The possibility of judgment aggregation under subjunctive implications

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April 2005, revised July 2006

Abstract. The new field of judgment aggregation aims to find group judgments on logically interconnected propositions. Recent impossibility results have established limitations on the possibility to vote independently on the propositions, hence notably on the possibility to use quota rules (rules with acceptance thresholds for the propositions). I show that, fortunately, the impossibilities fail to apply to a wide class of realistic agendas once connection rules (like "if a then b") are adequately modelled, namely as subjunctive rather than material implications. For these agendas, I characterise the set of all consistent quota rules, and I suggest ways to choose the acceptance thresholds. I also prove a characterisation result valid for arbitrary agendas in most logics.

Key words: judgment aggregation, material implication, subjunctive implication, possibility theorems.

JEL Classification Numbers: D70, D71, D79

1 Introduction

In judgment aggregation (JA), the objects of the group decision are not as usually (mutually exclusive) *alternatives*, but *propositions* representing interrelated questions the group faces. To ensure that these interrelations are well-defined, propositions are statements of formal logic. As a simple example, suppose the board of a central bank disagrees on which of the following propositions hold.

a :	GDP growth will pick up;
b :	inflation will pick up;
$a \rightarrow b$:	*if* GDP growth will pick up *then* inflation will pick up.

Of course, many other propositions could be added to the board's agenda: atomic ones like c : "the Euro-Dollor exchange rate will remain stable", and compound ones like $c \rightarrow (a \wedge b)$, $b \rightarrow c$, etc. But let us stick to the simple example.

The need to reach joint beliefs on the three propositions might come from the need of joint *actions*. For instance, the board members might unanimously agree that interest rates should be raised if and only if b holds. Or the joint beliefs might be needed in order to issue a joint report on the state of the economy, containing consistent judgments on a, $a \rightarrow b$ and b. I will come back later to the role of group beliefs.

Reaching an agreement is non-trivial. In the situation of Table 1, each board member holds consistent beliefs but the propositionwise majority beliefs are inconsistent.

	a	$a \to b$	b
1/3 of the board	Yes	Yes	Yes
1/3 of the board	No	Yes	No
1/3 of the board	Yes	No	No
Group under majority rule	Yes	Yes	No
Group under premise-based rule	Yes	Yes	Yes
Group under the (below-defined) quota rule	No	No	No

Table 1: Inconsistencies under majority voting

To achieve consistent judgments on all propositions, the group cannot use majority voting. One option is the *premise-based* rule: only a and $a \to b$ – the "premises" – are decided through majority votes, while b – the "conclusion" – is accepted if and only if a and $a \to b$ have been accepted; so, in the situation of Table 1, a and $a \to b$, and hence b, are accepted. While one might like that "premises" or "reasons" are given priority, the decision rule has its draw backs. Normatively, it might be undemocratic that the group belief on b ignores people's beliefs on b. From a strategic angle, the rule is agenda manipulable: replacing the premises a and $a \to b$ by other premises of b may reverse the outcome on b (see Dietrich 2004a); and it is manipulable by voters: voters who reject b may achieve group rejection of b by pretending that they reject *both* premises of b (see Dietrich and List 2004).

Both objections are avoided if the group uses a *quota rule*. Here, separate anonymous votes are taken on the propositions, using (proposition-specific) acceptance thresholds. The hope is that a suitable choice of thresholds prevents a simultaneous acceptance of $a, a \to b$ and $\neg b$ (whenever each individual is consistent). Obviously, a uniform majority threshold does not work. Now suppose a premise (a or $a \to b$) is accepted if and only if 3/4 of people believe it, and b is decided using a majority threshold. Then, in the situation of Table 1, a, $a \to b$ and b are all rejected, i.e. the outcome is $\{\neg a, \neg(a \to b), \neg b\}$. This leads us to the heart of this paper. Although intuitively $\{\neg a, \neg(a \to b), \neg b\}$ is perfectly consistent, it is declared *in*consistent in classical propositional logic because $\neg(a \to b)$ is modelled as equivalent to $a \wedge \neg b$ ("a and not-b"); more on this issue in Section 3. More generally, the use of classical logic lets it appear as if, for most relevant agendas, there is no appealing aggregation rule with independent votes on the propositions.

In this paper, I propose to use non-classical logic, and to interpret "$\to$" as a *subjunctive* implication, making $\{\neg a, \neg(a \to b), \neg b\}$ consistent. I show that consistent quota rules exist for a wide class of JA problems: all those with so-called interlinked agendas; and I characterise all consistent rules by inequalities on the thresholds. Specifically, after explaining the model and our non-classical logic (Sections 2 and 3), I focus first on so-called *simple* interlinked agendas (Section 4). Then I prove a general characterisation of consistent quota rules, valid for all agendas in most logics (Section 5). This result is then applied to general interlinked agendas (Section 6). I finish with a short conclusion (Section 7).

The JA problem – deciding which propositions to accept based on which ones the individuals accept – and its formal results are open to different interpretations of "accepting" and different sorts of propositions. This paper's examples and discussion focus on the case where "accepting" means "believing"/"considering as true",[1] and mostly on the case where the propositions are *descriptive* (like "GDP growth will pick up"), although Section 4 touches on *normative* propositions (like "peace is better than war").[2] So we focus on the formation of group beliefs. Beliefs are generally considered as an essential ingredient of rational agency in an uncertain world.[3]

The JA problem is discussed on a less formal basis in law (e.g. Kornhauser and Sager 1986, Chapman 2002) and political philosophy (e.g. Pettit 2001), and is formalised by List and Pettit (2002) using classical propositional logic. A series of results establish, for varying agendas, the impossibility of aggregating independently on the propositions in accordance with varying other conditions (e.g. List and Pettit 2002, Pauly and van Hees 2004, Dietrich 2004a, 2004b, Gärdenfors 2004, Nehring and Puppe 2004, van Hees 2004 and Mongin 2005). Further impossibilities and the exact agendas to which they apply follow from Nehring and Puppe's (2002) results on property spaces. To achieve possibility, one might for instance use *fusion operators* (Pigozzi 2004) or *sequential* decision rules (List 2004 and Dietrich and List 2005a); as these rules do not aggregate independently, they offer manipulation opportunities to agenda setters and voters. If this paper takes a different route – that of quota rules – it is not to generally advocate voting independently on the propositions but to show that *even then* consistent group judgments are realistically achievable. The results in Section 5 use the generalised JA model introduced in Dietrich (2004b), and the other results use for the first time possible-worlds semantics.

2 Definitions

We consider a group or persons $N = \{1, 2, \ldots, n\}$ ($n \geq 2$), who need group judgments on a set of propositions expressed in formal logic.

The language. Let $\mathbf{L}$ be a formal language, i.e. a non-empty set of expressions (called *propositions* or *statements*) closed under negation (i.e. if $p \in \mathbf{L}$ then $\neg p \in \mathbf{L}$). $\mathbf{L}$ is endowed with an *entailment* relation $\vDash$ ($\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L}$), where for all $A \subseteq \mathbf{L}$ and $p \in \mathbf{L}$ "$A \vDash p$" means "A entails p".[4] Dropping brackets, we write $p_1, ..., p_k \vDash p$ for

[1] JA is interpretable as the aggregation of belief sets if "accepting" is read as "believing", as the aggregation of desire sets if "accepting" is read as "desiring", as the aggregation of moral judgment sets if "desiring" is read as "considering as morally good", etc.

[2] Standard preference aggregation problems can be modelled as JA problems by interpreting the preference of x over y as the belief of the normative proposition "x is better than y" (see Dietrich and List 2005b; also List and Pettit 2004).

[3] By considering beliefs on possibly normative propositions, I use a broader "belief" notion than is common in economics, where beliefs usually apply to descriptive facts only. But suppose now all propositions considered are descriptive. Then a second difference to standard models remains: usually, beliefs (of individual or group agents) are modelled as probability functions over states of the world. If the outcome of an action is state-dependent, each action induces (given probabilistic beliefs) a lottery of outcomes. So, if the agent can rank such lotteries, she can rank her actions. By contrast, the beliefs in this paper are non-probabilistic. This might bring us closer to real applications. First, forming (deterministic) group beliefs via JA is less demanding on the side of individual inputs than forming a group probability function based on individual probability functions. Second, deterministic group beliefs allow the group to choose actions without having to rank *lotteries* of outcomes: a simple ranking of outcomes, e.g. formed via standard preference aggregation, suffices.

[4] We use a *semantic* notion of entailment, resulting in semantic notions of consistency and tautology. For syntactic notions, see Dietrich (2004b).

$\{p_1, ..., p_k\} \vDash p$. A set $A \subseteq \mathbf{L}$ is *inconsistent* if entails each member of some pair $p, \neg p \in \mathbf{L}$, and *consistent* otherwise. A $p \in \mathbf{L}$ is a *tautology* if $\{\neg p\}$ is inconsistent

In all sections except Section 5, $\mathbf{L}$ is the set of propositions constructible using $\neg$ ("not"), $\wedge$ ("and") and $\rightarrow$ ("if-then") from a set $\mathcal{A}$ of non-decomposable symbols (*atomic* propositions, representing simple statements like "inflation will pick up"). So $\mathbf{L}$ is the smallest set such that (i) $\mathcal{A} \subseteq \mathbf{L}$ and (ii) $p, q \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$, $(p \wedge q) \in \mathbf{L}$ and $(p \rightarrow q) \in \mathbf{L}$. The critical question, treated in the next section, is how (not) to define $\vDash$ on $\mathbf{L}$: some entailments are non-controversial (e.g. $a, b \vDash a \wedge b$ and $a, a \rightarrow b \vDash b$), but others aren't. Notationally, I drop brackets when there is no ambiguity, e.g. $c \rightarrow (a \wedge b \wedge c)$ stands for $(c \rightarrow ((a \wedge b) \wedge c))$. Further, $p \vee q$ ("p or q") stands for $\neg(\neg p \wedge \neg q)$, and $p \leftrightarrow q$ ("p if and only if q") stands for $(p \rightarrow q) \wedge (q \rightarrow p)$. For any conjunction $p = a_1 \wedge ... \wedge a_k$ of one or more atomic propositions $a_1, ..., a_k$ (called the *conjuncts* of p), let $\text{Conj}(p) := \{a_1, ..., a_k\}$ (e.g. $\text{Conj}(a) = \{a\}$ and $\text{Conj}(a \wedge b) = \text{Conj}(b \wedge a) = \{a, b\}$). In JA, the term "connection rule" commonly refers to an implicational relation between atomic propositions, like "*if* GDP growth continues *and* interest rates stay below X *then* inflation will rise". I now formalise this terminology. If each of p and q is a conjunction of one or more atomic propositions,

- $p \rightarrow q$ is a *unidirectional connection rule*, called *non-degenerate* if $\text{Conj}(q) \backslash \text{Conj}(p) \neq \emptyset$ (i.e. if $p \rightarrow q$ is not a tautology, under the classical or the non-classical entailment relation discussed later);
- $p \leftrightarrow q$ is a *bidirectional connection rule*, called *non-degenerate* if $\text{Conj}(q) \backslash \text{Conj}(p) \neq \emptyset$ and $\text{Conj}(p) \backslash \text{Conj}(q) \neq \emptyset$ (i.e. if neither $p \rightarrow q$ nor $q \rightarrow p$ is a tautology).

The set of connection rules, i.e. uni- or bidirectional ones, is denoted $\mathcal{R}$ $(= \{a \rightarrow b, a \leftrightarrow b, (a \wedge b) \rightarrow c, ...\})$.

The agenda. The *agenda* is the set of propositions on which decisions are needed. Formally, it is a non-empty set $X \subseteq \mathbf{L}$ of the form $X = \{p, \neg p : p \in X^+\}$, where X^+ is a set containing no negated proposition $\neg q$. Hereafter, if $q = \neg p \in X \backslash X^+$ then when I write "$\neg q$" I mean p rather than $\neg\neg p$ (this convention ensures $\neg q \in X$). In our central bank example, $X^+ = \{a, b, a \rightarrow b\}$, an example of an *interlinked agenda*. X is an *interlinked agenda* if X^+ consists of non-degenerate connection rules and the atomic propositions occurring in them; it is called *simple* if all its connection rules are unidirectional ones $p \rightarrow q$ in which p and q are atomic. Many standard examples of JA problems can be modelled with interlinked agendas. The atomic propositions represent (controversial) issues, and connection rules represent (controversial) links between issues. Any accepted connection rule establishes a constraint on how to decide the issues.

Interrelated agendas can always be represented graphically.[5] Figure 1 shows seven interlinked agendas, of which (1) and (5)-(7) are simple. Agenda (1) represents our central bank example. The *doctrinal paradox*, from which JA originated, is described by agenda (3), where a is "the defendant has broken the contract", b is "the contract is legally valid" and c is "the defendant is liable". An environmental expert commission might face the agenda (2), where a is "global warming will continue" and b is "the ozone hole has size X". A company board, wondering about the price policy of three rival firms A-C, might face the agenda (3) or (4) or (7), where a is "Firm A will raise

[5] Nodes contain atomic propositions. Arrows represent connection rules: bidirectional arrows indicate biimplications, and bifurcations indicate conjunctions of more than one atomic proposition.

