Obsolescence of Durable Goods and Optimal Consumption*

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Abstract

We study a model with a durable good subject to periodic obsolescence, and characterize analytically the optimal purchasing policy. The key result is that consumers optimally synchronize new purchases with the innovation cycle. Consequently, some agents react to wealth shocks by adjusting only the timing of durable purchases, while other agents adjust only their non-durable consumption. On aggregate, demand for the durable starts responding to a wealth shock only after a lag, and this response is larger for goods that are more lumpy and have steeper obsolescence profiles. This comparative statics is consistent with the observed expenditure volatility rankings across durable categories.

1 Introduction

The fall in the quality-adjusted price of a durable, or obsolescence, is the major reason for depreciation in markets with technological innovation. Since much of this innovation is incorporated in new durables, modeling obsolescence of durable goods is vital for our understanding of the properties of aggregate demand. Our goal is to characterize the aggregate demand for durables in a dynamic model that captures the essential distinctions between obsolescence and physical depreciation.

While physical depreciation is idiosyncratic and its aggregate effects are smooth, obsolescence affects all durables at the same time. Moreover, major obsolescence episodes may be anticipated, especially when the introduction of new products is periodic. For some goods, such as automobiles, redesigned models do appear periodically, every 4 or 5 years. Even when obsolescence is not deterministic, typically, obsolescence episodes are not independent events either. Innovation processes naturally have hazard rates that are negligible immediately after an innovation; after all, no one expects a new generation of products to appear immediately after the introduction of a new model. We therefore think of the obsolescence process as a series of large shocks that are correlated through time.

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In reality, obsolescence patterns have both discrete and continuous elements, but markets where discrete obsolescence is likely to be important are commonplace. The literature typically associates periodic obsolescence with a monopolistic producer whose timing of product introduction is a strategic variable (e.g. Swan, 1972, Rust,1986, Fishman and Rob, 2000). Our focus is on a different set of markets, where major innovations affect all the producers, but they are infrequent due to technological constraints rather than strategic reasons. These markets include several (overlapping) categories. (1) Markets where new products have a different format or standard. Format switching is typical for data recording or storage devices, such as disk drives, camcorders and digital cameras. (2) Goods that depend on a "bottleneck" (lagging) technology. For example, power supply has been a constraining factor in adding new features to many portable electronic devices. (3) Markets where technological constraints are imposed by periodically changing government regulation, such as cellular communications.

The analysis of this paper builds on the basic idea that consumer expectations about the timing of future innovations affect the current purchasing behavior. Suppose that individual units are expected to depreciate abruptly at some future date. Consumers who purchase their durables just before this date will enjoy a lower service flow than those who buy soon after. Hence, consumers have an incentive to buy a durable only when the design is sufficiently new and is not about to be changed soon. Thus demand for new durables should drop some time prior to the dates when the new models become available. These anticipatory drops in demand have been noted for DVD players (Dranove and Gandal, 2003) and large-screen TVs (Shapiro and Varian, 1999, p. 15). In automobiles, regular timing of model year changes induces strong seasonal fluctuations of auto sales (Cooper and Haltiwanger, 1993a, b).

We consider an economy with a durable and a non-durable good and a large number of heterogenous consumers. The durable good in our model represents a fairly narrow consumption category. Then the appropriate way to think about the non-durable is the "outside option" category that allows consumers to derive utility from expenditure outside of the durable market.

Agents are infinitely lived, differ in their permanent income level, and can borrow and lend at an exogenously given interest rate. The durable is produced by a competitive industry with CRS technology. There are no secondary markets for used durables; units that are replaced are thrown away.

We solve analytically for the optimal consumption paths of individuals. One key result is that, despite their different income levels, *all* consumers who purchase a particular model of the durable find it optimal to do so simultaneously, at the time when this model is first introduced. This behavior is a result of a trade-off between getting the new good earlier and paying for it earlier, which reduces one's wealth. In our model this trade-off depends on the interest rate and the (endogenous) marginal utility of wealth. If the interest rate is zero, consumers are indifferent as to when to pay for the good, so anyone who wants a new model buys it without delay. If the interest rate is high enough, everyone will delay. We show that when the interest rate equals the rate of time preference (the steady state benchmark), the incentive to purchase early takes over regardless of the consumer's current wealth level. Purchasing a durable in the middle of the design cycle is never optimal, because any consumer can be made better off by either buying the current model without delay or by waiting until the next innovation. Consequently, agents only choose holding periods for the durable that are multiples of the design cycle length. Since the relevant choices of holding periods are discrete, the consumers smooth consumption by alternating between two holding periods from time to time.

Consumers endogenously partition themselves into classes according to their wealth and the age of their durable goods, with each class following a different durable replacement rule. Two types of rules are optimal. One type, which we term a "fixed" rule, is an (s, S) policy with a constant replacement frequency. The other type is a "flexible" rule that alternates between two adjacent fixed rules at irregular intervals.

A key difference between the two types of rules is how the agents react to unexpected changes in wealth. Consumers that follow a fixed rule adjust *only* their non-durable consumption in response to a marginal windfall. By contrast, consumers that follow flexible rules adjust *only* their durable consumption.[1] In particular, if a fixed rule consumer receives a windfall, he will immediately change his non-durable consumption by the annuity value of this windfall. In contrast, a flexible rule consumer will save the windfall and spend it, after a delay, on purchasing some future durable one period earlier.

These results have implications for the impulse response of durable expenditure to an aggregate wealth shock. For example, if the shock happens in the middle of the design cycle, consumers will wait at least until the next model introduction to adjust their durable spending. Consequently, it appears that, on aggregate, the peak response of durable purchases to a wealth shock does not happen on impact, but comes with a lag.

Empirically, aggregate durable consumption is more volatile than non-durable consumption (e.g. Attanasio (1999), p. 746). The usual explanation for this regularity is that durables have more volatile consumption because they have a lower depreciation rate than non-durables. But, this logic also predicts that consumption of durables with *higher* depreciation rate should exhibit *lower* volatility. The empirical evidence seems to contradict this prediction. We compare measures of consumption volatility for several categories of durables and find that the most volatile categories are computers and autos, and these goods also have among the highest depreciation rates. Moreover, durable categories with close depreciation rates may have very different volatilities. Our model can qualitatively match these observations, because it features separate comparative statics on lumpiness and the rate of obsolescence. In the model, the overall magnitude of response of durable consumption to a wealth shock depends on the mass of consumers in fixed and flexible rule classes. We show analytically that, *ceteris paribus*, durable demand is more wealth elastic for goods that are more lumpy (e.g. autos) and are subject to faster obsolescence (e.g. computers).

Our model points out a reason why the demand for durables may exhibit strong fluctuations even in the absence of income or wealth shocks. In reality, recurrent idiosyncratic uncertainty will interfere with consumers' ability to perfectly coordinate their durable purchases with the obsolescence cycle. For example, consumers who did not plan a durable purchase but find themselves unexpectedly wealthy may find it worthwhile to buy the durable in the middle of the cycle. The implication is that higher income instability can have opposite

[1] Leahy and Zeira (2005) derive a related result in a framework where consumers buy the durable only once in their lifetime. They find what they call an "insulation effect": both non-durable consumption and the size of the durable are unaffected by wealth shocks, but the timing of purchases is. In our model with repeated durable purchases, agents in flexible rule classes insulate their non-durable consumption, while agents in fixed rule classes insulate their durable consumption.

effects on the volatility of durables and non-durables: it can lower the volatility of durable demand while increasing the volatility of non-durable consumption.

The basic model produces perfect synchronization between innovation dates and durable purchase dates, but similar results obtain when innovation is stochastic. The key feature of the obsolescence process that produces demand bunching (and a salient feature of reality as well) is the negligible hazard rate immediately after an innovation. When the hazard rate is initially negligible, consumers who purchase the durable early enjoy a longer time without the risk of obsolescence. To capture this effect, we assume that there is a minimum gestation period when no innovations can happen, and after that new models arrive stochastically at a constant Poisson rate. We show that when a new model arrives, consumers choose to buy it either with a very short delay or a very long delay. Therefore, on arrival, there is an initial burst in demand, followed by an interval where no purchases are made. This behavior matches the intuition from our basic model. If a consumer is not prepared to buy a new durable right away, he would rather wait, expecting to buy a later model. We show that the no-purchase interval is always longer than the minimum gestation period. Of course, if the period between arrivals turns out to be unexpectedly long, consumers will eventually accumulate too much wealth and purchase the existing model. Therefore, after some time, the consumers return to the market even if there is no innovation.[2]

Our work contributes to the literature on coordination, of which there are several branches. Coordination of decisions among firms has been studied theoretically and empirically in Cooper and Haltiwanger (1993a). They find that the US auto plants synchronize their re-tooling and cite strategic complementarities or seasonal preference shocks as candidate explanations. Shleifer (1986) and Francois and Lloyd-Ellis (2003) have demonstrated how coordination of innovation dates across producers can arise from agents' rational expectations about the timing of economic booms and create aggregate deterministic output cycles. We offer a different explanation for coordination: firms coordinate, because they use the same type of capital good that is subject to large and somewhat predictable improvements (The firm problem is a special case of our model, and we solve it in section 5.3).

Another strand of literature on coordination seeks to understand simultaneous technology adoption. A leading explanation is that coordination is generated by positive externalities, such as information spillovers (Bannerjee, 1992), learning by doing (Jovanovic and Lach, 1989) and consumption externalities (Farrell and Saloner, 1985). Our contribution to this literature is to show how simultaneous adoption can arise without a network externality, and this possibility gives rise to distinct policy implications. For example, the Senate Commerce Committee has recently approved a Digital TV bill that provides a $ 1.5 billion subsidy to consumers to facilitate the switch to HDTV. Currently there are about 80 million analog TVs in the US.[3] Our model suggests that many consumers may be already waiting for the switch of technology to replace their current analog sets, and may not need any additional

[2] A related paper by Balcer and Lippman (1984) analyzes the technology adoption problem under uncertainty. They find that expected arrival of a better technology limits the total number of adopters, but makes them act fast.

[3] Source: US Senate Committe on Commerce, Science and Transportation Press release, Dec 21, 2005. http://commerce.senate.gov/newsroom/printable.cfm?id=249937. See also Steven Labaton, "Digital TV bill gets Senate panel's OK. Switch from analog possible by April 2009", San Francisco Chronicle, October 21, 2005, p. C-6.

incentive to do so.

Our model is related to a large literature that studies optimal consumption problems with infrequent replacement of durable goods. Most of this literature considers optimal (s, S) replacement policies. There are two broad categories of related (s, S) models. The first category includes representative agent models with a budget constraint (e.g. Grossman and Laroque (1990), Eberly (1994), Damgaard et. al (2003)) and uncertainty driven by a Brownian motion. The second category includes replacement models with aggregate dynamics (e.g. Caballero and Engel (1999), Caplin and Leahy (1999), Adda and Cooper (2000)). These papers consider a replacement problem without an inter-period budget constraint. The model of Adda and Cooper (2000) includes durables and non-durables, but does not allow borrowing and lending. None of these models focuses on a detailed description of obsolescence. As explained above, a realistic obsolescence process does not fit into a Brownian or Poisson framework, because the shocks to durable price are serially correlated. We develop an approach to modeling obsolescence in continuous time, along with a solution methodology that can be used in other replacement problems with indivisibilities.

Section 2 describes the model. Section 3 separately solves the durable consumption problem. We construct optimal policies using a very simple geometric argument. Section 4 determines the optimal allocation of wealth between durable and non-durable consumption. Section 5 discusses our results. In Section 6 we analyze the model with stochastic innovation and discuss its implications. Section 7 concludes.

2 Model

We consider a dynamic economy with two goods, a durable and a non-durable good, and a continuum of agents that differ in their permanent income $y \in \left[\underline{y}, \bar{y}\right]$. Incomes are given exogenously, and they stay constant over time.

Goods, technology and preferences: The durable good is indivisible and is produced by a constant returns to scale technology that uses p_0 units of the non-durable good for each unit of the durable good. New durables (new models) are introduced regularly into the market at times $\tau \in \mathbf{N} = \{0, 1, \ldots\}$. Without loss of generality, we have normalized to 1 the length of a design cycle. We refer to the durable introduced at time τ as " model τ". The *technological age* of a durable good is the number of new models introduced since it was produced. The consumers are infinitely-lived and have a (common) discount rate ρ and a (common) separable flow utility function $v(\alpha, c) = x_\alpha + u(c)$, where $\alpha \in \{0, 1, \ldots, T\}$ denotes the technological age of the durable good, and c is the consumption flow for the non-durable. Durable goods of any age less than T are perfect substitutes and each agent consumes at most one unit (additional units provide no utility). We think of the non-durable good as money for the consumption of other goods, and of u as an indirect utility function. We assume that $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $x_0 \geq x_1 \geq \cdots x_{T-1} > x_T = 0$.

Obsolescence is the only form of depreciation in our model. A durable becomes useless when its technological age is T or more. A new model τ provides a flow service of x_0 in the period $[\tau, \tau + 1)$. When a new model is introduced at time $\tau + 1$, model τ's flow service decreases to x_1, and so on. The consumers can buy a new durable at any moment, but the durable is depreciated as soon as the new model is introduced. Thus, if a consumer buys a new durable at time $t \in [\tau, \tau + 1)$, he gets the flow service x_0 in the interval $[t, \tau + 1)$, and then the flow service x_1 in the interval $[\tau + 1, \tau + 2)$, and so on, as long as he doesn't replace the durable.

