Bandwidth Choice for Bias Estimators in Dynamic Nonlinear Panel Models

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Abstract

This paper considers bandwidth selection for spectral density estimators based on panel data sets. The spectral densities of greatest interest in this paper are the ones that appear in the bias expression for fixed effects estimators in nonlinear dynamic panel models obtained by Hahn and Kuersteiner. The bias estimation problem is different from the usual HAC estimation problem because the need for positive definiteness does not arise. As a consequence, the usual justification for kernel smoothing of spectral estimators does not apply to this case. However, without kernel smoothing the bandwidth selection problem is significantly more difficult because in this case not only the usual proportionality constants are data-dependent but also the optimal rate at which the bandwidth parameter grows with sample size. In this paper an infinite order VAR model is used to obtain an estimate of the approximate mean squared error of the spectral estimator. It is shown that selecting the bandwidth parameter based on the estimated mean squared error criterion is asymptotically equivalent to the optimal infeasible bandwidth choice. Monte Carlo simulations show that truncated spectral estimates significantly outperform kernel weighted estimates in terms of their effectiveness in reducing bias in the panel application.

1 Introduction

Hahn and Kuersteiner (2004) analyze the bias properties of general nonlinear panel models with fixed effects. Maximum likelihood estimators of these models suffer from what is know as the incidental parameter problem especially in short panel data sets with a limited amount of time series observations. One typically finds that conventional cross-sectional asymptotic approximations where n tends to infinity while T is fixed, lead to inconsistent parameter estimators. They adopt an alternative asymptotic approximation where n and T tend to infinity at the same rate. Under these alternative asymptotics it can be shown that the limiting distribution of the maximum likelihood estimator is Gaussian but is not centered at zero. The non-centrality parameter is viewed as an approximation to the finite sample bias and show that it is a function of the spectral density at zero frequency of a certain non-linear transformation of the data. Estimators of the bias can be used to bias correct the ML estimator but practical implementation of this procedure is complicated by the need to select a bandwidth parameter to estimate the spectrum at zero frequency.

This paper discusses estimators of the bias term. The approximate mean squared error (MSE) of the bias estimator is derived and used to select the bandwidth in an automated way. Since the higher order MSE of the bias corrected maximum likelihood estimator depends on the bandwidth in the same way as the higher order MSE of the bias estimator itself, minimizing the MSE criterion used in this paper is also minimizing the higher order MSE of the bias corrected maximum likelihood estimator. The procedure thus is optimal not only in terms of an intermediate criterion but also in terms of the properties of the final stage estimator of interest. As is well known from the literature of heteroskedasticity and autocorrelation consistent (HAC) covariance matrix estimation, the MSE depends on nuisance parameters that are themselves functions of the temporal dependence structure of the data. An autoregressive approximations to the spectral density of suitable transformations of the observed data is proposed as an automatic way of estimating the nuisance parameters.

In the literature on HAC estimation, such as Newey and West (1987, 1994) and Andrews (1991) and Andrews and Monahan (1992) the use of kernel weighting functions for spectral density estimation at the zero frequency is motivated and justified by the need to produce positive definite estimators of the spectral densities for all sample sizes. The bandwidth selection problem is then a relatively simple one, because the rate at which the bandwidth expands with the sample size is usually known and depends on the kernel used.

In the case of bias correction there is no need to enforce positivity and thus the justification of using kernel weighted spectral estimators is less clear. It is therefore proposed here to use a truncated estimate of the spectral density matrix at frequency zero. This has the advantage of faster rates of convergence of the higher order MSE for the bias estimator. However, using a truncated version of the spectral density matrix comes at the cost of a more difficult bandwidth selection problem. To the best of my knowledge this selection problem has not been solved in the time series literature, let allone for nonlinear dynamic panel models. A higher order approximation to the MSE of a certain spectral matrix at frequency zero is used to obtain a criterion function that depends on the bandwidth.The bandwidth is then chosen optimally by minimizing this approximate MSE.

In order to estimate the unknown higher order MSE expression, a parametric VAR(h) model is used where the parameter h is tending to infinity with the sample size. This procedure is shown to produce bandwidth choices that are asymptotically of the same order as the infeasible optimal bandwidth obtained from minimizing the unknown approximate MSE.

2 Bias Estimators

Suppose that we are given a panel data model with a common parameter of interest θ_0 and individual specific fixed effects γ_{i0}, $i = 1, \ldots, n$. As in Hahn and Kuersteiner (2004) a maximization estimator is defined by

$$(1) \quad \left(\widehat{\theta}, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_n\right) = \underset{\theta, \gamma_1, \ldots, \gamma_n}{\text{argmax}} \sum_{i=1}^{n} \sum_{t=1}^{T} \psi\left(x_{it}; \theta, \gamma_i\right)$$

for some criterion function $\psi(\cdot)$ that does not depend on T. Assume that ψ is a sensible function in the sense that, if n is fixed, and $T \to \infty$, the estimator $\left(\widehat{\theta}, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_n\right)$ is consistent for $(\theta_0, \gamma_{10}, \ldots, \gamma_{n0})$. For simplicity of notation, assume $\dim(\gamma_i) = 1$. Hahn and Kuersteiner (2004) show that when $n/T \to \kappa$, where $0 < \kappa < \infty$ then

$$\sqrt{nT}\left(\widehat{\theta} - \theta_0\right) \to N\left(\beta\sqrt{\kappa}, \mathcal{I}^{-1}\Omega\left(\mathcal{I}'\right)^{-1}\right)$$

where for $U_i(x_{it}; \theta, \gamma_i) \equiv \frac{\partial\psi(x_{it};\theta,\gamma_i)}{\partial\theta} - \rho_{i0} \cdot \frac{\partial\psi(x_{it};\theta,\gamma_i)}{\partial\gamma_i}$, $\quad V_i(x_{it}; \theta, \gamma_i) \equiv \frac{\partial\psi(x_{it};\theta,\gamma_i)}{\partial\gamma_i}$, the spectral quantities

$$f_i^{VU^\gamma} \equiv \textstyle\sum_{l=-\infty}^{\infty} \text{Cov}\left(V_{it}, U_{it-l}^{\gamma_i}\right), f_i^{VV} \equiv \sum_{l=-\infty}^{\infty} \text{Cov}\left(V_{it}, V_{it-l}\right)$$

are defined and

$$\varphi^{VU^\gamma} \equiv \lim n^{-1} \textstyle\sum_{i=1}^{n} \left(E\left[V_{it}^{\gamma_i}\right]\right)^{-1} f_i^{VU^\gamma}, \varphi^{VV} \equiv \frac{1}{2} \lim n^{-1} \sum_{i=1}^{n} \left(E\left[V_{it}^{\gamma_i}\right]\right)^{-2} E\left[U_{it}^{\gamma_i\gamma_i}\right] f_i^{VV},$$

$\Psi \equiv \varphi^{VU^\gamma} - \varphi^{VV}$, $\quad \mathcal{I} \equiv \lim n^{-1} \sum_{i=1}^{n} \mathcal{I}_i$, $\quad \beta \equiv \mathcal{I}^{-1}\Psi$. The bias β can be expressed as

$$\beta \equiv -\lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} \left[\frac{f_i^{VU^\gamma}}{E\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]} - \frac{E\left[U_i^{\gamma_i\gamma_i}\left(x_{it}; \theta, \gamma_i\right)\right] f_i^{VV}}{2\left(E\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]\right)^2}\right].$$

It is more convenient to express β as a weighted average of elements of a spectral density matrix at frequency zero. For this purpose define

$$(2) \quad k_{it} = k\left(x_{it}; \theta_0, \gamma_{i0}\right) = \begin{bmatrix} V_i\left(x_{it}; \theta_0, \gamma_{i0}\right) \\ U_i^{\gamma_i}\left(x_{it}; \theta_0, \gamma_{i0}\right) \end{bmatrix}, \hat{k}_{it} = k\left(x_{it}; \hat{\theta}, \hat{\gamma}_i\right)$$

and

$$(3) \quad a_{i,1} = -\frac{E\left[U_i^{\gamma_i\gamma_i}\left(x_{it}; \theta_0, \gamma_{i0}\right)\right]}{2\left(E\left[\frac{\partial V_i(x_{it};\theta_0,\gamma_{i0})}{\partial\gamma_i}\right]\right)^2}, a_{i,2} = \frac{1}{E\left[\frac{\partial V_i(x_{it};\theta_0,\gamma_{i0})}{\partial\gamma_i}\right]}$$

where $a_{i,1}$ is a $d-1\times 1$ vector where $d = \dim\theta + 1$ and $a_{i,2}$ is a scalar. Then define the $2(d-1)\times d$ matrix

$$\tilde{A}_i = \begin{bmatrix} a_{i,1} & 0_{d-1\times 1} & \cdots & 0_{d-1\times 1} \\ 0 & a_{i,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{i,2} \end{bmatrix}$$

and let

$$(4) \quad A_i = -\mathcal{I}^{-1}\left[I_{d-1}, I_{d-1}\right]\tilde{A}_i$$

where I_{d-1} is the $d-1$ dimensional identity matrix. Now define $\Gamma_{i,j}^{kk} = E\left[k_{it}k_{it-j}'\right]$, $f_i^{kk} = \sum_{l=-\infty}^{\infty}\Gamma_{i,j}^{kk}$ and $f^{kk} = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} A_i f_i^{kk}$. The bias β can then be expressed as

$$\beta \equiv f^{kk}B$$

where

$$B = \begin{bmatrix} 1 \\ 0_{d-1\times 1} \end{bmatrix}.$$

Estimation of β is done by replacing population moments with sample moments. In this formulation, the estimators $\widehat{\mathcal{I}}_i$, $\widehat{E}\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]$, $\widehat{E}\left[U_i^{\gamma_i\gamma_i}(x_{it};\theta,\gamma_i)\right]$ and $\widehat{E}\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]$ are simple sample averages over t that do not need any truncation parameters. They will be ignored in the following discussion which is mainly concerned with estimates of the spectral densities $\widehat{f}_i^{VU^\gamma}$ and $\widehat{f}_i^{VV}$. These estimators require truncation parameters because their population counterparts depend on infinitely many terms. The estimator for the bias is therefore defined as

$$(5) \qquad \hat{\beta}_m \equiv \hat{f}_m^{kk}B$$

where

$$\hat{f}_m^{kk} = \frac{1}{n}\sum_{i=1}^{n} A_i \sum_{|j|\le m}\hat{\Gamma}_{i,j}^{kk}.$$

and

$$\hat{\Gamma}_{i,j}^{kk} = \frac{1}{T}\sum_{t=\max(1,j)}^{\min(T,T+j)} \hat{k}_{it}\hat{k}_{it-j}', \hat{\Gamma}_{i,-j}^{kk} = \left(\hat{\Gamma}_{i,j}^{kk}\right)'.$$

The bandwidth parameter m is chosen by minimizing the approximate mean squared error of $\hat{\beta}_m$. For this purpose define $k_{it}^{\gamma}(x_{it};\theta,\gamma_i) = \partial k(x_{it};\theta,\gamma_i)/\partial\gamma$ with $k_{0,it}^{\gamma} = k_{it}^{\gamma}(x_{it};\theta_0,\gamma_{i0})$ and $\psi_{it}^{\gamma,v} = -\left(E\left[V_i^{\gamma}\right]\right)^{-1}\frac{\partial\psi(x_{it};\theta_0,\gamma_{i0})}{\partial\gamma}$. Then, it is shown in the Appendix that the largest order terms of $\hat{\beta}_m$ depending on m have an approximate MSE given by $\left\|\left(\mathfrak{T}_{\mathsf{o}}(m) + \frac{m}{T}(\mathfrak{T}_1+\mathfrak{T}_2)\right)B\right\|^2$ where

$$\mathfrak{T}_0(m) = \frac{1}{n}\sum_{i=1}^{n} A_i \sum_{|j|>m}\Gamma_{i,j}^{kk},$$

$$(6) \qquad \mathfrak{T}_1 = \frac{1}{n}\sum_{i=1}^{n} A_i\left(\left(E\left[k_{0,it}^{\gamma}\right]\sum_{u=-\infty}^{\infty}E\left[\psi_{i1}^{\gamma,v}k_{i1-u}\right]'\right) + \left(\sum_{u=-\infty}^{\infty}E\left[\psi_{i1}^{\gamma,v}k_{i1-u}\right]E\left[k_{0,it}^{\gamma}\right]'\right)\right)$$

and

$$(7) \qquad \mathfrak{T}_2 = \frac{1}{n}\sum_{i=1}^{n} A_i\sum_{u=-\infty}^{\infty}E\left[\psi_{i1}^{\gamma,v}\psi_{i1-u}^{\gamma,v}\right]\left(E\left[k_{0,it}^{\gamma}\right]E\left[k_{0,it}^{\gamma}\right]'\right)$$

Define m^* as the choice of m that minizes the approximate mean squared error

$$Q(m) = \left\|\left(\mathfrak{T}_{\mathsf{o}}(m) + \frac{2m}{T}(\mathfrak{T}_1+\mathfrak{T}_2)\right)B\right\|^2$$

of $\tilde{\beta}_m$, such that m^* solves

$$m^* = \underset{m\in\{0,1,\ldots,\bar{m}\}}{\arg\min}\ Q(m)$$

where $\bar{m}$ is a prespecified upper limit that does not depend on the data.

3 Nuisance Parameter Estimation

In order to obtain a feasible bandwidth parameter $\hat{m}^*$ the criterion function $Q(m)$ which depends on unknown nuisance parameters needs to be replaced with an empirical counterpart $\hat{Q}(m)$. This can be easily achieved for the components $\mathfrak{T}_1$ and $\mathfrak{T}_2$ which can be replaced by the sample averages

$$(8)\quad \hat{\mathfrak{T}}_1 = \frac{1}{n}\sum_{i=1}^{n} \hat{A}_i \left\{ \left(T^{-1}\sum_{t=1}^{T} \hat{k}_{it}^{\gamma}\right)\left(\sum_{u=-\check{m}}^{\check{m}} T^{-1}\sum_{t=\max(1,u+1)}^{\min(T,T+u)} \hat{\psi}_{it}^{\gamma,v}\hat{k}_{it-u}\right)' + \left(\sum_{u=-\check{m}}^{\check{m}}\left(T^{-1}\sum_{t=\max(1,u+1)}^{\min(T,T+u)} \hat{\psi}_{it}^{\gamma,v}\hat{k}_{it-u}\right)\right)\left(T^{-1}\sum_{t=1}^{T}\hat{k}_{it}^{\gamma}\right)' \right\}$$

$$(9)\quad \hat{\mathfrak{T}}_2 = \frac{1}{n}\sum_{i=1}^{n}\left\{ \hat{A}_i \left(\left(T^{-1}\sum_{t=1}^{T}\hat{k}_{it}^{\gamma}\right)\left(T^{-1}\sum_{t=1}^{T}\hat{k}_{it}^{\gamma}\right)'\right) \times \sum_{u=-\check{m}}^{\check{m}} T^{-1}\sum_{t=\max(1,u+1)}^{\min(T,T+u)}\left[\hat{\psi}_{it}^{\gamma,v}\hat{\psi}_{it}^{\gamma,v}\right]\right\}$$

Lemma 1 *Assume that Conditions 1-7 hold and $m/T \to 0$ as $m, T \to \infty$. Let $\hat{\mathfrak{T}}_1$ and $\hat{\mathfrak{T}}_2$ be defined in 8 and 9. Then*

$$\hat{\mathfrak{T}}_1 - \mathfrak{T}_1 = O_p\left(T^{-1/2}\right)$$
$$\hat{\mathfrak{T}}_2 - \mathfrak{T}_2 = O_p\left(T^{-1/2}\right).$$

Proof. The result follows from Lemmas 3-19. For $\hat{\mathfrak{T}}_1$ and $\hat{\mathfrak{T}}_2$ the same arguments can be applied by noting that the function $\psi_{i1}^{\gamma,v}$ has the same properties as k_{it}. ■

The term $\mathfrak{T}_o(m)$ poses a more difficult estimation problem. Because in many panel data sets of interest the time dimension is too short to estimate the function $\mathfrak{T}_0(m)$ directly we propose to use a restriced VAR. In other words we approximate the spectral density matrix of k_{it} with a finite order vector autoregression of order h, or VAR(h). The approximate model with VAR coefficient matrices $\pi_{i1,h}, ...\pi_{ih,h}$ is given by

$$(10)\quad \hat{k}_{it} = \hat{\mu}_{ik,h} + \hat{\pi}_{i1,h}\hat{k}_{it-1} + ... + \hat{\pi}_{ih,h}\hat{k}_{it-h} + \hat{v}_{it,h}.$$

Let $\hat{U}_{it,h} = \left(\hat{k}_{it}', ..., \hat{k}_{it-h+1}'\right)'$ and define $\hat{\Gamma}_{i1,h} = (T-h)^{-1}\sum_{t=h}^{T-1}\hat{U}_{it,h}\hat{k}_{it+1}'$ and $\hat{\Gamma}_{i,h} = (T-h)^{-1}\sum_{t=h}^{T-1}\hat{U}_{it,h}\hat{U}_{it,h}'$. The estimated error covariance matrix is $\hat{\Sigma}_{vi,h} = (T-h)^{-1}\sum_{t=h+1}^{T}\hat{v}_{it,h}\hat{v}_{it,h}'$ where $\hat{v}_{it,h} = \hat{k}_{it} - \hat{\pi}_{i1,h}\hat{k}_{it-1} - ... - \hat{\pi}_{ih,h}\hat{k}_{it-h}$ with coefficients

$$\hat{\pi}_i(h)' = \left(\hat{\pi}_{i1,h}', ..., \hat{\pi}_{ih,h}'\right) = \hat{\Gamma}_{i1,h}'\hat{\Gamma}_{i,h}^{-1}.$$

The population analoge of $\hat{\pi}_{i,h}$ is the infinite dimensional matrix $\pi_i = (\pi_{i1}', \pi_{i2}', ...)'$ defined as $\pi_i' = \Gamma_{i1,\infty}'\Gamma_{i,\infty}^{-1}$ where $\Gamma_{i1,\infty}' = \left(\Gamma_{i1}^{kk}, ..., \Gamma_{ij}^{kk}, ...\right)$ and $\Gamma_{i,\infty}$ is an infinite dimensional matrix with (m,n)th block element equal to $\Gamma_{i(n-m)}^{kk}$. If $U_{it,\infty} = (k_{it}, k_{it-1}, ...)$ then define $\Sigma_{vi,\infty} = \mathrm{Var}\left(k_{it} - \pi_i' U_{it,\infty}\right)$. In order to obtain an approximation to the autocovariance function $\Gamma_{i,j}^{kk}$ define matrices

$$H_{i(h)} = \begin{bmatrix} \pi_{i1,h} & \pi_{i2,h} & \cdots & \pi_{ih,h} \\ I_d & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & I_d & 0 \end{bmatrix}, H_{i(\infty)} = \begin{bmatrix} \pi_{i1} & \pi_{i2} & \cdots & \pi_{ih} & \cdots \\ I & 0 & \cdots & 0 & \cdots \\ 0 & I & & 0 & \cdots \\ \vdots & & \ddots & \vdots & \end{bmatrix}$$

where $H_{i(h)}$ is of dimension $dh \times dh$ and $H_{i(\infty)}$ is infinite dimensional. The autocovariance function $\Gamma_{i,j}^{kk}$ has the representation

(11) $\Gamma_{i,j}^{kk} = \sum_{l=0}^{\infty} E'_{\infty} H_{i(\infty)}^{l+j} E_{\infty} \Sigma_{vi,\infty} E'_{\infty} H_{i(\infty)}^{l} E_{\infty}$

where $E_{\infty} = (I_d, 0, ...)'$. Also, let $E_h = (I_d, 0..., 0)'$ be a $hd \times h$ matrix. Then an approximation to the autocovariance function Γ_{ij}^{kk} is obtained as

(12) $\hat{\Gamma}_{i,h}(j) = \sum_{l=0}^{k_{\max}} E'_h \hat{H}_{i(h)}^{l+j} E_h \hat{\Sigma}_{vi,h} E'_h \hat{H}_{i(h)}^{l} E_h$

where $\hat{H}_{i(h)}^{l}$ is defined in the same way as $H_{i(h)}^{l}$ except that the parameters $\pi_{ij,h}$ have been replaced by $\hat{\pi}_{ij,h}$. The estimator for the function $\mathfrak{T}_0(m)$ is based on the estimated autocovariance matrices implied by the VAR approximation and is formulated as

$$\hat{\mathfrak{T}}_0(m) = \frac{1}{n} \sum_{i=1}^{n} \hat{A}_i \left(\sum_{j=m+1}^{k_{\max}} \hat{\Gamma}_{i,h}(j) \right).$$

The criterion function for selecting m is then formed as before

$$\hat{\mathfrak{T}}(m) = \hat{\mathfrak{T}}_o(m) + \frac{2m}{T} \left(\hat{\mathfrak{T}}_1 + \hat{\mathfrak{T}}_2 \right).$$

Then an estimate of the approximate MSE of $\hat{\beta}$ based on a VAR(h) approximation to the spectral density matrix of k_{it} is $B'\hat{\mathfrak{T}}(m)' \hat{\mathfrak{T}}(m) B$.

Theorem 1 *Assume Conditions 1, 2, 3, 4, 5, 6 and 7 hold. Also assume that for* $q \geq (d+1)/2 + 2$, *some* $0 < \upsilon < (100q + 120)^{-1}$, *assume that* $h, k_{\max} \to \infty$ *such that* $T^{\upsilon}/h \to 0$, $T^{\upsilon}/k_{\max} \to 0, h = o\left(T^{1/5-\upsilon}\right)$ *and* $k_{\max} = o(T^{1/5-\upsilon})$. *Then, for a sequence* m_0 *such that* $m_0 \to \infty$ *as* $T \to \infty$ *and uniformly in* m *for* $m < m_0$, $\sum_{l=k_{\max}} \left\| \Gamma_{i,l}^{kk} \right\| / \sum_{l=m}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| \to 0$ *and* $\sum_{l=h} \|\pi_{il}\| / \sum_{l=m}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| \to 0$ *as* $T \to \infty$ *it follows that*

$$\frac{\left\| \hat{\mathfrak{T}}(m) - \mathfrak{T}(m) \right\|}{\|\mathfrak{T}(m)\|} = o_p(1).$$

Corollary 1 *For* $\hat{Q}(m) = B'\hat{\mathfrak{T}}(m)' \hat{\mathfrak{T}}(m) B$ *it follows that* $\left| \hat{Q}(m) - Q(m) \right| = Q(m) o_p(1)$.

As is shown in Hannan and Deislter (1988, p. 333) the result of Corollary 1 is enough to establish that

$$\hat{m}^*/m^* \to_p 1$$

where $\hat{m}^*$ minimizes $\hat{Q}(m)$. Hannan and Deistler also show that for models with exponentially decaying autocovariance functions the optimal m^* does not depend on the constants $\mathfrak{T}_1$ and $\mathfrak{T}_2$. It is therefore possible to specify two versions of the bandwidth selection rule. The first version of $\hat{m}^*$ is defined as $\hat{m}_1$ such that

$$\hat{m}_1^* = \arg\min_m B'\hat{\mathfrak{T}}(m)' \hat{\mathfrak{T}}(m) B.$$

A modified bandwidth selection rule 2 is defined as minimizing the following simplified criterion function

$$\hat{m}_2 = \arg\min_m B' \left(\hat{\mathfrak{T}}_o(m) + \frac{m}{T} \mathbf{1}_{d-1} \mathbf{1}'_d \right)' \left(\hat{\mathfrak{T}}_o(m) + \frac{m}{T} \mathbf{1}_{d-1} \mathbf{1}'_d \right) B$$

where $\mathbf{1}_{d-1}$ is a $d-1 \times 1$ matrix consisting of all elements equal to one. Note that $\mathbf{1}'_d B = 1$ such that the bias contribution to $\hat{\beta}_m$ that is increasing in m takes on the form $\frac{m}{T} \mathbf{1}_{d-1}$, ie. all elements of β are affected in the same way.

4 Conclusions

The problem of estimating the bias in a nonlinear dynamic panel model resulting form an incidental parameter is investigated. As was shown in earlier work in Hahn and Kuersteiner (2004), this bias depends on a spectral density matrix at frequency zero. It is argued that this spectral density matrix should be estimated by truncating lags, rather than kernel smoothing. A higher order analysis of the approximate MSE for the spectral density estimator leads to a criterion for selecting the bandwidth or lag truncation parameter. It is shown that a VAR approximation can be use to estimate the MSE criterion. The main result of this paper is to establish, that the truncation lag selected based on a feasible MSE criterion function asymptotically is equivalent to the truncation parameter selected using the infeasible criterion function.

Appendix

A Regularity Conditions

We assume the following:

Condition 1 *For each* $\eta > 0$, $\inf_i \left[G_{(i)}(\theta_0, \gamma_{i0}) - \sup_{\{(\theta,\gamma):|(\theta,\gamma)-(\theta_0,\gamma_{i0})|>\eta\}} G_{(i)}(\theta,\gamma)\right] > 0$, *where* $\widehat{G}_{(i)}(\theta,\gamma_i) \equiv T^{-1}\sum_{t=1}^{T}\psi(x_{it};\theta,\gamma_i)$ *and* $G_{(i)}(\theta,\gamma_i) \equiv E[\psi(x_{it};\theta,\gamma_i)]$.

Condition 2 $n, T \to \infty$ *such that* $\frac{n}{T} \to \kappa$, *where* $0 < \kappa < \infty$.

