Judgment aggregation without full rationality

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Abstract

Several recent results on the aggregation of judgments over logically connected propositions show that, under certain conditions, dictatorships are the only independent (i.e., propositionwise) aggregation rules generating fully rational (i.e., complete and consistent) collective judgments. A frequently mentioned route to avoid dictatorships is to allow incomplete collective judgments. We show that this route does not lead very far: we obtain (strong) oligarchies rather than dictatorships if instead of full rationality we merely require that collective judgments be deductively closed, arguably a minimal condition of rationality (compatible even with empty judgment sets). We derive several characterizations of oligarchies and provide illustrative applications to Arrowian preference aggregation and Kasher and Rubinstein's group identification problem.

1 Introduction

Sparked by the "discursive paradox", the problem of "judgment aggregation" has recently received much attention. The "discursive paradox", of which Condorcet's famous paradox is a special case, consists in the fact that, if a group of individuals takes majority votes on some logically connected propositions, the resulting collective judgments may be inconsistent, even if all group members' judgments are individually consistent (Pettit 2001, extending Kornhauser and Sager 1986; List and Pettit 2004). A simple example is given in Table 1.

	a	b	$a \wedge b$
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
Majority	True	True	False

Table 1: A discursive paradox

Several subsequent impossibility results have shown that majority voting is not alone in its failure to ensure rational (i.e., complete and consistent) collective judgments when propositions are interconnected (List and Pettit 2002, Pauly and van Hees 2006, Dietrich 2006, Gärdenfors 2006, Nehring and Puppe 2002, 2005, van Hees forthcoming, Dietrich forthcoming, Mongin 2005, Dokow and Holzman 2005, Dietrich and List forthcoming-a). The generic finding is that, under the requirement of proposition-by-proposition aggregation (independence), dictatorships are the only aggregation rules generating complete and consistent collective judgments and satisfying some other conditions (which differ from result to result). This generic finding is broadly analogous to Arrow's theorem for preference aggregation. (Precursors to this recent literature are Wilson's 1975 and Rubinstein and Fishburn's 1986 contributions on abstract aggregation theory.)

A frequently mentioned escape route from this impossibility is to drop the requirement of complete collective judgments and thus to allow the group to make no judgment on some propositions. Examples of aggregation rules that ensure consistency at the expense of incompleteness are unanimity and certain supermajority rules (List and Pettit 2002, List 2004, Dietrich and List forthcoming-b).

The most forceful critique of the completeness requirement has been made by Gärdenfors (2006), in line with his influential theory of belief revision (e.g., Alchourron, Gärdenfors and Makinson 1985). Describing completeness as a "strong and unnatural assumption", Gärdenfors has argued that neither individuals nor a group need to hold complete judgments and that, in his opinion, "the [existing] impossibility results are consequences of an unnaturally strong restriction on the outcomes of a voting function". Gärdenfors has also proved the first and so far only impossibility result on judgment aggregation without completeness, showing that, under certain conditions, any aggregation rule generating consistent and deductively closed (but not necessarily complete) collective judgments, while not necessarily dictatorial, is weakly oligarchic.

In this paper, we continue this line of research and investigate judgment aggregation without the completeness requirement. We drop this requirement, first at the collective level and later at the individual level, replacing it with the weaker requirement of merely deductively closed judgments. Our results do not need the requirement of collective consistency. Under standard conditions on aggregation rules and the weakest possible assumptions about the agenda of propositions under consideration, we provide the first characterizations of (strong) oligarchies (without a default)[1] and the first characterization of the unanimity rule[2] (the only anonymous oligarchy). As corollaries, we also obtain new variants of several characterizations of dictatorships in the literature (using no consistency condition).

Our results strengthen Gärdenfors's oligarchy results in three respects. First, they impose weaker conditions on aggregation rules. Second, they show that strong and not merely weak oligarchies are implied by these conditions and fully

[1] For truth-functional agendas, Nehring and Puppe (2005) have characterized *oligarchies with a default*, which are distinct from the *(strong or weak) oligarchies* considered by Gärdenfors (2006) and in this paper. Oligarchies with a default by definition generate complete collective judgments.

[2] Again without a default, thus with possibly incomplete outcomes.

characterize strong oligarchies. Third, they do not require the logically rich and infinite agenda of propositions Gärdenfors assumes. They reinforce Gärdenfors's arguments, however, in showing that, under surprisingly mild conditions, we are restricted to oligarchic aggregation rules.

In judgment aggregation, one can distinguish between *impossibility results* (like Gärdenfors's results) and *characterizations of impossibility agendas* (like the present results and the results cited below). The former show that, for certain agendas of propositions, aggregation in accordance with certain conditions is impossible or severely restricted (e.g., to dictatorships or oligarchies). The latter characterize the precise class of agendas for which such an impossibility or restriction arises (and hence the class of agendas for which it does not arise). Characterizations of impossibility agendas have the merit of identifying precisely which kinds of decision problems are subject to the impossibility results in question and which are free from them. (Notoriously, preference aggregation problems are subject to most such impossibility results.) There has been much recent progress on such characterizations. Nehring and Puppe (2002) were the first to prove such results. Subsequent results have been derived by Dokow and Holzman (2005), Dietrich (forthcoming) and Dietrich and List (forthcoming-a). But so far all characterizations of impossibility agendas assume fully rational collective judgments. We here give the first characterizations of impossibility agendas without requiring complete (nor even consistent) collective judgments.

2 The model

Consider a set of individuals $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) seeking to make collective judgments on some logically connected propositions. To represent propositions, we introduce a logic, using Dietrich's (forthcoming) general logics framework (generalizing List and Pettit 2002, 2004). A *logic (with negation symbol* $\neg$*)* is a pair $(\mathbf{L}, \vDash)$ such that

(i) $\mathbf{L}$ is a non-empty set of formal expressions (*propositions*) closed under negation (i.e., $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$), and

(ii) $\vDash$ is a binary (*entailment*) relation ($\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L}$), where, for each $A \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $A \vDash p$ is read as "A entails p".

A set $A \subseteq \mathbf{L}$ is *inconsistent* if $A \vDash p$ and $A \vDash \neg p$ for some $p \in \mathbf{L}$, and *consistent* otherwise. Our results hold for any logic $(\mathbf{L}, \vDash)$ satisfying four minimal conditions;[3] this includes standard propositional, predicate, modal and

[3] L1 (self-entailment): For any $p \in \mathbf{L}$, $\{p\} \vDash p$. L2 (monotonicity): For any $p \in \mathbf{L}$ and any $A \subseteq B \subseteq \mathbf{L}$, if $A \vDash p$ then $B \vDash p$. L3 (completability): $\emptyset$ is consistent, and each consistent set $A \subseteq \mathbf{L}$ has a consistent superset $B \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. L4 (non-paraconsistency): For any $A \subseteq \mathbf{L}$ and any $p \in \mathbf{L}$, if $A \cup \{\neg p\}$ is inconsistent then $A \vDash p$. In L4, the converse implication also holds given L1-L3. See Dietrich (forthcoming, Section 4) for the main properties of entailment and inconsistency under L1-L4.

conditional logics. For example, in standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \rightarrow b)$, and $\vDash$ satisfies $\{a, a \rightarrow b\} \vDash b$, $\{a\} \vDash a \vee b$, but not $a \vDash a \wedge b$.

Some definitions are useful. A proposition $p \in \mathbf{L}$ is a *tautology* if $\{\neg p\}$ is inconsistent, and a *contradiction* if $\{p\}$ is inconsistent. A proposition $p \in \mathbf{L}$ is *contingent* if it is neither a tautology nor a contradiction. A set $A \subseteq \mathbf{L}$ is *minimal inconsistent* if it is inconsistent and every proper subset $B \subsetneq A$ is consistent.

The *agenda* is a non-empty subset $X \subseteq \mathbf{L}$, interpreted as the set of propositions on which judgments are to be made, where X can be written as $\{p, \neg p : p \in X^*\}$ for a set $X^* \subseteq \mathbf{L}$ of unnegated propositions. For notational simplicity, double negations within the agenda cancel each other out, i.e., $\neg\neg p$ stands for p.[4] In the example above, the agenda is $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$ in standard propositional logic. Informally, an agenda captures a particular decision problem.

An *(individual or collective) judgment set* is a subset $A \subseteq X$, where $p \in A$ means that proposition p is accepted (by the individual or group). Different interpretations of "acceptance" can be given. On the standard interpretation, to accept a proposition means to believe it, so that judgment aggregation is the aggregation of (binary) belief sets. On an entirely different interpretation, to accept a proposition means to desire it, so that judgment aggregation is the aggregation of (binary) desire sets.

A judgment set $A \subseteq X$ is

(i) *consistent* if it is a consistent set in $\mathbf{L}$,
(ii) *complete* if, for every proposition $p \in X$, $p \in A$ or $\neg p \in A$,
(iii) *deductively closed* if, for every proposition $p \in X$, if $A \vDash p$ then $p \in A$.

Note that the conjunction of consistency and completeness implies deductive closure, while the converse does not hold (Dietrich forthcoming, List 2004). Deductive closure can be met by "small", even empty, judgment sets $A \subseteq X$. Hence deductive closure is a much weaker requirement than "full rationality" (the conjunction of consistency and completeness). Let $\mathcal{C}$ be the set of all complete and consistent (and hence also deductively closed) judgment sets $A \subseteq X$. A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of individual judgment sets.