If x is interpreted as service flow, then one has to assume that the durable becomes less useful when new models arrive. However, our model also admits the interpretation of x as *relative utility* of a durable with respect to the latest model. In Appendix 1 we show that if the flow service of a durable remains constant over its lifetime, we can re-normalize utility and define x_α as the relative service flow from a durable of age α. Because utility is assumed to be quasilinear, this normalization changes only the utility level, but not the consumer's ranking of different durable consumption trajectories. In the example of Appendix 1, $x_\alpha = g(T - \alpha)$, where g is the average rate of technical progress in durables. In this case, each subsequent model of the durable has e^g times higher service flow than the previous one, and x falls with α simply because better goods become available at the same price.

The consumers can borrow and lend, but there are no secondary markets for used durables.

PRICES: Since the production technology is CRS, the price ratio of the durable good to the non-durable good is equal to a constant p_0 at all times. We will assume that the interest rate is fixed and equal to the discount rate: $r(t) = \rho$ for all $t \geq 0$. We therefore perform a partial equilibrium analysis. We think of the market for durables as being a small part of the aggregate economy and hence ignore the effect of durable demand on the interest rate. Our choice of interest rate is consistent with stationary equilibria. In a general equilibrium model where income (resource) flow and production technology are constant over time, a stationary equilibrium would imply a constant interest rate equal to the discount rate. If $q(t)$ and $p(t)$ denote, respectively, the prices of the non-durable and durable goods at time t, our assumption of a constant interest rate implies that $q(t) = e^{-\rho t}$ and $p(t) = p_0 q(t)$ for all $t \geq 0$, where we have normalized so that $q(0) = 1$. Define the total discount rate for one period as $\beta = e^{-\rho}$.

CONSUMER PROBLEM: Given his initial *state* (α, w), where $\alpha \in \{1, \ldots, T\}$ is the age of his endowed durable and w is his total wealth, a consumer chooses a sequence of durable purchase dates and a non-durable consumption path to maximize his discounted lifetime utility, $\int_0^\infty e^{-\rho t}[x_{\alpha_t} + u(c_t)]dt$, subject to a lifetime budget constraint. An agent's current wealth is equal to the present discounted value of all his future earnings, y/ρ, minus the present discounted value of his debts (past borrowing minus lending).

Since $r(t) = \rho$ for all t and utility is additively separable, optimally, non-durable consumption must be constant over time. Indeed, the (necessary and sufficient) first-order condition for non-durable consumption is in this case $e^{-\rho t}u'(c(t)) = \lambda e^{-\rho t}$ for all t, where $\lambda > 0$ is the Lagrange multiplier on the budget constraint. This implies that $c(t) = c(0)$ for all $t > 0$.

Let $\hat{u}(c)$ be the discounted non-durable consumption utility over one period (of length 1) in which a consumer spends (optimally) a budget c. This budget affords the constant consumption flow $c\rho/(1 - \beta)$. Hence

$$\hat{u}(c) = \int_0^1 e^{-\rho t} u\left(\frac{\rho c}{1 - \beta}\right) dt = \left[\frac{1 - \beta}{\rho}\right] u\left(\frac{\rho c}{1 - \beta}\right).$$

Let the consumer spend a constant non-durable budget c per period. Then, his lifetime non-durable discounted utility and total budget are respectively $\hat{u}(c)/(1-\beta)$ and $c/(1-\beta)$, and his residual budget for the consumption of durables is $b = w - c/(1-\beta)$.

Let $V_\alpha(b)$ denote the optimal durable consumption utility of a consumer that is endowed with a good of age α and spends a total budget b on durables. Then the problem of an agent with initial state (α, w) is

$$U_\alpha(w) = \max_{c\in[0,w]} \frac{\hat{u}(c)}{1-\beta} + V_\alpha\left(w - \frac{c}{1-\beta}\right). \tag{1}$$

In Section 3, we explicitly construct the functions V_α, $\alpha \in \{1, \ldots, T\}$, and in Section 4 we obtain the full solution for problem (1).

3 Durable consumption problem

Discrete Time: As a preliminary step in analyzing the durable consumption problem, we study a discrete time problem where the consumers are arbitrarily constrained to make new purchases only at the beginning of every period, that is, at times $t \in \mathbf{N}$. We subsequently show that removing this restriction does not change the optimal durable purchasing policy.

A consumer must choose the periods when he purchases a (new) unit of the durable good. A *durable purchasing policy* $\delta = \{\delta_t\}_{t\geq 0}$ specifies the periods in which the agent buys a new unit ($\delta_t = 1$) or keeps the old unit he has ($\delta_t = 0$). For any $i, j \in \mathbf{N}$, let $i \oplus j = \min\{i+j, T\}$ and $i \ominus j = \max\{i-j, 0\}$. Given an initial unit of age α, a purchasing policy determines the age of the unit consumed in every period $t \geq 0$ recursively as follows: $\alpha_t = 0$ if $\delta_t = 1$ and $\alpha_t = \alpha_{t-1} \oplus 1$ if $\delta_t = 0$.[4]

The optimization problem of an agent that initially has a good of age α and durable budget b is

$$\begin{aligned} V_\alpha(b) = \max & \sum_{t\geq 0} \beta^t \hat{x}_{\alpha_t} \\ \text{s.t.} \quad & \alpha_{-1} = \alpha - 1, \quad \delta_t \in \{0,1\} \text{ and } \alpha_t = (1-\delta_t)[\alpha_{t-1} \oplus 1],\ t \geq 0 \\ & b = p_0 \sum_{t\geq 0} \beta^t \delta_t, \end{aligned}$$

where $\hat{x}_\alpha = x_\alpha(1-\beta)/\rho$ denotes the total discounted utility from the consumption of a durable of age α over one period.

We solve the potentially difficult integer programming problem above using a direct geometric argument focusing on a particularly simple class of policies.

Definition: For each $R = 1, \ldots, T$, a policy δ that replaces the durable every time it reaches age R is called an *R-fixed rule.* That is, δ is an R-fixed rule if for all t, $\delta_t = 1$ if and only if $\alpha_{t-1} = R - 1$. A $(T+1)$-fixed rule is to never replace the durable: $\delta_t = 0$ for all t.

[4] To deal with period 0 as with any other period, we specify the age that the endowed durable would have been in the "previous period" and allow $\alpha_{-1} = -1$ to include the case $\alpha = 0$.

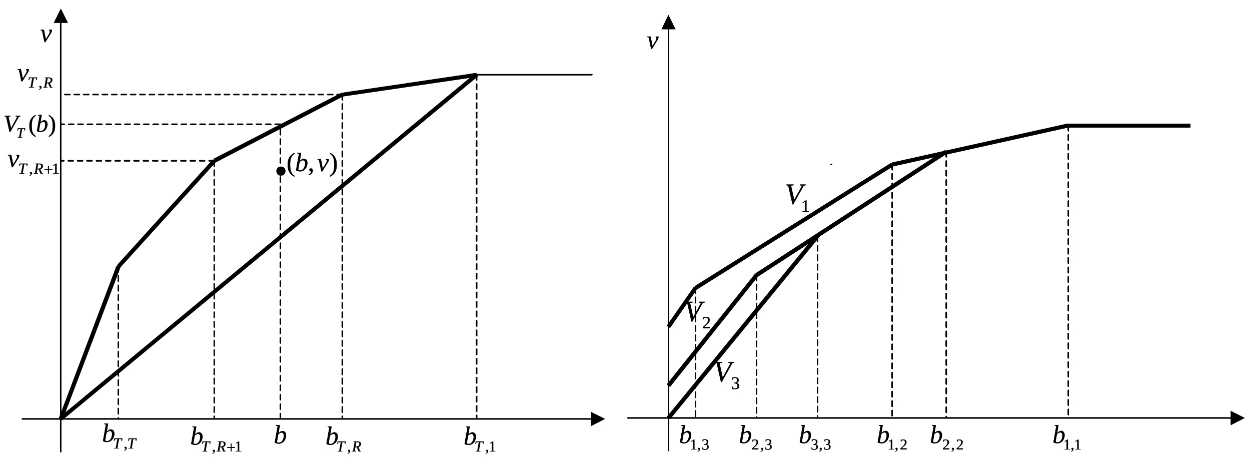

Figure 1: Optimal value function

Let $X_{\alpha,R}$ denote the total discounted utility from holding a durable from age α until age R:

$$X_{\alpha,R} = \begin{cases} \sum_{t=\alpha}^{R-1} \beta^{t-\alpha} \hat{x}_t & \alpha < R \\ 0 & \alpha \geq R. \end{cases}$$

For $R \leq T$, the value of following the R-fixed rule starting with a useless durable ($\alpha = T$) equals $v_{T,R} = X_{0,R}/(1-\beta^R)$, and its corresponding budget is $b_{T,R} = p_0/(1-\beta^R)$. The value and budget of the $(T+1)$-fixed rule are both zero.

Construct a piecewise linear function by joining the adjacent points $(b_{T,R+1}, v_{T,R+1})$ and $(b_{T,R}, v_{T,R})$ $(1 \leq R \leq T)$ with straight lines. Theorem 1 below states that this piecewise linear function is V_T. Moreover, V_T is concave (see the left frame of Figure 1).

Assume that $\alpha = T$ and for an arbitrary purchasing policy δ, group purchases by their "replacement age". That is, for each $R = 1, \ldots, T$, let L_R be the purchase dates of all durables that are later replaced at age R. Compute the weight $\lambda_R = (1-\beta^R)\sum_{t \in L_R} \beta^t$ and let $\lambda_{T+1} = 1 - \sum_{R=1}^{T} \lambda_R$. Roughly, the weight λ_R corresponds to the fraction of purchases that result in the replacement of a durable at age R. For example, if the policy is an R-fixed rule with $R < T+1$, then L_R contains all the periods t where $\delta_t = 1$, so that $\lambda_R = 1$ and $\lambda_k = 0$ for all $k \neq R$. Let (b, v) denote the budget and value of policy δ. It turns out that:

$$\begin{bmatrix} b \\ v \end{bmatrix} = \sum_{R=1}^{T+1} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix}.$$

Since the weights λ_R are nonnegative and add up to 1, the right-hand side is a convex combination of the points $\{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1}$. That is, the point (b, v) must be in the convex hull of $\{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1}$, as depicted in Figure 1. Note that the upper frontier of this set coincides with the graph of the posited optimal value function V_T. Hence, $v \leq V_T(b)$. The upper bound $V_T(b)$ is attained by a particular type of policy. Suppose R is such that $b \in [b_{T,R+1}, b_{T,R}]$, and let δ^* be a policy that replaces durables at age R or $R+1$ only. Such a policy is called an R-flexible rule. Its corresponding weights satisfy $\lambda_k^* = 0$ for all $k \notin \{R, R+1\}$. By appropriately choosing the periods when durables of age R or age $R+1$ are replaced, we can also ensure that $b = \lambda_R^* b_{T,R} + \lambda_{R+1}^* b_{T,R+1}$ (as we explain later, this is always possible provided that β is sufficiently large). Then, the value of δ^* is $\lambda_R^* v_{T,R} + \lambda_{R+1}^* v_{T,R+1} = V_T(b)$. That is, δ^* is optimal for the budget b.

For an arbitrary α now, let $b_{\alpha,R}$ and $v_{\alpha,R}$ denote the cost and the value of following the R-fixed rule when the endowed durable is of age α. Then

$$\begin{bmatrix} b_{\alpha,R} \\ v_{\alpha,R} \end{bmatrix} = \begin{bmatrix} 0 \\ X_{\alpha,R} \end{bmatrix} + \frac{\beta^{R\ominus\alpha}}{1-\beta^R} \begin{bmatrix} p_0 \\ X_{0,R} \end{bmatrix} \quad \text{for all } R \leq T$$

and $(b_{\alpha,T+1}, v_{\alpha,T+1}) = (0, X_{\alpha,T})$. It is also convenient to define $b_{T+1,T+1} = p_0$ and $b_{0,1} = \beta p_0/(1-\beta)$. Rules that replace goods more frequently require bigger budgets and have higher values. Hence $b_{\alpha,R} > b_{\alpha,R+1}$ and $v_{\alpha,R} > v_{\alpha,R+1}$.

The piecewise linear function obtained by joining the adjacent points $(b_{\alpha,R+1}, v_{\alpha,R+1})$ and $(b_{\alpha,R}, v_{\alpha,R})$ $(1 \leq R \leq T)$ with straight lines is the optimal value function V_α (see Theorem 1 below). Figure 1 (right frame) presents simultaneously the optimal value functions V_1, V_2 and V_3 for the case when $T = 3$.

Definition: Let $1 \leq R \leq T-1$ and $b \geq 0$. A policy δ is an (R, b)-*flexible rule* if it replaces durables only when they are of age R or age $R+1$ and spends the budget b exactly. If δ is an (R, b)-flexible rule then for all t, $\delta_t = 1$ implies that $\alpha_{t-1} \in \{R-1, R\}$.

Since an (R, b)-flexible rule sometimes replaces goods at age R, and sometimes at age $R+1$, it costs more than an $(R+1)$-fixed rule but less than an R-fixed rule. Hence, when the endowed good is of age α, b must be in the interval $[b_{\alpha,R+1}, b_{\alpha,R}]$. As we will see, for a certain range of b within this interval, there are multiple (R, b)-flexible rules. The R-fixed and the $(R+1)$-fixed rules are both special cases of the (R, b)-flexible rule for $b = b_{T,R}$ and $b = b_{T,R+1}$, respectively.

For $1 \leq \alpha, R \leq T$, let

$$A_R = \frac{v_{\alpha,R} - v_{\alpha,R+1}}{b_{\alpha,R} - b_{\alpha,R+1}} = \frac{1}{p_0} \left[X_{0,R} - \hat{x}_R \left[\frac{1-\beta^R}{1-\beta} \right] \right].$$

Note that A_R is independent of α and equals the slope of V_α on $[b_{\alpha,R+1}, b_{\alpha,R}]$. It is easy to check that $A_T > A_{T-1} > \cdots > A_1 > 0$, and therefore V_α is indeed a concave function.

