Equity Financing*
Preeliminary and Incomplete

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Abstract. This paper studies a model of corporate finance in which firms use stock issuance to finance investment. In contrast to the existing literature, we assume that the firm is "rational" and therefore recognizes the relationship between future dividends and stock prices. Under this assumption, future variables enter in the constraints of the firm, so that the problem is not recursive in a standard sense and the Bellman equation does not hold. This implies that the model has to be solved with recursive contracts methods such as the ones used, for example, in models of optimal macroeconomic policy or in risk sharing models with participation constraints. In addition, financial policy may be time inconsistent. First, we characterize several cases where time consistency arises. Second, we compare numerically the full commitment (and potentially time inconsistent) solution of a "rational" firm to the one of a "naive" firm that ignores the relationship between current price and future dividends. First, our results suggest that growing firms will pay lower dividends at the beginning and promise higher dividends in the future. This allows them to raise cheaper external funds through a higher value of stocks, accumulate more capital, and grow faster.

1. Introduction

Although there is an enormous amount of work on consumer choice in dynamic stochastic models, there is a lot less formal work on firms' choice of assets in order to finance investment in capital. This paper studies a general equilibrium model of corporate finance where firms use stock issuance to finance investment. The choice is, then, about how many stocks to issue, how much to pay out as dividends and when and how much to invest.

Some recent contributions such as Hopenhayn (1992), Cooley and Quadrini (2001), Covas and Den-Haan (2007), Quadrini and Jermann (2005), Gomes (2000) and Gomes et al (2006) among others have started to study interactions between firm financing, investment and output in dynamic stochastic models. These papers break the Modigliani Miller theorem by adding different types of financial costs. Our work differs from these papers in two important aspects. First, in order to break the Modigliani-Miller result, we assume that markets are incomplete and we impose that the objective of the manager is in conflict with the objective of market investors, the conflict arising from different degrees of risk aversion. Second, we focus on the following issue. Under rational expectations and non-bubble stock prices, market investors will only hold the stocks issued by the firms if the stock price is equal to the present discounted value of future dividends. Rational firms selling their stock in a

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competitive market should recognize this relationship between future dividends and current stock prices imposed by the market. In other words, they should realize that their own plans for future dividends influence the current stock price and, therefore, the amount of funds that can be raised by issuing stocks today. Note that this link has been previously ignored in the literature even though it can potentially affect the solution to the firm's problem. To see why this is the case, consider for example a growing firm, with an initial capital that is much lower than the long run capital. Higher future dividends imply a higher current stock price and therefore higher funds collected from equity issuance, allowing for higher investment and higher growth. A rational firm should take this trade-off into account and will have a tendency to lower today's dividends and increase future dividends in order to increase today's stock price.

Assuming that firms are rational poses a technical problem. If future variables (in this case, future dividends) enter the current firm's budget constraint, the problem of the firm is not recursive in the standard sense and the Bellman equation is not satisfied. In this case, the problem of the firm is of a similar nature to a problem of optimal macroeconomic policy, since it chooses today's dividend and stock issuance in a way consistent with promises about future dividends that induce shareholders to hold the stock. As in models of optimal policy, this financial policy is generally time inconsistent, the problem is not recursive in a standard sense and standard dynamic programming is not applicable. Using techniques of recursive contracts developed in Marcet and Marimon (2008) we formulate the rational choice recursively by adding a co-state variable that captures past promises about dividends to be issued today.

We argue that the lack of recursivity is a very general feature arising in most setups in which a firm uses stock issuance to finance its investment. It would also arise if a firm uses stock repurchases to distribute profits to the stockholders. In an incomplete markets setting, this is due to (i) either the disagreement among different types of stockholders (or among these and the manager) or (ii) sufficiently strong financial frictions. We find that financial frictions alone are not enough to generate time inconsistency. Time inconsistency arises when there is disagreement between the stockholders (or between these and the manager). For simplicity, we consider a firm that can not issue any kind of bond. Therefore, we do not address the equity/bond choice that a large part of the literature has focused on. However, the issues we mention would continue to arise if bonds were introduced as long as sufficient disagreement among stockholders or sufficient financing constraints are present.

We characterize analytically and numerically several special cases. In some cases, we find that financial policy is time consistent and in others it is not. This is not surprising, as the macroeconomics literature has also encountered many cases where the optimal fiscal or monetary policy under full commitment is time consistent. Even in these cases, however, the solution we find is often different from the one in which the firm does not take into account the relationship between future dividends and current prices. We denote this latter solution as "naive" because firms ignore that future dividends influence today's price.

Note that most of the literature has studied the "naive" solution in setups where this coincides with the rational solution that we study. One obvious case are the papers assuming either a fixed dividend rule or no equity issuance. Clearly, if no equity can be issued, there is no room for the stock price to play any role in influencing the firm's investment, since the price is irrelevant for firm funding. Another set of papers have assumed value maximization and financing frictions under which the solution follows the pecking order result (see e.g. Gomes et al (2006)). In this case, in any given period, either no dividends are paid or no new stock is issued. Under these assumptions, the naive solution often coincides with the rational solution and the rational solution is time consistent due to the implicit agreement between stockholders and managers. These papers represent an enormous progress relative to the previous literature and they have shed light on a number of issues. However, these are very special cases that are not validated by observed firm behavior, since many firms pay dividends and issue stocks in the same period. In order to go beyond these models, however, we argue that the equilibrium concept we discuss is the natural one.

As mentioned above, our paper is related to several strands in the literature. First, the empirical literature on firm financing is large but often does not use explicit modelling, making it hard to formulate hypotheses to be tested. Recently, the literature on firm dynamics mentioned above analyzes financial policy in a dynamic infinite horizon setting. In this literature, firms maximize their market value subject to financing frictions taking the stock price as given. In contrast, we assume conflicting objectives for the market investors and the managers and we abstract from issues such as firm heterogeneity and the size and age distribution of firms, which are central to those papers. Instead, we focus and explicitly model the dividend, stock and stock price interactions by allowing managers to anticipate the effects of dividend policy on the firm's stock price. Second, our paper is also related to a large strand of corporate finance literature. The possibility of time inconsistent financial policy is, in a way, implicitly recognized in the seminal work of Modigliani and Miller (1961), since their consideration that dividend policy is undetermined but that it gives today's value of the firm is taking into account the link we discuss. The difference is that, as we argue, this link influences choices every period because the stock price matters every period, while Modigliani and Miller only considered this in the price of the first period. The only explicit mention of time inconsistency in the literature of corporate finance we have found is in Miller and Rock (1985), in a model with two periods and private information. In contrast to this, our paper assumes full information but disagreement between stockholders and managers and we focus on the interaction between future dividends and current stock issuance. Given this, it provides an alternative reason for the presence of time inconsistency.

There are a number of issues that we do not address. Throughout the paper, we assume that firms have full commitment. That is, the manager announces in period zero all dividends that will be paid in the future under any realization and (due to costs of credibility loss or some other institutional arrangements that we do not model explicitly), this promise is fulfilled in the future. An alternative could have been to assume that firms follow time consistent policies and to use the solution techniques designed by Klein, Krusell and Ríos-Rull in a number of papers on optimal fiscal policy. Both solutions make extreme assumptions on the level of commitment of managers. But given the current knowledge of stochastic dynamic solution techniques, recursive contracts can be used in much larger problems, which is one reason for the researcher to assume full commitment. Furthermore, one interesting issue to study is what conditions and what institutions should be embedded in a firm so as to restore time consistency. For this purpose, a number of equilibrium concepts on how to sustain the full commitment solution have been developed in the macroeconomics literature and they could also be exported to the dynamic corporate finance problem that we address. We do not pursue these lines and simply assume full commitment.

Other important issues we do not address are why dividends are smooth or why dividends are paid at all. There is a very basic reason why firms pay dividends in our setup: under rational expectations and non-bubble prices zero dividends in all periods imply zero stock prices, so that the firm would be unable to finance itself with stock. As to dividend smoothing, many versions of our model simply impose that firm managers dislike dividend variability. This could be thought of as capturing an optimal contract where the firm's board has solved an agency problem determining that the payout to the manager should be according to dividend payments or the stock price of the firm. This has been justified in various ways in the literature of hidden information but we take it as given and discuss other issues. Finally, a potentially interesting extension that we do not address involves studying the case where the firm takes into account the effects of dividend policy on both stock prices and households' consumption. We leave this for future research.[1]

The paper is organized as follows. The model and its recursive formulation under full commitment is presented in Section 2. Section 3 presents several examples in the absence of uncertainty, some of which with an analytical solution. Section 4 discusses the issue of time inconsistency and it provides a proof of time inconsistency for the analytical example discussed previously. In addition, it presents several examples where financial policy turns out to be time consistent. Section 5 turns to numerical analysis of models with uncertainty, including a three period setting and an infinite horizon economy. Finally, Section 6 summarizes and concludes.

2. The Model

Time is discrete and indexed by $t = 0, 1, 2...$and the only source of uncertainty in the economy is an exogenous shock θ. The economy is populated by a continuum of identical investors and a firm. The cash flow of the representative firm is denoted by n. We study a production economy in which the cash flow is also a function of the aggregate capital stock k, which is determined endogenously. In this environment, cash flows are equal to total production minus investment, while production is achieved with capital accumulated in the last period and it is affected by a productivity shock, so $n_t = n(\theta_t, k_{t-1}, k_t) \equiv \theta_t f(k_{t-1}) - k_t + (1-d)k_{t-1}$.

The firm can only obtain financing by stock issuance. Every period t, the firm can issue new stocks that are traded at price p_t. Further, it distributes dividends d_t to the stockholders at the beginning of the period and it faces financial frictions that are represented by the function $\mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1})$. If we let s_t be the quantity of stocks outstanding at t, with $s_{-1} = 1$, the budget constraint of the firm is equal to:

$$d_t s_{t-1} + i_t \leq \theta_t f(k_{t-1}) + p_t (s_t - s_{t-1}) - \mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1}) \tag{1}$$

We also include a standard no-Ponzi game condition, requiring that total liabilities of the firm in the form of stocks, $(p_{t+1}+d_{t+1})s_t$, cannot grow faster than the interest rate. Formally, we require that in each period t,

$$\lim_{j\to\infty} E_t \delta^j \frac{u'(c_{t+j+1})}{u'(c_t)} (p_{t+j+1} + d_{t+j+1}) s_{t+j} = 0.$$

Notice that the budget constraint implies that the firm has to decide on investment under uncertainty subject to incomplete markets, since it can only issue stocks but cannot insure against shocks by issuing state contingent debt.

The stocks issued by the firm are bought by household-investors. Households can also trade in bonds that are assumed to be in zero net supply. They solve the following problem:[2]

$$E_0 \sum_{t=0}^{\infty} \delta^t u(c_t)$$

$$\text{s.t.} \quad c_t + p_t (s_{h,t} - s_{h,t-1}) + p_t^b b_{h,t} \leq d_t s_{h,t-1} + b_{h,t-1} \tag{2}$$

In the previous equation, s_h and b_h denote the holdings of stocks and bonds of the households. The price of the safe bond p_t^b is equal to:

$$p_t^b = \delta E_t \frac{u'(c_{t+1})}{u'(c_t)} \tag{3}$$

[1] Note that this case would be the closest one to the Ramsey optimal taxation literature. There is an important difference, however, since the Ramsey concept is typically associated with a benevolent government, while the firm is not benevolent in our framework.

[2] We implicitly assume that investors are subject to the natural borrowing limit.

Optimality also implies that the stock price depends on the stream of dividends according to the following equation:

$$p_t = \delta E_t \frac{u'(c_{t+1})}{u'(c_t)} [p_{t+1} + d_{t+1}] = E_t \sum_{j=1}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \tag{4}$$

We impose a non-bubble solution on stock prices requiring that $\lim_{j\to\infty} E_t \delta^j \frac{u'(c_{t+j})}{u'(c_t)} p_{t+j} = 0$.

Given the consumption process, equation (4) shows how the market maps a dividend process into a process for the stock price. In what follows we call this relationship the price-dividend mapping. It is important to note that this mapping reflects that the stock market is perfectly competitive. As stated in the introduction, we assume that the firm takes as given that the market imposes this mapping and it takes it into account when deciding on its financial policy. Authors who take stock prices as given ignore this interplay between dividends and stock prices.

We assume that firms take the consumption process as given. This is justified, for example, if there is a continuum of identical firms subject to the same shock or, more generally, if each firm has a minuscule impact on the consumption of the market stockholders.

We start by defining a competitive equilibrium where the cash flow and financial policy are taken as given. Next, we establish several results that will help us characterize how the cash flow and financial policy are determined optimally in a production equilibrium.

Definition 1. *A competitive equilibrium is given by a firm policy $\{d_t, s_t, n_t\}_{t=0}^{\infty}$, a vector of allocations for the households $x_h \equiv \{c_t, s_{h,t}, b_{h,t}\}_{t=0}^{\infty}$ and a vector of prices $\left\{p_t^b, p_t\right\}_{t=0}^{\infty}$ such that, (i) given the firm policy and price vectors, the vector x_h solves the problem of the households, (ii) the budget constraint of the firm is satisfied and (iii) markets clear. This implies that $c_t = n_t$, $s_{h,t} = s_t$ and $b_{h,t} = 0$ for all t and (4).*

The following results will be useful later on. Assume that there are no financing frictions. The budget constraint of the firm and the pricing equation in (4) imply:

$$\begin{aligned}(d_t + p_t) s_{t-1} &= p_t s_t + n_t \\ &= \delta E_t \frac{u(c_{t+1})}{u(c_t)} [(p_{t+1} + d_{t+1}) s_t] + n_t\end{aligned}$$

Substituting forward for $(p_{t+1} + d_{t+1}) s_t$ and using the no-Ponzi game condition and the fact that $(p_t + d_t) = E_t \sum_{j=0}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} d_{t+j}$, we obtain

$$(d_t + p_t) s_{t-1} = E_t \sum_{j=0}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} n_{t+j} \tag{5}$$

In addition, using again the budget constraint of the firm it follows that

$$p_t s_t = E_t \sum_{j=1}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} n_{t+j} \tag{6}$$

Given cash flows and dividend choices, we define the variables D_t and B_t as follows:

$$\begin{aligned}D_t &\equiv E_t \sum_{j=0}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} d_{t+j} \\ B_t &\equiv E_t \sum_{j=0}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} n_{t+j}\end{aligned}$$

where D_t represents the present value of dividends and B_t represents the present value of cash flows. Using this notation, Proposition 1 establishes two results regarding the firms's financial policy that we will use later on.

