Estimation of Static and Dynamic Models of Strategic Interactions

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This talk is based on the following papers

Patrick Bajari, Han Hong, John Krainer and Denis Nekipelov.
Estimating Static Models of Strategic Interactions.
Working paper, 2005.

Patrick Bajari and Victor Chernozhukov and Han Hong
Semiparametric Estimation of a Dynamic Game.
Working paper, 2005.

Motivation

- Empirical analysis of games in econometrics and industrial organization.
- Discrete choice model with other agent's actions entering as a right hand side variable.
- Often most straight forward to estimate a game in two steps.
- In a first step, the economist estimates the reduced forms implied by the model.
- In the second step, recover structural utility parameters that rationalize the observed reduced forms.
- Computation of multiple equilibria.

Literature

- Early examples, Vuong and Bjorn (1984) and Bresnahan and Reiss (1990,1991).
- Also, Aradillas-Lopez (2005), Ho(2005), Ishii (2005), Pakes, Porter, Ho and Ishii (2005), Rysman (2004), Seim(2004), Sweeting (2004), Tamer (2003) and Ciliberto and Tamer (2005).
- Single agent dynamic models: Rust (1987), Hotz and Miller (1993), Magnac and Thesmar (2003), Heckman and Navarro (2005), among many others.
- Dynamic games: Aguirregabiria and Mira (2003), Bajari, Benkard and Levin (2004), Pakes, Ovstrovsky and Berry (2004), and Pesendorfer and Schmidt-Dengler (2004).

Static Model: Notations

- "Estimating Static Models of Strategic Interactions"
- Players, $i = 1, ..., n$.
- Actions $a_i \in \{0, 1, \ldots, K\}$.
- $A = \{0, 1, \ldots, K\}^n$ and $a = (a_1, ..., a_n)$..
- $s_i \in S_i$: state for player i.
- $S = \Pi_i S_i$ and $s = (s_1, ..., s_n) \in S$.
- s is common knowledge and also observed by econometrician.
- For each agent i, $K + 1$ state variables $\epsilon_i(a_i)$
- $\epsilon_i (a_i)$: private information to each agent.
- $\epsilon_i = (\epsilon_i (0), \ldots, \epsilon_i (K))$.
- Density $f(\epsilon_i)$, i.i.d. across $i = 1, \ldots, n$.

Static model continued

- Period utility for player i with action profile a:

$$u_i(a, s, \epsilon_i; \theta) = \Pi_i(a_i, a_{-i}, s; \theta) + \epsilon_i(a_i)$$

- Example: the period profit of firm i for entering the market.
- Generalizes a standard discrete choice model.
- Agents act in isolation in standard discrete choice models.
- Unlike a standard discrete choice model, a_{-i} enters utility.
- Player i's decision rule is a function $a_i = \delta_i(s, \epsilon_i)$.
- Note that ϵ_{-i} does not enter.
- ϵ_{-i} is private information of other players.

Static model

- Conditional choice probability $\sigma_i(a_i|s)$ for player i:

$$\sigma_i(a_i = k|s) = \int 1\{\delta_i(s, \epsilon_i) = k\} f(\epsilon_i) d\epsilon_i.$$

- Choice probability is conditional s: public information.
- Choice specific expected payoff for player i:

$$\Pi_i(a_i, s; \theta) = \sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s; \theta) \sigma_{-i}(a_{-i}|s).$$

- Expected utility from choosing a_i, excluding preference shock.
- The optimal action for player i satisfies:

$$\sigma_i(a_i|s) = Prob \left\{ \begin{array}{c} \epsilon_i | \Pi_i(a_i, s; \theta) + \epsilon_i(a_i) \\ > \Pi_i(a_j, s; \theta) + \epsilon_i(a_j) \text{ for } j \neq i. \end{array} \right\}$$

- $\Pi_i(a_i, a_{-i}, s; \theta)$ is often a linear function, e.g.:

$$\Pi_i(a_i, a_{-i}, s) = \begin{cases} s' \cdot \beta + \delta \sum_{j \neq i} 1\{a_j = 1\} \text{ if } a_i = 1 \\ 0 \text{ if } a_i = 0 \end{cases}$$

- Mean utility from not entering normalized to zero.
- δ measures the influence of j's entry choice on i's profit.
- If firms compete with each other: $\delta < 0$.
- β measure the impact of the state variables on profits.
- $\varepsilon_i(a_i)$ capture shocks to the profitability of entry.
- Often $\varepsilon_i(a_i)$ are assumed to be i.i.d. extreme value distributed:

$$f(\varepsilon_i(k)) = e^{-\varepsilon_i(k)} e^{e^{-\varepsilon_i(k)}}.$$

- Choice specific expected payoff under linearity:

$$\Pi_i(a_i = 1, s; \theta) = s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s).$$

- Choice probability under the extreme value distribution

$$\sigma_i(a_i = 1|s) = \frac{\exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s))}{1 + \exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s))}$$

- Two approaches for estimation:
  - Full information maximum likelihood estimation.
  - Semiparametric two step estimation method.