Theorem 1: *Assume that*

$$\beta^{T-1}(1+\beta) > 1. \tag{2}$$

For each $\alpha = 1, \ldots, T$, the optimal value function V_α is

$$V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}], \; R = T, \ldots, 1,$$

and for any budget $b \geq 0$, a corresponding optimal purchasing policy is an (R, b)-flexible rule, where R is such that $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ (when $b = b_{\alpha,R}$, this policy coincides with the R-fixed rule). More precisely, the optimal purchasing policy is given by

$$\delta_\alpha^*(b) = \begin{cases} 0 & \text{for } b < b_{\alpha+1,\alpha+1} \\ \{0,1\} & \text{for } b_{\alpha+1,\alpha+1} \leq b \leq b_{\alpha-1,\alpha} \\ 1 & \text{for } b > b_{\alpha-1,\alpha}. \end{cases} \tag{3}$$

Proof: See Appendix 1.

Assumption (2) is equivalent to $\beta > \bar{\beta}$, where $\bar{\beta}$ is the (unique) root of $\beta^{T-1}(1+\beta) = 1$. This is the same as assuming that the period between subsequent model introductions is sufficiently short. When β is relatively small (and the period is long), there are budgets b that do not correspond to any durable purchasing policy. The intuition is clear. Suppose β is close to 0. Then the durable budget is almost fully determined by the timing of the first purchase. Let $\alpha < R$, δ be an R-flexible rule, and b be its corresponding budget. If the first purchase happens when the good is of age R, then $b \sim b_{\alpha,R}$ (even if all subsequent purchases replace durables of age $R+1$), and if it happens at age $R+1$, then $b \sim b_{\alpha,R+1}$ (even if all subsequent purchases replace durables of age R). Hence, budgets around the middle of the interval $(b_{\alpha,R+1}, b_{\alpha,R})$ are unattainable.

An agent that follows an R-flexible rule replaces goods of age R or $R+1$, but he is not always indifferent between these replacement ages. To follow an R-flexible rule requires that in each period the agent maintain a budget that is compatible with this rule. Assume that the durable has reached age R in the current period. Then, the current budget b must be in the interval $[b_{R,R+1}, b_{R,R}]$. Suppose b is close to $b_{R,R}$. If the agent keeps the good this period, his budget next period would be $b/\beta > b_{R+1,R}$, too large to follow the R-flexible rule from that point onward. Therefore, the agent can keep the durable this period only if $b \in [b_{R,R+1}, b_{R-1,R}]$; if $b > b_{R-1,R}$, the agent must replace now at age R. Now assume that b is close to $b_{R,R+1}$. If the agent replaces the durable now, his budget next period would be $(b - p_0)/\beta < b_{1,R+1}$, too small to follow the R-flexible rule from that point onward. Therefore, the agent can replace his durable of age R this period only if $b \in [b_{R+1,R+1}, b_{R,R}]$; if $b < b_{R+1,R+1}$, the agent must keep the durable for one more period. Assumption (2) also guarantees that $b_{R+1,R+1} < b_{R-1,R}$, and for $b \in [b_{R+1,R+1}, b_{R-1,R}]$ both keeping and replacing the durable this period are consistent with the R-flexible rule. For this interval of budgets, the agent is indifferent between replacing the durable now at age R and next period at age $R+1$.

Continuous Time: We now allow consumers to purchase durables at times other than $t \in \mathbf{N}$ and show that this does not change the optimal value function. For the continuous time replacement problem, we need a more detailed representation of the durable purchasing policy. Let τ'_k denote the period (or, equivalently, the model number) and $d_k \in [0, 1)$ be the "delay" of the k-th purchase, so the time of the k-th purchase is $\tau'_k + d_k$. The following theorem states that it is optimal to set $d_k = 0$ for all k.

Theorem 2: *For each $\alpha = 1, \ldots, T$, the optimal value function is*

$$V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}],\ R = T, \ldots, 1.$$

For any budget $b \geq 0$, the corresponding optimal purchasing policy $\{(\tau'_k, d_k)\}_{k\geq 1}$ has $d_k = 0$ for all k and is an (R, b)-flexible rule, where R is such that $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$.

Proof: See Appendix 1.

To see the intuition behind the proof of Theorem 2, suppose that an agent who wanted to replace his durable this period delayed replacement. Consider the costs and benefits from this delay, given that there are no delays in the future. If a consumer replaces his durable of age R dt periods later, he forgoes service flow $(x_0 - x_R)\,dt$ but increases his wealth by $p_0 \rho dt$, the amount of interest saved by purchasing the good later. Theorem 1 shows that the marginal utility of wealth for someone who prefers replacing the durable before if reaches age $R+1$ is less than equal to A_R. Simple algebra shows that

$$\rho p_0 A_R \leq \rho \frac{\hat{x}_0 - \hat{x}_R}{1-\beta} = x_0 - x_R, \tag{4}$$

so the financial gain from delay is less than the corresponding loss of service.

The proof of the theorem generalizes this argument by showing that similar reasoning applies also when more than one durable purchase is delayed. An arbitrary policy with delays can be modified recursively by eliminating one delay at a time while maintaining the same budget and improving its value.

Discussion: Notice that inequality (4), and Theorem 2 as well, hold for any values of β, T that satisfy (2) and any non-increasing sequence of $\hat{x}_\alpha$. That is to say that purchases with no delays are optimal regardless of the average rate of obsolescence, as long as obsolescence is periodic and the interest rate is not too high.

The results of Theorems 1 and 2 generalize in a straightforward fashion if we allow deterministic physical depreciation within a period. To see this, assume that for all α and all $s \in [0,1]$

$$x_\alpha(s,d) = \begin{cases} x_\alpha \cdot \Delta(\max\{0, s-d\}), & \alpha = 0 \\ x_\alpha \cdot \Delta(s), & \alpha > 0 \end{cases},$$

where α is the technological age of the good, s is the time elapsed since the beginning of the current period, $d < 1$ is the initial delay in the purchase of the durable, and $\Delta(\cdot)$ is a decreasing function that captures physical depreciation during a period. Let $\Delta(0) = 1$ and $\Delta(1) > \max_\alpha \left(\frac{x_{\alpha+1}}{x_\alpha}\right)$ to guarantee that the service flow from the durable is non-increasing. As before, define the average service flow per period

$$\hat{x}_\alpha = x_\alpha \int_0^1 e^{-\rho s} \Delta(s)\, ds.$$

By construction, $\hat{x}_\alpha$ is a non-increasing sequence, so Theorem 1 holds. To see if Theorem 2 will hold, consider costs and benefits of a single delay. The service flow forgone from delaying a durable purchase by dt now depends on d and equals $[x_0 \cdot \Delta(0) - x_R \cdot \Delta(d)]\,dt$. That is, since the current durable experiences physical depreciation, the forgone service flow rises with delay and is always *greater* than $(x_0 - x_R)\,dt$. Now it is even easier to satisfy inequality (4), because

$$\rho p_0 A_R \leq \rho \frac{\hat{x}_0 - \hat{x}_R}{1-\beta} < x_0 - x_R \leq x_0 - x_R \cdot \Delta(d).$$

We conclude, that, if anything, physical depreciation strengthens the incentive to avoid delays.

4 Optimal budget allocation

We now solve problem (1) for the optimal consumption of non-durables as a function of α and w. An agent with wealth w that spends b on durables optimally spends $c = (1-\beta)(w-b)$

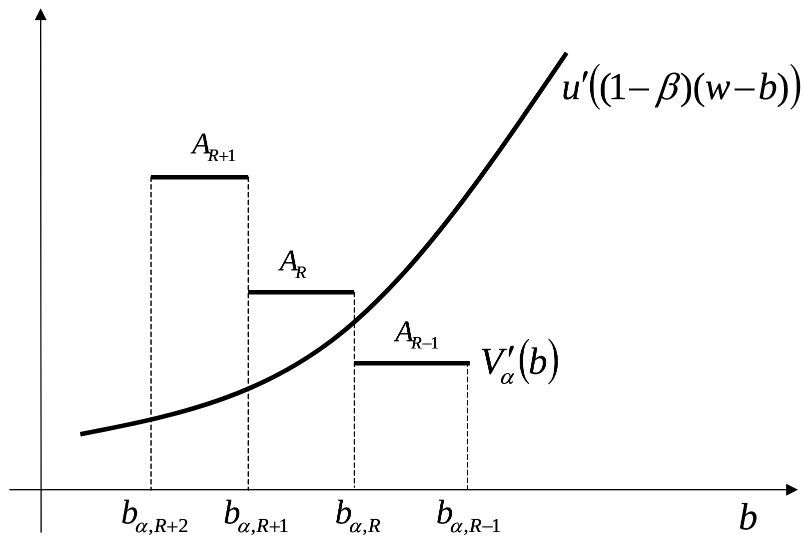

Figure 2: Marginal utilities of consumption and wealth as functions of durable budget.

per period on non-durables. Ideally, the agent should pick c (or, equivalently, b) so as to equate the marginal utility of consumption $\hat{u}'(c)$ and the marginal utility of wealth $V'_\alpha(b)$. Figure 2 depicts the marginal utility of wealth (the falling step-function because V_α is a concave piecewise linear function) and the marginal utility of consumption as functions of b (for given values of α and w). In the figure, $\hat{u}'$ crosses V'_α at a point of discontinuity. This depicts the situation when the optimal durable budget equals $b_{\alpha,R}$ and the corresponding durable purchasing policy is the R-fixed rule. Now decrease w by a small amount. The graph of $\hat{u}'((1-\beta)(w-b))$ will shift to the left, but it will still cross V'_α at $b = b_{\alpha,R}$. In other words, there is an *interval* of wealths w for which it is optimal to follow the R-fixed rule in the state (α, w). If we further decrease w, $\hat{u}'$ will eventually cross V'_α at a point where V'_α is flat and equal to A_R. This is the case when it is optimal to choose a budget corresponding to an R-flexible rule and pick the non-durable budget c_R, where $\hat{u}'(c_R) = A_R$. Hence, there is also an interval of wealths w for which it is optimal to follow the R-flexible rule and spend c_R in non-durables every period. For that range of wealths, the optimal non-durable budget remains *constant* and variations of wealth affect the durable consumption path only (higher wealths afford replacing durables at age R more frequently, while lower wealths require replacing durables at age $R+1$ more often). In contrast, when a fixed rule is optimal, a higher wealth leads to a higher level of non-durable consumption.

For a fixed α, if w varies continuously from infinity to zero, the intersection of $\hat{u}'$ with V'_α in Figure 2 moves monotonically to the left and maps out the optimal durable replacement rule (as a function of w). The wealthiest consumers use a 1-fixed rule. Next comes a group of consumers that follow 1-flexible rule, and then a group that follows the 2-fixed rule, and so on. The intervals of wealth where agents follow fixed rules are interlaced with the intervals of wealth where they follow flexible rules. The bounds of these intervals can be computed explicitly. Fix α and let

$$w_{\alpha,R}(c) = \frac{c}{1-\beta} + b_{\alpha,R}$$

be the wealth required to follow the R-fixed rule and spend a constant non-durable budget c per period when the initial durable is of age α. The wealthiest person that follows the R-flexible rule replaces his durable every R periods and consumes c_R. Hence his wealth is $w_{\alpha,R}(c_R)$. The poorest person that follows the $(R-1)$-flexible rule also replaces his durable every R periods but consumes $c_{R-1} > c_R$, so that his wealth is $w_{\alpha,R}(c_{R-1}) > w_{\alpha,R}(c_R)$. In between, there are consumers with wealth $w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$ that follow the R-fixed rule. Each one spends the same durable budget $b_{\alpha,R}$ and the non-durable budget per period

$$c_{\alpha,R}(w) = (1-\beta)(w - b_{\alpha,R}).$$

A consumer with more wealth than $w_{1,1}(c_1) = (c_1 + p_0)/(1-\beta)$ will replace his durable every period and spend more than c_1 per period in non-durables. We will assume that $\bar{y}/\rho \geq w_{1,1}(c_1)$, and define $\bar{w} = \bar{y}/\rho$ and $c_0 = (1-\beta)\bar{w} - p_0$. Similarly, a consumer with less wealth than $c_T/(1-\beta)$ will spend all his wealth in non-durable consumption. We will assume that $\underline{y}/\rho \leq c_T/(1-\beta)$, and define $\underline{w} = \underline{y}/\rho$ and $c_{T+1} = (1-\beta)\underline{w}$.

We can also express the optimal purchasing policy (3), stated in Theorem 1, as a function of wealth (and with abuse of notation denote this function by the same symbol δ^*_α). The following theorem states these results formally.

Theorem 3: *Let $c_0 = (1-\beta)\bar{w} - p_0$, $c_{T+1} = (1-\beta)\underline{w}$, and for each $R = 1,\ldots,T$, let c_R be such that $\hat{u}'(c_R) = A_R$. Denote by $c^*_\alpha(w)$ the optimal solution of problem (1). Then, for $\alpha = 1,\ldots,T$,*

$$c^*_\alpha(w) = \begin{cases} c_{\alpha,R}(w) & \text{for } w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})], \; R = T+1,\ldots,1 \\ c_R & \text{for } w \in [w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R)], \; R = T,\ldots,1, \end{cases}$$

and

$$\delta^*_\alpha(w) = \begin{cases} 0 & \text{for } w < w_{\alpha+1,\alpha+1}(c_\alpha) \\ \{0,1\} & \text{for } w_{\alpha+1,\alpha+1}(c_\alpha) \leq w \leq w_{\alpha-1,\alpha}(c_\alpha) \\ 1 & \text{for } w > w_{\alpha-1,\alpha}(c_\alpha). \end{cases} \tag{5}$$

Proof: See Appendix 1.