Proposition 1. *(a) For any sequence of cash flows, stocks and dividends, $\{d_t, n_t, s_t\}_{t=0}^{\infty}$, the period by period budget constraint of the firm in (1), the no-Ponzi game condition for the firm, and the market equilibrium condition (4) are satisfied at all t if and only if the following constraints are satisfied:*

$$s_{-1} E_0 \sum_{j=0}^{\infty} \delta^j \frac{u(c_j)}{u(c_0)} d_j = E_0 \sum_{j=0}^{\infty} \delta^j \frac{u(c_j)}{u(c_0)} n_j \tag{7}$$

$$\frac{B_t}{D_t} \text{ is measurable with respect to information up to } t-1 \text{ for all } t>0. \tag{8}$$

(b) In addition, given a cash flow process $\{n_t\}_{t=0}^{\infty}$, there are many feasible financial choices $\{d_t, s_t\}_{t=0}^{\infty}$ that are compatible with the firm budget constraint and the pricing equation (4). Further, under these different financial choices, the real allocations and firm value are unchanged.

Proof of Proposition 1. The proof of the proposition is provided in Appendix A.1.

The previous Proposition has several important implications. The first part implies that, in contrast to a framework where markets are complete, the period by period budget constraint of the firm is not equivalent to the period zero consolidated budget constraint in (7). Under incomplete markets, the measurability conditions (8) also need to be satisfied[3]. In other words, while many dividend sequences satisfy (7), not all of them are feasible, since they have to adjust so that equation (5) is satisfied at each period t.

To see this, assume, for example, that households are risk neutral and consider the constant stream of dividends $d_t = d = (1-\delta) E_0 \sum_{t=0}^{\infty} \delta^t n_t$. This clearly satisfies equation (7) under risk neutrality and the associated stock price is equal to $p = d\frac{\delta}{1-\delta}$. In such a case, the budget constraint of the firm would only be satisfied if:

$$s_{t-1} = \frac{E_t \sum_{j=0}^{\infty} \delta^j n_{t+j}}{d+p}$$

But if cash flows are stochastic, the right side of this equation depends on information up to t, while the left side can only be chosen contingent on information up to $t-1$ so that constant dividends are not feasible.

Second, the proposition establishes a Modigliani-Miller-like result in the absence of financing frictions, in the sense that many financial choices are feasible given a certain cash flow process. This will imply that financial policy will be irrelevant in certain economies. This result will be useful in determining what ingredients have to be introduced in the model so that it determines a unique policy for the firm.

2.1. The Problem of the Firm. This section discusses the problem of the firm in a production economy under different firm objectives that can have different implications regarding the relevance of financial policy. Such a model allows for the study of how stock issuance can be used to finance firm investment and how the latter is related to the firm's dividend policy.

[3]Notice that the proof of this part follows closely the reasoning of Proposition 1 in Aiyagari, Marcet, Sargent and Seppala (2002).

We assume that the firm owns and accumulates the capital stock that it uses for production each period. The cash flow of the firm is therefore given by:

$$n_t = \theta_t f(k_{t-1}) + (1-\eta)\, k_{t-1} - k_t \tag{9}$$

where η is the depreciation rate of capital. We assume that the firm maximizes the following objective:

$$\max_{\{d_t, s_t, k_t\}} \sum_{t=0}^{\infty} \delta^t v(d_t, d_{t-1}, k_t, k_{t-1}, s_t, s_{t-1}, p_t) \tag{10}$$

The previous objective encompasses several objective functions in the literature. The first one is value maximization, which is the usual objective in the literature. Note that the (cum-dividend) value of the firm V_0 at time $t = 0$ is equal to:

$$V_0 \equiv (p_0 + d_0)s_{-1} \tag{11}$$

Using the relationships derived earlier, we can re-write the value of the firm V_0 as:

$$V_0 = s_{-1} E_0 \sum_{t=0}^{\infty} \delta^t \frac{u'(c_t)}{u'(c_0)} d_t = E_0 \sum_{t=0}^{\infty} \delta^t \frac{u'(c_t)}{u'(c_0)} n_t \tag{12}$$

Here, we emphasize that this substitution implies using the price-dividend mapping that arises from the optimization problem of the investors in re-formulating the objective of the firm. This is one of the alternatives that we will consider. In this case, shareholders and managers will agree. In addition, to make stock issuance determined, we will have to assume financing frictions for the firm.

A second case is one in which the firm maximizes the value of dividend payments according to an increasing and concave utility function v:

$$W_0 \equiv E_0 \sum_{t=0}^{\infty} \delta^t v(d_t) \tag{13}$$

As we will show later, this second objective, which we label as a risk averse firm in what follows, can be interpreted as a case where a manager owns a fixed fraction of the firm and he cannot save. More generally, v can be justified as the contract that the manager has been offered to give him or her incentives to manage the firm properly. In a setting in which the optimal payout and investment are not observable, a manager who is restricted from overinvesting or keeping funds by linking his compensation to the payout. In other words, there may be a signalling problem, or hidden action mechanism in the background, that prompts the firm to offer a reward to the manager that is tied to the dividend of the firm. Indeed, many firms offer stocks or options as a form of payment to managers, and managers are not allowed to sell these assets for a long time. We can think of v as a reduced form of that incentive problem that we take as exogenous here but that, ideally, would be endogenized. Here we concentrate on the optimal stock issuance policy given v.

Another interpretation of this utility function is that there are two types of stockholders: market stockholders and internal stockholders. Market stockholders are the investors (households) described in the previous subsection. Internal stockholders are somehow tied to this firm, either because they founded the firm, or because their human capital is particularly useful in this firm; they run the firm and they decide how much to invest and how many stocks to issue, while the utility $v(d_t)$ represents their direct preferences on the firm's performance.

Finally, we will consider several cases regarding what the firm internalizes when it makes the financing decisions. In the benchmark economy, we assume that firms take into account the effects of dividend policy on prices, that is, firms know that (4) holds in all periods. We denote these firms as *rational.* It is important to note that papers in the literature assuming that the firm maximizes (12) and justifying this as value maximization already introduce the rational assumption, at least in part. To make the point clearer, notice that a manager who ignored (4) and who literally maximized the value of the firm taking stock prices as given would maximize (11). This manager would treat p_0 as outside of his control and decide that the optimum is to pay everything out as dividends today and close down the firm in one period. A manager could only go from (11) to (12) if he/she understood the link between future dividends and p_0.

In a later section, the rational equilibrium will be compared to the case where the firm is *naive*, in the sense that it does not take into account the effect of dividends on prices in the budget constraint. Some papers in the literature address issues of firm financing by bond issuance and restrict stocks to be constant (or shown to be constant in the optimum). In this case "naive" agents are the same a rational agents. But using the objective in (12) as representing value maximization and taking prices as given in the budget constraint would seem like an inconsistent assumption in models where stock issuance occurs in equilibrium.

2.2. The Rational Firm. The problem of a rational firm is given by:

$$\max_{\{p,d,s,k\}} V_0 \text{ or } W_0 \tag{14}$$

subject to

$$d_t s_{t-1} + k_t - (1-\eta)\, k_{t-1} = p_t\,(s_t - s_{t-1}) + \theta_t f(k_{t-1}) - \mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1}) \tag{15}$$

p and d satisfy the price-dividend mapping.

Equation (15) reflects that, in addition to stock issuance (external funds), the income of the firm is given by production (earnings or internal funds), which depends on past capital, today's productivity shock and the production function f. That the price and dividend sequences are competitive implies that (4) holds. In other words, the firm can use internal and external funds to face the investment expense $k_t - (1-\eta)\, k_{t-1}$ and pay dividends, knowing that the price at which it can sell the stock is related to today's price through (4). A naive firm will not take into account (4), since it does not internalize the effects of dividends on stock prices.

Definition 2. A rational competitive equilibrium is a vector of allocations for the households $x_h \equiv \{c_t, s_{h,t}, b_{h,t}\}_{t=0}^{\infty}$, a vector of allocations for the firms $x_f \equiv \{k_t, d_t, s_t\}_{t=0}^{\infty}$ and a vector of prices $\{p_t^b, p_t\}_{t=0}^{\infty}$ such that (i) x_h and x_f are a competitive equilibrium and (ii) x_f maximizes the objective of the firm.

The fact that (4) enters as a constraint in the problem of the firm means that this problem is not recursive, in the sense that the Bellman equation does not hold. This happens in many other models where future variables enter today's choice set. The famous time inconsistency problem studied by Kydland and Prescott points out that this occurs in models of fiscal or monetary optimal policy. That the Bellman equation does not hold in the optimum means that there is no ground to state that the optimal choice is a time invariant function of the natural state variables. This means that we should not expect that the optimal choice at time t, (s_t, d_t, p_t, k_t), is given by a time invariant function $F(s_{t-1}, k_{t-1}, \theta_t)$. There are additional complications arising from the fact that the Bellman equation does not hold. First, the value function is not a contraction mapping. Second, the optimal policy is time inconsistent, in the sense that the firm has incentives to promise a future path for dividends in period zero such that, if in a future period, say t', the firm is allowed to re-optimize (without any other institutional or reputational constraints that bind the firm), it will change the decision about variables dated $t > t'$ relative to the optimum stated in period zero. We have assumed that the firm is fully committed to the promises it makes in period zero and that it simply does not consider the possibility of re-optimizing in the future.[4]

The macroeconomic literature has recently devoted a large amount of effort to solve models of this sort and to study the time inconsistency issue. We first concentrate on the discussion of how to compute an optimum. To simplify the exposition, we focus on risk averse firms and no financing frictions. One can follow the approach of Marcet and Marimon (2008) and write the Lagrangian as:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \delta^t \Big[v(d_t) + \gamma_t u'(c_t) \left(\theta_t f(k_{t-1}) + (1-\eta) k_{t-1} - k_t - d_t s_{t-1} \right) \\ &\qquad + (s_t - s_{t-1}) E_t \sum_{j=1}^{\infty} \delta^j u'(c_{t+j}) d_{t+j} \Big] \end{aligned}$$

where γ_t is the multiplier associated with the period t budget constraint[5].

The presence of expectations of future variables in the expression above implies that the problem is not recursive yet. Nevertheless, the Lagrangian can be rewritten recursively by introducing a new state variable as follows:

$$L = E_0 \sum_{t=0}^{\infty} \delta^t \left[v(d_t) + \mu_{t-1} u'(c_t) d_t + \gamma_t u'(c_t) \left(\theta_t f(k_{t-1}) + (1-\eta) k_{t-1} - k_t - d_t s_{t-1} \right) \right]$$

where the co-state variable μ_t follows the law of motion:

$$\mu_t = \mu_{t-1} + \gamma_t (s_t - s_{t-1}) \text{ with } \mu_{-1} = 0. \tag{16}$$

After rewriting the problem in this way it is clear that future variables do not enter today's objective function and that now past μ's appear in the objective. This suggests that the optimal choice (with the additional assumption that θ is Markov) can now be found by looking for a policy function such that

$$(s_t, d_t, p_t, k_t, \gamma_t) = f(s_{t-1}, k_{t-1}, \theta_t, \mu_{t-1})$$

This policy rule, together with (16), determines the whole equilibrium path. Marcet and Marimon provide conditions guaranteeing that this is indeed the case and they show a saddle point functional equation that plays the role of the Bellman equation. In other words, the recursiveness of the equilibrium is recovered by introducing the new state variable μ and a new decision variable γ.

In the present setting, the multiplier μ_t captures the promises that have been made in the past about the dividend in period t, d_t. Since there are no past promises to be kept at the beginning of time, the optimal choice entails setting $\mu_{-1} = 0$. On the other hand, at $t = 1$, there is an inherited promise from period 0, $\mu_0 = \gamma_0 (s_0 - s_{-1})$, which arises from the fact that p_0 depends on the choice for future dividends and the firm (if it is fully committed to the optimal plan) will have to remember the promise made in all past periods about today's

[4] There is a recent literature studying time consistent equilibria in optimal policy in macroeconomics models where this equilibrium is different from the full commitment solution. See, for example, Klein, Krusell and Rios-Rull (2007). We will not consider these equilibria in this paper, while they would certainly be an interesting alternative to be studied.

[5] For convenience, we have multiplied the budget constraints by $u'(c_t)$, essentially renormalizing the multipliers γ_t.

dividend payments. Similarly, as we consider dividends further away in the future (d_2, d_3 etc.), these are linked with promises made in past periods. As reflected by its law of motion, the co-state μ_{t-1} adds up all of these past promises and summarizes them in a single number.

More intuition can be obtained by considering the first order conditions arising from this problem. These are given by:

$$v'(d_t) = \gamma_t s_{t-1} u'(c_t) - u'(c_t)\,\mu_{t-1} \tag{17}$$

$$\gamma_t u'(c_t)\,p_t = E_t[\gamma_{t+1} u'(c_{t+1})\,(d_{t+1} + p_{t+1})] \tag{18}$$

$$\gamma_t u'(c_t) = \delta E_t\left[\gamma_{t+1} u'(c_{t+1})\left(\theta_{t+1} f'(k_t) + 1 - \eta\right)\right] \tag{19}$$

The last two equations represent the stock Euler equation (18) and the capital Euler equation (19) respectively, which are fairly standard. We therefore focus on the condition describing the optimal dividend choice (17). As we see, a marginal increase in d_t yields a direct utility benefit of $v'(d_t)$ but it has a cost in terms of lost resources at t that is equal to $\gamma_t s_{t-1} u'(c_t)$. A naive firm that does not realize the relationship between its stock price and its dividend policy would only have to consider these two effects. In other words, a naive firm would have $\mu = 0$ in all periods.

On the other hand, a rational firm has to take into account the fact that the dividend choice at time t will affect stock prices in all previous periods. In particular, a marginal increase in d_t also implies increases in the stock prices of all previous periods and this in turn affects the resources available in all these periods. If the firm has been issuing stocks, this price effect is positive, since it implies more funds raised for the same level of stock issuance. Conversely, if the firm has been repurchasing stocks in the past, a price increase has a negative effect on resources.

