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Extension of Arrow's Theorem to Symmetric Sets of Tournaments

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Abstract

Arrow's impossibility theorem [1] shows that the set of acyclic tournaments is not closed to non dictatorial Boolean aggregation. In this paper we extend the notion of aggregation to general tournaments and we show that for tournaments with four vertices or more any proper symmetric (closed to vertex permutations) subset cannot be closed to non dictatorial monotone aggregation and to non neutral aggregation. We also demonstrate a proper subset of tournaments that is closed to parity aggregation for an arbitrarily large number of vertices. This proves a conjecture of Kalai [4] for the non neutral and the non dictatorial and monotone cases and gives a counter example for the general case.

Keywords: social choice; aggregation; tournament; Arrow.

1 Introduction

An acyclic tournament is a binary relation that defines a linear order; such relations are thought to represent individual preferences between alternatives. It is a basic problem in economics to find functions that map individual preference profiles to collective preferences satisfying two natural conditions: Independence of Irrelevant Alternatives (IIA) and Pareto efficiency (P). IIA requires that the collective preference on two alternatives depends only on the individual preference on these alternatives. Representing the individual preferences on two alternatives as Boolean variables, the collective preference is determined by a Boolean function on these variables. The P condition requires that identical individual preference on two alternatives imply the same for the

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collective preference, i.e. the Boolean functions that determine the aggregation are unanimous $f(x, \ldots, x) = x$.

This notion of aggregation can be generalized to general tournaments.

Definition 1 *Let Δ_X denote the set of all tournaments on X. An m-**place generalized aggregation** or simply **aggregation** is a function $f = \Delta_X^m \to \Delta_X$ satisfying for any $R_1, \ldots, R_m \in \Delta_X$ and $R = f(R_1, \ldots, R_m)$:*

IIA: *aRb depends only on $aR_1b, \ldots, aR_mb$ for all $a, b \in X$.*
P: *If aR_ib for $i = 1, \ldots, m$ then aRb.*

Arrow's impossibility theorem shows that for all non dictatorial aggregation functions there exists a profile of acyclic tournaments that cannot be aggregated into an acyclic tournament.

Definition 2 *A set of tournaments $\mathfrak{C} \subset \Delta_X$ is **closed** to an aggregation f iff $f(R_1, \ldots, R_m) \in \mathfrak{C}$ for $R_1, \ldots, R_m \in \mathfrak{C}$.*

Thus Arrow's theorem shows that the set of acyclic tournaments is not closed to any aggregation satisfying the IIA and P conditions. From a combinatorial point of view it is interesting to ask whether a similar result would hold on other sets of tournaments.

Much research has been done on sets of acyclic tournaments (also called restricted domains) that preserve acyclicity under aggregation such as Black's [2] work on single peaked tournaments.[1] It can be shown that the set of single peaked tournaments is closed to aggregation. The problem with the set of single peaked tournaments is the lack of symmetry between alternatives. For instance, alternatives in the middle[2] can never be the least preferred for any tournament in the set so they have an inherent advantage upon those at the beginning or the end.

The *image* of tournament R under a permutation π of X is a tournament R^π such that $aR^\pi b$ iff $\pi(a)R\pi(b)$ for $a, b \in X$. A set of tournaments is *symmetric* if it is closed to permutations i.e. $R \in \mathfrak{C}$ implies $R^\pi \in \mathfrak{C}$ for any permutation π. The group of permutations acts transitively on the set of acyclic tournaments, thus the set of acyclic tournaments is symmetric but has no proper symmetric subsets, hence no restricted domain is symmetric. The cyclic tournaments on three alternatives namely $\{R_1, R_2\}$ such that aR_1b, bR_1c, cR_1a and aR_2c, cR_2b,

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[1] A full characterization of acyclicity preserving sets was given by Kalai and Muller [3].
[2] The single peakedness is relative to some fixed linear order, middle in this case refers to alternatives that are neither the most or least preferred relative to the fixed order.

bR_2a are an example of a symmetric set of tournaments. Another example is the set of i-nearly acyclic tournaments $1 \leq i \leq \binom{n}{2}$, these are tournaments which disagree with acyclic tournament on no more then i pairs $a, b \in X$.