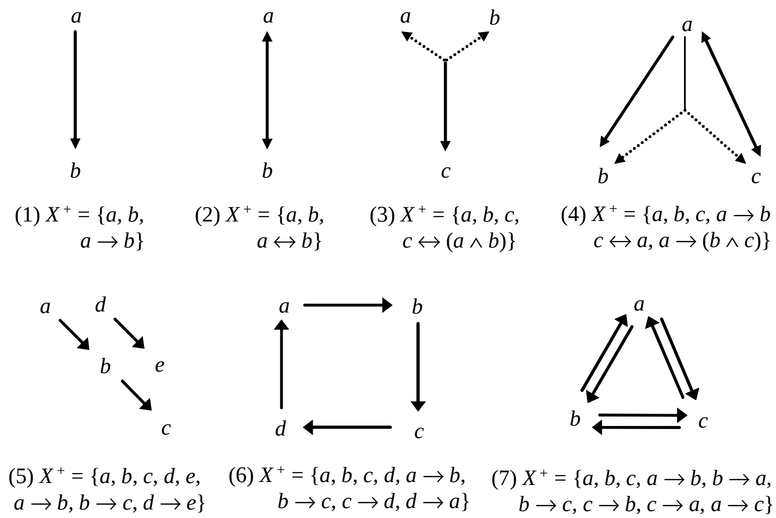

Figure 1: Examples of interlinked agendas

prices", b is "Firm B will raise prices", and c is "Firm C will raise prices". The three agendas differ in the type of connections between a, b, c deemed possible. In Section 4, I discuss two types of decision problems captured by interlinked agendas: reaching judgments on facts and their causal relations, and reaching judgments on hypotheses and their justificational/evidential relations.

But not all agendas are interlinked agendas. Counterexamples include $X^+ = \{a, b, a \wedge b\}$, the agenda needed to represent a preference aggregation problem (see Section 5), and interlinked agendas under a generalised definition of connection rules as (bi)implica-tions between *any* propositions built using $\neg$ and $\wedge$ (e.g. $(\neg(a \wedge \neg b) \rightarrow c)$.

Judgment sets. A *judgment set* (held by a person or the group) is a subset $A \subseteq X$, where $p \in A$ stands for "the person/group accepts proposition p". A judgment set A can be more or less rational. It is fully rational if it is *complete*, i.e. contains at least one member of each pair $p, \neg p \in X$, and *consistent* as defined above, i.e. A entails at most one member of each pair $p, \neg p \in \mathbf{L}$. A is *weakly consistent* if A contains at most one member of each pair $p, \neg p \in X$.

Aggregation rules. A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of (individual) judgment sets $A_i \subseteq X$. A *(judgment) aggregation rule* is a function F that maps each profile $(A_1, \ldots, A_n)$ in a given domain to a (group) judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. Often, the domain of F is *universal*, i.e. consists of all profiles of complete and consistent judgment sets. F is *complete/consistent/weakly consistent* if F generates a complete/consistent/weakly consistent judgment set for each profile in its domain. On the universal domain, *majority rule* (given by $F(A_1, ..., A_n) = \{p \in X :$ more persons i have $p \in A_i$ than $p \notin A_i\})$ is weakly consistent, and a *dictatorial rule* (given by $F(A_1, ..., A_n) = A_j$ for a fixed j) is consistent. We will focus on *quota rules* thus defined. To each family $(m_p)_{p \in X^+}$ of numbers in $\{1, ..., n\}$, the *quota rule with* *thresholds* $(m_p)_{p \in X^+}$ is the aggregation rule with universal domain given by

$$F_{(m_p)_{p\in X^+}}(A_1, ..., A_n) = \{p \in X : \text{at least } m_p \text{ persons } i \text{ have } p \in A_i\},$$

where $m_{\neg p} := n+1-m_p$ for all $p \in X^+$ (ensuring that exactly one member of each pair $p, \neg p \in X$ is accepted). So each family of thresholds $(m_p)_{p\in X^+}$ generates a quota rule. As one easily checks, an aggregation rule is a quota rule if and only if it has universal domain, is complete, weakly consistent, anonymous, responsive, independent and monotonic.[6] The important property missing here is consistency. We will investigate if and how the thresholds can be chosen so as to achieve consistency. The properties of independence and monotonicity are equivalent to *strategy-proofness* if each individual i holds *epistemic* preferences, i.e. would like the group to hold beliefs close (in a technical sense) to A_i, the set of propositions i considers true.[7]

3 A non-classical logic

For the language **L** defined above, how should the notion of entailment $\vDash$, hence of consistency, be specified? Although classical logic gets some entailments right (like $a, a \to b \vDash b$), its treatment of connection rules is inappropriate, or so I will argue.

Requirements on the representation of connection rules. To reflect the intended meaning of connection rules such as $a \to b, c \leftrightarrow a, a \to (b \wedge c)$, the logic should respect the following conditions.

(a) The *acceptance* of a connection rule r establishes exactly the intended logical constraints on atomic propositions, i.e. r is consistent with the "right" sets of atomic and negated atomic propositions. For instance, $a \to b$ is inconsistent with $\{a, \neg b\}$ but consistent with each of $\{a, b\}$, $\{\neg a, b\}$, $\{\neg a, \neg b\}$.

(b) The *negation* of a (non-degenerate) connection rule r does *not* constrain atomic propositions, i.e. $\neg r$ is consistent with *each* (consistent) set of atomic and negated atomic propositions. For instance, $\neg(a \to b)$ is consistent with each of $\{a, b\}$, $\{a, \neg b\}$, $\{\neg a, b\}$, $\{\neg a, \neg b\}$.

To illustrate (b), consider again the central bank example, where a is "GDP growth will pick up" and b is "inflation will pick up". Consider a board member who

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[6] *Anonymity*: $F(A_1, ..., A_n) = F(A_{\pi(1)}, ..., A_{\pi(n)})$ for all admissible profiles $(A_1, ..., A_n)$, $(A_{\pi(1)}, ..., A_{\pi(n)})$, where $\pi : N \mapsto N$ is a permutation. *Responsiveness*: for all $p \in X$ there are admissible profiles $(A_1, ..., A_n)$, $(A_1^*, ..., A_n^*)$ with $p \in F(A_1, ..., A_n)$ and $p \notin F(A_1^*, ..., A_n^*)$. *Independence*: for all $p \in X$ and all admissible profiles $(A_1, ..., A_n)$, $(A_1^*, ..., A_n^*)$, if $\{i : p \in A_i\} = \{i : p \in A_i^*)$ then $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A_1^*, ..., A_n^*)$. *Monotonicity* can be defined in two ways. *Proposition-wise monotonicity*: for all $p \in X$, individuals i, and admissible profiles $(A_1, ..., A_n)$, $(A_1, ..., A_i^*, ..., A_n)$ differing only in i's judgment set, if $p \notin A_i$, $p \in A_i^*$ and $p \in F(A_1, ..., A_n)$ then $p \in F(A_1, ..., A_i^*, ..., A_n)$. *Judgment-set-wise monotonicity* (which avoids a flavour of independence): for all $p \in X$, individuals i, and admissible profiles $(A_1, ..., A_n)$, $(A_1, ..., A_i^*, ..., A_n)$ differing only in i's judgment set, if $F(A_1, ..., A_n) = A_i^*$ then $F(A_1, ..., A_i^*, ..., A_n) = A_i^*$. Clearly, quota rules satisfy all the axioms. Conversely, independence and monotonicity imply that the group judgment on any given $p \in X$ depends only on the number $n_p := |\{i : p \in A_i\}|$. This dependence is positive by monotonicity, hence described by an acceptance threshold m_p, which is at least 1 and at most n by responsiveness. Finally, completeness and weak consistency imply $m_{\neg p} = n + 1 - m_p$.

[7] More precisely, i weakly prefers the group to hold judgment set A than judgment set B if for all $p \in X$ on which A_i agrees with B, A_i also agrees with A. This condition only partially fixes i's preferences, but it for instances requires i to most prefer i's own judgment set A_i (for details, see Dietrich and List 2004).

believes that $\neg(a \to b)$, i.e. that rising GDP does *not* imply rising inflation. This belief is intuitively perfectly consistent with any beliefs on a and b, i.e. on whether GDP will grow and whether inflation will rise.

The failure of the material implication. Material (bi)implications (used in classical logic) satisfy (a) but not (b). Consider $a \to b$. Interpreted materially, $a \to b$ is equivalent to $\neg a \vee b$ (not-a or b), and $\neg(a \to b)$ to $a \wedge \neg b$ (a and not-b); so:

- (a) holds because $a \to b$ is inconsistent with $\{a, \neg b\}$ (as desired) and consistent with each of $\{a, b\}, \{\neg a, b\}, \{\neg a, \neg b\}$ (as desired);
- (b) is violated because $\neg(a \to b)$, far from imposing no constraints, is inconsistent with all sets containing $\neg a$ or containing b.

It is well-known that the material reading misrepresents the intended meaning of most conditional statements in common language. The (in common language clearly false) statement "if the sun stops shining then we burn" is *true* materially because the sun does *not* stop shining. The material interpretation clashes with our intuition because, in common language, "if a then b" is not a statement about whether a and b hold in the *actual* world, but about whether b holds in world(s) where a holds (e.g. in worlds where the sun stops shining). "If a then b" thus means "if a *were* true ceteris paribus, then b *would be* true", not "a *is* false or b *is* true".

A conditional logic. A *subjunctive* reading of "$\to$", where the truth value of $a \to b$ depends on b's truth value in possibly non-actual worlds, has been formalised using *possible-worlds semantics*, and more specifically using *conditional logic* which originated from Stalnaker (1968) and D. Lewis (1973) and is now well-established in non-classical logic. I will use a standard version of conditional logic, often denoted C^+ (other versions could also be used). For further reference, e.g. Priest (2001).

For comparison, recall that in *classical* logic (not in C^+) $A \subseteq \mathbf{L}$ entails $p \in \mathbf{L}$ if and only if every interpretation that makes all $q \in A$ true makes p true, where an *interpretation* is a ("truth") function $v : \mathbf{L} \to \{T, F\}$, assigning to each proposition a truth value, such that, for all $p, q \in \mathbf{L}$,

($\neg$) $v(\neg p) = T$ if and only if $v(p) = F$,
($\wedge$) $v(p \wedge q) = T$ if and only if $v(p) = T$ and $v(q) = T$,
($\overset{\text{mat}}{\to}$) $v(p \to q) = T$ if and only if $v(p) = F$ or $v(q) = T$.

This leads to the counter-intuitive entailments $\neg a \vDash a \to b$ and $b \vDash a \to b$, the so-called paradoxes of material implication. By contrast, a (C^+-)*interpretation* is a triple $(W, (f_p), (v_w)) \equiv (W, (f_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$, where:

- W is a non-empty set, whose members are called *(possible) worlds*;
- $(f_p)_{p \in \mathbf{L}}$ is a family of functions $f_p : W \to \mathcal{P}(W)$ ($w' \in f_p(w)$ means "w' is similar to w and p holds in w'") such that, for all $w \in W$ and $p \in \mathbf{L}$,
  - if $w' \in f_p(w)$ then $v_{w'}(p) = T$ (ensuring p's truth in all worlds in $f_p(w)$),
  - if $v_w(p) = T$ then $w \in f_p(w)$ (a plausible demand as w is similar to itself);
- $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \to \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for each world $w \in W$ and any propositions $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies ($\neg$), ($\wedge$), and

($\overset{\text{sub}}{\rightarrow}$) $v_w(p \to q) = T$ if and only if $v_{w'}(q) = T$ for all worlds $w' \in f_p(w)$.

By ($\overset{\text{sub}}{\rightarrow}$), $p \to q$ holds in a world just if q holds in similar worlds with true p. This captures the intuitive meaning of implications. "If the sun stops shining then we burn" is false in our world: we do not burn in worlds similar to ours but without the sun shining.

By definition, $A \subseteq \mathbf{L}$ *(C^+-)entails* $p \in \mathbf{L}$ ($A \vDash p$) if, for all interpretations $(W, (f_p), (v_w))$ and all worlds $w \in W$, if all $q \in A$ hold in w then p holds in w (i.e. p holds "whenever" all $q \in A$ hold). For instance, $a, b, (a \wedge b) \to b \vDash b$, but $\neg a \nvDash a \to b$ and $b \nvDash a \to b$ (so C^+ does not suffer the paradoxes of material implication). Recall that $A \subseteq \mathbf{L}$ is consistent if and only if there is no $p \in \mathbf{L}$ with $A \vDash p$ and $A \vDash \neg p$. So

- A is (C^+-)consistent if and only if there is an interpretation $(W, (f_p), (v_w))$ and a world $w \in W$ in which all $q \in A$ hold (i.e. all $q \in A$ "can" hold simultaneously).

So $\{a, \neg a\}$ is inconsistent: if a holds in a word w, $\neg a$ is false in w by ($\neg$). And $\{a, \neg(a \to b), b\}$ is consistent (but classically inconsistent): let a and b both hold in w and let $f_a(w)$ contain a world w' in which b is false.