Over time, a consumer that follows an R-fixed rule has a constant holding time R and revisits the same points in the state space (α, w) every R periods. His wealth trajectory is cyclical. While the consumer keeps the current good, both α and w increase, as the consumer "saves" for the next purchase. When the new durable is purchased, both α and w go down, and the holding cycle starts again.

The time path for wealth of a consumer that follows an R-flexible rule is more erratic. Usually, his wealth trajectory is not cyclical: each time the durable is of age R, he has a different wealth level. For example, suppose that the consumer starts with a durable of age R and wealth level $w_0 \in (w_{R-1,R}(c_R), w_{R,R}(c_R))$. Then, he must replace the durable now, and the next time his good reaches age R, his wealth will be $w_R = [w_0 - p_0]\beta^R < w_0$. If $w_R > w_{R-1,R}$, he will have to replace the durable again. But eventually, if he continues to replace each time the durable reaches age R, he will reach a state (R, w), where $w < w_{R+1,R+1}(c_R)$. At this point, he is forced to wait one more period. Thus, the agent will switch replacement frequencies erratically, as each time that his state is of the form (R, w), his wealth level w is at a different place of the interval $[w_{R+1,R}(c_R), w_{R,R}(c_R)]$.

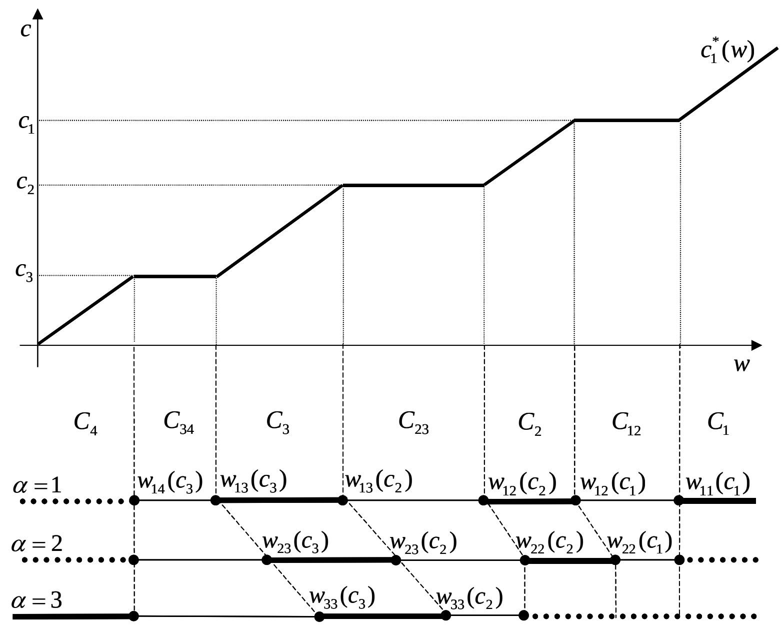

Figure 3: Consumption classes; optimal consumption function $c_1^*(w)$.

4.1 Consumption classes

The optimal policies partition the state space (α, w) into disjoint classes, with each class corresponding to a different durable replacement rule. All individuals in a class follow the same rule and the trajectories of their states stay forever in the same class. For every $R \in \{1, \ldots, T+1\}$ and $\alpha \in \{1, \ldots, R\}$, let

$$W_R^\alpha = [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$$

be the wealth levels of consumers that follow an R-fixed rule and currently have a durable of age α. Similarly, for every $R \in \{1, \ldots, T\}$ and $\alpha \in \{1, \ldots, \min\{R+1, T\}\}$ let

$$W_{R,R+1}^\alpha = (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$$

be the wealth levels of consumers that follow an R-flexible rule and currently have a durable of age α. Note that for each α, $\{W_R^\alpha\}_{R=1}^{T+1} \cup \{W_{R,R+1}^\alpha\}_{R=1}^{T}$ forms a partition of $[\underline{w}, \bar{w}]$. At the beginning of every period, agents with a state in $C_R = \bigcup_{\alpha=1}^{R} \{\alpha\} \times W_R^\alpha$ follow the R-fixed rule, and with a state in $C_{R,R+1} = \bigcup_{\alpha=1}^{R+1} \{\alpha\} \times W_{R,R+1}^\alpha$ follow the R-flexible rule. Note that after the initial period, nobody visits the states $\{\alpha\} \times W_R^\alpha$, $\alpha > R$, or the states $\{\alpha\} \times W_{R,R+1}^\alpha$, $\alpha > R+1$. A consumer with one of these initial states has been endowed with a durable that is "too old" for his initial wealth level. The classes C_R and $C_{R,R+1}$ are closed: if an agent follows the R-fixed rule (R-flexible rule) and his initial state is in C_R (in $C_{R,R+1}$), then his state remains in C_R (in $C_{R,R+1}$) forever. Figure 3 illustrates consumption classes for the case $T = 3$ and one of the corresponding optimal consumption function $c_1^*(w)$ described in Theorem 3. Three horizontal lines on the lower panel of figure 3 represent the state space $\{1, 2, 3\} \times [\underline{w}, \bar{w}]$. Bold lines indicate wealth intervals that belong to fixed rule classes, and thin lines indicate flexible rule classes. Class boundaries are marked by dashed lines. Dotted lines indicate the intervals in the state space that are empty in the long run.

4.2 Consumption response to a change in wealth

Aggregate durable and non-durable consumption both respond to aggregate changes in wealth. Consumers in class C_R have a fixed durable budget and a positive marginal propensity to consume non-durables (see Figure 3). Therefore, if any such consumer receives windfall income, he will spend it all on non-durable consumption. By contrast, consumers in a class $C_{R,R+1}$ have a zero marginal propensity to consume non-durables and a variable durable budget. The magnitude of the overall response of durable consumption to a change in wealth will depend on the mass of consumers in fixed and flexible rule classes. These masses, of course, are functions of the wealth distribution. To isolate the effect of the model's parameters on the sensitivity of durable consumption, we assume a uniform distribution over the set of recurrent states (i.e. the states marked by solid lines on Figure 3). Then the mass of consumers in classes C_R and $C_{R,R+1}$ ($R = 1, \ldots, T$) are respectively

$$\mu(C_R) = \sum_{\alpha=1}^{R} [w_{\alpha,R}(c_{R-1}) - w_{\alpha,R}(c_R)]$$

$$\mu(C_{R,R+1}) = \sum_{\alpha=1}^{\min\{R+1,T\}} [w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R)].$$

Also, define $\mu(C_{T+1}) = w_{T,T+1}(c_T) - \underline{w}$. Then, the fraction of consumers that follow flexible rules is

$$\theta = \frac{\Sigma_{R=1}^{T}\mu(C_{R,R+1})}{\Sigma_{R=1}^{T+1}\mu(C_R) + \Sigma_{R=1}^{T}\mu(C_{R,R+1})}.$$

Given a small change in wealth, approximately[5] θ consumers will adjust only their durable consumption and $1-\theta$ consumers will adjust only their non-durable consumption. The larger is θ, the more sensitive is durable consumption to changes in wealth.

Assume that $x_\alpha = g(T - \alpha)$, $\alpha = 0, \ldots, T$. This is the linear obsolescence pattern that arises when x_α is interpreted as a relative service flow (see Appendix 1). In this case, g represents the obsolescence rate – the speed at which relative service flow decays – and T represents the durability of the good – the length of its useful life. The following proposition states that under some restrictions on preferences faster obsolescence, and higher price all make durable consumption more sensitive to changes in wealth.

Proposition 1 *Let $x_\alpha = g(T - \alpha)$, $\alpha = 0, \ldots, T$. Then*

(i) An increase in the rate of obsolescence g increases θ if u has decreasing absolute risk aversion.

(ii) Assume that $u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$ with $\sigma \geq 1$. Then θ increases in p_0.

Proof: See Appendix 1.

[5] A small mass of consumers will change their consumption class as a result of change in wealth.

Higher obsolescence rate makes the service flow decline more steeply with age. As a result, durables are replaced more frequently, on average, and this makes aggregate durable demand more wealth elastic. Since we assumed that durables are only available in one size, a higher p_0 corresponds to a durable good that is more lumpy. When the durable is more expensive, the budgets corresponding to all fixed rules become larger. As a result, the kinks in $V_\alpha(b)$ become less pronounced, and the fixed rule classes become smaller in relation to flexible rule classes. A conventional assumption on preferences is constant relative risk aversion (which implies decreasing absolute risk aversion) and low intertemporal elasticity of substitution (which implies high σ). Therefore, under conventional assumptions on $u(\cdot)$ durable demand will be more sensitive to changes in aggregate wealth when g and p_0 are higher.

5 Results

5.1 Lag in durable consumption response

In response to a windfall, all fixed rule consumers immediately change their non-durable consumption by an amount equal to the annuity value of this windfall. Flexible rule consumers instead adjust their durable consumption, but only after a delay. The initial delay occurs because consumers synchronize replacement with new model introductions, so durable spending shows no response until the end of the current model cycle. Afterward, the adjustment of durable spending will be gradual and protracted, because flexible rule consumers save their marginal windfall until it can pay for replacing some future durable one period earlier. Depending on their flexible rule and their wealth, consumers choose different delays for spending the marginal windfall, which makes the adjustment of durable spending protracted.

The protracted adjustment of durable consumption to the shock is a standard feature of the (s, S) models. The non-standard part is that our model predicts the initial lag in adjustment, which has distinct (and testable) implications about the shape of the impulse response of durable expenditure to an aggregate wealth shock.

To make the distinction clear, consider this impulse response in a deterministic (s, S) model with a continuum of heterogenous consumers. On impact, the shock makes a mass of agents simultaneously reach their replacement point (lower s), generating a strong initial response. Afterwards, the response has to be weaker (unless the distribution of durable goods stocks has mass points), and its precise shape depends on the distribution of durable goods stocks before the shock. By contrast, in our model, the impulse response will initially be zero, as agents wait until the next model introduction to purchase a new durable.

Bar-Ilan and Blinder (1992, Tables 1, 2) estimate impulse responses to permanent income shocks for auto expenditure and unit sales. Their impulse responses have a hump shape followed by dampened oscillations, with the time of peak response typically 3 quarters after the shock. This suggests that the peak response is not on impact, but is delayed, consistent with our model.

One might object to this explanation of delayed consumption response for autos by arguing that there are multiple models of automobiles whose obsolescence cycles are staggered, and therefore at every instant there is a brand-new design available. The evidence from the auto industry shows otherwise. Cooper and Haltiwanger (1993 a,b) show that model year changes are coordinated across auto producers and that there is a large seasonal component in auto production and sales. Then if one estimates an impulse response of auto sales to a wealth shock with quarterly data (as Bar-Ilan and Blinder do), a delay in response is consistent with consumers waiting for the model year change to spend their windfall. Another objection might be that many of the model year changes for autos are purely cosmetic, so consumers are almost indifferent between buying the current and the next model year. In our model, the average rate of obsolescence will affect the average length of the replacement cycle, but not the purchase timing (recall the discussion following Theorem 2). That is, if obsolescence rate is low, consumers will skip several models and come back to the market only when the design changes sufficiently.

It is also true that *aggregate* durable purchases, and not just the autos, are slow to respond to aggregate wealth shocks (e.g. Caballero, 1990). Our model will have a harder time matching this fact. In the extreme case, suppose that there is a continuum of symmetric durables with perfectly staggered obsolescence cycles. Then, at any moment there is a brand-new model of some durable in the continuum. In this environment, each durable category may have periodic spikes in expenditure, but on aggregate there will be no lags in the impulse response of the durable expenditure to the wealth shock. However, in reality, consumers experience periodic large preference shocks as well, such as changes in location or family size. If these changes are infrequent and somewhat anticipated, consumers may forgo immediate adjustment. For example, a family that expects to move in the near future may delay their expenditure on furniture and appliances.

5.2 Volatility of durable consumption

It is hard to rigorously test a model as stylized as ours, but we can still see if the comparative statics of Proposition 1 match the evidence qualitatively. In this section we compare the volatility of expenditure for different categories of durable goods. We use two measures of expenditure on durable categories: expenditure share, which is the ratio of category expenditure to total consumption, and investment rate, which is the ratio of current category expenditure to the existing stock of durable in the category. To make volatility measures comparable across categories that may have different average expenditure shares and investment rates, we normalize expenditure shares and investment rate by dividing them by their HP-filter trend. That is, if expenditure in a particular year is right on its trend, the normalized share and the normalized investment rate are both equal to 1. The values of normalized expenditures that differ from 1 measure the deviation from the trend. For example, if normalized investment rate equals 1.2, this means that investment rate is 20 percent above its long-run trend. We measure the volatility of expenditure with either the standard deviation of normalized expenditure share or the standard deviation of the normalized investment rate. Table 1 compares depreciation rates, obsolescence rates and volatility measures for four categories of durables: automobiles, audio and video equipment, household appliances and computers (including software).[6] We can compare all four categories in the time inter-

[6]We measure the obsolescence rate with the annual rate of fall in the price index for the durable relative to PCE deflator (NIPA Table 2.4.4). The depreciation rate is the average ratio of depreciation to durable

val 1984-2005 when the BEA has the data on computers, and three categories for a longer period, 1953-2005. The last two columns of Table 1 show volatility measures by category for different time periods. Automobiles have high volatility, audio and video goods are in the middle, and appliances have the lowest volatility of expenditure. This volatility ranking is consistent across both measures and time periods, although the numbers vary slightly due to uneven depreciation and time-varying growth of consumption. The volatility of expenditure on computers during 1984-2005 is slightly bigger than it is for autos.

