Optimal Growth Through Product Innovation

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Abstract

In Lentz and Mortensen (2005), we formulate and estimate a market equilibrium model of endogenous growth through product innovation in the spirit of Klette and Kortum (2004). In this paper, we provide a quantitative solution to the social planner's problem in the modeled environment. We find that the optimal growth rate is over three times larger than its value in market equilibrium and that the associated welfare gains along a transition path from the market equilibrium solution to the optimal steady state is equivalent to about 47% of consumption.

JEL Classification: E22, E24, J23, J24, L11, L25

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1 Introduction

In Lentz and Mortensen (2006), we formulate and estimate a structural market equilibrium model of growth through product innovation. The model is an extended version of that proposed by Klette and Kortum (2004) originally designed to explain the relationships between innovation investment and the size distribution of firms. Their framework in turn is an elaboration of the Grossman and Helpman (1991) model of endogenous growth through creative-destruction. In our version of the model, firms differ with respect to the quality of the intermediate products they create as a consequence of investment in research and development (R&D). We find that heterogeneity in this sense is needed to explain the size distribution of firms and the distribution of labor productivity observed in our panel data of Danish firms. One important implication of this form of heterogeneity is that labor reallocation to faster growing firms that create more profitable higher quality intermediate products plays an important role in determining the aggregate growth rate.

The purpose of this paper is to explore the model's quantitative welfare implications. Namely, we formulate and compute the socially optimal R&D strategy for the modeled environment. The market equilibrium solution need not be socially optimal for three different reasons spelled out in Grossman and Helpman (1991). First, product innovators have monopoly power and use it to set prices above the marginal cost of production. Second, every innovation replace an older version of some product and by doing so truncates the stream of quasi rents accruing to its creator, which adversely affects the incentive to innovate. Finally, because each new improvement builds on past technology, innovation has a positive "spill-over" effect on future productivity which is not fully captured by the innovator in a market equilibrium. The net deviation of the equilibrium growth rate from that which is socially optimal is unclear. One of the contributions of a quantitative equilibrium model is its ability to reflect light on the relative magnitudes of these effects.

In this paper, we begin by formulating and characterizing the market solution to the estimated model and the solution to the obvious planner's problem in the environment modelled. We find that the planner's optimal innovation investment strategy differs significantly from that observed in equilibrium. Although firms that produce better products invest more in R&D, all types have an incentive to make a positive investment in market equilibrium. In contrast, almost all of the R&D investment in the optimal solution is make by an elite few firms, those that can create products of the highest quality. The result is the consequence of the fact that the "spill over" externality offset the "product destruction" externality only if the quality improvement embodied in an innovation is sufficiently large.

The optimal growth rate is over three times larger than the market equilibrium value. The welfare gains attributable to the optimal entry and R&D investment strategies are also large. Indeed, the typical household would be willing to forego up to almost 47% of consumption along the optimal path in order to adopt the optimal plan. Although the first best solution requires marginal cost pricing, we also show that there are substantial welfare gains to be had even if innovators are granted patent rights and allowed to set monopoly limit prices. Indeed, only about 5 percentage points of the total welfare gain attributable to the optimal R&D investment strategy can be attributed to marginal cost pricing.

2 Growth Through Product Innovation

Firms come in an amazing range of shapes and sizes. This fact cannot be ignored in any analysis of the relationship between firm size and factor productivity. Furthermore, an adequate theory must account for entry, exit, and firm evolution in order to explain observed size distributions. Klette and Kortum (2004) construct a stochastic model of firm product innovation and growth that is consistent with stylized facts regarding the firm size evolution and distribution. In Lentz and Mortensen (2005), we find that an extension of their model that allows for cross firm heterogeneity in the quality of innovations is needed to explain our Danish data.

2.1 Preferences and Technology

The utility of the representative household at time t is given by

$$U_t = \int_t^\infty \ln C_s e^{-\rho(s-t)} ds \tag{1}$$

where $\ln C_t$ denotes the instantaneous utility of aggregate consumption at date t and ρ represents the pure rate of time discount. Each household is free to borrow or lend at interest rate r_t. Nominal household expenditure at date t is $E_t = P_t C_t$. Optimal consumption expenditure must solve the differential equation $\dot{E}/E = r_t - \rho$. Following Grossman and Helpman (1991), we choose the numeraire so that $E_t = 1$ for all t without loss of generality, which implies $r_t = r = \rho$ for all t. Note that this choice of the numeraire also implies that price of the consumption good, P_t, falls over time at a rate equal to the rate of growth in consumption.

The quantity of consumption produced is determined by the quantity and quality of the economy's intermediate inputs. Specifically, there is a unit continuum of inputs and consumption is determined by the production function

$$\ln C_t = \int_0^1 \ln(A_t(j)x_t(j))dj = \ln A_t + \int_0^1 \ln x_t(j)dj \tag{2}$$

where $x_t(j)$ is the quantity of input $j \in [0,1]$ at time t, $A_t(j)$ is the productivity of input j at time t, and A_t represent aggregate productivity. The level of productivity of each input and aggregate productivity are determined by the number of technical improvements made in the past. Specifically,

$$A_t(j) = \Pi_{i=1}^{J_t(j)} q_i(j) \text{ and } \ln A_t \equiv \int_0^1 \ln A_t(j)dj. \tag{3}$$

where $J_t(j)$ is the number of innovations made in input j up to date t and $q_i(j) > 1$ denotes the quantitative improvement (step size) in productivity attributable to the i^{th} innovation in product j. Innovations arrive at rate δ which is endogenous but the same for all intermediate products.

The model is constructed so that a steady state growth path exists with the property that consumption output grows at a constant rate, equal to the rate of productivity growth, while the intermediate good quantities produced and the innovation frequencies are stationary. As a consequence of the law of large numbers, the assumption that the number of innovations to date is Poisson with arrival frequency δ for all intermediate goods implies

$$\begin{aligned} \ln C_t &= \ln A_t + \int_0^1 \ln x(j)dj = \int_0^1 \sum_{i=1}^{J_t(j)} \ln q_i(j)dj + \int_0^1 \ln x(j)dj \qquad (4) \\ &= \delta E \ln(q)t + \int_0^1 \ln x(j)dj. \end{aligned}$$

where $EJ_t(j) = \delta t$ for all j is the expected number of innovations per intermediate product over a time period of length t and $E\ln(q) \equiv \int_0^1 \frac{1}{J_t(j)} \sum_{i=1}^{J_t(j)} \ln q_i(j)dj$

is the expected improvement in productivity per innovation. In other words, consumption grows at the rate of growth in productivity which is the product of the creative-destruction rate and the expected log of the size of an improvement in productivity induced by an innovation.

