Global Uniqueness and Money Non-neutrality in a Walrasian Dynamics without Rational Expectations

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ABSTRACT.— We define a non-tâtonnement dynamics in continuous-time for pure-exchange economies with outside and inside *fiat* money. Traders are myopic, face a cash-in-advance constraint, and play dominant strategies in a short-run monetary strategic market game involving the limit-price mechanism. The profits of the Bank are redistributed to its private shareholders, but they can use them to pay their own debts only in the "next period". Provided there is enough inside money, monetary trade curves converge towards Pareto optimal allocations; money has a positive value along each trade curve, except on the optimal rest-point where it becomes a veil while trades vanish. Moreover, generically, given initial conditions, there is a piecewise globally unique trade-and-price curve not only in real, but also in nominal variables. Finally, money is locally neutral in the short-run and non-neutral in the long-run.

RÉSUMÉ.— On définit une dynamique de non-tâtonnement en temps continu pour les économies de consommation, avec *fiat* monnaie interne et externe. Les agents sont myopes, font face à une contrainte de liquidité, et jouent une stratégie dominante dans un jeu de marchés stratégique monétaire construit autour du mécanisme de prix limites de Mertens (2003). Les revenus de la Banque Centrale sont redistribués à ses actionnaires, mais ne peuvent être utilisés qu'à la période "suivante". S'il y a suffisamment de monnaie interne, les courbes d'échange monétaire convergent vers une allocation Pareto-optimale et la monnaie a une valeur positive le long de ces courbes d'échange, hormis à la limite, où elle devient inutile, tandis que les échanges cessent. De surcroît, génériquement, à dotations initiales fixées correspond une courbe d'échange et de prix unique par morceaux, non seulement en termes réels, mais encore en termes nominaux. Enfin, la monnaie est localement neutre ` court terme, mais n'est jamais neutre à long-terme.

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1 Introduction

The structure of a model reflects the practical purposes which drive the research in the first place. Standard macroeconomic models reduce the aggregate economy to manageable proportions, and frequently a common simplification is the representation of each sector by agents which behave identically; consequently, they are presented in "representative agent" format. On the other hand, standard general equilibrium with heterogenous agents quickly becomes intractable: Closed form solutions typically cannot be derived, and their results are often not robust. The main impediment lies on the multiplicity of equilibria. A second one is the static viewpoint, in effect, underlying both standard macroeconomics and general equilibrium theory: In most cases, theory is unable to describe in a sensible way what happens *out of equilibrium*.

In the present paper, while maintaining the basic ingredients of equilibrium analysis, namely market clearing and agent optimisation, we offer a dynamic continuous-time non-tâtonnement process for monetary economies with heterogenous agents. By considering boundedly rational households, we deviate from the rational expectations hypothesis. By building upon the strategic market games framework, and embedding the limit-price mechanism of Mertens (2003), we maintain heterogeneity, and we manage to build a model with a monetary sector where money has non-neutal effects. In addition, short-run equilibria converge towards Pareto-optimal allocations, and *both real and nominal variables* are generically piecewise unique.

1.1 Monetary short-run linear economies

More precisely, we consider an exchange economy whose dynamics is driven, at each time instant, by a monetary short-run (linear) economy in which agents maximize the first-order approximation of their current utilities subject to a cash-in-advance constraint *à la* Clower (1967).[1] Hence, our approach can be viewed alternatively as the monetary counterpart of Giraud (2004), which was itself a game-theoretic rewriting of Champsaur & Cornet (1990). We postulate a cash-in-advance constraint whereby receipts from commodity sales cannot be used contemporaneously for purchases. Therefore, traders borrow inside money from a loan market in anticipation of future income which is used to pay back their loans. We partly follow the monetary paradigm as set out by Dubey & Geanakoplos (1992, 2003a,b). Households are endowed with outside money and a government Bank injects inside money. We depart in as much as we allow agents to send limit-orders (and not just market orders) to the market, and profits of the Central Bank are redistributed to private shareholders (cf. Shubik & Tsomocos (1992)). Drèze & Polemarchakis (1999, 2000, 2001) in a related monetary framework assume, in addition, that the Bank distributes its profits to private shareholders. But, since they are in a static one-shot world, they can use them to pay their own debts to

[1] See also Grandmont & Younès (1972).

the Bank. As a consequence, there is no outside money in their model, and nominal indeterminacy of the static equilibria obtains.[2] Here, when a shareholder receives dividends at the end of time t, she can only use them as cash (outside money) at time $t + dt$. The consequence is that, at variance with Dubey & Geanakoplos (2003a,b), the exit of the quantity, $\overline{m}$, of outside money is incipient because this money (being equal, along each equilibrium trade curve, to the Bank's instantaneous profit) is reinjected in the economy one nano-second later. But, unlike Drèze & Polemarchakis (2001, strong indeterminacy) and Dubey & Geanakoplos (2003b, generic local uniqueness), we get global uniqueness of the outcome.

