$, and clearly $\mu_C(B) + \mu_C(B^c) = 1$, *i.e,* $B \in \mathcal{E}_{\mu_C}$. ■

The previous lemma says that a bet on the event A is attainable if and only if the event A is an unambiguous event. It suggests that we may interpret the lack of some bet on the financial market as consequence of a vague information concerning the likelihood of some events.

Remark 22 *Given a subadditive, positively affinely homogeneous, monotone and normalized function $C : \mathbb{R}^S \to \mathbb{R}$, we obtain that $\left\{B^* : B \in \mathcal{E}_{\mu_C}\right\} \subset F_C$: in fact, if B is such that $\mu_C(B) + \mu_C(B^c) = 1$ then*

$$\begin{aligned} C(B^*) + C(-B^*) &= C(B^*) + C\left((B^c)^* - S^*\right) = \\ \max_{P \in \mathcal{Q}} P(B) + \max_{P \in \mathcal{Q}} \left(P(B^c) - 1\right) &= \mu_C(B) + \mu_C(B^c) - 1 = 0. \end{aligned}$$

So, every porfolio with assets that are bets on unambiguous events are attanaible. Moreover, $\mathcal{Q}_C \subset \mathcal{Q}_{\mu_C}$.

Theorem 23 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*

(i) C is the cost function of a market of $\{0,1\}$-securities with no-arbitrage oportunity;

(ii) There exists a strictly positive probability P_0 belonging $\mathcal{Q}_{\mu_C}$ and for any contingent claim X,

$$C(X) = \max_{P \in \mathcal{Q}_{\mu_C}} E_P(X).$$

Furthemore, under (i) and (ii) $F_{\mathcal{E}_{\mu_C}}$ is the set of attanaible claims and $\mathcal{Q}_{\mu_C}$ is the set of risk-neutral probabilities of the underlying market.

Proof. $(i) \Rightarrow (ii)$

Our assumption says that there exist $B_0, B_1, ..., B_m \in 2^S$ with $B_0 = S$ and a striclty positive probability P_0 on 2^S such that $P_0(B_j) = C\left(B_j^*\right)$, for any $j \in \{0, 1, ..., m\}$ and $\forall X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P(B_j) = C\left(B_j^*\right),\ 0 \leq j \leq m\}$.

Let us to now prove that there exists a strictly positive probability P_0 belonging $\mathcal{Q}_{\mu_C}$. From Lemma 11 we know that if $B^* \in F$ then $P(B) = P_0(B)$ for any $P \in Q$, hence $\mu_C(B) = P_0(B)$. Since by Lemma 21 $B \in \mathcal{E}_{\mu_C}$ if and only if $B^* \in F$, it turns out that $P_0(B) = \mu_C(B)$, $\forall B \in \mathcal{E}_{\mu_C}$.

Now we need to show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$. Note that by Theorem 13 and the Remark 22 we have that $\mathcal{Q} = \mathcal{Q}_C \subset \mathcal{Q}_{\mu_C}$. For the other inclusion, taking $P \in \mathcal{Q}_{\mu_C}$ and B_j let us to show that $P(B_j) = C\left(B_j^*\right)$. Since B_j^* is an attanaible claim (in fact, it is a basic asset) by Lemma 21 we know that $B_j \in \mathcal{E}_{\mu_C}$, hence $P \in \mathcal{Q}_{\mu_C}$ so $\mathcal{Q}_{\mu_C} \subset \mathcal{Q}$.

So $\mathcal{Q}_{\mu_C}$ is actually the set of risk-neutral probabilities of the initial market. Remains to prove that $F_{\mathcal{E}_{\mu_C}} = F$. In fact, Lemma 21 says that $B \in \mathcal{E}_{\mu_C} \Leftrightarrow B^* \in F$, hence $F_{\mathcal{E}_{\mu_C}} = span\left\{B^* : B \in \mathcal{E}_{\mu_C}\right\} = span\left\{B^* : B^* \in F\right\} = F$.

$(ii) \Rightarrow (i)$

Since $B_0 = S$, let us consider the finite family of all unambiguous events $B_0, B_1, ..., B_m$. By assumption there exists a strictly positive probability P_0 such that $P(B_j) = C(B_j^*)$, $0 \leq j \leq m$. The proof will be completed if we show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$ and $F_{\mathcal{E}_{\mu_C}} = F$, where $\mathcal{Q}$ and F refer to the previous defined market of $\{0,1\}$-securities $\mathcal{M} = (B_0^*, B_1^*, ..., B_m^*; 1, \mu_C(B_1^*), ..., \mu_C(B_m^*))$. But it is straightforward by the equality $\mathcal{E}_{\mu_C} = \{B_0, B_1, ..., B_m\}$. ■

Example 24 *Consider the case with four states of nature and the function* $C : \mathbb{R}^4 \to \mathbb{R}$ *that satisfies:*

$$C(X) = \begin{cases} \frac{3}{8}x_1 + \frac{3}{8}x_2 + \frac{3}{8}x_3, & \text{if } x_2 + x_4 \geq x_1 + x_3 \\ \frac{3}{8}x_1 + \frac{5}{8}x_3, & \text{if } x_2 + x_4 < x_1 + x_3 \end{cases}$$