2.2 The Behavior of a Firm

As a consequence of enforceable patents, each individual firm is the sole supplier of the products it created in the past that have survived to the present. The price charged for each is limited by the ability of suppliers of previous versions to provide a substitute. In Nash-Bertrand equilibrium, any successful innovator takes over the market for its good type by setting the price just below that at which final good producers are indifferent between the new more productive product supplied by the innovator and the alternative supplied by the previous provider. The price charged is the product of the relative quality of the innovation and the previous producer's marginal cost of production. Given the symmetry of demands for the different good types and the assumption that future quality improvements are independent of the type of good, one can drop the good subscript without confusion. Because quantities along the equilibrium growth path are constant, the time subscript can be dropped as well.

Labor and capital, in fixed proportions, are used in the production of intermediate inputs to the final goods production process. Labor productivity is the same across all intermediate products and is set equal to unity. The required capital expressed in units of output, a constant κ, is also the same for all products. The operating profit per unit obtained from supplying an intermediate product is $p(1-\kappa)-w$ which implies that the lowest price that the previous supplier is willing to charge, that which yields no profit, is $w/(1-\kappa)$. The quality leader will charge $p = qw/(1-\kappa)$ because consumers are exactly indifferent between buying from the quality leader at this price and the zero profit price of the previous supplier. Hence, product output supply and employment demand are both equal to

$$x = \frac{1}{p} = \frac{1-\kappa}{wq}. \tag{5}$$

and the gross profit associated with supplying the good is

$$\pi(q) = p(1-\kappa)x - wx = (1-\kappa)\left(1-q^{-1}\right). \tag{6}$$

Following Klette and Kortum (2004), the discrete number of products supplied by a firm, denoted as k, is defined on the integers and its value evolves over time as a birth-death process reflecting product creation and destruction. In their interpretation, k reflects the firm's past successes in the product innovation process as well as current firm size. New products are generated by R&D investment. The firm's R&D investment flow generates new product arrivals at frequency γk. The total R&D investment cost is $wc(\gamma)k$ where $c(\gamma)k$ represents the labor input required in the research and development process. The function $c(\gamma)$ is assumed to be strictly increasing and convex. According to the authors, the implied assumption that the total cost of R&D investment is linearly homogenous in the new product arrival rate and the number of existing product, "captures the idea that a firm's knowledge capital facilitates innovation." In any case, this cost structure is needed to obtain firm growth rates that are independent of size as typically observed in the data.

The market for any current product supplied by the firm is destroyed by the creation of a new version by some other firm, which occurs at the rate δ. Below we refer to γ as the firm's product innovation rate and to δ as the aggregate creative-destruction rate faced by all firms. The firm chooses the creation rate γ to maximize the expected present value of its future net profit flow.

Firms differ with the respect to the expected quality improvement that their products offer. Specifically, there are $i = 1, 2, ...n$ types of firms and the distribution (c.d.f.) of quality, denoted as $F_i(q)$, is stochastically decreasing in the type index; that is, $F_i(q) \leq F_{i+1}(q)$ for all $q \geq 1$. Hence, the index reflects the rank order of the types by the expected quality of their innovations.

The value of the firm of type i that currently markets k products is the solution to the asset pricing equation

$$rV_k(i) = max_{\gamma \geq 0} \left\{ \begin{array}{c} \left[E_i \left\{\pi(q)\right\} - wc(\gamma)\right] k + \gamma k[V_{k+1}(i) - V_k(i)] \\ +\delta k[V_{k-1}(i) - V_k(i)] \end{array} \right\} \qquad (7)$$

where $E_i\{\pi(q)\} = \int \pi(q)dF_i(q)$ is the expected gross profit flow obtained per product line a firm of type i. Hence, the first term on the right side of (7) is the total net profit flow from supplying it current products. The second term is the expected capital gain associated with the arrival of a new product line. Finally, the last term represents the expected capital loss associated with the possibility that one among the existing product lines will be destroyed.

The unique solution to (7) is proportional to the number of product lines. Formally,

$$V_k(i) = k \max_{\gamma \geq 0} \left\{ \frac{E_i\{\pi(q)\} - wc(\gamma)}{r + \delta - \gamma} \right\} \tag{8}$$

as one can verify by substitution. Consequently, any positive optimal choice of the product creation rate for a type q firm must satisfy

$$wc'(\gamma_i) = V_{k+1}(i) - V_k(i) = \max_{\gamma \geq 0} \left\{ \frac{E_i\{\pi(q)\} - wc(\gamma)}{r + \delta - \gamma} \right\}. \tag{9}$$

The second order condition, $c''(\gamma) > 0$,the fact that profit π is increasing in q, and the assumption that the quality c.d.f. $F_i(q)$ is stochastically decreasing in i imply that the a firm's creation rate increases with its product quality rank. That is $\gamma_i \geq \gamma_{i+1}$, $i = 1, ..., n$.

2.3 Firm Entry and Labor Market Clearing

The entry of a new firm requires a successful innovation. Suppose that there are a constant measure m of potential entrants. The rate at which any one of them generates a new product is γ_0 and the total cost is $wc(\gamma_0)$ where the cost function is the same as that faced by an incumbent. The firm's type is unknown ex ante but is realized immediately after the arrival of an innovation. However, entry requires that the innovation provide a positive expected future income stream ex post. Since the aggregate entry rate is $\eta = m\gamma_0$, the entry rate satisfies the following free entry condition

$$\begin{aligned} wc'\left(\frac{\eta}{m}\right) &= \sum_{i=1}^{n} \left(\int_q \max \langle V_1(i), 0 \rangle \, dF_i(q) \right) \phi_i \\ &= \sum_{i=1}^{n} \max_{\gamma \geq 0} \left\{ \frac{E_i\{\pi(q)\} - wc(\gamma)}{r + \delta - \gamma} \right\} \phi_i \end{aligned} \tag{10}$$

where ϕ_i is the probability that the entrant will turn out to be of type i and the second equality is implied by the fact that the value of adoption, $V_1(q)$ as defined by equation (8), is always non-negative and linear in gross profit per product. Obviously, in this formulation learning one's type takes no time, which is unrealistic but a useful abstraction for the purposes of this paper.