A *(judgment) aggregation rule* is a function F that assigns to each admissible profile $(A_1, \ldots, A_n)$ a collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. The set of admissible profiles is denoted $Domain(F)$.

Call F *universal* if $Domain(F) = \mathcal{C}^n$; call it *consistent*, *complete*, or *deductively closed* if it generates a consistent, complete, or deductively closed collective judgment set $A = F(A_1, \ldots, A_n)$ for every profile $(A_1, \ldots, A_n) \in Domain(F)$; call it *unanimity-respecting* if $F(A, ..., A) = A$ for all unanimous

[4] To be precise, when we use the negation symbol $\neg$ hereafter, we mean a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is unnegated and $\sim p := q$ if $p = \neg q$ for some q.

profiles $(A, ..., A) \in Domain(F)$; and call it *anonymous* if, for any profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$ that are permutations of each other, $F(A_1, \ldots, A_n) = F(A_1^*, \ldots, A_n^*)$.

Examples of aggregation rules are *majority voting*, where, for each $(A_1, ..., A_n) \in \mathcal{C}^n$, $F(A_1, ..., A_n) = \{p \in X : |\{i \in N : p \in A_i\}| > |\{i \in N : p \notin A_i\}|\}$ and a *dictatorship* of some individual $i \in N$, where, for each $(A_1, ..., A_n) \in \mathcal{C}^n$, $F(A_1, ..., A_n) = A_i$. Majority voting and dictatorships are each universal and unanimity-respecting. Majority voting is anonymous while dictatorships are not. But, as the "discursive paradox" shows, majority voting is not consistent (or deductively closed) (and it is complete if and only if n is odd), while dictatorships are consistent, complete and deductively closed. For some agendas X, so-called premise-based and conclusion-based aggregation rules can be defined.

The model can represent various realistic decision problems, including Arrowian preference aggregation problems and Kasher and Rubinstein's group identification problem, as illustrated in Sections 4 and 5.

3 Characterization results

Are there any appealing aggregation rules F if we allow incomplete outcomes? Our results share with previous results the requirement of *propositionwise aggregation*: the group "votes" independently on each proposition, as captured by the following condition.

Independence. For any $p \in X$ and any $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow p \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow p \in F(A_1^*, \ldots, A_n^*)$.

Interpretationally, independence requires the group judgment on any given proposition $p \in X$ to "supervene" on the individual judgments on p (List and Pettit forthcoming). This reflects a "local" notion of democracy, which could for instance be viewed as underlying direct democratic systems that are based on referenda on various propositions. If we require the group not only to vote independently on the propositions, but also to use the same voting method for each proposition (a neutrality condition), we obtain the following stronger condition.

Systematicity. For any $p, q \in X$ and any $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow q \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow q \in F(A_1^*, \ldots, A_n^*)$.

Some of our results require systematicity (and not just independence), and some also require the following responsiveness property.

Monotonicity. For any $(A_1, ..., A_n) \in Domain(F)$, we have $F(A_1^*, ..., A_n^*) = F(A_1, ..., A_n)$ for all $(A_1^*, ..., A_n^*) \in Domain(F)$ arising from $(A_1, ..., A_n)$ by replacing one A_i by $F(A_1, ..., A_n)$.

Monotonicity states that changing one individual's judgment set towards the present outcome (collective judgment set) does not alter the outcome.[5]

We call an aggregation rule F a *(strong) oligarchy* (dropping "strong" whenever there is no ambiguity) if it is universal and given by

$$F(A_1, .., A_n) = \cap_{i \in M} A_i \text{ for all profiles } (A_1, ..., A_n) \in \mathcal{C}^n, \qquad (1)$$

where $M \subseteq N$ is fixed non-empty set (of *oligarchs*). A *weak oligarchy* is a universal aggregation rule F such that there exists a smallest winning coalition, i.e., a smallest non-empty set $M \subseteq N$ that satisfies (1) with "$=$" replaced by "$\supseteq$".[6] An oligarchy (respectively, weak oligarchy) accepts all (respectively, at least all) propositions unanimously accepted by the oligarchs.

Interesting impossibility results on judgment aggregation never apply to all agendas X (decision problems). Typically, impossibilities using the (strong) systematicity condition apply to most relevant agendas, while impossibilities using the (weaker) independence condition apply to a class of agendas that both includes and excludes many relevant agendas. Our present results confirm this picture.

We here use two weak agenda conditions (for our systematicity results) and one much stronger one (for our independence results). For any sets $Z \subseteq Y \subseteq X$, let $Y_{\neg Z}$ denote the set $(Y \backslash Z) \cup \{\neg p : p \in Z\}$, which arises from Y by negating the propositions in Z. The two weak conditions are the following.

(α) There is an inconsistent set $Y \subseteq X$ with pairwise disjoint subsets $Z_1, Z_2, \{p\}$ such that $Y_{\neg Z_1}$, $Y_{\neg Z_2}$ and $Y_{\neg \{p\}}$ are consistent.

(β) There is an inconsistent set $Y \subseteq X$ with disjoint subsets $Z, \{p\}$ such that $Y_{\neg Z}$, $Y_{\neg \{p\}}$ and $Y_{\neg (Z \cup \{p\})}$ are consistent.

These conditions are not *ad hoc*. As shown later, they are the weakest possible conditions needed for our results. Moreover, (α) and (β) are weaker than (and if X is finite or the logic compact, equivalent to), respectively, the

[5] This is a judgment-set-wise monotonicity condition, which differs from a proposition-wise one (e.g., Dietrich and List 2005). Similarly, our condition of unanimity-respectance is judgment-set-wise rather than proposition-wise. One may consider this as an advantage, since a flavour of independence is avoided, so that the conditions in the characterisation are in the inutitive sense "orthogonal" to each other.

[6] The term "oligarchy" (without further qualification) refers to a strong oligarchy, whereas in Gärdenfors (2006) it refers to a weak one. A distinct oligarchy notion is Nehring and Puppe's (2005) "oligarchy with a default", which always generates complete collective judgments by reverting to a default on each pair $p, \neg p \in X$ on which the oligarchs disagree.

following conditions.[7]

- ($\tilde{\alpha}$) There is a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$.
- ($\tilde{\beta}$) There is a minimal inconsistent set $Y \subseteq X$ with disjoint subsets $Z, \{p\}$ such that $Y_{\neg Z}$ and $Y_{\neg(Z \cup \{p\})}$ are consistent. (If X is finite or the logic compact, this is equivalent to a standard condition[8].)

The conditions ($\tilde{\alpha}$) and ($\tilde{\beta}$), and hence (α) and (β), hold for most standard examples of judgment aggregation agendas X. For instance, if X contains propositions $a, b, a \wedge b$ as in the example of Table 1, then in ($\tilde{\alpha}$) and ($\tilde{\beta}$) we can take $Y = \{a, b, \neg(a \wedge b)\}$, where in ($\tilde{\beta}$) $Z = \{a\}$ and $p = b$. If X contains propositions $a, a \rightarrow b, b$ then in ($\tilde{\alpha}$) and ($\tilde{\beta}$) we can take $Y = \{a, a \rightarrow b, \neg b\}$, where in ($\tilde{\beta}$) $Z = \{a\}$ and $p = \neg b$. In Sections 4 and 5, we show that the agendas for representing Arrowian preference aggregation or Kasher and Rubinstein's group identification problem also satisfy (α) and (β).

The stronger agenda condition, used in Theorem 2, is that of *path-connectedness*, a variant of Nehring and Puppe's (2002) *total blockedness* condition. For any $p, q \in X$, we write $p \vDash^* q$ (*p conditionally entails q*) if $\{p\} \cup Y \vDash q$ for some $Y \subseteq X$ consistent with p and with $\neg q$. For instance, for the agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$, we have $a \wedge b \vDash^* a$ (take $Y = \emptyset$) and $a \vDash^* \neg b$ (take $Y = \{\neg(a \wedge b)\}$). An agenda X is *path-connected* if, for every contingent $p, q \in X$, there exist $p_1, p_2, ..., p_k \in X$ (with $p = p_1$ and $q = p_k$) such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$.

The agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$ is *not* path-connected: for a negated proposition ($\neg a$ or $\neg b$ or $\neg(a \wedge b)$), there is no path to a non-negated proposition. By contrast, as discussed in Sections 4 and 5, the agendas for representing Arrowian preference aggregation problems or Kasher and Rubinstein's group identification problem are path-connected.

Theorem 1 *Let the agenda X satisfy (α) and (β).*

- (a) *The oligarchies are the only universal, deductively closed, unanimity-respecting and systematic aggregation rules.*
- (b) *Part (a) continues to hold if the agenda condition (β) is dropped and the aggregation condition of monotonicity is added.*

Theorem 2 *Let the agenda X satisfy path-connectedness and (β).*

- (a) *The oligarchies are the only universal, deductively closed, unanimity-respecting and independent aggregation rules.*

[7] For instance, ($\tilde{\alpha}$) implies (α) because, if Y is as in ($\tilde{\alpha}$), we may choose distinct $p, q, r \in Y$, put $Z_1 = \{p\}$, $Z_2 = \{q\}$ and $Z_3 = \{r\}$, and use the fact that a minimal inconsistent set becomes consistent by negating a single one of its members.