Note that: (a) $C(S^*) = 1$*; (b)* $C(\emptyset) = C(0S^*) = 0$*, so* $S^* \in \mathcal{E}_{\mu_C}$*; (c) Computing* μ_C*, we have*

$$\begin{aligned}
\mu_C(\emptyset) &= 0, \mu_C(\{s_1\}) = \frac{3}{8}, \mu_C(\{s_2\}) = \frac{3}{8}, \\
\mu_C(\{s_3\}) &= \frac{5}{8}, \mu_C(\{s_4\}) = \frac{3}{8}, \mu_C(\{s_1, s_2\}) = \frac{3}{8}, \\
\mu_C(\{s_1, s_3\}) &= 1, \mu_C(\{s_1, s_4\}) = \frac{3}{8}, \mu_C(\{s_2, s_3\}) = \frac{5}{8}, \\
\mu_C(\{s_2, s_4\}) &= \frac{6}{8}, \mu_C(\{s_3, s_4\}) = \frac{5}{8}, \mu_C(\{s_1, s_2, s_3\}) = 1, \\
\mu_C(\{s_1, s_3, s_4\}) &= 1, \mu_C(\{s_1, s_2, s_4\}) = \frac{6}{8}, \\
\mu_C(\{s_2, s_3, s_4\}) &= 1, \mu_C(S) = 1.
\end{aligned}$$

which entails that

$$\mathcal{E}_{\mu_C} = \{\emptyset, S, \{s_1, s_2\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_3, s_4\}\},$$

and

$$\mathcal{Q}_{\mu_C} = \left\{ \left(\frac{3}{8} - p, p, \frac{5}{8} - p, p \right) : 0 \leq p \leq \frac{3}{8} \right\} \ni \left(\frac{2}{8}, \frac{1}{8}, \frac{4}{8}, \frac{1}{8} \right) > 0;$$

(d) Since $C(X) = \max_{P \in \mathcal{Q}_{\mu_C}} E_P(X)$ *hence* C *is a cost function of* $\mathcal{M} = \left(S^*, \{s_1, s_2\}^*, \{s_2, s_3\}^*;\ 1, \frac{3}{8}, \frac{5}{8}\right)$.

A direct consequence of the Theorem 23 is the caracterization of makerts of $\{0,1\}$-securities through the notion of no-wateful assets and span of the bets on the unambiguous events, as in the following corollary:

Corollary 25 C *is a cost function of a market of* $\{0,1\}$*-securities with no-arbitrage oportunity if and only if* $C \in \mathcal{H}_0$ *and* $L_C = F_{\mathcal{E}_{\mu_C}}$.

3.2.1 Cost Functions, Capacities and Choquet Integral

Now we introduce some useful notation and definitions:

Definition 26 $\mu : 2^S \to [0,1]$ *is a capacity if,*
(i) $\mu(\emptyset) = 0$ *and* $\mu(S) = 1$,
(ii) $A \supseteq B \Rightarrow \mu(A) \geq \mu(B)$,
Moreover, μ *is concave if for any* $A, B \in 2^S$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

Remark 27 *Consider a subadditive, positively affinely homogeneous, monotone and normalized function* $C : \mathbb{R}^S \to \mathbb{R}$ *and the induced set-function* μ_C *on* 2^S *as we made previously. It is simple to see that* μ_C *is a capacity.*

Definition 28 *The anticore of a capacity* μ *is defined by*

$$acore(\mu) := \left\{ P \in \Delta : P(A) \leq \mu(A), \forall A \in 2^S \right\}.$$

Remark 29 *It is well known that any concave capacity* μ *on* 2^S *has the following representation:*

$$\mu(E) = \max_{P \in \mathcal{K}} P(E),$$

for some nonempty, convex and closed set of probabilities $\mathcal{K}$ *(actually,* $\mathcal{K} = acore\,(\mu)$*). See, for example, Chateauneuf and Jaffray (1989). But the converse is not true (examples can be found in Schmeidler (1972) or Huber and Strassen (1973)).*

In a complete markets setting a bet on the event A can be pricing by the unique risk neutral probability denoted by P and in this case the price of the bet on the event A is given by $P(A)$, i.e., there is no ambiguous concerning the price of the bet on the event A. On the other hand, if there exists ambiguity concerning the price of some event A we have implicity assuming an incomplete market struture with respective set of multiples risk neutral probabilities $\mathcal{Q}$, and in this case

$$\mu(A) := \max_{P \in \mathcal{Q}} P(A),$$

is the price of the bet on the event A. Note that $\mu(A) + \mu(A^c) > 1$, *i.e.*, due to the ambiguous pricing rule the sum of the cost of the bets on the events A and A^c is more expensive than the cost of the riskless bond.

Definition 30 *The outer capacity of* μ*, denoted by* μ^**, is defined by:*

$$A \in 2^S \mapsto \mu^*(A) = \min\{\mu(B) : B \in \mathcal{E}_\mu \text{ and } A \subset B\},$$

where $\mathcal{E}_\mu = \left\{ B \in 2^S : \mu(B) + \mu(B^c) = 1 \right\}$.

Remark 31 *Given a capacity* μ *on* 2^S*, since* $\mu^* \geq \mu$ *clearly* $acore\,(\mu_C) \subset acore\,(\mu_C^*)$.