Condition 3 *(i) For each* i, $\{x_{it}, t = 1, 2, \ldots\}$ *is a stationary mixing sequence; (ii)* $\{x_{it}, t = 1, 2, ...\}$ *are independent across* i; *(iii)* $\sup_i |\alpha_i(m)| \leq Ca^m$ *for some* a *such that* $0 < a < 1$ *and some* $C > 0$, *where* $\mathcal{A}_t^i \equiv \sigma(x_{it}, x_{it-1}, x_{it-2}, ...)$, $\mathcal{B}_t^i \equiv \sigma(x_{it}, x_{it+1}, x_{it+2}, ...)$, *and* $\alpha_i(m) \equiv \sup_t \sup_{A\in\mathcal{A}_t^i, B\in\mathcal{B}_{t+m}^i} |P(A\cap B) - P(A)P(B)|$.

Condition 4 *Let* $\psi(x_{it},\phi)$ *be a function indexed by the parameter* $\phi = (\theta,\gamma) \in \operatorname{int}\Phi$, *where* Φ *is a compact, convex subset of* $\mathbb{R}^p$, $p \equiv \dim(\phi) = d$, *and* $d - 1 = \dim(\theta)$. *Let* $\nu = (\nu_1, ..., \nu_k)$ *be a vector of non-negative integers* v_i, $|v| = \sum_{j=1}^{k} v_j$ *and* $D^v\psi(x_{it},\phi) = \partial^{|\nu|}\psi(x_{it},\phi) / (\partial\phi_1^{v_1} ... \partial\phi_k^{\nu_k})$. *There exists a function* $M(x_{it})$ *such that* $|D^v\psi(x_{it},\phi_1) - D^v\psi(x_{it},\phi_2)| \leq M(x_{it})\|\phi_1 - \phi_2\|$ *for all* $\phi_1, \phi_2 \in \Phi$ *and* $|v| \leq 5$. *The function* $M(x_{it})$ *satisfies* $\sup_{\phi\in\Phi}\|D^v\psi(x_{it},\phi)\| \leq M(x_{it})$ *and* $\sup_i E\left[|M(x_{it})|^{10q+12+\delta}\right] < \infty$ *for some integer* $q \geq p/2 + 2$ *and for some* $\delta > 0$.

Condition 5 *Let* λ_{iT} *denote the smallest eigenvalue of* $\Sigma_{iT} = \operatorname{Var}\left(T^{-1/2}\sum_{t=1}^{T} U_i(x_{it};\theta,\gamma_i)\right)$. *We assume that* $\inf_i \inf_T \lambda_{iT} > 0$.

Condition 6 $\inf_i |E[\partial V_i(x_{it};\theta_0,\gamma_{i0}) / \partial\gamma_i]| > 0$.

Condition 7 *Let* $\mu_{i1} \leq ... \leq \mu_{ik} \leq ... \leq \mu_{id-1}$ *be the eigenvalues of* $\mathcal{I}_i$ *in ascending order. Assume that (i)* $0 < \inf_i \mu_{i1} \leq \sup_i \mu_{id-1} < \infty$; *(ii)* $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\mathcal{I}_i$ *exists; (iii) letting* $\mathcal{I} \equiv \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\mathcal{I}_i$, *we assume that* $\mathcal{I}$ *is positive definite.*

Condition 8 *Let the Wold representation of* k_{it} *be given by* $k_{it} = \sum_{j=0}^{\infty} B_{ij} v_{it-j}$ *where* v_{it} *is a white noise sequence. Then* $B_i(z) = \sum_{j=0}^{\infty} B_{ij} z^j$ *has no roots for* $|z| \leq 1$ *where* $z \in \mathbb{C}$.

Condition 9 *Uniformly in* i *it follows that* $\left(\sum_{|k|>m}^{\infty}\sum_{|j|>k}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right) / \left(\sum_{|j|>m}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right) \to c_1$ *as* $m \to \infty$ *for some* $c_1 \in (0,\infty)$. *Moreover, uniformly in* m, $\frac{1}{n}\sum_{i=1}^{n}\|A_i\|\left(\sum_{j=m}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right) / \|\mathfrak{T}_0(m)\| < \infty$ *as* $n \to \infty$.

Remark 1 *The first part of the last condition seems to be a mild regularity condition in light of Condition 3 and the fact that* $\left\|\Gamma_{i,j}^{kk}\right\| \leq ca_*^j$ *for some* $a_* \in (0,1)$ *because of the mixing inequality. The secon part of the condition holds for any* Γ_i^{kk} *that can be represented as an ARMA process.*

B Proofs and Auxiliary Results for MSE Calculation

Let $r_1 = \max(1, l)$ and $r_2 = \min(T, T + l)$. Also let $r_{1,j} = \max(1, l_j)$ and $r_{2,j} = \min(T, T + l_j)$. The following Lemma establishes an approximation to $\hat{f}_m^{kk} = \frac{1}{n}\sum_{i=1}^{n} \hat{A}_i \sum_{|j|\leq m} \hat{\Gamma}_{i,j}^{kk}$.

Lemma 2 *Assume that Conditions 1-7 hold and $m/T \to 0$ as $m, T \to \infty$. Define Let $\hat{S}_m = \mathfrak{T}_0(m) + T_4 + T_8 + T_{14}$. Let $\hat{\beta}_m$ be as defined in 5. Then*

$$\hat{\beta}_m = \hat{S}_m B + o_p\left(\frac{m}{T}\right)$$

and

$$E\left\|\hat{S}_m B\right\|^2 = \left\|\left(\mathfrak{T}_{\mathfrak{o}}(m) + \frac{2m}{T}(\mathfrak{T}_1 + \mathfrak{T}_2)\right) B\right\|^2 + o\left(\left(\frac{m}{T}\right)^2\right).$$

Proof. Follows from Lemmas 3-20. ∎

To prove Lemmas 3-20 the following expansion for $\hat{f}_m^{kk}$ is obtained. Let $K_{i,m} = \frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2} k_{it}k'_{it-l}$ and define

$$\begin{aligned}
k_{it}^{\theta}(x_{it};\theta,\gamma_i) &= \partial k(x_{it};\theta,\gamma_i)/\partial\theta', \\
k_{it}^{\gamma}(x_{it};\theta,\gamma_i) &= \partial k(x_{it};\theta,\gamma_i)/\partial\gamma, \\
k_{it}^{\gamma\gamma}(x_{it};\theta,\gamma_i) &= \partial^2 k(x_{it};\theta,\gamma_i)/(\partial\gamma)^2, \\
k_{it}^{\gamma\theta}(x_{it};\theta,\gamma_i) &= \partial^2 k(x_{it};\theta,\gamma_i)/\left(\partial\gamma\partial\theta'\right), \\
k_{it}^{\theta\theta}(x_{it};\theta,\gamma_i) &= \partial\left(\operatorname{vec}\partial k(x_{it};\theta,\gamma_i)/\partial\theta'\right)/\partial\theta'
\end{aligned}$$

as well as

$$\begin{aligned}
k_{0,it}^{\theta} &= k_{it}^{\theta}(x_{it};\theta_0,\gamma_{i0}), \\
k_{0,it}^{\gamma} &= k_{it}^{\gamma}(x_{it};\theta_0,\gamma_{i0}), \\
k_{it}^{\gamma\gamma} &= k_{it}^{\gamma\gamma}\left(x_{it};\tilde{\theta},\tilde{\gamma}_i\right) \\
k_{it}^{\gamma\theta} &= k_{it}^{\gamma\theta}\left(x_{it};\tilde{\theta},\tilde{\gamma}_i\right) \\
k_{it}^{\theta\theta} &= k_{it}^{\theta\theta}\left(x_{it};\tilde{\theta},\tilde{\gamma}_i\right)
\end{aligned}$$

with $\tilde{\theta}, \tilde{\gamma}_i$ such that $\left\|\tilde{\theta} - \theta_0\right\| \leq \left\|\widehat{\theta} - \theta_0\right\|$, $\|\tilde{\gamma}_i - \theta_0\| \leq \|\hat{\gamma}_i - \gamma_{I0}\|$. Then use the multivariate mean value theorem to obtain the second order expansion

$$\begin{aligned}
\widehat{k}_{it} - k_{it} &= k_{0,it}^{\theta}\left(\widehat{\theta} - \theta_0\right) + k_{0,it}^{\gamma}(\widehat{\gamma}_i - \gamma_{i0}) + \frac{1}{2}\left(\left(\hat{\theta} - \theta_0\right)' \otimes I\right) k_{it}^{\theta\theta}\left(\hat{\theta} - \theta_0\right) \\
&\quad + \frac{1}{2}k_{it}^{\gamma\gamma}(\widehat{\gamma}_i - \gamma_{i0})^2 + k_{it}^{\gamma\theta}\left(\widehat{\theta} - \theta_0\right)(\widehat{\gamma}_i - \gamma_{i0}).
\end{aligned}$$

We consider

$$\operatorname{vec}\left(\hat{f}^{kk} - f^{kk}\right) = \sum_{j=1}^{J} T_j$$

where

$$(13)\quad T_1 = \text{vec}\left(\frac{1}{n}\sum_{i=1}^{n} A_i E\left(K_{i,m}\right) - f^{kk}\right)$$

$$T_2 = \text{vec}\left(\frac{1}{n}\sum_{i=1}^{n} A_i\left(K_{im} - E\left(K_{i,m}\right)\right)\right)$$

$$T_3 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2\prime}\left(I\otimes A_i\right)\left(k_{it-l}\otimes k_{0,it}^{\theta}\right)\left(\widehat{\theta}-\theta\right)$$

$$(14)\quad T_4 = \frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]$$

$$T_5 = \frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(0)}{T^{2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]$$

$$T_6 = \frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\tilde{\varepsilon})}{T^{5/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]$$

$$T_7 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{0,it-l}^{\theta}\otimes k_{it}\right)\left(\widehat{\theta}-\theta\right)$$

$$T_8 = \frac{1}{n}\sum_{i=1}^{n}\left(I\otimes A_i\right)\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{0,it-l}^{\gamma}\otimes k_{it}\right]$$

$$T_9 = \frac{1}{n}\sum_{i=1}^{n}\left(I\otimes A_i\right)\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(0)}{T^{2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{0,it-l}^{\gamma}\otimes k_{it}\right]$$

$$T_{10} = \frac{1}{n}\sum_{i=1}^{n}\left(I\otimes A_i\right)\frac{\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\tilde{\varepsilon})}{T^{5/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{0,it-l}^{\gamma}\otimes k_{it}\right].$$

$$T_{11} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{0,it-l}^{\theta}\otimes k_{0,it}^{\theta}\right)\left[\left(\widehat{\theta}-\theta\right)\otimes\left(\widehat{\theta}-\theta\right)'\right]$$

$$T_{12} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{0,it-l}^{\theta}\otimes k_{0,it}^{\gamma}\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)\left(\widehat{\theta}-\theta\right)$$

$$T_{13} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{0,it-l}^{\gamma}\otimes k_{0,it}^{\theta}\right)\left(\widehat{\theta}-\theta\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)$$

$$(15)\quad T_{14} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)^2\text{vec}\left(k_{0,it}^{\gamma}k_{0,it-l}^{\gamma'}\right)$$

$$T_{15} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)(k_{it-l}\otimes I)\operatorname{vec}\left(\left(\left(\hat{\theta}-\theta_0\right)'\otimes I\right)k_{it}^{\theta\theta}\left(\hat{\theta}-\theta_0\right)\right)$$
$$T_{16} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\left(\widehat{\theta}-\theta_0\right)(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{17} = \frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\left(\widehat{\theta}-\theta_0\right)$$
$$T_{18} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)(k_{it-l}\otimes k_{it}^{\gamma\gamma})(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{19} = \frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\left(\widehat{\theta}-\theta_0\right)$$
$$T_{20} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{0,it}^{\gamma\gamma}\right)(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{21} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(\left(\left(\hat{\theta}-\theta_0\right)'\otimes I\right)k_{it}^{\theta\theta}\otimes k_{it}\right)\left(\widehat{\theta}-\theta\right)$$
$$T_{22} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it}^{\gamma\theta}\otimes k_{it}\right)\left(\widehat{\theta}-\theta\right)(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{23} = \frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it}^{\gamma\theta}\otimes k_{it}\right)\left(\widehat{\theta}-\theta_0\right)$$
$$T_{24} = \frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\frac{\widehat{\gamma}_i^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(k_{it}^{\gamma\gamma}\otimes k_{it})(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{25} = \frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{0,it-l}^{\gamma\theta}\otimes k_{it}\right)\left(\widehat{\theta}-\theta_0\right)$$
$$T_{26} = \frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\frac{\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{0,it-l}^{\gamma\gamma}\otimes k_{it}\right)(\widehat{\gamma}_i-\gamma_{i0}).$$

For the last set of terms define

$$\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right) = \frac{1}{2}\left(\left(\hat{\theta}-\theta_0\right)'\otimes I\right)k_{it}^{\theta\theta}\left(\hat{\theta}-\theta_0\right)+\frac{1}{2}k_{it}^{\gamma\gamma}(\widehat{\gamma}_i-\gamma_{i0})^2+k_{it}^{\gamma\theta}\left(\widehat{\theta}-\theta_0\right)(\widehat{\gamma}_i-\gamma_{i0}).$$

Then

$$T_{27} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{0,it-l}^{\theta}\otimes\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\right)\left[\left(\widehat{\theta}-\theta\right)\otimes I\right]$$
$$T_{28} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\otimes k_{0,it-l}^{\theta}\right)\left[I\otimes\left(\widehat{\theta}-\theta\right)\right]$$
$$T_{30} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\otimes\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\right)$$
$$T_{31} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{0,it-l}^{\theta}\otimes\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\right)\left(\widehat{\theta}-\theta\right)$$
$$T_{32} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\otimes k_{0,it}^{\gamma}\right)(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{33} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{0,it-l}^{\gamma}\otimes\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\right)(\widehat{\gamma}_i-\gamma_{i0})$$
$$T_{34} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)\otimes k_{0,it}^{\theta}\right)\left(\widehat{\theta}-\theta\right)$$
$$T_{35} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)(\widehat{\gamma}_i-\gamma_{i0})\operatorname{vec}\left(k_{0,it}^{\gamma}\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)'\right)$$
$$T_{36} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)(\widehat{\gamma}_i-\gamma_{i0})\operatorname{vec}\left(\Delta k_{it}^{\gamma\theta}\left(\hat{\theta},\hat{\gamma}_i\right)'k_{0,it-l}^{\gamma'}\right).$$

The following Lemmas establish the mean and variances of T_j, $j=1,..,36$.

Lemma 3 $T_1 = \mathrm{vec}((\mathfrak{T}_0(m)) + O(T^{-1})$ *where*

$$(16) \quad \mathfrak{T}_0(m) = \frac{1}{n}\sum_{i=1}^{n} A_i \sum_{|l|>m} E\left[k_{it}k_{it-l}'\right]$$

and $\mathfrak{T}_0(m) = O\left(\left(a^{\frac{\delta}{2+\delta}}\right)^m\right)$.

Proof.

$$\begin{aligned} ET_1 &= \mathrm{vec}\left(\frac{1}{n}\sum_{i=1}^{n} A_i E(K_{i,m}) - f^{kk}\right) \\ &= \mathrm{vec}\left(\frac{1}{n}\sum_{i=1}^{n} A_i \left(\sum_{l=-m}^{m}\left(\frac{r_2-r_1}{T}-1\right)E\left[k_{it}k_{it-l}'\right] - \sum_{|l|>m} E\left[k_{it}k_{it-l}'\right]\right)\right) \\ &= O(T^{-1}) + O\left(\left(a^{\frac{\delta}{2+\delta}}\right)^m\right), \end{aligned}$$

where the last line follows from Condition 3 and

$$\begin{aligned} \|\mathrm{Var}(T_1)\| &= \left\| E\left(\mathrm{vec}\frac{1}{n}\sum_{i=1}^{n} A_i\left(K_{i,m} - E(K_{i,m})\right)\right)\left(\mathrm{vec}\frac{1}{n}\sum_{i=1}^{n} A_i\left(K_{i,m} - E(K_{i,m})\right)\right)'\right\| \\ &= \left\| n^{-1}\frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\,\mathrm{Var}(\mathrm{vec}\,K_{i,m})\,(I\otimes A_i')\right\| \\ &\leq n^{-1}\frac{1}{n}\sum_{i=1}^{n}\|A_i\|^2\,\|\mathrm{Var}(\mathrm{vec}\,K_{i,m})\| \\ &= O(n^{-1}). \end{aligned}$$

since $\mathrm{Var}(\mathrm{vec}\,K_{i,m})$ is uniformly bounded in i and m. ■

Lemma 4 $E(T_2) = 0$ *and* $\mathrm{Var}(T_2) = O\left(n^{-1}\right)$.

Proof. See Lemma (3). ■

Lemma 5 *Assume that Condition 2 holds and* $m/T \to 0$ *as* $m, T \to \infty$. *Then* $T_3 = O_p(n^{-1/2}T^{-1/2}) + O_p(n^{-1/2}T^{-1}m)$ *and* $T_7 = O_p(n^{-1/2}T^{-1/2}) + O_p(n^{-1/2}T^{-1}m)$.

Proof. For T_3 use the expansion

$$\sqrt{nT}\left(\hat{\theta} - \theta_0\right) = \sqrt{n}\theta^{\epsilon}(0) + \frac{1}{2}\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}(0) + o_p(1)$$

obtained by Hahn and Kuersteiner (2004) such that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2\prime}(I\otimes A_i)\left(k_{it-l}\otimes k_{0,it}^{\theta}\right)\left(\widehat{\theta}-\theta\right) \\ =\ &\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{0,it}^{\theta}\right)\sqrt{n}\theta^{\epsilon}(0) \\ &+\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\frac{1}{2}(I\otimes A_i)\left(k_{it-l}\otimes k_{0,it}^{\theta}\right)\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}(0) \\ &+o_p(1)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left(k_{it-l}\otimes k_{0,it}^{\theta}\right) \end{aligned}$$

where the last term is $o_p(n^{-1/2}m/T)$ and thus is neglected. Then for the first term define $\bar{k}^{\theta}_{0,it} = k^{\theta}_{0,it} - E\left[k^{\theta}_{0,it}\right]$ such that

$$\begin{aligned}
& E\left\|\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right)\sqrt{n}\theta^{\epsilon}\left(0\right)\right\| \\
\leq\ & \frac{1}{n^{3/2}}\sum_{i=1}^{n}\left(E\left\|\sqrt{n}\theta^{\epsilon}\left(0\right)\right\|^{2}\right)^{1/2}\left(E\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right)\right\|^{2}\right)^{1/2}\left\|I\otimes A_i'\right\| \\
=\ & O(n^{-1/2}T^{-1/2}).
\end{aligned}$$

The last equality follows from

$$E\left\|\sqrt{n}\theta^{\epsilon}\left(0\right)\right\|^{2} = \operatorname{tr}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\frac{1}{nT}\sum_{i_1,i_2=1}^{n}\sum_{t_1,t_2=1}^{T}E\left[U_{i_1t_1}U'_{i_2t_2}\right]\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1} = O(1)$$

because of Conditions (3) and (4) and

$$\begin{aligned}
& E\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right)\right\|^{2} \\
=\ & \frac{1}{T^{3}}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\operatorname{tr}E\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}^{\theta}_{0,it_1}\bar{k}^{\theta\prime}_{0,it_2}\right) \\
& \frac{1}{T^{3}}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(\operatorname{tr}\left(E\left[k_{it_1-l_1}k'_{it_2-l_2}\right]\otimes E\left[\bar{k}^{\theta}_{0,it_1}\bar{k}^{\theta\prime}_{0,it_2}\right]\right)\right. \\
& +\operatorname{tr}\left(E\left[k_{it_1-l_1}\otimes \bar{k}^{\theta}_{0,it_1}\right]E\left[k'_{it_2-l_2}\otimes \bar{k}^{\theta\prime}_{0,it_2}\right]\right)+\operatorname{tr}\left(E\left[k'_{it_2-l_2}\otimes \bar{k}^{\theta}_{0,it_1}\right]E\left[k_{it_1-l_1}\otimes \bar{k}^{\theta\prime}_{0,it_2}\right]\right) \\
& \left.+\operatorname{tr}\left[\mathcal{K}_4\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}^{\theta}_{0,it_1}\bar{k}^{\theta\prime}_{0,it_2}\right)\right]\right)
\end{aligned}$$

where $\mathcal{K}_4\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}^{\theta}_{0,it_1}\bar{k}^{\theta\prime}_{0,it_2}\right)$ is the matrix that has as its elements the forth order comulants of the elements of the matrix $\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}^{\theta}_{0,it_1}\bar{k}^{\theta\prime}_{0,it_2}\right)$. The largest order term in the above expression is

$$\frac{1}{T^{3}}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\operatorname{tr}\left(E\left[k_{it_1-l_1}\otimes \bar{k}^{\theta}_{0,it_1}\right]E\left[k'_{it_2-l_2}\otimes \bar{k}^{\theta\prime}_{0,it_2}\right]\right)' = O(T^{-1})$$

where the bound holds uniformly in i. Next consider

$$\begin{aligned}
& \frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\frac{1}{2}\left(I\otimes A_i\right)\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right)\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}\left(0\right) \\
=\ & -\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\frac{1}{2}\left(I\otimes A_i\right)\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right)\sqrt{\frac{n}{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\Psi \\
& +o_p\left(1\right)\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\frac{1}{2}\left(I\otimes A_i\right)\left(k_{it-l}\otimes \bar{k}^{\theta}_{0,it}\right) \\
=\ & O_p(n^{-1/2}T^{-1/2})
\end{aligned}$$

by the same arguments as before. Next

$$\begin{aligned}
& E\left\|\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(I\otimes A_i\right)\left(k_{it-l}\otimes E\left[k^{\theta}_{0,it}\right]\right)\sqrt{n}\theta^{\epsilon}\left(0\right)\right\| \\
\leq\ & \left(E\left\|\sqrt{n}\theta^{\epsilon}\left(0\right)\right\|^{2}\right)^{1/2}\frac{1}{n^{3/2}}\sum_{i=1}^{n}\left(E\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes E\left[k^{\theta}_{0,it}\right]\right)\right\|^{2}\right)^{1/2}\left\|I\otimes A_i'\right\| \\
=\ & O(n^{-1/2}T^{-1}m)
\end{aligned}$$

where

$$E\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes E\left[k^{\theta}_{0,it}\right]\right)\right\|^{2} = O(T^{-2}m^{2})$$

by an argument analoguous to the proof of Lemma 20. The results can be shown for terms involving $\theta^{\epsilon\epsilon}(0)$. This shows that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2\prime}(I\otimes A_i)\left(k_{it-l}\otimes E\left[k_{0,it}^{\theta}\right]\right)\left(\widehat{\theta}-\theta\right)=O_p(n^{-1/2}T^{-1}m).$$

The analysis for T_7 proceeds in the same way by noting that

$$E\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(E\left[k_{0,it-l}^{\theta}\right]\otimes k_{it}\right)\right\|^2=O(T^{-2}m^2)$$

as before. ■

Lemma 6 *Define*

(17) $$\psi_{it}^{\gamma,v}=-\left(E\left[V_i^{\gamma}\right]\right)^{-1}\frac{\partial\psi\left(x_{it};\theta_0,\gamma_{i0}\right)}{\partial\gamma}.$$

Then

$$E\left[T_4\right]=\frac{2m}{T}\frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\left(\left(\sum_{u=-\infty}^{\infty}E\left[\psi_{it}^{\gamma,v}k_{it-u}\right]\right)\otimes E\left[k_{0,i1}^{\gamma}\right]\right)+o(\frac{m}{T})=O(m/T)$$

and $\mathrm{Var}(T_4)=O(n^{-1}T^{-2}m^2)$. *Similarly,*

$$E\left[T_8\right]=\frac{2m}{T}\frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\left(Ek_{0,i1}^{\gamma}\otimes\sum_{u=-\infty}^{\infty}E\left[\psi_{it}^{\gamma,v}k_{it-u}\right]\right)+o\left(\frac{m}{T}\right)=O\left(\frac{m}{T}\right)$$

and $\mathrm{Var}(T_8)=O(n^{-1}T^{-2}m^2)$.

Proof. Use $\widehat{\gamma}_i^{\epsilon}(0)=-\left(E\left[V_i^{\gamma}\right]\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial\psi(x_{it};\theta_0,\gamma_{i0})}{\partial\gamma}+E\left[V_i^{\theta}\right]\theta^{\epsilon}(0)\right)$ where

$$\theta^{\epsilon}(0)=\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\frac{1}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}U_{it}\right)$$

and define

(18) $$\bar{k}_{it}^{\gamma}=k_{it}^{\gamma}-Ek_{it}^{\gamma}$$

and

(19) $$U_{js}^{\theta}=-\left(E\left[V_i^{\gamma}\right]\right)^{-1}E\left[V_i^{\theta}\right]\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}U_{js}$$

Define

(20) $$T_{4,1}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s=1}^{T}(I\otimes A_i)\psi_{is}^{\gamma,v}\left(k_{it-l}\otimes k_{0,it}^{\gamma}\right)$$

(21) $$T_{4,2}=\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{j=1}^{n}\sum_{s=1}^{T}(I\otimes A_i)U_{js}^{\theta}\left(k_{it-l}\otimes k_{0,it}^{\gamma}\right).$$

Then, for $ET_{4,1}$ consider

(22) $$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s=1}^{T}(I\otimes A_i)E\left[\psi_{is}^{\gamma,v}\left(k_{it-l}\otimes\bar{k}_{0,it}^{\gamma}\right)\right]\\ =\;&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s=1}^{T}(I\otimes A_i)\,\mathrm{cum}\left(\psi_{is}^{\gamma,v},k_{it-l},\bar{k}_{it}^{\gamma}\right)=O(T^{-2}m)\end{aligned}$$

where $\mathrm{cum}\left(\psi_{is}^{\gamma,v},k_{it-l},\bar{k}_{it}^{\gamma}\right)$ is the matrix of third order cross-cumulants of $\psi_{is}^{\gamma,v},k_{it-l}$ and $\bar{k}_{it}^{\gamma}$ and the relationship $E\left[\psi_{is}^{\gamma,v}k_{it-l}\bar{k}_{it}^{\gamma}\right]=\mathrm{cum}\left(\psi_{is}^{\gamma,v},k_{it-l},\bar{k}_{it}^{\gamma}\right)$ follows from Shiryaev (1996, p.293). By the same argument as in

Andrews (1991, Lemma 1) it follows that $\sum_{t=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left|\operatorname{cum}\left(\psi_{is}^{\gamma,v},k_{it-l},\bar{k}_{it}^{\gamma}\right)\right|<\infty$ uniformly in l such that the result follows. Next consider

$$
\begin{aligned}
(23)\qquad & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s=1}^{T}(I\otimes A_i)\left(E\left[\psi_{is}^{\gamma,v}k_{it-l}\right]\otimes E\left[k_{it}^{\gamma}\right]\right)\\
= & \frac{2m}{T}\frac{1}{n}\sum_{i=1}^{n}(I\otimes A_i)\left(\left(\sum_{u=-\infty}^{\infty}E\left[\psi_{it}^{\gamma,v}k_{it-u}\right]\right)\otimes E\left[k_{it}^{\gamma}\right]\right)+o\left(\frac{m}{T}\right)=O(mT^{-1})
\end{aligned}
$$

by the mixing properties which imply that $\sum_{u=-\infty}^{\infty}\left\|E\left[\psi_{it}^{\gamma,v}k_{it-u}\right]\right\|<\infty$.