[8] This condition is condition "(ii)" in Dietrich (forthcoming) and in Dietrich and List (fothcoming): there is a minimal inconsistent set $Y \subseteq X$ such that $Y_{\neg Z}$ is consistent for some subset $Z \subseteq Y$ of even size. The claimed equivalence is shown in Lemma 12. Another equivalent condition (if X is finite) is Dokow and Holzman's (2005) *non-affineness* condition.

(b) *Part (a) continues to hold if the agenda condition (β) is dropped and the aggregation condition of monotonicity is added.*

Proofs are given in the Appendix. Theorems 1 and 2 provide four characterizations of oligarchies. They differ in the conditions imposed on aggregation rules and the agendas permitted. Part (a) of Theorem 2 is perhaps the most surprising result, as it characterizes oligarchies on the basis of the logically weakest set of conditions on aggregation rules. We later apply this result to Arrowian preference aggregation problems and Kasher and Rubinstein's group identification problem.

In each characterization, adding the condition of anonymity eliminates all oligarchies except the *unanimity rule* (i.e., the oligarchy with $M = N$), and adding the condition of completeness eliminates all oligarchies except dictatorships (as defined above). So we obtain characterizations of the unanimity rule and of dictatorships.

Corollary 1 (a) *In each part of Theorems 1 and 2, the unanimity rule is the only aggregation rule satisfying the specified conditions and anonymity.*
(b) *In each part of Theorems 1 and 2, dictatorships are the only aggregation rules satisfying the specified conditions and completeness.*

Note that none of the characterizations of oligarchic, dictatorial or unanimity rules uses a collective consistency condition: consistency follows from the other conditions, as is seen from the consistency of oligarchic, dictatorial or unanimity rules.

As mentioned in the introduction, our results are related to (and strengthen) Gärdenfors's (2006) oligarchy results. We discuss the exact relationship in Section 6, when we relax the requirement of completeness not only at the collective level but also at the individual one.

Part (b) of Corollary 1 is also related to the characterizations of dictatorships by Nehring and Puppe (2002), Dokow and Holzman (2005) and Dietrich and List (forthcoming-a). To be precise, the dictatorship corollaries derived from parts (a) of Theorems 1 and 2 are variants (without a collective consistency condition) of Dokow and Holzman's (2005) and Dietrich and List's (forthcoming-a) characterizations of dictatorships.[9] The dictatorship corollaries derived from parts (b) of Theorems 1 and 2 are variants (again without a collective consistency condition) of Nehring and Puppe's (2002) characterizations of dictatorships.

As announced in the introduction, we seek to characterize impossibility agendas. While Theorems 1 and 2 establish the sufficiency of our agenda conditions for the present oligarchy results, we also need to establish their necessity. This

[9] Our agenda conditions are, in the general case, at least as strong as those of the mentioned other dictatorship characterizations; but they are equivalent to them if X is finite or belongs to a compact logic (because then (β) reduces to a standard condition; see footnote 8).

is done by the next result. The proof consists in the construction of appropriate non-oligarchic counterexamples, given in the Appendix.[10]

Theorem 3 *Suppose $n \geq 3$ (and X contains at least one contingent proposition).*

(a) *If the agenda condition (β) is violated, there is a non-oligarchic (in fact, non-monotonic) aggregation rule that is universal, deductively closed, unanimity-respecting and systematic.*
(b) *If the agenda condition (α) is violated, there is a non-oligarchic aggregation rule that is universal, deductively closed, unanimity-respecting, systematic and monotonic.*
(c) *If the agenda is not path-connected, and is finite or belongs to a compact logic, there is a non-oligarchic (in fact, non-systematic) aggregation rule that is universal, deductively closed, unanimity-respecting, independent and monotonic.*

4 Application I: preference aggregation

We apply Theorem 2 to the aggregation of (strict) preferences, specifically to the case where a profile of fully rational individual preference orderings is to be aggregated into a possibly partial collective preference ordering.

To represent this aggregation problem in the judgment aggregation model, consider the *preference agenda* (Dietrich and List forthcoming-a; see also List and Pettit 2004), defined as $X = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K \text{ with } x \neq y\}$, where

(i) $\mathbf{L}$ is a simple predicate logic, with
- a two-place predicate P (representing strict preference), and
- a set of constants $K = \{x, y, z, ...\}$ (representing alternatives);

(ii) for each $S \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $S \vDash p$ if and only if $S \cup Z$ entails p in the standard sense of predicate logic, with Z defined as the set of rationality conditions on strict preferences.[11]

We claim that strict preference orderings can be formally represented as judgments on the preference agenda. Call a binary preference relation $\succ$ on K a *strict partial ordering* if it is asymmetric and transitive, and call $\succ$ a *strict ordering* if it is in addition connected. Notice that (i) the mapping that assigns to each strict partial ordering $\succ$ the judgment set $A = \{xPy, \neg yPx \in X : x \succ_i y\} \subseteq X$ is a bijection between the set of all strict partial orderings and the set of all consistent and deductively closed (but not necessarily complete) judgment

[10] Part (c) still holds for $n = 2$. It also follows from a rule specified by Nehring and Puppe (2002); our proof uses a simpler (and non-complete) rule.

[11] Z contains $(\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1)$ (asymmetry), $(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3)$ (transitivity), $(\forall v_1)(\forall v_2)(\neg v_1{=}v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1))$ (connectedness) and, for each pair of distinct contants $x, y \in K$, $\neg x{=}y$.

sets; and (ii) the restriction of this mapping to strict orderings is a bijection between the set of all strict orderings and the set of all consistent and complete (hence deductively closed) judgment sets.

To apply Theorem 2, we observe that the preference agenda for three or more alternatives satisfies the agenda conditions of Theorem 2.

Lemma 1 *If $|K| \geq 3$, the preference agenda satisfies path-connectedness and (β).*

Proof. Let X be the *preference agenda* with $|K| \geq 3$. The path-connectedness of X is shown in Dietrich and List (forthcoming-a) (Nehring 2003 has proved this result for the weak preference agenda). In $(\tilde{\beta})$ (implying (β)), take $Y = \{xPy, yPz, zPx\}$ (for distinct alternatives $x, y, z \in K$), $Z = \{xPy\}$ and $p = yPz$. ■

Corollary 2 *For a preference agenda with $|K| \geq 3$, the oligarchies are the only universal, deductively closed (and also consistent), unanimity-respecting and independent aggregation rules.*

We have bracketed consistency since the result does not need the condition, although the interpretation offered above assumes it. In the terminology of preference aggregation, Corollary 2 shows that the oligarchies are the only preference aggregation rules with universal domain (of strict orderings) generating strict partial orderings and satisfying the weak Pareto principle and independence of irrelevant alternatives. Here an *oligarchy* is a preference aggregation rule such that, for each profile of strict orderings $(\succ_1, ..., \succ_n)$, the collective strict partial ordering $\succ$ is defined as follows: for any alternatives $x, y \in K$, $x \succ y$ if and only if $x \succ_i y$ for all $i \in M$, where $M \subseteq N$ is an antecedently fixed non-empty set of *oligarchs*.

This corollary is closely related to Gibbard's (1969) classic result showing that, if the requirement of transitive social orderings in Arrow's framework is weakened to that of quasi-transitive ones (requiring transitivity only for the strong component of the social ordering, but not for the indifference component), then oligarchies (suitably defined for the case of weak preference orderings) are the only preference aggregation rules satisfying the remaining conditions of Arrow's theorem. The relationship to our result lies in the fact that the strong component of a quasi-transitive social ordering is a strict partial ordering, as defined above.

5 Application II: group identification

Here we apply Theorem 2 to Kasher and Rubinstein's (1997) problem of "group identification". A set $N = \{1, 2, ..., n\}$ of individuals (e.g., a population) each make a judgment $J_i \subseteq N$ on which individuals in that set belong to a particular social group, subject to the constraint that at least one individual belongs to the group but not all individuals do (formally, each J_i satisfies $\varnothing \subsetneq J_i \subsetneq N$). The individuals then seek to aggregate their judgments $(J_1, ..., J_n)$ on who belongs to the social group into a resulting collective judgment J, subject to the same constraint ($\varnothing \subsetneq J \subsetneq N$). Thus Kasher and Rubinstein analyse the case in which the group membership status of all individuals must be settled definitively.

By contrast, we apply Theorem 2 to the case in which the membership status of individuals can be left undecided: i.e., some individuals are deemed members of the group in question, others are deemed non-members, and still others are left undecided with regard to group membership, subject to the very minimal "deductive closure" constraint that if all individuals except one are deemed non-members, then the remaining individual must be deemed a member, and if all individuals except one are deemed members, then the remaining individual must be deemed a non-member.

To represent this problem in our model (drawing on a construction in List 2006), consider the *group identification agenda*, defined as $X = \{a_1, \neg a_1, ..., a_n, \neg a_n\}$, where

(i) $\mathbf{L}$ is a simple propositional logic, with atomic propositions a_1, ..., a_n and the standard connectives $\neg$, $\wedge$, $\vee$;
(ii) for each $S \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $S \vDash p$ if and only if $S \cup Z$ entails p in the the standard sense of propositional logic, where $Z = \{a_1 \vee ... \vee a_n, \neg(a_1 \wedge ... \wedge a_n)\}$.