There is a fixed measure of available workers, denoted by ℓ, seeking employment at any positive wage. In equilibrium, these are allocated across production and R&D activities, those performed by both incumbent firms and potential entrants. Since the number of workers employed for production purposes per product of quality q is $x = 1/p = (1-\kappa)/qw$ from equations (5) and (6), the number of workers demanded for production of a product of quality q is $\ell_x(q) = (1-\kappa)/qw$. The number of R&D workers employed per product by incumbent firms of type i is $\ell_R(i) = c(\gamma_i)$. Because each potential entrant innovates at frequency η/m, the aggregate number of workers engaged by all m in R&D is $\ell_E = mc(\eta/m)$. Hence, the equilibrium wage satisfies the labor market clearing condition

$$\begin{aligned} \ell &= \sum_{i=1}^{n}\sum_{k=1}^{\infty}\left([E_i\ell_x(q)k\} + \ell_R(i)k\right] M_i(k))\,\phi_i + \ell_E \qquad (11) \\ &= \sum_{i=1}^{n}\left(E_i\left\{\frac{1-\kappa}{qw}\right\} + c(\gamma(i))\sum_{k=1}^{\infty}kM_i(k)\right)\phi_i + mc(\eta/m) \end{aligned}$$

where $M_i(k)$ represents the mass of firms of type i that supply k products.

2.4 The Distribution of Firm Size

Once a firm enters, its size as reflected in the number of product lines supplied, evolves as a birth-death process. As the set of firms with k products at a point in time must either have had k products already and neither lost nor gained another, have had $k-1$ and innovated, or have had $k+1$ and lost one to destruction over any sufficiently short time period, the equality of the flows into and out of the set of firms of type i with $k > 1$ products requires

$$\gamma_i(k-1)M_i(k-1) + \delta(k+1)M_i(k+1) = (\gamma_i + \delta)kM_i(k)$$

where $M_k(i)$ is the steady state mass of firms of type i that supply k products. Because an incumbent dies when its last product is destroyed by assumption but entrants flow into the set of firms with a single product at rate η,

$$\phi_i\eta + 2\delta M_i(2) = (\gamma_i + \delta)M_i(1)$$

where ϕ_i is the fraction of the new entrants who are of type i. Births must equal deaths in steady state and only firms with one product are subject to death risk. Therefore, $\phi_i \eta = \delta M_i(1)$ and

$$M_i(k) = \frac{k-1}{k}\gamma_i M_i(k-1) = \frac{\eta\phi_i}{\delta k}\left(\frac{\gamma_i}{\delta}\right)^{k-1} \tag{12}$$

by induction.

The size distribution of firms conditional on type can be derived using equation (12). Specifically, the total mass of firms of type i is

$$\begin{aligned} M_i &= \sum_{k=1}^{\infty} M_i(k) = \frac{\phi_i \eta}{\delta}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\gamma_i}{\delta}\right)^{k-1} \\ &= \frac{\eta}{\delta}\ln\left(\frac{\delta}{\delta-\gamma_i}\right)\frac{\delta\phi_i}{\gamma_i}. \end{aligned} \tag{13}$$

where convergence requires that the aggregate rate of creative destruction exceed the creation rate of every incumbent type, i.e., $\delta > \gamma_i \; \forall i$. Hence, the fraction of type i firm with k product is

$$\frac{M_i(k)}{M_i} = \frac{\frac{1}{k}\left(\frac{\gamma_i}{\delta}\right)^k}{\ln\left(\frac{\delta}{\delta-\gamma_i}\right)}. \tag{14}$$

This is the logarithmic distribution with parameter $\gamma(q)/\delta$.[1] Consistent with the observations on firm size distributions, that implied by the model is highly skewed to the right.

By equation (14), the mean of the firm size distribution conditional on product profitability is

$$E\{k|i\} = \sum_{k=1}^{\infty}\frac{kM_i(k)}{M_i} = \frac{\frac{\gamma_i}{\delta-\gamma_i}}{\ln\left(\frac{\delta}{\delta-\gamma_i}\right)}, \tag{15}$$

which one can show is an increasing function of γ_i. As the first order condition for optimal investment in R&D, equation (9), implies that the product creation rate γ_i increases with expected profitability $E_i\{\pi(q)\}$, the expected number of products supplied increases with expected productivity rank. That is $E\{k|i\} \geq E\{k|i+1\}$.

[1] This result is in Klette and Kortum (1992). We include the derivation here simply for completeness.

2.5 Creative-Destruction and Growth

Because the number of products is fixed, the rate of creative-destruction is the sum of the entry rate and the creation rates of all the incumbents. As the new product arrival rate of a firm of type q with k products is $\gamma(q)k$ and the measure of such firms is $M_k(q)$,

$$\delta = \eta + \sum_{i=1}^{n} \gamma_i \sum_{k=1}^{\infty} kM_i(k)\phi_i. \tag{16}$$

Finally, the contribution to growth of a new entrant and a current incumbent is $\ln q$, the growth rate in consumption is

$$g = \delta E\{\ln(q)\} = \eta \sum_{i=1}^{n} E_i\{\ln(q)\}\phi_i + \sum_{i=1}^{n} \gamma_i E_i\{\ln(q)\} \sum_{k=1}^{\infty} kM_i(k)\phi_i \tag{17}$$

2.6 Market Equilibrium

A steady state *market equilibrium* is a triple composed of a labor market clearing wage w, entry rate η, and creative destruction rate δ together with an optimal creation rate $\gamma(q)$ and a steady state size distribution $M_k(q)$ for each type that satisfy equations (9), (10), (11), (12), and (16) provided that $\gamma(q) < \delta$, for every q in the support of the entry distribution. Lentz and Mortensen (2004) provide a proof of existence.

3 The Social Planner's Problem

3.1 Formulation

As already stated, a firm's type is not known prior to entry but is revealed and common knowledge afterward. Although this assumption abstracts from the obvious learning problem, doing so allows for a much less complicated analysis. The social planner, then, chooses non-negative time paths for the production rate, x_i, and the rate of new product creation, γ_i, per product line for each firm type $i = 1, ..., n$, the rate of product innovation, γ_0, and the aggregate creative-destruction rate δ to maximize the present discounted utility of the representative household's consumption subject to the fact that there are a fixed number of intermediate products, a labor resource constraint, and laws of motion for the state variables. Under symmetric information, both the firm and the planner observes the realized productivity of any innovation. Given this information, the planner finds it in her interest to screen innovations before adoption. Let $\Phi_i(q)$ represent an indicator function which takes on the value unity if the planner adopts an innovation of realized quality q created by a firm of type i. Of course, the indicator is zero if the decision is not to adopt.