The previous discussion implies that the multiplier μ_{t-1} summarizes the effect of a marginal change in d_t on all previous periods' resources, and it can be positive or negative depending on the history of stock issuance and repurchase. Thus, despite having a maximization problem where the Bellman equation does not hold, by adding the co-state variable μ, we can make the solution recursive. This is due to the fact that, even though the whole past history is needed to make decisions at any point in time t, the recursive contracts formulation allows us to summarize all the relevant information in just one variable, μ_{t-1}. The nature of time inconsistency is that the firm will always be tempted to follow a policy where μ is re-set to zero and only the fact that the firm is fully committed will prevent this from happening.

The above first order conditions, together with the budget constraint and price equations below, characterize the equilibrium:

$$s_{t-1} E_t \sum_{j=0}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} = E_t \sum_{j=0}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} n_{t+j} \tag{20}$$

$$p_t = E_t[\frac{u'(c_{t+1})}{u'(c_t)}(d_{t+1} + p_{t+1})] \tag{21}$$

It is easy to see that in an exchange economy, where the cash flow is exogenous, the system of equations that would characterize the equilibrium allocations would be the same as above except that condition (19) would not be present.

Consider now a naive firm that does not exploit the interaction between dividends and stock prices. In this case, the equilibrium is characterized by equations (20), (21), (18), (19) and by the following first order condition:

$$v'(d_t) = \gamma_t u'(c_t)\, s_{t-1} \tag{22}$$

In principle one would expect that the law of motion of the system above can be written as a time invariant function in the natural state variables $(\theta_t, k_{t-1}, s_{t-1})$.[6] In this case, for a given law of motion for prices, the Bellman equation applies and the solution is time consistent.

The equilibrium system of equations reflects that financial policy is fully determined when the firm is risk averse. In particular, the investment and financial decisions are linked in the present setup. To provide a more intuitive explanation of why this is the case, we now show that our firm objective implies that firms care about both maximizing cash flows and smoothing dividend payments. This can be done by plugging (7) into the objective function of the firm. The problem can then be rewritten as:

$$\max_{\{d,s,k\}} E_0 \sum_{t=0}^{\infty} \delta^t \left[v\left(d_t\right) - \gamma_0 \frac{u'\left(c_t\right)}{u'\left(c_0\right)} d_t + \gamma_0 \frac{u'\left(c_t\right)}{u'\left(c_0\right)} n_t) \right] \tag{23}$$

$$\text{st.} \quad (8)$$

where γ_0 is the Lagrange multiplier of (7). Expressed in this way, it is clear that the manager would like to maximize the expected, discounted *weighted* sum of two elements. The first element is $v\left(d_t\right) - \gamma_0 \frac{u'(c_t)}{u'(c_0)} d_t$ and it depends only on dividends, whereas the second element is the cash flow weighted by $\gamma_0 \frac{u'(c_t)}{u'(c_0)}$. This illustrates that investment and financing decisions are linked with risk averse firms, in the sense that a given financial policy will have to come up with a dividend policy that balances these two objectives. In this case, the presence of the part $v\left(d_t\right) - \gamma_0 \frac{u'(c_t)}{u'(c_0)} d_t$ in the objective can be interpreted as the manager caring about minimizing the variability of dividends for a given cash flow. If the manager cared only about this part of the objective function, he would not choose capital efficiently (as under value maximization), since he would use it to smooth dividends. In fact, the optimal behavior of the manager has to balance the optimality of the capital choice that maximizes the cash flows and is best for the consumers with the desire to smooth dividends.

In contrast, suppose that firms would maximize their market value according to (12). Note that this would correspond to the case where managers just care about the last part of the previous objective. In this case, it is easy to show that they would set capital so that it satisfies the following equation:

$$1 = \delta E_t \left[\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} \left(\theta_{t+1} f'(k_t) + 1 - \eta\right) \right] \tag{24}$$

In what follows, we refer to this value of capital as the value maximizing level of capital. In the absence of financing frictions, it is also easy to see using the results of Proposition 1 that many financial policies would be consistent with that level of investment. In fact, since the level of investment is independent of the financial policy of the firm, financial policy is indeterminate.

2.3. A Modified Setting. In this section, we show that a risk averse firm can be interpreted as one where managers hold a (fixed) number of stocks s_m. To see this, we now write a slightly different version of the model where the manager's stock holding is explicit. As before, we assume that there are two kinds of agents: managers and investors. Investors are

[6] We say "one would expect" because in general one can not rule out equilibria with more state variables. The equilibrium here is given by a fixed point in the space of pricing functions and, even though $(\theta_t, k_{t-1}, s_{t-1})$ is a minimum set of state variables, there could be other equilibria with more state variables. In situations like this the papers on dynamic stochastic equilibria often simply *assume* that agents follow a Markov strategy with the above set of state variables. This is what we do in the computed examples. Since the naïve equilibrium is not the focus of our paper, we do not discuss this issue any further.

all alike and they can invest in stocks of the firm that is run by the manager. The problem of the investors is given by:

$$\max_{\{c_t,s_{h,t}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \delta^t u_h(c_t) \text{ st.}$$

$$c_t + p_t(s_{h,t} - s_{h,t-1}) = d_t s_{h,t-1}$$

All investors have the same utility and initial stock holdings. The usual first order condition for an interior solution for stocks is given by:

$$p_t = \delta \; E_t \frac{u'(c_{t+j})}{u'(c_t)} \left[p_{t+j} + d_{t+j}\right] = E_t \left(\sum_{j=1}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right)$$

The manager runs the firm and decides on the investment and financial policy. In particular, he decides how much to invest, how much stock to issue each period and how much to pay out as dividends. The income of the manager is tied to the dividend paid by the firm. This could be because the manager owns a fixed number of stocks or because the salary of the manager is proportional to the dividends. Both cases give rise to the same model outcome. The manager has no other means of saving or dissaving and his consumption is given by:

$$c_{m,t} = s_m d_t$$

where s_m can be interpreted as the fixed number of shares the manager owns or as the constant that determines his salary. The problem of the manager is therefore given by:

$$\max_{\{d,s,k\}} E_0 \sum_{t=0}^{\infty} \delta^t u_m(d_t) \text{ st.}$$

$$d_t s_{t-1} = p_t(s_t - s_{t-1}) + n_t$$

where $u_m(d_t) \equiv v\left(d_t s_m\right)$ and $s_t = s_{h,t} + s_m$ is the total number of stocks.

It is important to note that the way this model is presented, it assumes that the manager's stocks are fixed and the investors' stocks can change. Clearly, this implies that the total number of stocks is also variable. On the other hand, this setup can also be rewritten as one where managers and investors can both change their *proportion* of the firm owned, since the proportion of the manager in the firm changes with a change in the number of stocks. Since this modified version explicitly models the presence of two different agents, we stick to this version in what follows.

3. Examples

This section presents different examples in which the solution of a rational firm might potentially be time inconsistent and different from the naive solution. Throughout these examples, investors are assumed to be risk neutral and there is no uncertainty. All the examples are solved numerically assuming that the initial capital stock of the firm is lower than the steady state value, which is reached after a finite number of periods T. The rational solution is compared to the naive solution and time inconsistency is verified by computing the case in which the firm is allowed to deviate in a given period.[7]

[7] The solution to examples 1 and 2 is not reported but can provided by the authors upon request.

In the first example, we assume value maximization and introduce the friction of costly equity issuance. Formally, the problem of the firm is given by:

$$Max_{\{k_t, s_t, d_t\}} \sum_{t=0}^{\infty} \delta^t d_t \text{ s.t.}$$

$$\begin{aligned}
d_t s_{t-1} &= F(k_{t-1}) - i_t + p_t (s_t - s_{t-1}) - \tau p_t (s_t - s_{t-1})^2 \\
i_t &= k_t - (1-\delta) k_{t-1} \\
d_t s_{t-1} &\geq 0,\ p_t (s_t - s_{t-1}) \geq 0 \\
p_t &= \sum_{j=1}^{\infty} \delta^j d_{t+j}
\end{aligned}$$

As mentioned earlier, a naive firm would ignore the last constraint, representing the price dividend mapping. First, an important feature of the solution to this problem is the pecking order result. In particular, we find that firms only issue equity along the growth path, in which case they do not pay any dividends. In contrast, when firms have sufficient funds to pay dividends, no equity is issued. Second, we find that the rational solution is time consistent and equal to the naive solution.

In the second example, we assume value maximization with costs of equity issuance and costs of changing dividends. Formally, the firm solves:

$$\max_{\{d_t, k_t, s_t\}} \sum_{t=0}^{\infty} \delta^t d_t \text{ s.t.}$$

$$\begin{aligned}
d_t s_{t-1} &= F(k_t) - k_t + (1-\delta) k_{t-1} + p_t (s_t - s_{t-1}) \\
&\quad - \tau p_t (s_t - s_{t-1})^2 - \tau_d (d_t - d_{t-1})^2 \\
d_t s_{t-1} &\geq 0,\ p_t (s_t - s_{t-1}) \geq 0 \\
p_t &= \sum_{j=1}^{\infty} \delta^j d_{t+j}
\end{aligned}$$

First, in contrast to the solution to the previous example, this case does not result in a pecking order solution. In particular, the desire of firms to smooth dividends over time implies that both dividend payments and equity issuance occur simultaneously. Second, we find that the rational solution is time consistent but different from the naive solution.

In the third example, we assume that firms are risk averse and they are subject to a bound on issuance. Formally, firms solve:

$$Max_{\{k_t, s_t, d_t\}} E_0 \sum_{t=0}^{\infty} \delta^t v(d_t s_m) \text{ s.t.}$$

$$\begin{aligned}
d_t s_{t-1} &= F(k_{t-1}) - k_t + (1-\delta) k_{t-1} + p_t (s_t - s_{t-1}) \\
s_t - s_{t-1} &\leq \Delta \\
p_t &= E_t \sum_{j=1}^{\infty} \delta^j d_{t+j}
\end{aligned}$$

where s_m are the fixed stockholdings of the manager, which we normalize to one. First, as in example 2, the solution to this problem is not characterized by a pecking order result. In this case, the fact that the firm has a motive for dividend smoothing embedded in the objective implies that both equity issuance and dividend payments occur simultaneously. Second, we now find that the rational solution is both time inconsistent and different from the naive solution.

The results obtained in the previous examples are summarized by the following Table:

Table 1: Examples with different objectives and frictions

Objective	Costly Issuance	$s_t - s_{t-1} \leq \Delta$	Costly div. changes	N=R	TC
VM	Yes	No	No	Yes	Yes
VM	Yes	No	Yes	No	Yes
$v(d)$	No	Yes	No	No	No

As reflected by the last columns, the rational solution is time consistent in the presence of value maximization, whereas there is time inconsistency when firms are risk averse. This suggests that time inconsistency is a consequence of the disagreement between stockholders and managers. In addition, the table reflects that the naive solution is different to the rational solution in the two cases in which the pecking order result breaks. This suggests that one can ignore the price dividend mapping when the solution has the pecking order property. It is important to note that, even under value maximization, the pecking order result might not obtain for some type of financial frictions.

3.1. Analytical Solution to Example 3. To better illustrate the difference between rational and naive firms, we present the analytical solution to the third an example. We assume that initial capital is relatively low with respect to the steady state capital. This implies that the firm is growing over time. Thus, the role of stock issuance is, precisely, to provide funding to invest capital so that the firm can operate at the optimal level given by the golden rule. Clearly, both types of firms would achieve the optimal capital after one period in the absence of uncertainty. Given this, we also assume that there is a maximum amount of stock that can be issued in the first periods. This can be justified by the presence of transaction costs or due to the manager disliking that too many stocks are distributed, since this would cause a loss of his control.

As mentioned earlier, the naive manager solves:

$$\max_{\{d_t, s_t, k_t\}} \sum_{t=0}^{\infty} \delta^t v(d_t s_m) \quad \text{s.t.}$$

$$\begin{aligned} d_t s_{t-1} + k_t - (1-\eta)k_{t-1} &= p_t(s_t - s_{t-1}) + f(k_{t-1}) && (25) \\ s_t - s_{t-1} &\leq \Delta && (26) \\ k_{-1}, s_{-1} & \quad \text{given} && (27) \end{aligned}$$

where $s_t = s_{h,t} + s_m$ and $\Delta > 0$ is a fixed constant limiting the amount of stocks that can be issued. In the rational case, the manager also takes into account the following constraint:

$$p_t = \sum_{j=1}^{\infty} \delta^j d_{t+j} \tag{28}$$

As stated earlier, we assume that initial capital is much lower than the steady state capital. Formally, the steady state capital, which we denote by k^{GR} for 'golden rule', satisfies:

$$1 = \delta \left[f'(k^{GR}) + 1 - \eta \right] \tag{29}$$

and we assume that $k_{-1} < k^{GR}$.

No Bounds on Stock Issuance: $\Delta = \infty$. As mentioned above, in the absence of uncertainty, the rational firm would be able to achieve the complete market solution if the constraint (26) was not present. That is, if $\Delta = \infty$, the rational manager would be able to issue a sufficiently large amount of stocks in the first period to finance the desired accumulation of capital at $t = 0$, achieving the first best capital in one step. In fact, the manager would be able to complete the markets with stock issuance so that $k_t = k^{GR}$ for all $t \geq 0$. In contrast, we now show that the naive firm would not be able to achieve this allocation. We do this analytically for the case with $v(.) = \log(.)$.

Result 1. *When $\Delta = \infty$, $v(.) = \log(.)$ and $k_{-1} < k_s$, the naive firm allocations are*

$$\begin{aligned}
k_t^N &= k^{GR} = \left[\frac{\frac{1}{\delta} - 1 + \eta}{\alpha}\right]^{\frac{1}{\alpha-1}} \\
s_t^N &= \bar{s}^N \ \text{ for } t \geq 0 \\
d_t^N &= \bar{d}^N \text{ for } t \geq 1 \\
d_0^N &= \frac{n^{GR}}{s_{-1}} \\
p_t^N &= \bar{p}^N = \frac{\delta}{1-\delta}\bar{d}^N
\end{aligned}$$

where k^{GR} denotes the 'golden rule' level of capital and n^{GR} is the corresponding cash flow. Further, the rational firm allocations are

$$\begin{aligned}
k_t^{FR} &= k^{GR} = \left[\frac{\frac{1}{\delta} - 1 + \eta}{\alpha}\right]^{\frac{1}{\alpha-1}} \\
s_t^{FR} &= \bar{s}^{FR} \ \text{ for } t \geq 0 \\
d_t^{FR} &= \bar{d}^{FR} \text{ for } t \geq 0 \\
p_t^N &= \bar{p}^{FR} = \frac{\delta}{1-\delta}\bar{d}^{FR}
\end{aligned}$$

Result 2. *The allocations of the naive and rational firms satisfy the following relationships:*

$$d_0^{FR} < d_0^N, \ \bar{d}^{FR} > \bar{d}^N, \ \bar{s}^{FR} < \bar{s}^N, \ \bar{p}^{FR} > \bar{p}^N$$

Proof of Results 1 and 2. The proof of these results is provided in Appendix A.2.