Informally, a_j is the proposition that "individual j is a member of the social group", and $S \models p$ means that S implies p relative to the constraint that the disjunction of a_1, ..., a_n is true and their conjunction false. The mapping that assigns to each J (with $\varnothing \subsetneq J \subsetneq N$) the judgment set $A = \{a_j : j \in J\} \cup \{\neg a_j : j \notin J\} \subseteq X$ is a bijection between the set of all fully rational judgments in the Kasher and Rubinstein sense and the set of all consistent and complete judgment sets in our model. A merely deductively closed judgment set $A \subseteq X$ represents a judgment that possibly leaves the membership status of some individuals undecided, as outlined above and illustrated more precisely below.

To apply Theorem 2, we observe that the group identification agenda for three or more individuals ($n \geq 3$) satisfies the agenda conditions of Theorem 2.

Lemma 2 *If $n \geq 3$, the group identification agenda satisfies path-connectedness and (β).*

Proof. Let X be the *group identification agenda* with $n \geq 3$. The path-connectedness of X is shown in List (2006). In ($\tilde{\beta}$) (implying (β)), take $Y = \{a_j : j \in N\}$, and let Z and $\{p\}$ be arbitrary disjoint subsets of Y with $Z \cup \{p\} \neq Y$. ■

Corollary 3 *For a group identification agenda with $n \geq 3$, the oligarchies are the only universal, deductively closed (and consistent), unanimity-respecting and independent aggregation rules.*

In group identification terms, the oligarchies are the only group identification rules with universal domain generating possibly incomplete but deductively closed group membership judgments and satisfying unanimity and independence. Here an *oligarchy* is a group identification rule such that, for each profile $(J_1, ..., J_n)$ of fully rational individual judgments on group membership, the collective judgment is given as follows: the set of determinate group members is $\bigcap_{i \in M} J_i$, the set of determinate non-members is $\bigcap_{i \in M} (N \backslash J_i)$, and the set of individuals with undecided membership status is the complement of these two sets in N, where $M \subseteq N$ is an antecedently fixed non-empty set of *oligarchs*.[12]

6 The case of incomplete individual judgments

As argued by Gärdenfors (2006), it is natural to relax the requirement of completeness not only at the collective level, but also at the individual one. Do the above impossibilities disappear if individuals can withhold judgments on some or even all pairs $p, \neg p \in X$? Unfortunately, the answer to this question is negative, even if the conditions of independence or systematicity are weakened by allowing the collective judgment on a proposition $p \in X$ to depend not only on the individuals' judgments on p but also on those on $\neg p$. Such weaker independence or systematicity conditions are arguably more defensible than the standard conditions: $\neg p$ is intimately related to p, and thus individual judgments on $\neg p$ should be allowed to matter for group judgments on p. As the weakened conditions are equivalent to the standard ones under individual completeness, all the results in Section 3 continue to hold for the weakened independence and systematicity conditions.

Formally, let $\mathcal{C}^*$ be the set of all consistent and deductively closed (but not necessarily complete) judgment sets $A \subseteq X$, and call F *universal** if F has domain $(\mathcal{C}^*)^n$ (a superdomain of $\mathcal{C}^n$). An *oligarchy** is the universal* variant of an oligarchy as defined above.

Following Gärdenfors (2006), call F *weakly independent* if, for any $p \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow p \in A_i^*$ and $\neg p \in A_i \Leftrightarrow \neg p \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow p \in$

[12] In fact, the set of individuals whose group membership status is to be decided need not coincide with the set of individuals who submit judgments on who is a member. More generally, the set N can make judgments on which individuals in another set K ($|K| \geq 3$) belong to a particular social group, subject to the constraint stated above. K can be infinite. Corollary 3 continues to hold since the corresponding group identification agenda (for a suitably adapted logic) still satisfies path-connectedness and (β). Interestingly, if K is infinite the agenda belongs to a non-compact logic.

$F(A_1^*, \ldots, A_n^*)$. Likewise, call F *weakly systematic* if, for any $p, q \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow q \in A_i^*$ and $\neg p \in A_i \Leftrightarrow \neg q \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow q \in F(A_1^*, \ldots, A_n^*)$.

We now give analogues of parts (a) of Theorems 1 and 2, proved in the Appendix.

Theorem 1* *Let the agenda X satisfy (α) and (β). The oligarchies* are the only universal*, deductively closed, unanimity-respecting and weakly systematic aggregation rules.*

Theorem 2* *Let the agenda X satisfy path-connectedness and (β). The oligarchies* are the only universal*, deductively closed, unanimity-respecting and weakly independent aggregation rules.*

In analogy with Theorems 1 and 2, these characterizations of oligarchies* do not contain a collective consistency condition (but require individual consistency). In each of Theorems 1* and 2*, adding the collective completeness requirement (respectively, anonymity) narrows down the class of aggregation rules to dictatorial ones (respectively, the unanimity rule), extended to the domain $(\mathcal{C}^*)^n$. So Theorems 1* and 2* imply characterizations of the latter rules on the domain $(\mathcal{C}^*)^n$. Note, further, that our applications of Theorem 2 to the preference and group identification agendas in Sections 4 and 5 can accommodate the case of incomplete individual judgments by using Theorem 2* instead of Theorem 2.

We can finally revisit the relationship of our results with Gärdenfors's results. Theorem 2, Corollary 1 and Theorem 2* strengthen Gärdenfors's oligarchy results. First, they do not require Gärdenfors's "social consistency" condition.[13] Second, they show that the conditions on aggregation rules imply (and in fact fully characterize) strong and not merely weak oligarchies (respectively, oligarchies*). Third, they weaken Gärdenfors's assumption that the agenda has the structure of an atomless Boolean algebra, replacing it with the weakest possible agenda assumption under which the oligarchy result holds, i.e., path-connectedness and (β).[14]

Our results reinforce the observation that, if we seek to avoid the standard impossibility results on judgment aggregation by allowing incomplete judgments while preserving the requirements of deductive closure and (weak) independence, this route does not lead very far. To obtain genuine possibilities, deduc-

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[13]Gärdenfors's "social logical closure" is equivalent to our "deductive closure", where *entailment* in Gärdenfors' Boolean algebra agenda X should be defined as follows: a set $A \subseteq X$ *entails* $p \in X$ if and only if $(\wedge_{q \in A_0} q) \wedge \neg p$ is the contradiction for some finite $A_0 \subseteq A$.

[14]It is easily checked that Gärdenfors's agenda satisfies (β) and path-connectedness, where paths involving at most two conditional entailments exist between any two propositions. To be precise, our present generalization of Gärdenfors's Corollary 3 applies to the case of a finite number of individuals. A similar generalization can be given for the infinite case.

tive closure must be relaxed or – perhaps better – independence must be given up in favour of non-propositionwise aggregation rules.

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A Appendix: proofs

We first introduce some notation. For all $B \subseteq X$,

- if B is consistent let $A_B \subseteq X$ be any consistent *and complete* judgment set such that $B \subseteq A_B$ (A_B is a "completion" of B and exists by L1-L3);
- let $\overline{B} := \{p \in X : B \vDash p\}$ ($\overline{B}$ is the "deductive closure" of B);
- let $B^{\neg} := \{\neg p : p \in B\}$.

Further, when we consider a profile $(A_1, ..., A_n)$, we often write N_p for the set $\{i : p \in A_i\}$ of individuals accepting $p \in X$. Finally, for any $\mathcal{W} \subseteq \mathcal{P}(N)$ (which can be arbitrary, even empty), let $F_{\mathcal{W}}$ be the universal rule given by

$$F(A_1, ..., A_n) = \{p \in X : N_p \in \mathcal{W}\} \text{ for all profiles } (A_1, ..., A_n) \in \mathcal{C}^n.$$

The first two lemmas have simple proofs, which we leave to the reader.

Lemma 3 *The intersection of deductively closed judgment sets is deductively closed. In particular, oligarchies are deductively closed.*

Lemma 4 (a) *F is universal and systematic if and only if $F = F_{\mathcal{W}}$ for some $\mathcal{W} \subseteq \mathcal{P}(N)$.*

(b) *F is oligarchic if and only if $F = F_{\{C \subseteq N : M \subseteq C\}}$ for some $\emptyset \neq M \subseteq N$.*

(c) *Let X contain a contingent proposition. Then, for all $\mathcal{W}, \mathcal{W}' \in \mathcal{P}(N)$,*
- *if $\mathcal{W} \neq \mathcal{W}'$ then $F_{\mathcal{W}} \neq F_{\mathcal{W}'}$;*
- *$F_{\mathcal{W}}$ is unanimity-respecting if and only if $N \in \mathcal{W}$ and $\emptyset \notin \mathcal{W}$;*
- *$F_{\mathcal{W}}$ is monotonic if and only if*

$$C \in \mathcal{W} \& C \subseteq C^* \subseteq N \Rightarrow C^* \in \mathcal{W}, \tag{2}$$

The next two lemmas are the essential steps towards Theorem 1.