The logic C^+ meats the conditions (a)-(b) on the treatment of connection rules.[8]

4 Consistent quota rules for simple interlinked agendas

Given the logic C^+, which quota rules are consistent? I first discuss *simple* interlinked agendas; for them the general characterisation result of Section 6 is easily stated.

Theorem 1 *For a simple interlinked agenda X, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent if and only if*

$$m_b \leq m_a + m_{a \to b} - n \text{ for all } a \to b \in X. \tag{1}$$

The first remark is that consistent quota rules do exist: putting $m_p = n$ for all $p \in X^+$ validates (1). By contrast, they wouldn't exist in classical logic for most simple interlinked agendas, because additional inequalities would have to hold.[9] Although Theorem 1 is a corollary, I now sketch a (direct) proof, also indicating where non-classical logic is used.

Sketched proof. 1. First, (1) is necessary and sufficient to guarantee that no triple $a, a \to b, \neg b \in X$ is simultaneously accepted (this would also hold in classical logic). To discuss only sufficiency, suppose (1). Consider a profile for which $a, a \to b \in X$ are accepted; I show why $\neg b$ is rejected. Denote by n_p the number of persons holding $p \in X$. As $\neg a$ is rejected, $n_{\neg a} \leq m_{\neg a} - 1$, i.e. $n_{\neg a} \leq n - m_a$; similarly, $n_{\neg(a \to b)} \leq n - m_{a \to b}$. As all persons holding $\neg b$ hold $\neg a$ or hold $\neg(a \to b)$ (by consistency), we have $n_{\neg b} \leq n_{\neg a} + n_{\neg(a \to b)}$. So $n_{\neg b} \leq n - m_a + n - m_{a \to b}$, which by $n_{\neg b} = n - n_b$ implies $n_b \geq m_a + m_{a \to b} - n$. So by (1) $n_b \geq m_b$; i.e. b is accepted and $\neg b$ rejected.

2. By part 1, it suffices to show that a quota rule outcome $A \subseteq X$ is inconsistent *if and only if* it contains a triple $a, a \to b, \neg b$. Clearly, if A contains such a triple, A

[8] Requirement (b) holds by Lemma 6, applied to sets A consisting of possibly negated atomic proposition and of negated (non-degenerate) connection rules (where (7) holds by $A \cap \mathcal{R} = \emptyset$).

[9] As one may show, there is a *classically* consistent quota rule $F_{(m_p)_{p \in X^+}}$ if and only if (using the later terminology) X contains no path of length above 2; in which case there is only one such rule, given by $m_a = n$ and $m_{a \to b} = m_b = 1$ for all $a \to b \in X$.

is inconsistent (it would be so too in classical logic). Now assume A contains no triple $a, a \to b, \neg b$. I show that A is consistent. In classical logic, A need *not* be consistent: A is *classically* inconsistent once it contains, say, a pair $\neg(a \to b), b$. We would need additional inequalities to enforce classical consistency, e.g. the inequality $m_b \geq m_{a \to b}$ to prevent A from containing the pair $\neg(a \to b), b$. Back to C^+ now.

First one should show the consistency of the subset $A^* := A \backslash \{\neg(a \to b) : a, b \in \mathcal{A}\}$, in which all negated implications are removed. This can be done by specifying an interpretation $(W, (f_p), (v_w))$ with a world $w \in W$ in which all $p \in A^*$ hold. Simply let W contain a *single* world, w, in which the only true atomic propositions are those in A^*. This fully fixes the interpretation, because in a one-world interpretation the truth values $v_w(p) \in \{T, F\}$ and sets $f_p(w) \subseteq \{w\}$ for general $p \in \mathbf{L}$ are recursively generated from those for atomic propositions $a \in \mathcal{A}$ ($v_w(p)$ follows the rules ($\neg$), ($\wedge$) and ($\overset{\text{sub}}{\to}$); $f_p(w)$ is $\{w\}$ if $v_w(p) = T$ and $\emptyset$ otherwise). Now all atomic or negated atomic $p \in A^*$ obviously hold in w. All $a \to b \in A^*$ with false a hold in w by $f_a(w) = \emptyset$. For all $a \to b \in A^*$ with true a, we have $\neg b \notin A^*$ (otherwise A^* would contain the triple $a, a \to b, \neg b$); hence $b \in A^*$, implying that b holds in w, whence also $a \to b$ does. In summary, as there are no $\neg(a \to b) \in A^*$, all $p \in A^*$ hold in w, as desired.

By a lengthy proof (not sketched here), the consistency of A^* implies that of A, as desired. Informally, this proof step reflects that negated subjunctive implications impose no constraints on atomic propositions (see Section 3) nor, as one can show, on the non-negated subjunctive implications in A. ■

Consider a simple interlinked agenda X. Theorem 1 allows one to check easily whether a *given* quota rule $F_{(m_p)_{p \in X^+}}$ is consistent. I now discuss the construction of consistent quota rules, i.e. the choice of thresholds $(m_p)_{p \in X^+}$ in accordance with the inequalities (1). If $a \to b \in X$, a is a *parent* of b and b a *child* of a. The notions of *ancestor* and *descendant* are defined by taking the transitive closure of parenthood resp. childhood. A *path* (in X) is a sequence $(a_1, a_2, ..., a_k)$ $(k \geq 2)$ in which each a_j is a parent of a_{j+1} $(j < k)$. X is *acyclic* if it has no cycle, i.e. no path $(a_1, ..., a_k)$ with $a_1 = a_k$. The *depth* of X is $d_X := \sup\{k :$ there is a path in X of length $k\}$, and the *level* of an atomic proposition $a \in X$ is $l_a := \sup\{k :$ there is a path in X of length k ending with $a\}$, interpreted as 1 if no path ends with a. So $a \in X$ has level 1 if it has no parents, level 2 if it has parents all of which have level 1, etc. Figure 2 shows an acyclic simple interlinked with three levels.

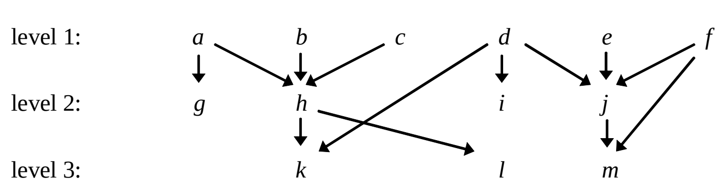

Figure 2: An acyclic simple interlinked agenda X of depth $d_X = 3$.

How free are we in choosing the thresholds $(m_p)_{p \in X^+}$? Clearly, by (1) the thresholds of atomic propositions must weakly decrease along any path. If X is acyclic and finite (hence of finite depth d_X), $(m_p)_{p \in X^+}$ can be chosen recursively in the following d_X steps.

Step l **(**$= 1, 2, ..., d_X$**):** for all $b \in X$ of level l, choose a threshold $m_b \in \{1, ..., n\}$ and thresholds $m_{a \to b} \in \{1, ..., n\}$ for the parents a of b, such that

$$m_b \leq m_a + m_{a \to b} - n \text{ for all parents } a \text{ of } b. \quad (2)$$

However, this procedure involves choosing potentially many thresholds: in Figure 2, the thresholds for 13 atomic propositions and 13 implications! To reduce the complexity, the group might use

- the same threshold $m = m_{a \to b}$ for all connection rules $a \to b \in X$, where m reflects how easily the group imposes constraints between issues,
- the same threshold m_l for all propositions in X with the same level l ($\in \{1, ..., d_X\}$), where m_l reflects how easily the group accepts level l propositions.

I write a quota rule of this type as $F_{m,m_1,...,m_{d_K}}$. Here, only $1+d_X$ parameters must be chosen, e.g., in Figure 2, $1 + 3 = 4$ parameters rather than 26. Applying Theorem 1 to quota rules of type $F_{m,m_1,...,m_{d_K}}$, we obtain the following characterisation, by a proof left to the reader.

Corollary 1 *For a finite acyclic simple interlinked agenda X, a quota rule $F_{m,m_1,...,m_{d_K}}$ is consistent if and only if*

$$m_l \leq m_{l-1} + m - n \text{ for all levels } l \in \{2, ..., d_X\}. \quad (3)$$

Consistent quota rules of type $F_{m,m_1,...,m_{d_K}}$ can be constructed as follows.

Step 0: choose $m \in \{1, ..., n\}$ such that (i) $m \geq n - (n-1)/(d_X - 1)$.

Step l **(**$= 1, 2, ..., d_X$**):** choose $m_l \in \{1, ..., n\}$ such that (ii) $m_l \geq 1 + (d_X - l)(n - m)$ and (iii) $m_l \leq m_{l-1} + m - n$ if $l > 1$.

The conditions (i)-(iii) follow from Corollary 1: (iii) is obvious, and (i) and (ii) are necessary and sufficient to enable the choices in future steps (for instance, without (i) there would be *no* choices of $m_1, ..., m_{d_X}$ satisfying (3)). For a group of size $n = 10$ and the agenda of Figure 2, a consistent quota rule F_{m,m_1,m_2,m_3} might be chosen as follows.

Step 0: $m = 8$ (note that $8 \geq n - (n-1)/(d_X - 1) = 10 - 9/2 = 5.5$).

Step 1: $m_1 = 8$ (note that $8 \geq 1 + (d_X - 1)(n - m) = 1 + 2 \times 2 = 5$).

Step 2: $m_2 = 6$ (note that $6 \geq 1 + (d_X - 2)(n - m) = 1 + 2 = 3$ and $6 \leq m_1 + m - n = 8 + 8 - 10 = 6$).

Step 3: $m_3 = 4$ (note that $4 \geq 1 + (d_X - 3)(n - m) = 1$ and $4 \leq m_2 + m - n = 6 + 8 - 10 = 4$).

Causal and justificational interpretation. I now offer two potential interpretations of connection rules, and hence of the kind of decision problems captured by interlinked agendas. Mainly for conceptual simplicity, I restrict myself to a *simple* interlinked agendas X (but most remarks could be generalised).

First, suppose the implications $a \to b \in X$ have a *causal* status: $a \to b$ is believed or disbelieved depending on whether a is believed to cause b. So X might contain "*if* the ozone hole has size X *then* global warming will continue" and "*if* global warming will continue *then* species Y will die out". One could thus view the decision problem captured by X as one of forming beliefs about facts and their causal links; a judgment set $A \subseteq X$ would then be consistent if the beliefs on facts obey the beliefs on causal links. A path $(a_1, ..., a_k)$ in X is a causal chain (provided all causal links $a_j \to a_{j+1}$, $j < k$, indeed hold), and the level of a proposition indicates how "causally fundamental" it is. By our results, to ensure group consistency causally more fundamental propositions need higher acceptance thresholds: they should be harder to accept.

Second, suppose the implications $a \to b \in X$ have a *justificational* (or *evidential* or *indicative*) status: $a \to b$ is believed or disbelieved depending on whether a is considered as a sufficient reason to believe b (this can be so without a causing b: a wet street indicates rain without causing it). One might thus view X as capturing a decision problem of forming beliefs about claims and their justificational links. To add applications, let us enlarge our focus for a moment and allow X to also contain *normative* propositions like "a multi-cultural society is desirable" or "option x is better than option y". For instance, the mentioned environmental panel might also have on its agenda the proposition $b:$ "tax T on kerosine should be introduced", and the implication $a \to b$ where $a:$ "the ozone hole has size X". A path $(a_1, ..., a_k)$ is an "argumentative" chain (for one who believes all links $a_j \to a_{j+1}$, $j < k$), and the level of a proposition reflects how "argumentatively fundamental" it is. Plausibly, high level propositions are more concrete and might state that certain collective acts should be taken (a road should be built, a firm downsized, a law amended, the country's own currency preserved, etc.), whereas their ancestors describe potential *reasons* or *arguments*, either of a descriptive kind (traffic will increase, demand will fall, etc.) or of a normative kind (the state should preserve its autonomy, a multi-cultural society is desirable etc.). Of course, the group may disbelieve these reasons (by rejecting them) or disbelieve their status as reasons (by rejecting the respective connection rules). Again, to ensure consistency reasons need (weakly) higher acceptance thresholds than their (argumentative) descendents, e.g. $\frac{3}{4}n$ versus $\frac{1}{2}n$.

5 General characterisation of consistent quota rules

Often, interlinked agendas are non-simple, for instance (taking up the two interpretations just discussed) because facts can be jointly caused by many facts or because a statement can be jointly justified by many reasons. I now provide a general technique to characterise consistent quota rules, valid for any agenda in a general logic. The application to general interlinked agendas will follow in the next section.