Define $\mathcal{C}_{is,t-l,t}=\psi_{is}^{\gamma,v}\left(k_{it-l}\otimes k_{0,it}^{\gamma}\right)$ and consider $\operatorname{Var}(T_4)$ where

$$
\begin{aligned}
\operatorname{Var}(T_{4,1}) =\ & \frac{1}{n^2T^4}\sum_{i=1}^{n}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,s_2=1}^{T}(I\otimes A_i)\\
& \times E\left[\left(\mathcal{C}_{is_1,t_1-l_1,t_1}-E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\right]\right)\left(\mathcal{C}_{is_2,t_2-l_2,t_2}-E\left[\mathcal{C}_{is_2,t_2-l_2,t_2}\right]\right)'\right](I\otimes A_i')\\
=\ & \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{T^4}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,s_2=1}^{T}(I\otimes A_i)\,E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\mathcal{C}_{is_2,t_2-l_2,t_2}'\right](I\otimes A_i')\\
& -\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{T^4}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,s_2=1}^{T}(I\otimes A_i)\,E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\right]E\left[\mathcal{C}_{is_2,t_2-l_2,t_2}\right]'(I\otimes A_i').
\end{aligned}
$$

The second term in the last display is $O(n^{-1}T^{-4}m^2)$ by 22. The matrix $E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\mathcal{C}_{is_2,t_2-l_2,t_2}'\right]$ is a matrix of sixth order cross moments of the elements in $\psi_{is}^{\gamma,v},k_{it-l}$ and $k_{0,it}^{\gamma}$. We use the index (a,b,c,d,e,f) to denote indiviual elements of the random vectors $\psi_{is_1}^{\gamma,v},k_{it_1-l_1},\ k_{0,it_1}^{\gamma},\psi_{is_2}^{\gamma,v},k_{it_2-l_2}$ and k_{0,it_2}^{γ} where $\psi_{a,is_1}^{\gamma,v}$ is the a-th element of $\psi_{is_1}^{\gamma,v}$ and so forth. For any combination of elements (a,b,c,d,e,f) consider the index $I\subseteq\{1,2,...,6\}$ and denote by

$$
\begin{aligned}
& E_{(a,b,c,d,e,f)}\left[I\right]\\
= & E\left[\left(\{1\in I\}\psi_{a,is_1}^{\gamma,v}+\{1\notin I\}\right)\left(\{2\in I\}k_{b,it_1-l_1}+\{2\notin I\}\right)\times...\times\left(\{6\in I\}k_{f,0,it_2}^{\gamma}+\{6\notin I\}\right)\right]
\end{aligned}
$$

with a similar definition holding for the cumulant $\operatorname{cum}_{(a,b,c,d,e,f)}[I]$. Each element of $E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\mathcal{C}_{is_2,t_2-l_2,t_2}'\right]$ is charactarized by a particular value for (a,b,c,d,e,f). From Shiryaev (1996, p. 292) it follows that for a typical element $E_{(a,b,c,d,e,f)}[\{1,..,6\}]$ of $E\left[\mathcal{C}_{is_1,t_1-l_1,t_1}\mathcal{C}_{is_2,t_2-l_2,t_2}'\right]$

$$
E_{(a,b,c,d,e,f)}\left[\{1,..,6\}\right]=\sum_{\cup_r^q I_r=\{1,2,...,6\}}\prod_{r=1}^{q}\operatorname{cum}_{(a,b,c,d,e,f)}\left[I_r\right]
$$

where the sum is over all possible decompositions of $\{1,2,...,6\}$ and $1\leq q\leq 6$. All terms involving cumulants of order higher than two are smaller when averaged across time periods because of the mixing properties. The largest terms of $E_{(a,b,c,d,e,f)}[\{1,..,6\}]$ are covariance terms of the form

$$
\begin{aligned}
& \frac{1}{T^4}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,s_2=1}^{T}\operatorname{Cov}\left(\psi_{a,is_1}^{\gamma,v},k_{b,it_1-l_1}\right)\left(\operatorname{Cov}\psi_{d,is_2}^{\gamma,v},k_{e,it_2-l_2}\right)E\left[k_{c,0,it_1}^{\gamma}\right]E\left[k_{f,0,it_2}^{\gamma}\right]\\
= & O\left(T^{-2}m^2\right)
\end{aligned}
$$

which shows that $\operatorname{Var}(T_{4,1})=O(n^{-1}T^{-2}m^2)$.

Next consider $T_{4,2}$ where

$$
\begin{aligned}
& \left\|\frac{1}{n}\sum_{i=1}^{n}\frac{-\left(E\left[V_i^{\gamma}\right]\right)^{-1}E\left[V_i^{\theta}\right]\theta^{\epsilon}(0)}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}(I\otimes A_i)\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]\right\|\\
\leq & \left\|\sqrt{n}\theta^{\epsilon}(0)\right\|\sup_i\left\|\left(E\left[V_i^{\gamma}\right]\right)^{-1}E\left[V_i^{\theta}\right]\right\|\frac{1}{n^{3/2}}\sum_{i=1}^{n}\left\|(I\otimes A_i)\right\|\left\|\frac{1}{T^{3/2}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]\right\|.
\end{aligned}
$$

From Lemma 20 it follows that

$$\left\| \frac{1}{T} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \left[k_{it-l} \otimes k_{0,it}^{\gamma} \right] \right\| = O\left(1\right) + O(m/T^{1/2})$$

such that $T_{4,2} = O_p\left(n^{-1/2}T^{-1/2}\right) + O_p\left(n^{-1/2}T^{-1}m\right)$. A more detailed analysis of $T_{4,2}$ is needed to obtain $E\left[T_4\right]$ and $\mathrm{Var}\left(T_4\right)$. Use the fact that U_{js} is independent of $\left(k_{it-l} \otimes k_{0,it}^{\gamma}\right)$ for $i \neq j$ and $EU_{js} = 0$ such that

$$\begin{aligned} & \frac{1}{n^{3/2}} \sum_{i=1}^{n} \frac{1}{T^2} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \sum_{j=1}^{n} \sum_{s=1}^{T} \left(I \otimes A_i\right) E\left[U_{js}^{\theta}\left(k_{it-l} \otimes \bar{k}_{0,it}^{\gamma}\right)\right] \\ = & \frac{1}{n^{3/2}} \sum_{i=1}^{n} \frac{1}{T^2} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \sum_{s=1}^{T} \left(I \otimes A_i\right) E\left[U_{is}^{\theta}\left(k_{it-l} \otimes \bar{k}_{0,it}^{\gamma}\right)\right] = O(mT^{-2}n^{-1/2}) \end{aligned}$$

by the same argument as in (22) while

$$\frac{1}{n^{3/2}} \sum_{i=1}^{n} \frac{1}{T^2} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \sum_{s=1}^{T} \left(I \otimes A_i\right) E\left[U_{is}^{\theta}\left(k_{it-l} \otimes E\left[k_{0,it}^{\gamma}\right]\right)\right] = O(mT^{-1}n^{-1/2})$$

by the same argument as in (23). These results show that $ET_4 = ET_{4,1} + o(mT^{-1})$. Similar arguments as for $\mathrm{Var}(T_{4,1})$ can be used to establish that $\mathrm{Var}(T_{4,2}) = O\left(n^{-1}T^{-2}m^2\right)$. From the Cauchy-Schwartz inequality it follows that $\mathrm{Cov}\left(T_{4,1}, T_{4,2}\right) = O\left(n^{-1}T^{-2}m^2\right)$. It thus follows that $\mathrm{Var}(T_4) = \left(n^{-1}T^{-2}m^2\right)$. The analysis of T_8 proceeds analogously. ■

Lemma 7 *Assume that Condition 2 holds and $m/T \to 0$ as $m, T \to \infty$. Then $T_5 = O_p(T^{-1}) + O_p(mT^{-3/2})$ and $T_9 = O_p(T^{-1}) + O_p(mT^{-3/2})$.*

Proof. Consider

$$E\left\|T_5\right\| \leq \frac{1}{nT} \sum_{i=1}^{n} \left\|I \otimes A_i\right\| \left(E\left|\widehat{\gamma}_i^{\epsilon\epsilon}\left(0\right)\right|^2\right)^{1/2} \left(E\left\|\frac{1}{T} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} k_{it-l} \otimes k_{0,it}^{\gamma}\right\|^2\right)^{1/2}$$

where by Lemma 20

$$\sup_i E\left\|\frac{1}{T} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} k_{it-l} \otimes k_{0,it}^{\gamma}\right\|^2 = O(1) + O(mT^{-1/2})$$

and straight forward but tedious calculations based on 31 show that $\sup_i E\left|\widehat{\gamma}_i^{\epsilon\epsilon}\left(0\right)\right|^2 = O(1)$. This establishes that $E\left\|T_5\right\| = O(T^{-1}) + O(mT^{-3/2})$. The analysis for T_9 follows the same arguments. ■

Lemma 8 $T_6 = o_p(T^{-12/10}) + o_p(mT^{-17/10})$ *and* $T_{10} = o_p(T^{-12/10}) + o_p(mT^{-17/10})$.

Proof. Note that

$$\left\|T_6\right\| \leq \frac{1}{T^{12/10}} \sup_i \frac{\left|\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}\left(\tilde{\varepsilon}\right)\right|}{T^{3/10}} \frac{1}{n} \sum_{i=1}^{n} \left\|I \otimes A_i\right\| \left\|\frac{1}{T} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} k_{it-l} \otimes k_{0,it}^{\gamma}\right\|$$

where

$$\sup_i \frac{\left|\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}\left(\tilde{\varepsilon}\right)\right|}{T^{3/10}} = o_p(1)$$

by Lemma 21 and

$$\frac{1}{n} \sum_{i=1}^{n} \left\|I \otimes A_i\right\| \left\|\frac{1}{T} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} k_{it-l} \otimes k_{0,it}^{\gamma}\right\| = O_p(1) + O_p(mT^{-1/2})$$

by the Markov inequality and Lemma 20. It then follows that $T_6 = o_p(T^{-12/10}) + o_p(mT^{-17/10})$. The analysis for T_9 follows the same arguments. ■

Lemma 9 $T_{11} = O_p(n^{-1}T^{-1}m)$.

Proof.

$$\|T_{11}\| \leq \left\|\widehat{\theta} - \theta\right\|^2 \frac{1}{n}\sum_{i=1}^n \|I \otimes A_i\| \frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left\|k_{0,it-l}^\theta\right\| \left\|k_{0,it}^\theta\right\|$$

where

$$\frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left(E\left\|k_{0,it-l}^\theta\right\|^2 E\left\|k_{0,it}^\theta\right\|^2\right)^{1/2} = O(m)$$

and $\left\|\widehat{\theta} - \theta\right\|^2 = O_p(n^{-1}T^{-1})$ such that the result follows. ■

Lemma 10 $T_{12} = O_p(n^{-1/2}T^{-9/10}m) = o_p(T^{-1}m)$. *The same result hold for* T_{13}.

Proof.

$$\|T_{12}\| \leq T^{-2/5}\left\|\widehat{\theta} - \theta\right\| \max_i \frac{\sqrt{T}\left|\widehat{\gamma}_i - \gamma_{i0}\right|}{T^{1/10}} \frac{1}{n}\sum_{i=1}^n \|I \otimes A_i\| \frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left\|k_{0,it-l}^\theta\right\| \left\|k_{0,it}^\gamma\right\|$$

where

$$\frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left(E\left\|k_{0,it-l}^\theta\right\|^2\right)^{1/2} \left(E\left\|k_{0,it}^\gamma\right\|^2\right)^{1/2} = O(m),$$

$$\max \frac{\sqrt{T}\left|\widehat{\gamma}_i - \gamma_{i0}\right|}{T^{1/10}} = o_p(1)$$

by Lemma 21 and $\widehat{\theta} - \theta = O_p(n^{-1/2}T^{-1/2})$ such that $T_{12} = O_p(n^{-1/2}T^{-9/10}m)$. ■

Lemma 11 *Let*

$$T_{14,1,1} = \frac{1}{n}\sum_{i=1}^n \frac{1}{T^3}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \sum_{s_1,s_2=1}^T \left(I \otimes A_i\right) \psi_{is_1}^{\gamma,v}\psi_{is_2}^{\gamma,v} \operatorname{vec}\left(k_{0,it}^\gamma k_{0,it-l}^{\gamma'}\right).$$

Then, $T_{14} = T_{14,1,1} + o_p(m/T)$. *Moreover,*

$$E\left\|T_{14,1,1}\right\|^2 = \frac{4m^2}{T^2}\left\|\frac{1}{n}\sum_{i_1=1}^n \left(I \otimes A_i\right)\left(E\left[k_{0,i1}^\gamma\right] \otimes E\left[k_{0,i1}^\gamma\right]\right)\sum_{u=-\infty}^\infty \operatorname{Cov}\left(\psi_{i1}^{\gamma,v}, \psi_{i1-u}^{\gamma,v}\right)\right\|^2 + o\left(\frac{m^2}{T^2}\right)$$

Proof. Write

$$\widehat{\gamma}_i - \gamma_{i0} = \frac{1}{\sqrt{T}}\widehat{\gamma}_i^\epsilon(0) + \frac{1}{2T}\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})$$

for some $\tilde{\epsilon} \in \left[0, 1/\sqrt{T}\right]$ such that

$$\left(\widehat{\gamma}_i - \gamma_{i0}\right)^2 = \frac{1}{T}\left(\widehat{\gamma}_i^\epsilon(0)\right)^2 + \frac{1}{4T^2}\left(\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})\right)^2 + \frac{1}{\sqrt{T}T}\widehat{\gamma}_i^\epsilon(0)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon}).$$

Define

$$\begin{aligned}
T_{14,1} &= \frac{1}{n}\sum_{i=1}^n \frac{1}{T^2}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left(I \otimes A_i\right)\left(\widehat{\gamma}_i^\epsilon(0)\right)^2 \operatorname{vec}\left(k_{0,it}^\gamma k_{0,it-l}^{\gamma'}\right) \\
T_{14,2} &= \frac{1}{n}\sum_{i=1}^n \frac{1}{4T^3}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left(I \otimes A_i\right)\left(\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})\right)^2 \operatorname{vec}\left(k_{0,it}^\gamma k_{0,it-l}^{\gamma'}\right) \\
T_{14,3} &= \frac{1}{n}\sum_{i=1}^n \frac{1}{T^2\sqrt{T}}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \left(I \otimes A_i\right)\widehat{\gamma}_i^\epsilon(0)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon}) \operatorname{vec}\left(k_{0,it}^\gamma k_{0,it-l}^{\gamma'}\right).
\end{aligned}$$

Then

$$\begin{aligned}\|T_{14,3}\| &\leq \frac{\sup_i |\widehat{\gamma}_i^{\epsilon}(0)| \sup_i |\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})|}{T^{3/10}} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^2 T^{1/5}} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \|I \otimes A_i\| \left\|k_{0,it}^{\gamma}\right\| \left\|k_{0,it-l}^{\gamma}\right\| \\ &= o_p\left(mT^{-6/5}\right)\end{aligned}$$

and

$$\begin{aligned}\|T_{14,2}\| &\leq \left(\frac{\sup_i |\widehat{\gamma}_i^{\epsilon\epsilon}(\tilde{\epsilon})|}{T^{2/10}}\right)^2 \frac{1}{n} \sum_{i=1}^{n} \frac{1}{4T^2 T^{3/5}} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \|I \otimes A_i\| \left\|k_{0,it}^{\gamma}\right\| \left\|k_{0,it-l}^{\gamma'}\right\| \\ &= o_p\left(mT^{-8/5}\right)\end{aligned}$$

by Lemma 21. For $T_{14,1}$ recall

$$\widehat{\gamma}_i^{\epsilon}(0) = \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_{it}^{\gamma} - \left(E\left[V_i^{\gamma}\right]\right)^{-1} E\left[V_i^{\theta}\right] \theta^{\epsilon}(0)\right)$$

where $\psi_{it}^{\gamma,v}$ is defined in (17) such that $T_{14,1}$ can be written as a sum of three terms

$$\begin{aligned}T_{14,1,1} &= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^3} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \sum_{s_1,s_2=1}^{T} (I \otimes A_i)\, \psi_{is_1}^{\gamma,v} \psi_{is_2}^{\gamma,v} \operatorname{vec}\left(k_{0,it}^{\gamma} k_{0,it-l}^{\gamma'}\right) \\ T_{14,1,2} &= -\frac{1}{n} \sum_{i=1}^{n} (I \otimes A_i) \left(E\left[V_i^{\gamma}\right]\right)^{-1} E\left[V_i^{\theta}\right] \theta^{\epsilon}(0) \frac{1}{T^{5/2}} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \sum_{s=1}^{T} \psi_{is}^{\gamma,v} \operatorname{vec}\left(k_{0,it}^{\gamma} k_{0,it-l}^{\gamma'}\right) \\ T_{14,1,3} &= \frac{1}{n} \sum_{i=1}^{n} (I \otimes A_i) \left(E\left[V_i^{\gamma}\right]\right)^{-2} \left(E\left[V_i^{\theta}\right] \theta^{\epsilon}(0)\right)^2 \frac{1}{T^2} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \operatorname{vec}\left(k_{0,it}^{\gamma} k_{0,it-l}^{\gamma'}\right).\end{aligned}$$

For $T_{14,1,3}$ we note that

$$T_{14,1,3} = \frac{1}{n^2} \sum_{i=1}^{n} (I \otimes A_i) \left(\frac{1}{\sqrt{nT}} \sum_{j=1}^{n} \sum_{s=1}^{T} U_{js}^{\theta}\right)^2 \frac{1}{T^2} \sum_{l=-m}^{m} \sum_{t=r_1}^{r_2} \operatorname{vec}\left(k_{0,it}^{\gamma} k_{0,it-l}^{\gamma'}\right)$$

such that

$$\begin{aligned}E\|T_{14,1,3}\|^2 &= \frac{1}{n^4} \sum_{i_1,i_2=1}^{n} \operatorname{tr} E\Bigg[(I \otimes A_i) \left(\frac{1}{\sqrt{nT}} \sum_{j=1}^{n} \sum_{s=1}^{T} U_{js}^{\theta}\right)^4 \frac{1}{T^4} \sum_{l_1,l_2=-m}^{m} \sum_{t_1,t_2=r_1}^{r_2} \\ &\quad \times \left(k_{0,i_1t_1}^{\gamma} k_{0,i_2t_2}^{\gamma\prime} \otimes k_{0,i_1t_1-l_1}^{\gamma} k_{0,i_2t_2-l_2}^{\gamma'}\right) (I \otimes A_i')\Bigg]\end{aligned}$$

where the eighth order moment matrix

$$E\left[\left(\sum_{j=1}^{n} \sum_{s=1}^{T} U_{js}^{\theta}\right)^4 \left(k_{0,i_1t_1}^{\gamma} k_{0,i_2t_2}^{\gamma\prime} \otimes k_{0,i_1t_1-l_1}^{\gamma} k_{0,i_2t_2-l_2}^{\gamma'}\right)\right]$$

can be expressed as a sum of products of cumulants (see Shiryaev (1996, p. 292)). The least summable terms of this expectation are of the form

$$E\left(\sum_{j=1}^{n} \sum_{s=1}^{T} U_{js}^{\theta}\right)^4 \left(E\left[k_{0,i_1t_1}^{\gamma}\right] E\left[k_{0,i_2t_2}^{\gamma\prime}\right] \otimes E\left[k_{0,i_1t_1-l_1}^{\gamma}\right] E\left[k_{0,i_2t_2-l_2}^{\gamma'}\right]\right)$$

because $E\left[k_{0,i_1t_1}^{\gamma}\right] \neq 0$. Note that

$$E\left(\sum_{j=1}^{n} \sum_{s=1}^{T} U_{js}^{\theta}\right)^4 = O\left(\left(\sum_{j=1}^{n} \sum_{u=-\infty}^{\infty} \operatorname{Cov}\left(U_{j1}^{\theta}, U_{j1-u}^{\theta}\right)\right)^2 T^2\right) = O(n^2T^2).$$

It then follows that $E\|T_{14,1,3}\|^2 = O(n^{-2}T^{-2}m^2)$.

For $T_{14,1,2}$ note that

$$E\left\|T_{14,1,2}\right\|^2 = \frac{1}{n^4}\sum_{i_1,i_2=1}^n \operatorname{tr} E\left[(I\otimes A_i)\frac{1}{T^6}\sum_{l_1,l_2=-m}^m\sum_{t_1,t_2=r_1}^{r_2}\left(\sum_{j=1}^n\sum_{s=1}^T U_{js}^\theta\right)^2 \times\left(\sum_{s_1,s_2=1}^T \psi_{i_1s_1}^{\gamma,v}\psi_{i_2s_2}^{\gamma,v}\right)\left(k_{0,i_1t_1}^{\gamma}k_{0,i_2t_2}^{\gamma\prime}\otimes k_{0,i_1t_1-l_1}^{\gamma}k_{0,i_2t_2-l_2}^{\gamma'}\right)(I\otimes A_i')\right]$$

where the least summable terms of

$$E\left(\sum_{j=1}^n\sum_{s=1}^T U_{js}^\theta\right)^2\left(\sum_{s_1,s_2=1}^T \psi_{i_1s_1}^{\gamma,v}\psi_{i_2s_2}^{\gamma,v}\right)\left(k_{0,i_1t_1}^{\gamma}k_{0,i_2t_2}^{\gamma\prime}\otimes k_{0,i_1t_1-l_1}^{\gamma}k_{0,i_2t_2-l_2}^{\gamma'}\right)$$

are of the form

$$\sum_{s_1,\ldots,s_4=1}^T\sum_{j_1,j_2=1}^n E\left[U_{j_1s_3}^\theta U_{j_2s_4}^\theta\psi_{i_1s_1}^{\gamma,v}\psi_{i_2s_2}^{\gamma,v}\right]\left(E\left[k_{0,i_1t_1}^{\gamma}\right]E\left[k_{0,i_2t_2}^{\gamma\prime}\right]\otimes E\left[k_{0,i_1t_1-l_1}^{\gamma}\right]E\left[k_{0,i_2t_2-l_2}^{\gamma'}\right]\right)$$

with

$$\begin{aligned}E\left[U_{j_1s_3}^\theta U_{j_2s_4}^\theta\psi_{i_1s_1}^{\gamma,v}\psi_{i_2s_2}^{\gamma,v}\right] &= \operatorname{Cov}\left(U_{j_1s_3}^\theta,U_{j_2s_4}^\theta\right)\operatorname{Cov}\left(\psi_{i_1s_1}^{\gamma,v},\psi_{i_2s_2}^{\gamma,v}\right)+\operatorname{Cov}\left(U_{j_1s_3}^\theta,\psi_{i_1s_1}^{\gamma,v}\right)\operatorname{Cov}\left(U_{j_2s_4}^\theta,\psi_{i_2s_2}^{\gamma,v}\right)\\&\quad+\operatorname{Cov}\left(U_{j_1s_3}^\theta,\psi_{i_2s_2}^{\gamma,v}\right)\operatorname{Cov}\left(U_{j_2s_4}^\theta,\psi_{i_1s_1}^{\gamma,v}\right)+\operatorname{cum}\left(U_{j_1s_3}^\theta,U_{j_2s_4}^\theta,\psi_{i_1s_1}^{\gamma,v},\psi_{i_2s_2}^{\gamma,v}\right).\end{aligned}$$

As always, the sum over the fourth order cumulant term is of lower order. The largest term then is

$$\begin{aligned}&\sum_{s_1,\ldots,s_4=1}^T\sum_{j_1,j_2=1}^n\operatorname{Cov}\left(U_{j_1s_3}^\theta,U_{j_2s_4}^\theta\right)\operatorname{Cov}\left(\psi_{i_1s_1}^{\gamma,v},\psi_{i_2s_2}^{\gamma,v}\right)\\&= O\left(T^2n\right)O\left(\sum_{j=1}^n\sum_{u=-\infty}^\infty\operatorname{Cov}\left(U_{j1}^\theta,U_{j1-u}^\theta\right)\sum_{u=-\infty}^\infty\operatorname{Cov}\left(\psi_{i,1}^{\gamma,v},\psi_{i1-u}^{\gamma,v}\right)\right)\end{aligned}$$

when $i_1=i_2$ and zero otherwise while

$$\sum_{s_1,\ldots,s_4=1}^T\sum_{j_1,j_2=1}^n\operatorname{Cov}\left(U_{j_1s_3}^\theta,\psi_{i_1s_1}^{\gamma,v}\right)\operatorname{Cov}\left(U_{j_2s_4}^\theta,\psi_{i_2s_2}^{\gamma,v}\right)=O\left(T^2\right)$$

for all i_1 and i_2 and the same holds for the sum over $\operatorname{Cov}\left(U_{j_1s_3}^\theta,\psi_{i_2s_2}^{\gamma,v}\right)\operatorname{Cov}\left(U_{j_2s_4}^\theta,\psi_{i_1s_1}^{\gamma,v}\right)$. This implies that $E\left\|T_{14,1,2}\right\|^2=O\left(n^{-2}m^2/T^2\right)$ which implies that $T_{14,1,2}=O_p\left(n^{-1}T^{-1}m\right)$.