Lemma 5 *Let X satisfy (β). For all $\mathcal{W} \subseteq \mathcal{P}(N)$, if $F_{\mathcal{W}}$ is unanimity-respecting and deductively closed, then* (2) *holds, i.e. $F_{\mathcal{W}}$ is monotonic by Lemma 4.*

Proof. Assume (β). Let $\mathcal{W} \subseteq \mathcal{P}(N)$, and suppose $F := F_{\mathcal{W}}$ is unanimity-respecting and deductively closed. We assume $C \in \mathcal{W} \& C \subseteq C^* \subseteq N$ and show $C^* \in \mathcal{W}$. Let Y, Z, p be as specified in (β). A profile $(A_1, ..., A_n)$ can be defined (using the above notation) by

$$A_i = \begin{cases} A_{Y_{\neg\{p\}}} & \text{if } i \in C \\ A_{Y_{\neg Z}} & \text{if } i \in N \backslash C^* \\ A_{Y_{\neg(Z \cup \{p\})}} & \text{if } i \in C^* \backslash C, \end{cases}$$

where we used that $Y_{\neg\{p\}}$, $Y_{\neg Z}$ and $Y_{\neg(Z\cup\{p\})}$ are consistent sets by (β). Now $F(A_1, ..., A_n)$ contains all $q \in Z$ by $N_q = C \in \mathcal{W}$, and all $q \in Y \backslash (Z \cup \{p\})$ by $N_q = N \in \mathcal{W}$. So $Y \backslash \{p\} \subseteq F(A_1, ..., A_n)$. By Y's inconsistency (and L4), $Y \backslash \{p\} \vDash \neg p$, whence $F(A_1, ..., A_n) \vDash \neg p$. So, by deductive closure, $\neg p \in F(A_1, ..., A_n)$. Hence $N_{\neg p} \in \mathcal{W}$, i.e. $C^* \in \mathcal{W}$, as desired. ■

Lemma 6 *Let X satisfy (α). For all $\emptyset \neq \mathcal{W} \subseteq \mathcal{P}(N)$ satisfying* (2)*, $F_{\mathcal{W}}$ is deductively closed if and only if $\mathcal{W} = \{C \subseteq N : M \subseteq C\}$ for some $M \subseteq N$.*

Proof. Let X and $\mathcal{W}$ be as specified. Let $F := F_{\mathcal{W}}$.

First, suppose $\mathcal{W} = \{C \subseteq N : M \subseteq C\}$ for some $M \subseteq N$. If $M \neq \emptyset$, then F is oligarchic by Lemma 4(b), hence deductively closed by Lemma 3. If $M = \emptyset$, then $\mathcal{W} = \mathcal{P}(N)$ by (2), whence F always generates the full set X, hence is again deductively closed.

Second, suppose F is deductively closed. Note that, to show that $\mathcal{W} = \{C \subseteq N : M \subseteq C\}$ for some $M \subseteq N$, it suffices (by $\mathcal{W} \neq \emptyset$ and (2)) to show that $\mathcal{W}$ is closed under taking finite intersections. Let $W, W' \in \mathcal{W}$, and let us show that $W \cap W' \in \mathcal{W}$. Let $Y, Z_1, Z_2, \{p\}$ be as in (α), and consider the profile $(A_1, ..., A_n)$ given (in the above notation) by

$$A_i = \begin{cases} A_{Y_{\neg Z_1}} & \text{if } i \in N \backslash W \\ A_{Y_{\neg Z_2}} & \text{if } i \in W \backslash W' \\ A_{Y_{\neg\{p\}}} & \text{if } i \in W \cap W', \end{cases}$$

where we use that $Y_{\neg Z_1}$, $Y_{\neg Z_2}$ and $Y_{\neg\{p\}}$ are each consistent by (α). Then $F(A_1, ..., A_n)$ contains all $q \in Z_1$ by $N_q = N \backslash (N \backslash W) = W \in \mathcal{W}$, contains all $q \in Z_2$ by $N_q = N \backslash (W \backslash W') \supseteq W' \in \mathcal{W}$ and (2), and contains all $r \in Y \backslash (Z_1 \cup Z_2 \cup \{p\})$ by $N_r = N \in \mathcal{W}$. So $Y \backslash \{p\} \subseteq F(A_1, ..., A_n)$. By Y's inconsistency (and L4), $Y \backslash \{p\} \vDash \neg p$. Hence $F(A_1, ..., A_n) \vDash \neg p$, so that by deductive closure $\neg p \in F(A_1, ..., A_n)$. Hence $N_{\neg p} \in \mathcal{W}$, i.e. $W \cap W' \in \mathcal{W}$, as desired. ■

Proof of Theorem 1. We prove first part (b) and then part (a).

(b) Let (α) hold. As noted above, oligarchies satisfy the specified conditions. Now suppose F satisfies the conditions. By Lemma 4(a), $F = F_{\mathcal{W}}$ for some $\mathcal{W} \subseteq \mathcal{P}(N)$, where by Lemma 4(c) $\mathcal{W}$ satisfies (2), $\emptyset \notin \mathcal{W}$ and $N \in \mathcal{W}$. Hence Lemma 6 applies, so that $\mathcal{W} = \{C \subseteq N : M \subseteq C\}$ for some $M \subseteq N$. As $\emptyset \notin \mathcal{W}$, $M \neq \emptyset$. So, by Lemma 4(b), F is oligarchic.

(a) Let (α) and (β) hold. Again, as noted, oligarchies have the specified properties. Suppose now that F has these properties. By Lemma 4(a), $F = F_{\mathcal{W}}$ for some $\mathcal{W} \subseteq \mathcal{P}(N)$. By Lemma 5, F is monotonic. So, by part (b), F is oligarchic. ■

Theorem 2 follows from Theorem 1 with the help of two further lemmas. The first lemma is similar to a proof step in Dietrich and List (2004), and the second lemma shows that a standard argument, first made by Nehring and Puppe (2002), requires neither completeness and consistency, nor monotonicity.

Lemma 7 *If X is path-connected and contains a contingent proposition, (α) holds.*

Proof. Let X be as specified. Then there are a contingent $q \in X$, and propositions $q = p_1, p_2, ..., p_k = \neg q \in X$ such that $p_t \vDash^* p_{t+1}$ for all $t \in \{1, ..., k-1\}$. We first show that $p_t \nvDash p_{t+1}$ for some $t \in \{1, ..., k-1\}$. Assume the contrary holds. As $\{p_1\} = \{q\}$ is consistent and $p_1 \vDash p_2$, $\{p_1, p_2\}$ is consistent. So, as $p_2 \vDash p_3$, $\{p_1, p_2, p_3\}$ is consistent. Repeating this procedure, $\{p_1, ..., p_k\}$ is consistent. But then $\{p_1, p_k\} = \{q, \neg q\}$ is consistent, a contradiction.

As just shown, there is a $t \in \{1, ..., k-1\}$ with $p_t \nvDash p_{t+1}$. As $p_t \vDash^* p_{t+1}$, we have $\{p_t\} \cup Y^* \vDash p_{t+1}$ for a $Y^* \subseteq X$ consistent with each of p_t and $\neg p_{t+1}$. It follows that

$$\{p_t, \neg p_{t+1}\} \cup Y^* \text{ is inconsistent,} \tag{3}$$

$$\{p_t, p_{t+1}\} \cup Y^* \text{ and } \{\neg p_t, \neg p_{t+1}\} \cup Y^* \text{ are each consistent..} \tag{4}$$

By $p_t \nvDash p_{t+1}$, we have $Y^* \neq \emptyset$. Since $\{p_t, \neg p_{t+1}\}$ is consistent, $\{p_t, \neg p_{t+1}\} \cup B$ is consistent for some set B consisting of exactly one member of each pair $r, \neg r$ in $\{r, \neg r : r \in Y^*\}$. Now we define $Y := \{p_t, \neg p_{t+1}\} \cup Y^*$, $p := p_t$, $Z_1 := \{\neg p_{t+1}\}$, and we let Z_2 be the subset of Y^* for which $Y^*_{\neg Z_2} = B$. Then, as required in (α), $Y = \{p_t, \neg p_{t+1}\} \cup Y^*$ is inconsistent (by (3)), and $Z_1, Z_2, \{p\}$ are pairwise disjoint subsets of Y, where the three sets $Y_{\neg\{p\}} = \{\neg p_t, \neg p_{t+1}\} \cup Y^*_t$, $Y_{\neg Z_1} = \{p_t, p_{t+1}\} \cup Y^*_t$ and $Y_{\neg Z_2} = \{p_t, \neg p_{t+1}\} \cup B$ are consistent (in the first two cases by (4)). ■

Call $C \subseteq N$ *semi-winning for* $p \in X$ (under F) if $p \in F(A_1, ..., A_n)$ for all profiles $(A_1, ..., A_n)$ in the domain with $\{i : p \in A_i\} = C$.

Lemma 8 *Let F be universal, deductively closed, independent and unanimity-respecting.*

(a) *For all* $p, q \in X$*, if* $C \subseteq N$ *is semi-winning for* p *and* $p \vDash^* q$ *then* C *is semi-winning for* q*.*
(b) *If* X *is path-connected,* F *is systematic.*

Proof. Let F be as specified.