Specifically, the planner's strategy determined $(x_i, \gamma_i, \Phi_i(q))$ for $i = 1, ..., n$, γ_0. and δ at each date. It maximizes the expected present value of the representative consumer's utility stream subject to a set of constraints. Formally, the criterion is

$$\int_0^\infty \ln C_t e^{-rt} dt = \int_0^\infty \left[\ln A_t + \sum_i \ln (x_i(t)) K_i(t) \right] e^{-rt} dt$$

where A represents aggregate productivity, $K_i = \sum_{k=1}^\infty k M_i(k)$ is the mass of products supplied by type i firms, and x_i is the quantity of each product supplied by a firm of type i and the t index indicates the date of each. The constraints follow: Employment cannot exceed the available labor supply,

$$\ell \geq \sum_i [x_i + c(\gamma_i)] K_i + mc(\gamma_0).$$

As entry requires both innovation and adoption, the aggregate entry rate is

$$\eta = m\gamma_0 \sum_i E_i\{\Phi_i(q)\}\phi_i$$

where $m\gamma_0$ is the frequency with which the mass of potential entrants innovated and $E_i\{\Phi_i(q)\}$ represents the fraction of the innovations by firm's of type i that are adopted. The assumption that there is a continuum of potential innovators of each type and the usual appeal to the law of large numbers justifies the substitution of the expected fraction for the realized one. The creative-destruction constraint is

$$\delta = \eta + \sum_i \gamma_i E_i\{\Phi_i(q)\} K_i$$

where K_i is the fraction of products supplied by firms of type i.The law of motion for aggregate productivity is

$$\frac{d \ln A}{dt} \equiv g = \gamma_0 m \sum_i E_i\{\ln (q) \, \Phi_i(q)\}\phi_i + \sum_i \gamma_i E_i\{\ln (q) \, \Phi_i(q)\} K_i,$$

where again the expectation is used by an appeal to the law of large numbers. The mass of products supplied by each firm type evolves according to

$$\frac{dK}{dt} = \dot{K}_i = m\gamma_0 E_i\{\Phi_i(q)\}\phi_i + (\gamma_i E_i\{\Phi_i(q)\} - \delta)K_i, i = 1, ..., n. \tag{18}$$

The planner's problem is a relatively standard one in dynamic control. The constraint augmented present value Hamiltonian for the problem can be written as

$$\begin{aligned} H &= \ln A + \sum_i \ln x_i K_i \\ &\quad +\omega \left(\ell - \sum_i \left[x_i + c(\gamma_i) \right] K_i - mc\left(\gamma_0\right) \right) \\ &\quad +\tau \left(\delta - m\gamma_0 \sum_i E_i\{\Phi_i(q)\}\phi_i - \sum_i \gamma_i E_i\{\Phi_i(q)\}K_i \right) \\ &\quad +\lambda \left(m\gamma_0 \sum_i E_i\{\ln(q)\, \Phi_i(q)\}\phi_i + \sum_i \gamma_i E_i\{\ln(q)\, \Phi_i(q)\}K_i \right) \\ &\quad +\sum_i v_i \left[m\gamma_0 E_i\{\Phi_i(q)\phi_i + \gamma_i E_i\{\Phi_i(q)\}K_i - \delta K_i \right] \end{aligned}$$

where ω and τ are multipliers associated respectively with the labor supply constraint and the creative-destruction constant while λ is the shadow price of the state variable $\ln A$ and v_i is the shadow price or co-state variable associated with K_i, $i = 1, ..., n$. In addition, the necessary transversality conditions require that λe^{-rt} and $v_i e^{-rt} \forall i$ converge to zero as $t \to \infty$.

As the optimal controls maximize the Hamiltonian given state and co-state variables, the first order necessary conditions for all the continuous choice variables are

$$\begin{aligned} \frac{\partial H}{\partial x_i} &= \left(\frac{1}{x_i} - \omega \right) K_i \leq 0 \text{ (with } = \text{ if } x_i > 0\text{)}, i = 1, ..., n. \\ \frac{\partial H}{\partial \gamma_i} &= \left[\lambda E_i\{\ln(q)\, \Phi_i(q)\} + (v_i - \tau)\, E_i\{\Phi_i(q)\} - \omega c'(\gamma_i) \right] K_i \leq 0 \\ \text{(with} &= \text{ if } \gamma_i > 0\text{)}, i = 1, .., n. \end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial \gamma_0} &= \left[\sum_i \left(\lambda E_i\{\ln(q)\,\Phi_i(q)\} + (v_i - \tau)\right) E_i\{\Phi_i(q)\}\phi_i - \omega c'(\gamma_0)\right] m \leq 0 \\
(\text{with} &= \text{if } \eta > 0) \\
\frac{\partial H}{\partial \delta} &= \tau - \sum_i v_i K_i = 0 \leq 0 \text{ (with } = \text{ if } \delta > 0)
\end{aligned}$$

The assumption that the cost of R&D is convex in the innovation rate is sufficient to guarantee that the second order necessary conditions are satisfied. Finally, the optimality condition for the discrete choice of whether or not to adopt an innovation can be represent as

$$\begin{aligned}
\Phi_i(q) &= 1 \text{ if and only if} \\
\frac{\partial H}{\partial \Phi_i(q)} &= \left[\lambda \ln(q) + v_i - \tau\right]\left[m\gamma_0\phi_i + \gamma_i K_i\right] F_i'(q) \geq 0 \text{ for all } i \text{ and } q.
\end{aligned}$$

The co-state (Euler) equations are

$$\begin{aligned}
\frac{\partial H}{\partial \ln A} &= 1 = r\lambda - \dot{\lambda}. \\
\frac{\partial H}{\partial K_i} &= \ln x_i - \omega\left[x_i + c(\gamma_i)\right] + \lambda\gamma_i E_i\{\ln(q)\,\Phi_i(q)\} \\
&\quad + (v_i - \tau)E_i\{\Phi_i(q)\}\gamma_i - \delta v_i \\
&= r\ v_i - \dot{v}_i,\ i = 1, ..., n.
\end{aligned}$$

The fact that the criterion, laws of motion, and constraints are all linear in the states is sufficient to guarantee that a unique solution to the problem exists. (See Kamien and Schwartz (1991).)