Full information maximum likelihood

- Choice probability equations define fixed point mappings for

$$\sigma_j(a_j = 1|s).$$

- For each $\theta = (\beta, \delta)$, solve for

$$\sigma_j(a_j = 1|s; \theta)$$

  for all $j = 1, \ldots, n$.
- Maximize likelihood function:

$$L(\theta) = \prod_{t=1}^{T} \prod_{i=1}^{n} (\sigma_i (a_i = 1|s; \theta))^{1\{a_{i,t}=1\}} (1 - \sigma_i (a_i = 1|s; \theta))^{1\{a_{i,t}=0\}}$$

Two step semiparametric approach

- At the true parameter θ, $\sigma_j(a_j = 1|s;\theta))$ can be recovered from the data nonparametrically:

$$\hat{\sigma}_j(a_j = 1|s).$$

- Pseudo likelihood easy to maximize: (multinomial) logit

$$\prod_{t=1}^{T}\prod_{i=1}^{n}\left(\frac{\exp(s'\cdot\beta + \delta\sum_{j\neq i}\hat{\sigma}_j(a_j = 1|s))}{1+\exp(s'\cdot\beta + \delta\sum_{j\neq i}\hat{\sigma}_j(a_j = 1|s))}\right)^{1\{a_{i,t}=1\}}$$
$$\left(1-\frac{\exp(s'\cdot\beta + \delta\sum_{j\neq i}\hat{\sigma}_j(a_j = 1|s))}{1+\exp(s'\cdot\beta + \delta\sum_{j\neq i}\hat{\sigma}_j(a_j = 1|s))}\right)^{1\{a_{i,t}=0\}}$$

- Both the first stage estimates $\widehat{\sigma}_i(a_i = 1|s)$ and the term $s'\cdot\beta$ depend on the vector of state variables s.
- Colinearity and identification: Need a covariate that enters the first stage, but not the second stage.

Nonparametric Identification

A1 Assume that the error terms $\epsilon_i(a_i)$ are distributed i.i.d. across actions a_i and agents i, and come from a known parametric family.

- Not possible to allow nonparametric mean utility and error terms at once, even in simple single agent problems (e.g. a probit).
- In Bajari, Hong and Ryan (2005)- even a single agent model is not identified without an independence assumption.
- Well known that $\Pi_i(0, s)$ are not identified.
- $\sigma_i(a_i|s)$ only functions of $\Pi_i(a_i, s) - \Pi_i(0, s)$.
- Suppose $\epsilon_i(a_i)$ is extreme value,

$$\sigma_i(a_i|s) = \frac{\exp(\Pi_i(a_i, s) - \Pi_i(0, s))}{\sum_{k=0}^{K} \exp(\Pi_i(k, s) - \Pi_i(0, s))}$$

A2 For all i and all a_{-i} and s, $\Pi_i(a_i = 0, a_{-i}, s) = 0$.

- Can only learn choice specific value functions up to a first difference. Need normalization
- Similar to "outside good" assumption in single agent model.
- Entry: the utility from not entering is normalized to zero.

- Hotz and Miller (1993) inversion, for any k, k':

$$\log\left(\sigma_i(k|s)\right) - \log\left(\sigma_i(k'|s)\right) = \Pi_i(k,s) - \Pi_i(k',s).$$

- More generally let $\Gamma : \{0, ..., K\} \times S \rightarrow [0,1]$:

$$(\sigma_i(0|s), ..., \sigma_i(K|s)) = \Gamma_i\left(\Pi_i(1,s) - \Pi_i(0,s), ..., \Pi_i(K,s) - \Pi_i(0,s)\right)$$

- And the inverse Γ^{-1}:

$$(\Pi_i(1,s) - \Pi_i(0,s), ..., \Pi_i(K,s) - \Pi_i(0,s)) = \Gamma_i^{-1}\left(\sigma_i(0|s), ..., \sigma_i(K|s)\right)$$

- Invert equilibrium choice probabilities to nonparametrically recover $\Pi_i(1,s) - \Pi_i(0,s), ..., \Pi_i(K,s) - \Pi_i(0,s)$.
- $\Pi_i(a_i, s)$ is known by our inversion and probabilites σ_i can be observed by econometrician.

- Next step: how to recover $\Pi_i(a_i, a_{-i}, s)$ from $\Pi_i(a_i, s)$.
- Requires inversion of the following system:

$$\Pi_i(a_i, s) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\Pi_i(a_i, a_{-i}, s),$$

$$\forall i = 1, \dots, n, a_i = 1, \dots, K..$$

- Given s, $n \times K \times (K+1)^{n-1}$ unknowns utilities of all agents.
- Only $n \times (K)$ known expected utilities.
- Obvious solution: impose exclusion restrictions.

- Partition $s = (s_i, s_{-i})$, and suppose

$$\Pi_i(a_i, a_{-i}, s) = \Pi_i(a_i, a_{-i}, s_i)$$

depends only on the subvector s_i.

$$\Pi_i(a_i, s_{-i}, s_i) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s_{-i}, s_i)\Pi_i(a_i, a_{-i}, s_i).$$

- Identification: Given each s_i, the second moment matrix of the "regressors" $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$,

$$E\sigma_{-i}(a_{-i}|s_{-i}, s_i)\sigma_{-i}(a_{-i}|s_{-i}, s_i)'$$

is nonsingular.

- Needs at least $(K+1)^{n-1}$ points in the support of the conditional distribution of s_{-i} given s_i.