Theorem 32 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*

(i) C is the cost function of a market of $\{0,1\}$-securities with no-arbitrage oportunity;

(ii) C satisfies,

(a) $acore(\mu_C)$ contains a strictly positive probability P_0,

(b) $acore(\mu_C) = acore(\mu_C^*)$,

(c) For any contingent claim X,

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X).$$

Furthemore, under (i) and (ii) $F_{\mathcal{E}_{\mu_C}}$ is the set of attanaible claims and $acore(\mu_C)$ is the set $\mathcal{Q}$ of risk-neutral probabilities of the underlying market

Proof. $(i) \Rightarrow (ii)$

From the Definition 8 we have that for any $A \subset S$,

$$\mu_C(A) = C(A^*) = \max_{P \in \mathcal{Q}} P(A),$$

hence μ_C is a antiexact capacity and the $acore(\mu_C)$ contains at least one strictly positive probability, namely P_0.

Let us now show that

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X), \ \forall X \in \mathbb{R}^S.$$

Note that it is enough to show that $\mathcal{Q} = acore(\mu_C)$:

Consider $P \in acore(\mu_C)$, hence $P(B_j) \leq \mu_C(B_j), 0 \leq j \leq m$. But, in fact, B_j is unambiguous (Lemma 21) which entails $\mu_C(B_j) + \mu_C(B_j^c) = 1$. Also, $P(B_j^c) \leq \mu_C(B_j^c), 0 \leq j \leq m$ and then

$$P(B_j) + P(B_j^c) = 1 = \mu_C(B_j) + \mu_C(B_j^c) = 1,$$

allows us to obtain $P(B_j) = \mu_C(B_j), 0 \leq j \leq m$, *i.e.*, $P \in \mathcal{Q}$.

Now, setting $P \in Q$ and $A \subset S$, since our assumption says that

$$\mu_C(A) = \max_{P \in \mathcal{Q}} P(A),$$

clearly $P(A) \leq \mu_C(A)$, *i.e.*, $P \in acore(\mu_C)$.

For (b) it is enough to show that $acore(\mu_C^*) \subset acore(\mu_C)$, or else from the previous idendity $\mathcal{Q} = acore(\mu_C)$ that $acore(\mu_C^*) \subset \mathcal{Q}$. So let $P \in acore(\mu_C)$ and let B_j be chosen. By definition of μ_C^*, one has $\mu_C^*(B_j) = \mu_C(B_j)$ therefore $P(B_j) \leq \mu_C^*(B_j)$ implies $P(B_j) \leq \mu_C(B_j)$; as we notice before $B_j \in \mathcal{E}_{\mu_C}$, hence $\mu_C^*(B_j) = \mu_C(B_j)$ and $P(B_j^c) \leq \mu_C^*(B_j^c)$ implies $P(B_j^c) \leq \mu_C(B_j^c)$ from $P(B_j) + P(B_j^c) = 1 = \mu_C(B_j) + \mu_C(B_j^c)$, it turns out that $P(B_j) \leq \mu_C(B_j)$.

$(ii) \Rightarrow (i)$

We need to prove that there exist $B_0,\ B_1,...,\ B_m \in 2^S$ with $B_0 = S$ and a striclty positive probability P_0 on 2^S such that $P_0(B_j) = C\left(B_j^*\right)$, for any $j \in \{0,1,...,m\}$ and $\forall\ X \in \mathbb{R}^S$,

$$C(X) = \max_{P\in\mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P(B_j) = C\left(B_j^*\right),\ 0 \leq j \leq m\}$.

Note that C is well defined since $acore(\mu_C) \neq \emptyset$ (by assumption (a)) and compact, moreover for any $A \subset S$

$$\mu_C(A) = C(A^*) = \max_{P\in acore(\mu_C)} P(A).$$

Clearly $B_0 := S \in \mathcal{E}_{\mu_C}$, and $\mathcal{E}_{\mu_C}$ is formed with a finite number of events $B_0, B_1, ..., B_m$. Note that for any $B \in \mathcal{E}_{\mu_C}$ and for any $P \in acore(\mu_C)$ it is true that $P(B) = \mu_C(B)$: actually $P \in acore(C)$ implies that $P(B) \leq \mu_C(B)$, $P(B^c) \leq \mu_C(B^c)$ and $P(B)+P(B^c) = 1 = \mu_C(B_j)+\mu_C\left(B_j^c\right)$, gives the desired equality (note that its implies that $\mathcal{Q} \supset acore(\mu_C)$). Since, by hypothesis there exists a strictly positive probability $P_0 \in acore(\mu_C)$, it turns out that the first requeriment is satisfied. So the formula

$$C(X) = \max_{P\in\mathcal{Q}} E_P(X),$$

holds for any $X \in \mathbb{R}^S$ if and only if $\mathcal{Q} = acore(\mu_C)$. Just above we proved that $\mathcal{Q} \supset acore(\mu_C)$. By our assumption (b) we only have to show that $\mathcal{Q} \subset acore(\mu_C^*)$. Let $P \in \mathcal{Q}$ and $A \subset S$, from the definition of μ_C^* we have that there exists $B \in \mathcal{E}_{\mu_C}$ such that $A \subset B$ and $\mu_C^*(A) = \mu_C(B)$, hence

$$P(A) \leq P(B) = C(B) = \mu_C^*(A),$$

i.e., $P \in acore(\mu_C^*)$.