The transversality condition $\lim_{t\to\infty} \lambda e^{-rt} = 0$ requires that the shadow price of log productivity equal the inverse of the discount rate at all dates,

$$\frac{1}{\lambda} = r. \tag{19}$$

The optimal strategy for screening new innovations reflects the externalities present. As v_i is the shadow value of a product developed by a firm of type i and τ is the expected value of the product it replaced, the optimal adoption policy requires that sum of the value of it contribution to future productivity growth, represented by $\lambda E_i\{\ln(q)\}$, plus the present value of the inventor's future profit from both production and innovation activity must exceed the expected value of the product that it will replace. As a consequence, the optimal adoption strategy has the following reservation property

$$\Phi_i(q) = 1 \text{ if and only if } \ln q \geq \frac{\tau - v_i}{\lambda} = r(\tau - v_i) \tag{20}$$

where

$$\tau = \sum_i v_i K_i \tag{21}$$

Since $\tau - v_i > 0$ for at least one i is $n > 1$, it follows that some innovations by firm types with low product value v_i should not be adopted even when they represent improvements in quality in the sense that $\ln(q) > 0$. The screening policy implies that the aggregate entry, creative-destructions and growth rates are

$$\begin{aligned} \eta &= m\gamma_0 \sum_i \int_1^\infty \Phi_i(q) dF_i(q) \phi_i \qquad (22) \\ \delta &= \eta + \sum_{i=1}^n \int_1^\infty \Phi_i(q) dF_i(q) \gamma_i K_i. \\ g &= \sum_i (\gamma_0 m \phi_i + \gamma_i K_i) \int_1^\infty \ln(q)\, \Phi_i(q) dF_i(q). \end{aligned}$$

Under the reasonable regularity conditions $c'(0) = c(0) = 0$, one can rewrite the first order necessary conditions for a solution as

$$\begin{aligned} \omega x_i &= 1,\ i = 1, ..., i.. \qquad (23) \\ \omega c'(\gamma_i) &= \frac{1}{r} \int_1^\infty \max \langle \ln q - r(\tau - v_i), 0 \rangle\, dF_i(q),\ i = 1, ..., n. \\ \omega c'(\gamma_0) &= \frac{1}{r} \sum_i \int_1^\infty \max \langle \ln q - r(\tau - v_i), 0 \rangle\, dF_i(q) \phi_i = \sum_i \omega c'(\gamma_i) \phi_i \end{aligned}$$

because

$$\begin{aligned} & E_i\{\lambda \ln(q)\, \Phi_i(q)\} + (v_i - \tau)\, E_i\{\Phi_i(q)\} \qquad (24) \\ = & \frac{1}{r} E_i\{\Phi_i(q)\} \left[E\{\ln q | \ln q \geq r(\tau - v_i)\} - r(\tau - v_i) \right] \\ = & \frac{1}{r} \int_1^\infty \max \langle \ln q - r(\tau - v_i), 0 \rangle\, dF_i(q) \geq 0 \end{aligned}$$

This term, the product of the probability of adoption and the expected present value of the future social gain conditional on adoption, is the expected gain in social surplus attributable to an innovation created by a firm of type i. Hence, the second equation of (23) requires that the marginal cost of investment in R&D by an incumbent of type i equal the present value of expected future social surplus conditional on type and the third requires that the analogous condition hold for potential entrants under the assumption that firm type is not yet known. As a corollary, it follows that the marginal cost of R&D investment by a potential entrant is equal to the expected marginal cost of investment by incumbents taken with respect to the initial distribution of types at entry, which is the interpretation of the last equality.

The first equation of (23) implies that all inputs should be supplied at the same rate. From the labor supply constraint it follows that the common rate of production is

$$x_i = x = \frac{1}{\omega} = \ell - \sum_i c(\gamma_i)K_i - mc\left(\frac{\eta}{m}\right) \text{ for all } i. \tag{25}$$

The optimality of equal production rates reflects the fact that the utility function is symmetric across product types and that the marginal costs of production are identical. By implication, higher quality intermediate inputs are generally under produced relative to lower quality inputs because demands are equal while suppliers of higher quality products charge a higher limit price.

Under the assumption that $c'(0) = c(0) = 0$, the co-state equation for the value of product supplied by each firm type can be written as

$$\dot{v}_i = (r+\delta)v_i + 1 - \ln(x) - \omega\left(\gamma_i c'(\gamma_i) - c(\gamma_i)\right), \ i = 1, ..., n \tag{26}$$

by using the second equation of (23) and equation (21) to substitute the marginal cost the innovation rate for the expected return to an innovation. Because $\ln(x) - \omega x = \ln(x) - 1$ is the profit earned by supplying a product expressed in utility terms and $\omega\left(\gamma_i c'(\gamma_i) - c(\gamma_i)\right)$ is the net flow return to R&D activity expressed in terms of utility, the instantaneous utility of a product is the sum of these two sources of producer surplus and v_i is the present value of this future utility stream discounted as the rate $r + \delta$.

The planner's problem can be decentralized as follows: The planner controls product adoption by offering to buy the rights to each innovation created by a type i firm with a payment equal to $\ln q/r + v_i$ less a "destruction tax" equal to τ. If this difference is negative, the innovator can choose not to sell but cannot adopt by law. The production rights to any innovation of positive value are then resold at auction to many different firms in order to induce competition in the supply of each good. These firms sell at prices equal to marginal cost, ω, in competitive equilibrium. Under this scheme, potential entrants and incumbents have the correct incentives for an optimal investment in R&D choice.

4 Numerical Solutions

In this section, we use the model's parameter estimates reported by Lentz and Mortensen (2005) to compare the quantitative properties of the market and the planner's solutions. We conclude the section by computing the welfare gain that could be achieved by a switch to the optimal solution when the market steady state solution characterizes the initial conditions.