Are there any symmetric sets of tournaments that are closed under some aggregation? These questions appeared in the wider framework of choice functions. A *choice function* on a set of alternatives X is a function specifying for each subset of X the most preferred alternative, the only requirement being that it belongs to the subset. This is another way to model preference, through a function that specifies the most preferred alternative from any set. Linear orders are embedded in this class through maximizing functions, but this class is wider [3]. As with tournaments, the notions of symmetry and aggregation can be generalized to choice functions and it is natural to ask whether there exist symmetric sets of choice function that are closed to aggregation. It was conjectured by Kalai [4] that the only symmetric set of choice functions on at least four alternatives closed to non dictatorial aggregation is the set of all choice functions. A variant of this definition is to look at constant size subsets of X (a tournament is a choice function on 2-sets). k-set choice functions were studied by Shelah [6] who proved the conjecture for $7 \leq k \leq |X| - 7$. In this paper we explore Kalai's conjecture for tournaments, we show that the general conjecture is false but it becomes true with some additional conditions.

Definition 3 *An aggregation f is* ***neutral*** *if $R = f(R_1, \ldots, R_m)$ implies $R^\pi = f(R_1^\pi, \ldots, R_m^\pi)$ for any $R_1, \ldots, R_m$ and permutation π.*

We prove the following theorem:

Theorem 1 *If $\mathfrak{C}$ is a symmetric set of tournaments on at least four alternatives closed to an aggregation function that is either non neutral or non dictatorial and monotone then $\mathfrak{C}$ is the set of all tournaments.*

This results introduces new directions in the study of the Arrow impossibility phenomena in classical social choice theory and shows that the phenomena is further reaching than indicated by previous results. A natural resolution to Arrow's paradox would be to relax the transitivity requirement a little. Had the domain of i-nearly acyclic tournaments been closed to aggregation, it would have been possible to create an almost ideal voting scheme by tolerating a bounded amount of irrationality. The main theorem shows, however, that neither this domain nor any other domain obtained by relaxing the transitivity requirement, is closed to monotone aggregation. Thus, there is no bound on the irrationality generated through aggregation.

In section 5 we show that the impossibility results do not necessarily apply to

[3] for instance a choice function choosing the second best relative to some linear order cannot be a maximizing function.

non monotone aggregation functions. We prove this by constructing a domain of tournaments that is both symmetric and closed to a non monotone aggregation function. This shows that Kalai's conjecture is false in the general case and that monotinicity is an essential part of the impossibility phenomena. It is left to future research to characterize symmetric domains that are closed to non monotone aggregation.

2 Preliminaries

We identify X with a set of $|X|$ vertices and we identify tournament R with an orientation (X, E) of the full graph $K_{|X|}$ such that $< a, b > \in E$ iff aRb.

An edge $\{o, a\}$ is out of o if oRa and it is into o if aRo. The *out degree* of $o \in X$ in tournament R is $V^R_{out}(o) := |\{a \in X - \{o\} : \ oRa\}|$. If $V^R_{out}(o) = 0$ then o is called a *sink* and if $V^R_{out}(o) = n - 1$ then o is called a *source*.

Denote R^o the tournament induced on $X - \{o\}$ by R. For a set of tournaments $\mathfrak{C}$ let $\mathfrak{C}^o_k$ be the set of tournaments in $\mathfrak{C}$ with out degree k at o, thus $\Delta^o_{n,k}$ is the set of all tournaments on n vertices with out degree k at o.

The orientations of choice functions $R_1, \dots, R_m \in \Delta_n$ on the pair $< a, b >$ are identified with a Boolean tuple $\alpha_1, \dots, \alpha_m$ such that

$$\alpha_j = \begin{cases} 0 \ bR_ja \\ 1 \ aR_jb \end{cases}$$

Condition I implies that f is characterized by a family of Boolean functions $\{f_{<a,b>}\}_{a,b\in X}$ such that $f_{<a,b>}(\alpha_1, \dots, \alpha_m) = 0$ iff bRa. It follows that

$$f_{<b,a>}(x_1, \dots, x_m) = 1 - f_{<a,b>}(1 - x_1, \dots, 1 - x_m)$$

A *Boolean aggregation function* is a Boolean function satisfying $f(x, \dots, x) = x$. Condition P shows that $f_{<a,b>}$ is a Boolean aggregation function for all $a, b \in X$. By definition an aggregation f is neutral iff $f_{<a,b>} = f_{<c,d>}$ for all $a, b, c, d \in X$ (including cases where $< c, d >=< b, a >$).

Aggregation closure is passed from $\mathfrak{C}$ to $\mathfrak{C}^o_{n-1}$ (tournaments with o as a source) and $\mathfrak{C}^o_0$ (tournaments with o as a sink) for all $o \in X$ due to the Pareto principle.