Futhermore, under (i) and (ii) $acore(\mu_C)$ is the set of risk-neutral probabilities and by Theorem 23 $F_{\mathcal{E}_{\mu_C}}$ is the set of attanaible claims. ■

Example 33 *Consider the same function as in Example 15 given by*

$$C(X) = \begin{cases} x_3,\ \text{if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2),\ \text{if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}.$$

We already proved that C is a cost function. Note that for any $A \neq \emptyset$,

$$\mu_C(A) \in \left\{\frac{1}{2}, 1\right\} \text{ with } \mu_C(A) = \frac{1}{2} \text{ iff } A \in \{\{s_1\}, \{s_2\}\},$$

which implies that $\mathcal{E}_{\mu_C} = \{\emptyset, S\}$, hence for any $A \neq \emptyset$,we have that $\mu_C^*(A) = 1$ and $acore(\mu_C^*) = \Delta$. Since $\delta_{\{s_1\}} \notin acore(\mu_C)$ we obtain that

$$acore(\mu_C) \neq acore(\mu_C^*).$$

Hence, C is not a cost function of a market with $\{0,1\}$-securities.

Now, we study the possibility of cost functions to be a Choquet integral, which is the natural extention of the usual integral for capacities.

Definition 34 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then C is a Choquet integral if*

(a) μ_C defined by $\mu_C(A) = C(A^*)$ for any $A \in 2^S$ is a capacity,

(b) For any $X \in \mathbb{R}^S$, $C(X) = \int X d\mu_C$ where,

$$\int X d\mu_C := \int_{-\infty}^{0} [\mu_C(\{X \geq t\}) - 1] \, dt + \int_{0}^{\infty} \mu_C(\{X \geq t\}) \, dt$$

Definition 35 *A Riesz subspace of $\mathbb{R}^S$ is a linear subspace F of $\mathbb{R}^S$ such that $X, Y \in F$ implies that $X \vee Y \in F$ and $X \wedge Y \in F$.*

Lemma 36 *If the cost function C of a market without arbitrage opportunity is a Choquet integral then the capacity μ_C is concave and the subspace F of attainable claims is a Riesz-space.*

Proof. First, we note that from Proposition 3 given by Schmeidler (1986) we have that if C is a subadditive Choquet integral with respect to the capacity μ_C then μ_C is a concave capacity.

Let now prove that F is a Riesz space:

Let $X, Y \in F$, then by Lemma 11 we have that for any $P \in \mathcal{Q}$, $E_P(X) + E_P(Y) = C(X) + C(Y)$. Since C is a Choquet Integral with respect to a concave capacity, it turns out that[16]

$$C(X) + C(Y) \geq C(X \vee Y) + C(X \wedge Y).$$

Therefore, using the previous equality

$$E_P(X \vee Y) + E_P(X \wedge Y) = E_P(X) + E_P(Y) \geq C(X \vee Y) + C(X \wedge Y).$$

But $E_P(X \vee Y) \leq C(X \vee Y)$ and $E_P(X \wedge Y) \leq C(X \wedge Y)$ for any $P \in \mathcal{Q}$. Hence, $E_P(X \vee Y) = C(X \vee Y)$ and $E_P(X \wedge Y) = C(X \wedge Y)$ for any $P \in \mathcal{Q}$ which implies by Lemma 11 that $X \vee Y$ and $X \wedge Y$ belongs to F. ■

Lemma 37 [17] *Let F be a Riesz subspace of $\mathbb{R}^n$ containing the unit vector $1_{\mathbb{R}^n} = (1, ..., 1) \in \mathbb{R}^n$ then F is a "partition" linear subspace of $\mathbb{R}^n$,* i.e.*, up to permutation:*

$$x \in F \text{ iff } x = (x_1, ...x_1, ..., x_j, ..., x_j, ..., x_m, ..., x_m).$$

Proof. The proof is by induction on the cardinality $\#S$ of $S \geq 1$. Clearly the result is true if $\#S = 1$, now assume that the result is true for $\#S = k$ and let us to show that it remains true for $\#S = k + 1$.

[16] See, for instance, Huber (1981) pages 260 and 261.

[17] For sake of completeness we give a direct proof of this result, which in fact has been obtained indepently by Polyrakis (1996, 1999).

So let F a subspace of $\mathbb{R}^{k+1}$ contaning $1_{\mathbb{R}^{k+1}}$, and let G defined by[18]:

$$G := \left\{ y = (x_1, ..., x_k) \in \mathbb{R}^k : \exists x_{k+1} \text{ s.t. } (y, x_{k+1}) \in F \right\}.$$

It is straighforward to check that G is a Riesz-subspace of $\mathbb{R}^k$ containing $1_{\mathbb{R}^k}$, therefore by the induction hypothesis and up to a permutation $y \in G$ is equivalent to $y = (x_1, ..., x_1, ...x_j, ..., x_j, ..., x_m, ..., x_m)$ where $x_j \in \mathbb{R}$, $1 \leq j \leq m$. Clearly, if $x \in F$ then $x \in \widetilde{G} \oplus \widetilde{H}$ the direct sum of the linear subspaces of $\mathbb{R}^{k+1}$ given by

$$\begin{aligned} \widetilde{G} &= \left\{ (y, 0) \in \mathbb{R}^{k+1} : y \in G \right\} \\ \widetilde{H} &= \left\{ (0, ..., 0, x_{k+1}) \in \mathbb{R}^{k+1} : x_{k+1} \in \mathbb{R} \right\}. \end{aligned}$$