Finally, consider $T_{14,1,1}$ where

$$E\left\|T_{14,1,1}\right\|^2 = \frac{1}{n^2}\sum_{i_1,i_2=1}^n\operatorname{tr}E\left[(I\otimes A_i)\frac{1}{T^6}\sum_{l_1,l_2=-m}^m\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,\ldots,s_4=1}^T\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\psi_{i_2s_3}^{\gamma,v}\psi_{i_2s_4}^{\gamma,v}\cdots\times\left(k_{0,i_1t_1}^{\gamma}k_{0,i_2t_2}^{\gamma\prime}\otimes k_{0,i_1t_1-l_1}^{\gamma}k_{0,i_2t_2-l_2}^{\gamma'}\right)(I\otimes A_i')\right]$$

where the largest contribution to the expectation over the multiple sums comes from terms of the form

$$E\left[\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\psi_{i_2s_3}^{\gamma,v}\psi_{i_2s_4}^{\gamma,v}\right]\left(E\left[k_{0,i_1t_1}^{\gamma}\right]E\left[k_{0,i_2t_2}^{\gamma\prime}\right]\otimes E\left[k_{0,i_1t_1-l_1}^{\gamma}\right]E\left[k_{0,i_2t_2-l_2}^{\gamma'}\right]\right)$$

where

$$\begin{aligned}E\left[\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\psi_{i_2s_3}^{\gamma,v}\psi_{i_2s_4}^{\gamma,v}\right] &= \operatorname{Cov}\left(\psi_{i_2s_3}^{\gamma,v},\psi_{i_2s_4}^{\gamma,v}\right)\operatorname{Cov}\left(\psi_{i_1s_1}^{\gamma,v},\psi_{i_1s_2}^{\gamma,v}\right)+\operatorname{Cov}\left(\psi_{i_2s_4}^{\gamma,v},\psi_{i_1s_1}^{\gamma,v}\right)\operatorname{Cov}\left(\psi_{i_2s_3}^{\gamma,v},\psi_{i_2s_2}^{\gamma,v}\right)\\&\quad+\operatorname{Cov}\left(\psi_{i_2s_4}^{\gamma,v},\psi_{i_1s_2}^{\gamma,v}\right)\operatorname{Cov}\left(\psi_{i_2s_4}^{\gamma,v},\psi_{i_1s_2}^{\gamma,v}\right)+\operatorname{cum}\left(\psi_{i_1s_1}^{\gamma,v},\psi_{i_1s_2}^{\gamma,v},\psi_{i_2s_3}^{\gamma,v},\psi_{i_2s_4}^{\gamma,v}\right).\end{aligned}$$

with the dominating term

$$\sum_{s_1,\ldots,s_4=1}^T\operatorname{Cov}\left(\psi_{i_2s_3}^{\gamma,v},\psi_{i_2s_4}^{\gamma,v}\right)\operatorname{Cov}\left(\psi_{i_1s_1}^{\gamma,v},\psi_{i_1s_2}^{\gamma,v}\right)=O\left(T^2\right)$$

for all i_1 and i_2 such that $E\left\|T_{14,1,1}\right\|^2=O\left(m^2T^{-2}\right)$,

$$E\left\|T_{14,1,1}\right\|^2=\frac{4m^2}{T^2}\left\|\frac{1}{n}\sum_{i_1=1}^n(I\otimes A_i)\left(E\left[k_{0,it}^\gamma\right]\otimes E\left[k_{0,it}^\gamma\right]\right)\sum_{u=-\infty}^\infty\operatorname{Cov}\left(\psi_{i1}^{\gamma,v},\psi_{i1-u}^{\gamma,v}\right)\right\|^2+o\left(\frac{m^2}{T^2}\right)$$

and $T_{14,1,1}=O_p\left(mT^{-1}\right)$. Two other products of covariances are zero except when $i_1=i_2$ which implies that the grand sum over all these terms is $O\left(n^{-1}m^2T^{-2}\right)$ which is of smaller order. ■

Lemma 12 $T_{15} = T_{21} = O_p\left(n^{-1}T^{-1}m\right).$

Proof. Write

$$\operatorname{vec}\left(\left(\left(\hat{\theta}-\theta_0\right)'\otimes I\right)k_{it}^{\theta\theta}\left(\hat{\theta}-\theta_0\right)\right) = \left(\left(\left(\hat{\theta}-\theta_0\right)'\otimes I\right)\otimes\left(\hat{\theta}-\theta_0\right)\right)\operatorname{vec}k_{it}^{\theta\theta}$$

and use $\left\|\operatorname{vec}k_{it}^{\theta\theta}\right\|^2 = \operatorname{tr}\left(\operatorname{vec}k_{it}^{\theta\theta}\left(\operatorname{vec}k_{it}^{\theta\theta}\right)'\right) = \left(\operatorname{vec}k_{it}^{\theta\theta}\right)'\operatorname{vec}k_{it}^{\theta\theta} = \operatorname{tr}\left(k_{it}^{\theta\theta}k_{it}^{\theta\theta\prime}\right) = \left\|k_{it}^{\theta\theta}\right\|^2$ such that

$$\|T_{15}\| \leq \left\|\hat{\theta}-\theta_0\right\|^2 \frac{1}{n}\sum_{i=1}^n \frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2} \|I\otimes A_i\| \|k_{it-l}\| \left\|k_{it}^{\theta\theta}\right\| \|I\| = O_p\left(n^{-1}T^{-1}m\right)$$

because $\left\|k_{it}^{\theta\theta}\right\| \leq \sup_{(\theta,\gamma)}\left\|\partial\left(\operatorname{vec}\partial k\left(x_{it};\theta,\gamma\right)/\partial\theta'\right)/\partial\theta'\right\| \leq M(x_{it})$ by Condition 4. The term T_{21} has the same structure as T_{15} and can be analyzed in the same way. ■

Lemma 13 $T_{16} = T_{22} = O_p\left(n^{-1/2}T^{-9/10}\right) + O_p\left(n^{-1/2}T^{-15/10}m\right).$

Proof.

$$\|T_{16}\| = \frac{1}{T^{4/10}}\left\|\widehat{\theta}-\theta_0\right\| \max_i \frac{\sqrt{T}\left\|(\widehat{\gamma}_i-\gamma_{i0})\right\|}{T^{1/10}} \frac{1}{n}\sum_{i=1}^n \|I\otimes A_i\| \left\|\frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\|$$

where $\left\|\widehat{\theta}-\theta_0\right\| = O_p\left(n^{-1/2}T^{-1/2}\right),$

$$\max_i \frac{\sqrt{T}\left\|(\widehat{\gamma}_i-\gamma_{i0})\right\|}{T^{1/10}} = o_p(1)$$

from Lemma 21 and

$$\left\|\frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\| = O_p(1) + O_p\left(mT^{-1/2}\right)$$

by the same arguments as in proof of Lemma 14. Then,

$$T_{16} = O_p\left(n^{-1/2}T^{-9/10}\right) + O_p\left(n^{-1/2}T^{-15/10}m\right)$$

■

Lemma 14 $T_{17} = T_{23} = o_p\left(n^{-1/2}T^{-12/5}m\right) + o_p\left(n^{-1/2}T^{-9/10}\right)$

Proof.

$$\|T_{17}\| \leq \left\|\widehat{\theta}-\theta_0\right\| \max_i \frac{\left|\widehat{\gamma}_i^{\epsilon}(0)\right|}{T^{1/10}} \frac{1}{n}\sum_{i=1}^n \|I\otimes A_i\| \left\|\frac{1}{T^{3/2-1/10}}\sum_{l=-m}^m \sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\|$$

where $\left\|\widehat{\theta}-\theta_0\right\| = O_p(n^{-1/2}T^{-1/2})$ and $\max_i \frac{|\widehat{\gamma}_i^{\epsilon}(0)|}{T^{1/10}} = o_p(1)$. Furthermore

$$E\left\|\frac{1}{T}\sum_{l=-m}^m \sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\|^2 = \frac{1}{T^2}\sum_{l_1,l_2=-m}^m \sum_{t_1,t_2=r_1}^{r_2} E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes k_{it_1}^{\gamma\theta}k_{it_2}^{\gamma\theta\prime}\right)$$

Let $\bar{k}_{it}^{\gamma\theta} = k_{it}^{\gamma\theta} - E\left[k_{it}^{\gamma\theta}\right]$ such that

$$\begin{aligned} E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes k_{it_1}^{\gamma\theta}k_{it_2}^{\gamma\theta\prime}\right) &= E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes \bar{k}_{it_1}^{\gamma\theta}\bar{k}_{it_2}^{\gamma\theta\prime}\right) + E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes \bar{k}_{it_1}^{\gamma\theta}E\left[k_{it_2}^{\gamma\theta\prime}\right]\right) \\ &\quad + E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes E\left[k_{it_1}^{\gamma\theta}\right]\bar{k}_{it_2}^{\gamma\theta\prime}\right) - E\left(k_{it_1-l_1}k_{it_2-l_2}'\otimes E\left[k_{it_1}^{\gamma\theta}\right]E\left[k_{it_2}^{\gamma\theta\prime}\right]\right). \end{aligned}$$

The terms of the form $E\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes E\left[k^{\gamma\theta}_{it_1}\right]\bar{k}^{\gamma\theta\prime}_{it_2}\right)$ only contain third order cumulant terms which are of lower order when summed over t_1,t_2. The largest terms involving $E\left(k_{it_1-l_1}k'_{it_2-l_2}\otimes\bar{k}^{\gamma\theta}_{it_1}\bar{k}^{\gamma\theta\prime}_{it_2}\right)$ are

$$\begin{aligned}&\left\|\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\operatorname{Cov}\left(k_{it_1-l_1},k_{it_2-l_2}\right)\otimes\operatorname{Cov}\left(\bar{k}^{\gamma\theta}_{it_1},\bar{k}^{\gamma\theta}_{it_2}\right)\right\|\\ \leq\ &\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left\|\operatorname{Cov}\left(k_{it_1-l_1},k_{it_2-l_2}\right)\right\|\left\|\operatorname{Cov}\left(\bar{k}^{\gamma\theta}_{it_1},\bar{k}^{\gamma\theta}_{it_2}\right)\right\|=O\left(m^2/T\right)\end{aligned}$$

because

$$\left\|\operatorname{Cov}\left(\bar{k}^{\gamma\theta}_{it_1},\bar{k}^{\gamma\theta}_{it_2}\right)\right\|\leq 2E\left\|k^{\gamma\theta}_{i1}\right\|^2\leq E\left|M(x_{it})\right|^2<\infty$$

by Condition 4. Next

$$\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}E\left[k_{it_1-l_1}\otimes\bar{k}^{\gamma\theta}_{it_1}\right]E\left[k'_{it_2-l_2}\otimes\bar{k}^{\gamma\theta\prime}_{it_2}\right]=O\left(1\right)$$

because

$$E\left[k_{it_1-l_1}\otimes\bar{k}^{\gamma\theta}_{it_1}\right]\leq Ca^{\frac{\delta}{2+\delta}|l_1|}$$

by the mixing ineqaulity. Finally,

$$\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}E\left[k_{it_1-l_1}k'_{it_2-l_2}\right]\otimes E\left[k^{\gamma\theta}_{it_1}\right]E\left[k^{\gamma\theta\prime}_{it_2}\right]=O(m^2/T)$$

such that $\|T_{17}\|=O_p(n^{-1/2}T^{-1/2})o_p(1)O\left(T^{-2/5}\right)\left(O_p\left(m/T^{-1/2}\right)+O_p\left(1\right)\right)=o_p\left(n^{-1/2}T^{-12/5}m\right)+o_p\left(n^{-1/2}T^{-9/10}\right).$
■

Lemma 15 $T_{18}=T_{24}=o_p\left(T^{-13/10}m\right)+o_p\left(T^{-8/10}\right).$

Proof.

$$\|T_{18}\|\leq\max_i\frac{\left|\widehat{\gamma}^{\epsilon}_i\left(0\right)\right|}{T^{1/10}}\max_i\frac{\sqrt{T}\left|\widehat{\gamma}_i-\gamma_{i0}\right|}{T^{1/10}}\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\left\|I\otimes A_i\right\|\left\|\frac{1}{T^{2-2/10}}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k^{\gamma\gamma}_{it}\right)\right\|$$

where $\max_i\frac{|\widehat{\gamma}^{\epsilon}_i(0)|}{T^{1/10}}=o_p\left(1\right)$, $\max_i\frac{\sqrt{T}|\widehat{\gamma}_i-\gamma_{i0}|}{T^{1/10}}=o_p\left(1\right)$ by Lemma 21 and

$$E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k^{\gamma\gamma}_{it}\right)\right\|^2=O(1)+O\left(m^2/T\right)$$

by the same arguments as in the proof of Lemma 14. Then

$$T_{18}=o_p\left(T^{-13/10}m\right)+o_p\left(T^{-8/10}\right).$$

■

Lemma 16 $T_{19}=T_{25}=o_p\left(n^{-1/2}T^{-18/10}m\right)+o_p\left(n^{-1/2}T^{-13/10}\right)$

Proof.

$$\|T_{19}\|\leq\frac{1}{T^{8/10}}\left\|\widehat{\theta}-\theta_0\right\|\max_i\sup_{\epsilon\in\left[0,1/\sqrt{T}\right]}\frac{\left|\widehat{\gamma}^{\epsilon\epsilon}_i\left(\epsilon\right)\right|}{T^{2/10}}\frac{1}{n}\sum_{i=1}^{n}\left\|I\otimes A_i\right\|\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k^{\gamma\theta}_{it}\right)\right\|$$

where $\left\|\widehat{\theta}-\theta_0\right\| = O_p\left(n^{-1/2}T^{-1/2}\right),$

$$\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \frac{\left|\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon)\right|}{T^{2/10}} = o_p(1)$$

by Lemma 21 and

$$\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\| = O_p(1) + O_p\left(mT^{-1/2}\right)$$

by the same arguments as in the proof of Lemma 14. Then the result follows. ■

Lemma 17 $T_{20} = T_{26} = o_p\left(T^{-12/10}\right) + o_p\left(mT^{-17/10}\right).$

Proof.

$$\begin{aligned}\|T_{20}\| &\leq \frac{1}{T^{12/10}}\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \frac{\left|\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon)\right|}{T^{2/10}} \max_i \frac{\sqrt{T}\left|\widehat{\gamma}_i-\gamma_{i0}\right|}{T^{1/10}} \\ &\quad\times\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\|I\otimes A_i\|\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{it}^{\gamma\theta}\right)\right\|\end{aligned}$$

where the result follows again by Lemma 21 and the same argument as in the proof of Lemma 14. ■

Lemma 18 *The terms* $T_{27},...,T_{36} = o_p(m/T).$

Proof. The proof follows by the same arguments as the proof of Lemmas 9-11. ■

Lemma 19 $ET_4T_{14}' = (\operatorname{vec}\mathfrak{T}_1)(\operatorname{vec}\mathfrak{T}_2)' + o\left(m^2/T^2\right).$

Proof. Only consider the largest order terms of T_4 and T_{14},

$$T_{4,1} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s=1}^{T}(I\otimes A_i)\,\psi_{is}^{\gamma,v}\left(k_{it-l}\otimes k_{0,it}^{\gamma}\right)$$

and

$$T_{14,1,1} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^3}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\sum_{s_1,s_2=1}^{T}(I\otimes A_i)\,\psi_{is_1}^{\gamma,v}\psi_{is_2}^{\gamma,v}\operatorname{vec}\left(k_{0,it}^{\gamma}k_{0,it-l}^{\gamma'}\right)$$

such that

$$\begin{aligned}ET_{4,1}T_{14,1,1}' &= \frac{1}{n^2}\sum_{i_1,i_2=1}^{n}\frac{1}{T^5}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,..,s_3=1}^{T}(I\otimes A_{i_1}) \\ &\quad\times E\left[\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\psi_{i_2s_3}^{\gamma,v}\left(k_{i_2t_2-l_2}k_{0,i_1t_1}^{\gamma}\otimes k_{0,i_2t_2}^{\gamma}k_{0,i_1t_1-l_1}^{\gamma'}\right)\right](I\otimes A_{i_2}')\end{aligned}$$

where the least summable terms of

$$E\left[\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\psi_{i_2s_3}^{\gamma,v}\left(k_{i_2t_2-l_2}k_{0,i_1t_1}^{\gamma}\otimes k_{0,i_2t_2}^{\gamma}k_{0,i_1t_1-l_1}^{\gamma'}\right)\right]$$

are of the form

$$\begin{aligned}&\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\sum_{s_1,..,s_3=1}^{T}E\left[\psi_{i_1s_1}^{\gamma,v}\psi_{i_1s_2}^{\gamma,v}\right]\left(E\left[\psi_{i_2s_3}^{\gamma,v}k_{i_2t_2-l_2}\right]E\left[k_{0,i_1t_1}^{\gamma}\right]\otimes E\left[k_{0,i_2t_2}^{\gamma}\right]E\left[k_{0,i_1t_1-l_1}^{\gamma'}\right]\right) \\ =\;& O(m^2T^3)\end{aligned}$$

such that

$$E\left[T_{4,1}T'_{14,1,1}\right] = O\left(m^2/T^2\right).$$

It then follows that $ET_{4,1} = \frac{m}{T}\mathfrak{T}_1 + o(m/T)$ and $ET_{14,1,1} = \frac{m}{T}\mathfrak{T}_2 + o(m/T)$ and also

$$ET_{4,1}T'_{14,1,1} = \frac{m^2}{T^2}\mathfrak{T}_1\mathfrak{T}'_2 + o\left(m^2/T^2\right).$$

■

Lemma 20 *Assume that Condition 2 holds and $m/T \to 0$ as $m, T \to \infty$. Then the following bounds hold for the spectral matrix $\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2} k_{it-l} \otimes k_{0,it}^{\gamma}$:*

$$\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes \bar{k}_{0,it}^{\gamma} - E\left[k_{it-l}\otimes \bar{k}_{0,it}^{\gamma}\right]\right) = O_p(mT^{-1}),$$

$$\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes E\left[\bar{k}_{0,it}^{\gamma}\right]\right) = O_p(mT^{-1/2}),$$

and

$$\sup_i E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left(k_{it-l}\otimes k_{0,it}^{\gamma}\right)\right\|^2 = O(1) + O(mT^{-1/2}).$$

Furthermore, the same bounds hold for the vector $\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2} k_{0,it-l}^{\gamma} \otimes k_{it-l}$.

Proof. Let $\bar{k}_{0,it}^{\gamma} = k_{0,it}^{\gamma} - E\left[k_{0,it}^{\gamma}\right]$. Then

$$\begin{aligned}
\left(E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{it-l}\otimes k_{0,it}^{\gamma}\right]\right\|^2\right)^{1/2} &\leq \left(E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{it-l}\otimes \bar{k}_{0,it}^{\gamma}\right]\right\|^2\right)^{1/2} \\
&\quad + \left(E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{it-l}\otimes E\left[k_{0,it}^{\gamma}\right]\right]\right\|^2\right)^{1/2} \\
&= O(1) + O(mT^{-1/2})
\end{aligned}$$

To see this note that

$$\begin{aligned}
(24) \qquad & E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2} k_{it-l}\otimes \bar{k}_{0,it}^{\gamma}\right\|^2 \\
= {} & \frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2} E\left[k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}_{0,it_1}^{\gamma}\bar{k}_{0,it_2}^{\gamma\prime}\right] \\
= {} & \frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(E\left[k_{it_1-l_1}k'_{it_2-l_2}\right]\otimes E\left[\bar{k}_{0,it_1}^{\gamma}\bar{k}_{0,it_2}^{\gamma\prime}\right]\right) \\
& + \frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(E\left[k_{it_1-l_1}\bar{k}_{0,it_2}^{\gamma\prime}\right]\otimes E\left[\bar{k}_{0,it_1}^{\gamma}k'_{it_2-l_2}\right]\right)K_{pp} \\
& + \frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\operatorname{vec}\left(E\left[k_{it_1-l_1}\bar{k}_{0,it_1}^{\gamma\prime}\right]\right)\operatorname{vec}\left(E\left[\bar{k}_{0,it_2}^{\gamma}k'_{it_2-l_2}\right]\right)' \\
& + \frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\operatorname{cum}\left(k_{it_1-l_1}, k'_{it_2-l_2}, \bar{k}_{0,it_1}^{\gamma}, \bar{k}_{0,it_2}^{\gamma\prime}\right)
\end{aligned}$$

where the last equality follows from Kuersteiner (2005, Corollary 4.3), K_{pp} is the commutation matrix of Magnus and Neudecker (1988) and $\operatorname{cum}\left(k_{it_1-l_1}, k'_{it_2-l_2}, \bar{k}_{0,it_1}^{\gamma}, \bar{k}_{0,it_2}^{\gamma\prime}\right)$ is the matrix of fourth order cumulants of the of the elements of the matrix $k_{it_1-l_1}k'_{it_2-l_2}\otimes \bar{k}^{\gamma}_{0,it_1}\bar{k}^{\gamma\prime}_{0,it_2}$. By previous arguments the cumulant term is of smaller order while the term

$$\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\text{vec}\left(E\left[k_{it_1-l_1}\bar{k}^{\gamma\prime}_{0,it_1}\right]\right)\text{vec}\left(E\left[\bar{k}^{\gamma}_{0,it_2}k'_{it_2-l_2}\right]\right)'=O(1)$$

where this bound holds uniformly in i because of the uniform mixing and moment conditions and

$$\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(E\left[k_{it_1-l_1}k'_{it_2-l_2}\right]\otimes E\left[\bar{k}^{\gamma}_{0,it_1}\bar{k}^{\gamma\prime}_{0,it_2}\right]\right)=O\left(\frac{m}{T}\right)$$

Next consider

$$\begin{aligned}&E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[k_{it-l}\otimes E\left[k^{\gamma}_{0,it}\right]\right]\right\|^2\\ =\;&\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(E\left[k_{it_1-l_1}k'_{it_2-l_2}\right]\otimes E\left[k^{\gamma}_{0,it_1}\right]E\left[k^{\gamma\prime}_{0,it_2}\right]\right)\\ =\;&O(m^2T^{-1}).\end{aligned}$$

Finally for the last part of the Lemma note that for

$$\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\bar{k}^{\gamma}_{0,it-l}\otimes k_{it-l}$$

the analysis goes through as before while

$$\begin{aligned}&E\left\|\frac{1}{T}\sum_{l=-m}^{m}\sum_{t=r_1}^{r_2}\left[E\left[k^{\gamma}_{0,it-l}\right]\otimes k_{it-l}\right]\right\|^2\\ =\;&\frac{1}{T^2}\sum_{l_1,l_2=-m}^{m}\sum_{t_1,t_2=r_1}^{r_2}\left(E\left[k^{\gamma}_{0,it_1-l_1}\right]E\left[k^{\gamma\prime}_{0,it_2-l_2}\right]\otimes E\left[k_{it_1}k'_{it_2}\right]\right)\\ =\;&O(m^2T^{-1})\end{aligned}$$

as before. ■

B.1 Expansion for $\widehat{\gamma}_i$

Let

$$\widehat{\gamma}_i(\epsilon)\equiv\underset{a}{\text{argmax}}\int\psi\left(x_{it};\widehat{\theta}(\epsilon),a\right)dF_i(\epsilon)$$

From the first order condition

$$0=\int V_i\left(\widehat{\theta}(\epsilon),\widehat{\gamma}_i(\epsilon)\right)dF_i(\epsilon).$$

Using the same arguments as earlier, we are looking for the expansion

$$\widehat{\gamma}_i(\epsilon)-\gamma_{i0}=\frac{1}{\sqrt{T}}\widehat{\gamma}^{\epsilon}_i(0)+\frac{1}{2T}\widehat{\gamma}^{\epsilon\epsilon}_i(0)+\frac{1}{6T^{3/2}}\widehat{\gamma}^{\epsilon\epsilon\epsilon}_i(\tilde{\epsilon})$$

for some $\tilde{\epsilon}\in\left[0,T^{-1/2}\right]$. Let

$$v_i(\cdot,\epsilon)\equiv V_i\left(\theta\left(F(\epsilon)\right),\gamma_i\left(F_i(\epsilon)\right)\right).$$

The first order condition may be written as

$$0=\int v_i(\cdot,\epsilon)\,dF_i(\epsilon)$$

Differentiating repeatedly with respect to ϵ, we obtain

$$(25)\quad 0 = \int \frac{dv_i(\cdot,\epsilon)}{d\epsilon} dF_i(\epsilon) + \int v_i(\cdot,\epsilon)\, d\Delta_{iT}$$

$$(26)\quad 0 = \int \frac{d^2 v_i(\cdot,\epsilon)}{d\epsilon^2} dF_i(\epsilon) + 2\int \frac{dv_i(\cdot,\epsilon)}{d\epsilon} d\Delta_{iT}$$

$$(27)\quad 0 = \int \frac{d^3 v_i(\cdot,\epsilon)}{d\epsilon^3} dF_i(\epsilon) + 3\int \frac{d^2 v_i(\cdot,\epsilon)}{d\epsilon^2} d\Delta_{iT}$$

where $\Delta_{iT} \equiv \sqrt{T}\left(\widehat{F}_i - F_i\right)$.