Throughout this paper let R, P denote tournaments, o, u, a, b, c, d vertices, x, y Boolean variables and α, δ Boolean values.

x_1	x_2	D^1	D^2	AND	OR
0	0	0	0	0	0
0	1	0	1	0	1
1	0	1	0	0	1
1	1	1	1	1	1

Table 1
The two place aggregation functions

3 The non neutral case

Proposition 1 *If $\mathfrak{C}$ is closed to an m-place non neutral aggregation then it is closed to a 2-place non neutral aggregation.*

proof: Let f be a non neutral aggregation for which $\mathfrak{C}$ is closed. Non neutrality is a consequence of $f_{<a,b>} \neq f_{<c,d>}$ for some $a, b, c, d \in X$ thus $f_{<a,b>}(x_1, \ldots, x_m) \neq f_{<c,d>}(x_1, \ldots, x_m)$. It follows that $f_{<a,b>}(\alpha_1, \ldots, \alpha_m) \neq f_{<c,d>}(\alpha_1, \ldots, \alpha_m)$ for some tuple $\alpha_1, \ldots, \alpha_m \in \{0, 1\}$. We define a 2-place aggregation as follows:

$$g_{<a,b>}(x, y) = f_{<a,b>}(x_1, \ldots, x_m) \text{ for all } a, b \in X \quad \text{where } x_j = \begin{cases} x & \alpha_j = 0 \\ y & \alpha_j = 1 \end{cases}$$

By definition $g_{<a,b>}(0, 1) = f_{<a,b>}(\alpha_1, \ldots, \alpha_m) \neq f_{<c,d>}(\alpha_1, \ldots, \alpha_m) = g_{<c,d>}(0, 1)$ hence $g_{<a,b>} \neq g_{<c,d>}$ and g is non neutral □

Corollary 1 *$\mathfrak{C}$ is closed to a non neutral aggregation if it is closed to aggregation by an f such that $f_{<a,b>}$ is one of the functions D^1, D^2, AND, OR(see table 1) for all $a, b \in X$.*

proof: It follows from the proposition that if $\mathfrak{C}$ is closed to a non neutral aggregation it is closed to an aggregation f such that $f_{<a,b>}$ is a two place Boolean aggregation function. There are four such functions: D^1, D^2, AND, OR □

Definition 4 *Let f be a 2-place aggregation. A triple $(a, o, b) \in X^3$ is* ***increasing*** *in f iff $f_{<o,a>}(1, 0) = f_{<o,b>}(0, 1) = 1$. f is increasing if there exists an increasing triple in f. A triple $(a, o, b) \in X^3$ is* ***decreasing*** *in f iff $f_{<o,a>}(1, 0) = f_{<o,b>}(0, 1) = 0$. f is decreasing if there exists a decreasing triple in f.*

By this definition f is increasing if there exists a triple $(a, o, b) \in X^3$ such that $f_{<o,a>}$ and $f_{<o,b>}$ are two different dictators, two ORs or a dictator and an OR, and decreasing if they are two different dictators two ANDs or a dictator and an AND.

Lemma 1 *For a non neutral aggregation f there exists $o, a, b \in X$ such that $f_{<o,a>} \neq f_{<o,b>}$.*

proof: Non neutrality implies $f_{<a,b>} \neq f_{<c,d>}$ for some $a, b, c, d \in X$. If $<c, d>=<b, a>$ then take any $o \neq a, b$, if $f_{<o,a>} = f_{<o,b>}$ then either $f_{<a,o>} \neq f_{<a,b>}$ or $f_{<b,o>} \neq f_{<b,a>}$ otherwise $f_{<a,b>} = f_{<a,o>} = 1 - f_{<o,a>} = 1 - f_{<o,b>} = f_{<b,o>} = f_{<b,a>}$. If $\{c, d\}$ and $\{a, b\}$ have exactly one common vertex then there is nothing to prove. If $\{c, d\} \cap \{a, b\} = \emptyset$ then either $f_{<a,b>} \neq f_{<a,d>}$ or $f_{<d,a>} \neq f_{<d,c>}$ otherwise $f_{<c,d>} = 1 - f_{<d,c>} = 1 - f_{<d,a>} = f_{<a,d>} = f_{<a,b>}$ □