Nonparametric Estimation

- Step 1: Estimation of choice probabilities (e.g. sieve)

$$\widehat{\sigma}_i(k|s) = \sum_{t=1}^{T} 1(a_{it} = k) q^{\kappa(T)}(s_t)(Q_T' Q_T)^{-1} q^{\kappa(T)}(s).$$

where

$$Q_T = (q^{\kappa(T)}(s_1), \ldots, q^{\kappa(T)}(s_T),$$

and

$$q^{\kappa(T)}(s) = (q_1(s), \ldots, q_{\kappa(T)}(s)).$$

- Step 2: Inversion of expected utilities

$$\left(\widehat{\Pi}_i(1, s_t) - \widehat{\Pi}_i(0, s_t), ...\widehat{\Pi}_i(K, s_t) - \widehat{\Pi}_i(0, s_t)\right) = \Gamma_i^{-1}\left(\widehat{\sigma}_i(0|s_t), ..., \widehat{\sigma}_i(K|s_t)\right)$$

In the logit model

$$\hat{\Pi}_i(k, s_t) - \hat{\Pi}_i(0, s_t) = \log\left(\widehat{\sigma}_i(k|s_t)\right) - \log\left(\widehat{\sigma}_i(0|s_t)\right)$$

- Step 3: Recovering structural parameters.
- Infer $\Pi_i(a_i, a_{-i}, s_i)$ from $\hat{\Pi}_i(k, s)$.
- Use empirical analog of

$$\Pi_i(a_i, s_{-i}, s_i) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s_{-i}, s_i)\Pi_i(a_i, a_{-i}, s_i).$$

- For given s_i and given $a = (a_i, a_{-i})$, run local regression

$$\sum_{t=1}^{T} \left(\hat{\Pi}_i(a_i, s_{-it}, s_i) - \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{-it}, s_i)\, \Pi_i(a_i, a_{-i}, s_i) \right)^2 w(t, s_i).$$

Nonparametric weights can take different forms, e.g.

$$w(t, s_i) = k\left(\frac{s_{it} - s_i}{h}\right) / \sum_{\tau=1}^{T} k\left(\frac{s_{i\tau} - s_i}{h}\right).$$

- Asymptotically identified.

Semiparametric Estimation

- Linear model of deterministic utility

$$\Pi_i\left(a_i, a_{-i}, s_i\right) = \Phi_i\left(a_i, a_{-i}, s_i\right)' \theta.$$

- Choice specific value function

$$\Pi_i\left(a_i, s\right) = E\left[\Pi_i\left(a_i, a_{-i}, s_i\right) | s, a_i\right] = \Phi_i\left(a_i, s\right)' \theta.$$

where

$$\Phi_i\left(a_i, s\right) = \sum_{a_{-i}} \Phi_i\left(a_i, a_{-i}, s_i\right) \prod_{j \neq i} \sigma(a_j = k_j | s).$$

- Estimated parameter choice probabilities:

$$\sigma_i\left(a_i | s, \hat{\Phi}, \theta\right) = \frac{\exp\left(\hat{\Phi}_i\left(a_i, s\right)' \theta\right)}{1 + \sum_{k=1}^{K} \exp\left(\hat{\Phi}_i\left(k, s\right)' \theta\right)}$$

- Semiparametric method of moment estimator

$$\frac{1}{T} \sum_{t=1}^{T} \hat{A}\left(s_t\right)\left(y_t - \sigma\left(s_t, \hat{\Phi}, \hat{\theta}\right)\right) = 0.$$

where

$$\sigma\left(s_t, \hat{\Phi}, \theta\right) = \left(\sigma_i\left(k | s_t, \hat{\Phi}, \theta\right), \forall k = 1, \ldots, K, \forall i = 1, \ldots, n\right)$$

- Nonparametric vs semiparametric methods
- Typically $\widehat{\sigma}_i(k|s)$ will converge to the true $\sigma_i(k|s)$ at a nonparametric rate which is slower than $T^{1/2}$.
- Alternative estimators: kernel smoothing or local polynomial
- Nonparametric method is more robust against misspecification.
- May suffer from curse of dimensionality.
- Semiparametric method more practical in applications.
- θ can be estimated at parametric rates despite first stage nonparametric estimation.
- Pseudo MLE can be used in place of Method of moments.

Linear probability model

- Binary choice model $K = 1$:

$$\Pi_i(0, a_{-i}, s) = \epsilon_i(0) \equiv 0, \quad , \Phi_i(1, a_{-i}, s; \theta) = \Phi_i^1(s_i)' \beta + \Phi_i^1(a_{-i}, s)' \gamma$$

- Action 1 is chosen if and only if

$$\Phi_i^1(s_i)' \beta + E\left[\Phi_i^2(a_{-i}, s) | s\right]' \gamma + \epsilon_i(1) > 0.$$

- Assume uniform distribution of $\epsilon_i(1)$:

$$a_i = \Phi_i^1(s_i)' \beta + E\left[\Phi_i^2(a_{-i}, s) | s\right]' \gamma + \eta_{it}(1),$$

where $E(\eta_{it}(1) | s_t) = 0$. Alternative representation:

$$a_i = \Phi_i^1(s_i)' \beta + \Phi_i^2(a_{-i}, s)' \gamma + E\left[\Phi_i^2(a_{-i}, s) | s\right]' \gamma - \Phi_i^2(a_{-i}, s)' \gamma + \eta_{it}(1),$$

- Valid instruments, function of s_t are mean independent of

$$E\left[\Phi_i^2(a_{-i}, s) | s\right]' \gamma - \Phi_i^2(a_{-i}, s)' \gamma + \eta_{it}(1),$$

- Can be estimated by 2SLS (ivreg in Stata)