	Average rate of depreciation	Average rate of obsolescence	Standard deviation of normalized investment rate	Standard deviation of normalized expenditure share
New autos, 1984-2005	0.23	0.014	0.079	0.088
New autos, 1953-2005	0.26	0.015	0.094	0.096
Audio and video equipment, 1984-2005	0.25	0.064	0.050	0.040
Audio and video equipment, 1953-2005	0.22	0.054	0.062	0.054
Appliances, 1984-2005	0.16	0.033	0.028	0.021
Appliances, 1953-2005	0.16	0.030	0.042	0.033
Personal computers, 1984-2005	0.44	0.24	0.081	0.105

Table 1. Rate of depreciation, rate of obsolescence and measures of expenditure volatility for categories of durable goods, 1953-2005. Source: BEA NIPA Tables 2.4.4, 2.4.5, 8.1, 8.4.

It is well-known that aggregate durable consumption is more volatile than aggregate non-durable consumption. The standard PIH model (e.g. Mankiw, 1982, see also Appendix 2 for details) can explain this. In the standard model, the short-run wealth elasticity of demand for a durable is inversely proportional to its rate of economic depreciation. Therefore, demand for durables (with depreciation rates less than 100%) should be more volatile than demand for non-durables (with depreciation rate of 100%). However, the model also implies that the smaller is the rate of depreciation of a good, the more volatile is its demand. The data in Table 1 seems to contradict this explanation. For example, appliances have both the lowest depreciation rate and the lowest volatility. Audio/video equipment and autos have similar depreciation rates, but different volatilities.

On the other hand, Proposition 1 can explain some of the volatility rankings in Table 1. In particular, according to our model, among the goods with similar size, those with higher obsolescence rates will have more volatile expenditure. This result agrees with volatility

stock (NIUPA Tables 8.1 and 8.4).

rankings for appliances, audio/video equipment and computers, all of which arguably have similar sizes. Proposition 1 also says that among the goods with similar obsolescence rates, those that are more lumpy will have more volatile expenditure. This agrees with volatility rankings for appliances versus autos, as they have similar obsolescence rates, but different sizes.

5.3 Optimal decision rule in an investment problem

Investment problems usually do not have budget constraints. We can eliminate the budget constraint from our model by assuming that $u(c)$ is linear and that consumers can afford any replacement sequence. Then the investment problem becomes a special case of our model. To generate differences in optimal replacement rules, we assume that consumers have different preferences over the durable. In particular, we assume that a consumer of type h derive utility xh from a durable with the service flow x. Then, his maximization problem is

$$\max_{\{\tau'_k, d_k\}} (vh - b)$$

where v is the value of a durable purchasing policy and b is the budget. The following corollary to Theorem 2 shows that the optimal policy of any consumer is a fixed rule without delays.

Corollary For any h, the optimal policy is a R^*-fixed rule without delays, where

$$R^* = \arg\max_R (v_{T,R} \cdot h - b_{T,R})$$

Proof: See Appendix.

Note that flexible rules disappear in the model without the budget constraint because the utility function is linear in non-durable consumption, and therefore there is no incentive for consumption-smoothing.

If we think of consumers as firms and consumer types as firm-specific productivities, then our model predicts that different firms coordinate their investment decisions with the obsolescence cycle. Investment is spiky not just on firm level, but on the industry level as well. This explanation for coordination across firms is similar in spirit to that in Bertola and Caballero (1990) who argue that the extent of coordination depends on the magnitude of idiosyncratic versus aggregate shocks. In our framework, there is no idiosyncratic uncertainty, and the movements in the price of the durable are perfectly anticipated. Our analysis suggests that one might look at the uneven technological progress in producer durables as a source of aggregate shocks that drive coordination in industries that use similar types of equipment.

6 Random Period Length

In this section, we solve the durable replacement problem with random arrival times. In our basic framework with certainty, the consumers can perfectly predict when new models will arrive to the market. By contrast, in a model with Poisson arrivals, the hazard rate stays constant, and the expected service flow of a new durable is independent of the time of purchase. Therefore, consumers have no incentive to time their purchases near the model introduction dates. We adopt a model that combines these two extreme cases. We assume that the development times of new models are i. i. d. random variables distributed on $[S, \infty)$, where $S \geq 0$ is the *minimum gestation period.* If τ is the time it takes to develop and introduce a new model into the market, then $\tau - S$ has an exponential distribution with parameter λ. Thus, the average arrival time is $S + 1/\lambda$. Note that our deterministic model is the limit case when $S = 1$ and $\lambda \to \infty$, and that the pure Poisson arrival model corresponds to the case when $S = 0$ and $\lambda > 0$.

For tractability, we assume that there is only one good, the durable, and that each agent has a lifetime budget b to spend on durables.[7] This model focuses exclusively on the intertemporal trade-off of purchasing the durable at different points in time, and excludes the possibility of an on-going substitution between the durable and other goods. Since we are primarily interested in the timing of durable purchases, this simplified specification seems appropriate.

In many durable goods markets, the quality-adjusted price for the durable falls over time because of manufacturing efficiency improvements. Thus, we now also let the price of the durable fall exponentially over time: $p_t = p_0 e^{-\gamma t}$. With falling prices it may become attractive to buy a durable with delay: though its expected service flow decreases, the durable also becomes cheaper.

The state variables for the consumer are α - the technological age of the endowed durable, $\bar{b}_t = b_t/p_t$ - the "purchasing power" of the consumer, and s - the time since the last arrival of a new model (i.e. the age of the current model). The law of motion for the purchasing power is

$$\bar{b}_{t+\Delta t} = \left(\bar{b}_t - \delta_t\right) e^{(r+\gamma)\Delta t},$$

where, as before, $\delta_t = 1$ if a new durable is purchased at date t and $\delta_t = 0$ otherwise.

Let $V_\alpha\left(\bar{b}, s\right)$ be a consumer's total discounted value of holding a purchasing power $\bar{b}$ and a durable of technological age α at the moment when a time s has elapsed since the last arrival of a new model. We restrict attention to the case $T = 1$, so $\alpha \in \{0, 1\}$. This captures the main insights from the extended model without making the proofs excessively complicated.

When the new model reaches age S, innovations start arriving at a constant Poisson rate λ and s becomes uninformative about the time of the next arrival. Therefore, the value function ceases to depend on s:

$$V_\alpha\left(\bar{b}, s\right) = V_\alpha\left(\bar{b}, S\right),\ s \geq S.$$

$V_0\left(\bar{b}, S\right)$ is the value of the service flow x_0 until the next Poisson event and the continuation

[7] Because we assumed that durable budgets are exogenously determined, there is not an exact correspondence between the durable purchasing policy and the solution to an optimal consumption problem with two goods. In a dynamic optimization problem with two goods, the agents will optimally reallocate their budgets between durables and non-durables when new information becomes available. For example, if the design cycle has been unexpectedly long, an agent may want to start using part of his durable budget to increase his non-durable consumption.

value of an old model afterwards:

$$V_0\left(\bar{b}, S\right) = \int_0^\infty dt\lambda e^{-\lambda t}\left[\int_0^t d\tau e^{-\rho\tau}x_0 + e^{-\rho t}V_1\left(\bar{b}e^{(r+\gamma)t}, 0\right)\right]$$
$$= \frac{x_0}{\lambda+\rho} + \int_0^\infty \lambda e^{-(\lambda+\rho)t}V_1\left(\bar{b}e^{(r+\gamma)t}, 0\right)dt. \quad (6)$$

For $s < S$,

$$V_0\left(\bar{b}, s\right) = \int_0^{S-s} x_0 e^{-\rho\tau}d\tau + e^{-\rho(S-s)}V_0\left(\bar{b}e^{(r+\gamma)(S-s)}, S\right).$$

Replacement decision: The agent chooses the delay in replacement $d\left(\bar{b}\right)$ of a depreciated good (x_1) since the last arrival of a new model. It is convenient to distinguish between two cases: $d < S$ and $d > S$, and separate the optimization problems over these intervals. Call the corresponding value functions $V_1^L\left(\bar{b}\right)$ and $V_1^R\left(\bar{b}\right)$. Both of those value functions are measured at the point where $s = 0$. Then

$$V_1\left(\bar{b}, 0\right) = \max\left\{V_1^L\left(\bar{b}\right), V_1^R\left(\bar{b}\right)\right\},$$

where

$$V_1^L\left(\bar{b}\right) = \max_{0\leq d\leq S}\left(\frac{x_1(1-e^{-\rho d})}{\rho} + e^{-\rho d}V_0\left(\bar{b}e^{(r+\gamma)d} - 1, d\right)\right) \quad (7)$$

and

$$V_1^R\left(\bar{b}\right) = \max_{d\geq S}\int_0^S x_1 e^{-\rho\tau}d\tau + e^{-\rho S}\int_0^{d-S} dt\lambda e^{-\lambda t}\left(\int_0^t x_1 e^{-\rho\tau}d\tau + e^{-\rho t}V_1\left(\bar{b}e^{(r+\gamma)(t+S)}, 0\right)\right)$$
$$+\ e^{-\lambda(d-S)}\left(\int_S^d x_1 e^{-\rho\tau}d\tau + e^{-\rho d}V_0\left(\bar{b}e^{(r+\gamma)d} - 1, S\right)\right). \quad (8)$$

The first term in the above expression is the value of holding the depreciated good until the new model reaches age S, the second term is the expected value of holding the depreciated good between S and d and the third term is the expected value of the replacement at d. (Note here that when a new good is bought at the time when $s > S$, the corresponding continuation value of a new durable is as if it were of age S.) The following proposition characterizes the optimal delay.

Proposition 2: Let $\rho = r + \gamma$ and let $S > 0$. Then there is an interval $\left[\underline{S}, \bar{S}\right]$ with $0 \leq \underline{S} < S < \bar{S}$ such that no durable purchases are made in the interval $\left[\underline{S}, \bar{S}\right]$ after new model arrivals. That is

$$d\left(\bar{b}\right) \leq \underline{S} \text{ or } d\left(\bar{b}\right) \geq \bar{S} \text{ for all } \bar{b} \geq 0.$$

Moreover, all consumers who can afford the new model either purchase it right away (when $s = 0$) or after date S:

$$d\left(\bar{b}\right) = 0 \text{ or } d\left(\bar{b}\right) \geq \bar{S} \text{ for all } \bar{b} \geq 1.$$

Furthermore, if $S = 0$, then $\underline{S} = \bar{S} = 0$.

Proof: See Appendix.

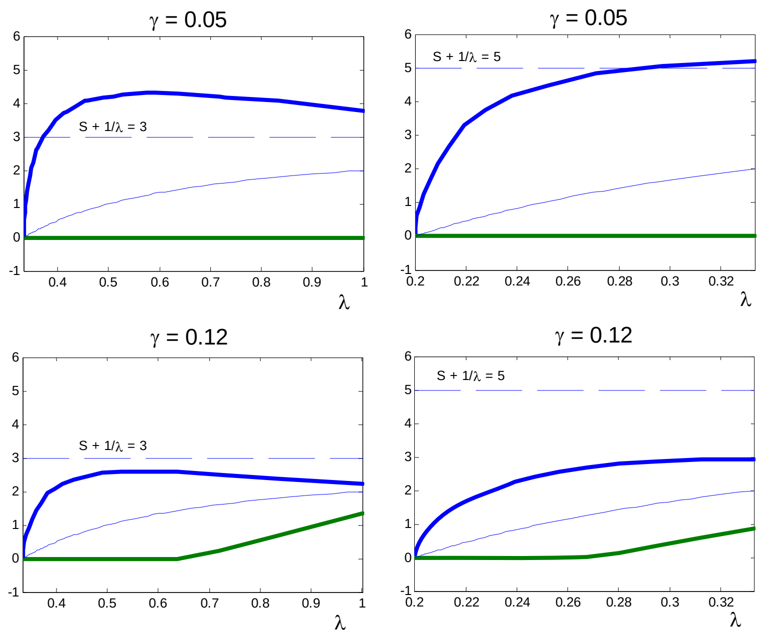

Figure 4: No-purchase intervals for various values of γ, λ and S.

Proposition 2 illustrates how predictability of innovations affects the properties of demand. When $S = 0$ and the hazard rate of innovation is constant (i.e. arrival is unpredictable), demand for the durable stays positive from the time of innovation until the time when all consumers have purchased the current model. By contrast, when $S > 0$, demand for the durable falls to zero as the date of possible new arrivals (that is, date S) draws sufficiently close and stays at zero for some time after date S. The latter result mirrors our Theorem 2, where we showed that new models are always purchased without delay for $r = \rho$ and $\gamma = 0$. By continuity, Proposition 2 should also hold when $r = \rho$ and $\gamma > 0$ is sufficiently small. We investigate numerically whether the no-purchase intervals are quantitatively important, especially when the price of the durable is falling.

For the simulations, we choose parameter values that we think are representative of markets for high-tech products. If x_0 and x_1 are the relative service flows of the durables in a detrended model, then the corresponding absolute service flows are $z_\tau = e^{x_\tau}$, $\tau = 0, 1$ (see the Appendix). Hence, $z_0/z_1 = e^{(x_0 - x_1)}$ is the relative advantage of a new model. We set $x_0 = 0.4$ and $x_1 = 0$, which corresponds to a new model providing 50 percent more service than old models. We perform simulations for two average introduction times: $A = S + 1/\lambda = \{3, 5\}$ years. We set parameters $\rho = r = 0.04$ and experiment with different values of γ, S and λ.

Figure 4 shows our results. In each panel, purchase delay is on the vertical axis and λ is on the horizontal axis. The average introduction time A is 3 years for the left panels and 5 years for the right panels, while $\gamma = 0.05$ for the top panels and $\gamma = 0.12$ for the bottom panels. Since in each panel the average introduction time is kept constant, as λ varies, S needs to be adjusted accordingly. Let $S(\lambda) = A - 1/\lambda$. This function is plotted as a thin solid line in each panel. The thick lines show the ends of the no-purchase interval, $\underline{S}(\lambda)$ and

$\bar{S}(\lambda)$. In all cases, $\underline{S}(\lambda) \leq S(\lambda) \leq \bar{S}(\lambda)$. All three lines cross at $S = 0$, which corresponds to the case of pure Poisson arrival times.