4.1 The Market Equilibrium Solution

A steady state competitive equilibrium solution to the model satisfies the following conditions: The innovation rate for each firm type solves the FOC

$$wc'(\gamma_i) = \max_{\gamma \geq 0} \left\{ \frac{E_i\{\pi(q)\} - wc(\gamma)}{r + \delta - \gamma} \right\} \tag{27}$$

where the profit rate for each type is

$$\pi(q) = (1 - \kappa)(1 - q^{-1}), \tag{28}$$

the market wage w satisfies the labor market clearing conditions

$$\begin{aligned} \frac{L}{Z} &= \sum_i E_i \left\{ \frac{1 - \kappa}{qw} \right\} + \sum_i c(\gamma_i) + mc\left(\frac{\eta}{m}\right) \\ &= \sum_i \frac{1 - \kappa - E_i\{\pi(q)\}}{w} + \sum_i c(\gamma_i) + mc\left(\frac{\eta}{m}\right) \end{aligned} \tag{29}$$

the entry rate η satisfies its own FOC

$$wc'\left(\frac{\eta}{m}\right) = \sum_i \max_{\gamma \geq 0} \left\{ \frac{E_i\{\pi(q)\} - wc(\gamma)}{r + \delta - \gamma} \right\} \phi_i = \sum_q wc'(\gamma_i)\phi_i. \tag{30}$$

The steady state number of products supplied by each type is given by

$$K_i = \frac{\eta \phi_i}{\delta - \gamma_i}. \tag{31}$$

The rate of creative-destruction and the growth rate are

$$\delta = \eta + \sum_i \gamma(q_i) K_i. \tag{32}$$

and

$$g = \eta \sum_i E_i\{\ln(q)\}\phi_i + \sum_i \gamma_i E_i\{\ln(q)\} K_i. \tag{33}$$

The parameter estimates obtained by Lentz and Mortensen (2005) derived from Danish firm data using a simulated method of moments are reported in Table 1. In the version of the model estimated, the R&D cost function is assumed to take the power form $c(\gamma) = c_0 \gamma^{1+c_1}$. The parameter Z is the average real value added per product line and L is the total labor force. Since real value added per product is normalized at unity and the total number of product is set to unity as well in the model, the total labor supply per product line is $\ell = L/Z$. These estimates are conditional on only three types ($n = 3$) of firms. The fitted distributions of product quality conditional on type is a three-parameter Weibull. The implied conditional mean profit per product line and log quality are reported in the table as well as the parameter values. The case of four firm types was considered but added little to the fit.

Table 1: Model Parameter Values

Cost scale parameter	c_0	9.340	Interest rate	r	0.0500
Cost curvature parameter	c_1	2.892	Wage rate	w	190.24
Creative-destruction rate	δ	0.0798	Labor supply	L	44.89
Capital cost per product	κ	0.4250	Value added	Z	15801.14
Incumbent mass	M	0.689	Entrant mass	m	1.358
Firm Types	i	ϕ_i	$E_i\{\pi(q)\}$		$E_i\{\ln q\}$
	1	0.0958	0.2254		0.6964
	2	0.3201	0.0162		0.0302
	3	0.5841	0.0054		0.0097
Weibull Parameters	i	gamma	beta		eta
	1	1.1160	0.4228		0.7224
	2	1.0052	0.4228		0.0097
	3	1.0017	0.4228		0.0029

The extreme right skew in the distribution of firm types at entry reflects the fact few entrants create products of the highest quality. Indeed, over 90% of the entrants can be expected to create innovation of only marginal value. These firms might be interpreted as simply "imitators."

The associated vector of equilibrium innovation rates per year and vector of steady state shares of products by type implied by the parameter values in Table 1 and equations (27) - (31) are

$$\gamma = \begin{pmatrix} 0.06736 \\ 0.02410 \\ 0.01622 \end{pmatrix} \text{ and } K_0 = \begin{pmatrix} 0.34055 \\ 0.25375 \\ 0.40570 \end{pmatrix}. \tag{34}$$

Finally, the growth rate obtained using equation (33) is 1.985%,

The differences between the distribution of firm types at entry reported in Table 1 and the steady state distribution of products across firm types reported in equation (34) reflect the fact that firms that create higher quality products are more profitable and grow faster as a consequence. Specifically, those that are expected to create higher quality products eventually supply relatively more product lines than at entry because they innovate more frequently. This selection process reflects a continual reallocation of employment from dying firms to younger fast growing firms that create products of high quality.

The impact of this reallocation process on growth can be measured using the following decomposition of the growth rate:

$$g = \eta \sum_i \ln(q_i)\phi_i + \sum_i \gamma_i \ln(q_i)\phi_i + \sum_i \gamma_i \ln(q_i)(K_i - \gamma_i). \tag{35}$$

Obviously, the first term reflects the contribution of entry to the growth rate. The second term represents the contribution to the growth rate of incumbent firms if there were no change in the relative number of products supplied and, consequently, no need to reallocate workers across incumbent firms. Finally, the last term, the contribution to growth attributable to the fact that more profitable firms grow faster, is associated with worker reallocation. The values of the parameters and the steady state values of the innovation rates and the distribution of products across firm types imply that entry accounts for 18.26% of the overall rate of growth while the magnitude of the last term reflects the fact that the process of reallocation accounts for 57.46% of growth. In sum, over three-quarters of the growth rate is the consequence of entry and reallocation.

4.2 The Steady State Solution to the Planner's Problem

A steady state solution to the planner's problem is a vector of product values and a distribution of products across types that satisfy the steady state conditions

$$v_i = \frac{\ln(x) - 1 + \omega\left(\gamma_i c'(\gamma_i) - c(\gamma_i)\right)}{r+\delta},\ i = 1, .., n. \tag{36}$$

and

$$K_i = \frac{m\gamma_0\phi_i}{\delta - \gamma_i}, i = 1, ..., n. \tag{37}$$

Of course, the production rate, the innovation rates, the adoption indicator, and the "destruction tax" satisfy equations (23), (20) and (21). Table 2 provides a comparison of the market equilibrium and the planner's steady state solutions.

Because low quality innovations are not adopted in the optimal solution, the entry rate is much smaller than in the market equilibrium and most of the entering firms are those that are expected to create high quality products in the future. The large difference between the expected quality of product created by the highest quality firm type and the others implies that the innovation frequency of the highest quality type is higher than in equilibrium while the innovation frequencies of the other two types is trivial. As a consequence, the highest quality firm type, type 1, supplies virtually all the products in steady state and account for almost all of the growth.[2]

The rate of entry in the planner's solution is much lower than in market equilibrium because few potential entrants can create high quality products and because potential entrants that create low quality products are not allowed to enter. Although the optimal rate of creative-destruction is larger than its market equilibrium value, most of it is accounted for by the R&D investment of incumbent firms. Finally, the optimal steady state growth rate is *over three times larger* than its equilibrium value because almost all innovations are by the highest quality type firms which have a larger impact on productivity growth. As total aggregate investment in R&D is also larger under the optimal policy, the output rate per product line is necessarily smaller than its equilibrium value as a consequence of the labor supply constraint.