Lemma 2 *A non dictatorial two place aggregation on at least four vertices is both increasing and decreasing.*

proof: Corollary 1 shows that $\{f_{<a,b>} : \ a, b \in X\} \subset \{D^1, D^2, OR, AND\}$. If f is non dictatorial then either $f_{<a,b>} = AND$ for some pair $a, b \in X$(and then $f_{<a,b>} \neq f_{<b,a>}$) or two edges with $f_{<a,b>} = D^1$ and $f_{<c,d>} = D^2$. In both cases f is non neutral. Lemma 1 shows that there exists $o, a, b \in X$ such that $f_{<o,a>} \neq f_{<o,b>}$. If $f_{<o,a>}$ and $f_{<o,b>}$ are two different dictators then we are done. If $f_{<o,a>} = OR$ and $f_{<o,b>} = f_{<a,b>} = D^1$ then (a, o, b) is increasing and (o, a, b) is decreasing because $f_{<a,o>} = AND$. This proves the lemma in the case f has a triangle with two dictator and one OR/AND edges. If $f_{<o,a>} = f_{<o,b>} = OR$ and $f_{<a,b>} = D^1$ then (a, o, b) is increasing and (o, a, b) is decreasing. If $f_{<o,a>} = f_{<o,b>} = AND$ then (a, o, b) is decreasing and (o, a, b) is increasing. If $f_{<o,a>} = AND$, $f_{<o,b>} = OR$ and $f_{<a,b>} = D^1$ then (o, a, b) is increasing and (o, b, a) is decreasing. This proves the lemma in the case f has a triangle with one dictator and two OR/AND edges.

Finally suppose $f_{<o,a>} = f_{<a,b>} = f_{<b,o>} = OR$, let $u \neq o, a, b$ (this is where we use the assumption that there are at least four vertices). If one of the edges from u to $\{o, a, b\}$ is a dictator then f has a triangle with dictators and AND/ORs and we are done. Otherwise, $f_{<u,o>}, f_{<u,a>}, f_{<u,b>} \in \{OR, AND\}$. W.l.g we assume $f_{<u,a>} = f_{<u,b>}$, if $f_{<u,a>} = f_{<u,b>} = OR$ then (a, u, b) is increasing and (u, a, o) decreasing since $f_{<a,u>} = f_{<a,o>} = AND$, otherwise $f_{<u,a>} = f_{<u,b>} = AND$ and (a, u, b) is decreasing and (u, a, o) is increasing since $f_{<a,u>} = f_{<a,b>} = OR$ □

Lemma 3 *Let f be a 2-place aggregation and let $\mathfrak{C}$ be a symmetric set of tournaments. If f is increasing then for all $0 < k < n - 1$ and $o \in X$:*

(1) $\mathfrak{C}^o_k \neq \emptyset$ *implies* $\mathfrak{C}^o_{k+1} \neq \emptyset$
(2) $\mathfrak{C}^o_k = \Delta^o_{n,k}$ *implies* $\mathfrak{C}^o_{k+1} = \Delta^o_{n,k+1}$

If f is decreasing then for all $0 < k < n - 1$ and $o \in X$:

(1) $\mathfrak{C}^o_k \neq \emptyset$ *implies* $\mathfrak{C}^o_{k-1} \neq \emptyset$
(2) $\mathfrak{C}^o_k = \Delta^o_{n,k}$ *implies* $\mathfrak{C}^o_{k-1} = \Delta^o_{n,k-1}$

proof: Symmetry implies that $\mathfrak{C}_k^o \neq \emptyset$ for all $o \in X$ iff $\mathfrak{C}_k^u \neq \emptyset$ for at least one $u \in X$ and the same follows for $\mathfrak{C}_k^o = \Delta_{n,k}^o$. Thus w.l.g we may assume (a, o, b) is increasing for f. The assumption $\mathfrak{C}_k^o \neq \emptyset$ for $0 < k < n-1$ implies the existence of a tournament R with at least one edge into o and one edge out of o. Since $\mathfrak{C}_k^o$ is closed to permutations of $X - \{o\}$ we may assume oRa and bRo. Let R' be the image of R under permutation of a and b thus $aR'o$ and $oR'b$. Let $\tilde{R} = f(R, R') \in \mathfrak{C}$, then $f_{<o,a>}(1, 0) = f_{<o,b>}(0, 1) = 1$ imply by definition that $o\tilde{R}a$ and $o\tilde{R}b$. $oR's \equiv oRs$ for any $s \in X - \{o, a, b\}$ and $f_{<o,s>}(x, x) = x$ hence $o\tilde{R}s \equiv oRs$. This shows that $V_{out}^{\tilde{R}}(o) = V_{out}^{R}(o) + 1 = k + 1$ therefore $\mathfrak{C}_{k+1}^o \neq \emptyset$.