Note that when $\gamma = 0.05$, $\underline{S} = 0$. When the average introduction time is 3 (top-left panel), note also that $\bar{S} > 3$ for $\lambda \geq 0.4$. For those parameters, the time between new arrivals will often be shorter than $\bar{S}$. That is, the consumers who wanted to delay the purchase beyond $\bar{S}$ are frequently 'surprised' with a new arrival before the time when they were prepared to buy a new model. Obviously, when this happens, the consumers begin the new cycle with a higher purchasing power; and those with a high enough $\bar{b}$ will buy the new model right away. That is, for some consumers, the surprise provokes an earlier purchase than what was 'scheduled'. In reality, then, the majority of the purchases will be perfectly synchronized with the arrivals of new models, as in the deterministic setting. When $\gamma = 0.12$, $\bar{S} > 0$ for high values of λ. Here, because prices are falling rapidly, some consumers are prepared to wait for a while after a new model arrives before purchasing it. So the initial purchase spike, produced by those consumers with $d\left(\bar{b}\right) = 0$, is followed by a period of positive demand (until $\underline{S}$). Afterwards, demand drops to zero and purchases resume only if the period turns out to be sufficiently long.

7 Conclusions

We have presented a model of durable goods that highlights the difference between obsolescence and physical wear and tear. The basic model is simple and it can be solved analytically. We identify discrete obsolescence as a distinct source of demand fluctuations, and explain how it affects the technology adoption decisions and transmission of wealth shocks. The key implications of the basic model carry over to the case when obsolescence is stochastic and the relative price of the durable is falling.

Our model offers a building block for a general equilibrium analysis of an investment problem with capital obsolescence. Periodic obsolescence makes investment spiky even at the aggregate level, although interest rate adjustments will partially smooth out these spikes. Our framework can generate cyclical investment patterns and suggests a relationship between technological innovations in capital goods and the business cycle.

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Appendix 1: Proofs

Detrending: Our model can be viewed as the detrended version of a fully dynamic model with a constant rate of technical progress. Suppose that a model τ provides a *constant* service flow z_τ for the duration of its useful life, in the interval $[\tau, \tau+T)$, and that $z_\tau = e^{g\tau}$, where g is the rate of technical progress, or, equivalently, the rate of decrease of the quality-adjusted price for the durable. Now assume that the consumers' utility function is $\hat{v}(z,c) = \ln(z) + u(c)$, where z is the service flow of the durable good and c is the flow of non-durable consumption. This dynamic model corresponds to the stationary model we propose when

$$x_\alpha = g(T-\alpha) \quad \text{for } \alpha = 0, \ldots, T.$$

Indeed, let $\alpha : \mathbf{R}_+ \to \{0, \ldots, T\}$ and $c : \mathbf{R}_+ \to \mathbf{R}_+$ be two measurable functions representing the consumption trajectory of a consumer (where $\alpha(t)$ is the technological age of the durable being consumed at time t). For any $r \in \mathbf{R}$, let $\lfloor r \rfloor$ denote the largest integer less than or equal to r. Note that along that trajectory, the model being consumed at time t is $\tau(t) = \lfloor t \rfloor - \alpha(t)$. Thus, the total discounted utility for the trajectory (α, c) is

$$\begin{aligned} U(\alpha,c) &= \int_0^\infty e^{-\rho t}[\ln(z_{\tau(t)}) + u(c(t))]dt = \int_0^\infty e^{-\rho t}[g(\lfloor t \rfloor - T + T - \alpha(t)) + u(c(t))]dt \\ &= K + \int_0^\infty e^{-\rho t}[x_{\alpha(t)} + u(c(t))]dt, \end{aligned}$$

where

$$K = \int_0^\infty e^{-\rho t} g(\lfloor t \rfloor - T)dt = \sum_{k=0}^{\infty} gk \int_0^1 e^{-\rho(k+t)}dt - \frac{gT}{\rho} = \frac{g}{\rho}\left[\frac{e^{-\rho}}{1-e^{-\rho}} - T\right].$$

Arbitrarily, we can re-normalize utility to set $K = 0$ without changing the consumer's preferences over consumption paths. Then, the total discounted utility coincides with that of a consumer with utility function $v(\alpha, c) = x_\alpha + u(c)$.

Proof of Theorem 1: Suppose the agent is endowed with a durable of age α and follows an arbitrary purchasing policy $\tau = \{\tau_k\}_{k=1}^\infty$. We first show that the total cost and value (b, v) of policy τ can be represented as a convex combination of the points $\{(b_{T,R}, v_{T,R})\}_{R=1}^{T+1}$. Let $\tau_0 = -\alpha$ and $r_k = \min\{\tau_{k+1} - \tau_k, T\}$ for all $k \geq 0$. Then

$$b = p_0 \sum_{k \geq 1} \beta^{\tau_k} \quad \text{and} \quad v = X_{\alpha, r_0} + \sum_{k\geq 1} \beta^{\tau_k} X_{0, r_k}.$$

Define $K_R = \{k \geq 1 \mid r_k = R\}$ for $R = 1, \ldots, T$. Then

$$\beta^{\tau_1} = [\beta^{\tau_1} - \beta^{\tau_2}] + [\beta^{\tau_2} - \beta^{\tau_3}] + \cdots \geq \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R),$$

where the inequality is strict if for some $k \in K_T$, $\tau_{k+1} - \tau_k > T$. Let $\lambda_R = \sum_{K_R} \beta^{\tau_k}(1-\beta^R)$ for $R = 1, \ldots, T$, and let $\lambda_{T+1} = 1 - \sum_{R=1}^T \lambda_R$. Thus $\lambda_R \geq 0$ for all R, $\sum_{R=1}^{T+1} \lambda_R = 1$, and since $b_{T,T+1} = v_{T,T+1} = 0$,

$$b = p_0 \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k} = \sum_{R=1}^{T} \sum_{k \in K_R} \beta^{\tau_k} (1-\beta^R) b_{T,R} = \sum_{R=1}^{T+1} \lambda_R b_{T,R}$$

$$v - X_{\alpha,r_0} = \sum_{R=1}^{T} X_{0,R} \sum_{k \in K_R} \beta^{\tau_k} = \sum_{R=1}^{T} \frac{X_{0,R}}{1-\beta^R} \sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R) = \sum_{R=1}^{T+1} \lambda_R v_{T,R}.$$

Put differently, $(b, v - X_{\alpha,r_0}) = \sum_R \lambda_R (b_{T,R}, v_{T,R})$ is a convex combination of the two-dimensional vectors $(b_{T,R}, v_{T,R})$. Note that when $\alpha = T$, $X_{\alpha,r_0} = 0$ for all r_0.[8]

We next deduce an optimal policy for the case where $\alpha = T$ (i.e., when the agent is endowed with a useless durable). If $b \geq b_{T,1}$, the agent can afford to replace the durable every period and $V_T(b) = v_{T,1}$ (moreover, if $b > b_{T,1}$, it is not possible for the agent to spend the budget b in durables). For what follows assume that $b < b_{T,1}$. Let R and $\lambda_R^* \in [0,1]$ be such that $b = \lambda_R^* b_R + (1-\lambda_R^*) b_{R+1}$. Since $(b, V_T(b)) = \sum \lambda_R (b_{T,R}, v_{T,R})$ for some nonnegative weights λ_R adding to 1, we have that $V_T(b) \leq \lambda_R^* v_{T,R} + (1-\lambda_R^*) v_{T,R+1}$. To conclude, we only need to show that this bound is attained. For this we need to show that there exists a policy τ such that $\sum_{k \in K_R} \beta^{\tau_k}(1-\beta^R) = \lambda_R^*$ and $\sum_{k \in K_{R+1}} \beta^{\tau_k}(1-\beta^{R+1}) = 1-\lambda_R^*$. Put differently, we need to show that there exists an R-flexible rule with budget b.

Assume that $R < T$ and let B_R^* denote the set of budgets $b(\tau)$ corresponding to policies τ that are R-flexible rules and satisfy $\tau_1 = 0$ (that is, τ makes a purchase in the first period). Let τ be such a policy and τ' be its continuation policy from the period of the second purchase onward: $\tau'_t = \tau_{t+1} - \tau_1$ for all $t \geq 1$. Then, τ' is also an R-flexible rule and $\tau'_1 = 0$ and its corresponding budget $b(\tau') \in B_R^*$. Now, either $b(\tau) = p_0 + \beta^R b(\tau')$ (if $\tau_2 = R$) or $b(\tau) = p_0 + \beta^{R+1} b(\tau')$ (if $\tau_2 = R+1$). Therefore, B_R^* is the largest set B such that $B = [p_0 + \beta^R B] \cup [p_0 + \beta^{R+1} B]$. Observe that $p_0 + \beta^{R+1} b_{T,R+1} = b_{T,R+1}$ and $p_0 + \beta^R b_{T,R} = b_{T,R}$, and that $p_0 + \beta^R b_{T,R+1} < p_0 + \beta^{R+1} b_{T,R}$ when $\beta^{T-1}(1+\beta) > 1$. Therefore $B = [b_{T,R+1}, b_{T,R}]$ is a fixed point of the above equation. Since $p_0 + \beta^R d < d$ for all $d > b_{T,R}$ and $p_0 + \beta^{R+1} d > d$ for all $d < b_{T,R+1}$, B is also the largest such fixed point, and thus $B_R^* = B$. That is, for each budget $b \in B_R^* = [b_{T,R+1}, b_{T,R}]$ there exists a (R,b)-flexible rule (that spends the budget b exactly). The proof for $R = T$ is similar (here $b_{T,T+1} = 0$ and we must consider policies τ where $\tau_{k+1} - \tau_k > T+1$ for some k).

Finally, observe that if (T,b) is the initial state and τ and $\hat{\tau}$ are two (R,b)-flexible rules (they spend the same budget b), then their corresponding λ_R (and $1-\lambda_R$) must coincide, and therefore they must have the same value as well. In particular, if $b \in [b_{T,R+1}, b_{T,R}]$, then *any* R-flexible rule that spends the budget b exactly is an optimal policy.

By construction, the value of following an (R,b)-flexible rule starting from a durable of age T is given by

$$V_T(b) = v_{T,R+1} + A_R (b - b_{T,R+1}), \; b \in [b_{T,R+1}, b_{T,R}], \; R = T, \ldots, 1.$$

When the endowed durable is of age $\alpha < T$, the corresponding optimal value function $V_\alpha(b)$ can be deduced from $V_T(b)$ from the observation that the continuation of an optimal

[8]For each $R = 1, \ldots, R$, we could define instead $L_R = \{\tau_k \mid k \in \mathbf{N} \text{ and } r_k = R\}$, as we did in Section 3. Then, $\lambda_R = \sum_{t \in L_R} \beta^t$. While K_R contains the purchase numbers, L_R contains the purchase periods of durables that are disposed at age R. However, for other purposes, the set K_R is more convenient.

policy is an optimal policy for the subproblem that arises in the second period after following the policy in the first period.

If starting with a budget $b \in [p_0, b_{T,1}] = [b_{T+1,T+1}, b_{T,1}]$, a consumer buys a durable in the first period and then keeps it for the next $\alpha - 1$ periods, his budget at the beginning of period $\alpha \geq 1$ is $\theta_\alpha(b) = (b - p_0)/\beta^\alpha$. Moreover, for any $1 \leq R \leq T+1$ and $1 \leq \alpha \leq \min\{R, T\}$, $\theta_\alpha(b_{T,R}) = b_{\alpha,R}$.

Assume that the initial state is (α, b), where $1 \leq \alpha < T$ and $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ for some $\alpha \leq R \leq T$. Let $\tilde{b} = p_0 + \beta^\alpha b$. Then $b = \theta_\alpha(\tilde{b})$. Since $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$, it must be that $\tilde{b} \in [b_{T,R+1}, b_{T,R}]$. Therefore, starting at state $(T, \tilde{b})$, it is optimal to follow an R-flexible rule. Assume he does so. Then, after α periods his state becomes (α, b), and from state (α, b) he must be following an R-flexible rule as well. Hence, the agent must keep the durable for another $R - \alpha$ periods (at least). At that point, he arrives at state $(R, b/\beta^{R-\alpha})$. Note that $(1/\beta^{R-\alpha})[b_{\alpha,R+1}, b_{\alpha,R}] = [b_{R,R+1}, b_{R,R}]$ and that $\beta^{R-\alpha} b_{R+1,R+1} = b_{\alpha+1,R+1} \in (b_{\alpha,R+1}, b_{\alpha,R})$. Hence, if $b/\beta^{R-\alpha} \in [b_{R,R+1}, b_{R+1,R+1})$ he must keep the durable this period and buy a new durable next period, so his continuation value is $\hat{x}_R + V_T(b/\beta^{R+1-\alpha})$. If $b/\beta^{R-\alpha} \in [b_{R+1,R+1}, b_{R,R}]$ he can optimally buy a new durable this period, and his continuation value is $V_T(b/\beta^{R-\alpha})$. Therefore

$$V_\alpha(b) = \begin{cases} X_{\alpha,R+1} + \beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha}) & \text{for } b \in [b_{\alpha,R+1}, b_{\alpha+1,R+1}) \\ X_{\alpha,R} + \beta^{R-\alpha} V_T(b/\beta^{R-\alpha}) & \text{for } b \in [b_{\alpha+1,R+1}, b_{\alpha,R}]. \end{cases}$$

Suppose that $b \in [b_{\alpha,R+1}, b_{\alpha+1,R+1})$. Then $b/\beta^{R+1-\alpha} \in [b_{R+1,R+1}, b_{R+1,R+1}/\beta) \subset [b_{T,R+1}, b_{T,R}]$. Therefore, $V_T(b/\beta^{R+1-\alpha}) = v_{T,R+1} + A_R(b/\beta^{R+1-\alpha} - b_{T,R+1})$, and

$$X_{\alpha,R+1} + \beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha}) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}).$$