Fixed effect model of unobserved heterogeneity

- Smooth function of state variables s: $\alpha(a_i, s)$.
- Estimatable belief of players only a function of s.
- Nonparametric Identification:

$$\Pi_i(a_i, a_{-i}, s; \theta) = \alpha(a_i, s) + \tilde{\Pi}_i(a_i, a_{-i}, s_i; \theta).$$

- Expected utility nonparametric identified:

$$\Pi_i(a_i, s) = \quad \alpha(a_i, s) + \textstyle\sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\tilde{\Pi}_i(a_i, a_{-i}, s_i).$$

- Differencing between pairs of players

$$\Pi_i(k, s) - \Pi_j(k, s) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\, \tilde{\Pi}_i(k, a_{-i}, s_i) - \sum_{a_{-j}} \sigma_{-j}(a_{-j}|s)\, \tilde{\Pi}_j(k, a_{-j}, s_j)$$

- Given s_i and s_j, use variations in $\sigma_{-i}(a_{-i}|s)$ and $\sigma_{-j}(a_{-j}|s)$ to identify $\tilde{\Pi}_i(k, a_{-i}, s_i)$ and $\tilde{\Pi}_j(k, a_{-j}, s_j)$.
- Under symmetry:

$$\Pi_i(k, s) - \Pi_j(k, s) = \sum_{a_{-ij}} \sigma_{-ij}(a_{-ij}|s) \left[\tilde{\Pi}_{ij}(k, a_{-ij}, s_i) - \tilde{\Pi}_{ij}(k, a_{-ij}, s_j)\right].$$

- Parametric mean utilities

$$\tilde{\Pi}_i\left(a_i, a_{-i}, s_i\right) = \Phi_i\left(a_i, a_{-i}, s_i\right)' \theta.$$

- Conditional logit estimation, $\log L$

$$\sum_{t=1}^{T}\left(\log\left[\exp\left(\theta' \sum_{i=1}^{n} a_{it}\hat{\Phi}_i\left(1, s_{it}\right)\right)\right] - \log\left[\sum_{d_t \in B_t} \exp\left(\theta' \sum_{i=1}^{n} d_{it}\hat{\Phi}_i\left(1, s_{it}\right)\right)\right]\right)$$

where

$$B_t \equiv \{d_t : \sum_{i=1}^{n} d_{it} = \sum_{i=1}^{n} a_{it}\}$$

- Conditional likelihood given the number of entrants.
- Rank estimation

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j\neq i} 1\left(a_{it} > a_{jt}\right)\rho_{-}\left(\left(\hat{\Phi}_i\left(1, s_{it}\right) - \hat{\Phi}_j\left(1, s_{jt}\right)\right)' \theta\right).$$

- Examples of penalty functions

$$\rho_{-}\left(x\right) = 1\left(x < 0\right) \quad \text{or} \quad \rho_{-}\left(x\right) = 1\left(x < 0\right)x^2.$$

Computing Multiple Equilibria

- Known distribution for the error term $F(\epsilon_i)$
- Known mean utility functions $\Pi_i(a_i, a_{-i}, \theta)$.
- Conditional choice probability fixed point mappings:

$$\sigma_i(a_i|s) = \Gamma_i\left(\sum_{-i} \sigma_{a_{-i}}(a_{-i}|s)\left[\Pi_i(k, a_{-i}, s; \theta) - \Pi_i(0, a_{-i}, s; \theta)\right], \forall k\right).$$

- Given linear mean utility, fixed point mappings:

$$\sigma_i(a_i|s) = \Gamma_i\left(\sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\Phi_i(a_i, a_{-i}, s)'\theta,\ a_i = 1, \dots, K\right),\ i = 1, \dots, n.$$

- $K \times n$ equations and $K \times n$ unknown variables

$$\sigma_i(a_i|s), \forall a_i = 1, \dots, K, i = 1, \dots, n,$$

- Possibility of multiple solutions.

Homotopy Method

- Find multiple and possibly all solutions
- The fixed point system, for $\sigma = \sigma(s)$:

$$\sigma - \Gamma(\sigma) = 0,$$

- Homotopy: linear mapping between two spaces of functions

$$H(\sigma, \tau) = \tau G(\sigma) + (1 - \tau)(\sigma - \Gamma(\sigma)), \quad \tau \in [0, 1],$$

- $H(\sigma, \tau)$ and $G(\sigma)$: vectors of $n \times K$ component functions

$$H_{i,a_i}(\sigma, \tau) \quad \text{and} \quad G_{i,a_i}(\sigma) \quad i = 1, \ldots, n \quad a_i = 1, \ldots, K.$$

- Initial system: $\tau = 1$, $H(\sigma, 0) = G(\sigma)$.
- End system: $\tau = 0$, $H(\sigma, 0) = \Gamma(\sigma)$.

- Typically initial system $G(\sigma)$ easy to solve.
- The solution path $\sigma(\tau)$:

$$H(\sigma(\tau), \tau) = 0.$$

- Differentiating this homotopy with respect to τ:

$$\frac{d}{\partial \tau} H(\sigma(\tau), \tau) = \frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \tau}.$$

- Trace the path $\tau = 1$ to $\tau = 0$.
- Algorithms for numerically tracing this differential equation system.