Assume $\mathfrak{C}_k^o = \Delta_{n,k}^o$ and let $R \in \Delta_{n,k+1}^o$ such that oRa and oRb. Take R_1, R_2 to be a tournament such that oR_1a, bR_1o, aR_2o, oR_2b and $R_1 = R_2 = R$ for all other edges thus $R_1, R_2 \in \Delta_{n,k}^o = \mathfrak{C}_k^o \subset \mathfrak{C}$. Let $\tilde{R} = f(R_1, R_2) \in \mathfrak{C}$ then once again $f_{<o,a>}(1, 0) = f_{<o,b>}(0, 1) = 1$ hence $o\tilde{R}a$ and $o\tilde{R}b$. For any other edge $R_1 = R_2 = R$ thus from unanimity it follows that $\tilde{R} = R$. This shows that $\mathfrak{C}_{k+1}^o$ contains all the tournaments in which a and b are vertices of edges out of o, symmetry implies that $\mathfrak{C}_{k+1}^o = \Delta_{n,k+1}^o$. The same argument mutatis mutandis works when f is decreasing $\square$

Proposition 2 *If $\mathfrak{C}$ is a nonempty symmetric set of tournaments on at least four vertices closed to a non neutral 2-place aggregation function f then $\mathfrak{C} = \Delta_X$.*

proof: It follows from lemma 2 that f is increasing and decreasing. Any tournament has some vertex $o \in X$ which is neither a sink nor a source, hence $\mathfrak{C}_{k'}^o \neq \emptyset$ for some $0 < k' < n-1$. Lemma 3 implies that $\mathfrak{C}_k^o \neq \emptyset$ for all $0 \leq k \leq n-1$ and in particular this shows $\mathfrak{C}$ contains tournaments with o as a source and as a sink. Having made these observations we proceed by induction.

The induction base $n = 4$: In this case Δ_4 consists of four symmetry orbits with out degrees $(0, 1, 2, 3)$ (linear order), $(1, 1, 1, 3)$, $(0, 2, 2, 2)$ and $(1, 1, 2, 2)$. It follows from the definition of increasing in f that subtracting one from one entry and adding it to another takes us from one tournament in $\mathfrak{C}$ to another one in $\mathfrak{C}$. This shows that if $\mathfrak{C}$ contains a tournament from any one of the orbits it contains a tournament from all the other orbits. Since $\mathfrak{C}$ is non empty and symmetric it follows that $\mathfrak{C} = \Delta_4$.

The induction step: Since $\mathfrak{C}_k^u \neq \emptyset$ for at least one $u \in X$ implies $\mathfrak{C}_k^o \neq \emptyset$ for all $o \in X$ we may choose a vertex $u \in X$ such that $\mathfrak{C}_k^u \neq \emptyset$ for $k \in \{0, n-1\}$ and $f|_{X-\{u\}}$ is non neutral. The tournaments induced by $\mathfrak{C}_0^u$ and $\mathfrak{C}_{n-1}^u$ on $X - \{u\}$ are symmetric sets of tournaments on $n-1$ vertices closed to the aggregation induced by f. The inductive assumption maintains that both sets are Δ_{n-1} hence $\Delta_{n,0}^u = \mathfrak{C}_0^u \subset \mathfrak{C}$ and $\Delta_{n,n-1}^u = \mathfrak{C}_{n-1}^u \subset \mathfrak{C}$. Symmetry implies that the same is true for all $o \in X$.

Lemma 1 shows that there exist $o, a, b \in X$ such that $f_{<o,a>}(x, y) \neq f_{<o,b>}(x, y)$, thus $f_{<o,a>}(\alpha_1, \alpha_2) \neq f_{<o,b>}(\alpha_1, \alpha_2)$ for some pair $\alpha_1, \alpha_2 \in \{0, 1\}$. For an arbitrary $P \in \Delta_{n-1}$ take $R_1, R_2 \in \mathfrak{C}$ such that $R_i^o = P$ and o is a source of R_i if $\alpha_i = 1$ and a sink of R_i if $\alpha_i = 0$. Let $\tilde{R} = f(R_1, R_2) \in \mathfrak{C}$ and $k = V_{out}^{\tilde{R}}(o)$. Since $f_{<o,a>}(\alpha_1, \alpha_2) \neq f_{<o,b>}(\alpha_1, \alpha_2)$ the edges $\{o, a\}$ and $\{o, b\}$ are directed differently thus $0 < k < n - 1$. This shows that $\mathfrak{C}_k^o$ contains all the tournaments with out degree k for which $\{o, a\}$ and $\{o, b\}$ are directed differently. Every tournament of out degree $0 < k < n - 1$ is a permutation of such a tournaments, since $\mathfrak{C}_k^o$ is closed to permutations of $X - \{o\}$ it follows that $\mathfrak{C}_k^o = \Delta_{n,k}^o$. f is both increasing and decreasing thus lemma 3 completes the proof $\square$

Note that the set of cyclic tournaments on three vertices is closed to the aggregation defined by $f_{<o,a>} = f_{<a,b>} = f_{<b,o>} = OR$ hence the proposition is not true for less than four vertices.