The above allocations imply that both firms issue stocks in the first period and invest enough to jump to the optimal level of capital immediately. As a result, real allocations and the value of the firm are the same in both cases. However, financial policy and manager's welfare differ. Compared to the naive firm, the rational firm pays less dividends in the initial period and more in all future periods. This implies smoother dividends under rational firms with the added benefit that the stock price is always higher. This is irrelevant from period 1 onwards since there is no stock issuance, but it is important in period 0 in which both firms issue stocks.

This 'cheaper' external finance under rational firms that arises from credible promises about future dividends, allows the rational firm to implement the complete markets allocation and achieve both smooth dividends and optimal capital accumulation[8]. In contrast, the naive firm has to sacrifice dividend smoothing to achieve the optimal investment. Even more

[8] As will be shown in example 2 of section 4, the allocations of this example are time-consistent.

interestingly, the naive firm needs to issue more stocks to achieve the optimal investment. This provides a hint that, in the presence of financial frictions (costly or limited external finance), the naive firm would be unable to invest as much as the rational firm. We explore this below by choosing a finite and relatively tight bound Δ.

Bounds on Stock Issuance: $\Delta < \infty$. Suppose there is a bound on stock issuance for the first T periods

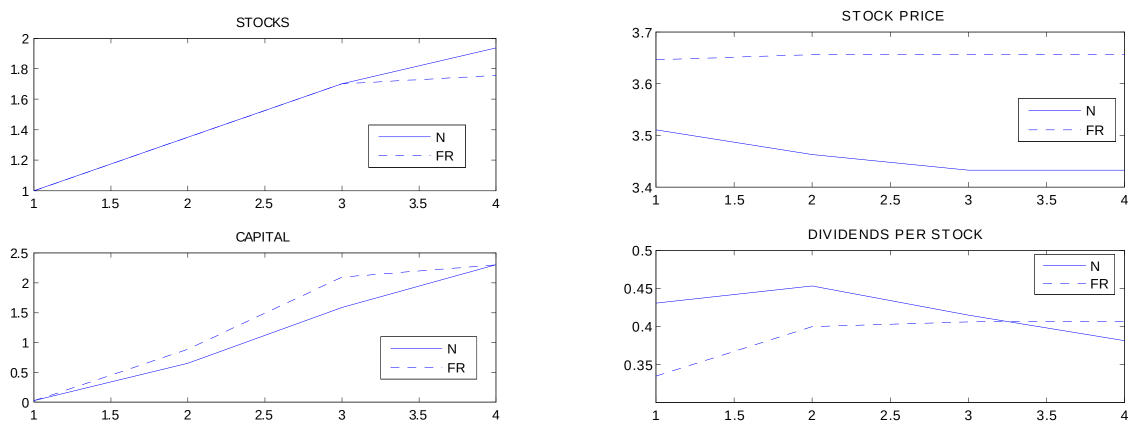
$$s_t - s_{t-1} \leq \Delta \text{ for } 0 \leq t \leq T-1$$

and no bound after that. We choose T and Δ so that the bound is binding for all these initial periods. Starting at period $t = T$ and given s_{T-1} and k_{T-1}, the continuation problem is one without bounds on stock issuance with the solution given by the one in the previous section. We now analyze the naive and rational firm policies for this setup, where the policy for the rational firm turns out to be time inconsistent[9].

Since the bounds on stock issuance are binding for both the naive and rational firms, the levels of stock issuance are equal for both. However, the rational firm uses the price dividend mapping to obtain higher levels of external finance. In particular, by promising high dividends in the future, it ensures that the competitive price for its stock is higher and thus its external finance is higher for the same level of stock issuance. In turn, higher funds raised externally imply that the rational firm can grow faster and reach the optimal level of capital earlier.

In what follows, we provide an example with $T = 2$ and $\Delta = 0.35$, in which the limit on stock issuance binds in both periods. The rest of the parameters are as follows: $\delta = 0.9$, $\eta = 0.1$, $\alpha = 0.36$, $\gamma = 1$ and $s_m = 1$. Initially, investor's stock holdings are one half of the total stocks, that is, $s_{-1} = s_{h,-1} + s_m = 2$. We consider a startup firm, that is, a firm that starts at a very low level of capital, $k_{-1} = 0.01k^{GR}$. The results are displayed in figures 1 and 2, where the solid lines depict the growth paths for the naive firm and the dashed lines depict those for the rational firm.

Figure 1



Stocks and Capital Stock | Stock Price and Dividends per Stock

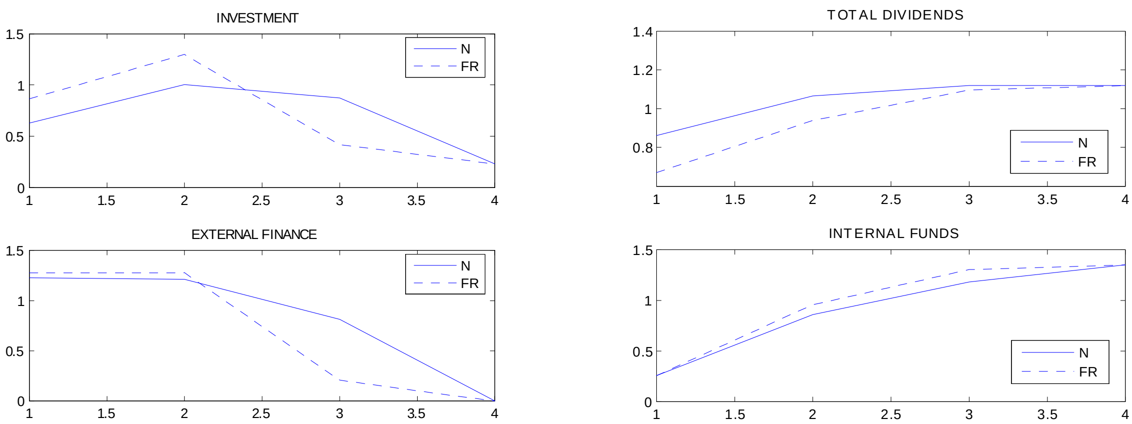
The left panel of figure 1 depicts the stocks held by households starting at $s_{h,-1} = 1$. As we see, they grow at the same speed for both firms and for the first two periods. This illustrates that the limit on issuance binds for both firms, who are able to issue only Δ stocks in each of the two periods. Capital starts at a very low level and steadily grows towards the

[9] This is proven in Section 4.

steady state. The steady state is reached in the first period in which the limits don't bind, which is period 3. But the rational firm manages to invest more initially and approach the steady state faster than the naive firm. This is for two reasons. First, the firm pays less dividends in the beginning, releasing more funds for investment. Second, the dividend policy includes a promise of high future dividends, which has the additional effect that stock prices are higher. As a result, a higher level of external funds is raised, allowing the firm to invest more initially and grow faster than a naive firm. Since the limit on stock issuance is not binding any more in the third period, both firms reach the golden rule level of capital and stop issuing any additional stocks.

The figures also reflect that the jump to the optimal capital requires much less stock issuance both because it is closer to the steady state and because its stock price is higher. The two firms remain at this level of capital and stocks from then on. Figure 2 depicts the levels of investment, external finance, total dividends and internal funds. As we see, the total dividend payout from then on is constant and equal for the two firms. However, having used less stock issuance to grow, the rational firm is in a position to pay higher dividends per stock from then on and maintain a higher price forever. Clearly, the rational manager achieves a higher welfare by using the additional knowledge from the price-dividend mapping. More interestingly, the value of the firm is also higher under rational managers, implying that investors are also better off.

Figure 2



Investment and External Finance — Total Dividends and Internal Funds

4. Time Consistency

This section discusses the issue of time inconsistency that can potentially arise under rational firms. As shown in the previous section, it seems that a rational firm can in general improve its stance by credibly promising a certain path for dividends. In this way, the firm can achieve a higher price today for a certain number of stocks issued. In particular, when the firm needs to raise external funds, it can do so by increasing the amount of stocks outstanding. Given this, the firm might want to drive today's price up by paying a low dividend today and promising at the same time a stream of high future dividends. However, when tomorrow comes and investors have already bought the firm's stock, it is likely that the manager has an incentive to deviate if he is not fully committed. This is due to the fact that adjusting the dividends downwards will not affect the stock price, which depends only on future dividends. In other words, if the institutions allow him to do so, it seems that it is better for the manager to renege on past promises. In other words, the policy under rational firms is likely to be time inconsistent.

To analyze this issue, we first describe a standard definition of time consistency. Let the full commitment solution be given by $x_0^* \equiv \{c_t^*, d_t^*, s_t^*, p_t^*, n_t^*\}_{t=0}^{\infty}$. Given a time period $\bar{t}$, define the "time-$\bar{t}$" continuation problem as:[10]

$$\max_{\{d_t, s_t\}_{t=\bar{t}}^{\infty}} E_{\bar{t}} \sum_{t=\bar{t}}^{\infty} \delta^{t-\bar{t}} v\left(d_t s_m\right) \text{ st.}$$

$$\text{(1)-(4) for all } t \geq \bar{t}$$
$$s_{\bar{t}-1} = s_{\bar{t}-1}^*$$

Denote the solution to this problem by $x_{\bar{t}}^{**} \equiv \{c_t^{**}, p_t^{**}, d_t^{**}, s_t^{**}, n_t^{**}\}_{t=\bar{t}}^{\infty}$. Note that this is the solution that would arise if, having followed the full commitment solution up to time $\bar{t}$, the manager decided to re-optimize and choose the best solution from then on, ignoring the plans that were involved in the solution x_0^* that was optimal from the standpoint of period zero.

Definition 3. *We say that the problem is time consistent at time $\bar{t}$ if*

$$\begin{bmatrix} c_t^{**} \\ p_t^{**} \\ d_t^{**} \\ s_t^{**} \end{bmatrix} = \begin{bmatrix} c_t^{*} \\ p_t^{*} \\ d_t^{*} \\ s_t^{*} \end{bmatrix} \quad \text{for all } t \geq \bar{t} \tag{30}$$

Also, we say that the problem is time consistent if it is time consistent for all $\bar{t} > 0$.

As already noted before, the fact that time inconsistency may arise in the present setup is reflected formally in the recursive formulation of the last section. As we see, the same (time-invariant) policy function F has to be used for all periods with μ_{t-1} as an argument. Further, μ_{t-1} is determined endogenously every period from past actions and it captures promises that have been made about today's dividends. The fact that there are no past commitments in the first period is reflected in $\mu_{-1} = 0$. Further, that a manager is tempted to re-optimize is reflected in the fact that, if he was allowed to do so in period t' without being restricted to honor past commitments, he would want to follow a policy that implies re-setting $\mu_{t'-1} = 0$ and following the optimal policy F from then onwards. If the manager is fully committed to following the announced policy, however, he will plug in the actual $\mu_{t'-1}$ in the policy function.

Although it is well known that time-inconsistency can arise in models when a constraint such as (4) that involves values of future choice variables, it is sometimes the case that the solution displays time consistency in these models. In what follows, we discuss a case where the solution is time inconsistent and several cases where the full commitment solution is time consistent. We begin by showing that the solution is time inconsistent in the absence of uncertainty but with limits on stock issuance and a capital that starts below steady state. This corresponds to the example discussed in Section 3.

Proposition 2. *In a production economy with no uncertainty, bounds on stock issuance for the initial T periods and initial capital less than the steady state, the problem is time inconsistent.*

The previous proposition shows that the example described in Section 3 exhibits time inconsistency. Further, Proposition 3 characterizes several other cases where the solution is time consistent, implying that the plans the manager makes for future dividends and stock

[10] Throughout the analysis, the list of variables would include k in the production economy.

issuances will indeed be fulfilled in the future, even if the manager is offered the opportunity to re-optimize. In particular, the proposition shows that time consistency arises in exchange economies, where the cash flow is exogenous, or in production economies when the value maximizing level of capital is implemented.

Proposition 3: *Under VM, the solution is time consistent.* [11]

The previous proposition confirms our conjecture, that the solution is time consistent in the absence of disagreement between shareholders and managers, as is the case if firms maximize their market value. The intuition for this result is simple. Under value maximization, the price dividend mapping is equal to the objective of the firm. Given this, if the solution to the continuation problem implies a higher value (or a higher stock price), this implies that the original solution is not optimal. In other words, under value maximization, there is no possible deviation that increases the objective of the manager.

Proposition 4. *The solution is time consistent in economies where the capital allocation is at the first best, that is, it satisfies*

$$u'(c_t^*) = \delta E_t \left[u'(c_{t+1}^*) \left(\theta_{t+1} f'(k_t^*) + 1 - \eta \right) \right]$$

Proposition 4 shows that the dividend policy is time consistent in exchange economies or in production economies where the value maximizing level of capital is chosen under full commitment. In these cases, there is no conflict between the objective of smoothing dividends and maximizing income. In general, however, the value maximizing level of capital is not the full commitment solution. In this case, time consistency requires that the gamma increments in (44) of the "continuation problem" are the same as in the full commitment case, since this always ensures that the first order conditions with respect to dividends and stocks of the "continuation problem" are satisfied. But these gammas cannot satisfy the first order condition with respect to capital (equation (19)) in general, as shown for example in Proposition 2.

In what follows, we discuss some additional variants of the model with production where time consistency arises. For simplicity of exposition, these cases are discussed assuming that investors are risk neutral.

Example 1. Constant dividends. Assume that, for some reason, the full commitment solution implies that dividends are constant, that is,

$$d_t = \overline{d} \text{ for all } t$$

This would occur, for example, if there was no uncertainty. In addition, it would occur if we introduced a full array of contingent claims. If dividends are constant, it is easy to see from (17) that γ_t will be constant, and the first order condition for capital in (19) implies that it will be set at the value maximizing level. In this case, the arguments of Proposition 3 apply and we have time consistency.