Therefore, $\dim F \leq \dim \widetilde{G} \oplus \widetilde{H} = m + 1$. It is also immediate to see that $\dim F \geq m$: in fact, $y \in G$ is equivalent to

$$y = \sum_{j=1}^{m} x_j V_j^*,$$

where each $V_j^* \in \mathbb{R}^k$ (*i.e.*, $V_j \subset \{1, ..., k\}$) and $\{V_1^*, ..., V_m^*\}$ is a basis of G. Let $z_j \in \mathbb{R}$ be such that $\left(V_j^*, z_j\right) \in F$, $1 \leq j \leq m$; it is immediate to see that $\{\{V_1^*\}, ..., \{V_m^*\}\}$ linearly independent in G implies $\{\{V_1^*, z_1\}, ..., \{V_m^*, z_m\}\}$ linearly independent in F, hence $\dim F \geq m$.

Two cases have to be examined:

1. $\dim F = m + 1$: Clearly since $F \subset \widetilde{G} \oplus \widetilde{H}$, this implies that $F = \widetilde{G} \oplus \widetilde{H}$ and F is a "partition" space.

2. $\dim F = m$: In such a case since $\left\{ W_j^* := \left\{ V_j^*, z_j \right\}, 1 \leq j \leq m \right\}$ is linearly independent in F, $\left\{ W_j^* :, 1 \leq j \leq m \right\}$ is a basis of F. Hence, we obtain that $x \in F$ if and only if there exists x_j, $1 \leq j \leq m$ such that $x = \sum_{j=1}^{m} x_j W_j^*$, in particular,

$$x_{k+1} = \sum_{j=1}^{m} x_j z_j, \qquad (\Gamma).$$

So, it remains to show that there exists $j_0 \in \{1, ..., m\}$ such that for any $x \in F$ it is possible to write $x = \sum_{j=1}^{m} x_j V_j^* + x_{j_0}$. Note that is enough to

[18] For $y = (x_1, ..., x_k) \in \mathbb{R}^k$ and $x_{k+1} \in \mathbb{R}$ we use the following notation:

$$(y, x_{k+1}) := (x_1, ..., x_k, x_{k+1}) \in \mathbb{R}^{k+1}.$$

show that all the z_j´s equal to zero except $z_{j_0} = 1$. Since $1_{\mathbb{R}^{k+1}} \in F$ by the property above (Γ), we obtain that $\sum_{j=1}^{m} z_j = 1$.

Now take $j \neq i,\ j, i \in \{1, ..., m\}$. Since F is a Riesz space, $W_j^*, W_i^* \in F$ implies that $W_j^* \wedge W_i^* \in F$, but $W_j^* \wedge W_i^* = \left((V_j \cap V_i)^*, z_j \wedge z_i\right)$ and $V_j \cap V_i = \emptyset$, hence by the property (Γ) we obtain that $0 = \sum_{j=1}^{m} x_j z_j = z_j \wedge z_i$, therefore $z_j \geq 0$. On the other hand, the Riesz space struture implies also that $W_j^* \vee W_i^* \in F$, but $W_j^* \vee W_i^* = (1_{\mathbb{R}^{k+1}}, z_j \vee z_i)$, hence by the property (Γ) we obtain that $z_j \vee z_i = z_j + z_i$. Summing up, we have

$$\begin{aligned} \sum_{j=1}^{m} z_j &= 1, \text{ therefore for any } j \neq i,\ j, i \in \{1, ..., m\} : \\ z_j \wedge z_i &= 0 \text{ and } \ z_j \vee z_i = z_j + z_i; \end{aligned}$$

it implies that there exists a unique $j_0 \in \{1, ..., m\}$ such that $z_{j_0} = 1$ and for any $j \in \{1, ..., m\} \setminus \{j_0\}$ it is true that $j = 0$, the desired result.

■

Definition 38 *The mapping $C : X \in \mathbb{R}^S \to C(X) \in \mathbb{R}$ is the cost function of a "partition market" of $\{0, 1\}$-securities without arbitrage opportunities if:*

1) *There exist a list of events $B_1, ..., \ B_m \in 2^S$ forming a partition of S and there exists a striclty positive probability P_0 on 2^S such that $P_0(B_j) = C\left(B_j^*\right)$, for any $j \in \{1, ..., m\}$;*

2) *Denoting, $\mathcal{Q} = \left\{P \in \Delta : P(B_j) = C(B_j^*),\ 1 \leq j \leq m\right\}$ it turn out that $\forall$ $X \in \mathbb{R}^S$*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Theorem 39 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then the following assertions are equivalent:*

(i) C is the cost function of a market of securities with no-arbitrage opportunity, which is a Choquet integral;

(ii) C is the cost function of a "partition" market of $\{0, 1\}$-securities without arbitrage opportunities;

(iii) There exists a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S and such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s);$$

(iv) μ_C is concave, $\mu_C = \mu_C^*$, there exists at least a strictly positive probability $P_0 \in acore(\mu_C)$, and $\forall\ X \in \mathbb{R}^S$