So consider an arbitrary agenda X and language $\mathbf{L}$ (see Section 2); and let the entailment relation $\vDash$ satisfy mild conditions guaranteeing a well-behaved consistency notion.[10] Let $\mathcal{I}$ be the set of all inconsistent sets $Y \subseteq X$, and let $\mathcal{MI}$ be the set of all minimal inconsistent sets $Y \subseteq X$. As every set $A \in \mathcal{I}$ has a subset in $\mathcal{MI}$ (if $|A| = \infty$ by compactness, defined in footnote 10), the outcome of a quota rule $F_{(m_p)_{p \in X+}}$ is consistent if and only if it has no subset $Y \in \mathcal{MI}$. The latter could be ensured by requiring that for all $Y \in \mathcal{MI}$ the thresholds $(m_p)_{p \in Y}$ satisfy a particular

[10] These conditions (valid for C^+ and most familiar propositional, predicate, modal or conditional logics) are, in Dietrich's (2004b) terminology: L1: for any $p \in \mathbf{L}$, $p \vDash p$ (*self-entailment*); L2: for any $p \in \mathbf{L}$ and $A \subseteq B \subseteq \mathbf{L}$, if $A \vDash p$ then $B \vDash p$ (*monotonicity*); L3: the empty set $\emptyset$ is consistent, and each consistent set $A \subseteq \mathbf{L}$ has a consistent superset $B \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$ (*completability*); L5: for any $p \in \mathbf{L}$ and $A \subseteq \mathbf{L}$, if $A \vDash p$ then $B \vDash p$ for some finite subset $B \subseteq A$ (*compactness*).

inequality. But this system of inequalities is of little help if the class $\mathcal{MI}$ is very rich and complex, which is so for general interlinked agendas. Can consistent quota rules be characterised by a simpler system? I now introduce a notion of *irreducible* inconsistent sets $Y \subseteq X$, and show that consistent quota rules can be characterised by requiring an inequality for each *irreducible* inconsistent set, rather than for each *minimal* inconsistent set.

The irreducibility notion depends on a parameter that should be chosen such that, for the agenda at hand, irreducible sets look simple.[11] This parameter is a (binary) relation $<$ on $\mathcal{I}$. But not *any* relation on $\mathcal{I}$ will do. Call $<$ *admissible* if

- for all $Y, Z \in \mathcal{I}$, if $Z \subsetneq_{\text{finite}} Y$ then $Z < Y$ ($Z \subsetneq_{\text{finite}} Y$ means $Z \subsetneq Y \& |Z| < \infty$; if $|X| < \infty$, "$\subsetneq_{\text{finite}}$" could be replaced throughout by "$\subsetneq$");
- $<$ is *well-founded*, i.e. there is no infinite sequence $(Y_k)_{k=1,2,...}$ in $\mathcal{I}$ that is decreasing (i.e. $Y_{k+1} < Y_k$ for all $k = 1, 2, ...$).[12]

For instance, the relations $Z < Y :\Leftrightarrow Z \subsetneq_{\text{finite}} Y$ and $Z < Y :\Leftrightarrow |Z| < |Y|$ are each admissible. Later I will use a lexicographically defined relation $<$ to study interlinked agendas. An admissible relation $<$ induces a reduction relation on $\mathcal{I}$ as follows.

Definition 1 *Let an admissible relation $<$ on $\mathcal{I}$ be given.*

(i) $Z \in \mathcal{I}$ *is a* ($<$-)reduction of $Y \in \mathcal{I}$ *(and Y is* ($<$-)reducible *to Z) if $Z < Y$ and moreover each $p \in Z \backslash Y$ is entailed by some $V \subseteq Y$ with $V \cup \{\neg p\} < Y$.*

(ii) $Y \in \mathcal{I}$ *is* ($<$-)irreducible *if it has no reduction; we put $\mathcal{IR}_< := \{Y \in \mathcal{I} : Y$ is $<$-irreducible$\}$ (the set of of minimal elements of the reduction relation).*

Let me illustrate the concept using the two above examples.

1. Let $<$ be $\subsetneq_{\text{finite}}$. Then Z is a reduction of Y if and only if $Z \subsetneq_{\text{finite}} Y$, and Y is irreducible if and only if Y is minimal inconsistent. Hence $\mathcal{IR}_< = \mathcal{MI}$. It can be shown that if X is a *simple* interlinked agenda $\mathcal{IR}_<$ consists of all sets $Y \subseteq X$ of type $Y = \{p, \neg p\}$ or type

$$Y = \{a_1, a_1 \rightarrow a_2, ..., a_{k-1} \rightarrow a_k, \neg a_k\},\ a_1, ..., a_k \text{ all distinct, } k \geq 2. \tag{4}$$

2. Let $Z < Y :\Leftrightarrow |Z| < |Y|$. Then reducibility is equivalent to the special reducibility notion in Dietrich and List (2006). If X is again a simple interlinked agenda, $\mathcal{IR}_<$ can be shown to consist of all sets $Y \subseteq X$ of type $\{p, \neg p\}$ or of type (4) with $k = 2$. To see why for $k > 2$ a set of type (4) is not irreducible, note that such a set Y is reducible (for instance) to $Z := \{a_{k-1}, a_{k-1} \rightarrow a_k, \neg a_k\}$: we have $|Z| < |Y|$, and a_{k-1} is entailed by $V := \{a_1, a_1 \rightarrow a_2, ..., a_{k-2} \rightarrow a_{k-1}\}$ where $|V \cup \{\neg a_{k-1}\}| < |Y|$. As a second agenda, consider a standard (strict) preference aggregation problem with a set of options K. This can be represented by the agenda $X_K := \{xPy, \neg xPy : x, y \in K, x \neq y\}$ in a suitable predicate logic with a binary *predicate* P for strict preference, a set of *constants* K for options, and a set of *axioms* containing the rationality conditions on strict linear orders, e.g. the transitivity axiom

[11] This approach generalises Dietrich and List's (2006) special definition of irreducibility which is suitable only for some agendas (see example 2 below). Lemma 3 generalises one of their results.

[12] $<$ need not be complete, nor even transitive (if $<$ also satisfies these conditions, $<$ is a well-order). Note that well-foundedness implies asymmetry (i.e. if $Z < Y$ then $Y \not< Z$), hence irreflexivity. Further, given asymmetry and transitivity, $<$ is well-founded if and only if every set $\emptyset \neq \mathcal{J} \subseteq \mathcal{I}$ on which $<$ is complete has a least element (i.e. a $Z \in \mathcal{J}$ with $Z < Y$ for all $Y \in \mathcal{J} \backslash \{Z\}$).

$(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \to v_1 P v_3)$ (e.g. Dietrich and List 2005b). Dietrich and List (2006) call a set $Y \subseteq X_K$ a "k-cycle" ($k \geq 2$) if it has type

$$Y = \{x_1 P x_2, x_2 P x_3, ..., x_{k-1} P x_k, x_k P x_1\},\ x_1, ..., x_k \text{ all distinct,} \tag{5}$$

or arises from such a set by replacing one or more of the members xPy by $\neg yPx$. They show that $\mathcal{MI}$ consists of all k-cycles with arbitrary k, and that $\mathcal{IR}_<$ consists of all 2- or 3-cycles. To see why for $k \geq 4$ a k-cycle is not irreducible, note that a set Y of type (5) with $k \geq 4$ is reducible to $Z := \{x_1 P x_2, x_2 P x_3, x_3 P x_1\}$ (a 3-cycle): $|Z| < |Y|$, and $x_3 P x_1$ is entailed by $V := \{x_3 P x_4, x_4 P x_5, ..., x_k P x_1\}$ where $|V \cup \{\neg x_3 P x_1\}| < |Y|$.

Lemma 1 *Let $<$ be any admissible relation on $\mathcal{I}$. Then the reduction relation is again admissible; that is:*

(i) *For all $Y, Z \in \mathcal{I}$, if $Z \subsetneq_{\text{finite}} Y$ then Z is a reduction of Y.*

(ii) *There is no infinite sequence $(Y_k)_{k=1,2,...}$ in $\mathcal{I}$ such that Y_{k+1} is a reduction of Y_k for all $k = 1, 2, ...$*

Proof. Both (i) and (ii) follow immediately from the admissibility of $<$. ■

Lemma 2 *Let $<$ be an admissible relation on $\mathcal{I}$.*

(i) *$\mathcal{IR}_< \subseteq \mathcal{MI}$, and if $< = \subsetneq_{\text{finite}}$ then $\mathcal{IR}_< = \mathcal{MI}$.*

(ii) *If $<'$ is an (admissible) subrelation of $<$ then $\mathcal{IR}_< \subseteq \mathcal{IR}_{<'}$.*

Proof. (i) For all $Y \in \mathcal{I}$, if $Y \notin \mathcal{MI}_<$ then Y has an inconsistent proper subset Z, which we can choose finite by compactness of the logic. By Lemma 1 Y is reducible to Z, whence $Y \notin \mathcal{IR}_<$.

(ii) If $<'$ is an (admissible) subrelation of $<$, then $<'$-reduction is a subrelation of $<$-reduction. So $\mathcal{IR}_< \subseteq \mathcal{IR}_{<'}$. ■

Lemma 2 gives a general idea about how the set of irreducible sets $\mathcal{IR}_<$ depends on the choice of $<$. The finer the relation $<$ is (i.e. the more pairs it can compare), the more reductions are possible, and so the smaller $\mathcal{IR}_<$ is (see part (ii)). The coarsest admissible choice of $<$ is $\subsetneq_{\text{finite}}$; then the only reductions are those to finite proper subsets, and $\mathcal{IR}_<$ is maximal: $\mathcal{IR}_< = \mathcal{MI}$, whereas in general $\mathcal{IR}_< \subseteq \mathcal{MI}$ (see part (i)).

Lemma 3 *For any admissible relation $<$ on $\mathcal{I}$, every inconsistent and complete judgment set $A \subseteq X$ has a subset in $\mathcal{IR}_<$.*

This crucial property will later be applied to outcomes A of quota rules, which are indeed complete. By Lemma 3 and example 2 above, if X is a simple interlinked agenda then any inconsistent and complete judgment set $A \subseteq X$ has a subset of type $\{p, \neg p\}$ or $\{a, a \to b, \neg b\}$; and if X is instead the preference agenda X_K, A has a subset that is a 2- or 3-cycle – a well-known result of social choice since A corresponds to a connected strict preference relation $\succ$ on K with rationality violation.

Proof. Let $<$ and A be as specified. Assume for a contradiction that A has no subset in $\mathcal{IR}_<$. I recursively define a sequence $(Y_k)_{k=1,2,...}$ of inconsistent subsets of A such that $Y_{k+1} < Y_k$ for all k. This contradicts the well-foundedness of $<$.

First, put $Y_1 := A$, which is indeed an inconsistent subset of A.

Second, suppose Y_k is already defined. As Y_k is an inconsistent subset of A, Y_k is by assumption reducible, say to $Z \in \mathcal{I}$. First assume $Z \subseteq Y_k$. Letting $Y_{k+1} := Z$, it is true that Y_{k+1} is a subset of A (as $Y_k \subseteq A$) and is inconsistent with $Y_{k+1} < Y_k$ (as Y_{k+1} is a reduction of Y_k). Now suppose $Z \nsubseteq A$. Then there is a $p \in Z \backslash A$. As Z is a reduction of Y_k, there is a $V \subseteq Y_k$ that entails p with $V \cup \{\neg p\} < Y_k$. Letting $Y_{k+1} := V \cup \{\neg p\}$, Y_{k+1} is obviously inconsistent with $Y_{k+1} < Y_k$. To see why Y_{k+1} is also a subset of A, note that $V \subseteq A$ (by $Y_k \subseteq A$) and that $\neg p \in A$ since A is complete and $p \notin A$. ■

I can now state and prove the desired characterisation of consistent quota rules.

Theorem 2 *For any admissible relation $<$ on $\mathcal{I}$, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent if and only if*

$$\sum_{p \in Y} (n - m_p) < n \text{ for all } Y \in \mathcal{IR}_< \text{ (where } m_{\neg p} := n + 1 - m_p \ \forall p \in X^+\text{).}$$

The finer the relation $<$, the smaller $\mathcal{IR}_<$, and hence the "slimmer" the characterisation (since fewer inequalities are needed). Often *no* consistent quota rule exists because the inequalities have no solution. Theorem 2 generalises the anonymous case of Nehring and Puppe's (2005) *intersection property* and Dietrich and List's (2005a) Theorem 1(c); these results arise from Theorem 2 by letting $<$ be $\subsetneq_{\text{finite}}$. Further, a non-anonymous version of Theorem 2 can be shown similarly, generalising the mentioned intersection property.[13]

Proof. Let $<$ be admissible. Suppose first that F has an inconsistent outcome $F(A_1, ..., A_n)$. By Lemma 3, there is a $Y \subseteq F(A_1, ..., A_n)$ with $Y \in \mathcal{IR}_<$. I show that the inequality for Y is violated. For all $p \in Y$, put $n_p := |\{i : p \in A_i\}|$; hence $|\{i : p \notin A_i\}| = n - n_p$. So

$$|\{(p, i) \in Y \times N : p \notin A_i\}| = \sum_{p \in Y} (n - n_p).$$

As Y is inconsistent, no A_i contains all $p \in Y$. So $|\{(p, i) \in Y \times N : p \notin A_i\}| \geq n$, i.e. $\sum_{p \in Y} (n - n_p) \geq n$. Hence, as for all $p \in Y$ we have $n_p \geq m_p$ (by $p \in F(A_1, ..., A_n)$), $\sum_{p \in Y} (n - m_p) \geq n$, as desired.