Condition 1 (Regularity) *Let $\nabla(\tau)$ denote the Jacobian of the Homotopy functions with respect to σ at the solution path $\sigma(\tau)$:*

$$\nabla(\tau) = \frac{\partial}{\partial \sigma} Re\{H(\sigma, \tau)\}\bigg|_{\sigma=\sigma(\tau)},$$

where $Re\{H(\sigma, \tau)\}$ denotes the real component of the homotopy functions. The jacobian $\sigma(\tau)$ has full rank for almost all τ.

Condition 2 (Path Finiteness) *Define $H^{-1}(\tau)$ to be the set of solutions $\sigma(\tau)$ to the homotopy system at τ. $H^{-1}(\tau)$ is bounded for all $0 < \tau \leq 1$. In other words, for all $\tau > 0$.*

$$\lim_{||\sigma|| \to \infty} H(\sigma, \tau) \neq 0.$$

Multiple Equilibria in Static Discrete Games

- $\Gamma(\cdot)$ a linear function in a linear probability model.
- May not have multiple equilibria if argument to $\Gamma(\cdot)$ is linear in choice probabilities.
- For example if profit depends on number of competitors.
- But nonlinear interactions of choice probabilities also possible.
- Functional form of $\Gamma(\cdot)$ depends on distribution of error terms.
- In general, $\Gamma(\sigma)$ nonlinear function of a polynomial index of choice probabilities.
- Polynomial error distributions difficult to justify.
- Choice of initial system:

$$G_{i,a_i}(\sigma) = \sigma_i(a_i)^{q_{i,a_i}} - 1 = 0, \quad i = 1, \ldots, n, \quad a_i = 1, \ldots, K.$$

Theorem 2 *Define the sets* $H^{-1} = \{(\sigma_r, \sigma_i, \tau) \mid H(\sigma_r, \sigma_i, \tau) = 0\}$ *amd*

$$H^{-1}(\tau) = \{(\sigma_r, \sigma_i) \mid H(\sigma, \tau) = 0\} \quad \textit{for} \quad \sigma_r \in \mathbb{R}^{nK}, \quad \textit{and} \quad \sigma_i \in \mathbb{R}^{nK}.$$

Also define $\wp_\epsilon = \cup_{i,a_i}\{|\sigma_{r,i,a_i}| \leq \epsilon\}$ *to be the area around the imaginary axis.*

1) *The set* $H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \wp_\epsilon \times [0,1]\}$ *consists of closed disjoint paths.*
2) *For any* $\tau \in (0,1]$ *there exists a bounded set such that* $H^{-1}(\tau) \cap \mathbb{R}^{2nK} \setminus \wp_\epsilon$ *is in that set.*
3) *For* $(\sigma_r, \sigma_i, \tau) \in H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \wp_\epsilon \times [0,1]\}$ *the homotopy system allows parametrization* $H(\sigma_r(s), \sigma_i(s), \tau(s)) = 0$. *Moreover,* $\tau(s)$ *is a monotone function.*

Theorem 2 *For given* τ *one can pick the power* q_{i,a_i} *of the initial function (1) such that the homotopy system is regular and path finite outside any sequence of converging polyhedra* $\wp_\epsilon$, $\epsilon \longrightarrow 0$.

Monte Carlo analysis

- Entry game with a small number of players
- Multiple equilibria computation, not about identification.
- Payoff to player i a linear function of the indicator of the rival's entry $(a_i = 1)$, market covariates and random term:

$$U_i\left(1, a_{-i}\right) = \theta_1 - \theta_2 \left(\sum_{j \neq i} 1\left(a_j = 1\right) \right) + \theta_3 x_1 + \theta_4 x_2 + \epsilon_i\left(a\right),$$
$$i = 1, \ldots, n.$$

- Symmetric model, ex-ante probability of entry:

$$P_i = \frac{e^{\theta_1 - \theta_2\left(\sum_{j \neq i} P_j\right) + \theta_3 x_1 + \theta_4 x_2}}{1 + e^{\theta_1 - \theta_2\left(\sum_{j \neq i} P_j\right) + \theta_3 x_1 + \theta_4 x_2}}$$

Table: Characteristics of the parameters

Parameter	Mean	Variance	Distribution
θ_1	2.45	1	Normal
θ_2	5.0	1	Normal
θ_3	1.0	1	Normal
θ_4	-1.0	1	Normal
x_1	1.0	0.33	Uniform
x_2	1.0	0.33	Uniform

Table: Results of Monte-Carlo Simulations.

Parameter	Mean	Std Dev	Max	Min
$n = 3$				
# of equilibria	1.592	1.175	7	1
P_1	0.366	0.362	0.998	0
P_2	0.360	0.367	0.995	0
P_3	0.363	0.348	0.993	0.003
$n = 4$				
# of equilibria	1.292	0.777	5	1
P_1	0.278	0.328	0.981	0.001
P_2	0.246	0.320	0.981	0.003
P_3	0.276	0.338	0.999	0.001
P_4	0.280	0.338	0.987	0.002
$n = 5$				
# of equilibria	1.106	0.505	5	1
P_1	0.104	0.201	0.964	0
P_2	0.138	0.252	0.975	0
P_3	0.315	0.338	0.992	0
P_4	0.356	0.385	0.983	0
P_5	0.319	0.344	0.982	0

Application: equity analyst's recommendations

- Build on previous work by Bajari and Krainer.
- Study the set of recommendations on a stock can be viewed as the outcome of a game.
- Payoffs may depend on recommendations made by peers since they are benchmarked against peer recommendations.
- Also, inherent indeterminacy in system of rankings.
- Focus on interesting behavior during run-up and subsequent collapse of the NASDAQ in 2000.
- Previous studies: least recommended stocks earned an average abnormal return of 13% in 2000-2001.
- Most highly recommended stocks earned average abnormal returns of -7%.