This dynamic process of infinitesimal trades continues till the resulting system of differential equation converges to a rest-point. If there is enough inside money, this rest-point is a Pareto-optimal point. Otherwise, the economy remains stuck at a non-efficient point. As Giraud (2004) shows, this limit-price process induces a discontinuous vector field. This discontinuity is unavoidable, and can be traced back to the fundamental lack of continuity that characterizes both strategic market games and double-auctions (Mertens' (2003) limit-price mechanism is an extension of both, and therefore inherits the discontinuity). We thus invoke Filippov's solution concept to differential equations with a discontinuous right-hand side in order to establish existence of monetary trade curves and convergence of the state of the economy to a long-run (non-monetary, i.e., Walrasian) price equilibrium.

The advantage of the formulation we adopt in this paper is that it enables us to introduce a Central Bank providing inside money, keeps a cash-in-advance constraint, and formulate all these ingredients within a dynamics which is driven *via* a full-blown local game associated to each short-run economy. Intraperiod nominal interest rates are endogenously determined at each instant, and serve as market-clearing prices between bonds and inside money. The feasibility constraint on each money-selling order is tantamount to a cash-in-advance constraint. Hahn's (1965) famous puzzle is solved because agents who borrow M inside money from the Central Bank at interest rate $r(t) > 0$ must return $(1 + r(t))M(t) > M(t)$. If $r(t) = \frac{\overline{m}}{M(t)}$ — and this will turn out to be the case along each equilibrium monetary trade curve—, all the outside money $\overline{m}$ will exit the economy at time t along with inside money (before being reinjected at time $t + dt$). Fisher's quantity theory of money takes the following form in each short-run economy (except at a rest-point of the dynamics where trades collapse):

$$\overline{m} + M(t) = \sum_i \sum_c p_c(t) q_c^i(t, p_c), \tag{1}$$

where $\overline{m}$ (resp. $M(t)$) is the aggregate amount of outside (resp. inside) money put on the market at time t, $p_c(t)$ is the (endogenous) current price of commodity c, and $q_c^i(t, p_c)$ is the (endogenous as well) current quantity of commodity c put up for sale at p_c. $\overline{m}$ is not indexed by t because it will turn out to remain constant across time, whatever being the dynamics of inside money. We show that the "classical dichotomy" holds in the short-run, provided there is enough inside money. Indeed, provided there are

[2] The exception is when the government budget constraint is violated, in which case Drèze & Polemarchakis'model would also induce nominal determinacy. Violation of the government budget constraint is equivalent to existence of outside money. Notice that, in the present paper, the governement violates its budget constraint only during a nano-second, since the profits earned by the Central Bank at time t are redistributed (as outside money) at time $t + dt$. This suffices to recover determinacy.

enough gains to trade, i.e., $M(t) > \frac{\overline{m}}{\gamma(e(t))}$,[3], the old quantity theory of money, defended by the neo-classical school holds water: One can separate the real and nominal sides of the economy, solving the real side for relative prices, and fixing their levels by the stock of nominal money. But this holds only *in the short-run*. Indeed, the amount of money $M(t)$ *must* change over time in order to keep the preceding inequality. How it changes will then necessarily affect both nominal and real variables along every trade curve. Moreover, the (implicit) price of money (to be distinguished both from the interest rate) is always equal to 1, and its velocity is a decreasing function of time (and of $r(t)$), bounded from above by 1. Finally, not only does money have value in our model, its value is determinate: For generic economies, the dynamics of interest rates, price levels and commodity allocations are uniquely defined (within a certain time interval). Monetary policy being non-neutral in the long-run, its effects can in principle be tracked because of the global uniqueness of solution paths to our dynamics.[4]