Example 2. No uncertainty after T. Consider a special case, where productivity is previously known after T, that is,

$$\theta_t = \overline{\theta} \text{ for } t \geq T$$

[11] This result has been indicated to us by Wouter Den Haan and the proof in the appendix follows his derivations.

for a predetermined constant $\overline{\theta}$. Using the previous arguments, it is easy to see that the value maximizing level of capital will be implemented from period T onwards. In particular, notice that the following solution satisfies all the first order conditions for $t \geq T$:

$$\begin{aligned} k_t &= \overline{k}, \text{ where } 1 = \delta\left[\overline{\theta} f'(\overline{k}) + 1 - \eta\right] \text{ for } t \geq T \\ \gamma_t &= \gamma_T \text{ for } t \geq T \\ d_t &= \overline{d}(k_{T-1}, s_{T-1}) \\ &\equiv \frac{1-\delta}{s_{T-1}}\left(\overline{\theta} f(k_{T-1}) - \overline{k} + (1-\eta)k_{T-1} + \delta\frac{\overline{\theta} f(\overline{k}) - \eta\overline{k}}{1-\delta}\right) \text{ for all } t \geq T \end{aligned} \tag{31}$$

where the third equation gives the constant level of dividends that satisfies the budget constraint in (20) given the states k_{T-1} and s_{T-1} in that period. In addition, the levels for stocks after T will be given by:

$$s_t = s_T = \frac{\overline{\theta} f(\overline{k}) - \eta\overline{k}}{\overline{d}} \quad \text{for all } t \geq T$$

because this guarantees (20) for $t \geq T$. In other words, the value maximizing capital stock, dividends and stocks will be constant from period T onwards.

Here, we have used the quotes because these dividends are only "optimal" contingent on the state variables k_{T-1}, s_{T-1}, which are themselves random. This makes the dividends $\overline{d}(k_{T-1}, s_{T-1})$, in principle, random. Note also that the capital stock is set at the risk neutral level from period T onwards, as it would have been under complete markets. Note also that, even though the dividend is constant, it is not the one that would have been achieved with complete markets for the actual initial condition, since the shocks up to period T will influence the realized k_{T-1}, s_{T-1}. In other words, the long run level of d is stochastic and will be different from the one with complete markets.

By the previous arguments, it is clear that the model will be time consistent after T. In Appendix B we provide a proof that the value maximizing capital will also be chosen at $T-1$, implying that the model is also time consistent at $T-1$.

Example 3. Constant Productivity after T. Consider another special case where randomness stops at period T, so that:

$$\theta_t = \theta_T \text{ for } t \geq T$$

In such a case, it is clear that the risk neutral capital level will be implemented from period $T+1$ onwards. To see this, notice that the following solution satisfies all the first order conditions for $t \geq T$:

$$\begin{aligned} k_t &= \overline{k} \quad \text{where } 1 = \delta\left[\theta_T f'(\overline{k}) + 1 - \eta\right] \text{ for all } t \geq T+1 \\ \gamma_t &= \gamma_T \text{ for all } t \geq T+1 \\ d_t &= \overline{d}(k_T, s_T) \\ &\equiv \frac{1-\delta}{s_T}\left(\overline{\theta} f(k_T) - \overline{k} + (1-\eta)k_T + \delta\frac{\overline{\theta} f(\overline{k}) - \eta\overline{k}}{1-\delta}\right) \text{ for all } t \geq T+1 \end{aligned} \tag{32}$$

As before, $\overline{k}$ is random, since it is a function of θ_T. Using similar arguments to the ones used in the previous example, it is possible to show that the model is time consistent after period T, but again it is unlikely that it is the case in earlier periods.

Example 4. Finite Horizon and Full Capital Depreciation. Assume that the economy only lasts for a finite number of periods and capital depreciates fully. In this case, the risk neutral level of capital will be chosen and the solution will be time consistent. To see this, consider the last period. The first order condition with respect to dividends in (17) can be rewritten as:

$$v'(d_T) = v'(d_{T-1}) + s_{T-1}(\gamma_{T-1} - \gamma_T)$$

implying that:

$$\gamma_T = \gamma_{T-1} + \frac{v'(d_{T-1}) - v'(d_T)}{s_{T-1}}$$

Substituting for γ_T into the capital Euler equation in (19), which is given by:

$$\gamma_{T-1} = E_{T-1}\left[\gamma_T \delta\theta_T f'(k_{T-1})\right]$$

we obtain:

$$\gamma_{T-1}\left[1 - \delta\theta_T f'(k_{T-1})\right] = E_{T-1}\left[\frac{v'(d_{T-1}) - v'(d_T)}{s_{T-1}}\delta\theta_T f'(k_{T-1})\right]$$

Note that just need to show that the right hand side of the previous equation is equal to zero, that is,

$$E_{T-1}\left[\frac{v'(d_{T-1}) - v'(d_T)}{s_{T-1}}\delta\theta_T f'(k_{T-1})\right] = 0 \tag{33}$$

To see that this is the case, note first that the last period budget constraint is given by:

$$d_T s_{T-1} = \theta_T f(k_{T-1})$$

implying that

$$\theta_T f'(k_{T-1}) = \frac{d_T s_{T-1}}{k_{T-1}}$$

Replacing the previous expression in (33), it follows that we just need to show that:

$$E_{T-1}\left[v'(d_{T-1}) - v'(d_T)d_T\right] = 0$$

On the other hand, the first order condition with respect to stocks implies that:

$$E_{T-1}(\gamma_T d_T) = \gamma_{T-1}E_{T-1}d_T$$

and replacing γ_T in terms of γ_{T-1} we get

$$E_{T-1}\left[\frac{u'(d_{T-1}) - u'(d_T)}{s_{T-1}}\right]d_T) = 0$$

This proves that:

$$1 = \delta f'(k_{T-1})\, E_{T-1}(\theta_T)$$

Following the same steps for the case where depreciation is not equal to one, it is easy to show that we would also need to have that $E_{T-1}\left[v'(d_{T-1}) - v'(d_T)\right] = 0$, which is unlikely to be satisfied. Given that the solution is likely to be time inconsistent with partial depreciation, we study this case numerically in the next sections.

5. Numerical Examples

This section analyzes numerically several examples with uncertainty where the solution is likely to be time inconsistent. We start with a three-period version of the model and a later section analyzes the infinite horizon economy. As stated earlier, we compare the financial and real allocations as well as the prices for the cases with naive and rational firms. Even though our solution is the one under full commitment, we also discuss the time-inconsistency issues. This serves to confirm that the solution is time inconsistent in the case that the friction arises from lack of full insurance.

5.1. A Three Period Example. For this three period example, we make the following assumptions on the functional forms: $f(k) = k^{\alpha}$, $u(c) = \frac{c^{1-\gamma_h}}{1-\gamma_h}$ and $v(d) = \frac{d^{1-\gamma_m}}{1-\gamma_m}$, where γ_h and γ_m are the risk aversion values for the household and the manager respectively. Regarding the parameterization, we assume that $\gamma_h = 0.5$ (almost risk neutral investors)[12], $\gamma_m = 5$, $\alpha = 0.4$, $\delta = 0.9$, $\eta = 0$ and $s_m = s_{h,-1} = 1$. Finally, we assume that there is no uncertainty in period $t = 1$, while the productivity shock θ can only take two possible values $\theta^L = 0.7$ and $\theta^H = 1.3$ in periods $t = 2$ and $t = 3$.

Exchange Economy. We begin by analyzing an exchange economy and then proceed to the economy with capital accumulation. To explain the equilibrium prices and allocations more clearly, it is best to begin by considering the role of the financial asset (the stock). In the absence of stock trading ($s_{h,t} = s_{h,t-1} = s_h$), investors and managers would get the following consumptions:

$$c_t = \frac{s_m}{s_m + s_h}\theta_t$$

$$d_t = \frac{1}{s_m + s_h}\theta_t$$

This implies that each agent gets a constant fraction of the earnings realization under autarky. On the other hand, the ability to trade in stocks allows managers and investors to smooth the consumption and dividend processes against earnings fluctuations. Here, it is important to note that the agents have conflicting objectives. For example, when earnings are low, both investors and managers would like to use stock trade to smooth c_t and d_t, but both cannot do so simultaneously. It turns out that the agent with the highest level of risk aversion obtains more insurance, while the autarkic equilibrium obtains if the two agents are equally risk averse.

With our benchmark parameterization, the manager is more risk averse, implying that he can smooth dividends using stock trade. In turn, investors are only willing to provide insurance and withstand a higher consumption *volatility* only if they receive a higher *level* of consumption as a compensation. In other words, there is a level versus smoothness trade-off in consumption. This results in the more risk averse party getting smoothing but less level, while the less risk averse party gets less smoothing and a higher level. With our parameterization, this implies that stocks will be issued when θ is low and repurchased when θ is high, since this is the profile that smooths dividend payments.

This basic idea is enough to understand the allocations in the 3-period model with naive firms, which is shown in Table 1. Throughout the Table, we denote the equilibrium with naive firms with N and the one with rational firms with FR. In the first period, the investor is allowed to consume more than the manager ($c_1 > d_1$). This way, the manager transfers some of his resources to the investor through stock repurchases and he pays a lower dividend per stock. As explained before, this is the payment for the insurance that the manager has bought. In period two, if bad times come (θ_2 is low), the investor 'agrees' not to try to smooth his consumption. On the contrary, we see that external funds are positive and consumption is hit especially hard by the shock. In contrast, the manager maintains a relatively stable level of dividends. The exact opposite happens in the case of good times (θ_2 is high).

Finally, the last period allocations are dictated by the budget constraints and by the fact that $s_3 = p_3 = 0$. Since there is really no choice to be made in the last period, given any realization of internal funds θ_3, total dividends are equal to this realization, while the dividends per stock d_3 are equal to $\frac{\theta_3}{s_2+s_m}$. As a result, whenever there is a lot of equity issued in period two, d_3 is low in the last period and vice versa. The previous observations

[12] Similar results can be obtained with $\gamma_h = 0$. Since consumption turns out to be negative in this case, however, we have decided to report the results with almost risk neutral investors.

allow us to explain price behavior using:

$$p_t = E_t \sum_{j=1}^{T} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \tag{34}$$

According to the previous equation, we expect prices to be high when expected future dividends are high. As a result, we expect period two prices to be high when equity is bought back and low when new equity is issued. The only thing that could reverse this relationship is a strong opposite movement in the intertemporal marginal rate of substitution[13].

If we compare these results with the allocations under rational firms, all of the above mechanisms are in operation too. However, the manager realizes now the effect that dividends can have on the stock price. In this case, the questions of interest are: Can managers use this knowledge to improve their situation? What prices would they choose if they had this option? How could they actually implement these prices?

Table 1: Exchange Economy

(t, θ) :	1	$(2, l)$	$(2, h)$	$(3, ll)$	$(3, lh)$	$(3, hl)$	$(3, hh)$
N d_t	**0.456**	**0.394**	**0.550**	**0.338**	**0.628**	**0.439**	**0.816**
FR d_t	0.458	0.391	0.560	0.340	0.631	0.433	0.804
N c_t	**0.543**	**0.305**	**0.749**	**0.361**	**0.671**	**0.260**	**0.483**
FR c_t	0.541	0.308	0.739	0.359	0.668	0.266	0.495
N ef_t	**−0.087**	**0.053**	**−0.249**				
FR ef_t	−0.082	0.048	−0.227				
N s_t^h	**0.907**	**1.069**	**0.592**				
FR s_t^h	0.912	1.058	0.615				
N p_t	**0.950**	**0.330**	**0.792**				
FR p_t	0.943	0.334	0.766				
	U_c	U_m	PV_c	PV_d	std_c	std_d	
N	**3.815**	**−17.28**	**1.378**	**1.331**	**0.173**	**0.106**	
FR	3.813	−17.23	1.375	1.334	0.167	0.107	

It turns out that the answer to the first question is positive. The FR allocation is different from the competitive equilibrium and it implies higher welfare for the manager and the same welfare for the investor. This can be seen in the last two rows of Table 1, reporting welfare values for the investor (U_c) and the manager (U_m), as well as the standard deviation of consumption (std_c) and dividends (std_d). In addition, we report a measure of the level of dividends and consumption that corresponds to their present value, PV_d and PV_c respectively, using risk neutral valuations. As we see, the welfare of the manager is increased through a higher level of dividends and despite their higher variability.

Proceeding to the second question, we observe the following. Whenever new stocks are issued, the manager would like the price to be high and whenever stocks are being repurchased the manager would like the price to be low. Comparing the naive and rational firm allocations clearly shows this pattern. Here, it is important to clarify that the manager does not really 'choose' prices as if it were a monopolist. In particular, the stock price has to follow competitive market prices according to the price-dividend mapping in (34) and the manager cannot influence other firms with his actions. On the other hand, they can choose dividend payments and, in a sense, this means that they can choose the 'quality' of their stock, while the price is determined competitively given this quality choice. Nevertheless, the price-dividend mapping is exploited and it allows external funds to be raised with favorable

[13] This channel is very small because of our assumption on investors being almost risk neutral.

prices. This is achieved by promising higher (lower) future dividends when they want to have higher (lower) stock prices.

Finally, recall that proposition 2 shows that the rational allocation is time consistent in the exchange economy. We postpone the discussion of this issue until the end of this section. In what follows, we discuss the production economy, where time inconsistency arises.

Production Economy. As in the exchange economy, we start by analyzing a simple benchmark where managers and investors have the same level of risk aversion. This implies that there is no scope for stock trading. In the absence of stock trading, the investor's budget constraint implies that consumption and dividends are always equal. In turn, by the manager's budget constraint, output net of investment is equally distributed between the investor and the manager. The only issue to be decided is therefore capital accumulation. Note also that the gamma multipliers are constant since,

$$\gamma_t = \frac{s_m v'(d_t s_m)}{(s_{h,t-1} + s_m) u'(c_t)} = \frac{1}{2}$$

where we have used the fact that $d_t = c_t$, $s_m = s_{h,t-1} = 1$ and u and v are the same. In this particular case, it also follows that capital accumulation is efficient, in the sense that it is equal to the value maximizing capital satisfying the following equation:

$$u'(c_t) = E_t \left[u'(c_{t+1})(\alpha\theta_{t+1} k_{t+1}^{\alpha-1} + 1 - \eta) \right]$$

As usual in finite period models, the optimal choice of capital dictates that it is run down to zero in the last period. Further, the optimal allocation dictates choosing high capital/investment in periods of high productivity and low capital/investment in periods of low productivity. Capital is then 'optimal', in the sense that it is the one that investors would choose if they had control of investment. Here there is a perfect alignment of manager and investor objectives since they have the same risk aversion and dividends equal consumption, therefore the manager chooses investment 'optimally' in the above sense.