B.2 $\widehat{\gamma}^{\epsilon}(0)$

Because

$$\frac{dv_i(\cdot,\epsilon)}{d\epsilon} = \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i}\frac{\partial\gamma_i}{\partial\epsilon}$$

we may rewrite (25) as

$$0 = \int \left(\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i}\frac{\partial\gamma_i}{\partial\epsilon}\right) dF_i(\epsilon) + \int v_i(\cdot,\epsilon)\, d\Delta_{iT}$$

Evaluating at $\epsilon = 0$ we obtain

$$(28)\quad \widehat{\gamma}_i^{\epsilon}(0) = -\left(E\left[V_i^{\gamma}\right]\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial\psi(x_{it};\theta_0,\gamma_{i0})}{\partial\gamma} + E\left[V_i^{\theta}\right]\theta^{\epsilon}(0)\right)$$

where $\theta^{\epsilon}(0)$ is defined in Hahn and Kuersteiner (2004). It also follows that

$$(29)\quad \widehat{\gamma}_i^{\epsilon}(\epsilon) = -\left(\int \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} dF_i(\epsilon)\right)^{-1}\left[\int\left(\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\right) dF_i(\epsilon)\,\theta^{\epsilon}(\epsilon) + \int v_i(\cdot,\epsilon)\, d\Delta_{iT}\right].$$

Next, consider

$$\begin{aligned}\frac{d^2 v_i(\cdot,\epsilon)}{d\epsilon^2} &= \frac{\partial\theta'}{\partial\epsilon}\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial^2\theta}{(\partial\epsilon)^2} + 2\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i}\frac{\partial\theta}{\partial\epsilon}\frac{\partial\gamma_i}{\partial\epsilon} \\ &\quad + \frac{\partial^2 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^2}\left(\frac{\partial\gamma_i}{\partial\epsilon}\right)^2 + \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i}\frac{\partial^2\gamma_i}{(\partial\epsilon)^2}\end{aligned}$$

such that $\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon)$ is characterized by

$$\begin{aligned}0 &= \theta^{\epsilon}(\epsilon)'\int\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'} dF_i(\epsilon)\,\theta^{\epsilon}(\epsilon) + \int\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'} dF_i(\epsilon)\,\theta^{\epsilon\epsilon}(\epsilon) \\ &\quad + 2\int\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i} dF_i(\epsilon)\,\theta^{\epsilon}(\epsilon)\,\widehat{\gamma}_i^{\epsilon}(\epsilon) + \int\frac{\partial^2 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^2} dF_i(\epsilon)\left(\widehat{\gamma}_i^{\epsilon}(\epsilon)\right)^2 \\ (30)\qquad &\quad + \int\frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} dF_i(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon) + \int\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'} d\Delta_{iT}\theta^{\epsilon}(\epsilon) + \int\frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} d\Delta_{iT}\widehat{\gamma}_i^{\epsilon}(\epsilon).\end{aligned}$$

It follows that

$$\begin{aligned}(31)\quad \widehat{\gamma}_i^{\epsilon\epsilon}(0) &= -\left(E\left[V_i^{\gamma}\right]\right)^{-1}\Big(\theta^{\epsilon}(0)' E\left[V_i^{\theta\theta}\right]\theta^{\epsilon}(0) + E\left[V_i^{\theta}\right]\theta^{\epsilon\epsilon}(0) \\ &\quad + 2E\left[V_i^{\gamma\theta}\right]\theta^{\epsilon}(0)\,\widehat{\gamma}_i^{\epsilon}(0) + E\left[V_i^{\gamma\gamma}\right]\left(\widehat{\gamma}_i^{\epsilon}(0)\right)^2 \\ &\quad + \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\partial\psi(x_{it};\theta_0,\gamma_{i0})/\partial\gamma\partial\theta' - E\left[\partial\psi(x_{it};\theta_0,\gamma_{i0})/\partial\gamma\partial\theta'\right]\right)\theta^{\epsilon}(0) \\ &\quad + \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\partial\psi(x_{it};\theta_0,\gamma_{i0})/(\partial\gamma)^2 - E\left[\partial\psi(x_{it};\theta_0,\gamma_{i0})/(\partial\gamma)^2\right]\right)\widehat{\gamma}_i(0)\Big).\end{aligned}$$

Next consider

$$
\begin{aligned}
\frac{d^3 v_i(\cdot,\epsilon)}{d\epsilon^3} &= \left(\frac{\partial\theta}{\partial\epsilon}' \otimes \frac{\partial\theta}{\partial\epsilon}'\right) \frac{\partial \operatorname{vec}\left(\frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'}\right)}{\partial\theta'} \frac{\partial\theta}{\partial\epsilon} + 2\frac{\partial^2\theta}{(\partial\epsilon)^2}' \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'} \frac{\partial\theta}{\partial\epsilon} \\
&+ \frac{\partial\theta}{\partial\epsilon}' \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'} \frac{\partial^2\theta}{(\partial\epsilon)^2} + \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'} \frac{\partial^3\theta}{(\partial\epsilon)^3} \\
&+ 2\frac{\partial\theta}{\partial\epsilon}' \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'\partial\gamma_i} \frac{\partial\theta}{\partial\epsilon} \frac{\partial\gamma_i}{\partial\epsilon} \\
&+ 2\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i} \frac{\partial^2\theta}{(\partial\epsilon)^2} \frac{\partial\gamma_i}{\partial\epsilon} + 2\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i} \frac{\partial\theta}{\partial\epsilon} \frac{\partial^2\gamma_i}{(\partial\epsilon)^2} \\
&+ \frac{\partial\theta}{\partial\epsilon}' \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\theta(\partial\gamma_i)^2} \left(\frac{\partial\gamma_i}{\partial\epsilon}\right)^2 + 2\frac{\partial^2 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^2} \frac{\partial^2\gamma_i}{(\partial\epsilon)^2} \\
&+ \frac{\partial\theta}{\partial\epsilon}' \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\gamma_i} \frac{\partial^2\gamma_i}{(\partial\epsilon)^2} + \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} \frac{\partial^3\gamma_i}{(\partial\epsilon)^3} \\
&+ \frac{\partial\theta}{\partial\epsilon}' \frac{\partial^3 v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'\partial\gamma_i} \frac{\partial\theta}{\partial\epsilon} \frac{\partial\gamma_i}{\partial\epsilon} + \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\gamma_i\partial\theta'} \frac{\partial^2\theta}{(\partial\epsilon)^2} \frac{\partial\gamma_i}{\partial\epsilon} \\
&+ 2\frac{\partial^3 v_i(\cdot,\epsilon)}{\partial\theta'(\partial\gamma_i)^2} \frac{\partial\theta}{\partial\epsilon} \left(\frac{\partial\gamma_i}{\partial\epsilon}\right)^2 + \frac{\partial^3 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^3} \left(\frac{\partial\gamma_i}{\partial\epsilon}\right)^3 \\
&+ \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial^2\gamma_i} \frac{\partial\gamma_i}{\partial\epsilon} \frac{\partial^2\gamma_i}{(\partial\epsilon)^2}
\end{aligned}
$$

such that $\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\epsilon)$ is characterized by

$$
\begin{aligned}
0 &= \left(\theta^\epsilon(\epsilon)' \otimes \theta^\epsilon(\epsilon)'\right) \int \frac{\partial \operatorname{vec}\left(\frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'}\right)}{\partial\theta'} dF_i(\epsilon)\,\theta^\epsilon(\epsilon) + 2\frac{\partial^2\theta}{(\partial\epsilon)^2}' \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'} dF_i(\epsilon)\,\theta^\epsilon(\epsilon) \\
&+ \theta^\epsilon(\epsilon)' \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'} dF_i(\epsilon)\,\theta^{\epsilon\epsilon}(\epsilon) + \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'} dF_i(\epsilon)\,\theta^{\epsilon\epsilon\epsilon}(\epsilon) \\
&+ 2\theta^\epsilon(\epsilon)' \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'\partial\gamma_i} dF_i(\epsilon)\,\theta^\epsilon(\epsilon)\,\widehat{\gamma}_i^\epsilon(\epsilon) \\
&+ 2\int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i} dF_i(\epsilon)\,\theta^{\epsilon\epsilon}(\epsilon)\,\widehat{\gamma}_i^\epsilon(\epsilon) + 2\int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'\partial\gamma_i} dF_i(\epsilon)\,\theta^\epsilon(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon) \\
&+ \theta^\epsilon(\epsilon)' \int \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\theta(\partial\gamma_i)^2} dF_i(\epsilon)\left(\widehat{\gamma}_i^\epsilon(\epsilon)\right)^2 + 2\int \frac{\partial^2 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^2} dF_i(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon) \\
&+ \theta^\epsilon(\epsilon)' \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\theta\partial\gamma_i} dF_i(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon) + \int \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} dF_i(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\epsilon) \\
&+ \theta^\epsilon(\epsilon)' \int \frac{\partial^3 v_i(\cdot,\epsilon)}{\partial\theta\partial\theta'\partial\gamma_i} dF_i(\epsilon)\,\theta^\epsilon(\epsilon)\,\widehat{\gamma}_i^\epsilon(\epsilon) + \int \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial\gamma_i\partial\theta'} dF_i(\epsilon)\,\theta^{\epsilon\epsilon}(\epsilon)\,\widehat{\gamma}_i^\epsilon(\epsilon) \\
&+ 2\int \frac{\partial^3 v_i(\cdot,\epsilon)}{\partial\theta'(\partial\gamma_i)^2} dF_i(\epsilon)\,\theta^\epsilon(\epsilon)\left(\widehat{\gamma}_i^\epsilon(\epsilon)\right)^2 + \int \frac{\partial^3 v_i(\cdot,\epsilon)}{(\partial\gamma_i)^3} dF_i(\epsilon)\left(\widehat{\gamma}_i^\epsilon(\epsilon)\right)^3 \\
&+ \int \frac{\partial^2 v_i(\cdot,\epsilon)}{\partial^2\gamma_i} dF_i(\epsilon)\,\widehat{\gamma}_i^\epsilon(\epsilon)\,\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon) + 3\int \frac{d^2 v_i(\cdot,\epsilon)}{d\epsilon^2} d\Delta_{iT}
\end{aligned}
$$

Lemma 21 *Let Conditions (2),(3),(4) and (5) be satisfied. Then*

$$\begin{aligned}
\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left|\widehat{\gamma}_i^{\epsilon}(\epsilon)\right| > T^{\frac{1}{10}-\upsilon}\right] &= o\left(T^{-1}\right),\\
\Pr\left[\max_i \left|\widehat{\gamma}_i^{\epsilon}(0)\right| > T^{\frac{1}{10}-\upsilon}\right] &= o\left(T^{-1}\right),\\
\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left|\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon)\right| > \left(T^{\frac{1}{10}-\upsilon}\right)^2\right] &= o\left(T^{-1}\right),\\
\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left|\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\epsilon)\right| > \left(T^{\frac{1}{10}-\upsilon}\right)^3\right] &= o\left(T^{-1}\right),\\
\Pr\left[\max_i \left|\sqrt{T}\left(\widehat{\gamma}_i - \gamma_{i0}\right)\right| > T^{1/10-\upsilon}\right] &= o\left(T^{-1}\right),
\end{aligned}$$

for $0 < \upsilon < (100q+120)^{-1}$.

Proof. Note that the last claim follows as a consequence of the first three claims. In order to prove the first claim, we note that

$$\Pr\left[\sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left\|\theta^{\epsilon}(\epsilon)\right\| \geq T^{\frac{1}{10}-\upsilon}\right] = o\left(T^{-1}\right)$$

by a result in Hahn and Kuersteiner (2004). There it is also shown that

$$\begin{aligned}
\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left\|\int\left(\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\right) dF_i(\epsilon) - E\left[\frac{\partial v_i(\cdot,\epsilon)}{\partial\theta'}\right]\right\| > \eta\right] &= o\left(T^{-1}\right),\\
\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left|\int \frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i} dF_i(\epsilon) - E\left[\frac{\partial v_i(\cdot,\epsilon)}{\partial\gamma_i}\right]\right| > \eta\right] &= o\left(T^{-1}\right).
\end{aligned}$$

as well as

$$\Pr\left[\max_i \sup_{\epsilon\in\left[0,1/\sqrt{T}\right]} \left|\int v_i(\cdot,\epsilon)\, d\Delta_{iT}\right| > T^{1/10-\upsilon}\right] = o\left(T^{-1}\right).$$

This proves the result for $\widehat{\gamma}_i^{\epsilon}(\epsilon)$. For $\widehat{\gamma}_i(0)$ the result follows directly from (28) and corresponding results in Hahn and Kuersteiner (2004). For $\widehat{\gamma}_i^{\epsilon\epsilon}(\epsilon)$ and $\widehat{\gamma}_i^{\epsilon\epsilon\epsilon}(\epsilon)$ the result follows from representation (30). ■

C Proofs and Auxiliary Results for Bandwidth Selection

Proof of Theorem 1. By Lemma 1 $\left\|\hat{\mathfrak{T}}_1 - \mathfrak{T}_1\right\| = O_p\left(T^{-1/2}\right)$, $\left\|\hat{\mathfrak{T}}_2 - \mathfrak{T}_2\right\| = O_p\left(T^{-1/2}\right)$. Then

$$\begin{aligned}
\left\|\hat{\mathfrak{T}}(m) - \mathfrak{T}(m)\right\| &\leq \left\|\hat{\mathfrak{T}}_0(m) - \mathfrak{T}_0(m) + \left[\left(\hat{\mathfrak{T}}_1 - \mathfrak{T}_1\right) + \left(\hat{\mathfrak{T}}_2 - \mathfrak{T}_2\right)\right]\frac{m}{T}\right\|\\
&= \left\|\hat{\mathfrak{T}}_0(m) - \mathfrak{T}_0(m) + \frac{m}{T}O_p\left(T^{-1/2}\right)\right\|
\end{aligned}$$

and

$$\begin{aligned}\left\|\hat{\mathfrak{T}}_0(m)-\mathfrak{T}_0(m)\right\| &\le \frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i\right\|\left(\sum_{j=m}^{k_{\max}}\left\|\hat{\Gamma}_{i,h}(j)-\Gamma_{i,j}^{kk}\right\|\right)\\ &+\frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i-A_i\right\|\left(\sum_{j=m}^{k_{\max}}\left\|\Gamma_{i,j}^{kk}\right\|\right)\\ &+\frac{1}{n}\sum_{i=1}^n\left\|A_i\right\|\left(\sum_{j=k_{\max}+1}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right)\end{aligned}$$

where for the first term

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i\right\|\left(\sum_{j=m}^{k_{\max}}\left\|\hat{\Gamma}_{i,h}(j)-\Gamma_{i,j}^{kk}\right\|\right) &\le o_p(1)\frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i\right\|\left(\sum_{j=m}^{k_{\max}}\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\right)\\ &= o_p(1)\frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i\right\|\left(\sum_{l=m}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\right)\end{aligned}$$

by Lemma 36 and Condition 9. By Lemma 25 it follows that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\left\|\hat{A}_i-A_i\right\|\left(\sum_{j=m}^{k_{\max}}\left\|\Gamma_{i,j}^{kk}\right\|\right) &\le \max_i\left\|\hat{A}_i-A_i\right\|\frac{1}{n}\sum_{i=1}^n\left(\sum_{j=m}^{k_{\max}}\left\|\Gamma_{i,j}^{kk}\right\|\right)\\ &\le o_p(1)\frac{1}{n}\sum_{i=1}^n\left(\sum_{j=m}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right).\end{aligned}$$

By the conditions of the Theorem it follows that $\sum_{j=k_{\max}+1}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|/\sum_{j=m}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\to 0$ as m, n, $T\to\infty$. This establishes that

$$\left\|\hat{\mathfrak{T}}_0(m)-\mathfrak{T}_0(m)\right\|\le o_p(1)\frac{1}{n}\sum_{i=1}^n\left\|A_i\right\|\left(\sum_{j=m}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\|\right)$$

which by Condition 9 is equvalent to $\left\|\hat{\mathfrak{T}}_0(m)-\mathfrak{T}_0(m)\right\|\le o_p(1)\left\|\mathfrak{T}_0(m)\right\|$. It therefore follows that

$$\left\|\hat{\mathfrak{T}}(m)-\mathfrak{T}(m)\right\|\le o_p(1)\left(\left\|\mathfrak{T}_0(m)\right\|+\frac{m}{T}\right)$$

where $o_p(1)\mathfrak{T}(m)\le o_p(1)\left(\left\|\mathfrak{T}_0(m)\right\|+\frac{m}{T}\right)$. ■

C.1 Auxiliary Results

Lewis and Reinsel (1985) and Hannan and Deistler (1988) show that estimators of the VAR(h) approximation to the VAR(∞) model converge in distribution to a normal distribution when evaluated in an appropriate sense. Their results assume that the VAR(∞) representation of the observed process has martingale difference innovations. In this Appendix similar results are established for the panel case when the underlying data are general mixing processes, and the data to which the VAR's are fit are based on first stage estimates.

We consider a modification of Lahiri's (1992) Lemma 5.1, which is stated for bounded zero mean random variables.

Lemma 22 *Assume that* $\{W_t, t=1,2,\ldots\}$ *is a stationary, mixing sequence with* $E[W_t]=0$ *and* $E\left[|W_t|^{4r+2\delta}\right]<\infty$ *for any positive integer* r*, some* $\delta>0$ *and all* t*. Let* $\mathcal{A}_t=\sigma(W_t,W_{t-1},W_{t-2},...)$*,* $\mathcal{B}_t=\sigma(W_t,W_{t+1},W_{t+2},...)$ *and* $\alpha(\tau)=\sup_t\sup_{A\in\mathcal{A}_t,B\in\mathcal{B}_{t+\tau}}|P(A\cap B)-P(A)P(B)|$*. Define* $Y_t=W_tW_{t-j}$ *for* j *fixed. Then, for any* τ *such that* $1\le\tau<C(r)n$

$$E\left[\left(\sum_{i=1}^n Y_i\right)^{2r}\right]\le C(r)E\left[|Y_i|^{2r+\delta}\right]\left[n^r\tau^{2r}+n^{2r}\alpha(\tau-j)^{\frac{\delta}{2r+\delta}}\right]$$

where $C(r)$ *is a constant that depends on* r*.*

Proof. The proof follows in the same way as the proof of Lemma 4 in Hahn and Kuersteiner (2004) by noting that Y_i and $Y_{i+\tau}$ contain variables $W_i, W_{i-j}, W_{i-\tau}$ and $W_{i-j+\tau}$ which are separated at least by $\tau - j$ periods. The remaining arguments of the proof are unchanged. ■

Lemma 23 *Suppose that, for each i, $\{\xi_{it,j}, t = 1, 2, \ldots\}$ is a mixing sequence with $E[\xi_{it,j}] = 0$ for all i, t. Let $\mathcal{A}_t^i = \sigma(\xi_{it,j}, \xi_{it-1,j}, \xi_{it-2,j}, \ldots)$ and $\mathcal{B}_t^i = \sigma(\xi_{it,j}, \xi_{it+1,j}, \xi_{it+2,j}, \ldots)$. Let*

$$\alpha_i(\tau - j) = \sup_t \sup_{A \in \mathcal{A}_t^i, B \in \mathcal{B}_{t+\tau}^i} |P(A \cap B) - P(A) P(B)|.$$

Assume that $\sup_i |\alpha_i(\tau)| \leq Ca^\tau$ for some a such that $0 < a < 1$ and some $0 < C < \infty$. We assume that $\{\xi_{it,j}, t = 1, 2, 3, \ldots\}$ are independent across i. We also assume that $n = O(T)$. Finally, assume that $E\left[|\xi_{it,j}|^{10+\delta}\right] < \infty$ for some $\delta > 0$. We then have

$$\Pr\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\frac{1}{T} \sum_{t=1}^T \xi_{it,j}\right| > \eta\right] = o\left(T^{-1}\right)$$

for every $\eta > 0$ and h such that $h \to \infty$ and $h/T^{1/4-\varepsilon} \to 0$ for some $\varepsilon > 0$. Now assume in addition that $E\left[|\xi_{it}|^{10q+12+\delta}\right] < \infty$ for some $\delta > 0$ and some integer $q \geq 1$. Then,

$$\Pr\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_{it,j}\right| > \eta T^{\frac{3}{10}-\upsilon}\right] = o\left(T^{-q+1/4}\right)$$

for every $\eta > 0$ and $0 < \upsilon < (100q + 120)^{-1}$.

Proof. By the Markov inequality

$$\Pr\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\frac{1}{T} \sum_{t=1}^T \xi_{it,j}\right| > \eta\right] = \Pr\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\sum_{t=1}^T \xi_{it,j}\right|^{10} > \eta^{10} T^{10}\right] \leq T^{-10} \eta^{-10} E\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\sum_{t=1}^T \xi_{it,j}\right|^{10}\right]$$

and by a an inequality for the Orlicz norm of a maximum of random variables (van der Vaart and Wellner, 1996, p.96) one obtains

$$E\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\sum_{t=1}^T \xi_{it,j}\right|^{10}\right] \leq nh \max_{i,j} E\left[\left|\sum_{t=1}^T \xi_{it,j}\right|^{10}\right].$$

From Lemma (22) it follows that

$$E\left[\left|\sum_{t=1}^T \xi_{it,j}\right|^{10}\right] \leq CE\left[|\xi_{it,j}|^{10+\delta}\right] \left(T^4 \tau^{10} + T^{10} \alpha_i(\tau - j)^{\frac{\delta}{10+\delta}}\right)$$

for any τ such that $1 \leq \tau \leq CT$. Choose $\tau = T^\gamma$ and some γ such that $0 < \gamma \leq 1$. Then, for $\gamma < \frac{1}{5}$, and h such that $h \to \infty$ and $h/\tau \to 0$

$$\begin{aligned} \Pr\left[\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left|\frac{1}{T} \sum_{t=1}^T \xi_{it,j}\right| > \eta\right] &\leq nT^{-10} \eta^{-10} C\left(T^{5+10\gamma} + T^{10} a^{\frac{\delta}{10+\delta}(T^\gamma - h)}\right) \\ &= O(T^{-4+10\gamma} + T^{10} a^{\frac{\delta}{10+\delta}(T^\gamma - h)}) = o\left(T^{-3/2}\right). \end{aligned}$$

For the second part of the Lemma, note that by previous arguments

$$\begin{aligned}
& T^{q-1/4}\Pr\left[\max_{0\le j\le h}\max_{1\le i\le n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}\right|>\eta T^{\frac{3}{10}-\upsilon}\right] \\
= \; & T^{q-1/4}\Pr\left[\max_{0\le j\le h}\max_{1\le i\le n}\left|\sum_{t=1}^{T}\xi_{it}\right|>\eta T^{\frac{4}{5}-\upsilon}\right] \\
= \; & T^{q-1/4}\Pr\left[\max_{0\le j\le h}\max_{1\le i\le n}\left|\sum_{t=1}^{T}\xi_{it}\right|^{10q+12}>\eta^{(10q+12)}T^{\left(\frac{3}{5}-\upsilon\right)(10q+12)}\right] \\
\le \; & T^{q-1/4}T^{-\left(\frac{4}{5}-\upsilon\right)(10q+12)}\eta^{-(10p+12)}E\left[\max_{0\le j\le h}\max_{1\le i\le n}\left|\sum_{t=1}^{T}\xi_{it}\right|^{10q+12}\right] \\
= \; & O\left[T^{-1/4-7q-\frac{48}{5}+10\upsilon q+12\upsilon}nh\cdot C\left(T^{5q+6+\gamma(10q+12)}+T^{(10q+12)}a^{\frac{\delta}{10q+12+\delta}T^{\gamma}-h}\right)\right] \\
= \; & o\left(T^{-1/4+10\upsilon q+12\upsilon}\right)=o(1)
\end{aligned}$$

for $\gamma>0$ sufficiently small. ■

Lemma 24 *Let $\xi(x_{it},\phi)$ be a function indexed by the parameter $\phi\in\Phi$ where Φ is a convex subset of $\mathbb{R}^p$. Assume that there exists a function $\mathbf{M}(x_{it})$ such that $|\xi(x_{it},\phi_1)-\xi(x_{it},\phi_2)|\le\mathbf{M}(x_{it})\|\phi_1-\phi_2\|$ for all $\phi_1,\phi_2\in\Phi$ and $\sup_\phi|\xi(x_{it},\phi)|\le\mathbf{M}(x_{it})$. For each i, let x_{it} be a α-mixing process with exponentially decaying mixing coefficients $\underline{\alpha}_i(\tau)$ satisfying $\sup_i|\underline{\alpha}_i(\tau)|\le Ca^\tau$ for some a such that $0<a<1$ and some $0<C<\infty$. Let q denote a positive integer such that $q\ge\frac{2p+10}{4}$, where $p=\dim\phi$. We also assume that $E\left[|\mathbf{M}(x_{it})|^{10q+12+\delta}\right]<\infty$ for some $\delta>0$. Finally, assume that $n=O(T)$. We then have*

$$\Pr\left[\max_{0\le j\le h}\max_{0\le i\le n}\sup_{\phi}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\xi(x_{it},\phi)\,\xi(x_{it-j},\phi)-E\left[\xi(x_{it},\phi)\,\xi(x_{it-j},\phi)\right]\right)\right|>T^{\frac{3}{10}-\upsilon}\right]=o\left(T^{-1}\right)$$

for $0<\upsilon<(100q+120)^{-1}$ and h such that $h\to\infty$ and $h/T^{1/5-\upsilon-\varepsilon}\to0$ for some $\varepsilon>0$. Here, $\{\phi_i\}$ is an arbitrary nonstochastic sequence in Φ.