Corollary 2 *If $\mathfrak{C}$ is a nonempty symmetric set of tournaments on at least four vertices closed to a non neutral m-place aggregation f then $\mathfrak{C} = \Delta_X$.*

4 The neutral monotone case

We identify between Boolean functions and neutral aggregations they define. Let $\mathfrak{C}$ be a non empty symmetric set of tournaments on more than four vertices closed to a neutral aggregation f. For aggregations on two voters the theorem is trivial since the only possible aggregations are the two dictator functions.

Proposition 3 *If $\mathfrak{C}$ is closed to an m-place non dictatorial neutral aggregation then it is closed to some 3-place non dictatorial neutral aggregation.*

proof: Let r be the minimal integer for which there is an r-placed non dictatorial aggregation and let f be such an aggregation. Assume $r \geq 4$ and let $[r] = \{1, \ldots, r\}$.

Claim 1 *There exists $j \in [r]$ such that $f(x_1, \ldots, x_r) = x_j$ if $x_1, \ldots, x_r$ has some repetition of variables.*

proof: For two indices $l \neq k \in [r]$ if $x_k = x_l$ (identical variables) then $f(x_1, \ldots, x_r) = x_j$ is an $r - 1$ aggregation function, hence the minimality assumption on r implies that $f(x_1, \ldots, x_r)$ is dictatorial, thus $f(x_1, \ldots, x_r) = x_{h(l,k)}$ for some $h(l, k) \in [r]$. We may assume $h(l, k) \neq k$ since x_k and x_l are identical. If f is non dictatorial then $h(l, k)$ is not constant on $Q = \{(l, k) : 1 \leq$

x_1	x_2	x_3	Maj_3	P_3	D^1	D^2	D^3	$AntiD^1$	$AntiD^2$	$AntiD^3$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	1	1	1	0
0	1	0	0	1	0	1	0	1	0	1
0	1	1	1	0	0	1	1	1	0	0
1	0	0	0	1	1	0	0	0	1	1
1	0	1	1	0	1	0	1	0	1	0
1	1	0	1	0	1	1	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1

Table 2
The three place neutral aggregation functions

$l < k \leq r\}$. Suppose for some $l, k \in [r]$ we have $x_l = x_k$ and $f(x_1, \ldots, x_r) \neq x_l$. W.l.g we may assume $l = r - 1$, $k = r$ and $f(x_1, \ldots, x_r) = x_1$ thus $f(1-x, x, x, \ldots, x) = 1-x$ hence $h(l, k) = 1$ for any $l \neq k \in \{2, \ldots, r\}$. Since h is non constant on Q it follows that there is some k such that $h(1, k) \neq 1$. W.l.g we may assume $h(1, 2) = 3$. This implies that $f(1 - x, 1 - x, x, x, \ldots, x) = x$, but on the other hand $h(3, 4) = 1$ thus $f(1 - x, 1 - x, x, x, \ldots, x) = 1 - x$. The contradiction shows that $h(l, k) = l$ for all $k, l \in [r]$. Consequently $h(1, 2) = 1$ therefore $f(1 - x, 1 - x, x, x, \ldots, x) = 1 - x$ but also $h(3, 4) = 3$ thus $f(1 - x, 1 - x, x, x, \ldots, x) = x$. The contradiction proves the claim.

For any $a, b \in X$ and $R_1, \ldots, R_r \in \mathfrak{C}$ let $\alpha_1, \ldots, \alpha_r \in \{0, 1\}$ be a tuple corresponding to $aR_1b, \ldots, aR_rb$. Since $r > 2$ there must be some repetition thus w.l.g $\alpha_1 = \alpha_2$. It follows from the claim that $f(x_1, x_1, x_3, \ldots, x_r) = x_j$ for variables $x_1, \ldots x_r$ hence $f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = f(\alpha_1, \alpha_1, \alpha_3, \ldots, \alpha_r) = \alpha_j$. Since this holds for any tuple it follows that $f = D^j$, thus f is dictatorial contrary to the assumption □

It follows that $\mathfrak{C}$ is closed to non dictatorial neutral aggregation only if it is closed to three voter non dictatorial neutral aggregation. There are five Boolean aggregation functions that define a non dictatorial neutral aggregation: Maj_3 - majority, $Prty_3$ - parity and $AntiD^1 \div AntiD^3$ - anti dictator (see table 2). Monotonicity is preserved under repetition of variables $x_1, \ldots, x_m$ thus the proposition is true for non dictatorial neutral and monotone functions. This implies that $\mathfrak{C}$ is closed to such an aggregation only if it is closed to Maj_3.