Now assume that $Y \in \mathcal{IR}_<$ with $\sum_{p \in Y} (n - m_p) \geq n$. I define a profile $(A_1, ..., A_n)$ with $Y \subseteq F(A_1, ..., A_n)$, implying that $F(A_1, ..., A_n)$ is inconsistent. By $\sum_{p \in Y} (n - m_p) \geq n$, there is a partition of N into $|Y|$ (possibly empty) sets, denoted N^p, $p \in Y$, such that $|N^p| \leq n - m_p$ for all $p \in Y$. For each individual i, say in N^p, let A_i be any complete and consistent superset of $Y \backslash \{p\}$, which exists since $Y \backslash \{p\}$ is consistent by $Y \in \mathcal{MI}$ (see Lemma 2). For each $p \in Y$,

$$|\{i : p \in A_i\}| = |N \backslash N^p| = n - |N^p| \geq n - (n - m_p) = m_p,$$

whence $p \in F(A_1, ..., A_n)$. So $Y \subseteq F(A_1, ..., A_n)$, as desired. ■

[13] Specifically, an independent, monotonic, responsive, and complete aggregation rule with universal domain is consistent *if and only if*, for all $Y \in \mathcal{MI}_<$, $\cap_{p \in Y} C_p \neq \emptyset$ whenever $C_p \subseteq N$ is a winning coalition for p for all $p \in Y$.

Call an inconsistent set $Y \subseteq \mathbf{L}$ *trivial* if it contains a pair $p, \neg p$ or contains an inconsistent p (i.e. a *contradiction*, e.g. $a \wedge \neg a$). If $Y \in \mathcal{IR}_<$ is trivial then by minimal inconsistency $Y = \{p, \neg p\}$ or $Y = \{p\}$, and the inequality $\sum_{p \in Y}(n - m_p) < n$ holds automatically whatever the thresholds $(m_p)_{p \in X^+} \in \{1, ..., n\}^{X^+}$. Removing these redundant inequalities, we obtain a slightly slimmer characterisation:

Corollary 2 *Theorem 2 still holds if $\mathcal{IR}_<$ is replaced by $\mathcal{IR}_<^* := \{Y \in \mathcal{IR}_< : Y$ is non-trivial$\}$.*

As an illustration, consider a simple interlinked agenda X. By Theorem 1, $F_{(m_p)_{p \in X^+}}$ is consistent if and only if

$$m_a + m_{a \to b} - m_b \geq n \text{ for all } a \to b \in X. \quad (6)$$

This characterisation is equivalent to that of Corollary 2 if $<$ is defined by $Z < Y :\Leftrightarrow |Z| < |Y|$. For $\mathcal{IR}_<^* = \{\{a, a \to b, \neg b\} : a \to b \in X\}$ (see example 2 above), whence the inequalities $\sum_{p \in Y}(n - m_p) < n$, $Y \in \mathcal{IR}_<^*$, are equivalent to the inequalities (6). Alternatively, if we define $<$ as $\subsetneq_{\text{finite}}$, then (using example 1 above) $\mathcal{IR}_<^*$ consists of all sets of type (4); thus $\mathcal{IR}_<^*$ is now much larger, and the resulting characterisation of Corollary 2 contains redundant inequalities.

Determining the set $\mathcal{IR}_<$ (or $\mathcal{IR}_<^*$) may be hard in practice, e.g. for general interlinked agendas. Determining a *superset* of it can be simpler – and it suffices by the next corollary, obtained by combining Corollary 2 with Theorem 2 applied to the (coarsest admissible) relation $< = \subsetneq_{\text{finite}}$ for which $\mathcal{IR}_< = \mathcal{MI}$.

Corollary 3 *Theorem 2 still holds if $\mathcal{IR}_<$ is replaced by any $\mathcal{Y}$ with $\mathcal{IR}_<^* \subseteq \mathcal{Y} \subseteq \mathcal{MI}$.*

So, to find out for a concrete agenda which quota rules are consistent, it suffices to define a suitable admissible relation $<$ and determine *some* set $\mathcal{Y}$ with $\mathcal{IR}_<^* \subseteq \mathcal{Y} \subseteq \mathcal{MI}$. Precisely this we shall do for interlinked agendas.

6 Consistent quota rules for general interlinked agendas

We are now back to our particular language $\mathbf{L}$ (with set of atomic propositions $\mathcal{A}$ and set of connection rules $\mathcal{R}$), endowed with our non-classical notions of entailment and inconsistency. Let X ($\subseteq \mathbf{L}$) be a (general) interlinked agenda. For all $S \subseteq \mathbf{L}$ let $S^\neg := \{\neg p : p \in S\}$ and $\bar{S} := S \cup S^\neg$. We wish to apply Theorem 2 to X – but with which relation $<$? Defining $<$ as $\subsetneq_{\text{finite}}$ gives a very complicated set $\mathcal{IR}_< = \mathcal{MI}$ (containing sets like $Y = \{a, a \to b, a', a' \to b', b \wedge b' \to a \wedge c, \neg c\}$). Also the finer relation $Z < Y :\Leftrightarrow |Z| < |Y|$ is only appropriate for *simple* interlinked agendas. We will get a grip on $\mathcal{IR}_<$ by defining $<$ as follows. For all inconsistent $Z, Y \subseteq X$,

$$Z < Y :\Leftrightarrow (|Z \cap \bar{\mathcal{R}}|, |Z \cap \bar{\mathcal{A}}|) \text{ is lexicographically smaller than } (|Y \cap \bar{\mathcal{R}}|, |Y \cap \bar{\mathcal{A}}|),$$

i.e. $|Z \cap \bar{\mathcal{R}}| < |Y \cap \bar{\mathcal{R}}|$ or $|Z \cap \bar{\mathcal{R}}| = |Y \cap \bar{\mathcal{R}}| \& |Z \cap \bar{\mathcal{A}}| < |Z \cap \bar{\mathcal{A}}|$. For instance, $\{a, \neg b\} < \{a \to b\}$ as $(0, 2)$ is lexicographically smaller than $(1, 0)$. The following is easily shown (using that the "lexicographically smaller than" relation on $(\{0, 1, ...,\} \cup \{\infty\})^2$ is well-founded).

Lemma 4 *For an interlinked agenda X, the above relation $<$ is admissible.*[14]

To identify the $<$-irreducible sets, we first need to understand better which entailments and inconsistencies hold in interlinked agendas; hence the next two technical lemmas. Notationally, for all $p \in \mathbf{L}$ and all $R \subseteq \mathbf{L}$ let

$$R_p := \{s \in \mathbf{L} : p \to s \in R \text{ or } p \leftrightarrow s \in R \text{ or } s \leftrightarrow p \in R\},$$

the set of propositions "reachable" from p via (bi)implications in R. I first establish a plausible fact about entailments between connection rules: namely, for instance, that $R = \{p \to b, p \to (c \wedge d)\} \vDash p \to (b \wedge c)$ because each conjunct of $b \wedge c$ (i.e. b and c) is a conjunct of some $s \in R_p = \{b, c \wedge d\}$.

Lemma 5 *For all $R \subseteq \mathcal{R}$ and $p \to q \in \mathcal{R}$,*

$$R \vDash p \to q \Leftrightarrow \mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \bigcup_{s \in R_p} \mathrm{Conj}(s).$$

Note that this characterisation of $A \vDash p \to q$ implies one of $A \vDash p \leftrightarrow q$ (for $A \subseteq \mathcal{R}$ and $p \leftrightarrow q \in \mathcal{R}$), since $A \vDash p \leftrightarrow q$ if and only if $A \vDash p \to q$ and $A \vDash q \to p$.

Proof. Let $R \subseteq \mathcal{R}$ and $p \to q \in \mathcal{R}$.

1. First let $\mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \bigcup_{s \in R_p} \mathrm{Conj}(s)$. Suppose all $r \in R$ hold in world w of interpretation $(W, (f_r), (v_w))$. We have to show that $p \to q$ holds in w, i.e. that all $a \in \mathrm{Conj}(q)$ hold in all $w^* \in f_p(w)$. Let $a \in \mathrm{Conj}(q)$ and $w^* \in f_p(w)$. By assumption, $a \in \mathrm{Conj}(p)$ or $a \in \mathrm{Conj}(s)$ for some $s \in R_p$. In the first case, a holds in w^* as p does (by $w^* \in f_p(w)$). In the second case, a holds in w^* as s does (by $v_w(p \to s) = T$ and $w^* \in f_p(w)$).

2. Conversely, suppose that $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$ but $a \notin \bigcup_{s \in R_p} \mathrm{Conj}(s)$. To show $R \nvDash p \to q$, consider an interpretation $(W, (f_p), (v_w))$ such that: (i) W contains at least two distinct worlds w, w^*, (ii) all atomic propositions hold in w, (iii) all atomic propositions except a hold in w^*, (iv) $f_p(w) = \{w, w^*\}$ (which is allowed as p holds in w and w^*), and (v) for all $t \in \mathbf{L} \backslash \{p\}$ $f_t(w) \subseteq \{w\}$. To complete the proof, I show that all $r \in R$ hold in w but $p \to q$ doesn't. First, $v_w(p \to q) = F$ by (iv) and as $v_{w^*}(q) = F$ by (iii). To show the truth in w of all $r \in R$, I show that of every implication $t \to s$ with $t \to s \in R$ or $t \leftrightarrow s \in R$ or $s \leftrightarrow t \in R$. For such $t \to s$, if $t \neq p$ then $v_w(t \to s) = T$ by (v) and (ii); and if $t = p$ then $v_w(t \to s) = T$ by (iv) and (ii)-(iii) and using that $a \notin \mathrm{Conj}(s)$. ■

The next technical lemma shows that there are broadly two ways in which a subset A of the interlinked agenda X can be inconsistent (the second way, (7), hods for instance if $\neg(a \to (b \wedge c)), a \to b, a \to c \in A$.).

Lemma 6 *If $A \subseteq \bar{\mathcal{A}} \cup \bar{\mathcal{R}}$ is inconsistent, then either already $A \backslash \mathcal{R}^{\neg}$ is inconsistent or*

$$A \text{ contains a } \neg r \in \mathcal{R}^{\neg} \text{ such that } A \cap \mathcal{R} \vDash r. \tag{7}$$

[14] More generally, for any agenda X and any finite partition of X into $X_1, ..., X_k$, an admissible relation on $\mathcal{I}$ is defined by $Z < Y :\Leftrightarrow (|Z \cap X_1|, ..., |Z \cap X_n|)$ is lexicographically smaller than $(|Y \cap X_1|, ..., |Y \cap X_n|)$.

Proof. Suppose $A \subseteq \bar{\mathcal{A}} \cup \bar{\mathcal{R}}$. Assume $A_* := A \backslash \mathcal{R}^{\neg}$ is consistent and (7) does not hold. I show that A is consistent. For all $\neg(p \to q) \in A$,

(α) there is $a_{p\to q} \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$ with $a_{p\to q} \notin \mathrm{Conj}(q')$ for all $q' \in A_p$,

as otherwise $\mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \cup_{q' \in A_p} \mathrm{Conj}(q')$, whence by Lemma 5 $A \cap \mathcal{R} \vDash p \to q$ (take $R := A \cap \mathcal{R}$ and note that $R_p = A_p$), implying (7). Further, for all $\neg(p \leftrightarrow q) \in A$, *either*

($\beta 1$) there is $a^1_{p\leftrightarrow q} \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$ with $a^1_{p\leftrightarrow q} \notin \mathrm{Conj}(q')$ for all $q' \in A_p$

or

($\beta 2$) there is $a^2_{p\leftrightarrow q} \in \mathrm{Conj}(p) \backslash \mathrm{Conj}(q)$ with $a^2_{p\leftrightarrow q} \notin \mathrm{Conj}(p')$ for all $p' \in A_q$,

as otherwise $\mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \cup_{q' \in A_p} \mathrm{Conj}(q')$ and $\mathrm{Conj}(p) | \mathrm{Conj}(q) \subseteq \cup_{p' \in A_q}$, whence again by Lemma 5 $A \cap \mathcal{R} \vDash p \to q$ and $A \cap \mathcal{R} \vDash p \to r$, i.e. $A \cap \mathcal{R} \vDash p \leftrightarrow q$, implying (7).

To prove A's consistency, I construct an interpretation and show that in a world all $r \in A$ hold. Notationally, for any $r \in \mathcal{R}$ let r^{mat} be r's material counterpart: $(p \to q)^{\mathrm{mat}}$ is $\neg p \vee q$, and $(p \leftrightarrow q)^{\mathrm{mat}}$ is $(p \to q)^{\mathrm{mat}} \wedge (q \to p)^{\mathrm{mat}}$. Let A^{mat}_* be the set arising from A_* by replacing all $r \in A_* \cap \mathcal{R}$ by r^{mat}. Since A_* is consistent and $r \models r^{\mathrm{mat}}$ for all $r \in \mathcal{R}$, A^{mat}_* is also consistent. So there exists an interpretation $(W, (f_p), (v_w))$ and a world w such that

(w1) all members of A^{mat}_* are true in w.