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X),$$

(v) C satisfies,
(a) $\mathcal{E}_{\mu_C}$ is a Boolean algebra[19]*,*
(b) There exists a strictly positive probability P_0 belonging to $\mathcal{Q}_{\mu_C}$,
(c) For any contingent claim X,

$$C(X) = \max_{P \in \mathcal{Q}_{\mu_C}} E_P(X).$$

In any case, the set of attanaible claims is generated by the P_0-atoms[20] *of the Boolean algebra $\mathcal{E}_{\mu_C}$ and the set of all risk neutral probabilities are given by acore(μ_C).*

Proof. $(i) \Rightarrow (ii)$

By Lemma 36 we know that the set of attanaible claims F is a Riesz subspace of $\mathbb{R}^S$ containing the riskless bond S^*. Therefore, by Lemma 37 we obtain that F is a "partition" linear subspace of $\mathbb{R}^S$, hence C is the cost function of a "partition" market of $\{0,1\}$-securities without arbitrage opportunities.

$(ii) \Rightarrow (iii)$

By assumption we have a partition $\{B_1, ..., B_m\}$ of the state space S and a strictly positive probability P_0 such that $P_0(B_j) = C(B_j^*)$ for any $j \in \{1, ..., m\}$. Recall that,

$$\mathcal{Q} = \{P \in \Delta : P(B_j) = P_0(B_j), 1 \leq j \leq m\}$$

and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

hence since $E_P(X) = \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s)$. Now, denote by Q the risk neutral probability such that for any $j \in \{1, ..., m\}$,

$$Q(B_j) = Q(\{\widehat{s} \in B_j : X(\widehat{s}) = \max X(B_j)\}).$$

Hence,

$$\begin{aligned} C(X) &= \max_{P \in \mathcal{Q}} \left\{ \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \\ &= \sum_{j=1}^{m} \max_{P \in \mathcal{Q}} \left\{ \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \sum_{j=1}^{m} Q(B_j) \max X(B_j). \end{aligned}$$

[19] A family $\mathcal{E}$ of subsets of S is called a Boolean algebra if $\mathcal{E}$ contains S, it is closed for (finite) intersection and complement.

[20] Let $\mathcal{E}$ a Boolean algebra of subsets of S and P a probability measure over E, we say that an event $E \in \mathcal{E}$ is a P-atom if $P(E) > 0$ and for any $F \in \mathcal{E}$ such that $F \subset E$, $P(F) = P(E)$ or $P(F) = 0$. If P is strictly positive on the finite Boolean algebra $\mathcal{E}$, E is a P-atom iff $P(E) > 0$ and if $F \subset E$ and $F \neq \emptyset$ then $F \notin \mathcal{E}$.

Which allows us to write,

$$C(X) = \sum_{j=1}^{m} P_0(B_j) \max_{s \in B_j} X(s).$$

$(iii) \Rightarrow (i)$

By our assumption we have that there exists a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S and such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s).$$

Hence,

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j),$$

and it is well know that

$$C(X) = \int X d\mu_C,$$

which completes this part of the proof.

Remark 40 *Note that we proved that* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(ii) \Rightarrow (iv)$

From Theorem 32, it remains to prove that C is concave and that $\mu_C = \mu_C^*$. Take $A \subset S$, since $(ii) \Leftrightarrow (iii)$, it comes from (iii) that

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j);$$

since $P_0(B_j) > 0$ and $\sum P_0(B_j) = 1$, as it is well-known μ_C is plausibility function (*i.e.*, the dual of a belief function), hence μ_C is concave.

Remains to show that $\mu_C^* \leq \mu_C$. From Nehring (1999), we know that μ_C is concave implies that $\mathcal{E}_{\mu_C}$ is a Boolean algebra; let us show that it entails that μ_C^* is concave: Let A_1, A_2 be subsets of S, by definition of μ_C^* there exists $B_1 \supset A_1$ and $B_2 \supset A_2$, $B_i \in \mathcal{E}_{\mu_C}$ such that $\mu_C^*(A_i) = \mu_C(B_i)$, $i = 1, 2$. Hence, $\mu_C^*(A_1) + \mu_C^*(A_2) = \mu_C(B_1) + \mu_C(B_2) \geq \mu_C(B_1 \cup B_2) + \mu_C(B_1 \cap B_2)$. Since $B_1 \cup B_2, B_1 \cap B_2 \in \mathcal{E}_{\mu_C}$, $B_1 \cup B_2 \supset A_1 \cup A_2$ and $B_1 \cap B_2 \supset A_1 \cap A_2$, it turns out that $\mu_C^*(A_1) + \mu_C^*(A_2) \geq \mu_C^*(A_1 \cup A_2) + \mu_C^*(A_1 \cap A_2)$. Let $A \subset S$, μ_C^* concave implies that there exists a probability $P \in acore(\mu_C^*)$, but Theorem 32 guarantees that $acore(\mu_C) = acore(\mu_C^*)$ hence $P \in acore(\mu_C)$, therefore:

$$\mu_C^*(A) = P(A) \leq \mu_C(A),$$

which completes this part of the proof.

$(iv) \Rightarrow (v)$

Note that (a) comes from μ_C concave and the previously quoted result of Nehring (1999).