(a) Consider $p, q \in X$. Suppose $C \subseteq N$ is semi-winning for p and $p \vDash^* q$. By $p \vDash^* q$, there is a $Y \subseteq X$ such that $\{p\} \cup Y \vDash q$, and $\{p\} \cup Y$ and $\{\neg q\} \cup Y$ are consistent. So, as $\{p, \neg q\} \cup Y$ is inconsistent, $\{p, q\} \cup Y$ and $\{\neg p, \neg q\} \cup Y$ are each consistent. Let $(A_1, ..., A_n)$ be the profile given (in the above notation) by

$$A_i = \begin{cases} A_{\{p,q\} \cup Y} & \text{if } i \in C \\ A_{\{\neg p, \neg q\} \cup Y} & \text{if } i \notin C. \end{cases}$$

As $N_p = C$ and C is semi-winning for p, $p \in F(A_1, ..., A_n)$. From unanimity-respectance and independence it follows that $Y \subseteq F(A_1, ..., A_n)$. So $\{p\} \cup Y \subseteq F(A_1, ..., A_n)$. Hence, by $\{p\} \cup Y \vDash q$ and deductive closure, $q \in F(A_1, ..., A_n)$. So, by $N_q = C$ and independence, C is semi-winning for q, as desired.

(b) Let X be path-connected. To show systematicity, consider any $p, q \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in \mathcal{C}^n$ such that $C := \{i : p \in A_i\} = \{i : q \in A_i^*\}$. We suppose that $p \in F(A_1, ..., A_n)$ and prove that $q \in F(A_1^*, ..., A_n^*)$. The latter holds if $C = N$: if $C = N$ then, using unanimity-respectance and independence, it follows that $q \in F(A_1^*, ..., A_n^*)$, as desired. Now let $C \neq N$. We have $C \neq \emptyset$, because otherwise, again by unanimity-respectance and independence, we have $p \notin F(A_1, ..., A_n)$, a contradiction. As C is neither N nor $\emptyset$, p and q are each contingent (by individual rationality). Hence, by path-connectedness, there are $p = p_1, p_2, ..., p_k = q \in X$ such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$. By $C = \{i : p \in A_i\}$, $p \in F(A_1, ..., A_n)$ and independence, C is semi-winning for $p = p_1$. So a simple induction using part (a) tells us that C is semi-winning for $p_k = q$, as desired. ■

We base come to the proof of Theorems 1*, which we derive from Theorem 1 using two lemmas.

Lemma 9 *For all* $A \subseteq X$*, the "deductive closure"* $\overline{A}$ *(*$= \{r \in X : A \vDash r\}$*) is deductively closed, and it is consistent if* A *is consistent.*

Proof. Let $A \subseteq X$.

To show that $\overline{A}$ is deductively closed suppose for a contradiction that $r \in X$ with $\overline{A} \vDash r$ but $r \notin \overline{A}$. Then $A \nvDash r$. So, by L4, $\{\neg r\} \cup A$ is consistent, hence extendible to a complete and consistent $B \subseteq X$ with $\{\neg r\} \cup A \subseteq B$. As B is deductively closed, $\overline{A} \subseteq B$. So $\{\neg r\} \cup \overline{A} \subseteq B$. So $\{\neg r\} \cup \overline{A}$ is consistent. Hence $\overline{A} \nvDash r$, a contradiction.

Now let A be consistent. Then A is extendible to a complete and consistent set $B \subseteq X$. As B is deductively closed, $\overline{A} \subseteq B$. So $\overline{A}$ is consistent. ■

For all $C, C' \subseteq N$, we call C *semi-winning against* C' *for* $p \in X$ (under F) if $p \in F(A_1, ..., A_n)$ for all profiles $(A_1, ..., A_n)$ in the domain with $\{i : p \in A_i\} = C$ and $\{i : \neg p \in A_i\} = C'$; and we call C simply *semi-winning against* C' (under F) if C is semi-winning against C' for every $p \in X$. Note that a weakly systematic rule F is uniquely given by its set of pairs $(C, C') \in (\mathcal{P}(N))^2$ for which C is semi-winning against C'.

Lemma 10 *Let F be universal*, deductively closed, unanimity-respecting and weakly systematic. Let $C \subseteq N$ be semi-winning against $\widetilde{C} \subseteq N$, with $C \cap \widetilde{C} = \emptyset$.*

(a) *If X satisfies (β), C is semi-winning against all $C' \subseteq \widetilde{C}$.*

(b) *If X satisfies (α)-(β), C is semi-winning against all $C' \subseteq N$, i.e. is semi-winning.*

Proof. Let $X, F, C, \widetilde{C}$ be as specified.

(a) Assume (β) holds, and consider any $C' \subseteq \widetilde{C}$. By ($\beta$) there are pairwise disjoint sets $Y^*, Z, \{p\} \subseteq X$ such that

(*) $Y^* \cup Z \cup \{p\}$ is inconsistent,

(**) $Y^* \cup Z \cup \{\neg p\}$, $Y^* \cup Z^\neg \cup \{p\}$ and $Y^* \cup Z^\neg \cup \{\neg p\}$ are consistent.

Consider the profile $(A_1, ..., A_n)$ given (in our notation) by

$$A_i = \begin{cases} \overline{Y^* \cup Z \cup \{\neg p\}} & \text{if } i \in C \\ \overline{Y^* \cup Z^\neg} & \text{if } i \in \widetilde{C} \backslash C' \\ \overline{Y^* \cup Z^\neg \cup \{p\}} & \text{if } i \in C' \\ \overline{Y^*} & \text{if } i \notin C \cup \widetilde{C}. \end{cases}$$

This profile is in $(\mathcal{C}^*)^n$, by Lemma 9 and (**). We have $Y^* \subseteq F(A_1, ..., A_n)$ because N is winning against $\emptyset$ by unanimity-respectance and weak systematicity. Further, for all $z \in Z$, as by (**) Y^* is consistent with z and with $\neg z$, $\overline{Y^*}$ contains neither z nor $\neg z$; and so $N_z = C$ and $N_{\neg z} = \widetilde{C}$, whence $Z \subseteq F(A_1, ..., A_n)$ as C is winning against $\widetilde{C}$. By $Y^* \cup Z \subseteq F(A_1, ..., A_n)$ and (*), $F(A_1, ..., A_n) \vDash \neg p$, whence by deductive closure $\neg p \in F(A_1, ..., A_n)$. As by (**) Y^* and $Y^* \cup Z^\neg$ are each consistent with p and with $\neg p$, none of $\overline{Y^*}$ and $\overline{Y^* \cup Z^\neg}$ contains p or $\neg p$; and so $N_p = C'$ and $N_{\neg p} = C$. So, using weak systematicity, C is semi-winning against C', as desired.

(b) Let X satisfy (α)-(β), and consider any $C' \subseteq N$. We show that C is semi-winning against C'. This is vacuously true if $C \cap C' \neq \emptyset$ (using universality*). Now suppose $C \cap C' = \emptyset$. As $C' \subseteq N \backslash C$, it suffices by part (a) to show that C is winning against $N \backslash C$.

By (α) there are pairwise disjoint sets $Y^*, Z_1, Z_2, \{p\} \subseteq X$ such that

(*) $Y^* \cup Z_1 \cup Z_2 \cup \{p\}$ is inconsistent;

(**) $Y^* \cup Z_1^\neg \cup Z_2 \cup \{p\}$, $Y^* \cup Z_1 \cup Z_2^\neg \cup \{p\}$ and $Y^* \cup Z_1 \cup Z_2 \cup \{\neg p\}$ are consistent.

Let $(A_1, ..., A_n)$ be the profile given by

$$A_i = \begin{cases} \overline{Y^* \cup Z_1 \cup Z_2 \cup \{\neg p\}} & \text{if } i \in C \\ \overline{Y^* \cup \{p\}} & \text{if } i \in N \backslash C. \end{cases}$$

As in part (a), this profile belongs to $(\mathcal{C}^*)^n$ (using Lemma 9 and (**)), and $Y^* \subseteq F(A_1, ..., A_n)$ (as N is winning against $\emptyset$ by unanimity-respectance and weak systematicity). Further, for all $z \in Z_1 \cup Z_2$, $Y^* \cup \{p\}$ is by (**) consistent with z and with $\neg z$, whence $\overline{Y^* \cup \{p\}}$ contains neither z nor $\neg z$, and so $N_z = C$ and $N_{\neg z} = \emptyset$. So, as C is by part (a) winning against $\emptyset$, $Z_1 \cup Z_2 \subseteq F(A_1, ..., A_n)$. By $Y^* \cup Z_1 \cup Z_2 \subseteq F(A_1, ..., A_n)$ and (*), $F(A_1, ..., A_n) \vDash \neg p$, so that by deductive closure $\neg p \in F(A_1, ..., A_n)$. So, by $N_{\neg p} = C$ and $N_p = N \backslash C$ and by weak systematicity C is winning against $N \backslash C$, as desired. ■

*Proof of Theorem 1**. Let X be as specified. Oligarchies satisfy all properties mentioned (using Lemma 3). Now let F have these properties. As F is weakly systematic, F is given, for all $(A_1, ..., A_n) \in (\mathcal{C}^*)^n$, by

$$F(A_1, ..., A_n) = \{p \in X : N_p \text{ is semi-winning against } N_{\neg p}\}.$$

So F is oligarchic* if there is a non-empty set $M \subseteq N$ such that

$$C \text{ is semi-winning against } C' \Leftrightarrow M \subseteq C, \text{ for all disjoint } C, C' \subseteq N. \quad (5)$$

To show this, note first that the rule $F|_{\mathcal{C}^n}$, obtained by restricting F to the domain $\mathcal{C}^n$, is by part (a) of Theorem 2 oligarchic, say with set of oligarchs M. We show that this set M satisfies (5). For any disjoint $C, C' \subseteq N$, C is semi-winning against C' if and only if C is semi-winning against $N \backslash C$, by Lemma 10 (and using that (α) holds by Lemma 7). The latter is equivalent to C being semi-winning under $F|_{\mathcal{C}^n}$ (using that $N_{\neg p} = N \backslash N_p$ for all $(A_1, ..., A_n) \in \mathcal{C}^n$ and all $p \in X$), which is in turn equivalent to $M \subseteq C$ as $F|_{\mathcal{C}^n}$ is the M-oligarchy. ■

Theorem 2* follows from Theorem 1* with the help of Lemma 7 (which ensures that X satisfies (α)) and the following lemma (which ensures that F is weakly systematic).