As the propositions in A^{mat}_* contain no subjunctive (bi)implications, their truth values in w depend neither on other worlds nor on the functions $f_p, p \in \mathbf{L}$. So we may assume the following w.l.o.g.

(w2) For all $\neg(p \to q) \in A$, there is a world $w_{p\to q} \in W \backslash \{w\}$ in which all atomic proposition except $a_{p\to q}$ hold; and $w_{p\to q} \in f_p(w)$ but $w_{p\leftrightarrow q} \notin f_s(w)$ $\forall s \in \mathbf{L} \backslash \{p\}$.

(w3) For all $\neg(p \leftrightarrow q) \in A$ with ($\beta 1$), there is a world $w^1_{p\leftrightarrow q} \in W \backslash \{w\}$ in which all atomic propositions except $a^1_{p\leftrightarrow q}$ hold; and $w^1_{p\leftrightarrow q} \in f_p(w)$ but $w^1_{p\leftrightarrow q} \notin f_s(w)$ $\forall s \in \mathbf{L} \backslash \{p\}$.

(w4) For all $\neg(p \leftrightarrow q) \in A$ with ($\beta 2$), there is a world $w^2_{p\leftrightarrow q} \in W \backslash \{w\}$ in which all atomic propositions except $a^2_{p\leftrightarrow q}$ hold; and $w^2_{p\leftrightarrow q} \in f_q(w)$ but $w^2_{p\leftrightarrow q} \notin f_s(w)$ $\forall s \in \mathbf{L} \backslash \{q\}$.

(w5) Worlds $w' \in W$ other than those defined in (w1)-(w4) are not reachable from w: $w' \notin f_r(w)$ $\forall r \in \mathbf{L}$.

To complete the proof, I consider any $r \in A$ and show that r holds in w.

Case 1: r is atomic or negated atomic. Then $r \in A^{\mathrm{mat}}_*$. So r holds in w by (w1).

Case 2: r is an implication $s \to t$. Let $w' \in f_s(w)$. I have to show that t holds in w'. If $w' = w$, s holds in w by $w \in f_s(w)$; so, as $(s \to t)^{\mathrm{mat}} = \neg s \vee t$ holds in w by (w1), t holds in w. Now let $w' \neq w$. Then by (w5), w' is one of the worlds defined in (w2)-(w4). Assume $w' = w_{p\to q}$, a world defined in (w2) (proofs for (w3) and (w4) are similar). By $w_{p\to q} \in f_s(w)$ and (w2), $p = s$. By (w2), all atomic propositions except $a_{p\to q}$ hold in $w_{p\to q}$, where $a_{p\to q}$ isn't a conjunct of t by (α). So t holds in $w_{p\to q} = w'$.

Case 3: r is a biimplication $s \leftrightarrow t$. $s \leftrightarrow t$ holds in w if $s \to t$ and $t \to s$ are true in w. The latter can be shown by a procedure analogous to that in case 2.

Case 4: r is a negated implication $\neg(p \to q)$. To show that r holds in w, I show that $p \to q$ fails in w. This is so because, by (w2), $w_{p\to q} \in f_p(w)$ where q fails in $w_{p\to q}$ as its conjunct $a_{p\to q}$ fails.

Case 5: r is a negated biimplication $\neg(p \leftrightarrow q)$. To show that r holds in w, I show that $p \leftrightarrow q$ is false in w, i.e. that $p \to q$ or $q \to p$ is false in w. Under ($\beta 1$) $p \to q$ is false in w (consider the world $w^1_{p\leftrightarrow q}$ and use (w3)), and under ($\beta 2$) $q \to p$ is false in w (consider the world $w^2_{p\leftrightarrow q}$ and use (w4)). ■

We now define a class of inconsistent sets $\mathcal{Y}$ and show, using Lemmas 5 and 6, that $\mathcal{IR}_< \subseteq \mathcal{Y} \subseteq \mathcal{MI}$. Notationally, for any conjunctions p, q of atomic propositions,

$$X_{pq} := \{S \subseteq X_p : S \text{ is minimal subject to } \mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \cup_{s \in S} \mathrm{Conj}(s)\}.$$

For instance, if $p \to q \in X$ then X_{pq} always contains $\{q\}$, but perhaps no other set. If $a \to (b \wedge c) \in X$ and also $a \to b, a \to (c \wedge d) \in X$, then $X_{a,b\wedge c}$ contains not only $\{b \wedge c\}$ but also $\{b, c \wedge d\}$. For our interlinked agenda X, define $\mathcal{Y} := \mathcal{Y}_\to \cup \mathcal{Y}_\leftrightarrow \cup \mathcal{Y}_{\neg\to} \cup \mathcal{Y}_{\neg\leftrightarrow}$, where $\mathcal{Y}_\to$, $\mathcal{Y}_\leftrightarrow$, $\mathcal{Y}_{\neg\to}$ and $\mathcal{Y}_{\neg\leftrightarrow}$ are the sets that consist, respectively, of

- all $Y \subseteq X$ of type $\{p \to q, \neg a\} \cup \mathrm{Conj}(p)$ where $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$;
- all $Y \subseteq X$ of type $\{p \leftrightarrow q, \neg a\} \cup \mathrm{Conj}(p)$ or $\{q \leftrightarrow p, \neg a\} \cup \mathrm{Conj}(p)$ where $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$;
- all $Y \subseteq X$ of type $\{\neg(p \to q)\} \cup \{p_s : s \in S\}$ where $S \in X_{pq}$ and $\forall s \in S$ $p_s \in \{p \to s, p \leftrightarrow s, s \leftrightarrow p\}$;
- all $Y \subseteq X$ of type $\{\neg(p \leftrightarrow q)\} \cup \{p_s : s \in S\} \cup \{q_s : s \in S'\}$ where $S \in X_{pq}$, $\forall s \in S$ $p_s \in \{p \to s, p \leftrightarrow s, s \leftrightarrow p\}$, $S' \in X_{qp}$, $\forall s \in S'$ $q_s \in \{q \to s, q \leftrightarrow s, s \leftrightarrow q\}$, and the sets $\{p_s : s \in S\}$, $\{q_s : s \in S'\}$ are either each disjoint with $\{p \leftrightarrow q, q \leftrightarrow p\}$ or each equal to $\{q \leftrightarrow p\}$ (the latter is only possible if $S = \{q\} \& S' = \{p\}$; the former holds automatically if $S \neq \{q\} \& S' \neq \{p\}$ as then $q \notin S \& p \notin S'$).

Lemma 7 *For an interlinked agenda X and $\mathcal{Y}$ as defined above, $\mathcal{Y} \subseteq \mathcal{MI}$.*

Proof. Let X and $\mathcal{Y}$ be as specified. Consider any $Y \in \mathcal{Y}$. I show that $Y \in \mathcal{MI}$ by going through the four possible cases.

1. Let $Y \in \mathcal{Y}_\to$, i.e. $Y = \{p \to q, \neg a\} \cup \mathrm{Conj}(p)$ where $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$. Y is inconsistent because, by $\mathrm{Conj}(p) \vDash p$ and $\{p \to q, p\} \vDash q$, we have $\{p \to q\} \cup \mathrm{Conj}(p) \vDash q$. Moreover, for any $y \in Y$, the consistency of $Y \backslash \{y\}$ can be checked by finding an interpretation with a world w in which all $z \in Y \backslash \{y\}$ hold. Specifically, $\{p \to q\} \cup \mathrm{Conj}(p)$ is consistent: let all atomic propositions hold in w and in all other worlds; $\{\neg a\} \cup \mathrm{Conj}(p)$ is consistent: let all atomic propositions except a hold in w; and, for any $y \in \mathrm{Conj}(p)$, $\{p \to q, \neg a\} \cup \mathrm{Conj}(p) \backslash \{y\}$ is consistent: let the only atomic propositions true in w be those in $\mathrm{Conj}(p) \backslash \{y\}$, and put $f_p(w) = \emptyset$ (which is allowed as p fails in w).

2. If $Y \in \mathcal{Y}_\leftrightarrow$ then $Y \in \mathcal{MI}$ by a proof similar to that under 1.

3. Now let $Y \in \mathcal{Y}_{\neg\to}$, say (in the earlier notation) $Y = \{\neg(p \to q)\} \cup \{p_s : s \in S\}$. We have $\{p_s : s \in S\}_p = S \in X_{pq}$, whence by Lemma 5 $\{p_s : s \in S\} \vDash p \to q$. So Y is inconsistent. To check *minimal* inconsistency, consider any $Z \subsetneq Y$. If $\neg(p \to q) \notin Z$, Z is consistent, as seen from an interpretation such that all atomic propositions hold in all worlds. If $\neg(p \to q) \in Z$, then $Z = \{\neg p \to q)\} \cup R^*$ with $R^* = \{p_s : s \in S^*\}$ and $S^* \subsetneq S$. Note that $R^*_p = S^*$. So $R^*_p \subsetneq S$. This and $S \in X_{pq}$ imply that $\mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \nsubseteq \cup_{s \in R^*_p} \mathrm{Conj}(s)\}$, whence by Lemma 5 $R^* \nvDash p \to q$. So Z $(= \{\neg p \to q)\} \cup R^*)$ is consistent.

4. Finally, let $Y \in \mathcal{Y}_{\neg\leftrightarrow}$, say (in the earlier notation) $Y = \{\neg(p \leftrightarrow q)\} \cup \{p_s : s \in S\} \cup \{q_s : s \in S'\}$. It can be shown like under 3 that $\{p_s : s \in S\} \vDash p \to q$ and $\{q_s : s \in S'\} \vDash q \to p$. So $\{p_s : s \in S\} \cup \{q_s : s \in S'\} \vDash p \leftrightarrow q$. Hence Y is inconsistent.

Now consider any $Z \subsetneq Y$. If $\neg(p \leftrightarrow q) \notin Z$, Y is consistent by an argument like in case 3. If $\neg(p \leftrightarrow q) \in Z$, then $Z = \{\neg(p \leftrightarrow q)\} \cup R^*$ with $R^* = \{p_s : s \in S^*\} \cup \{q_s : s \in S'^*\}$ and $S^* \subseteq S$, $S'^* \subseteq S'$, where $S^* \subsetneq S$ or $S'^* \subsetneq S'$. Note that $R^*_p = S^*$ and $R^*_q = S'^*$. So $R^*_p \subsetneq S$ or $R^*_q \subsetneq S'$. Hence $R^* \nvDash p \to q$ or $R^* \nvDash q \to p$, by an argument like that under 3. So $R^* \nvDash p \leftrightarrow q$. Hence Z $(= \{\neg(p \leftrightarrow q)\} \cup R^*)$ is consistent. ■

Lemma 8 *For an interlinked agenda X and $<$ and $\mathcal{Y}$ as defined above, $\mathcal{IR}^*_< \subseteq \mathcal{Y}$.*

Proof. Let X, $<$ and $\mathcal{Y}$ $(= \mathcal{Y}_\to \cup \mathcal{Y}_\leftrightarrow \cup \mathcal{Y}_{\neg\to} \cup \mathcal{Y}_{\neg\leftrightarrow})$ be as specified. Consider a $Y \in \mathcal{IR}^*_<$. I show that $Y \in \mathcal{Y}$. I will use that $Y \in \mathcal{MI}$ by Lemma 2, and that (*) Y contains no pair $t, \neg t$ by non-triviality.

Case 1: $Y \cap \mathcal{R}^\neg = \emptyset$. Then (i) Y has a subset of type $\{p \to q\} \cup \mathrm{Conj}(p)$, *or* (ii) Y has a subset of type $\{p \leftrightarrow q\} \cup \mathrm{Conj}(p)$ or $\{q \leftrightarrow p\} \cup \mathrm{Conj}(p)$. Otherwise Y would be consistent, as seen from an interpretation with a world w in which the only true atomic propositions are those in Y and such that $f_t(w) = \emptyset$ if $t \in \mathbf{L}$ is false in w: in w, all $y \in Y \cap \bar{\mathcal{A}}$ hold by construction (and by (*)), all $p \to q \in Y$ hold by $f_p(w) = \emptyset$ (as p is false by not-(i)), all $p \leftrightarrow q \in Y$ hold by $f_p(w) = f_q(w) = \emptyset$ (as p and q are false by not-(ii)), and there are no $y \in Y \cap \mathcal{R}^\neg$.