Proof. First note that

$$\begin{aligned}
\left|\xi(x_{it},\phi_1)\,\xi(x_{it-j},\phi_1)-\xi(x_{it},\phi_2)\,\xi(x_{it-j},\phi_2)\right| \le \; & \left|\xi(x_{it},\phi_1)-\xi(x_{it},\phi_2)\right|\left|\xi(x_{it-j},\phi_1)\right| \\
& +\left|\xi(x_{it},\phi_2)\right|\left|\xi(x_{it-j},\phi_1)-\xi(x_{it-j},\phi_2)\right| \\
\le \; & 2\mathbf{M}(x_{it})\mathbf{M}(x_{it-j})\|\phi_1-\phi_2\|
\end{aligned}$$

and

$$\left|E\left[\xi(x_{it},\phi_1)\,\xi(x_{it-j},\phi_1)\right]-E\left[\xi(x_{it},\phi_2)\,\xi(x_{it-j},\phi_2)\right]\right|\le 2E\left[\mathbf{M}(x_{it})\mathbf{M}(x_{it-j})\right]\|\phi_1-\phi_2\|.$$

Now define

$$\zeta(x_{it},x_{it-j},\phi_i)=\xi(x_{it},\phi_i)\,\xi(x_{it-j},\phi_i)-E\left[\xi(x_{it},\phi_i)\,\xi(x_{it-j},\phi_i)\right]$$

such that

$$\left|\zeta(x_{it},x_{it-j},\phi_1)-\zeta(x_{it},x_{it-j},\phi_2)\right|\le 2\left(\mathbf{M}(x_{it})\mathbf{M}(x_{it-j})+E\left[\mathbf{M}(x_{it})\mathbf{M}(x_{it-j})\right]\right)\|\phi_1-\phi_2\|$$

Note that we have

$$T\Pr\left[\max_j\max_i\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta(x_{it},x_{it-j},\phi)\right|>T^{\frac{3}{10}-\upsilon}\right]\le T\sum_{j=0}^{h}\sum_{i=1}^{n}\Pr\left[\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta(x_{it},x_{it-j},\phi)\right|>T^{\frac{3}{10}-\upsilon}\right]$$

Adapting an argument in Hall and Horowitz (1996) we chose $\varepsilon > 0$ and divide Φ into subsets $\Phi_1, \Phi_2, ...\Phi_{M(\varepsilon)}$ such that $\|\phi_1 - \phi_2\| < \frac{\varepsilon}{\sqrt{T}}$ whenever ϕ_1, ϕ_2 are in the same subset Φ_k. Then

$$\Pr\left[\sup_{\phi\in\Phi}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi\right)\right| > T^{\frac{3}{10}-\upsilon}\right] \leq \sum_{k=1}^{M(\varepsilon)}\Pr\left[\sup_{\phi\in\Phi_k}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi\right)\right| > T^{\frac{3}{10}-\upsilon}\right]$$

Then, for some $\phi_k \in \Phi_k$ and any $\phi \in \Phi_k$

$$\begin{aligned}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi\right)\right| &\leq \left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi_k\right)\right| + \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left|\zeta\left(x_{it}, x_{it-j}, \phi\right) - \zeta\left(x_{it}, x_{it-j}, \phi_k\right)\right| \\ &\leq \left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi_k\right)\right| + 2\left|\frac{\varepsilon}{T}\sum_{t=1}^{T}\left(M\left(x_{it}\right)M\left(x_{it-j}\right) - E\left[M\left(x_{it}\right)M\left(x_{it-j}\right)\right]\right)\right| \\ &\quad +4\varepsilon E\left[M\left(x_{it}\right)M\left(x_{it-j}\right)\right]\end{aligned}$$

such that

$$\begin{aligned}\Pr\left[\sup_{\phi\in\Phi_k}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi\right)\right| > T^{\frac{3}{10}-\upsilon}\right] &\leq \Pr\left[\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi_k\right)\right| > \frac{T^{\frac{3}{10}-\upsilon}}{3}\right] \\ &\quad + \Pr\left[\frac{1}{T}\left|\sum_{t=1}^{T}\left(M\left(x_{it}\right)M\left(x_{it-j}\right) - E\left[M\left(x_{it}\right)M\left(x_{it-j}\right)\right]\right)\right| > \frac{T^{\frac{3}{10}-\upsilon}}{3}\right].\end{aligned}$$

By Lemma (23), it follows that both terms on the right are of order $o\left(T^{-q+1/4}\right)$, where the orders are uniform in i. Since $M\left(\varepsilon\right) = O\left(T^{p/2}\right)$ it follows that

$$T\Pr\left[\max_j\max_i\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta\left(x_{it}, x_{it-j}, \phi_i\right)\right| > T^{\frac{3}{10}-\upsilon}\right] = hnT\cdot o\left(T^{-q+1/4}\right)\cdot O\left(T^{p/2}\right) = o\left(T^{9/4-q+1/4+p/2}\right) = o\left(1\right).$$

■

Corollary 2 *Under the assumptions of Lemma (24) it follows that for any* $\eta > 0$

$$\Pr\left[\max_{0\leq j\leq h}\max_{0\leq i\leq n}\left|\frac{1}{T}\sum_{t=1}^{T}\left(\xi\left(x_{it}, \phi_i\right)\xi\left(x_{it-j}, \phi_i\right) - E\left[\xi\left(x_{it}, \phi_i\right)\xi\left(x_{it-j}, \phi_i\right)\right]\right)\right| > T^{-\frac{1}{5}-\upsilon}\right] = o\left(T^{-1}\right)$$

Lemma 25 *Let* A_i *be defined in 4. Let* $\hat{A}_i$ *be an estimator of* A_i *where* $a_{i,1}$ *and* $a_{i,2}$ *as defined in 3 are replaced by*

$$\hat{a}_{i,1} = -\frac{\hat{E}\left[U_i^{\gamma_i\gamma_i}\left(x_{it};\theta,\gamma_i\right)\right]}{2\left(\hat{E}\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]\right)^2}, \hat{a}_{i,2} = \frac{1}{\hat{E}\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right]}$$

where $\hat{E}\left[U_i^{\gamma_i\gamma_i}\left(x_{it};\theta,\gamma_i\right)\right] = T^{-1}\sum_{t=1}^{T}U_i^{\gamma_i\gamma_i}\left(x_{it};\hat{\theta},\hat{\gamma}_i\right)$, $\hat{E}\left[\frac{\partial V_i(x_{it};\theta,\gamma_i)}{\partial\gamma_i}\right] = T^{-1}\sum_{t=1}^{T}\frac{\partial V_i\left(x_{it};\hat{\theta},\hat{\gamma}_i\right)}{\partial\gamma_i}$ *and* $\widehat{\mathcal{I}}=\frac{1}{n}\sum_{i=1}^{n}\widehat{\mathcal{I}}_i$ *with* $\widehat{\mathcal{I}}_i = -T^{-1}\sum_{t=1}^{T}\frac{\partial U_i\left(x_{it};\hat{\theta},\hat{\gamma}_i\right)}{\partial\theta'}$. *Then,* $\max_i\left\|\hat{A}_i - A_i\right\| = o_p\left(1\right)$.

Proof. The proof follows from Lemma 3 in Hahn and Kuersteiner (2004). ■

Lemma 26 *Assume that Condition 8 holds. Then, for all* i,

$$\Gamma_{ij}^{kk} = O\left(\sum_{l=j}^{\infty}\|B_{il}\|\right)$$

and $\left\|\Gamma_{ij}^{kk}\right\| \leq \sum_{l=j}^{\infty}\|B_{il}\|$.

Proof. Note that $\Gamma_{ij}^{kk} = \sum_{l=0}^{\infty} B_{il}\Sigma_v B'_{il+j}$ such that

$$\begin{aligned} \left\|\Gamma_{ij}^{kk}\right\| &\leq \|\Sigma_v\| \textstyle\sum_{l=0}^{\infty} \|B_{il}\| \|B_{il+j}\| \\ &\leq \|\Sigma_v\| \left(\textstyle\sum_{l=0}^{\infty} \|B_{il}\|^2\right)^{1/2} \left(\textstyle\sum_{l=0}^{\infty} \|B_{il+j}\|^2\right)^{1/2} \\ &\leq \|\Sigma_v\| \left(\textstyle\sum_{l=0}^{\infty} \|B_{il}\|^2\right)^{1/2} \textstyle\sum_{l=0}^{\infty} \|B_{il+j}\| \end{aligned}$$

where the second inequality uses Hölder's Inequality (see Magnus and Neudecker, 1988, p.230). By the Wold representation theorem $\sum_{l=0}^{\infty} \|B_{il}\|^2$ is bounded such the result follows. ■

Lemma 27 *Assume that Condition 8 holds. Then, for all i,*

$$\|B_{ij}\| \leq C \sum_{l=j}^{\infty} \left\|\Gamma_{il}^{kk}\right\|.$$

where $C = \max\left(1, \sum_{j=1}^{\infty} \|\pi_{ij}\|\right) < \infty$. Also,

$$\sum_{k=0}^{\infty} \|B_{ij+k}\| \leq (1+C) \sum_{k=0}^{\infty} \left\|\Gamma_{ij+k}^{kk\prime}\right\|.$$

Proof. Note that $k_{it} = \sum_{j=1}^{\infty} \pi_{ij} k_{it-j} + v_{it}$. Then, for $j > 0$

$$E\left(k_{it}k'_{it+j}\right) - \sum_{l=1}^{\infty} \pi_{il} E\left(k_{it-l}k'_{it+j}\right) = E\left(v_{it}k'_{it+j}\right)$$

or, after using $k'_{it+l} = \sum_{j=0}^{\infty} v'_{it-j+l} B'_{ij}$,

$$\Gamma_{i(-j)}^{kk} - \sum_{l=1}^{\infty} \pi_{il} \Gamma_{i-(j+l)}^{kk} = B'_{ij}$$

which is equivalent to $\Gamma_{ij}^{kk\prime} - \sum_{l=1}^{\infty} \pi_{il} \Gamma_{i(j+l)}^{kk'} = B_{ij}$.Then,

$$\begin{aligned} \|B_{ij}\| &\leq \left\|\Gamma_{ij}^{kk\prime}\right\| + \sum_{l=1}^{\infty} \|\pi_{il}\| \left\|\Gamma_{i(j+l)}^{kk}\right\| \\ &\leq C \sum_{l=j}^{\infty} \left\|\Gamma_{il}^{kk}\right\|. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{k=0}^{\infty} \|B_{ij+k}\| &\leq \sum_{k=0}^{\infty} \left\|\Gamma_{ij+k}^{kk}\right\| + \sum_{l=1}^{\infty} \|\pi_{il}\| \sum_{k=0}^{\infty} \left\|\Gamma_{i(j+l+k)}^{kk}\right\| \\ &\leq \sum_{k=0}^{\infty} \left\|\Gamma_{ij+k}^{kk\prime}\right\| + \sum_{k=0}^{\infty} \left\|\Gamma_{i(j+k)}^{kk}\right\| \sum_{l=1}^{\infty} \|\pi_{il}\| = (1+C) \sum_{k=0}^{\infty} \left\|\Gamma_{ij+k}^{kk\prime}\right\|. \end{aligned}$$

■

Lemma 28 *Let $\pi_{i1,h}, ..., \pi_{ih,h}$ be the projection coefficients of projecting $k_{it} - Ek_{it}$ onto $k_{it-1} - Ek_{it-1}, ..., k_{it-h} - Ek_{it-h}$. Let $\pi_{ih}(L) = I - \pi_{i1,h}L - ... - \pi_{ih,h}L^h$. Then, for all $h > h_0$ and some $h_0 < \infty$ all the roots of $\pi_{ih}(L)$ are outside the unit cricle and $\sum_{l=1}^{h} \|\pi_{il,h}\| < \infty$ for all h.*

Proof. Choose h arbitrary. By assumption, $B_i(L) = I + \sum_{j=1}^{\infty} B_{ij} L^j$ has no roots in or on the unit circle. Therefore $\pi_i(z) = B_i(z)^{-1}$ exists on a radius of convergence $|z| \leq 1 + \varepsilon/2$ for some $\varepsilon > 0$. Then, for $|z| \leq 1$

$$\|\pi_{ih}(z) - \pi_i(z)\| \leq \sum_{j=1}^{h} \|\pi_{ij,h} - \pi_{ij}\| \leq \sum_{j=h+1}^{\infty} \|\pi_{ij}\| \to 0$$

as $h \to \infty$ by Deistler and Hannan (1988, Theorem 6.6.12). From

$$0 < \|\pi_i(z)\| \leq \|\pi_{ih}(z) - \pi_i(z)\| + \|\pi_{ih}(z)\|$$

for all $|z| \leq 1$ it follows that for all $h > h_0$ and some $h_0 < \infty$ that $\pi_{ih}(z)$ has no zeros on or inside the unit circle. The process $v_{it,h}$ has the representation $v_{it,h} = \pi_{ih}(L) B_i(L) v_{it}$. The last part of the Lemma follows from

$$\|\pi_{ih}(z)\| \leq \|\pi_{ih}(z) - \pi_i(z)\| + \|\pi_i(z)\| < \infty$$

for all h such that $\|\pi_{ih}(z)\|$ is bounded. ■

Lemma 29 *Let $\pi_{ih}(L)$ be as defined in Lemma 28. Then, $B_{ih}(L) = \pi_{ih}(L)^{-1}$ exists for all $h > h_0$ and some $h_0 < \infty$ and for $z \in \mathbb{C}$, $B_{ih}(z)$ has no zeros in or on the unit circle.*

Proof. By Lemma 28 and Brockwell and Davis (1991, p.85) $\pi_{ih}(z)^{-1}$ has a convergent power series expansion for all $|z| < 1 + \varepsilon/2$ for some $\varepsilon > 0$, all $h > h_0$ and some $h_0 < \infty$. This implies that the coefficients $B_{ij,h}$ of $B_{ih}(L)$ satisfy $\sum_{j=0}^{\infty} \|B_{ij,h}\| < \infty$. Then, for all $|z| < 1 + \varepsilon/2$

$$1 = B_{ih}(z)\pi_{ih}(z) \leq \|B_{ih}(z)\| \|\pi_{ih}(z)\|$$

such that

$$0 < \frac{1}{\|\pi_{ih}(z)\|} \leq \|B_{ih}(z)\|$$

for all $|z| \leq 1$ because $\infty > \|\pi_{ih}(z)\| > 0$ by Lemma 28. This shows that $\|B_{ih}(z)\|$ has no zeros on $|z| \leq 1$ for $h > h_0$ and some $h_0 < \infty$. ■

Lemma 30 *Assume that $h, k_{\max} \to \infty$ such that $T^{\upsilon}/h \to 0$, $T^{\upsilon}/k_{\max} \to 0$, $h = o\left(T^{1/5-\upsilon}\right)$ and $k_{\max} = o(T^{1/5-\upsilon})$ where υ is defined in Lemma 23. Then, uniformly in j for $0 < j < h$ it follows that*

$$\max_{0 \leq j \leq h} \max_{1 \leq i \leq n} \left\| \hat{\Gamma}_{i,j}^{kk} - \Gamma_{i,j}^{kk} \right\| = O_p\left(T^{-1/5+\upsilon}\right)$$

where $\hat{\Gamma}_{i,j}^{kk} = \frac{1}{T}\sum_{t=j+1}^{T} \hat{k}_{it}\hat{k}_{it-j}'$.

Proof. Let $k_{it}^{\theta} = k_{it}^{\theta}\left(x_{it}; \tilde{\theta}, \tilde{\gamma}_i\right)$ and $k_{it}^{\gamma} = k_{it}^{\gamma}\left(x_{it}; \tilde{\theta}, \tilde{\gamma}_i\right)$ where $\tilde{\theta}, \tilde{\gamma}_i$ are such that $\left\|\tilde{\theta} - \theta_0\right\| \leq \left\|\widehat{\theta} - \theta_0\right\|$, $\left\|\tilde{\theta}' - \theta_0\right\| \leq \left\|\widehat{\theta} - \theta_0\right\|$. Use a first order mean value expansion to obtain

$$\widehat{k}_{it} - k_{it} = k_{it}^{\theta}\left(\widehat{\theta} - \theta_0\right) + k_{it}^{\gamma}\left(\widehat{\gamma}_i - \gamma_{i0}\right).$$

Then, for $\tilde{\Gamma}_{i,j}^{kk} = \frac{1}{T}\sum_{t=j+1}^{T} k_{it}k_{it-j}'$ consider

$$(32) \qquad \max_{1 \leq j \leq h} \max_{1 \leq i \leq n} \left\| \hat{\Gamma}_{i,j}^{kk} - \tilde{\Gamma}_{i,j}^{kk} \right\|$$
$$\leq \max_{1 \leq j \leq h} \max_{1 \leq i \leq n} \left\| \frac{1}{T} \sum\nolimits_{t=j+1}^{T} \left(\hat{k}_{it} - k_{it}\right)\left(\hat{k}_{it-j} - k_{it-j}\right)' \right\|$$
$$+ \max_{1 \leq j \leq h} \max_{1 \leq i \leq n} \left\| \frac{1}{T} \sum\nolimits_{t=j+1}^{T} k_{it}\left(\hat{k}_{it-j} - k_{it-j}\right)' \right\| + \frac{1}{n}\sum\nolimits_{i=1}^{n} \left\| \frac{1}{T}\sum\nolimits_{t=j+1}^{T} \left(\hat{k}_{it} - k_{it}\right) k_{it-j}' \right\|$$

where

$$\begin{aligned}
&\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(\hat{k}_{it}-k_{it}\right)k'_{it-j}\right\| \\
\le\ & \left\|\widehat{\theta}-\theta_0\right\|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it-j}\otimes k_{it}^{\theta}-E\left[k_{it-j}\otimes k_{it}^{\theta}\right]\right)\right\| \\
&+\left\|\widehat{\theta}-\theta_0\right\|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}E\left[k_{it-j}\otimes k_{it}^{\theta}\right]\right\| \\
&+\max_{1\le i\le n}\left|\widehat{\gamma}_i-\gamma_{i0}\right|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it-j}\otimes k_{it}^{\gamma}-E\left[k_{it-j}\otimes k_{it}^{\gamma}\right]\right)\right\| \\
&+\max_{1\le i\le n}\left|\widehat{\gamma}_i-\gamma_{i0}\right|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}E\left[k_{it-j}\otimes k_{it}^{\gamma}\right]\right\|
\end{aligned}$$

where by Lemmas 21,24 and Corollary 2 it follows that

$$\max_{1\le i\le n}\left|\widehat{\gamma}_i-\gamma_{i0}\right|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it-j}\otimes k_{it}^{\gamma}-E\left[k_{it-j}\otimes k_{it}^{\gamma}\right]\right)\right\|=O_p(T^{-1/5+\upsilon})$$

is bounded uniformly in j and i with the same result holding for $\left\|\frac{1}{T}\sum_{t=j+1}^{T}\left(k_{it-j}\otimes k_{it}^{\theta}-E\left[k_{it-j}\otimes k_{it}^{\theta}\right]\right)\right\|$. Also, by the mixing inequality and Condition 3 it follows that $\max_{1\le i\le n}\left\|\frac{1}{T}\sum_{t=j+1}^{T}E\left[k_{it-j}\otimes k_{it}^{\gamma}\right]\right\|=O\left(T^{-1}\right)$ with the same result holding for $\max_{1\le i\le n}\left\|\frac{1}{T}\sum_{t=j+1}^{T}E\left[k_{it-j}\otimes k_{it}^{\theta}\right]\right\|$. It then follows that

$$\max_{1\le i\le n}\left|\widehat{\gamma}_i-\gamma_{i0}\right|\sum\nolimits_{j=0}^{h}n^{-1}\sum\nolimits_{i=1}^{n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}k_{it-j}\otimes k_{it}^{\gamma}\right\|=O_p\left(T^{-2/5+\upsilon}\right)O_p\left(T^{-1/2}h\right)=o_p\left(T^{-7/10+\upsilon}\right)$$

and

$$\left\|\widehat{\theta}-\theta_0\right\|\max_{1\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it-j}\otimes k_{it}^{\theta}-E\left[k_{it-j}\otimes k_{it}^{\theta}\right]\right)\right\|=O_p\left(n^{-1/2}T^{-1/2}\right)O_p\left(T^{-1/5+\upsilon}\right)=o_p\left(T^{-6/5+\upsilon}\right).$$

The remaining terms of (32) can be analyzed in the same way and are of order not larger than $o_p\left(T^{-1/5+\upsilon}\right)$. Next turn to

$$\begin{aligned}
\max_{0\le j\le h}\max_{1\le i\le n}\left\|\tilde{\Gamma}_{i,j}^{kk}-\Gamma_{i,j}^{kk}\right\| \le\ & \max_{1\le i\le n}\max_{0\le j\le h}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it}k'_{it-j}-E\left[k_{it}k'_{it-j}\right]\right)\right\| \\
&+\max_{1\le i\le n}\left(\left\|\Gamma_{i,0}^{kk}\right\|+\sum\nolimits_{j=1}^{h}\frac{|j|}{T}\left\|\Gamma_{i,j}^{kk}\right\|\right)
\end{aligned}$$

where for some constant c, uniformly in h

$$\sum\nolimits_{j=1}^{h}\frac{|j|}{T}\left\|\Gamma_{i,j}^{kk}\right\|\le c\sum\nolimits_{j=1}^{\infty}\frac{|j|}{T}\left(a^{\frac{\delta}{2+\delta}}\right)^{j}=O(T^{-1})$$

while

$$\max_{0\le j\le h}\max_{1\le i\le n}\left\|\frac{1}{T}\sum\nolimits_{t=j+1}^{T}\left(k_{it}k'_{it-j}-E\left[k_{it}k'_{it-j}\right]\right)\right\|=O_p\left(T^{-1/5+\upsilon}\right)$$

by Corollary 2. ■

Lemma 31 *Assume that* $h\to\infty$ *such that* $T^{\upsilon}/h\to 0$*, and* $h=T^{1/5-\delta-\upsilon}$ *where* υ *is defined in Lemma 23. Then*

$$\max_{1\le i\le n}\left\|\hat{\Gamma}_{ih}^{-1}-\Gamma_{ih}^{-1}\right\|_2=o_p(1).$$

where $\|.\|_2$ *is defined as* $\|A\|_2^2=\sup_{l\neq 0}l'A'Al/l'l$.