Lemma 4 *Let $\mathfrak{C}$ be a symmetric set of tournaments closed to Maj_3 and $o \in X$.*

(1) If $1 < k < n - 1$ then $\mathfrak{C}^o_k \neq \emptyset$ implies $\mathfrak{C}^o_{k+1} \neq \emptyset$

(2) If $0 < k < n-2$ then $\mathfrak{C}_k^o \neq \emptyset$ implies $\mathfrak{C}_{k-1}^o \neq \emptyset$

proof: Since $1 < k < n-1$ any $R \in \mathfrak{C}_k^o$ has at least two edges out of o and one edge into o thus there exist $o, a, b, c \in X$ such that aRo, oRb and oRc. Let R_1 be the image of R under permutation of a and b and let R_2 be the image under permutation of a and c thus oR_1a, bR_1o, oR_1c, oR_2a, oR_2b, cR_2o and $oRd \equiv oR_1d \equiv oR_2d$ for any $d \in X - \{o, a, b, c\}$. Take $\tilde{R} = Maj_3(R, R_1, R_2) \in \mathfrak{C}$. Since $Maj_3(0,1,1) = Maj_3(1,0,1) = Maj_3(1,1,0) = 1$ it follows by definition that $o\tilde{R}a$, $o\tilde{R}b$ and $o\tilde{R}c$ furthermore $Maj_3(x,x,x) = x$ for Boolean variable x therefore $o\tilde{R}d \equiv oRd$ for $d \in X - \{o, a, b, c\}$. This shows that $\tilde{R} \in \mathfrak{C}_{k+1}^o$ thus $\mathfrak{C}_{k+1}^o \neq \emptyset$. A similar argument proves the second part of the lemma □

Proposition 4 *If $\mathfrak{C}$ is a nonempty symmetric set of tournaments closed to Maj_3 then $\mathfrak{C} = \Delta_X$.*

proof: Any tournament on more than four vertices has out degree $1 < k' < n-2$ on at least one vertex. Since $\mathfrak{C} \neq \emptyset$ it follows from lemma 4 that for some $0 \in X$ $\mathfrak{C}_k^o \neq \emptyset$ for all $0 \leq k \leq n-1$, in particular this shows that $\mathfrak{C}$ contains tournaments with o as a source and as a sink. We proceed by induction on the number of vertices.

The induction base $n = 4$: As we have already mentioned Δ_4 has four symmetry orbits. Each row in figure 1 shows three tournaments in one orbit that are aggregated into a tournament in another orbit yielding the following set of implications:

$$\begin{aligned}(0,2,2,2) \subset \mathfrak{C} &\Rightarrow (0,1,2,3) \subset \mathfrak{C} \Rightarrow (1,1,1,3) \subset \mathfrak{C} \\ &\Rightarrow (1,1,2,2) \subset \mathfrak{C} \Rightarrow (0,2,2,2) \subset \mathfrak{C}\end{aligned}$$

hence if $\mathfrak{C}$ is nonempty then $\mathfrak{C} = \Delta_4$. The induction step: Let $o \in X$ be as above a vertex such that $\mathfrak{C}_k^o \neq \emptyset$ for all $0 \leq k \leq n-1$. The tournaments induced by $\mathfrak{C}_0^o$ and $\mathfrak{C}_{n-1}^o$ on $X - \{o\}$ are nonempty symmetric sets of tournaments on $n-1$ vertices closed to the aggregation induced by Maj_3. The inductive assumption maintains that both sets are Δ_{n-1} hence $\Delta_{n,0}^o = \mathfrak{C}_0^o \subset \mathfrak{C}$ and $\Delta_{n,n-1}^o = \mathfrak{C}_{n-1}^o \subset \mathfrak{C}$. For any $0 < k < n-1$ and $P \in \Delta_k^o$ take $R \in \mathfrak{C}_k^o$ such that $oPa \equiv oRa$ for all $a \in X - \{o\}$ and take $R_1 \in \mathfrak{C}_0^o$ and $R_2 \in \mathfrak{C}_{n-1}^o$ such that $R_1^o = R_2^o = P^o$. Let $\tilde{R} = Maj_3(R, R_1, R_2) \in \mathfrak{C}$, then $Maj_3(x,0,1) = x$ implies that $o\tilde{R}a \equiv oRa \equiv oPa$ for all $a \in X - \{o\}$ and $Maj_3(y,x,x) = x$ implies that $a\tilde{R}b \equiv aR_1b \equiv aPb$ for all $a, b \in X - \{o\}$ hence $P = \tilde{R} \in \mathfrak{C}$. Consequently $\mathfrak{C} = \Delta_n$ □

The set of cyclic tournaments on three vertices is closed to permutations and to Maj_3 aggregation hence the proposition is not true for less than four vertices.