Lemma 11 *Let F be universal*, deductively closed, unanimity-respecting and weakly independent.*

(a) *For all $p, q \in X$, if $C \subseteq N$ is semi-winning against $C' \subseteq N$ for p, and $p \vDash^* q$, then C is semi-winning against C' for q.*
(b) *If X is path-connected, F is weakly systematic.*

Proof (with similarities to the proof of Lemma 8). Let F be as specified.

(a) Consider $p, q \in X$. Suppose $C \subseteq N$ is semi-winning for p against $C' \subseteq N$ and $p \vDash^* q$. If $C \cap C' \neq \emptyset$, it is vacuously true that C is semi-winning against C'

for q. Now let $C \cap C' = \emptyset$. By $p \vDash^* q$, there is a $Y \subseteq X$ such that $\{p\} \cup Y \vDash q$, and $\{p\} \cup Y$ and $\{\neg q\} \cup Y$ are consistent. So, as $\{p, \neg q\} \cup Y$ is inconsistent,

(*) $\{p, q\} \cup Y$ and $\{\neg p, \neg q\} \cup Y$ are consistent.

Let $(A_1, ..., A_n)$ be the profile given (in the above notation) by

$$A_i = \begin{cases} \overline{\{p, q\} \cup Y} & \text{if } i \in C \\ \overline{\{\neg p, \neg q\} \cup Y} & \text{if } i \in C' \\ \overline{Y} & \text{if } i \notin C \cup C'. \end{cases}$$

This profile is in $(\mathcal{C}^*)^n$, by (*) and Lemma 9. Further, $\overline{Y}$ contains none of $p, \neg p, q, \neg q$: otherwise Y would be inconsistent with (another) one of them, violating (*). It follows that $N_p = N_q = C$ and $N_{\neg p} = N_{\neg q} = C'$. So, as C is semi-winning against C' for p, $p \in F(A_1, ..., A_n)$. By unanimity-respectance and weak independence, $Y \subseteq F(A_1, ..., A_n)$. So $\{p\} \cup Y \subseteq F(A_1, ..., A_n)$. Hence, by $\{p\} \cup Y \vDash q$ and deductive closure, $q \in F(A_1, ..., A_n)$. So, as $N_q = C$ and $N_{\neg q} = C'$, and by weak independence, C is semi-winning against C' for q, as desired.

(b) Let X be path-connected. To show weak systematicity, consider any $p, q \in X$ and $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in (\mathcal{C}^*)^n$ such that $C := \{i : p \in A_i\} = \{i : q \in A_i^*\}$ and $C' := \{i : \neg p \in A_i\} = \{i : \neg q \in A_i^*\}$. We suppose that $p \in F(A_1, ..., A_n)$ and prove that $q \in F(A_1^*, ..., A_n^*)$ (the converse being analogous).

First suppose that p or q is non-contingent, i.e. a tautology or contradiction. Then, as all A_i and A_i^* are consistent and deductively closed, one of C, C' is N and the other one is $\emptyset$. It is not possible that $C = \emptyset$ and $C' = N$: otherwise $p \notin F(A_1, ..., A_n)$, since $\emptyset$ is not semi-winning against N for p by unanimity-respectance and weak independence. So $C = N$ and $C' = \emptyset$. Then, as desired, $q \in F(A_1^*, ..., A_n^*)$, because N is semi-winning against $\emptyset$ for q, again by unanimity-respectance and weak independence.

Now let p and q be contingent. Then, by path-connectedness, there are $p = p_1, p_2, ..., p_k = q \in X$ such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$. By $p \in F(A_1, ..., A_n)$ and weak independence, C is semi-winning against C' for $p = p_1$. So a simple induction using part (a) tells us that C is semi-winning against C' for $p_k = q$. Hence $q \in F(A_1^*, ..., A_n^*)$, as desired. ■

We now give constructive proofs of each part of Theorem 3.

Proof of Theorem 3. Let $n \geq 3$ and let X contain a contingent proposition.

(a) Let F be $F_{\mathcal{W}}$ where $\mathcal{W} := \{N, N \backslash \{1, 2\}\}$. By Lemma 4, F is non-monotonic (hence non-oligarchic), universal, systematic, and unanimity-respecting (the latter uses that $\emptyset \notin \mathcal{W}$ by $n \geq 3$). The crucial claim is that, if (β) is violated, F is deductive closed. We suppose F is *not* deductively closed and prove (β).

By assumption, there is a profile $(A_1, ..., A_n) \in \mathcal{C}^n$ and a $q \in X\backslash F(A_1, ..., A_n)$, such that $F(A_1, ..., A_n) \vDash q$. We prove that (β) hods for

$$Y := \{r \in X : N_r = N\backslash\{1,2\} \text{ or } N_r = N\} \cup \{\neg q\} \ (= F(A_1, ..., A_n) \cup \{\neg q\})$$
$$Z := \{r \in X : N_r = N\backslash\{1,2\}\},\ p := \neg q.$$

First, Y is inconsistent as $F(A_1, ..., A_n) \vDash q$.

Second, we show that $\{p\}$ $(= \{\neg q\})$ and Z are disjoint. Note that

$$F(A_1, ..., A_n) \text{ is consistent,} \tag{6}$$

as $F(A_1, ..., A_n) \subseteq \cap_{k \in N\backslash\{1,2\}} A_k$. If $\{p\}$ and Z were not disjoint, we would have $p \in Z$, hence $p \in F(A_1, ..., A_n)$; so $F(A_1, ..., A_n)$ would entail both p $(= \neg q)$ and q, violating (6).

Finally, we show that $Y_{\neg Z}$ and $Y_{\neg(Z \cup \{\neg q\})}$ are consistent. Note that $\cap_{k \in N\backslash\{1,2\}} A_k \vDash q$ by $F(A_1, ..., A_n) \vDash q$ and $F(A_1, ..., A_n) \subseteq \cap_{k \in N\backslash\{1,2\}} A_k$. Hence for each $k \in N\backslash\{1,2\}$, A_k entails q, hence contains q. So $N\backslash\{1,2\} \subseteq N_q$. Hence, as N_q is (by $q \notin F(A_1, ..., A_n)$) neither N nor $N\backslash\{1,2\}$, N_q is either $N\backslash\{1\}$ or $N\backslash\{2\}$. We assume that

$$N_q = N\backslash\{1\}, \text{ and hence } N_{\neg q} = \{1\} \tag{7}$$

(the case of $N_q = N\backslash\{2\}$ being analogous). Note that

$$\begin{aligned} Z \cup \{p\} &= \{\neg q\} \cup \{r \in X : N_r = N\backslash\{1,2\}\} \\ Y &= \{\neg q\} \cup \{r \in X : N_r = N\backslash\{1,2\}\} \cup \{r \in X : N_r = N\}, \end{aligned}$$

where these are unions of pairwise disjoint sets by $N_{\neg q} = \{1\}$. So

$$\begin{aligned} Y_{\neg Z} &= (\{\neg q\} \cup \{r \in X : N_r = N\backslash\{1,2\}\} \cup \{r \in X : N_r = N\})_{\neg\{r \in X : N_r = N\backslash\{1,2\}\}} \\ &= \{\neg q\} \cup \{\neg r \in X : N_r = N\backslash\{1,2\}\} \cup \{r \in X : N_r = N\}, \\ Y_{\neg(Z \cup \{p\})} &= (\{\neg q\} \cup \{r \in X : N_r = N\backslash\{1,2\}\} \cup \{r \in X : N_r = N\})_{\neg(\{\neg q\} \cup \{r \in X : N_r = N\backslash\{1,2\}\})} \\ &= \{q\} \cup \{\neg r \in X : N_r = N\backslash\{1,2\}\} \cup \{r \in X : N_r = N\}. \end{aligned}$$

It follows that $Y_{\neg Z} \subseteq A_1$ and $Y_{\neg(Z \cup \{p\})} \subseteq A_2$, in both cases using (7) and $N_r = N\backslash\{1,2\} \Leftrightarrow N_{\neg r} = \{1,2\}$. So $Y_{\neg Z}$ and $Y_{\neg(Z \cup \{p\})}$ are consistent.