Subcase 1a: (i) holds, say $\{p \to q\} \cup \mathrm{Conj}(p) \subseteq Y$. I show that $Y \in \mathcal{Y}_\to$. If there is an $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$ with $\neg a \in Y$, then $\{p \to q, \neg a\} \cup \mathrm{Conj}(p) \subseteq Y$, hence $\{p \to q, \neg a\} \cup \mathrm{Conj}(p) = Y$ (as $Y \in \mathcal{MI}$), and so $Y \in \mathcal{Y}_\to$. Hence it suffices to prove that such an a exists. For a contradiction, suppose (**) $\neg a \notin Y$ for all $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$. I show that Y is reducible to $Z := Y \cup \mathrm{Conj}(q) \backslash \{p \to q\}$, a contradiction. First, Z is indeed inconsistent: otherwise there would exist an interpretation with a world w in which all $z \in Z$ hold, where by $Z \cap \mathcal{R}^\neg = \emptyset$ we may assume w.l.o.g. that $f_p(w)$ contains no world other than w; thus $p \to q$ also holds in w, so that $Z \cup \{p \to q\} = Y \cup \mathrm{Conj}(q)$ is consistent, a contradiction. Second, we have $Z < Y$ by $|Z \cap \bar{\mathcal{R}}| = |Y \cap \bar{\mathcal{R}}| - 1$ (and by our lexicographic definition of $<$). Finally, any $y \in Z \backslash Y$ belongs to $\mathrm{Conj}(q)$, hence is entailed by $Z := \mathrm{Conj}(p) \cup \{p \to q\}$ $(\subseteq Y)$; it remains to show $Z \cup \{\neg y\} < Y$, which I do by proving that $|(Z \cup \{\neg y\}| \cap \bar{\mathcal{R}}| < |Y \cap \bar{\mathcal{R}}|$, i.e. that $|Y \cap \bar{\mathcal{R}}| > 1$. Suppose the contrary. Then $Y = \{p \to q\} \cup \mathrm{Conj}(p) \cup Y'$ for some $Y' \subseteq \bar{\mathcal{A}}$. By $Y \in \mathcal{MI}$, $\mathrm{Conj}(p) \cup Y'$ is consistent. So there is an interpretation with a world w in which all $a \in \mathrm{Conj}(p) \cup Y'$ hold, where by $\mathrm{Conj}(p) \cup Y' \subseteq \bar{\mathcal{A}}$ we may assume w.l.o.g. that $f_p(p)$ contains no world other than w, and that all $a \in \mathcal{A}$ with $\neg a \notin Y$ hold in w. All $a \in \mathrm{Conj}(q)$ satisfy $\neg a \notin Y$: if $a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$ by (**), and if $a \in \mathrm{Conj}(q) \cap \mathrm{Conj}(p)$ by (*). So, in w, all $a \in \mathrm{Conj}(q)$ and hence q hold; so $p \to q$ holds. But then all $y \in Y$ hold in w, contradicting Y's inconsistency.

Subcase 1b: (ii) holds, say $\{p \leftrightarrow q\} \cup \mathrm{Conj}(p) \subseteq Y$ (the proof is analogous if $p \leftrightarrow q$ is replaced by $q \leftrightarrow p$). To show that $Y \in \mathcal{Y}_\leftrightarrow$, it suffices to slightly adapt the proof in Subcase 1a: replace "$\to$" by "$\leftrightarrow$", and in both interpretations assume w.l.o.g. that $f_q(w)$ (in addition to $f_p(w)$) contains no world other than w.

Case 2: $Y \cap \mathcal{R}^\neg \neq \emptyset$. Then $Y \backslash \mathcal{R}^\neg \subsetneq Y$, whence $Y \backslash \mathcal{R}^\neg$ is consistent by $Y \in \mathcal{MI}$. So by Lemma 6 Y contains a $\neg r \in \mathcal{R}^\neg$ such that $Y \cap \mathcal{R} \vDash r$. Let $R := Y \cap \mathcal{R}$. As Y is *minimal* inconsistent, $Y = \{\neg r\} \cup R$. I consider two subcases.

Subcase 2a: r is an implication $p \to q$. I show that $Y \in \mathcal{Y}_{\neg\to}$. As $Y = \{\neg(p \to q)\} \cup R \in \mathcal{MI}$, R is minimal subject to entailing $p \to q$. So, by Lemma 5, R is minimal subject to $\mathrm{Conj}(q) \backslash \mathrm{Conj}(p) \subseteq \bigcup_{s \in R_p} \mathrm{Conj}(s)$. This implies that $R_p \in X_{pq}$

and that $R = \{p_s : s \in R_p\}$ for some $p_s \in \{p \to s, p \leftrightarrow s, s \leftrightarrow p\}$, $s \in R_p$. So Y $(= \{\neg(p \to q)\} \cup R)$ is in $\mathcal{Y}_{\neg\to}$.

Subcase 2b: r is an biimplication $p \leftrightarrow q$. I show $Y \in \mathcal{Y}_{\neg\leftrightarrow}$. Write $R = R^1 \cup R^2 \cup T$ with $R^1 := R \cap \{p \to s, p \leftrightarrow s, s \leftrightarrow p : s \in \mathbf{L}\}$, $R^2 := R \cap \{q \to s, q \leftrightarrow s, s \leftrightarrow q : s \in \mathbf{L}\}$ and $T := R \backslash (X_p \cup X_q)$. As $Y = \{\neg(p \leftrightarrow q)\} \cup R$ is minimal inconsistent, R is minimal subject to entailing $p \leftrightarrow q$, i.e. minimal subject to entailing each of $p \to q$ and $q \to p$. So, by Lemma 5 and using that $R_p = R^1_p$ and $R_q = R^2_q$, the set R is minimal subject to satisfying both (a) $\text{Conj}(q) \backslash \text{Conj}(p) \subseteq \bigcup_{s \in R^1_p} \text{Conj}(s)$ and (b) $\text{Conj}(p) \backslash \text{Conj}(q) \subseteq \bigcup_{s \in R^2_q} \text{Conj}(s)$. It follows that $R = R^1 \cup R^2$ (i.e. $T = \emptyset$).

First suppose $q \leftrightarrow p \in R^1$ or $q \leftrightarrow p \in R^2$. Then $Y = \{\neg(p \leftrightarrow q)\} \cup R \supseteq \{\neg(p \leftrightarrow q), q \leftrightarrow p\}$, hence by minimal inconsistency $Y = \{\neg(p \leftrightarrow q), q \leftrightarrow p\}$. So $Y \in \mathcal{Y}_{\neg\leftrightarrow}$, as desired.

Now suppose $q \leftrightarrow p \notin R^1$ and $q \leftrightarrow p \notin R^2$. As also $p \leftrightarrow q \notin R^1$ and $p \leftrightarrow q \notin R^2$ by (*), we have $R^1 \cap R^2 = \emptyset$. This and the fact that the set $Y = R^1 \cup R^2$ is minimal subject to (a)&(b) imply that R^1 is minimal subject to (a) and that R^2 is minimal subject to (b). So (like in Subcase 2a) $R^1_p \in X_{pq}$ with $R^1 = \{p_s : s \in R_p\}$ for some $p_s \in \{p \to s, p \leftrightarrow s, s \leftrightarrow p\}$, $s \in R_p$, and $R^2_q \in X_{qp}$ with $R^2 = \{q_s : s \in R_q\}$ for some $q_s \in \{q \to s, q \leftrightarrow s, s \leftrightarrow q\}$, $s \in R_q$. So Y $(= \{\neg(p \to q)\} \cup R^1 \cup R^2)$ is in $\mathcal{Y}_{\neg\leftrightarrow}$, as desired.

By Lemmas 7 and 8, we can apply Corollary 3 to characterise consistent quota rules. After a series of simplifications (performed in the proof below), the characterisation takes the following form (where $A \Delta B := (A \backslash B) \cup (B \backslash A)$).

Theorem 3 *For an interlinked agenda X, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent if and only if the thresholds satisfy the following relations:*

(a) *for every $p \to q \in X$,*

$$\sum_{a \in \text{Conj}(p)} (n - m_a) + \max_{b \in \text{Conj}(q) \backslash \text{Conj}(p)} m_b \leq m_{p \to q} \leq n - \max_{S \in X_{pq}} \sum_{s \in S : p \to s \in X} (n - m_{p \to s});$$

(b) *for every $p \leftrightarrow q \in X$, we have $m_{p \leftrightarrow q} = n$, all $a \in \text{Conj}(p) \cap \text{Conj}(q)$ have threshold $m_a = n$, and all $a \in \text{Conj}(p) \Delta \text{Conj}(q)$ have the same threshold, which equals n if $|\text{Conj}(p) \Delta \text{Conj}(q)| \geq 3$.*

Theorem 3 characterises consistency of quota rules by a computationally easily verifiable, yet rather complicated system of (in)equalities. Before giving the proof, I note three corollaries. First, a possibility result follows.[15]

Corollary 4 *For any interlinked agenda X, there exists*

(i) *a consistent quota rule $F_{(m_p)_{p \in X^+}}$ (hence a consistent, complete, independent, anonymous, monotonic and responsive aggregation rule with universal domain);*

(ii) *a single consistent quota rule $F_{(m_p)_{p \in X^+}}$ with identical thresholds m_p, $p \in X^+$, namely the quota rule with the unanimity threshold n for all $p \in X^+$.*

[15] See footnote 6 for the conditions listed in part (i). By a different proof, part (i) holds more generally for any agenda $X \subseteq \bar{\mathcal{R}} \cup \bar{\mathcal{A}}$ (X can contain an $r \in \bar{\mathcal{R}}$ without containing the atomic propositions occuring in r).

Proof. As (ii) implies (i), I only show (ii). Let X be an interlinked agenda and $F_{(m_p)_{p \in X+}}$ a quota rule with identical thresholds $m_p = m$ $(\in \{1, ..., n\})$. If $m = n$ then (a)&(b) hold, implying consistency. Conversely, assume consistency, hence (a)&(b). X contains a $p \to q$ or a $p \leftrightarrow q$ (otherwise X would be empty, hence not an agenda). In the second case, $m = n$ by (b). In the first case, the LHS inequality in (a) implies $\sum_{a \in \mathrm{Conj}(p)} (n - m) + m \leq m$, whence again $m = n$. ■

In practice, the system (a)&(b) often looks much simpler. If X contains no biimplications, (b) drops out. If X is even simple, we obtain exactly Theorem 3, as (b) drops out and in (a) the RHS inequality is trivial (by $X_{pq} = \{\{q\}\}$) and the LHS inequality becomes $n - m_p + m_q \leq m_{p \to q}$. Generalising this argument yields the following.

Corollary 5 *For an interlinked agenda X all of whose connection rules are implications $p \to b$ with atomic b, a quota rule $F_{(m_p)_{p \in X+}}$ is consistent if and only if*

$$\sum_{a \in \mathrm{Conj}(p)} (n - m_a) + m_b \leq m_{p \to b} \text{ for all } p \to b \in X.$$

By (b) in Theorem 3 there is only little freedom in the choice of thresholds of biimplications $p \leftrightarrow q \in X$ and their contained atomic propositions. More generally, let me show that *cycles* in X severely restrict the possibility. Extending earlier definitions to general interlinked agendas, I call an atomic proposition a a *parent* of another one b if there is a $p \to q \in X$ or $p \leftrightarrow q \in X$ or $q \leftrightarrow p \in X$ such that $a \in \mathrm{Conj}(p)$ and $b \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$. Parenthood yields the notion of an *ancestor*. A *path* is a sequence $(a_1, ..., a_k)$ $(k \geq 2)$ where a_j is a parent of a_{j+1} $\forall j < k$; it is a *cycle* if $a_1 = a_k$.

Corollary 6 *Let $F_{(m_p)_{p \in X+}}$ be a consistent quota rule for an interlinked agenda X.*
(i) *If $a \in X$ is an ancestor of $b \in X$, then $m_a \geq m_b$.*
(ii) *If $a, b \in X$ occur in a cycle (i.e. are ancestors of each other), then $m_a = m_b$ and $m_{p \to q} = n$ for all $p \to q \in X$ with $a \in \mathrm{Conj}(p)$ and $b \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$.*

Proof. Let X and $F_{(m_p)_{p \in X+}}$ be as specified.

(i) Let $a \in X$ be a parent of $b \in X$ (obviously it suffices to consider this case). Then $a \in \mathrm{Conj}(p)$ and $b \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p)$, where $p \to q \in X$ or $p \leftrightarrow q \in X$ or $q \leftrightarrow p \in X$. In the last two cases, (b) implies $m_a \geq m_b$. In the first case, the LHS inequality in (a) implies $(n - m_a) + m_b \leq m_{p \to q}$, so $m_b \leq m_{p \to q} - n + m_a \leq m_a$.

(ii) Let a, b be as specified. By (i) $m_a \leq m_b$ and $m_b \leq m_a$, hence $m_a = m_b$. Now let $p \to q$ be as specified. By the LHS inequality in (a), $(n - m_a) + m_b \leq m_{p \to q}$, hence (by $m_a = m_b$) $m_{p \to q} = n$. ■

Proof of Theorem 3. Let X be an interlinked agenda, $F_{(m_p)_{p \in X+}}$ a quota rule, and $\mathcal{Y}$, $\mathcal{Y}_{\to}, \mathcal{Y}_{\leftrightarrow}$, $\mathcal{Y}_{\neg\to}$, $\mathcal{Y}_{\neg\leftrightarrow}$ the sets defined above. By Corollary 3 and Lemmas 4, 7 and 8, I have to show that (a)&(b) hold iff for all $Y \in \mathcal{Y}$ $(= \mathcal{Y}_{\to} \cup \mathcal{Y}_{\leftrightarrow} \cup \mathcal{Y}_{\neg\to} \cup \mathcal{Y}_{\neg\leftrightarrow})$

$$\sum_{y \in Y} (n - m_y) < n. \tag{8}$$

I will build up this equivalence in the following four steps.