It is important to point out that the risk averse manager would like to smooth dividends but cannot do so without trading in stocks. We now move to a case where the investor is less risk averse and thus willing to provide some insurance to the manager. Inevitably, investment will move away from the value maximizing level in this case. Table 2 reports results for this case where the manager is more risk averse than investors. We begin by focusing on the naive firm, indexed by N. The manager is issuing equity both in the first and second periods which allows for dividend smoothing across time. In the second period, he issues more when the shock is low, which means that dividends are also smoothed across states of nature. Dividend smoothing is also enhanced by the use of investment, which is now less than the value maximizing level in the beginning (intertemporal smoothing). The overall effect is smooth dividends and volatile consumption, as can be seen in the last two rows of Table 2. The investor is compensated for this higher consumption volatility with more consumption 'level' compared to dividend levels (see PV_c and PV_d). Obviously, stock trade makes both agents better off, since after all they are not forced to trade. As in the exchange economy, prices are negatively related to contemporaneous stock issuance. To be more precise, the more stocks $s_{h,t}$, the less dividends per stock d_t are expected to be tomorrow so the lower the price.

Armed with a clear understanding of the naive firm equilibrium prices and allocations and the mechanisms that drive them, we are now in a position to explain how that equilibrium changes when the firm realizes the price-dividend mapping. Table 2 also reports the prices and allocations for the rational firm, indexed by FR.

The first thing to notice is that the FR allocations lead to higher welfare for the manager but lower welfare for the investor. For the manager, this comes as a result of higher dividend level and despite higher dividend volatility (the opposite is observed for the investors).

Compared to the naive equilibrium, capital is higher in the beginning and thus closer to the value-maximizing level of capital. This inevitably comes with less smoothing of available resources across time.

Table 2: Production Economy

$(t,\theta):$	1	$(2,l)$	$(2,h)$	$(3,ll)$	$(3,lh)$	$(3,hl)$	$(3,hh)$
N d_t	**2.371**	**2.232**	**2.564**	**2.073**	**2.525**	**2.388**	**2.874**
FR d_t	2.359	2.230	2.564	2.089	2.545	2.404	2.893
N c_t	**2.349**	**1.456**	**2.704**	**1.129**	**2.096**	**1.202**	**2.231**
FR c_t	2.349	1.457	2.705	1.130	2.099	1.203	2.234
N ef_t	**0.176**	**0.172**	**0.162**				
FR ef_t	0.161	0.160	0.150				
N s_t	**1.046**	**1.137**	**1.120**				
FR s_t	1.042	1.126	1.110				
N y_t	**2.349**	**1.456**	**2.704**				
FR y_t	2.349	1.457	2.705				
N k_t	**6.238**	**3.301**	**3.860**				
FR k_t	6.247	3.311	3.869				
N p_t	**3.861**	**1.879**	**2.187**				
FR p_t	3.876	1.896	2.205				
	U_c	U_m	PV_c	PV_d	std_c	std_d	
N	**8.4116**	**−0.02133**	**6.5531**	**6.5259**	**0.3179**	**0.1641**	
FR	8.4114	−0.02132	6.5523	6.5275	0.3134	0.1674	

Allowing the manager to understand and exploit the price-dividend relationship leads him to 'choose' prices that are higher, since equity is being issued. The 'choice of price' here is only indirect, since the price increase is achieved through a promise of higher future dividends.[14] So the idea is to reduce dividend payments now, increase investment instead and then use the proceeds from this investment to pay higher dividends tomorrow. It is best to increase dividends in the third period because they would affect both first *and* second period prices.

Perhaps a clearer explanation comes when one looks at the firm's financing equality (budget constraint) to explain how this new dividend profile will affect policy. We have

$$\begin{aligned} Internal + External &= Investment + Dividends \\ \theta_t k_{t-1}^{\alpha} + p_t(s_{h,t} - s_{h,t-1}) &= k_t - k_{t-1} + d_t(s_{h,t-1} + s_m) \end{aligned}$$

In the first period, internal funds are given by past history and the current productivity shock and, as a result, are outside the control of the firm. Suppose the firm is considering lowering dividends now and raising them in the future. A reduction in dividends can be used either to increase investment or decrease external funds or both. For a firm that is naive, this means decreasing stocks and increasing investment (both will happen in an interior solution). Both the increase in investment and the decrease in stocks will allow the firm to have higher dividends per stock d_t in the future. The optimum is decided by weighing these benefits against the cost of low d_t today and the actual optimal value is reported in the above table. Now let us allow the firm to realize that the change in dividend policy will also have an additional effect through prices. The additional effect is that prices will

[14] Dividends are indeed higher in the third period, but not so in the second. But the movement in second period dividend is very small and obviously dominated by the third period increase. Note also that there is an effect through the intertemporal marginal rates of substitution which we abstract from because it is small in this example.

be higher, so the marginal benefit of reducing dividends today is actually higher than the manager thought before. The resulting allocation will have the manager choosing even lower dividends d_t today, higher investment and lower stocks which will allow the payment of even higher dividends per stock in the future. This is borne out in the allocations in Table 2.

This discussion is also at the heart of the time inconsistency problem. The reason is that the above arguments rely on the fact that today's dividends do not affect past prices, simply because the past has already happened and the price has been paid. Looking at the dividend first order condition

$$s_m v'(d_t s_m) = -\mu_{t-1} u'(c_{h,t}) + \gamma_t u'(c_{h,t})(s_{h,t-1} + s_m) \tag{35}$$

we can trace the above intuition. On the left we have the 'utility' cost of reducing d_t. On the right the second term is the benefit through the increase in today's resources. The first term is the effect of the decrease in d_t, that comes through reducing stock prices, on all previous periods' resources. This would be a cost in periods where equity is issued and a benefit in periods where equity is bought back. For periods two and three, the increase in dividends will have an additional benefit which is the increase in prices in period 1. The important asymmetry here is that for period 1, the reduction in d_1 has no effect since $\mu_0 = 0$. So, in a sense, today's dividend reduction comes 'for free' at least with regard to the price effect and that is why the dividend profile is tilted towards the future. But if the manager could re-optimize in period 2, it is now those dividends that can be freely reduced without affecting any price, since in re-optimizing we would now have $\mu_1 = 0$. So we would expect that, despite the promise of higher dividends tomorrow, when tomorrow comes dividends will be low and a new promise of high future dividends will be made. This will further increase the price as well as investment.

Note also that time inconsistency does not arise in exchange economies, precisely because the change in the value of resources reflected by γ is exactly offset by the effect of μ. In other words, firms cannot gain anything by deviating from the FR full commitment equilibrium.

The intuition for this time inconsistency is therefore very similar to the standard optimal taxation case. In that framework, what matters for current investment decisions are expected future capital taxes not current capital taxes and that creates the opportunity for manipulation of the level of investment through promises about the future. In our setup, it is dividends that determine the return to investing in the firm, but current decisions on buying stocks depend on promises about future dividends.

5.2. Infinite Horizon with uncertainty. To be added.

6. Extensions

The previous analysis has assumed that firms take into account the effects of financial policy on prices but not on consumption. A possible extension of our work is to study the Stackelberg leader firm, assuming that firms also internalize the consumption effects. In this sense, this firm is the closest to a Ramsey government in the optimal taxation literature.

Under this assumption, the solution is likely to be time inconsistent, even in the exchange economy. To see this, consider the more general equilibrium of the model with investors and managers that we have described earlier. The problem of a Stackelberg leader firm is given by:

$$\max_{\{d,s\}} E_0 \sum_{t=0}^{\infty} \delta^t v(d_t s_m) \text{ st.}$$

$$d_t s_{t-1} + \theta_t = p_t (s_t - s_{t-1})$$

$$s_{t-1} = s_{ht-1} + s_m$$

$$p_t = \delta E_t \left(\frac{u'(c_{ht+1})}{u'(c_{ht})} \left[(p_{t+1} + d_{t+1} \right] \right) \equiv E_t \left(\sum_{j=1}^{\infty} \delta^j \frac{u'(c_{ht+j})}{u'(c_{ht})} d_{t+j} \right)$$

$$c_{ht} = \theta_t - d_t s_m$$

As before, we can apply recursive contracts and introduce the co-state variable $\{\mu\}$, with law of motion given by:

$$\mu_t = \mu_{t-1} + \gamma_t (s_t - s_{t-1})$$

where γ_t is the multiplier on the budget constraint of the firm. The conditions that characterize the equilibrium of the previous problem are:

$$\begin{aligned} v'(d_t) &= -\mu_{t-1} \left[u''(c_{ht}) d_t s_m - u'(c_{ht}) \right] \\ &\quad + \gamma_t \left[u''(c_{ht}) s_m (d_t s_{t-1} - \theta_t) - u'(c_{ht}) s_{t-1} \right] \end{aligned}$$

$$\gamma_t u'(c_{ht}) p_t = \delta E_t \gamma_{t+1} u'(c_{ht+1}) \left[p_{t+1} + d_{t+1} \right]$$

It is easy to see that the proof of Proposition 3 does not apply to the present setup unless households are risk neutral ($u''(c) = 0$). Given this, the solution to this problem is likely to be time inconsistent.

7. Conclusions

We have provided a way to formulate and solve a stochastic general equilibrium dynamic model of dividend and stock policy. The aim was to provide a framework within which a number of important issues can be addressed. The model proposed makes explicit the distinction between dividends and stock issuance or repurchases. It is thus well suited to analyze payout policy. In addition, the framework is also available for the analysis of questions regrading the interplay between payout policy and investment.

As a first implication of the theoretical analysis presented in the main section of this paper, we highlight the behavior of growing firms with regard to dividend payments. Typically, startup firms pay little or no dividends, while they funnel resources towards the available productive projects that lead to firm growth. One obvious theoretical explanation of this observation points at financial frictions that do not allow for unlimited funds being raised from external sources. Our framework provides another, complementary mechanism that can explain this observation. The idea is that young firms lack the burden of past promises about dividends and can therefore pay little now, while promising a lot of dividends for the future. This strategy allows them to raise external funds at more favorable prices by inflating the price of their stock. Using the cheaper external funds, they can also grow faster.

Our framework also provides a rationale for why a firm would prefer to use dividends as opposed to repurchases if the full commitment solution is taken as the benchmark case. As mentioned above, the reason is that dividend promises can be used to influence prices towards achieving cheaper external finance, while the same objective cannot be achieved through announcements in stock repurchases.

Finally, our work identifies a potential for time inconsistency in financial policy even in the absence of asymmetric information of the type considered by Miller and Rock (1985). We point out the complications arising from the need for commitment and we provide examples where the full commitment policy is time consistent and others where it is not. This raises the question of how the time consistent policy would look like, its efficiency properties and the arrangements that can be used to implement more efficient policies. We leave these questions for future research.

Appendix A.1. Proof of Proposition 1

(a) To prove the first part of Proposition 1, we first show that the period-by period constraints in (1) and the price Euler equation from the consumers' problem in (4), together with the No-Ponzi scheme assumption, imply (7), (8). As we have already stated above, these imply (5). Since this holds for all $t \geq 0$, the equation evaluated at $t = 0$ implies (7). In addition, using the definitions of B_t and D_t, equation (5) implies $\frac{B_t}{D_t} = s_{t-1}$ so that (8) is satisfied.

To prove the converse, we show that given (7), (8) and (4), we can construct a sequence of stock holdings such that (1) is satisfied. First, define S_t as follows:

$$S_t \equiv \frac{B_t}{D_t}$$

so that S_t is measurable with respect to to information up to $t-1$. Then

$$\begin{aligned} D_t S_t &= n_t + E_t \sum_{j=1}^{\infty} \delta^j \frac{u(c_{t+j})}{u(c_t)} n_{t+j} \\ &= n_t + \delta E_t \left[E_{t+1} \sum_{j=0}^{\infty} \delta^j \ \frac{u(c_{t+j})}{u(c_t)} n_{t+j+1} \right] \\ &= n_t + \delta E_t \left[n_{t+1} + E_{t+1} \sum_{j=1}^{\infty} \delta^j \ \frac{u(c_{t+j})}{u(c_t)} n_{t+j+1} \right] \\ &= n_t + \delta E_t \left[D_{t+1} S_{t+1} \right] \end{aligned}$$

But S_{t+1} is measurable with respect to information up to t, so that

$$D_t S_t = n_t + \delta S_{t+1} E_t \left[D_{t+1} \right]$$

Finally, noticing that $D_t = p_t + d_t$, we see that period-by-period budget constraint is satisfied for $s_{t-1} = S_t = \frac{B_t}{D_t}$.

(b) The second part of the proposition directly follows from the first. In particular, assume that the cash flow of the firm is given by $\{n_t\}_{t=0}^{\infty}$ and consider the equilibrium consumption process $\{c_t\}_{t=0}^{\infty} = \{n_t\}_{t=0}^{\infty}$. Consider any choice of stocks $\{\widetilde{s}_t\}_{t=0}^{\infty}$ such that $\widetilde{s}_t \neq 0$ almost surely and let $\left\{\widetilde{s}_t^h\right\}_{t=0}^{\infty} = \{\widetilde{s}_t\}_{t=0}^{\infty}$. Consistent with this choice of $\widetilde{s}$ we find the associated price to satisfy the following equation:

$$\widetilde{p}_t \widetilde{s}_t = E_t \left(\sum_{j=1}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} n_{t+j} \right)$$

and the divided process $\left\{\widetilde{d}\right\}$ to satisfy:

$$\widetilde{d}_t \widetilde{s}_{t-1} + \widetilde{p}_t \widetilde{s}_{t-1} = E_t \sum_{j=0}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} n_{t+j}$$

Now we have to show that such a stock, price and dividend processes satisfy budget constraints and pricing equations. First, notice that

$$\widetilde{p}_t \widetilde{s}_t = E_t \left(E_{t+1} \left(\sum_{j=1}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} n_{t+j} \right) \right) = E_t \left(\delta \frac{u'(c_{t+1})}{u'(c_t)} E_{t+1} \sum_{j=0}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_{t+1})} n_{t+j+1} \right)$$

Using the definition of $\widetilde{d}_t \widetilde{s}_{t-1} + \widetilde{p}_t \widetilde{s}_{t-1}$ we also have that

$$\widetilde{p}_t \widetilde{s}_t = E_t \left(\delta \frac{u'(c_{t+1})}{u'(c_t)} \left(\widetilde{d}_{t+1} \widetilde{s}_t + \widetilde{p}_{t+1} \widetilde{s}_t \right) \right)$$

so $\widetilde{s}_t$ cancels out and (4) holds. It is easy to see also that the above choices satisfy the budget constraint of the firm. We can find many other equilibria by changing $\{\widetilde{s}_t\}_{t=0}^{\infty}$.■

Appendix A.2. Proof of Results 1 and 2.