$(v) \Rightarrow (ii)$

By hyphotesis, there exists a strictly positive probability $P_0 \in \mathcal{Q}_{\mu_C}$ and $\mathcal{E}_{\mu_C}$ is a Boolean algebra. Let $\{B_1, ..., B_m\}$ be the collection of P_0-atoms of the Boolean algebra $\mathcal{E}_{\mu_C}$, hence $\{B_1, ..., B_m\}$ is a partition of S. Of course, $P_0(B_j) = C(B_j^*)$, for any $j \in \{1, ..., m\}$ and $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$. For $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$, note that if $P \in \Delta$ is such that $P(B_j) = \mu_C(B_j)$ for any $j \in \{1, ..., m\}$ then if $B \in \mathcal{E}_{\mu_C}$ and $B \notin \{B_1, ..., B_m\}$ hence there exist $\Lambda \subset \{1, ..., m\}$ such that $B = \cup_{j \in \Lambda} B_j$, therefore $P(B) = \sum_{j \in \Lambda} P(B_j) = \sum_{j \in \Lambda} \mu_C(B_j) = \mu_C(\cup_{j \in \Lambda} B_j) = \mu_C(B)$. Hence, C is a cost function of a "partition" market. ■

An immediate corollary follow as:

Corollary 41 *A cost function C is a Choquet integral if and only if the set of attanaible claims is a Riesz space.*

Remark 42 *We note that the class of cost functions that can be written as a Choquet Integral is linked to financial markets where* derivative markets *(in the sense of Aliprantis, Brown and Werner (2000)) are complete. A* derivative contingent claim *is any contingent claim that has the same payoff in states in which the payoffs of all securities are the same. A restatement of the result due to Ross (1976) provided by Aliprantis, Brown and Werner (2000) says that derivative markets are complete if and only if the vector space of attanaible claims is a Riesz subspace. Hence, by the previous proposition we have that* Choquet cost functions *describe the minimum-cost of superreplication in markets where derivative markets are complete.*

Example 43 *The most two simple examples of cost functions follow from the complete market case and the* "most incomplete" *market case under the existence of the bond. The first case is the market characterized by a probability measure $P \in \Delta$ such that*

$$C_P(X) := E_P(X) \text{ for any claim } X.$$

The second case, on the other hand, presents as available trade only the riskless asset 1_S. Of course, for any claim X

$$C_{\max}(X) = \max_{s \in S} X(s).$$

Moreover, we have the following market space

$$F_{C_P} = \mathbb{R}^S \text{ and } F_{C_{\max}} = \{k1_S : k \in \mathbb{R}\}.$$

One of the most well study class of markets are the *Arrow securities markets*. For these markets strutures the following definition is very natural for our analysis,

Definition 44 *Given a market* $\mathcal{M} = \{X_j, q_j;\ 0 \leq j \leq m\}$ *without arbitrage oportunities, we say that a state* $s^* \in S$ *is a Arrow state if* $\{s^*\}$ *is an unambiguous set. We denote by* E_0 *the union of all Arrow state,*

$$E_0 = \bigcup_{\{s^*\} \in \mathcal{E}} \{s^*\}.$$

Example 45 *Consider the following cost function*

$$C_A(X) = \sum_{s \in E_0} X(s)\, Q(\{s\}) + Q(E_o^c) \underset{s \in E_o^c}{max} X(s),$$

where $Q(E) \in (0,1)$*. Note that the cost of betting on the event* E *is given by the following capacity,*

$$\mu_{C_A}(E) = \begin{cases} Q(E), & E \subseteq E_0 \\ Q(E \cap E_0) + Q(E_0^c), & \text{otherwise.} \end{cases}$$

One possible underlying market of securities for this cost function is the market of Arrow securities and one bond given by:

$$\mathcal{M} = \left\{1_S, \left(1_{\{s_k\}}\right)_{k=1,\ldots,K}; 1, (q_k)_{k=1,..,K}\right\},$$

were E_0 *is the set of all Arrow states and* $q_k = Q(\{s_k\})$*. We dub* C_A *as a "Arrow cost function".*

Example 46 *Now, we give an example of market of a* $\{0,1\}$*-securities for which the cost function is not a Choquet integral*

$$\mathcal{M} = \left\{1_S, 1_{\{s_1,s_2\}}, 1_{\{s_2,s_3\}}; 1, q_1, q_2\right\}, \text{ where } q_1, q_2 > 0 \text{ and } q_1 + q_2 < 1.$$

For the cost function of this market the functional $C : \mathbb{R}^S \rightarrow \mathbb{R}$ *induces a capacity* μ_C *where*

$$\mu_C(\{s_1, s_2, s_3\}) + \mu_C(\{s_2\}) = (q_1 + q_2) + (q_1 \wedge q_2),$$

and

$$\mu_C(\{s_1, s_2\}) + \mu_C(\{s_2, s_3\}) = q_1 + q_2,$$

i.e., μ_C *is not concave. Moreover, the set of unambiguous events*

$$\mathcal{E}_{\mu_C} = \{\emptyset, S, \{s_1, s_2\}, \{s_3, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}\},$$

is not a Boolean algebra because the event $\{s_2\} = \{s_1, s_2\} \cap \{s_2, s_3\}$ *does not belong to* $\mathcal{E}_{\mu_C}$*.*