Four factors that could have influenced recommendations.

1. Recommendations must depend on public info about the future profitability of a firm (stock and time effects).
2. Second, analysts may have heterogeneous forecasts (merge earnings forecasts with recommendations).
3. Third, conflicts of interest (dummy variable for investment banking business in quarters before and after recommendation was made).
4. Finally, recommendations of other analysts matter.

Variables in data set

- RELATION-A dummy variable that is one if the analyst's brokerage engages in investment banking business with the company to which the recommendation applies.
- SPITDUM-A dummy variable that is equal to one after the quarter starting in June of 2001. Based on a comprehensive search of Wall Street Journal articles, this is when Elliot Spitzer began making very public criticisms of industry practices.
- IBANK-A dummy variable that is equal to one if the brokerage does any investment banking business with stocks in the NASDAQ 100.
- SBANK-the share of analysts that issued recommendations for a particular stock during a particular quarter where IBANK was one.

Table 1: Recommendation Variables.

Recommendation	*Numerical Value Recorded by I/B/E/S*
Strong Buy	1
Buy	2
Hold	3
Underperform	4
Sell	5

Table 2: Summary Statistics.

Variable	*Mean*	*Std.*	*Min.*	*Max.*	*Nobs*
Recommendation	2.210	0.9168	1	5	12719
Relation	0.0350	0.1839	0	1	12719
Ibank	0.8155	0.3878	0	1	12719
Earnings	0.1111	0.2439	-3.010	1.720	12719

Table 3: Tabulation of Recommendations by Quarter.

Variable/Time Period	*Q1 1998*	*Q1 2000*	*Q2 2003*
% Recs. Equal to 1	30.51	46.73	11.65
% Recs. Equal to 2	30.51	41.46	18.12
% Recs. Equal to 3	37.62	11.81	53.07
% Recs. Equal to 4	1.02	0.00	12.62
% Recs. Equal to 5	0.34	0.00	4.53

- Utility is modeled as an ordered logit.
- The strategic interaction is a best response to the expected recommendation of other analysts.
- Two sources of identification.
- The first is characteristics of other firms (e.g. whether or not they are IBANKS).
- The second is Elliot Spitzer.

Empirical Findings

- Reduced form ordered logit regression.
- High correlation between quarterly effects and market indexes.
- Conflict of Interest: coef on RELATION
- IBANK: general investment banking firms generally more conversative.
- Companies select banking firms favorable to them.
- Peer effects in the interaction model.
- Conformable peer effects important (1.8-2.3 IVBELIEF).
- Explains most of variation in the data.
- Parametric vs semiparametric first stage: similar results.

Table 7.4 Ordered Logit Estimates of the Effect of Fundamentals.

VARIBLE	COEF.	COEF.	COEF.	COEF.
%DEV	-.0539 (-0.276)	-.1030 (-0.519)	-	-
ABS. DEV			-.1030 (-0.519)	-
Log Likelihood	-16171.589	-14861.218	-14861.218	-14861.352
Psueo-R^2	0.0000	0.0810	0.0810	0.0810
Fixed Effects	none	quarterly,stock	quarterly, stock	quarterly, stock

In the ordered logit model, the dependent variable is the analyst's recommendation as coded by IBES. This takes on discrete values from one to five. In the table above, t-statistics are included in parentheses. Most of the quarterly and stock fixed effects are significant in the specifications that we study.

Table 7.6 Regression of Dummies on Market Indexes.

VARIABLE	COEFFICIENT	COEFFICIENT
Constant	.8208896 (3.965)	0.6270 (4.3)
Nasdaq Index	-.0003467 (-4.960)	-
QQQ Price	-	-0.007 (-4.7)
Nobs	21	18
R^2	0.48	0.6830

Table 7.7 Ordered Logit Estimates of the Effect of Conflicts of Interest.

VARIBLE	COEF.	COEF.	COEF.	COEF.
RELATION	-.3231 (-6.19)	-.1108 (-2.06)	.05397 (0.94)	.03932 (0.68)
IBANK				.1080 (4.18)
Log Likelihood	-16152.389	-15314.605	-14860.888	-14855.94
Psueo-R^2	0.0012	0.0530	0.0811	0.0814
Fixed Effects	none	quarterly	quarterly, stock	quarterly, stock

In the ordered probit model, the dependent variable is the analyst's recommendation as coded by IBES. This takes on discrete values from one to five. In the table above, t-statistics are included in parentheses. We do not report ancillary parameters, such as the cut values and values of the fixed effects.

Table 7.8 Ordered Logit Estimates including Peer Effects (Parametric First Stage)

VARIABLE	*COEF.*	*COEF.*	*COEF.*	*COEF.*
IVBELIEF	1.121982 (24.66)	1.108108 (7.90)	.2531818 (5.80)	.2531353 (5.75)
IBANK	-	-	-	.2938029 (4.41)
RELATION	-	-	-	-.0879437 (-1.16)
%DEV	-	-	-	.0073782 (0.71)
Log Likelihood	-15113.967	-14930.088	-14823.868	-14535.476
Psueo- R^2	0.0472	0.0588	0.0830	0.0837
Fixed Effects	none	stock	quarterly, stock	quarterly, stock

In the ordered logit model, the dependent variable is the analyst's recommendation as coded by IBES. This takes on discrete values from one to five. In the table above, t-statistics are included in parentheses (the t-statistic for the variable IVBELIEF is corrected using bootstrap). IVBELIEF is constructed by subtracting the individual fitted values (divided by the number of the other analysts for a specific stock in a specific quarter) from the conditional expectations of fitted values (conditional on quarter and stock), with these fitted values being from the parametric first stage regression. Most of the quarterly and stock fixed effects are significant in the specifications that we study.