[2] The contribution to the growth rate is 8.74×10^{-8},

Table 2: A Comparison of Equilibrium and Planner's SS Solutions

Innovation Rates by Type	Equilibrium	Planner's
γ_1	0.06736	0.08579
γ_2	0.02410	0.01011
γ_3	0.01622	0.00286
Product Fractions by Type		
K_1	0.34055	0.99812
K_2	0.25375	0.00198
K_3	0.40570	0.00012
Entry rate η	0.04415	0.00513
Creative-destruction rate δ	0.07979	0.09076
Growth Rate g	0.01985	0.06316
Production rate x	0.00273	0.00215

5 Transition Dynamics

The comparative results reported in Table 2 suggest the adoption of the planner's innovation strategy will yield large welfare gains. To obtain a quantitative measure of the gain, one must take account of the entire transition path from the market equilibrium to the optimal steady state. The transition dynamic is likely to be important for two reasons. First, an initial consumption sacrifice will be required to achieve the higher steady state growth promised by the optimal R&D investment strategy. Second, the fact that only a small fraction of new firm births have the ability to create products of the highest quality and that firms of this type supply almost all the products in steady state, suggest that the transition may take a long time. Hence, sufficient patience is required to realize a significant welfare gain from adopting the entry and R&D investment policies.

5.1 The ODE System

In general, the transition path is a solution to the system of ordinary differential equations

$$\begin{aligned} \dot{K}_i &= m\gamma_0\phi_i + (\gamma_i - \delta) K_i \\ (r+\delta)v_i - \dot{v}_i &= \ln x - 1 + \omega\left(\gamma_i c'(\gamma_i) - c(\gamma_i)\right) \end{aligned} \tag{38}$$

$i = 1, .., n$. The boundary conditions are the initial values of the state variables, which we will take to be the equilibrium distribution of product shares across firm types and the steady state values of the co-states.

As we know, only firms able to create products of the highest quality contribute to the supply of products and growth in the steady state. If the same is true in transition, then the dynamics can be approximated by the solution to a system of only two differential equations involving a single decision relevant state, the measure of products of the highest quality, K_1, and the difference between its value and that of any product supplied by one of the other types. As the solution for v_i is the same for all $i > 1$ under the assumption that $\gamma_i = 0$, $\tau = \sum_i v_i K_i = v_n K_n + v_0(1 - K_n)$ where $v_i = v_0$ for all $i > 1$. Hence, given the definition

$$y_1 \equiv v_1 - v_0, \tag{39}$$

the transitory dynamics can be described by the ODE system

$$\begin{aligned} \dot{K}_1 &= m\gamma_0\phi_1 + (\gamma_1 - \delta) K_1 = \delta(1 - K_1) \\ \dot{y}_1 &= (r + \delta)y_1 - \omega (\gamma_1 c'(\gamma_1) - c(\gamma_1)) \end{aligned} \tag{40}$$

where $\gamma_1, \gamma_0, \delta, x,$ and ω are determined by the following equations:

$$\begin{aligned} \omega c'(\gamma_1) &= \ln q_1/r + y_1(1 - K_1) \\ c'(\gamma_0) &= c'(\gamma_1)\phi_1 \\ \delta &= m\gamma_0\phi_1 + \gamma_1 K_1 \\ x &= \frac{1}{\omega} = \ell - c(\gamma_1)K_1 - mc(\gamma_0). \end{aligned} \tag{41}$$

Because the state of the system, the fraction of products supplied, K_1, converges to unity, its unique positive steady state, and because the other characteristic roots of the system exceed the interest rate r, the steady state solution is a saddle and the transversality conditions imply that the solution of interest is the stable manifold, the unique trajectory that converges to the steady state associated with the given initial value of K_1. One can easily find the numerical solution to the problem using any ordinary differential equation solver.

5.2 The Transitory Solution

The time paths of the creative-destruction and growth rates for the solution are all plotted in Figure 1 and the associated time paths of the fraction of

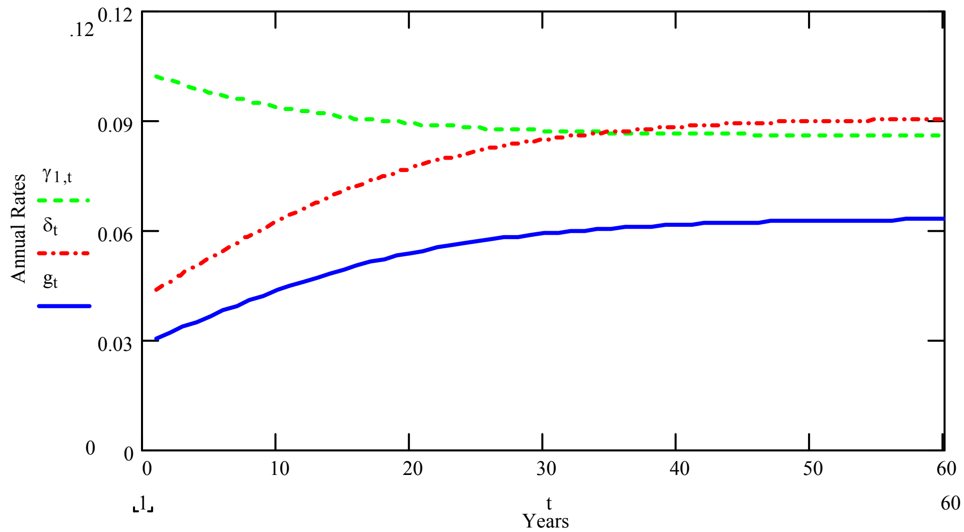

Figure 1: Optimal Innovation, Creative-Destruction, and Growth Rates

products supplied by the three firm types are illustrated in Figure 2. In Figure 1, γ_{1t} represents the innovation frequency of type 1 firms. The annual innovation frequency of the highest quality firm type jumps up to over 10% initially. As illustrated in Figure 2, the effect of the high innovation rate early in the transition is a continual increase in the fraction of the products supplied by type 1 firms. Although the innovation rate falls toward its steady state value throughout the transition phase, the rate of creative-destruction and the growth rate both rise in response to the rising product share of type 1 firms. Finally, note that the transition to the steady state takes 60 years with about half of the adjustment taking place in the first 15 years.