Example 47 *An example of Choquet integral that can not be a cost function is the Choquet integral w.r.t. epsilon-contaminated concave capacity. For instance,* *consider a strictly positive probability $Q \in \Delta$, a level $\varepsilon \in (0,1)$, and the following capacity:*

$$\lambda(A) = \begin{cases} (1-\varepsilon) Q(A) + \varepsilon, & A \neq \emptyset \\ 0, & A = \emptyset. \end{cases}.$$

Note that $acore(\lambda) = \{(1-\varepsilon) Q(A) + \varepsilon P : P \in \Delta\}$. *Consider the function* $C : \mathbb{R}^S \ni X \to C(X) = \int X d\lambda$. *In fact, for any contingent claim X it is true that*

$$C(X) = (1-\varepsilon) E_Q(X) + \varepsilon \max X(S).$$

The set of unambiguous events if given by $\mathcal{E}_\lambda = \{\emptyset, S\}$ *and by Theorem 39,*

$$C(X) = \max X(S),$$

and, of course, it is possible if and only if $\varepsilon = 1$. *Also, note that* $L_C = \mathbb{R}^S$ *and* $F_C = \{\alpha 1_S : \alpha \in \mathbb{R}\}$. *Hence, for any* $Q \in \Delta$ *and* $\varepsilon \in (0,1)$ *the set* $\{(1-\varepsilon) Q(A) + \varepsilon P : P \in \Delta\}$ *can not be a set of all risk neutral probabilites of any frictionless market.*

References

[1] Aliprantis, C.D. and R. Tourky (2002): *Markets that don´t replicate any option.* **Economics Letters** 76, 443-447.

[2] Aliprantis, C.D., D.J. Brown and J. Werner (2000): *Minimun-cost portfolio insurance.* **Journal of Economics Dynamics & Control** 24, 1703-1719.

[3] Arrow. K. (1953): *Le rôle des valeurs bousières pour la repartition la meillure des risques*, Colloq. Internat. Centre National de la Recherche Scientifique (Paris, 1952), Econometrie, 40, 41-47/ discussion, pp. 47-48, *CNRS* Paris, 1953. English translation: (1964) **Review of Economic Studies** 31, 91-96.

[4] Arrow, K. and G. Debreu (1954): *Existence of an equilibrium for a competitive economy.* **Econometrica** 22: 265-290.

[5] Baptista, A.M. (2007): *On the non-existence of redundant options.* **Economic Theory** 31, 205-212.

[6] Castagnoli, E., F. Maccheroni and M. Marinacci (2002): *Insurance premia consistent with the market.* **Insurance: Mathematical and Economics** 31, 267-284.

[7] Chateauneuf A. and J.-Y. Jaffray (1989): *Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion.* **Mathematical Social Sciences** 17: 263-283.

[8] Chateauneuf, A.(1991): *On the use of capacities in modeling uncertainty aversion and risk of aversion.* **Journal of Mathematical Economics**, 20, 343-369.

[9] Dunford, N. and J. T. Schwartz.(1988): **Linear Operators, Part I: General Theory**. A Wiley-Interscience Publication.

[10] Florenzano, M. (1999): *General equilibrium of financial markets: An introduction*. mimeo.

[11] Föllmer, H. and A. Schied (2004): **Stochastic Finance. An Introduction in Discrete Time**, 2nd Edition. de Gruyter Studies in Mathematics 27.

[12] Gale, D. (1960): **The Theory of Linear Economic Models**, McGraw-Hill, New York, NY.

[13] Huber, P. J. (1981): **Robust Statistics**. Wiley Series in Probabilities and Mathematical Statistics, John Wiley & Sons.

[14] Huber, P.J. and V. Strassen (1973): *Minimax tests and the Neymann-Pearson lemma for capacities*. **The Annals of Statistics** 1: 251-263.

[15] Lehrer, E. (2007): *Partially-specified probabilities: decision and games*. mimeo.

[16] Magill, M. and M. Quinzii (1996): **Theory of Incomplete Markets**. **Vol 1**. The MIT Press.

[17] Magill, M. and W. Shafer (1991): *Incomplete Markets*. in Chapter 30 of **Handbook of Mathematical Economics, Vol. IV**, Edited by W. Hildenbrand and H. Sonnenchein.

[18] McKenzie, L. (1954): *On equilibrium in Graham's model of world trade and other competitive systems*. **Econometrica** 22: 147-161.

[19] Nehring, K. (1999): *Capacities and probabilistic beliefs: a precarious coexistence*. **Mathematical Social Sciences** 38: 197-213.

[20] Polyrakis, I.A. (1996): *Finite-dimensional lattice-subspaces of* $C(\Omega)$ *and curves in* $\mathbb{R}^n$. **Transactions of the American Mathematical Society**, 348, 2793-2810.

[21] Polyrakis, I.A. (1999): *Minimal lattice-subspaces*. **Transactions of the American Mathematical Society**, 351, 4183-4203.

[22] Ross, S.A. (1976): *Options and efficiency*. **Quartely Journal of Economics** 90, 75-89.

[23] Schmeidler, D. (1972): *Cores of exact games I*. **Journal of Mathematical Analysis and Applications** 40: 214-225.

[24] Schmeidler, D. (1986): *Integral Representation without Additivity*. **Proceedings of the American Mathematical Society** 97(2): 255-261.