Table 7.9 Ordered Logit Estimates including Peer Effects (Semiparametric First Stage)

VARIABLE	*COEF.*	*COEF.*	*COEF.*	*COEF.*
IVBELIEF	1.121863 (10.73)	1.108009 (23.20)	.2542394 (5.95)	.2540114 (5.90)
IBANK	-	-	-	.2935598 (4.41)
RELATION	-	-	-	-.0880207 (-1.16)
%DEV	-	-	-	.0072834 (0.70)
Log Likelihood	-15113.55	-14929.674	-14545.766	-14535.35
Psueo- R^2	0.0472	0.0588	0.0830	0.0837
Fixed Effects	none	stock	quarterly, stock	quarterly, stock

In the ordered logit model, the dependent variable is the analyst's recommendation as coded by IBES. This takes on discrete values from one to five. In the table above, t-statistics are included in parentheses (the t-statistic for the variable IVBELIEF is corrected using bootstrap). IVBELIEF is constructed by subtracting the individual fitted values (divided by the number of the other analysts for a specific stock in a specific quarter) from the conditional expectations of fitted values (conditional on quarter and stock), with these fitted values being from the first stage semiparametric sieve regression. Most of the quarterly and stock fixed effects are significant in the specifications that we study.

Table 7.10 Random Effect Estimates including Peer Effects (Semiparametric First Stage)

VARIABLE	*COEF.*	*COEF.*	*COEF.*	*COEF.*	*COEF.*
IVBELIEF	.766924 (11.53)	-	-	.7631302 (11.51)	.7672082 (10.63)
IBANK	-	.6669054 (7.94)	-	-	.6559327 (7.87)
RELATION	-	-	-.2539996 (-1.61)	-.1754189 (-1.22)	-.2416273 (-1.67)
Log Likelihood	-6454.5547	-6687.9966	-6717.4526	-6453.8047	-6423.6011

The dependent variable is equal to zero if the analyst's recommendation is recorded as 1 (i.e. strong buy) in IBES and one otherwise. IVBELIEF is constructed by subtracting the individual fitted values (divided by the number of analysts for a specific stock in a specific quarter) from the conditional expectations of fitted values (conditional on quarter and stock), with these fitted values being from the first stage semiparametric sieve regression. In the table above, t-statistics are included in parentheses (the t-statistic for the variable IVBELIEF is corrected using bootstrap). There was a unique fixed effect for each stock during each quarter. In the data, 3447 observations were dropped due to all positive or negative recommendations.

Computing Multiple Equilibria

Table 7.11 Equilibrium Simulations.

	REC=1	REC=2	REC=3	REC=4	REC=5
IBANK=0 QUARTER=9	0.6353	0.2785	0.08113	0.4650E-03	0.3400E-04
IBANK=1 QUARTER=9 FIRST EQUILBRIUM	0.06847	0.68742	0.22397	0.014376	0.5753E-02
IBANK=1 QUARTER=9 SECOND EQUILBRIUM	0.4042	0.4009	0.1821	0.01183	0.87246E-04
IBANK=0 QUARTER=21	0.2096	0.4076	0.3507	0.2967E-02	0.2235E-03
IBANK=1 QUARTER=21	0.09775	0.2936	0.5316	0.7127E-02	0.5653E-03

Introducing Dynamics

- Players are forward looking.
- Infinite Horizon, Stationary, Markov Transition
- Now players maximize expected discounted utility using discount factor β.

$$W_i(s, \epsilon_i; \sigma) = \max_{a_i \in A_i} \Big\{ \Pi_i(a_i, s) + \epsilon_i (a_i) \\ + \beta \int \sum_{a_{-i}} W_i(s', \epsilon_i'; \sigma) g(s' | s, a_i, a_{-i}) \sigma_{-i}(a_{-i}|s) f(\epsilon_i') d\epsilon_i' \Big\}$$

- Definition: A Markov Perfect Equilibrium is a collection of $\delta_i(s, \epsilon_i)$, $i = 1, ..., n$ such that for all i, all s and all ϵ_i, $\delta_i(s, \epsilon_i)$ maximizes $W_i(s, \epsilon_i; \sigma_i, \sigma_{-i})$.