Proof. First, note that from Lewis and Reinsel (1985, p.396) it follows that for two matrices A and B, the inequalities $\|AB\|^2 \leq \|A\|_2^2 \|B\|^2$ and $\|AB\|^2 \leq \|A\|^2 \|B\|_2^2$ hold. By Condition 8 it follows that $\left\|\Gamma_{ih}^{-1}\right\|_2$ is uniformly bounded in h by a positive constant c_Γ (see Berk, 1974, p. 493 for the univariate case). Then, as in Lewis and Reinsel (1985, p. 397) it follows that $\left\|\hat{\Gamma}_{ih}^{-1}\right\|_2 \leq \left\|\Gamma_{ih}^{-1}\right\|_2 + \left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2$ and

$$\left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2 \leq c_\Gamma \left(\left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2 + c_\Gamma\right) \left\|\hat{\Gamma}_{ih} - \Gamma_{ih}\right\|_2$$

such that

$$0 \leq W_{ih,T} = \frac{\left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2}{c_\Gamma \left(\left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2 + c_\Gamma\right)} \leq \left\|\hat{\Gamma}_{ih} - \Gamma_{ih}\right\|_2$$

and

$$\begin{aligned} (33) \quad \max_{1\leq i\leq n} W_{ih,T} &\leq \max_{1\leq i\leq n} \left\|\hat{\Gamma}_{ih} - \Gamma_{ih}\right\|_2 \\ &\leq \max_{1\leq i\leq n} \left(\sum_{j,l=1}^{h} \operatorname{tr}\left(\hat{\Gamma}_{ij}^{kk} - \Gamma_{ij}^{kk}\right)\left(\hat{\Gamma}_{il}^{kk} - \Gamma_{il}^{kk}\right)'\right)^{1/2} \\ &\leq h \max_{1\leq i\leq n} \max_{0\leq j\leq h} \left\|\hat{\Gamma}_{ij}^{kk} - \Gamma_{ij}^{kk}\right\| \\ &= O_p\left(hT^{-1/5+v}\right) = o_p(1) \end{aligned}$$

because $\max_{0\leq j\leq h} \left\|\hat{\Gamma}_{ij}^{kk} - \Gamma_{ij}^{kk}\right\| = O_p\left(T^{-1/5+v}\right)$ by Lemma 30. Note that

$$\left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2 = \frac{c_\Gamma^2 W_{ih,T}}{(1 - W_{ih,T} c_\Gamma)}$$

and thus for any $1 > \varepsilon > 0$,

$$\begin{aligned} P\left(\max_{1\leq i\leq n} \left\|\hat{\Gamma}_{ih}^{-1} - \Gamma_{ih}^{-1}\right\|_2 > \eta\right) &\leq P\left(\max_{1\leq i\leq n} \frac{c_\Gamma^2 W_{ih,T}}{(1-\varepsilon)} > \eta, \max_{1\leq i\leq n} W_{ih,T} < \varepsilon/c_\Gamma\right) \\ &\quad + P\left(\max_{1\leq i\leq n} W_{ih,T} \geq \varepsilon/c_\Gamma\right) \end{aligned}$$

where both terms go to zero by (33). ■

Lemma 32 *Assume that $h, k_{\max} \to \infty$ such that $T^v/h \to 0$, $T^v/k_{\max} \to 0$, $h = o(T^{1/5-v})$ and $k_{\max} = o(T^{1/5-v})$ where v is defined in Lemma 23. Then, uniformly in j for $j < j_0$ such that $j_0 \to \infty$ as $n \to \infty$, $\sum_{l=k_{\max}} \left\|\Gamma_{i,l}^{kk}\right\| / \sum_{l=j}^{\infty} \left\|\Gamma_{i,l}^{kk}\right\| \to 0$ and $\sum_{l=h} \|\pi_{il}\| / \sum_{l=j}^{\infty} \left\|\Gamma_{i,l}^{kk}\right\| \to 0$ as $T \to \infty$ it follows that*

$$\max_{1\leq i\leq n} \left\|\hat{\Sigma}_{vi,h} - \Sigma_{vi,h}\right\| = O_p\left(T^{-1/5+v}\right)$$

$\hat{\Sigma}_{vi,h} = (T-h)^{-1} \sum_{t=h+1}^{T} \hat{v}_{it,h} \hat{v}_{it,h}'$ *where* $\hat{v}_{it,h} = \hat{k}_{it} - \hat{\pi}_{i1,h}\hat{k}_{it-1} - ... - \hat{\pi}_{ih,h}\hat{k}_{it-h}$ *and*

$$(34) \quad \Sigma_{vi,h} = E\left[\left(k_{it} - \pi_{i1,h}k_{it-1} - ... - \pi_{ih,h}k_{it-h}\right)\left(k_{it} - \pi_{i1,h}k_{it-1} - ... - \pi_{ih,h}k_{it-h}\right)'\right].$$

Proof. First consider $\tilde{v}_{it,h} = k_{it} - \hat{\pi}_{i1,h}k_{it-1} - ... - \hat{\pi}_{ih,h}k_{it-h}$ where

$$\begin{aligned} \hat{v}_{it,h} - \tilde{v}_{it,h} &= \hat{k}_{it} - k_{it} - (\hat{\pi}_{i1,h} - \pi_{i1})\left(\hat{k}_{it-1} - k_{it-1}\right) - ... - (\hat{\pi}_{ih,h} - \pi_{ih})\left(\hat{k}_{it-h} - k_{it-h}\right) \\ &\quad - \pi_{i1}\left(\hat{k}_{it-1} - k_{it-1}\right) - ... - \pi_{ih}\left(\hat{k}_{it-h} - k_{it-h}\right) \end{aligned}$$

and

$$\tilde{v}_{it,h} = k_{it} - (\hat{\pi}_{i1,h} - \pi_{i1}) k_{it-1} - ... - (\hat{\pi}_{ih,h} - \pi_{ih}) k_{it-h}$$
$$-\pi_{i1} k_{it-1} - ... - \pi_{ih} k_{it-h}$$

such that

$$(35)\quad \hat{\Sigma}_{vi,h} - \tilde{\Sigma}_{vi,h} = (T-h)^{-1} \sum_{t=h+1}^{T} (\hat{v}_{it,h} - \tilde{v}_{it,h}) (\hat{v}_{it,h} - \tilde{v}_{it,h})' + (T-h)^{-1} \sum_{t=h+1}^{T} \tilde{v}_{it,h} (\hat{v}_{it,h} - \tilde{v}_{it,h})' + (T-h)^{-1} \sum_{t=h+1}^{T} (\hat{v}_{it,h} - \tilde{v}_{it,h}) \tilde{v}'_{it,h}.$$

Let $k_{it}^{\theta} = k_{it}^{\theta}\left(x_{it}; \tilde{\theta}, \tilde{\gamma}_i\right)$ and $k_{it}^{\gamma} = k_{it}^{\gamma}\left(x_{it}; \tilde{\theta}, \tilde{\gamma}_i\right)$ where $\tilde{\theta}, \tilde{\gamma}_i$ are such that $\left\|\tilde{\theta} - \theta_0\right\| \leq \left\|\widehat{\theta} - \theta_0\right\|$, $\left\|\tilde{\theta}' - \theta_0\right\| \leq \left\|\widehat{\theta} - \theta_0\right\|$. Use a first order mean value expansion to obtain

$$\widehat{k}_{it} - k_{it} = k_{it}^{\theta}\left(\widehat{\theta} - \theta_0\right) + k_{it}^{\gamma}\left(\widehat{\gamma}_i - \gamma_{i0}\right).$$

Then

$$(36)\quad (T-h)^{-1} \sum_{t=h+1}^{T} \tilde{v}_{it,h} (\hat{v}_{it,h} - \tilde{v}_{it,h})' = (T-h)^{-1} \sum_{t=h+1}^{T} k_{it} (\hat{v}_{it,h} - \tilde{v}_{it,h})' - (T-h)^{-1} \sum_{t=h+1}^{T} \sum_{j=1}^{h} (\hat{\pi}_{ij,h} - \pi_{ij}) k_{it-j} (\hat{v}_{it,h} - \tilde{v}_{it,h})' - (T-h)^{-1} \sum_{t=h+1}^{T} \sum_{j=1}^{h} \pi_{ij} k_{it-j} (\hat{v}_{it,h} - \tilde{v}_{it,h})'$$

where the first term in (36) is

$$(37)\quad (T-h)^{-1} \sum_{t=h+1}^{T} \operatorname{vec}\left(k_{it} (\hat{v}_{it,h} - \tilde{v}_{it,h})'\right)$$
$$= (T-h)^{-1} \sum_{t=h+1}^{T} \left[\left(k_{it}^{\theta} \otimes k_{it}\right)\left(\widehat{\theta} - \theta_0\right) + \left(k_{it}^{\gamma} \otimes k_{it}\right)\left(\widehat{\gamma}_i - \gamma_{i0}\right)\right]$$
$$+ (T-h)^{-1} \sum_{j=1}^{h} \sum_{t=h+1}^{T} \left[\left((\hat{\pi}_{ij,h} - \pi_{ij}) k_{it-j}^{\theta} \otimes k_{it}\right)\left(\widehat{\theta} - \theta_0\right) + \left((\hat{\pi}_{ij,h} - \pi_{ij}) k_{it-j}^{\gamma} \otimes k_{it}\right)\left(\widehat{\gamma}_i - \gamma_{i0}\right)\right]$$
$$- (T-h)^{-1} \sum_{j=1}^{h} \sum_{t=h+1}^{T} \left[\left(\pi_{ij} k_{it-j}^{\theta} \otimes k_{it}\right)\left(\widehat{\theta} - \theta_0\right) + \left(\pi_{ij} k_{it-j}^{\gamma} \otimes k_{it}\right)\left(\widehat{\gamma}_i - \gamma_{i0}\right)\right]$$

such that

$$\max_i \left\|(T-h)^{-1} \sum_{t=h+1}^{T} \left[\left(k_{it}^{\theta} \otimes k_{it}\right)\left(\widehat{\theta} - \theta_0\right) + \left(k_{it}^{\gamma} \otimes k_{it}\right)\left(\widehat{\gamma}_i - \gamma_{i0}\right)\right]\right\|$$
$$\leq \left\|\widehat{\theta} - \theta_0\right\| \max_i \left((T-h)^{-1} \sum_{t=h+1}^{T} \left\|k_{it}^{\theta}\right\| \left\|k_{it}\right\|\right) + \max_i \left|\widehat{\gamma}_i - \gamma_{i0}\right| \max_i \left((T-h)^{-1} \sum_{t=h+1}^{T} \left\|k_{it}^{\gamma}\right\| \left\|k_{it}\right\|\right)$$

where $\left\|k_{it}^{\theta}\right\| \leq \sup_{(\theta,\gamma)} \left\|\operatorname{vec} \partial k\left(x_{it}; \theta, \gamma\right) / \partial \theta'\right\| \leq M(x_{it})$ and similarly for $\left\|k_{it}^{\gamma}\right\|$ and $\left\|k_{it}\right\|$ by Condition 4. By Lemma 5 in Hahn and Kuersteiner (2004) it follows that $\max_i (T-h)^{-1} \sum_{t=h+1}^{T} \left\|k_{it}^{\theta}\right\| \left\|k_{it}\right\| = O_p(1)$ and $\max_i (T-h)^{-1} \sum_{t=h+1}^{T} \left\|k_{it}^{\gamma}\right\| \left\|k_{it}\right\| = O_p(1)$. By Lemma 21, $\max_i \left|\widehat{\gamma}_i - \gamma_{i0}\right| = o_p\left(T^{-2/5+\upsilon}\right)$ and $\left\|\widehat{\theta} - \theta_0\right\| = O_p\left((nT)^{-1/2}\right)$. For the second term in 37 focus on the part depending on $\left(\widehat{\gamma}_i - \gamma_{i0}\right)$ while the part depending on $\left(\widehat{\theta}-\theta_0\right)$ can be handled in the same way but the argument is slightly easier and the term is of smaller order. Using Lemma 33 where it is shown that $\max_{1\leq i\leq n}\sup_j \|\hat{\pi}_{ij,h}-\pi_{ij}\| = O_p\left(T^{-1/5+v}\right)$ the former can be bounded as

$$
\begin{aligned}
(38)\qquad & \max_i \left\|(T-h)^{-1}\sum\nolimits_{j=1}^{h}\sum\nolimits_{t=h+1}^{T}\left[\left(\left(\hat{\pi}_{ij,h}-\pi_{ij}\right)k_{it-j}^{\gamma}\otimes k_{it}\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)\right]\right\| \\
\leq\ & \max_i \sup_j \|\hat{\pi}_{ij,h}-\pi_{ij}\| \sup_i |\widehat{\gamma}_i-\gamma_{i0}| \sum\nolimits_{j=1}^{h}\max_i \left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}k_{it-j}^{\gamma}\otimes k_{it}-E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\| \\
& +\max_i \sup_j \|\hat{\pi}_{ij,h}-\pi_{ij}\| \sup_i |\widehat{\gamma}_i-\gamma_{i0}| \sum\nolimits_{j=1}^{h}\max_i \left\|E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\| \\
=\ & O_p\left(\sqrt{n/T}hT^{-3/5+v}\right) = O_p\left(hT^{-3/5+v}\right) = o_p\left(T^{-2/5+v}\right)
\end{aligned}
$$

where $\sum_{j=1}^{h}\max_i \left\|E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\| \leq \sum_{j=1}^{h} c\max_i\left(E\left|M\left(x_{it}\right)\right|^{2+\delta}\right)^2\left(a^{\frac{\delta}{2+\delta}}\right)^{|j|}$ for some constant c is uniformly bounded in h which implies that this term is of lower order than the first term after the inequality. Turning to the first term after the inequality in (38), now consider

$$
\begin{aligned}
(39)\qquad & \sum\nolimits_{j=1}^{h} E\left[\max_i \left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}k_{it-j}^{\gamma}\otimes k_{it}-E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\|^2\right] \\
\leq\ & \sum\nolimits_{j=1}^{h} n\max_i\left((T-h)^{-2}\sum\nolimits_{t_1,t_2=h+1}^{T}\operatorname{tr}\operatorname{Cov}\left(k_{it_1-j}^{\gamma}\otimes k_{it_1}, k_{it_2-j}^{\gamma}\otimes k_{it_2}\right)\right) \\
=\ & \sum\nolimits_{j=1}^{h} n\max_i\left((T-h)^{-2}\sum\nolimits_{t_1,t_2=h+1}^{T}\operatorname{tr}\left(E\left[k_{it_1-j}^{\gamma}k_{it_2-j}^{\gamma\prime}\right]\otimes E\left[k_{it_1}k_{it_2}'\right]\right)\right) \\
& +\sum\nolimits_{j=1}^{h} n\max_i\left((T-h)^{-2}\sum\nolimits_{t_1,t_2=h+1}^{T}\operatorname{tr}\left(E\left[k_{it_1-j}^{\gamma}k_{it_2}'\right]\otimes E\left[k_{it_1}k_{it_2-j}^{\gamma\prime}\right]\right)K_{pp}\right) \\
& +\sum\nolimits_{j=1}^{h} n\max_i\left((T-h)^{-2}\sum\nolimits_{t_1,t_2=h+1}^{T}\operatorname{tr}\operatorname{cum}\left(k_{it_1-j}^{\gamma}, k_{it_2-j}^{\gamma}, k_{it_1}, k_{it_2}\right)\right) \\
=\ & O(nh/T)
\end{aligned}
$$

where the inequality follows from van der Vaart and Wellner (1996, p. 96) and for a typical element

$$
\begin{aligned}
\left|E\left[k_{it_1-j}^{\gamma}k_{it_2}'\right]_{l,m}\right| &\leq E\left|\left[k_{it_1-j}^{\gamma}\right]_l\right|^{2+\delta} E\left|\left[k_{it_2}\right]_l\right|^{2+\delta}\left(a^{\frac{\delta}{2+\delta}}\right)^{|t_1-t_2-j|} \\
&\leq E\left|M\left(x_{it}\right)\right|^{2+\delta} E\left|M\left(x_{it}\right)\right|^{2+\delta}\left(a^{\frac{\delta}{2+\delta}}\right)^{|t_1-t_2-j|}
\end{aligned}
$$

which is unifromly bounded by Condition 4. This establishes the order in (38). The largest component of the third term in (37) can be bounded as

$$
\begin{aligned}
& \max_i \left\|(T-h)^{-1}\sum\nolimits_{j=1}^{h}\sum\nolimits_{t=h+1}^{T}\left(\pi_{ij}k_{it-j}^{\gamma}\otimes k_{it}\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)\right\| \\
\leq\ & \max_i\left(\widehat{\gamma}_i-\gamma_{i0}\right)\max_i\sum\nolimits_{j=1}^{h}\|\pi_{ij}\|\left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\left(k_{it-j}^{\gamma}\otimes k_{it}\right)-E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\| \\
& +\max_i\left(\widehat{\gamma}_i-\gamma_{i0}\right)(T-h)^{-1}\sum\nolimits_{j=1}^{h}\sum\nolimits_{t=h+1}^{T}\max_i\left\|E\left(\pi_{ij}k_{it-j}^{\gamma}\otimes k_{it}\right)\right\|
\end{aligned}
$$

where the second part is $O_p\left(T^{-2/5+\upsilon}\right)$ by previous arguments. For the first term note that

$$\begin{aligned}
&\left(E\left(\max_i \left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\left(k_{it-j}^{\gamma}\otimes k_{it}\right)-E\left(k_{it-j}^{\gamma}\otimes k_{it}\right)\right\|\right)^2\right)^{1/2}\\
\leq\ &\left(n\max_i (T-h)^{-2}\sum\nolimits_{t_1,t_2=h+1}^{T}\operatorname{tr}\operatorname{Cov}\left(k_{it_1-j}^{\gamma}\otimes k_{it_1},k_{it_2-j}^{\gamma}\otimes k_{it_2}\right)\right)^{1/2}\\
=\ &O\left(\sqrt{n/T}\right)
\end{aligned}$$

uniformly in j. This implies that

$$\max_i \left\|(T-h)^{-1}\sum\nolimits_{j=1}^{h}\sum\nolimits_{t=h+1}^{T}\left(\pi_{ij}k_{it-j}^{\gamma}\otimes k_{it}\right)\left(\widehat{\gamma}_i-\gamma_{i0}\right)\right\| = O_p\left(T^{-2/5+\upsilon}\right).$$

Now turning to the second term in 36,

$$\begin{aligned}
&\left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\sum\nolimits_{j=1}^{h}\left(\hat{\pi}_{ij,h}-\pi_{ij}\right)k_{it-j}\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)'\right\|\\
\leq\ &\sum\nolimits_{j=1}^{h}\left\|\hat{\pi}_{ij,h}-\pi_{ij}\right\|\left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}k_{it-j}\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)'\right\|\\
=\ &o_p\left(hT^{-1/4}T^{-2/5+\upsilon}\right)=o_p\left(T^{-2/5+\upsilon}\right).
\end{aligned}$$

where $\max_i \left\|(T-h)^{-1}\sum_{t=h+1}^{T}k_{it-j}\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)'\right\| = O_p\left(T^{-2/5+\upsilon}\right)$ uniformly in j by the same arguments as for (37) and $\left\|\hat{\pi}_{ij,h}-\pi_{ij}\right\| = O_p\left(T^{-1/5+\upsilon}\right)$ uniformly in $j=1,...,h$ by Lemma 33. which establishes the order of probability in the last display. The third term in 36 is bounded by

$$\sum\nolimits_{j=1}^{h}\left\|\pi_{ij}\right\|\left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}k_{it-j}\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)'\right\| = O_p\left(T^{-2/5+\upsilon}\right)$$

by the same arguments as before. The term $(T-h)^{-1}\sum_{t=h+1}^{T}\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)\left(\hat{v}_{it,h}-\tilde{v}_{it,h}\right)'$ in (35) can be analyzed in the same way as (36) but is of smaller order. The details are omitted. Next consider

$$v_{it,h}=k_{it}-\pi_{i1,h}k_{it-1}-...-\pi_{ih,h}k_{it-h}$$

and

$$\begin{aligned}
(40)\quad \tilde{\Sigma}_{vi,h}-(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}v_{it,h}v_{it,h}' \ =\ &(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\left(\tilde{v}_{it,h}-v_{it,h}\right)\left(\tilde{v}_{it,h}-v_{it,h}\right)'\\
&+(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}v_{it,h}\left(\tilde{v}_{it,h}-v_{it,h}\right)'\\
&+(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\left(\tilde{v}_{it,h}-v_{it,h}\right)v_{it,h}'
\end{aligned}$$

where $\tilde{v}_{it,h}-v_{it,h}=\left(\pi_{i1,h}-\hat{\pi}_{i1,h}\right)k_{it-1}-...-\left(\pi_{ih,h}-\hat{\pi}_{ih,h}\right)k_{it-h}$. Then

$$\begin{aligned}
&\max_i \left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}\left(\tilde{v}_{it,h}-v_{it,h}\right)v_{it,h}'\right\|\\
\leq\ &\max_i \sum\nolimits_{j_1,j_2=1}^{h}\left\|\hat{\pi}_{ij_1,h}-\pi_{ij_1}\right\|\left\|\pi_{ij_2}\right\|\max_i \left\|(T-h)^{-1}\sum\nolimits_{t=h+1}^{T}k_{it-j_1}k_{it-j_2}'-E\left[k_{it-j_1}k_{it-j_2}'\right]\right\|\\
&+\max_i \sum\nolimits_{j_1,j_2=1}^{h}\left\|\hat{\pi}_{ij_1,h}-\pi_{ij_1}\right\|\left\|\pi_{ij_2}\right\|\max_i E\left[k_{it-j_1}k_{it-j_2}'\right]\\
=\ &O_p\left(T^{-1/5+\upsilon}\right)
\end{aligned}$$

where $\left(E\max_i\left\|(T-h)^{-1}\sum_{t=h+1}^{T}k_{it-j_1}k'_{it-j_2}-E\left[k_{it-j_1}k'_{it-j_2}\right]\right\|^2\right)^{1/2}=O\left(\sqrt{n/T}\right)$ as before, such that the stochastic order term of the previous display follows by the Markov inequality. The remaining terms in 40 can be handled in the same way. Finally, for $\Sigma_{vi,h}=E\left[v_{it,h}v'_{it,h}\right]$, set $\pi_{i0}=I$ such that

$$v_{it,h}v'_{it,h}=\sum_{j_1,j_2=0}^{h}\pi_{ij_1}k_{it-j_1}k'_{it-j_2}\pi_{ij_2}$$

and

$$(T-h)^{-1}\sum_{t=h+1}^{T}v_{it,h}v'_{it,h}-\Sigma_{vi,h}=\sum_{j_1,j_2=0}^{h}\pi_{ij_1}\left((T-h)^{-1}\sum_{t=h+1}^{T}\left(k_{it-j_1}k'_{it-j_2}-E\left[k_{it-j_1}k'_{it-j_2}\right]\right)\right)\pi_{ij_2}.$$

Then

$$\begin{aligned}&\max_i\left\|(T-h)^{-1}\sum_{t=h+1}^{T}v_{it,h}v'_{it,h}-\Sigma_{vi,h}\right\|\\ \leq\ &\max_i\sum_{j_1,j_2=0}^{h}\|\pi_{ij_1}\|\,\|\pi_{ij_2}\|\max_i\left\|(T-h)^{-1}\sum_{t=h+1}^{T}\left(k_{it-j_1}k'_{it-j_2}-E\left[k_{it-j_1}k'_{it-j_2}\right]\right)\right\|\end{aligned}$$

where

$$\text{(41)}\quad\left(E\max_i\left\|(T-h)^{-1}\sum_{t=h+1}^{T}\left(k_{it-j_1}k'_{it-j_2}-E\left[k_{it-j_1}k'_{it-j_2}\right]\right)\right\|^4\right)^{1/4}=O\left(n^{1/4}/T^{2/4}\right)=o(T^{-1/4}).$$

This can be seen from defning $z_t=k_{it-j_1}k'_{it-j_2}-E\left[k_{it-j_1}k'_{it-j_2}\right]$ such that $Ez_t=0$. Without loss of generality assume that z_t is univariate. Then

$$\begin{aligned}&\left(E\max_i\left\|(T-h)^{-1}\sum_{t=h+1}^{T}z_t\right\|^4\right)^{1/4}\\ \leq\ &\left(n\max_i E\left\|(T-h)^{-1}\sum_{t=h+1}^{T}z_t\right\|^4\right)^{1/4}\\ =\ &n^{1/4}\left[\max_i(T-h)^{-4}\sum_{t_1,\dots,t_4=h+1}^{T}\operatorname{Cov}(z_{t_1},z_{t_2})\operatorname{Cov}(z_{t_3},z_{t_4})\right.\\ &\left.+\operatorname{Cov}(z_{t_1},z_{t_3})\operatorname{Cov}(z_{t_2},z_{t_4})+\operatorname{Cov}(z_{t_1},z_{t_4})\operatorname{Cov}(z_{t_2},z_{t_3})+\operatorname{cum}(z_{t_1},z_{t_2},z_{t_3},z_{t_4})\right]\end{aligned}$$

where $\operatorname{Cov}(z_{t_1},z_{t_2})$ has a representation in terms of covariances of k_{it-j} and fourth order cumulant terms. Using the mixing inequality it can be established that all the terms are summable over at least two of the four indices. Thus the order in (41) follows. The details are omitted. The final result then follows form the fact that $O_p\left(T^{-2/5+v}\right)=o_p\left(T^{-1/4}\right)$ by the Condition of Lemma 21. ■

Lemma 33 *Assume that $h,k_{\max}\to\infty$ such that $T^v/h\to 0$, $T^v/k_{\max}\to 0$, $h=o\left(T^{1/5-v}\right)$ and $k_{\max}=o\left(T^{1/5-v}\right)$ where v is defined in Lemma 23. Let $\pi_i(h)=(\pi_{i1},...,\pi_{ih})'$. Then, uniformly in j for $0<j<j_0$ such that $j_0\to\infty$ as $n\to\infty$, $\sum_{l=k_{\max}}\left\|\Gamma_{i,l}^{kk}\right\|/\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\to 0$ and $\sum_{l=h}\|\pi_{il}\|/\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\to 0$ as $T\to\infty$ it follows that*

$$\max_{1\leq i\leq n}\|\hat{\pi}_i(h)-\pi_i(h)\|=O_p\left(T^{-1/5+v}\right).$$

Proof. As in the proof of Theorem 2.1 in Hannan and Kavalieris (1986, p.39), we have

$$\max_{1\leq i\leq n}\left\|\sum_{j=1}^{h}\left(\hat{\pi}_{ij,h}-\pi_{ij,h}\right)\hat{\Gamma}_{ij-k}^{kk}\right\|\leq\max_{1\leq i\leq n}\sum_{j=0}^{h}\|\pi_{ij}\|\left\|\hat{\Gamma}_{i,j-k}^{kk}-\Gamma_{j-k}^{kk}\right\|+\max_{1\leq i\leq n}\sum_{j=1}^{h}\|\pi_{ij}-\pi_{ij,h}\|\left\|\hat{\Gamma}_{i,j-k}^{kk}-\Gamma_{i,j-k}^{kk}\right\|$$

for $k=1,...,h$ where

$$\max_{1\le i\le n}\sum_{j=1}^{h}\|\pi_{ij}-\pi_{ij,h}\|\left\|\hat{\Gamma}^{kk}_{i,j-k}-\Gamma^{kk}_{i,j-k}\right\| \le \max_{0\le j\le h}\max_{1\le i\le n}\left\|\hat{\Gamma}^{kk}_{i,j}-\Gamma^{kk}_{i,j}\right\|\sum_{j=1}^{h}\|\pi_{ij}-\pi_{ij,h}\|$$

$$= O_p\left(T^{-1/5+v}\right)\sum_{j=h}^{\infty}\|\pi_{ij}\|$$

by Hannan and Kavalieris (1986) and Lemma (30). Moreover, $\sum_{j=1}^{h}\|\pi_{ij,h}-\pi_{ij}\| = O(\sum_{j=h+1}^{\infty}\|\pi_{ij}\|)$ by Hannan and Deistler (1988, Theorem 6.6.12). Again, by Hannan and Deistler (1988, Theorem 7.4.3) it follows that

$$\max_{1\le i\le n}\sum\nolimits_{j=0}^{h}\|\pi_{ij}\|\left\|\hat{\Gamma}^{kk}_{i,j-k}-\Gamma^{kk}_{i,j-k}\right\| \le \max_{1\le i\le n}\max_{0\le j\le h}\left\|\hat{\Gamma}^{kk}_{i,j}-\Gamma^{kk}_{i,j}\right\|\max_{1\le i\le n}\sum\nolimits_{j=0}^{\infty}\|\pi_{ij}\|$$

$$= O_p\left(T^{-1/5+v}\right).$$

Since $\max_{1\le i\le n}\left\|\hat{\Gamma}^{-1}_{ih}\right\|_2 = O_p(1)$ by Lemma 31 it follows that

$$\max_{1\le i\le n}\left\|\sum\nolimits_{j=1}^{h}\left(\hat{\pi}_{ij,h}-\pi_{ij,h}\right)\hat{\Gamma}^{kk}_{ij-k}\right\| = O_p\left(T^{-1/5+v}\right).$$