Corollary 3 *If $\mathfrak{C}$ is a nonempty symmetric set of tournaments on at least*

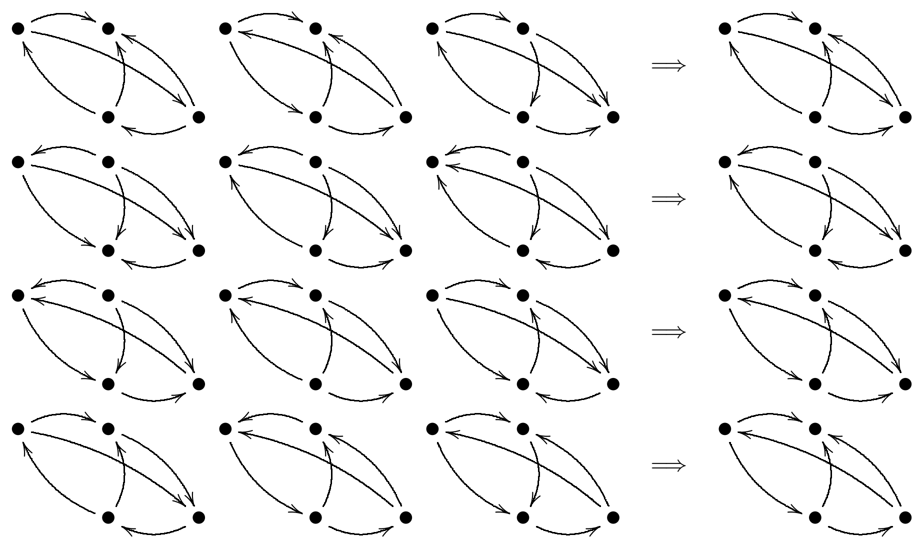

Fig. 1. Proof of the case $n = 4$

four vertices closed to a neutral non dictatorial monotone m-place aggregation f then $\mathfrak{C} = \Delta_X$.

This completes the proof of the theorem.

5 The neutral non monotone case

In this section we explicitly construct a proper symmetric set of tournaments that is closed to parity aggregation. This shows that without monotonicity an Arrow type theorem does not hold.

Lemma 5 *For n odd let $\mathfrak{C}$ be the set of all tournaments $R \in \Delta_n$ such that $V^R_{out}(o)$ is odd for every $o \in X$. Then $\mathfrak{C}$ is a symmetric set of tournaments that is closed to parity.*

proof: Permutations shift the out degrees between the vertices thus leaving the tournament within the set hence $\mathfrak{C}$ is symmetric. Let m be odd, then for $o \in X$, $R_1, \ldots, R_m \in \mathfrak{C}$ and $R = Prty(R_1, \ldots, R_m)$. It suffices to show that $V^R_{out}(o)$ is odd.

Assume $X = \{o, a_1, \ldots, a_{n-1}\}$ and let $\alpha^j_1, \ldots \alpha^j_{n-1}$ be the Boolean values corresponding to $oR_ja_1, \ldots, oR_ja_{n-1}$ for $j = 1, \ldots, m$. Take $\delta_i = Parity(\alpha^1_i, \ldots, \alpha^m_i) = \sum_{j=1}^{m} \alpha^j_i$ Mod 2 for $i = 1, \ldots n-1$ then by definition $\delta_1, \ldots, \delta_{n-1}$ are the values corresponding to $oRa_1, \ldots, oRa_{n-1}$. It follows from the assumption on $\mathfrak{C}$ that

$\sum_{i=1}^{n-1} \alpha_i^j = 1$ Mod 2. But then

$$\sum_{i=1}^{n-1} \delta_i = \sum_{i=1}^{n-1} \sum_{j=1}^{m} \alpha_i^j = \sum_{j=1}^{m} \sum_{i=1}^{n-1} \alpha_i^j = m = 1 \bmod 2$$

thus $V_{out}^R(o)$ is odd $\square$

Proposition 5 *For n arbitrarily large there exists a proper symmetric subset of Δ_n that is closed to non monotone aggregation.*

proof: For every n there exists a tournament $R \in \Delta_{4n+3}$ such that $V_{out}^R(o) = 2n + 1$ for all $o \in X$, hence the set of all tournaments $R \in \Delta_{4n+3}$ with $V_{out}^R(o)$ odd is nonempty. Lemma 5 shows that there exists a proper subset of tournaments that is closed to parity on m voters if m is odd, if m is even we take parity on $m - 1$ voters $\square$

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