Claim 1. The LHS inequalities in (a) hold iff (8) holds for all $Y \in \mathcal{Y}_{\rightarrow}$.

Claim 2. Given (b), the RHS inequalities in (a) hold iff (8) holds for all $Y \in \mathcal{Y}_{\neg\rightarrow}$.

Claim 3. (b) holds iff (8) holds for all $Y \in \mathcal{Y}_{\leftrightarrow}$.

By Claims 1-3, (a)&(b) holds iff (8) holds for all $Y \in \mathcal{Y}_{\rightarrow} \cup \mathcal{Y}_{\leftrightarrow} \cup \mathcal{Y}_{\neg\rightarrow}$; which is the case iff (8) holds for all $Y \in \mathcal{Y}_{\rightarrow} \cup \mathcal{Y}_{\leftrightarrow} \cup \mathcal{Y}_{\neg\rightarrow} \cup \mathcal{Y}_{\neg\leftrightarrow}$, because of our last claim which completes the proof.

Claim 4. If (8) holds for all $Y \in \mathcal{Y}_{\rightarrow} \cup \mathcal{Y}_{\leftrightarrow}$ then it holds for all $Y \in \mathcal{Y}_{\neg\leftrightarrow}$ (hence the inequalities for $Y \in \mathcal{Y}_{\neg\leftrightarrow}$ are redundant in the system).

Proof of Claim 1. The inequalities (8) for all $Y \in \mathcal{Y}_{\rightarrow}$ are given by

$$(n - m_{\neg a}) + (n - m_{p\rightarrow q}) + \sum_{a' \in \mathrm{Conj}(p)} (n - m_{a'}) < n \;\forall p \rightarrow q \in X \;\forall a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p).$$

Using that $n - m_{\neg a} = m_a - 1$, these inequalities can be rewritten as

$$m_a + \sum_{a' \in \mathrm{Conj}(p)} (n - m_{a'}) \leq m_{p\rightarrow q} \;\forall p \rightarrow q \in X \;\forall a \in \mathrm{Conj}(q) \backslash \mathrm{Conj}(p),$$

which by taking the maximum over a is equivalent to the LHS inequalities in (a).

Proof of Claim 2. Suppose (b). The inequalities (8) for all $Y \in \mathcal{Y}_{\neg\rightarrow}$ are given by

$$n - m_{\neg(p\rightarrow q)} + \sum_{s \in S} (n - m_{p_s}) < n \quad \begin{array}{l} \forall p \rightarrow q \in X \;\forall S \in X_{pq} \\ \forall (p_s)_{s \in S} \in (\{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X)^S. \end{array}$$

These inequalities can (by $n - m_{\neg(p\rightarrow q)} = m_{p\rightarrow q} - 1$) be rewritten as

$$m_{p\rightarrow q} + \sum_{s \in S} (n - m_{p_s}) \leq n \quad \begin{array}{l} \forall p \rightarrow q \in X \;\forall S \in X_{pq} \\ \forall (p_s)_{s \in S} \in (\{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X)^S, \end{array}$$

or equivalently as

$$m_{p\rightarrow q} + \max_{(p_s)_{s \in S} \in (\{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X)^S} \sum_{s \in S} (n - m_{p_s}) \leq n \;\forall p \rightarrow q \in X \;\forall S \in X_{pq}. \quad (9)$$

Note that

$$\max_{(p_s)_{s \in S} \in (\{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X)^S} \sum_{s \in S} (n - m_{p_s}) = \sum_{s \in S} (n - \min_{p_s \in \{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X} m_{p_s}). \quad (10)$$

For all $s \in S$ and all $p_s \in \{p \leftrightarrow q, q \leftrightarrow p\}$ we have $m_{p_s} = n$ by (b). So, for all $s \in S$, $\min_{p_s \in \{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X} m_{p_s}$ is n if $p \rightarrow s \notin X$ and $m_{p\rightarrow s}$ if $p \rightarrow s \in X$. Hence in (10) the term $(n - \min_{p_s \in \{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X} m_{p_s})$ drops out if $p \rightarrow s \notin X$ and equals $(n - m_{p\rightarrow s})$ if $p \rightarrow s \in X$. Therefore (10) implies

$$\max_{(p_s)_{s \in S} \in (\{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\} \cap X)^S} \sum_{s \in S} (n - m_{p_s}) = \sum_{s \in S : p \rightarrow s \in X} (n - m_{p\rightarrow s}).$$

Using this, the inequalities (9) are equivalent to

$$m_{p\rightarrow q} + \sum_{s \in S : p \rightarrow s \in X} (n - m_{p\rightarrow s}) \leq n \;\forall p \rightarrow q \in X \;\forall S \in X_{pq},$$

and hence, as desired, to

$$m_{p\to q} + \max_{S\in X_{pq}} \sum_{s\in S: p\to s\in X} (n - m_{p\to s}) \le n \;\forall p\to q \in X.$$

Proof of Claim 3. 1. First assume (8) holds for all $Y \in \mathcal{Y}_{\leftrightarrow}$, and let $p \leftrightarrow q \in X$.

1.1. Here I show $m_{p\leftrightarrow q} = n$. As $p \leftrightarrow q$ is non-degenerate, there exist $a \in \text{Conj}(p)\backslash\text{Conj}(q)$ and $b \in \text{Conj}(q)\backslash\text{Conj}(p)$. By assumption,

$$\begin{aligned}
(n - m_{p\leftrightarrow q}) + (n - m_{\neg b}) + \sum_{a'\in\text{Conj}(p)} (n - m_{a'}) &< n, \\
(n - m_{p\leftrightarrow q}) + (n - m_{\neg a}) + \sum_{b'\in\text{Conj}(q)} (n - m_{b'}) &< n.
\end{aligned}$$

Rewriting this (by using that $m_{\neg s} = n + 1 - m_s$ for all $s \in X$), we obtain

$$\begin{aligned}
m_{p\leftrightarrow q} + 1 - m_b &> \sum_{a'\in\text{Conj}(p)} (n - m_{a'}) \ge n - m_a, \\
m_{p\leftrightarrow q} + 1 - m_a &> \sum_{b'\in\text{Conj}(q)} (n - m_{b'}) \ge n - m_b.
\end{aligned} \tag{11}$$

So

$$m_{p\leftrightarrow q} \ge n - m_a + m_b \text{ and } m_{p\leftrightarrow q} \ge n - m_b + m_a. \tag{12}$$

Adding both inequalities, we get $2m_{p\leftrightarrow q} \ge 2n$, whence $m_{p\leftrightarrow q} = n$.

1.2. Next I show that all $a \in \text{Conj}(p)\Delta\text{Conj}(q)$ have the same threshold. As $\text{Conj}(p)\Delta\text{Conj}(q)$ is the union of the non-empty sets $\text{Conj}(p)\backslash\text{Conj}(q)$ and $\text{Conj}(q)\backslash \text{Conj}(p)$, it is sufficient to show that $m_a = m_b$ for all $a \in \text{Conj}(p)\backslash\text{Conj}(q)$ and $b \in \text{Conj}(q)\backslash\text{Conj}(p)$. Consider such a, b. The argument in 1.1 yields (12), which by $m_{p\leftrightarrow q} = n$ implies $m_a \ge m_b$ and $m_b \ge m_a$, whence $m_a = m_b$.

1.3. Let m be the common threshold of all $a \in \text{Conj}(p)\Delta\text{Conj}(q)$. I suppose $m < n$ and show that $|\text{Conj}(p)\Delta\text{Conj}(q)| \le 2$. The first inequality in (11) (where $b \in \text{Conj}(q)\Delta\text{Conj}(p)$) implies

$$m_{p\leftrightarrow q} + 1 - m_b > \sum_{a'\in\text{Conj}(p)\backslash\text{Conj}(q)} (n - m_{a'}),$$

which after substituting $m_{p\leftrightarrow q} = n$ and $m_b = m_{a'} = m$ gives

$$n - m \ge |\text{Conj}(p)\backslash\text{Conj}(q)|(n - m), \text{ i.e. } |\text{Conj}(p)\backslash\text{Conj}(q)| \le 1.$$

It can be shown similarly that $|\text{Conj}(q)\backslash\text{Conj}(p)| \le 1$. So $|\text{Conj}(p)\Delta\text{Conj}(q)| \le 2$.

1.4. Finally, let $a'' \in \text{Conj}(p) \cap \text{Conj}(q)$. I show that $m_{a''} = n$. Let a, b be as in 1.1. The first inequality in (11) implies

$$m_{p\leftrightarrow q} + 1 - m_b > (n - m_{a''}) + (n - m_a),$$

which by $m_{p\leftrightarrow q} = n$ and $m_a = m_b$ implies $1 > (n - m_{a''})$, i.e. $m_{a''} = n$.

2. Conversely, assume (b). Consider any $Y \in \mathcal{Y}_{\leftrightarrow}$, say $Y = \{r, \neg a\} \cup \text{Conj}(p)$ where $r \in \{p \leftrightarrow q, q \leftrightarrow p\}$ and $a \in \text{Conj}(q) \backslash \text{Conj}(p)$, and let me show (8). Using (b) and $n - m_{\neg a} = m_a - 1$,

$$\sum_{y \in Y} (n - m_y) = m - 1 + |\text{Conj}(p) \backslash \text{Conj}(q)|(n - m),$$

where m denotes the common threshold of all $a' \in \text{Conj}(p) \Delta \text{Conj}(q)$. Note that if $|\text{Conj}(p) \backslash \text{Conj}(q)| \geq 2$ then $|\text{Conj}(p) \Delta \text{Conj}(q)| \geq 3$, hence $m = n$. So, as desired,

$$\sum_{y \in Y} (n - m_y) = \begin{cases} m - 1 + n - m < n & \text{if } |\text{Conj}(p) \backslash \text{Conj}(q)| = 1 \\ n - 1 + 0 < n & \text{if } |\text{Conj}(p) \backslash \text{Conj}(q)| \geq 2. \end{cases}$$

Proof of Claim 4. Suppose (8) holds for all $Y \in \mathcal{Y}_{\rightarrow} \cup \mathcal{Y}_{\leftrightarrow}$. Consider any $Y \in \mathcal{Y}_{\neg\leftrightarrow}$, say (in the earlier notation) $Y = \{\neg(p \leftrightarrow q)\} \cup \{p_s : s \in S\} \cup \{q_s : s \in S'\}$. To prove the corresponding inequality,

$$(n - m_{\neg(p \leftrightarrow q)}) + \sum_{s \in S} (n - m_{p_s}) + \sum_{s \in S'} (n - m_{q_s}) < n,$$

I show that $m_{p_s} = n \; \forall s \in S$ and that $m_{q_s} = n \; \forall s \in S'$; in fact, I only show the former as the latter holds analogously. Let $s \in S$. Recall that $S \in X_{pq}$ and $p_s \in \{p \rightarrow s, p \leftrightarrow s, s \leftrightarrow p\}$.

If $p_s \in \{p \leftrightarrow s, s \leftrightarrow p\}$ then already by Claim 3 $m_{p_s} = n$, as desired.

Now assume $p_s = p \rightarrow s$. Since $s \in S \in X_{pq}$, $\text{Conj}(q) \backslash \text{Conj}(p)$ is a subset of $\cup_{s^* \in S} \text{Conj}(s^*)$ but not of $\cup_{s^* \in S \backslash \{s\}} \text{Conj}(s^*)$. So there is a $b \in \text{Conj}(s) \cap \text{Conj}(q) \backslash \text{Conj}(p)$. Moreover, as $p \leftrightarrow q$ is non-degenerate, there is an $a \in \text{Conj}(p) \backslash \text{Conj}(q)$. As $a, b \in \text{Conj}(p) \Delta \text{Conj}(q)$, we have $m_a = m_b$ by Claim 3. Using Claim 1,

$$m_{p \rightarrow q} \geq \sum_{a' \in \text{Conj}(p)} (n - m_{a'}) + \max_{b' \in \text{Conj}(q) \backslash \text{Conj}(p)} m_{b'} \geq n - m_a + m_b = n,$$

whence $m_{p \rightarrow q} = n$, as desired. ■

7 Conclusion

Connection rules, of the unidirectional kind $p \rightarrow q$ or bidirectional kind $p \leftrightarrow q$, are at the heart of judgment aggregation. They express links that may be accepted or rejected, for instance causal links between facts or justificational links between claims. Once we interpret (bi)implications subjunctively, we obtain consistent group judgment sets by taking independent votes on the propositions of an interlinked agenda, provided that the (proposition-specific) thresholds are chosen adequately. I have shown exactly which thresholds guarantee collective consistency (see Theorems 1 and 3). The result is an instance of a general characterisation of consistent quota rules, valid for all agendas in most logics (see Theorem 2). This characterisation could be applied to classes of agendas other than interlinked agendas, leading to new insights on possibilities of voting independently on the propositions. But *non-independent* forms of aggregation may be appealing too. A systematic account of non-independent judgment aggregation has yet to be developed.

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