5.3 Welfare Gain

The paths of instantaneous utility for a market equilibrium and the planner's solution are illustrated in Figure 3. Specifically, $\ln C_{0t}$ represents the time path of the instantaneous utility, the log of consumption, associated with a market equilibrium solution while $\ln C_{1t}$ is the utility obtained in the dynamic solution to the planner's problem where

$$\ln C_{jt} = g_j t + \ln(x_{jt}) \tag{42}$$

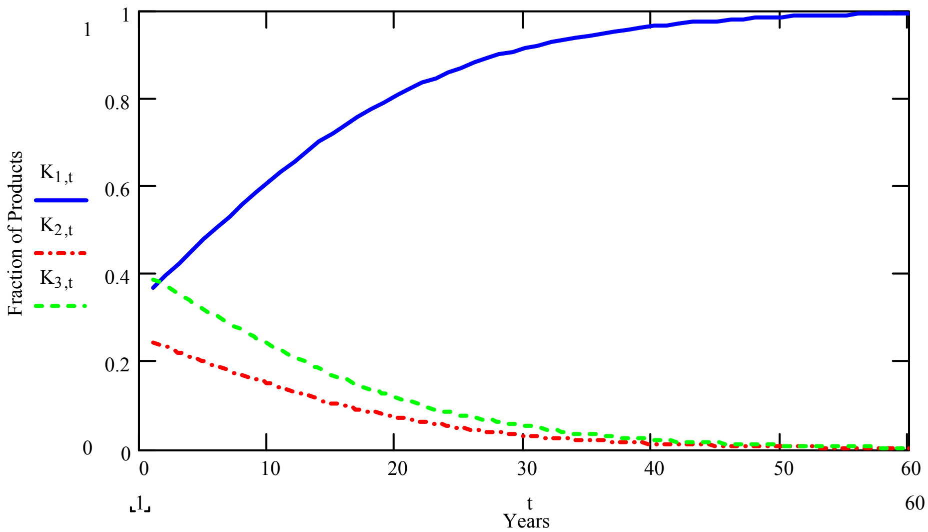

Figure 2: The Distribution of Products Supplied Over Firm Types

from equations (4), (17), and (25). Note that it takes about ten years for the utility provided by the planner's strategy to dominate that of the steady state market equilibrium solution. Still, because the optimal growth rate converges to a value over three times that in the market equilibrium solution implies, the eventual welfare gain is considerable.

Of course, a contemporaneous comparison between the two utility paths ignores time discounting. The fraction of consumption that the typical household would be willing to forego at every point along the optimal transition path in order to adopt the planner's strategy is a standard way to measure the welfare gain. Given that θ represents the fraction of consumption foregone, the resulting compensated utility path realized in each period is

$$\ln C_{2t} = g_1 t + \ln(x_{1t}) + \ln(1-\theta) \tag{43}$$

where θ is chosen so that

$$\int_0^\infty \ln C_{2t} e^{-rt} = \int_0^\infty \ln C_{1t} e^{-rt} + \frac{\ln(1-\theta)}{r} = \int_0^\infty \ln C_{0t} e^{-rt}. \tag{44}$$

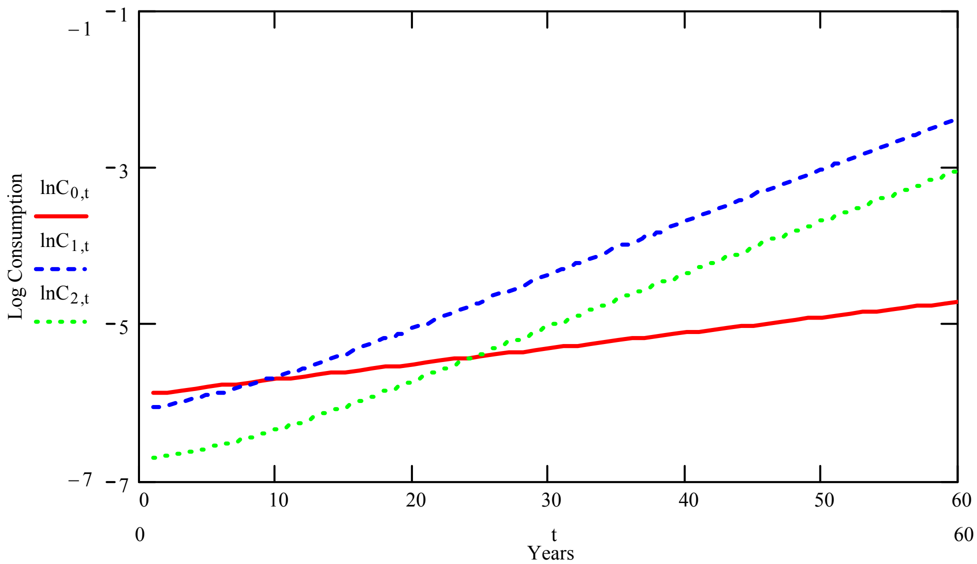

Figure 3: Equilibrium ($\ln C_0$), Optimal ($\ln C_1$), and Compensated ($\ln C_1$) Utility

The computed value of the fraction is $\theta = 0.4687$ and the associated compensated utility path is that represented as C_{2t} in Figure 4. In other words, the welfare gain attributable to the optimal R&D strategy is equivalent to an additional 46.87% of consumption per period along the optimal transition path. Figure 4 implies that the utility path equivalent in present value to that obtained in equilibrium yields more consumption only after the 24^{th} year into the transition to steady state! In sum, the welfare gain from adopting the optimal entry and R&D investment policy is huge.

6 Conclusion

In our companion paper, Lentz and Mortensen (2005), we show that a fully articulated structural model of firm evolution through creative destruction can explain the observed distributions of value added, wage bill, and employment across firms found in Danish panel data and their shifts over time. Here we show that the parameter estimates and aggregate equilibrium conditions have important implications for growth. The implied equilibrium growth rate is 1.98% annually. Because firms that create intermediate good innovations of higher quality grow faster, reallocation of labor resources to the relatively more rapidly growing firms accounts for 57% of aggregate growth in market equilibrium. However, the optimal growth rate is much larger (6.32%) because the social planner weeds out "imitations," innovations with values that do not compensate for the destruction of the products they replace. Even after accounting for the transition from the market equilibrium to the new steady state associated with the optimal strategy, the implied welfare gains, 47% of optimal consumption, are large.

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