Now suppose that $b \in [b_{\alpha+1,R+1}, b_{\alpha,R}]$. Then $b/\beta^{R-\alpha} \in [b_{R,R+1}, b_{R,R}/\beta) \subset [b_{T,R+1}, b_{T,R}]$. Therefore, $V_T(b/\beta^{R-\alpha}) = v_{T,R+1} + A_R\left(b/\beta^{R-\alpha} - b_{T,R+1}\right)$, and tedious algebra shows again that

$$X_{\alpha,R} + \beta^{R-\alpha} V_T(b/\beta^{R-\alpha}) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}).$$

Therefore, for all $\alpha \leq R \leq T$ and $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$, $V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1})$. It remains to find $V_\alpha(b)$ for $b > b_{\alpha,\alpha}$. We claim that $V_\alpha(b) = V_T(b)$ for all $b > b_{\alpha,\alpha}$. Since $b_{\alpha,R} = b_{T,R}$ for all $R \leq \alpha$, we have that $V_T(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1})$ for all $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ and $1 \leq R < \alpha$, and the claim would complete the proof. To prove our claim, we show that

$$V_T(b) > X_{\alpha,s+\alpha} + \beta^s V_T(b/\beta^s) \text{ for all } s > 0 \text{ and } b > b_{\alpha,\alpha}.$$

That is, when $b > b_{\alpha,\alpha}$, the consumer strictly prefers to replace the durable immediately than to replace it at any later time. One can check that the above inequality holds when $b = b_{\alpha,\alpha}$. Also, since V_T is concave, the function $V_T(b)$ has a higher slope than the function on the right hand side for any $b > 0$. Hence, the inequality holds for every $b > b_{\alpha,\alpha}$. ∎

Proof of Theorem 2: Consider an arbitrary purchasing policy $\{(\tau'_k, d_k)\}_{k=1}^\infty$, where $\tau'_k + d_k$ denotes the time of the k-th purchase and $\tau'_k \in \mathbf{N}$ its corresponding period (so $d_k \in [0, 1)$ denotes its "delay"). Let $r_0 = \alpha + \tau'_1$, and for all $k \geq 1$, let $\tau_k = \tau'_k - \tau'_1$ and

$r_k = \min\{\tau_{k+1} - \tau_k, T\}$. Then the continuation budget and value of such a policy, at the beginning of period τ'_1, are respectively

$$b = p_0 \sum_{k\geq 1} \beta^{\tau_k} e^{-\rho d_k} = p_0 \sum_{k\geq 1} \beta^{\tau_k} - p_0 \sum_{k\geq 1} \beta^{\tau_k}(1 - e^{-\rho d_k})$$
$$v = \sum_{k\geq 1} \beta^{\tau_k} X_{0,r_k} - \sum_{k\geq 1} \beta^{\tau_k}(x_0 - x_{r_{k-1}})(1 - e^{-\rho d_k})/\rho.$$

Let $I_{r_0} = 1$ and $I_R = 0$ for $R \neq r_0$. For $1 \leq R \leq T$, let $K_R = \{k \geq 1 \mid r_k = R\}$,

$$\lambda_R = \sum_{k\in K_R} \beta^{\tau_k}(1 - \beta^R), \quad \bar{\gamma}_R = I_R \beta^0 + \sum_{k\in K_R} \beta^{\tau_{k+1}} \quad \text{and}$$
$$\gamma_R = I_R \beta^0 \left[\frac{1 - e^{-\rho d_1}}{1 - \beta}\right] + \sum_{k\in K_R} \beta^{\tau_{k+1}} \left[\frac{1 - e^{-\rho d_{k+1}}}{1 - \beta}\right], \tag{9}$$

so that

$$\begin{bmatrix} b \\ v \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \hat{v} \end{bmatrix} - \sum_{R=1}^{T} \gamma_R \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_R \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} \hat{b} \\ \hat{v} \end{bmatrix} = \sum_{R=1}^{T} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix}.$$

The coefficient λ_R incorporates the discounting of all the periods in which a purchase is made for a good that will be replaced at age R. By contrast, γ_R incorporates the discounting of all the periods in which a purchase is made to replace a good of age R. The adjustment, reflected in the factor multiplying I_{r_0}, of γ_{r_0} (and $\bar{\gamma}_{r_0}$) is required to take into account the first purchase that replaces the endowed good (that in our accounting, was *not* previously purchased). Observe that for $1 \leq R \leq T-1$, $k \in K_R$ implies that $\tau_{k+1} = \tau_k + R$ (if $k \in R_T$ then $\tau_{k+1} \geq \tau_k + T$, where strict inequality holds when a useless good is not replaced for one or more periods). Therefore

$$\bar{\gamma}_R = I_R + \left[\frac{\beta^R}{1 - \beta^R}\right] \lambda_R \quad \text{for all } 1 \leq R \leq T-1, \text{ and}$$
$$\sum_{R=1}^{T} \frac{\lambda_R}{1-\beta^R} = \sum_{k\geq 1} \beta^{\tau_k} = \sum_{R=1}^{T} \bar{\gamma}_R = \sum_{R=1}^{T-1} \left[\frac{\beta^R}{1-\beta^R}\lambda_R + I_R\right] + \bar{\gamma}_T.$$

Hence, $\bar{\gamma}_T = \sum_{R=1}^{T-1} \lambda_R + \lambda_T/(1-\beta^T) + I_T - 1 = I_T + \lambda_T \beta^T/(1-\beta^T)$. Let $\Lambda = \{\lambda \in \mathbf{R}^T_+ \mid \sum_{R=1}^{T} \lambda_R \leq 1\}$, and

$$\Gamma = \{(\lambda, \gamma) \in \Lambda \times \mathbf{R}^T_+ \mid \gamma_R < \bar{\gamma}_R \quad \text{for } 1 \leq R \leq T\}.$$

CLAIM 1: Let $\{(\tau'_k, d_k)\}$ be an arbitrary purchasing policy and (λ, γ) be the weights defined by (9). Then $(\lambda, \gamma) \in \Gamma$. Conversely, for any $(\lambda, \gamma) \in \Gamma$ (and $\tau'_1 \geq 1$), there exists a purchasing policy $\{(\tau'_k, d_k)\}$ that satisfies (9). Though this policy is usually not unique, all such policies have the same budget and value. Thus, with abuse of notation we will also refer to a $(\lambda, \gamma) \in \Gamma$ as a purchasing policy.

The argument above essentially contains the proof of this claim.

CLAIM 2: Suppose that the policy corresponds to an R-flexible rule where $\tau_1 = 0$ and the replacement of durables of age $R+1$ is never delayed but the replacement of durables of age R is sometimes delayed. Then, the policy is suboptimal: there exists another R-flexible rule without delays that costs the same and has a strictly higher value.

Proof: For such a policy, $\lambda_R + \lambda_{R+1} = 1$, $\gamma_R > 0$, $\gamma_{R+1} = 0$, and $\lambda_k = \gamma_k = 0$ for all $k \notin \{R, R+1\}$. Moreover, since $\gamma_R < \lambda_R \beta^R/(1-\beta^R)$, we also have $\lambda_R > 0$. In this case, $(\hat{b}, \hat{v})$ is on the "Pareto frontier" (i.e., $\hat{v} = V_T(\hat{b})$). The vector $(\hat{b}, \hat{v}) - (b, v) = (p_0(1-\beta), \hat{x}_0 - \hat{x}_R)$ has "slope" $\sigma = [\hat{x}_0 - \hat{x}_R]/[p_0(1-\beta)]$, and

$$A_R = \frac{1}{p_0}\left[X_{0,R} - \hat{x}_R \frac{1-\beta^R}{1-\beta}\right] \leq (1-\beta^R)\frac{\hat{x}_0 - \hat{x}_R}{p_0(1-\beta)} < \sigma.$$

So, as the delays increase (γ_R increases), (b, v) moves away of $(\hat{b}, \hat{v})$, below the Pareto frontier. But, if $\sigma < A_{R+1}$, the delays may eventually take (b, v) back above the Pareto frontier. This could happen only if $b < b_{T,R+1}$. But even if every durable of age R is replaced with delay, the cost of the policy is more than replacing the durables at age $R+1$ all the time. That is, $b \geq b_{T,R+1}$. Therefore $b_{T,R+1} \leq b \leq b_{T,R}$ and $v < V_T(b)$, and there exists another R-flexible rule with no delays that costs b and has value $V_T(b)$.

CLAIM 3: Suppose that the policy $\{(\tau_k, d_k)\}$ is such that $\gamma_R > 0$ for some R. Then the policy is suboptimal: there exists another policy without delays that uses the same budget but has strictly higher value.

Assume that the policy has delays. We now recursively modify the policy by eliminating delays while maintaining the same budget and improving its value in every step. Let $h = \lambda_1 + \lambda_2$, $\hat{\lambda}_k = \lambda_k/h$ for $k = 1, 2$, and $\hat{\gamma}_1 = \gamma_1/h$. Then

$$\begin{bmatrix} b \\ v \end{bmatrix} = h\left[\hat{\lambda}_1 \begin{bmatrix} b_{T,1} \\ v_{T,1} \end{bmatrix} + \hat{\lambda}_2 \begin{bmatrix} b_{T,2} \\ v_{T,2} \end{bmatrix} - \hat{\gamma}_1 \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_1 \end{bmatrix}\right] + \sum_{R=2}^{T}\left[\lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix} - \gamma_R \begin{bmatrix} p_0(1-\beta) \\ \hat{x}_0 - \hat{x}_R \end{bmatrix}\right].$$

The weights $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\gamma}_1)$ represent a 1-flexible rule with delays (and $\hat{\lambda}_1 + \hat{\lambda}_2 = 1$). If $\gamma_1 > 0$ (so $\hat{\gamma}_1 > 0$), then by Claim 3 there exists another 1-flexible rule with weights $(\tilde{\lambda}_1, \tilde{\lambda}_2, 0)$ that is better. Let $\lambda'_k = h\tilde{\lambda}_k$ for $k = 1, 2$, $\gamma'_1 = 0$, $\lambda'_k = \lambda_k$ for $k \geq 3$, and $\gamma'_k = \gamma_k$ for $k \geq 2$. The policy (λ', γ') is better than the policy (λ, γ) and has $\gamma'_1 = 0$. Now, let $h = \lambda'_2 + \lambda'_3$, $\hat{\lambda}_k = \lambda'_k/h$ for $k = 2, 3$, and $\hat{\gamma}_2 = \gamma'_2/h$. The weights $(\hat{\lambda}_2, \hat{\lambda}_3, \hat{\gamma}_2)$ represent a 2-flexible rule with delays. Again, if $\hat{\gamma}_2 > 0$, Claim 3 implies that there exists a better 2-flexible rule without delays that can be used to modify (λ', γ') and construct a new policy (λ'', γ'') that is better, uses the same budget, and has $\gamma''_1 = \gamma''_2 = 0$. Continuing this way, after T steps, we will have constructed a policy (λ^*, γ^*) with $\gamma^* = 0$, that uses the same budget and has a better value than (λ, γ).

Finally, by Claim 2 (or Theorem 1), for any $\tau'_1 \geq 1$ and any weights λ^*, there exist R and an R-flexible rule that uses the same budget $\beta^{\tau'_1} b = \beta^{\tau'_1} \sum_k \lambda^*_k b_{T,k}$ (from period 0 onward) and delivers a (weakly) better value. Therefore, the optimal value function V_T for the continuous-time economy coincides with that for the discrete-time economy (as defined in Theorem 1). ∎

Proof of Corollary to Theorem 2: From Theorem 2, we can write

$$vh - b = X_{\alpha,r_0}h + \sum_{R=1}^{T} \lambda_R \left(v_{T,R} \cdot h - b_{T,R}\right) + \sum_{R=1}^{T} \gamma_R \left(p_0(1-\beta) - h \cdot (\hat{x}_0 - \hat{x}_R)\right),$$

where $r_0 = \alpha + \tau_1'$ and τ_1' is the period of the first purchase. Since the consumer chooses $r_0 \geq \alpha$ and $(\lambda, \gamma) \in \Gamma$ (where Γ was defined in the proof of Theorem 2), and the objective function is linear in (λ, γ), clearly $\lambda_R \in \{0, 1\}$ and $\gamma_R \in \{0, \bar{\gamma}_R\}$. Since $\sum \lambda_R \leq 1$, $\lambda_R = 1$ implies $\lambda_k = 0$ for all $k \neq R$, and consequently $\gamma_k = 0$ for all $k \neq R$ as well. So, in the optimal solution, one and only one replacement frequency R is used, and we only need to show that the corresponding $\gamma_R = 0$.