Naive firms. We start with the problem of a naive firm. This is given by:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \delta^t \log(d_t s_m) \\ \text{s.t. } d_t s_{t-1} &= p_t (s_t - s_{t-1}) + k_{t-1}^{\alpha} + (1-\eta) k_{t-1} - k_t \\ & s_{-1}, k_{-1} \text{ given} \end{aligned}$$

and the first order conditions are given by:

$$\begin{aligned} \frac{1}{d_t} &= \gamma_t s_{t-1} \\ \gamma_t p_t &= \delta \left[\gamma_{t+1} (d_{t+1} + p_{t+1}) \right] \\ \gamma_t &= \delta \gamma_{t+1} \left(1 - \eta + \alpha k_t^{\alpha-1} \right) \\ p_t &= \delta (d_{t+1} + p_{t+1}) \\ d_t s_{t-1} &= p_t (s_t - s_{t-1}) + k_{t-1}^{\alpha} + (1-\eta) k_{t-1} - k_t \end{aligned}$$

The stock Euler equation together with the price equation imply:

$$\gamma_t = \gamma_{t+1}$$

The fact that the multipliers γ_t are constant has two implications. First, the capital Euler equation implies

$$k_t = k^{GR} = \left[\frac{\frac{1}{\delta} - 1 + \eta}{\alpha} \right]^{\frac{1}{\alpha - 1}}$$

Second, the dividend first order conditions give

$$d_t s_{t-1} = d_{t+1} s_t$$

We now show that the following dividends and stocks satisfy the above equilibrium conditions

$$\begin{aligned} s_t &= \bar{s} \text{ for } t \geq 0 \\ d_t &= \bar{d} \text{ for } t \geq 1 \\ d_0 &= \frac{n^{GR}}{s_{-1}} \end{aligned}$$

These allocations satisfy the price-dividend mapping as long as

$$\bar{p} = \frac{\delta}{1-\delta} \bar{d}$$

and the dividend first order conditions as long as

$$d_0 s_{-1} = \bar{d} \bar{s}$$

Thus, all that remains is to find $\bar{d}$, d_0 and $\bar{s}$ so that the budget constraints are also satisfied. From the budget constraint from $t = 1$ onwards we can find

$$d_t s_{t-1} = \bar{d}\bar{s} = n^{GR}$$

where

$$n^{GR} = \left(k^{GR}\right)^{\alpha} - \eta k^{GR}$$

This also means that

$$d_0 = \frac{n^{GR}}{s_{-1}}$$

will ensure the dividend first order conditions and the budget constraints from period $t = 1$ onwards are satisfied. The period 0 budget constraint is:

$$\begin{aligned} d_0 s_{-1} &= p_0(s_0 - s_{-1}) + k_{-1}^{\alpha} + (1-\eta)k_{-1} - k_0 \\ n^{GR} &= \frac{\delta}{1-\delta}\bar{d}(\bar{s} - s_{-1}) + k_{-1}^{\alpha} + (1-\eta)k_{-1} - k_0 \end{aligned}$$

Using $\bar{d}\bar{s} = n^{GR}$, we can find $\bar{d}$ consistent with this budget constraint to be

$$\bar{d} = \frac{(1-\delta)\left(k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}\right) + (2\delta - 1)\, n^{GR}}{\delta s_{-1}}$$

and therefore

$$\bar{s} = \frac{\delta n^{GR}}{(1-\delta)\left(k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}\right) + (2\delta - 1)\, n^{GR}} s_{-1}$$

Note that if $k_{-1} = k^{GR}$, then $\bar{s} = s_{-1}$. But if $k_{-1} < k^{GR}$ then $\bar{s} > s_{-1}$ and $d_0 > \bar{d}$.

Rational firms. We now look at the problem of a rational firm. In this case, we don't need logarithmic utility to prove the result. The problem of the rational firm is:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \delta^t v\left(d_t s_m\right) \\ \text{s.t. } d_t s_{t-1} &= p_t\left(s_t - s_{t-1}\right) + k_{t-1}^{\alpha} + (1-\eta)k_{t-1} - k_t \\ p_t &= \delta\left(d_{t+1} + p_{t+1}\right) \\ & s_{-1}, k_{-1} \text{ given} \end{aligned}$$

The recursive Lagrangian is

$$L = \sum_{t=0}^{\infty} \delta^t \left[v\left(d_t s_m\right) + \mu_{t-1} d_t + \gamma_t\left(k_{t-1}^{\alpha} + (1-\eta)k_{t-1} - k_t - d_t s_{t-1}\right)\right]$$

and the equilibrium conditions are now

$$\begin{aligned} s_m v'(d_t s_m) &= \gamma_t s_{t-1} - \mu_{t-1} \\ \gamma_t p_t &= \delta\left[\gamma_{t+1}\left(d_{t+1} + p_{t+1}\right)\right] \\ \gamma_t &= \delta\gamma_{t+1}\left(1 - \eta + \alpha k_t^{\alpha-1}\right) \\ p_t &= \delta\left(d_{t+1} + p_{t+1}\right) \\ d_t s_{t-1} &= p_t\left(s_t - s_{t-1}\right) + k_{t-1}^{\alpha} + (1-\eta)k_{t-1} - k_t \\ \mu_t &= \mu_{t-1} + \gamma_t(s_t - s_{t-1}) \end{aligned}$$

We provide an analytical solution to these conditions. The stock Euler together with the price equation imply

$$\gamma_t = \gamma_{t+1}$$

so the stock Euler implies

$$k_t = k^{GR} = \left[\frac{\frac{1}{\delta} - 1 + \eta}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

just like under naive firms. Using the fact that $\gamma_t = \gamma_{t-1}$ for all $t \geq 1$ and the dividend first order conditions we have

$$\begin{aligned} s_m v'(d_t s_m) - s_m v'(d_{t-1} s_m) &= \gamma_t s_{t-1} - \mu_{t-1} - \gamma_{t-1} s_{t-2} + \mu_{t-2} \\ &= (\gamma_t - \gamma_{t-1}) s_{t-1} = 0 \end{aligned}$$

so $d_t = d_{t-1}$ for all $t \geq 1$. The constant dividend level is found from the time 0 budget constraint

$$d_t = \bar{d} = \frac{(1-\delta)\left(k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}\right) + \delta n^{GR}}{s_{-1}} \text{ for } t \geq 0$$

Given that, we can use the period 0 dividend first order condition to find γ_t :

$$\gamma_t = \gamma_0 = \frac{s_m v'(\bar{d} s_m)}{s_{-1}}$$

and the price is also constant and equal to

$$p_t = \bar{p} = \frac{\delta}{1-\delta}\bar{d}$$

We can now compute the stocks from the intertemporal budget constraints for $t \geq 1$

$$\begin{aligned} (\bar{d} + \bar{p}) s_{t-1} &= \sum_{j=t}^{\infty} \delta^{j-t} n^{GR} = \frac{n^{GR}}{1-\delta} \Rightarrow \\ s_{t-1} &= \bar{s} = \frac{n^{GR}}{\bar{d}} \text{for } t \geq 1 \end{aligned}$$

It is straightforward to see that

$$\bar{s} = \frac{n^{GR}}{(1-\delta)\left(k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}\right) + \delta n^{GR}} s_{-1} > s_{-1}$$

as long as $k_{-1} < k^{GR}$. Finally, the multipliers μ_t are constant after period 0 and equal to μ_0

$$\mu_t = \gamma_0(\bar{s} - s_{-1}) > 0 \text{ for } t \geq 0$$

Comparison. In what follows, we compare the allocations under naive and rational firms. We assume throughout that $k_{-1} < k^{GR}$. If the opposite holds, that is, if $k_{-1} > k^{GR}$, all the relationships are reversed. If $k_{-1} = k^{GR}$ allocations are trivially the same in the two setups.

Dividends are lower for the rational firm initially:

$$d_0^{FR} = \frac{(1-\delta)\left(k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}\right) + \delta n^{GR}}{s_{-1}} < \frac{n^{GR}}{s_{-1}} = d_0^N$$

but higher from then onwards:

$$\begin{aligned} d_0^{FR} + \frac{\delta}{1-\delta}\bar{d}^{FR} &= \frac{\sum_{t=0}^{\infty}\delta^t n_t}{s_{-1}} = d_0^N + \frac{\delta}{1-\delta}\bar{d}^N \Rightarrow \\ \bar{d}^{FR} - \bar{d}^N &= \frac{1-\delta}{\delta}\left(d_0^N - d_0^{FR}\right) > 0 \end{aligned}$$

This last result implies that the stock price is always higher under rational firms

$$p_t^{FR} = \frac{\delta}{1-\delta}\bar{d}^{FR} > \frac{\delta}{1-\delta}\bar{d}^N > p_t^N$$

In turn, this implies that the naive firm needs to issue more stocks to achieve the optimal investment. This can be seen using the expressions for s_0 in the two cases. Letting $n_0 = k_{-1}^{\alpha} + (1-\eta)k_{-1} - k^{GR}$

$$\begin{aligned} s_0^N &= \bar{s}^N = \frac{\delta n^{GR}}{(1-\delta)\, n_0 + (2\delta - 1)\, n^{GR}} s_{-1} \\ s_0^{FR} &= \bar{s}^{FR} = \frac{n^{GR}}{(1-\delta)\, n_0 + \delta n^{GR}} s_{-1} \end{aligned}$$

$$\begin{aligned} s_0^N &> s_0^{FR} \Leftrightarrow \\ \delta\left[(1-\delta)\, n_0 + \delta n^{GR}\right] &> (1-\delta)\, n_0 + (2\delta-1)\, n^{GR} \Leftrightarrow \\ (\delta - 1)\left[(1-\delta)\, n_0\right] + \left(\delta^2 - 2\delta + 1\right) n^{GR} &> 0 \Leftrightarrow \\ (1-\delta)^2\left(n^{GR} - \left[(1-\delta)\, n_0\right]\right) &> 0 \end{aligned}$$

which is true for $k_{-1} < k^{GR}$.■

Appendix A.3. Proof of Proposition 2.

The first order conditions for the time 0 problem are given by:

$$\mu_t = \mu_{t-1} + \gamma_t(s_t - s_{t-1}) \text{ with } \mu_{-1} = 0$$

$$s_m v'(s_m d_t) = \gamma_t s_{t-1} - \mu_{t-1}$$

along with

$$\begin{aligned} \gamma_t &= \gamma_{t+1}\delta(f'(k_t) + 1 - \eta) \\ s_t &= s_{-1} + (t+1)\Delta, \quad \text{for } 0 \leq t \leq T-1 \end{aligned}$$

We now consider whether a re-optimization in future periods would lead the firm to deviate from the dividend plans announced in period zero. We use the superscript R to denote the solution if the firm re-optimizes in period $t = 1$. The conditions for capital and the stock are the same as before. On the other hand, we have

$$\begin{aligned} \mu_t^R &= \mu_{t-1}^R + \gamma_t^R(s_t^R - s_{t-1}^R) \text{ for } t \geq 1 \\ \mu_0^R &= 0 \end{aligned}$$

$$s_m v'(s_m d_t^R) = \gamma_t^R s_{t-1}^R - \mu_{t-1}^R$$

This implies that the following equation holds for $t > 1$:

$$v'(d_t^R) = v'(d_{t-1}^R) + (\gamma_t^R - \gamma_{t-1}^R)s_{t-1}^R$$

In addition, since the firm re-optimizes at $t = 1$, we have

$$v'(d_1^R) = \gamma_1^R s_0$$

Suppose that the re-optimization choices are the same as the original ones, i.e. $d_t^R = d_t$, $s_t^R = s_t$ and $k_t^R = k_t$ for $t \geq 1$. We now show that this leads to a contradiction. If the re-optimized choices are the same as originally, the following must hold

$$v'(d_1) = \gamma_1^R s_0 \tag{36}$$

$$\gamma_2 - \gamma_1 = \frac{v'(d_2) - v'(d_1)}{s_1} = \gamma_2^R - \gamma_1^R \tag{37}$$

In addition, for these choices of γ to be compatible with the same choice for capital in period 1, the following equation must also be satisfied:

$$\gamma_2^R \delta(f'(k_1) + 1 - \eta) = \gamma_1^R$$

but this cannot happen. In fact, if (37) holds, we have $\gamma_2^R = \gamma_1^R - \gamma_1 + \gamma_2$ so that we need the following to be true

$$\begin{aligned} \gamma_1^R &= \gamma_2^R \delta(f'(k_1) + 1 - \eta) = (\gamma_1^R - \gamma_1 + \gamma_2)\delta(f'(k_1) + 1 - \eta) \\ &= (\gamma_1^R - \gamma_1)\delta(f'(k_1) + 1 - \eta) + \gamma_1 \end{aligned}$$

The last expression can only be equal to γ_1^R if either $\delta(f'(k_1) + 1 - \eta) = 1$ or $\gamma_1^R = \gamma_1$. The first condition arises when capital is optimal, a case which gives rise to time consistency as shown in Proposition 3 below, but which we have excluded above by the choice of a low initial capital and an upper bound on issuance Δ that is binding for at least two periods (period 0 and 1). The second case can be excluded by the formulae for γ_1^R in (36) and for γ_1 in the original problem, since $\mu_0 \neq 0$. Therefore the re-optimized solution cannot be the same as the original one and the time zero policy is time inconsistent in this example.■

Appendix A.4. Proof of Proposition 3.