■

Lemma 34 *For all i and uniformly in l,*

$$\left\|E'_h H^l_{i(h)}E_h - E'_\infty H^l_{i(\infty)}E_\infty\right\| = O\left(\sum\nolimits_{j=h+1}^{\infty}\|\pi_{ij}\|\right)$$

Proof. Let

$$H_{i(h),\infty} = \begin{bmatrix} \pi_{i1,h} & \pi_{i2,h} & \cdots & \pi_{ih,h} & 0 & \cdots \\ I & 0 & \cdots & 0 & 0 & \cdots \\ 0 & I & \ddots & \vdots & & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots & \\ & \vdots & \ddots & I & 0 & \cdots \\ & & & 0 & \ddots & \end{bmatrix}$$

and note that $E'_h H^j_{i(h)}E_h = E'_\infty H^j_{i(h),\infty}E_\infty$. Noting that

$$H^l_{i(h),\infty} - H^l_{i(\infty)} = \sum_{k=0}^{l-1} H^k_{i(h),\infty}\left(H_{i(h),\infty}-H_{i(\infty)}\right)H^{l-k-1}_{i(\infty)}$$

leads to

$$\begin{aligned}\left\|E_h' H_{i(h)}^l E_h - E_\infty' H_{i(\infty)}^l E_\infty\right\| &= \left\|E_\infty'\left(H_{i(h),\infty}^l - H_{i(\infty)}^l\right)E_\infty\right\| \\ &\le \sum_{k=0}^{l-1}\left\|E_\infty' H_{i(h),\infty}^k E_\infty E_\infty'\left(H_{i(h),\infty} - H_{i(\infty)}\right)H_{i(\infty)}^{l-k-1}E_\infty\right\| \\ &\le \sum_{k=0}^{l-1}\left\|E_\infty' H_{i(h),\infty}^k E_\infty\right\|\left\|H_{i(\infty)}^{l-k-1}E_\infty\right\|\left\|E_\infty'\left(H_{i(h),\infty} - H_{i(\infty)}\right)\right\| \\ &\le \sum_{k=0}^{l-1}\left\|E_\infty' H_{i(h),\infty}^k E_\infty\right\|\left\|H_{i(\infty)}^{l-k-1}E_\infty\right\|\left(\sum_{j=1}^{h}\left\|\pi_{ij,h}-\pi_{i,j}\right\|^2 + \sum_{j=h+1}^{\infty}\left\|\pi_{ij}\right\|^2\right)^{1/2} \\ &\le \sqrt{2}\sum_{j=h+1}^{\infty}\left\|\pi_{ij}\right\|\sum_{k=0}^{l-1}\left\|E_\infty' H_{i(h),\infty}^k E_\infty\right\|\left\|H_{i(\infty)}^{l-k-1}E_\infty\right\|.\end{aligned}$$

where $\sum_{j=1}^{h}\|\pi_{ij,h}-\pi_{i,j}\| \le \sum_{j=h+1}^{\infty}\|\pi_{ij}\|$ by Hannan and Deistler (1985, Theorem 6.6.12). Since

$$H_{i(\infty)}^{l-k-1}E_\infty = \sum\nolimits_{j=0}^{l-k-1} B_{ij}$$

it follows that

$$\left\|H_{i(\infty)}^{l-k-1}E_\infty\right\| \le \sum\nolimits_{j=0}^{l-k-1}\|B_{ij}\| \le \sum\nolimits_{j=0}^{\infty}\|B_{ij}\|$$

and

$$\sum_{k=0}^{l-1}\left\|E_\infty' H_{i(h),\infty}^k E_\infty\right\| = \sum_{k=0}^{l-1}\|B_{ik,h}\| \le \sum_{k=0}^{\infty}\|B_{ik,h}\| < \infty$$

by Lemma 29 such that for every i

$$\left\|E_h' H_{i(h)}^l E_h - E_\infty' H_{i(\infty)}^l E_\infty\right\| = O\left(\sum\nolimits_{j=h+1}^{\infty}\|\pi_{ij}\|\right)$$

uniformly in l. ■

Lemma 35 *Assume that $h \to \infty$ such that $T^{\upsilon}/h \to 0$ and $h = o\left(T^{1/5-\upsilon}\right)$ where υ is defined in Lemma 23. Let $\pi_i(h) = (\pi_{i1}, ..., \pi_{ih})'$. Then, uniformly in j for $0 < j < j_0$ such that $j_0 \to \infty$ as $n \to \infty$, $\sum_{l=k_{\max}}\left\|\Gamma_{i,l}^{kk}\right\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ and $\sum_{l=h}\|\pi_{il}\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ as $T \to \infty$ it follows that*

$$\max_{1\le i\le n} E_h'\left(\hat{H}_{i(h)}^l - H_{i(h)}^l\right)E_h = O_p\left(T^{-1/5+\upsilon}\right).$$

Proof. From the proof of Lewis and Reinsel (1985, p.404) it follows that

$$\begin{aligned}\operatorname{vec}\left(E_h'\left(\hat{H}_{i(h)}^l - H_{i(h)}^l\right)E_h\right) &= \sum_{j=0}^{l-1}\left(E_h' H_{i(h)}'^{l-j-1}\otimes B_{ij}\right)\operatorname{vec}\left(\hat{\pi}_i(h) - \pi_i(h)\right) \\ &\quad + \sum_{j=0}^{l-1}\left(E_h' H_{i(h)}'^{l-j-1}\otimes\left(\hat{B}_{ij,h} - B_{ij}\right)\right)\operatorname{vec}\left(\hat{\pi}_i(h) - \pi_i(h)\right)\end{aligned}$$

with $\hat{B}_{ij,h} = E_h'\hat{H}_{i(h)}^j E_h$ and $E_h' H_{i(h)}'^{l-j-1} = \sum_{s=1}^{\min(h,l-j-1)} B_{il-j-1-s,h}$ Choose $\ell_1, \ell_2 \in \mathbb{R}^d$ arbitrary. Then, define

$$\ell_i(h)' = \sum_{j=0}^{l-1}\left(\ell_1' E_h' H_{i(h)}'^{l-j-1}\otimes \ell_2' B_{ij}\right)$$

where

$$\ell_i(h)'\ell_i(h) = \sum_{j_1,j_2=0}^{l-1}\left(\ell_1' E_h' H_{i(h)}'^{l-j_1-1} H_{i(h)}^{l-j_2-1} E_h \ell_1 \otimes \ell_2' B_{j_1} B_{j_2}' \ell_2\right).$$

Then let $\gamma(l)$ be a sequence such that $\ell_i(h)'\ell_i(h)/\gamma_i(l) = O(1)$ as $l \to \infty$ uniformly in h. From Lemma 33 it follows that

$$\begin{aligned} & \left\|(\ell_1 \otimes \ell_2)' \textstyle\sum_{j=0}^{l-1}\left(E_h' H_{i(h)}^{l-j-1} \otimes B_j\right) \operatorname{vec}(\hat{\pi}_i(h) - \pi_i(h))\right\| \\ \leq \;& \left(\ell_i(h)'\ell_i(h)\right)^{1/2} \max_{1\leq i\leq n} \|\hat{\pi}_i(h) - \pi_i(h)\| = O_p\left(T^{-1/5+\upsilon}\right). \end{aligned}$$

Next note that $\max_i \sum_{l=1}^{h} \left\|\hat{B}_{ilh}\right\|$ is bounded with probability tending to one. It thus follows that

$$\begin{aligned} & \left\|\sum_{j=0}^{l-1}\left(E_h' H_{i(h)}^{l-j-1} \otimes \left(\hat{B}_{jh} - B_{ij}\right)\right) \operatorname{vec}(\hat{\pi}_i(h) - \pi_i(h))\right\| \\ \leq \;& \sum_{j=0}^{l-1}\left\|E_h' H_{i(h)}^{l-j-1}\right\| \left\|\hat{B}_{ijh} - B_{ij}\right\| \max_{1\leq i\leq n}\|\hat{\pi}_i(h) - \pi_i(h)\| \\ = \;& O_p\left(T^{-1/5+\upsilon}\right) \end{aligned}$$

since $\sum_{j=0}^{l-1}\left\|E_h' H_{i(h)}^{l-j-1}\right\| = O(1)$ by Lemma 29. ■

Lemma 36 *Let $\Gamma_{i,j}^{kk}$ be defined in (11) and $\hat{\Gamma}_{i,h}(j)$ in (12). Assume that Assume that $h, k_{\max} \to \infty$ such that $T^{\upsilon}/h \to 0$, $T^{\upsilon}/k_{\max} \to 0$, $h = o\left(T^{1/5-\upsilon}\right)$ and $k_{\max} = o\left(T^{1/5-\upsilon}\right)$ where υ is defined in Lemma 23. Then, uniformly in j for $j < j_0$ such that $j_0 \to \infty$ as $n \to \infty$, $\sum_{l=k_{\max}}\left\|\Gamma_{i,l}^{kk}\right\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ and $\sum_{l=h}\|\pi_{il}\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ as $T \to \infty$ it follows that*

$$\left\|\hat{\Gamma}_{i,h}(j) - \Gamma_{i,j}^{kk}\right\| = \textstyle\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| o_p(1)$$

where the $o_p(1)$ term is uniform in i.

Proof. Define $\Gamma_{i,h}(j) = \sum_{l=0}^{k_{\max}} E_h' H_{i(h)}^{l+j} E_h \Sigma_{vi,h} E_h' H_{i(h)}^{l} E_h$ where $\Sigma_{vi,h}$ is defined in (34). Consider

$$\begin{aligned} (42)\quad \left\|\Gamma_{i,h}(j) - \Gamma_{i,j}^{kk}\right\| \;=\;& \left\|\sum_{l=0}^{k_{\max}}\left(E_\infty' H_{i(h),\infty}^{l+j} E_\infty \Sigma_{vi,h} E_\infty' H_{i(h),\infty}^{l} E_\infty - E_\infty' H_{i(\infty)}^{l+j} E_\infty \Sigma_{vi,\infty} E_\infty' H_{i(\infty)}^{l} E_\infty\right)\right\| \\ & + \left\|\sum_{l=k_{\max}+1}^{\infty} E_\infty' H_{i(\infty)}^{l+j} E_\infty \Sigma_{vi,\infty} E_\infty' H_{i(\infty)}^{l} E_\infty\right\|. \end{aligned}$$

where the last term is, using $E_\infty' H_{i(\infty)}^{l} E_\infty = B_{il}$,

$$\begin{aligned} \left\|\sum_{l=k_{\max}+1}^{\infty} B_{il+j}\Sigma_{vi,\infty}B_{il}\right\| \;\leq\;& \|\Sigma_{vi,\infty}\| \sum_{l=k_{\max}+1}^{\infty}\|B_{il}\| \sum_{l=k_{\max}+1}^{\infty}\|B_{il+j}\| \\ =\;& o(1) \textstyle\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \end{aligned}$$

since $\sum_{l=k_{\max}+1}^{\infty}\|B_{il+j}\| \leq \sum_{l=1}^{\infty}\|B_{il}\| < \infty$ and $\sum_{l=k_{\max}+1}^{\infty}\|B_{il}\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ as $n \to \infty$ for all $j < j_0$ by the conditions of the lemma and Lemma 27. The first term in (42) can be written as

$$\begin{aligned}(43)\qquad & \sum_{l=0}^{k_{\max}} E_{\infty}'\left(H_{i(h),\infty}^{l+j} - H_{i(\infty)}^{l+j}\right) E_{\infty}\Sigma_{vi,h}E_{\infty}'H_{i(h),\infty}^{l}E_{\infty} \\ & + \sum_{l=0}^{k_{\max}} E_{\infty}'H_{i(\infty)}^{l+j}E_{\infty}\Sigma_{vi,h}E_{\infty}'\left(H_{i(h),\infty}^{l} - H_{i(\infty)}^{l}\right)E_{\infty} \\ & + \sum_{l=0}^{k_{\max}} E_{\infty}'H_{i(\infty)}^{l+j}E_{\infty}\left(\Sigma_{vi,h} - \Sigma_{vi,\infty}\right)E_{\infty}'H_{i(\infty)}^{l}E_{\infty}.\end{aligned}$$

Then

$$\begin{aligned}\left\|\sum_{l=0}^{k_{\max}} E_{\infty}'H_{i(\infty)}^{l+j}E_{\infty}\left(\Sigma_{vi,h} - \Sigma_{vi,\infty}\right)E_{\infty}'H_{i(\infty)}^{l}E_{\infty}\right\| &\leq \sum_{l=0}^{k_{\max}}\|B_{il+j}\|\|B_{il}\|\|\Sigma_{vi,h} - \Sigma_{vi,\infty}\| \\ &= O\left(\sum_{l=0}^{k_{\max}}\|B_{il+j}\|\|B_{il}\|\right)O\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right) \\ &= o(1)\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\end{aligned}$$

by the fact[1] that $\|\Sigma_{vi,h} - \Sigma_{vi,\infty}\| = O\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right)$ and $\sum_{l=0}^{k_{\max}}\|B_{il+j}\|\|B_{il}\| = O(1)$ uniformly in j and i. The last equality follows from the fact that $\sum_{j=h}^{\infty}\|\pi_{ij}\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,j}^{kk}\right\| \to 0$ as $n \to \infty$. From Lemma 34 it follows that the second term of 43 is bounded as

$$\begin{aligned}(44)\qquad & \left\|\sum_{l=0}^{k_{\max}} E_{\infty}'H_{i(\infty)}^{l+j}E_{\infty}\Sigma_{vi,h}E_{\infty}'\left(H_{i(h),\infty}^{l} - H_{i(\infty)}^{l}\right)E_{\infty}\right\| \\ \leq\ & \sum_{l=0}^{k_{\max}}\|B_{il+j}\|\|\Sigma_{vi,h}\| O\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right) \\ =\ & O\left(\sum_{l=0}^{\infty}\|B_{il+j}\|\right)O\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right) \\ =\ & o(1)\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\end{aligned}$$

where the second equality follows from Lemma 27, together with the fact that $\|\Sigma_{vi,h}\| = O(1)$ uniformly in i and h. Finally, the first term in 43 needs to be further expanded such that

$$\sum_{l=0}^{k_{\max}} E_{\infty}'\left(H_{i(h),\infty}^{l+j} - H_{i(\infty)}^{l+j}\right)E_{\infty}\Sigma_{vi,h}E_{\infty}'\left(H_{i(h),\infty}^{l} - H_{i,\infty}^{l}\right)E_{\infty} + \sum_{l=0}^{k_{\max}} E_{\infty}'\left(H_{i(h),\infty}^{l+j} - H_{i(\infty)}^{l+j}\right)E_{\infty}\Sigma_{vi,h}E_{\infty}'H_{i,\infty}^{l}E_{\infty}$$

where the second term is $O\left(\sum_{l=0}^{k_{\max}}\|B_{il+j}\|\right)O\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right)$ by (44) and the first part can be bounded as

$$\begin{aligned}& \left\|\sum_{l=0}^{k_{\max}} E_{\infty}'\left(H_{i(h),\infty}^{l+j} - H_{i(\infty)}^{l+j}\right)E_{\infty}\Sigma_{vi,h}E_{\infty}'\left(H_{i(h),\infty}^{l} - H_{i,\infty}^{l}\right)E_{\infty}\right\| \\ \leq\ & \sum_{l=0}^{k_{\max}}\left\|E_{\infty}'\left(H_{i(h),\infty}^{l+j} - H_{i(\infty)}^{l+j}\right)E_{\infty}\right\|\left\|E_{\infty}'\left(H_{i(h),\infty}^{l} - H_{i,\infty}^{l}\right)E_{\infty}\right\|\|\Sigma_{vi,h}\| \\ =\ & O\left(k_{\max}\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right)^2\right) \\ =\ & o(1)\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|.\end{aligned}$$

since $k_{\max}\left(\sum_{j=h}^{\infty}\|\pi_{ij}\|\right) \to 0$ as $n \to \infty$ and $\sum_{j=h}^{\infty}\|\pi_{ij}\| / \sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\| \to 0$ for $j < j_0$. For $\left\|\hat{\Gamma}_{i,h}(j) - \Gamma_{i,h}(j)\right\|$

[1] This result follows from Theorem 6.6.12 of Hannan and Deistler (1988) and elementary calculations.

consider

$$\begin{aligned}
\hat{\Gamma}_{i,h}(j)-\Gamma_{i,h}(j) &= \sum_{l=0}^{k_{\max}} E_h'\hat{H}_{i(h)}^{l+j}E_h\hat{\Sigma}_{vi,h}E_h'\hat{H}_{i(h)}^{l}E_h - E_h'H_{i(h)}^{l+j}E_h\Sigma_{vi,h}E_h'H_{i(h)}^{l}E_h \\
&= \sum_{l=0}^{k_{\max}} E_h'\left(\hat{H}_{i(h)}^{l+j}-H_{i(h)}^{l+j}\right)E_h\hat{\Sigma}_{vi,h}E_h'\hat{H}_{i(h)}^{l}E_h \\
&\quad +\sum_{l=0}^{k_{\max}} E_h'H_{i(h)}^{l+j}E_h\left(\hat{\Sigma}_{vi,h}-\Sigma_{vi,h}\right)E_h'\hat{H}_{i(h)}^{l}E_h \\
&\quad +\sum_{l=0}^{k_{\max}} E_h'H_{i(h)}^{l+j}E_h\Sigma_{vi,h}E_h'\left(\hat{H}_{i(h)}^{l}-H_{i(h)}^{l}\right)E_h
\end{aligned}$$

By Lemma 35 it follows that

$$\begin{aligned}
&\left\|\sum_{l=0}^{k_{\max}} E_h'H_{i(h)}^{l+j}E_h\Sigma_{vi,h}E_h'\left(\hat{H}_{i(h)}^{l}-H_{i(h)}^{l}\right)E_h\right\| \\
\leq\ & \sum_{l=0}^{k_{\max}}\left\|E_h'H_{i(h)}^{l+j}E_h\right\|\left\|\Sigma_{vi,h}\right\|\left\|E_h'\left(\hat{H}_{i(h)}^{l}-H_{i(h)}^{l}\right)E_h\right\| \\
\leq\ & \max_{1\leq i\leq n}\left\|E_h'\left(\hat{H}_{i(h)}^{l}-H_{i(h)}^{l}\right)E_h\right\|\sum_{l=0}^{k_{\max}}\left\|E_h'H_{i(h)}^{l+j}E_h\right\|\left\|\Sigma_{vi,h}\right\| \\
\leq\ & O_p\left(T^{-1/5+\upsilon}\right)\sum_{l=0}^{k_{\max}}\left\|E_h'H_{i(h)}^{l+j}E_h-B_{ij+l}\right\|\left\|\Sigma_{vi,h}\right\| \\
& +O_p\left(T^{-1/5+\upsilon}\right)\sum_{l=0}^{k_{\max}}\left\|B_{ij+l}\right\|\left\|\Sigma_{vi,h}\right\|
\end{aligned}$$

By Lemma 34 it follows that uniformly in $l+j$, $\left\|E_h'H_{i(h)}^{l+j}E_h-B_{ij+l}\right\|=O\left(\sum_{j=h+1}^{\infty}\left\|\pi_{ij}\right\|\right)$ such that the first term is

$$\begin{aligned}
O_p\left(T^{-1/5+\upsilon}\right)\sum_{l=0}^{k_{\max}}\left\|E_h'H_{i(h)}^{l+j}E_h-B_{ij+l}\right\|\left\|\Sigma_{vi,h}\right\| &= O_p\left(T^{-1/5+\upsilon}\right)k_{\max}\sum_{j=h+1}^{\infty}\left\|\pi_{ij}\right\| \\
&= o_p(1)\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|
\end{aligned}$$

because $k_{\max}T^{-1/5+\upsilon}\to 0$ and $\sum_{j=h+1}^{\infty}\left\|\pi_{ij}\right\|=o\left(\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\right)$. The second term is

$$O_p\left(T^{-1/5+\upsilon}\right)\sum_{l=0}^{k_{\max}}\left\|B_{ij+l}\right\|\left\|\Sigma_{vi,h}\right\|=O_p\left(T^{-1/5+\upsilon}\right)\sum_{l=0}^{\infty}\left\|B_{ij+l}\right\|$$

where $\sum_{l=0}^{\infty}\left\|B_{ij+l}\right\|=O\left(\sum_{l=j}^{\infty}\left\|\Gamma_{i,l}^{kk}\right\|\right)$ by Lemma 27. Next consider

$$\begin{aligned}
\sum_{l=0}^{k_{\max}} E_h'H_{i(h)}^{l+j}E_h\left(\hat{\Sigma}_{vi,h}-\Sigma_{vi,h}\right)E_h'\hat{H}_{i(h)}^{l}E_h &= \sum_{l=0}^{k_{\max}} E_h'H_{i(h)}^{l+j}E_h\left(\hat{\Sigma}_{vi,h}-\Sigma_{vi,h}\right)E_h'\left(\hat{H}_{i(h)}^{l}-H_{i(h)}^{l}\right)E_h \\
&\quad +\sum_{l=0}^{k_{\max}}\left(E_h'H_{i(h)}^{l+j}E_h-B_{ij+l}\right)\left(\hat{\Sigma}_{vi,h}-\Sigma_{vi,h}\right)E_h'H_{i(h)}^{l}E_h \\
&\quad +\sum_{l=0}^{k_{\max}} B_{ij+l}\left(\hat{\Sigma}_{vi,h}-\Sigma_{vi,h}\right)E_h'H_{i(h)}^{l}E_h
\end{aligned}$$

where the first term can be bounded as

$$\begin{aligned}
& \left\| \sum_{l=0}^{k_{\max}} E_h' H_{i(h)}^{l+j} E_h \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) E_h' \left(\hat{H}_{i(h)}^l - H_{i(h)}^l \right) E_h \right\| \\
\leq & \sum_{l=0}^{k_{\max}} \left\| E_h' H_{i(h)}^{l+j} E_h \right\| \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \left\| E_h' \left(\hat{H}_{i(h)}^l - H_{i(h)}^l \right) E_h \right\| \\
= & \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \max_{1 \leq i \leq n} \left\| E_h' \left(\hat{H}_{i(h)}^l - H_{i(h)}^l \right) E_h \right\| \left(k_{\max} \sum\nolimits_{j=h+1}^{\infty} \|\pi_{ij}\| + \sum_{l=0}^{\infty} \|B_{ij+l}\| \right) \\
= & o_p(1) \sum\nolimits_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| .
\end{aligned}$$

by Lemmas 32, 34 and 35 and using $k_{\max} T^{-1/5+v} \to 0$, $\sum_{j=h+1}^{\infty} \|\pi_{ij}\| = O\left(\sum_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| \right)$ and $\sum_{l=0}^{\infty} \|B_{ij+l}\| = O\left(\sum_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| \right)$ by Lemma 27 establishes the last equality. Next,

$$\begin{aligned}
\left\| \sum_{l=0}^{k_{\max}} \left(E_h' H_{i(h)}^{l+j} E_h - B_{ij+l} \right) \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) E_h' H_{i(h)}^l E_h \right\| & \leq \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \sum\nolimits_{j=h+1}^{\infty} \|\pi_{ij}\| \sum_{l=0}^{k_{\max}} \left\| E_h' H_{i(h)}^l E_h \right\| \\
& = \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \sum\nolimits_{j=h+1}^{\infty} \|\pi_{ij}\| \\
& = o_p(1) \sum\nolimits_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\|
\end{aligned}$$

by Lemmas 29 and 34 and the last equality is established as before. Finally,

$$\begin{aligned}
\sum_{l=0}^{k_{\max}} B_{ij+l} \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) E_h' H_{i(h)}^l E_h & = \sum_{l=0}^{k_{\max}} B_{ij+l} \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) \left(E_h' H_{i(h)}^l E_h - B_{il} \right) \\
& \quad + \sum_{l=0}^{k_{\max}} B_{ij+l} \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) B_{il}
\end{aligned}$$

where

$$\begin{aligned}
\left\| \sum_{l=0}^{k_{\max}} B_{ij+l} \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) B_{il} \right\| & \leq \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \sum_{l=0}^{\infty} \|B_{ij+l}\| \|B_{il}\| \\
& = o_p(1) \sum\nolimits_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\|
\end{aligned}$$

and

$$\begin{aligned}
\left\| \sum_{l=0}^{k_{\max}} B_{ij+l} \left(\hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right) \left(E_h' H_{i(h)}^l E_h - B_{il} \right) \right\| & \leq \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \sum_{l=0}^{k_{\max}} \|B_{ij+l}\| \left\| E_h' H_{i(h)}^l E_h - B_{il} \right\| \\
& = \max_{1 \leq i \leq n} \left\| \hat{\Sigma}_{vi,h} - \Sigma_{vi,h} \right\| \sum\nolimits_{j=h+1}^{\infty} \|\pi_{ij}\| \sum_{l=0}^{\infty} \|B_{ij+l}\| \\
& = o_p(1) \sum\nolimits_{l=j}^{\infty} \left\| \Gamma_{i,l}^{kk} \right\| .
\end{aligned}$$

■

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