- Conditional independence:
  - ϵ distributed i.i.d. over time.
  - State variables evolve according to $g\left(s'|s, a_i, a_{-i}\right)$.
- Define choice specific value function

$$V_i(a_i, s) = \Pi_i(a_i, s) + \beta E\left[V_i(s')|s, a_i\right].$$

- Players chooose a_i to maximize $V_i\left(a_i, s\right) + \epsilon_i\left(a_i\right)$,
- Ex ante value function (Social surplus function)

$$\begin{aligned} V_i\left(s\right) =& E_{\epsilon_i} \max_{a_i}\left[V_i(a_i, s) + \epsilon_i\left(a_i\right)\right] \\ =& G\left(V_i(a_i, s), \forall a_i = 0, \ldots, K\right) \\ =& G\left(V_i(a_i, s) - V_i\left(0, s\right), \forall a_i = 1, \ldots, K\right) + V_i\left(0, s\right) \end{aligned}$$

- When the error terms are extreme value distributed

$$\begin{aligned} V_i(s) &= \log \sum_{k=0}^{K} \exp\left(V_i(k,s)\right) \\ &= \log \sum_{k=0}^{K} \exp\left(V_i(k,s) - V_i(0,s)\right) + V_i(0,s). \end{aligned}$$

- Relationship between $\Pi_i(a_i, s)$ and $V_i(a_i, s)$:

$$\begin{aligned} V_i(a_i,s) =& \Pi_i(a_i,s) + \beta E\left[G\left(V_i(a_i,s'), \forall a_i = 0,\ldots,K\right)|s,a_i\right] \\ =& \Pi_i(a_i,s) + \beta E\left[G\left(V_i(k,s') - V_i\left(0,s'\right), \forall k = 1,\ldots,K\right)|s,a_i\right] \\ &+ \beta E\left[V_i\left(0,s'\right)|s,a_i\right] \end{aligned}$$

- With extreme value distributed error terms

$$\begin{aligned} V_i(a_i,s) =& \Pi_i(a_i,s) + \beta E\left[\log \sum_{k=0}^{K} \exp\left(V_i\left(k,s'\right) - V_i\left(0,s'\right)\right)|s,a_i\right] \\ &+ \beta E\left[V_i\left(0,s'\right)|s,a_i\right] \end{aligned}$$

- Hotz and Miller (1993): one to one mapping between $\sigma_i\left(a_i|s\right)$ and differences in choice specific value functions:

$$\left(V_i(1,s)-V_i(0,s),\ldots V_i(K,s)-V_i(0,s)\right)=\Omega_i\left(\sigma_i(0|s),\ldots,\sigma_i(K|s)\right)$$

- Example: i.i.d extreme value $f(\epsilon_i)$:

$$\sigma_i(a_i|s)=\frac{\exp(V_i(a_i,s)-V_i(0,s))}{\sum_{k=0}^{K}\exp(V_i(k,s)-V_i(0,s))}$$

- Inverse mapping:

$$\log\left(\sigma_i(k|s)\right)-\log\left(\sigma_i(0|s)\right)=V_i(k,s)-V_i(0,s)$$

- Since we can recover $V_i\left(k,s\right)-V_i\left(0,s\right)$, we only need to know $V_i\left(0,s\right)$ to recover $V_i\left(k,s\right),\forall k$.
- If we know $V_i\left(0,s\right)$, $V_i\left(a_i,s\right)$ and $\Pi_i\left(a_i,s\right)$ is one to one.

- Identify $V_i\left(0,s\right)$ first. Set $a_i = 0$:

$$V_i(0,s) = \Pi_i(0,s) + \beta E\left[\log \sum_{k=0}^{K} \exp\left(V_i\left(k,s'\right) - V_i\left(0,s'\right)\right)|s,0\right] + \beta E\left[V_i\left(0,s'\right)|s,0\right]$$

- This is a single contraction mapping unique fixed point iteration.
- Add $V_i\left(0,s\right)$ to $V_i\left(k,s\right) - V_i\left(0,s\right)$ to identify all $V_i\left(k,s\right)$.
- Then all $\Pi_i\left(k,s\right)$ calculated from $V_i\left(k,s\right)$ through

$$\Pi_i\left(k,s\right) = V_i\left(k,s\right) - \beta E\left[V_i\left(s'\right)|s,k\right].$$

- Why normalize $\Pi_i(0, s) = 0$?
- Why not $V_i(0, s) = 0$?
- If a firm stays out of the market in period t, current profit 0, but option value of future entry might depend on market size, number of other firms, etc.
- These state variables might evolve stochastically.
- Rest of the identification arguments: identical to the static model.

- Nonparametric and Semiparametric Estimation
- Hotz-Miller inversion recovers $V_i\left(k,s\right)-V_i\left(0,s\right)$ instead of $\Pi_i\left(k,s\right)-\Pi_i\left(0,s\right)$.
- Nonparametrically compute $V_i\left(0,s\right)$ using

$$\begin{aligned}\hat{V}_i(0,s)=&\beta\hat{E}\left[\log\sum_{k=0}^{K}\exp\left(\hat{V}_i\left(k,s'\right)-\hat{V}_i\left(0,s'\right)\right)|s,0\right]\\&+\beta\hat{E}\left[\hat{V}_i\left(0,s'\right)|s,0\right]\end{aligned}$$

- Obtain and $\hat{V}_i\left(k,s\right)$ and forward compute $\hat{\Pi}_i\left(k,s\right)$.
- The rest is identical to the static model.

- In semiparametric models, $\hat{\theta}$ converges at a $T^{1/2}$ rate and has normal asymptotics.
- Apply the results of Newey (1994)-derive appropriate "influence functions".
- The asymptotic distribution is invariant to the choice of method used to estimate the first stage.
- With proper weighting function (need to estimate nonparametrically), can achieve the same efficiency as full information maximum likelihoood.
- These results hold for both static and dynamic models.

Conclusion

- Static and dynamic interaction models.
- Incomplete information assumption.
- Nonparametric identification.
- Nonparametric and Semiparametric estimation.
- Need for computation of multiple equilibria.
- Equilibria computation important for model simulation.
- Extension: parametric estimation method that allows for multiple equilibria.