We now prove that the solution under VM is time inconsistent. Consider the problem of a firm that maximizes its market value subject to some financing frictions. The firm solves

$$V_1(k_0, s_0) = \max_{\{s_t, d_t, k_t\}_{t=1}^{\infty}} s_0 \left(d_1 + \sum_{t=1}^{\infty} \delta^t d_{t+1} \right)$$

$$\text{s.t.}$$

$$k_t + s_{t-1} d_t \leq f(k_{t-1}) + (s_t - s_{t-1})p_t - \mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1})$$

$$(s_t - s_{t-1})p_t \leq (s_t - s_{t-1}) \sum_{t=1}^{\infty} \delta^t d_{t+1}$$

$$d_t \geq 0,\ s_{t-} s_{t-1} \geq 0$$

where the amount of outstanding shares at the beginning of period 1, s_0, is given. Furthermore, d_t are the dividends per share, p_t is the price of the share (after dividends), and k_t is the end-of-period t capital stock. The subscript of the function V indicates that the maximization problem takes place at period 1. Since s_0 is given this problem is equivalent to

$$V_1(k_0, s_0) = \max_{\{s_t, d_t, k_t\}_{t=1}^{\infty}} d_1 + \sum_{t=1}^{\infty} \delta^t d_{t+1}$$

s.t.

$$k_t + s_{t-1}d_t \leq f(k_{t-1}) + (s_t - s_{t-1})p_t - \mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1})$$

$$(s_t - s_{t-1})p_t \leq (s_t - s_{t-1}) \sum_{t=1}^{\infty} \delta^t d_{t+1}$$

$$d_t \geq 0$$

$$s_{t-} s_{t-1} \geq 0$$

Let

$$\tilde{V}_1(s_1, k_1) = \sum_{t=1}^{\infty} \delta^{t-1} d_{t+1}$$

and let $\{p_{1,t}, s_{1,t}, d_{1,t}, k_{1,t}\}_{t=1}^{\infty}$ be the optimal sequence chosen in period 1. Now consider the period-2 problem.

$$V_2(k_{1,1}, s_{1,1}) = \max_{\{s_t, d_t, k_t\}_{t=2}^{\infty}} \left(d_2 + \sum_{t=2}^{\infty} \delta^{t-1} d_{t+1} \right)$$

s.t.

$$k_t + s_{t-1}d_t \leq f(k_{t-1}) + (s_t - s_{t-1})p_t - \mathcal{C}(k_t, k_{t-1}, s_t, s_{t-1}, d_t, d_{t-1})$$

$$(s_t - s_{t-1})p_t \leq (s_t - s_{t-1}) \sum_{t=1}^{\infty} \delta^t d_{t+1}$$

$$d_t \geq 0$$

$$s_{t-} s_{t-1} \geq 0$$

Agents would like to reoptimize if

$$V_2(s_{1,1}, k_{1,1}) > \tilde{V}_1(s_{1,1}, k_{1,1})$$

This cannot happen, however. If this condition is fulfilled then $\{p_{1,t}, s_{1,t}, d_{1,t}, k_{1,t}\}_{t=1}^{\infty}$ is not the solution to the period-1 optimization problem. It is easy to see why. Assume to the contrary that this condition is fulfilled and let $\{p_{2,t}, s_{2,t}, d_{2,t}, k_{2,t}\}_{t=2}^{\infty}$ be the solution to the period-2 optimization problem. But note that this sequence could have been chosen in period 1 as well. It clearly satisfies the budget constraint. Moreover, it also satisfies the participation constraint. If $V_2(s_{1,1}, k_{1,1}) > \tilde{V}_1(s_{1,1}, k_{1,1})$ then the outside investors also get more. So $\{p_{2,t}, s_{2,t}, d_{2,t}, k_{2,t}\}_{t=2}^{\infty}$ satisfies the period-1 constraints and leads to a higher value for the objective function. Consequently, $\{p_{1,t}, s_{1,t}, d_{1,t}, k_{1,t}\}_{t=1}^{\infty}$ cannot have been the solution to the period-1 problem.■

Appendix A.5 Proof of Proposition 4.

Proof of Proposition 4. (i) We start by proving that the solution is time consistent in exchange economies. In an exchange economy, we can ignore the first order condition with respect to capital in (19). Further, the first order conditions with respect to dividends and stocks corresponding to the full commitment solution imply that:

$$0 = E_t \left[\left(\gamma_t^* - \gamma_{t+1}^* \right) u' \left(c_{t+1}^* \right) \left(d_{t+1}^* + p_{t+1}^* \right) \right] \tag{38}$$

$$s_m v' \left(d_t^* \right) = \gamma_t^* s_{t-1}^* u' \left(c_t^* \right) - u' \left(c_t^* \right) \mu_{t-1}^* \text{ for } t > 0$$

$$s_m v' \left(d_0^* \right) = \gamma_0^* s_{-1}^* u' \left(c_0^* \right) \tag{39}$$

We argue now that these first order conditions imply that the first order conditions for the time-$\bar{t}$ problem are satisfied for the same dividend and stock processes but for different multiplier variables γ and μ. The conditions characterizing the solution for the time-$\bar{t}$ problem are:

$$\begin{aligned} s_m v'\left(d_t^{**}\right) &= \gamma_t^{**} s_{t-1}^{**} u'\left(c_t^{**}\right) - u'\left(c_t^{**}\right)\mu_{t-1}^{**} \text{ for } t > \bar{t} \\ s_m v'\left(d_{\bar{t}}^{**}\right) &= \gamma_{\bar{t}}^{**} s_{\bar{t}-1}^{**} u'\left(c_{\bar{t}}^{**}\right) \end{aligned} \tag{40}$$

$$0 = E_t\left[\left(\gamma_t^{**} - \gamma_{t+1}^{**}\right) u'\left(c_{t+1}^{**}\right)\left(d_{t+1}^{**} + p_{t+1}^{**}\right)\ \right] \text{ for } t \geq \bar{t} \tag{41}$$

$$p_t^{**} = \delta E_t \frac{u'\left(c_{t+1}^{**}\right)}{u'\left(c_t^{**}\right)}\left[d_{t+1}^{**} + p_{t+1}^{**}\right] \quad \text{for } t > \bar{t} \tag{42}$$

$$s_{t-1}^{**} E_t \sum_{j=0}^{\infty} \delta^j \frac{u\left(c_{t+j}^{**}\right)}{u\left(c_t^{**}\right)} d_{t+j}^{**} = E_t \sum_{j=0}^{\infty} \delta^j \frac{u\left(c_{t+j}^{**}\right)}{u\left(c_t^{**}\right)} n_{t+j}^{**} \quad \text{for } t > \bar{t} \tag{43}$$

To show that we can find multipliers so that the previous first order conditions are satisfied for (30), we can set

$$\begin{aligned} \gamma_{\bar{t}}^{**} &= -\frac{s_m v'(d_{\bar{t}}^*)}{s_{\bar{t}-1}^* u'\left(c_{\bar{t}}^*\right)} \\ \mu_{\bar{t}-1} &= 0 \end{aligned}$$

and we can then derive γ_t^{**} and μ_t^{**} for $t > \bar{t}$ using (40) and the law of motion of μ in (16) for (30). Since, equations (42)-(43) are satisfied for (30), we only need to show that the following condition is satisfied so that (41) also holds for (30):

$$\gamma_t^{**} - \gamma_{t+1}^{**} = \gamma_t^* - \gamma_{t+1}^* \text{ for } t > \bar{t}.$$

To see that this is the case note that equation (40) implies that

$$\begin{aligned} \gamma_t^{**} s_{t-1}^* &= \frac{s_m v'\left(d_t^*\right)}{u'\left(c_t^*\right)} + \mu_{t-1}^{**} \\ \gamma_{t+1}^{**} s_t^* &= \frac{s_m v'\left(d_{t+1}^*\right)}{u'\left(c_{t+1}^*\right)} + \mu_t^{**} \end{aligned}$$

It therefore follows that:

$$\begin{aligned} \gamma_t^{**} s_{t-1}^* - \gamma_{t+1}^{**} s_t^* &= x_t - \left[\mu_t^{**} - \mu_{t-1}^{**}\right] \\ &= x_t - \gamma_t^{**}\left[s_t^* - s_{t-1}^*\right] \end{aligned}$$

where $x_t = \frac{s_m v'(d_t^*)}{u'(c_t^*)} - \frac{s_m v'\left(d_{t+1}^*\right)}{u'\left(c_{t+1}^*\right)}$ and the last equality uses the law of motion for μ. Finally, rearranging the previous equation, we obtain that:

$$\gamma_t^{**} - \gamma_{t+1}^{**} = \frac{x_t}{s_t^*}. \tag{44}$$

Since the right hand side only depends on variables that are assumed to be the same in the two allocations, our initial claim is true.

(ii) We now prove that the solution is time consistent if capital satisfies condition (24). To do this, assume that the value maximizing level of capital is actually implemented in the full commitment solution, that is, (24) satisfied for all t. We now show that, in this case, the manager will not want to re optimize even if given the chance.

To prove this statement, we observe the following. The first part of the proposition has shown that all the first order conditions of the "continuation problem" of an exchange economy are satisfied for different multipliers. Since the system of equations that characterizes the equilibrium in a production economy is the same, except that it includes also the capital Euler equation in (19), all that is left to show is that this last condition will be satisfied for the new gammas we have found. We can rewrite the optimality condition with respect to capital in the full commitment solution (equation (19)) as follows:

$$\gamma_t^* \delta E_t \left[u'\left(c_{t+1}^*\right)\left(\theta_{t+1} f'(k_t^*) + 1 - \eta\right)\right] = \delta E_t \left[\gamma_{t+1}^* u'\left(c_{t+1}^*\right)\left(\theta_{t+1} f'(k_t^*) + 1 - \eta\right)\right]$$

But this implies that:

$$E_t[\left(\gamma_{t+1}^* - \gamma_t^*\right) u'\left(c_{t+1}^*\right)\left(\theta_{t+1} f'(k_t^*) + 1 - \eta\right)] = 0$$

As we see, the series for γ that satisfy the first order conditions in the continuation problem of the exchange economy also satisfy this first order condition with respect to capital in the production economy as long as the capital is at the value maximizing level. The solution is therefore time consistent.■

APPENDIX B: Time Consistency in Example 3

Consider the first order conditions in period $T-1$. These imply that:

$$E_{T-1}[\gamma_T(d_T + p_T)] = \gamma_{T-1} E_{T-1}[d_T + p_T]$$

$$E_{T-1}[\gamma_T(\overline{\theta} f'(k_{T-1}) + 1 - \eta)] = \gamma_{T-1}$$

Since $d_T + p_T = \frac{\overline{d}(k_{T-1}, s_{T-1})}{1-\delta}$, we have that $d_T + p_T$ is known at $T-1$ and the first equation implies that:

$$E_{T-1}[\gamma_T] = \gamma_{T-1} \tag{45}$$

Further, the second equation implies that

$$E_{T-1}[\gamma_T(\overline{\theta} f'(k_{T-1}) + 1 - \eta)] = E_{T-1}[\gamma_T](\overline{\theta} f'(k_{T-1}) + 1 - \eta) = \gamma_{T-1}$$

implying that:

$$\overline{\theta} f'(k_{T-1}) + 1 - \eta = 1$$

As we see, it follows that $k_{T-1} = \overline{k}$, while equation (45) and the first order condition for dividends gives:

$$E_{T-1} v'(d_T) = v'(d_{T-1})$$

However, since d_T is known with information up to period $T-1$, this implies that $E_{T-1} v'(d_T) = v'(d_T)$ and $d_T = d_{T-1}$. Thus, even one period before there is no uncertainty, dividends are constant and the risk neutral level of capital will be chosen. Plugging $k_{T-1} = \overline{k}$ into (31), we obtain a cleaner expression for the dividends:

$$\overline{d}(\overline{k}, s_{T-1}) \equiv \frac{\overline{\theta} f(\overline{k}) - \eta \overline{k}}{s_{T-1}}$$

Further, we can determine s_{T-1} by just plugging in (1) evaluated at $t = T-1$, the risk neutral level for k_{T-1} and the dividends to find that

$$\overline{d}(\overline{k}, s_{T-1}) s_{T-2} + \overline{k} - (1-\eta) k_{T-2} = \frac{\delta \overline{d}(\overline{k}, s_{T-1})}{1-\delta}(s_{T-1} - s_{T-2}) + \theta_{T-1} f(k_{T-2})$$

Given s_{T-2}, θ_{T-1} and k_{T-2}, the previous equation gives the solution for s_{T-1}. Simplifying
,

$$\frac{\overline{d}(\overline{k}, s_{T-1})}{1-\delta}(s_{T-2} - \delta s_{T-1}) = \theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}$$

which means that the jump to the risk neutral level of capital at $T-1$ (from k_{T-2} to $k_{T-1} = \overline{k}$) is financed by stock issuance in periods $T-1$ and T. To obtain a more explicit solution, note that the previous equations imply that:

$$\frac{\overline{\theta} f(\overline{k}) - \eta \overline{k}}{(1-\delta)s_{T-1}}(s_{T-2} - s_{T-1}) = \theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}$$

$$\frac{s_{T-2}}{s_{T-1}} = \left((1-\delta)\frac{\theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}}{\overline{\theta} f(\overline{k}) - \eta \overline{k}} + 1\right)$$

$$s_{T-1} = s_{T-2}\frac{\overline{\theta} f(\overline{k}) - \eta \overline{k}}{(1-\delta)\left[\theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}\right] + \overline{\theta} f(\overline{k}) - \eta \overline{k}}$$

Consequently, the long run dividends are given by

$$\overline{d}(\overline{k}, s_{T-1}) \equiv \frac{(1-\delta)\left[\theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}\right] + \overline{\theta} f(\overline{k}) - \eta \overline{k}}{s_{T-2}}$$

The previous arguments imply that we actually will have time consistency after $T-1$. However, we can not extrapolate this to previous periods. For example, while we also have that $d_{T-1} + p_{T-1} = \frac{\overline{d}(k_{T-1}, s_{T-1})}{1-\delta}$, s_{T-1} is not known at $T-2$, since it is determined by θ_{T-1}. Given this, we do not have an analog of (45). Instead, we have that

$$E_{T-2}[\gamma_{T-1}\overline{d}(k_{T-1}, s_{T-1})] = \gamma_{T-2} E_{T-2}[\overline{d}(k_{T-1}, s_{T-1})]$$

For $T > 2$, using the above formula for long run dividends, we obtain

$$\begin{aligned} & E_{T-2}[\gamma_{T-1}((1-\delta)\left[\theta_{T-1} f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}\right] + \overline{\theta} f(\overline{k}) - \eta \overline{k})] \\ = \; & \gamma_{T-2}[(1-\delta)\left[E_{T-2}(\theta_{T-1}) f(k_{T-2}) - \overline{k} + (1-\eta)k_{T-2}\right] + \overline{\theta} f(\overline{k}) - \eta \overline{k}] \end{aligned}$$

In this case, the solution is likely to be time inconsistent up to period $T-2$, since the gammas that satisfy (44) are not likely to satisfy the previous condition.

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