Suppose that $\lambda_R = 1$. Then type h must prefer the R-fixed rule to any other fixed rule. In particular, he must prefer it over the $R+1$-fixed rule: $v_{T,R} \cdot h - b_{T,R} \geq v_{T,R+1} \cdot h - b_{T,R+1}$. This implies that

$$h \geq \frac{b_{T,R} - b_{T,R+1}}{v_{T,R} - v_{T,R+1}} = \frac{1}{A_R}$$

For $\gamma_R = 0$, it must be the case that $p_0(1-\beta) < h(\hat{x}_0 - \hat{x}_R)$. We show that this inequality holds when $h = 1/A_R$, and therefore it also holds for any $h \geq 1/A_R$. Indeed

$$p_0(1-\beta) < \frac{1}{A_R}\left(\hat{x}_0 - \hat{x}_R\right) \Leftrightarrow A_R p_0 \frac{1}{\hat{x}_0 - \hat{x}_R} < \frac{1}{1-\beta} \quad \text{and}$$

$$A_R p_0 \frac{1}{\hat{x}_0 - \hat{x}_R} = \sum_{i=0}^{R-1} \beta^i \frac{\hat{x}_i - \hat{x}_R}{\hat{x}_0 - \hat{x}_R} < \sum_{i=0}^{R-1} \beta^i < \sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}.$$

Thus, given the choice of r_0, the optimal policy is a fixed rule without delays. To conclude, we need to show that r_0 is also consistent with this rule. That is, for all $h \in \left[\frac{1}{A_R}, \frac{1}{A_{R-1}}\right]$ (i.e. all consumer types that choose the R-fixed rule), $r_0 = R$. Let $J(h) = v_{T,R} \cdot h - b_{T,R}$ be the continuation value of the optimal policy from the first purchase onward. We must show that if the good is of age $R-1$ (or less), holding it for one more period is better than replacing it right away, and the opposite is true if the good is of age R (or older):

$$\hat{x}_{R-1}h + \beta J(h) \geq J(h),$$
$$\hat{x}_R h + \beta J(h) \leq J(h).$$

It suffices to show that the first inequality holds for $h = 1/A_{R-1}$ and the second one holds for $h = 1/A_R$, that is

$$A_{R-1} b_{T,R} \geq v_{T,R} - \frac{\hat{x}_{R-1}}{1-\beta},$$
$$A_R b_{T,R} \leq v_{T,R} - \frac{\hat{x}_R}{1-\beta}.$$

After some algebra, both expressions above are actually equalities, which completes the proof. ∎

Proof of Theorem 3: Recall that we defined $c_0 = \bar{w}(1-\beta) - p_0$ and $c_{T+1} = \underline{w}(1-\beta)$, so that $w_{1,1}(c_0) = \bar{w}$ and $w_{T,T+1}(c_{T+1}) = \underline{w}$.

Let $B(w,c) = w - c/(1-\beta)$ be the budget left for durables when the total wealth is w and the agent consumes a constant per period budget c on non-durables. For fixed α and w, the function $\varphi(c) = \hat{u}(c)/(1-\beta) + V_\alpha(B(w,c))$ is concave. Thus $\hat{c}$ maximizes $\varphi(c)$ if and only if $0 \in \partial\varphi(\hat{c})$ (that is, 0 is a subdifferntial of φ at $\hat{c}$) or equivalently, if and only if $\hat{u}'(\hat{c}) \in \partial V_\alpha(B(w,\hat{c}))$. There are two cases corresponding to the situations where (1) V_α is differentiable at $B(w,\hat{c})$; and (2) V_α has a kink at $B(w,\hat{c})$.

Case 1: Observe that $B(w,c_R) \in (b_{\alpha,R+1}, b_{\alpha,R})$ if and only if $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$. Now, if $B(w,c_R) \in (b_{\alpha,R+1}, b_{\alpha,R})$ for some R, then $\hat{u}'(c_R) = A_R = V'_\alpha(B(w,c_R))$, and c_R is the optimal solution of problem (1). That is, when $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$, it is optimal to consume a constant flow c_R of non-durables and follow an R-flexible purchasing rule for the durable good. One can check that $B(w,c_\alpha) = b_{\alpha+1,\alpha+1} \Leftrightarrow w = w_{\alpha+1,\alpha+1}(c_\alpha)$ and $B(w,c_\alpha) = b_{\alpha-1,\alpha} \Leftrightarrow w = w_{\alpha-1,\alpha}(c_\alpha)$, and $w_{\alpha,\alpha+1}(c_\alpha) < w_{\alpha+1,\alpha+1}(c_\alpha) < w_{\alpha-1,\alpha}(c_\alpha) < w_{\alpha,\alpha}(c_\alpha)$. Therefore, $\delta^*(w)$ is given by (5).

Case 2: Observe that $A_{R-1} \le \hat{u}'(c_{\alpha,R}(w)) \le A_R$ if and only if $c_R \le c_{\alpha,R}(w) \le c_{R-1}$, or alternatively, if and only if $w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$. Since $B(w, c_{\alpha,R}(w)) = b_{\alpha,R}$ and $\partial V_\alpha(b_{\alpha,R}) = [A_{R-1}, A_R]$, if $\hat{u}'(c_{\alpha,R}(w)) \in [A_{R-1}, A_R]$ for some R, then $c_{\alpha,R}(w)$ is the optimal solution of problem (1). That is, it is optimal to consume a constant flow $c_{\alpha,R}(w)$ of non-durables and follow the R-fixed purchasing rule for the durable good. In particular, $\delta^*(w) = 1$ if $R \le \alpha$ (or equivalently, if $w \ge w_{\alpha,\alpha}(c_\alpha)$) and $\delta^*(w) = 0$ if $R > \alpha$, as stated in (5).

For a fixed α, the intervals corresponding to case 1 alternate with those corresponding to case 2. Moreover, collectively, they are mutually exclusive and cover the whole wealth range. ■

Proof of Proposition 1:
(i) Since $w_{\alpha,R}(c_{R-1}) - w_{\alpha,R}(c_R) = (c_{R-1} - c_R)/(1-\beta)$, the total size of the fixed-rule classes is

$$\varphi_T = \frac{1}{1-\beta}\sum_{R=1}^{T} R[c_{R-1} - c_R] + \frac{c_T}{1-\beta} - \underline{w} = \frac{1}{1-\beta}\sum_{R=1}^{T-1}[c_R - c_T] + \bar{w} - \underline{w} - \frac{p_0}{1-\beta}.$$

(Recall that $c_0 = (1-\beta)\bar{w} - p_0$ and $c_{T+1} = (1-\beta)\underline{w}$.) Similarly, since $w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R) = b_{\alpha,R} - b_{\alpha,R+1}$, we have that $\mu(C_{R,R+1}) = p_0/(1-\beta^R)$ for $1 \le R \le T-1$, and $\mu(C_{T,T+1}) = p_0/(1-\beta)$. Therefore, the total size of the flexible-rule classes is

$$\psi_T = \sum_{R=1}^{T-1} \frac{p_0}{1-\beta^R} + \frac{p_0}{1-\beta}.$$

Note that $\theta = \psi_T/[\psi_T + \varphi_T]$ and that ψ_T does not depend on g. Therefore,

$$\frac{\partial\theta}{\partial g} = \frac{-\psi_T}{(\psi_T + \varphi_T)^2}\frac{\partial\varphi_T}{\partial g} \qquad \text{and} \qquad \frac{\partial\varphi_T}{\partial g} = \frac{1}{1-\beta}\sum_{R=1}^{T-1}\left(\frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g}\right).$$

A simple computation shows that

$$A_R = \frac{1}{p_0}\sum_{\alpha=0}^{R-1}\beta^\alpha(\hat{x}_\alpha - \hat{x}_R) = g\left[\frac{1-\beta}{\rho p_0}\right]\sum_{\alpha=0}^{R-1}\beta^\alpha(R-\alpha).$$

Since $\hat{u}'(c_R) = A_R$,

$$\frac{\partial c_R}{\partial g} = \frac{1}{\hat{u}''(c_R)}\frac{\partial A_R}{\partial g} = \frac{1}{\hat{u}''(c_R)}\frac{A_R}{g} = \frac{1}{g}\frac{\hat{u}'(c_R)}{\hat{u}''(c_R)}.$$

If u has decreasing (increasing) absolute risk aversion then so does $\hat{u}$, and since $c_1 > c_2 > \cdots > c_T$, for every $R = 1, \ldots, T-1$,

$$\frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g} = \frac{1}{g}\left[\frac{\hat{u}'(c_R)}{\hat{u}''(c_R)} - \frac{\hat{u}'(c_T)}{\hat{u}''(c_T)}\right] < 0 \ (> 0).$$

(ii) Let $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma \geq 1$. Then $\hat{u}(c) = [(1-\beta)/\rho]^{\sigma}[c^{1-\sigma}/(1-\sigma)]$ and $c_R = (1-\beta)A_R^{-1/\sigma}/\rho$. Differentiating c_R with respect to p_0 gives

$$\frac{\partial c_R}{\partial p_0} = -\frac{1}{\sigma}\frac{c_R}{A_R}\frac{\partial A_R}{\partial p_0} = \frac{1}{\sigma}\frac{c_R}{p_0},$$

so that

$$\frac{\partial \varphi_T}{\partial p_0} = \frac{1}{\sigma p_0}\left(\frac{1}{1-\beta}\sum_{R=1}^{T-1}[c_R - c_T] - \frac{p_0\sigma}{1-\beta}\right) = \frac{1}{\sigma p_0}\left(\varphi_T - (\bar{w} - \underline{w}) - \frac{p_0}{1-\beta}(\sigma - 1)\right)$$

Then

$$\frac{1}{\varphi_T}\frac{\partial \varphi_T}{\partial p_0} < \frac{1}{p_0} = \frac{1}{\psi_T}\frac{\partial \psi_T}{\partial p_0}$$

The size of fixed rule classes grows at a slower rate than the size of flexible rule classes, hence

$$\frac{\partial \theta}{\partial p_0} > 0.$$

■

Proof of Proposition 2: We first construct and upper bound $W(\bar{b})$ for $V_1(\bar{b}, 0)$. Let d be such that $e^{\rho d}\bar{b} = 1$. The agent has to wait at least until d before he can afford to purchase a new durable. If the agent consumes an old durable until d and a new durable forever after, his total discounted value is

$$W(\bar{b}) = \frac{x_1}{\rho}\left(1 - e^{-\rho d}\right) + \frac{x_0}{\rho}e^{-\rho d} = \frac{x_1}{\rho}\left(1 - \bar{b}\right) + \frac{x_0}{\rho}\bar{b}.$$

Clearly, $V_1(\bar{b}, 0) \leq W(\bar{b})$ for all $\bar{b} \in [0, 1)$. Also, $V_1(0, 0) = W(0) = x_1/\rho$. Therefore

$$\frac{\partial}{\partial \bar{b}}V_1(0, 0) \leq W'(0) = \frac{x_0 - x_1}{\rho}.$$

Differentiating both sides of the Bellman equation (6) with respect to $\bar{b}$ and evaluating the derivative at $\bar{b} = 0$ gives

$$\frac{\partial}{\partial \bar{b}}V_0(0, S) = \frac{\partial}{\partial \bar{b}}V_1(0, 0).$$

For convenience, let $G\left(\bar{b}, d\right)$ denote the right hand side of (7). This function is defined for all $\left(\bar{b}, d\right)$ where a purchase with delay $d \leq S$ is feasible. That is, when $\bar{b} \geq e^{-(r+\gamma)S}$ and $\max\left\{0, \frac{1}{r+\gamma}\ln\left(\frac{1}{\bar{b}}\right)\right\} \leq d \leq S$. Now we show that when $\rho = r+\gamma$, $G\left(\bar{b}, d\right)$ is strictly decreasing in d for all $\bar{b} \geq e^{-(r+\gamma)S}$. Differentiating G with respect to d and setting $\rho = r+\gamma$, we get

$$\begin{aligned}\frac{\partial G}{\partial d}\left(\bar{b}, d\right) &= e^{-\rho d}\rho\left[\frac{\partial}{\partial \bar{b}}V_0\left(e^{\rho S}\left(\bar{b} - e^{-\rho d}\right), S\right) - \frac{(x_0 - x_1)}{\rho}\right] \\ &< e^{-\rho d}\rho\left[\frac{\partial}{\partial \bar{b}}V_0\left(0, S\right) - \frac{(x_0 - x_1)}{\rho}\right] \leq 0.\end{aligned}$$

If $\bar{b} \geq 1$, any delay $d \in [0, S]$ is possible. Since $G\left(\bar{b}, d\right)$ is strictly decreasing in d for all $d \in [0, S]$, the optimal delay must be either $d = 0$ or $d > S$.

If $\bar{b} < 1$, delays $d < \frac{1}{r+\gamma}\ln\left(1/\bar{b}\right)$ are not feasible, but since $G\left(\bar{b}, d\right)$ is decreasing in d, the maximum is either $d\left(\bar{b}\right) = \frac{1}{r+\gamma}\ln\left(1/\bar{b}\right) < S$ or $d\left(\bar{b}\right) > S$. ■

Appendix 2: The frictionless PIH model

The stock of durable good, K_t, evolves according to

$$K_{t+1} = (1-\delta)K_t + E_t,$$

where $\delta < 1$ is the rate of economic depreciation and E_t is the current expenditure on durable goods. Service flow from the durable is proportional to K_t. Assume the interest rate r is constant and satisfies $\beta = 1/(1+r)$, where β is the discount factor. A consumer's problem is[9]

$$\max_{(c_t, K_t)} \sum_{t\geq 0} \beta^t \left[\ln c_t + \ln K_t\right] \quad \text{subject to} \quad \sum_{t\geq 0} \frac{1}{(1+r)^t}\left(c_t + K_{t+1} - (1-\delta)K_t\right) = w.$$

The optimal solution is $K_t = Aw$ and $c_t = Bw$ for all t, where $A = (1-\beta)/[r+2\delta]$ and $B = (r+\delta)A$. Therefore, the optimal durable expenditure every period is $E = \delta Aw$. Suppose that a shock changes the wealth from w to $(1+\epsilon)w$. Then, the non-durable consumption level changes from Bw to $(1+\epsilon)Bw$ and desired durable stock changes from Aw to $(1+\epsilon)Aw$. Therefore, the current period durable expenditure is $(1+\epsilon)Aw - (1-\delta)Aw = (1+\epsilon/\delta)\delta Aw$. That is, the *short-run* wealth elasticity of demand is 1 for the non-durable good and $1/\delta$ for the durable good.

[9] The results are similar for a more general class of preferences and other values of r. See, for example, Carroll and Dunn (1997). The results are also similar under uncertainty. See Damgaard *et. al.* (2003).