(b) Now let $F := F_{\mathcal{W}}$ where $\mathcal{W}$ is defined as $\mathcal{W} = \{C \subseteq N : \{1,3\} \subseteq C$ or $\{2,3\} \subseteq C\}$. Then, by Lemma 4, F is non-oligarchic, universal, systematic, unanimity-respecting, and monotonic. We assume that F is *not* deductively closed, i.e. there is a profile $(A_1, ..., A_n) \in \mathcal{C}^n$ and a $q \in X\backslash F(A_1, ..., A_n)$, such that $F(A_1, ..., A_n) \vDash q$. We prove that (α) holds for

$$Y := F(A_1, ..., A_n) \cup \{\neg q\},\ p := \neg q,$$
$$Z_i := \{r \in X : N_r \cap \{1,2,3\} = \{i,3\}\} \text{ for } i = 1,2.$$

First, Y is inconsistent, as $F(A_1, ..., A_n) \vDash q$.

Second, we show the pairwise disjointness of the sets $Z_1, Z_2, \{p\}$. Obviously, $Z_1 \cap Z_2 = \emptyset$. As $F(A_1, ..., A_n) \subseteq A_3$, we have (6). Now $\{p\}$ is disjoint with each Z_i, because otherwise $p \in Z_i$, hence $p \in F(A_1, ..., A_n)$, so that $F(A_1, ..., A_n)$ would entail p and also entail $q = \neg p$, in contradiction to (6).

Finally, we have to show the consistency of each of $Y_{\neg Z_1}$, $Y_{\neg Z_2}$ and $Y_{\neg\{p\}}$. As $Y = F(A_1, ..., A_n) \cup \{p\}$ is a disjoint union (by an argument like the previous one),

$$Y_{\neg\{p\}} = F(A_1, ..., A_n) \cup \{\neg p\} = F(A_1, ..., A_n) \cup \{q\}.$$

By $F(A_1, ..., A_n) \subseteq A_3$ and $F(A_1, ..., A_n) \vDash q$, we have $F(A_1, ..., A_n) \cup \{q\} \subseteq A_3$, i.e. $Y_{\neg\{p\}} \subseteq A_3$. Hence $Y_{\neg\{p\}}$ is consistent. Further, as $3 \in N_q$ and (by $q \notin F(A_1, ..., A_n)$) $N_q \notin \mathcal{W}$, we have $1, 2 \notin N_q$, whence

$$1, 2 \in N_{\neg q} = N_p. \tag{8}$$

Letting $Z_3 := \{r \in X : N_r \cap \{1, 2, 3\} = \{1, 2, 3\}\}$, we have $Y = Z_1 \cup Z_2 \cup Z_3 \cup \{p\}$, where this is a disjoint union (by an argument like the one above). So

$$Y_{\neg Z_1} = \{\neg r : r \in Z_1\} \cup Z_2 \cup Z_3 \cup \{p\}. \tag{9}$$

Here, $r \in Z_1$ implies $r \notin A_2$, which implies $\neg r \in A_2$. Using this and (8), the relation (9) implies that $Y_{\neg Z_1} \subseteq A_2$, whence $Y_{\neg Z_2}$ is consistent. For analogous reasons, $Y_{\neg Z_2}$ is consistent.

(c) Suppose X is not path-connected. Then there is a contingent $r \in X$ with no $\vDash^*$-path to some $s \in X$. Write $X = X_1 \cup X_2$, where

$$X_1 := \{s \in X : \text{there is a } \vDash^* \text{-path from } r \text{ to } s\} \text{ and } X_2 := X \backslash X_1.$$

Let F be the universal aggregation rule given, for all $(A_1, ..., A_n) \in \mathcal{C}^n$, by

$$F(A_1, ..., A_n) := (X_1 \cap A_1) \cup [X_2 \cap (\cap_{i \in N} A_i)];$$

i.e. within X_1 person 1 is a dictator and within X_2 the unanimity rule is used. F is non-oligarchic (by $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$), universal, unanimity-respecting, and independent.

To see monotonicity, let $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in \mathcal{C}^n$ be such that $A_i^* = A_i$ for all individuals i except from, say, individual j, who has $A_j^* = F(A_1, ..., A_n)$. To show that $F(A_1^*, ..., A_n^*)$ and $F(A_1, ..., A_n)$ are identical, we show that they have the same intersections with X_1 and with X_1. Regarding the intersection with X_2, we have

$$\begin{aligned} X_2 \cap F(A_1^*, ..., A_n^*) &= X_2 \cap (\cap_{i \in N} A_i^*) \\ &= X_2 \cap F(A_1, ..., A_n) \cap \left(\cap_{i \in N \backslash \{j\}} A_i\right) \\ &= X_2 \cap F(A_1, ..., A_n), \end{aligned}$$

as desired. Regarding the intersection with X_1, we have again

$$X_1 \cap F(A_1^*, ..., A_n^*) = X_1 \cap A_1^* = X_1 \cap F(A_1, ..., A_n),$$

where the last equality follows from $A_1^* = F(A_1, ..., A_n)$ if $j = 1$, and from $X_1 \cap A_1^* = X_1 \cap A_1 = X_1 \cap F(A_1, ..., A_n)$ if $j \neq 1$.

We finally show deductive closure. We suppose for a contradiction that there is a profile $(A_1, ..., A_n) \in \mathcal{C}^n$ and a $q \in X \backslash F(A_1, ..., A_n)$, such that $F(A_1, ..., A_n) \vDash q$. By $F(A_1, ..., A_n) \subseteq A_1$, we have (6), and we have $A_1 \vDash q$, hence $q \in A_1$. So $q \in X_2$: otherwise q would be in $X_1 \cap A_1$, hence in $F(A_1, ..., A_n)$, hence entailed by $F(A_1, ..., A_n)$. As X is finite or the logic compact, $F(A_1, ..., A_n)$ has a minimal subset Z that entails q. There is a $p \in Z \cap X_1$: otherwise $Z \subseteq X_2$, hence $Z \subseteq \cap_{i \in N} A_i$, so that $\cap_{i \in N} A_i \vDash q$, whence (by Lemma 3) $q \in \cap_{i \in N} A_i \subseteq F(A_1, ..., A_n)$, a contradiction.

We show that $p \vDash^* q$, a contradiction by $p \in X_1$ and $q \in X_2$. Putting $Y := Z \backslash \{p\}$, we have $\{p\} \cup Y = Z \vDash q$, where Y is consistent with $\neg q$ (otherwise $Y \vDash q$) and with p (as Z is consistent by $Z \subseteq F(A_1, ..., A_n)$). ■

Finally, we prove an earlier claim about the agenda condition (β).

Lemma 12 *If X is finite or belongs to a compact logic, X satisfies (β) (or $(\tilde{\beta})$) if and only if there is a minimal inconsistent set $Y \subseteq X$ such that $Y_{\neg Z}$ is consistent for some subset $Z \subseteq Y$ of even size.*

Proof. Let X be finite or the logic compact; so (β) and $(\tilde{\beta})$ are equivalent.

First, (β) implies the condition because $(\tilde{\beta})$ implies it: to see why, simply note that in $(\tilde{\beta})$ one of Z and $Z \cup \{p\}$ must have even size.

Conversely, let $Y \subseteq X$ be minimal inconsistent with an even-sized subset Z such that $Y_{\neg Z}$ is consistent. Choose a $Z \subseteq Y$ of *smallest* even size such that $Y_{\neg Z}$ is consistent. If $Y_{\neg Z'}$ is consistent for a $Z' \subseteq Z$ of size $|Z| - 1$, one easily checks that (β) holds for Y with disjoint subsets $Z', \{p\} = Z \backslash Z'$. Now assume

$$Y_{\neg Z'} \text{ is inconsistent for all } Z' \subseteq Z \text{ of size } |Z| - 1. \qquad (10)$$

Then $|Z| \geq 4$, as $|Z|$ is even, not zero (otherwise $Y_{\neg Z} = Y$, which is inconsistent) and not 2 (otherwise, by Y's minimal inconsistency, $Y_{\neg Z'}$ would be consistent for subsets $Z' \subseteq Z$ of size $|Z| - 1 = 1$). So Y contains no pair $r, \neg r$ (something we will implicitly use), and contains distinct $p, q \in Z$. Let

$$\widetilde{Z} := (Z \backslash \{p, q\})^{\neg}, \; Y' := (Y \backslash Z) \cup \widetilde{Z} \cup \{p\}.$$

We show (β) for the set Y' with disjoint subsets $\{p\}, \widetilde{Z}$.

First, Y' is inconsistent as $Y' \cup \{q\}$ and $Y' \cup \{\neg q\}$ are inconsistent: $Y' \cup \{q\} = Y_{\neg (Z \backslash \{p,q\})}$ by Z's minimality property, and $Y' \cup \{\neg q\} = Y_{\neg (Z \backslash \{p\})}$ by (10).

Second, $Y'_{\neg \widetilde{Z}} = Y \backslash \{q\}$ and $Y'_{\neg (\{p\} \cup \widetilde{Z})} = Y_{\neg \{p\}} \backslash \{q\}$ are consistent by Y's minimal inconsistency; and $Y'_{\neg \{p\}}$ is so by $Y'_{\neg \{p\}} \subseteq Y_{\neg Z}$ and $Y_{\neg Z}$'s consistency. ■