1.2 A central clearing house for inside and outside money

To elaborate our own definition of a short-run outcome, we recall in section 3.2. *infra* the ingredients of Dubey & Geanakoplos' (2003a) definition of a monetary equilibrium. As every Walrasian-like equilibrium concept, this monetary equilibrium does not guarantee, in general, global uniqueness of its corresponding allocation, even when restricted to linear short-run economies. On the other hand, it heavily rests on the perfect foresight assumption. Indeed, each "period" t is divided in three subperiods t_α, t_β and t_γ. At time t_α, agents borrow money from a Central Bank by selling bonds; at t_β, they exchange commodities for money; at t_γ, they repay their loans to the Central Bank. When selling bonds in the first subperiod t_α, households need to perfectly anticipate both t_β-commodity prices and interest rates (the latter being charged at time t_γ) in order not to default. And from a game-theoretic viewpoint, this no-default constraint also induces the use of a generalized game, and not a full-blown normal-form game, or the introduction of penalty rules in order to guarantee that, at a Nash equilibrium, nobody will go bankrupt.

In this paper, we keep the spirit (originating in Shubik & Wilson (1977)) of introducing money in general equilibrium theory *via* a central Bank. However, we drop the rational expectations hypothesis, which, though commonly made in the general equilibrium literature, would be at odds with the postulated myopia of our boundedly rational agents. The key insight of our analysis consists in extending (along the lines of Mertens (2003)) the paradigm introduced by Dubey & Geanakoplos (2003a) to limit-price orders — i.e., orders that are conditional on the realization of prices (or interest rates). This enables to relax the rational expectations hypothesis, to end-up with a classical one-shot normal-form game, and to recover the uniqueness of the short-run outcome associated to each state of the economy (equivalently, to get a *vector* field, and not just a *cone* field).

For this purpose, we equip each individual with a certain amount of bonds with which she starts afresh at each time t. Before entering the market for commodities,

[3]Where $\gamma(e(t))$ is a (short-run version of the) measure of gains to trade introduced in Dubey & Geanakoplos (1992).

[4]The exploration of this fascinating topic is left for further research.

traders can trade their bonds against inside money. In stage t_β, they then play a dominant strategy in the limit-price mechanism, taking into account the fact that, afterwards, they will have to fully deliver on their bonds.

Fiat money in our model corresponds to the paper used as cash in everyday life. All cash of course looks the same, regardless of whether it was originally injected as inside money or as outside money. Nevertheless, in our model as in Dubey & Geanakoplos (2003a), the origin of the money plays a critical role in determining the endogenous real and monetary variables. Hence, and this is of course specific to this paper, it is pivotal in the determination of the whole dynamics. Since it is the interplay between today's outside and inside money (or, equivalently, yesterday's profits of the Central Bank and today's inside money) that is responsible for the results obtained here, a serious justification is needed for the distinction between both types of money. When the State injects money into the private sector in exchange for assets promising the future delivery of money, its arrival foreshadows its departure, and we call it inside money. Money injected by the State as a transfer, or in exchange for real assets, commodities or labor (which gives no claim on future repayment) is called outside money. In the absence of a Central Bank providing liquidity, there would be no inside money in our model — which at least contradicts the fact that *fiat* money is a creature of the State, and immediately prompts the question as to where outside money comes from. On the other hand, in a model with just outside money used as a medium of exchange, nominal prices would be indeterminate. To realize this, just consider Mertens' (2003) limit-price mechanism associated to a graph of trades that is star-shaped with respect to (and only to) some worthless *numéraire* (called outside money). Then, nominal prices would be indeterminate (as they are in standard GET). By contrast, in a situation involving only inside money, our model reduces to the Walrasian (i.e., non-monetary) one introduced in Giraud (2004a). As a consequence, real trades are determinate, but there is nominal indeterminacy in prices (and $r \equiv 0$)

Finally, co-existence of inside and outside money does not suffice *per se*, however, to drive our results. Suppose, indeed, that at each time t, inside money can be exchanged against outside money, but that outside infinitesimal trades are performed thanks to the classical Shapley-Shubik (1977) model of trading-posts, and that there are separate trading-posts for each type of money. The variable r is the relative price of outside to inside money. Then, as soon as $M > m$, all the individuals will sell their whole endowment of outside money against inside money, and perform all their trades in commodities solely with outside money. Player i will end up with $\frac{Mm_i}{\sum_h m_h} = \frac{m_i}{\overline{m}} M$, and will spend this amount of cash to buy commodities. The quantity theory of money will then look like

$$M = \sum_i \sum_c p_c \dot{x}_c^i, \tag{2}$$

and the endogenous variables p_c of the real sector of the economy will again be indeterminate (and depend exclusively on M). Only r will depend both on M and m. Consequently, a slight change in M, sufficiently small to keep $M > m$, will not affect the real terms of trade, so that money will again be essentially neutral. Thus, the last ingredient for our recipe to hold water is to organize trades not according to the decentralized trading-posts *à la* Shapley-Shubik (called TP-mechanism in the sequel), but according to Shapley windows model.[5]

The next section presents the model in details. Section 3 is devoted to our main results. A last section offers some concluding remarks.

2 The model

2.1 The fundamentals

Commodities

The long-run economy $\mathcal{E}$ is defined by $C \geq 1$ consumption goods $c \in \{1, ..., C\}$ and $I \geq 1$ types of households $i = 1, ..., N$, characterized by $(u_i, \omega_i)_i$.[6] Each type i is represented by a *continuum* of clones, say $[0, 1]$, together with the (restriction of the) Lebesgue measure.

For each i, $\mathbb{R}_+^C$ represents type i's trading set. The function $u_i : \mathbb{R}_+^C \to \mathbb{R}$ is the (long-term) utility function of type i. The vector $\omega_i \in \mathbb{R}_+^C \setminus \{0\}$, is her/his initial endowment.

Definition (i) An **allocation** is an integrable, type-symmetric map $x : [0, 1]^N \to \mathbb{R}^{C+1}$ belonging to the *feasible set* τ:

$$\tau := \Big\{ x \in L^1([0,1]^N, \mathbb{R}_{++}^C) \ : \ \int_0^1 x_i di = \overline{\omega} := \int_0^1 \omega_i di = \frac{1}{N} \sum_i \omega_i \Big\}.$$

(ii) An allocation x is **individually rational** whenever $u_i(x_i) \geq u_i(\omega_i)$ for a.e. $i \in [0, 1]^N$.

Because of type-symmetry of allocations, we identify τ with:

$$\tau := \{ x \in \left(\mathbb{R}_+^C\right)^N \mid \sum_i (x_i - \omega_i) = 0 \}.$$

For every household i, $\hat{X}_i$ (resp. $\hat{X}_i^*$) is the subset of consumption bundles $x_i >> 0$ such that $x_i \leq \overline{\omega} := \sum_h \omega_h$ (resp. the subset of points x_i in $\hat{X}_i$ such that $u_i(x_i) \geq u_i(\omega_i)$).

> Assumption (C)(i) For each i, the restriction of u_i to $\hat{X}_i^*$ is $\mathcal{C}^1$, verifies $\nabla u_i(x_i) > 0$, and is quasi-concave.
>
> (ii) For each i and every $x_i \in \partial X_i \cap \hat{X}_i^*$, $\nabla u_i(x_i) \cdot x_i > 0$. Moreover, the restriction of u_i to $\hat{X}_i^*$ is strictly quasi-concave.

This assumption is weak in the sense that it makes no use of any boundary condition in order to keep away the dynamics from the boundary $\partial\tau$ of the feasible set. Similarly, preferences are not assumed to be strictly monotone.

A map $x : [0, 1]^N \to \mathbb{R}^C$ is said to be *type-symmetric* whenever each restriction, $x_{[0,1]}$, on the i^{th} element of the Cartesian product $[0, 1]^N$ is a.e. constant for all $i =$

[5] See Giraud (2003) for the distinction between both market games.

[6] Throughout the paper, $\mathbb{N}_C$ designates the finite set $\{1, ..., C\}$, $\nabla u(x)$ is the gradient of u at x.

$1, ..., N$. We denote by $x_i \in \mathbb{R}^C$, the equivalence class of this restriction. Hence, we use the notation x_i in three different senses: As the vector in $\mathbb{R}^C$, which is the common individual allocation attributed to each household of type i, as the constant function which maps each agent in $[0,1]$ to the vector x_i, and describes the symmetric allocation-selection of households of type i, and as the integral of this constant function on the unit interval $[0,1]$, which gives the aggregate allocation of agents of type i. This sense will be clear from the context.

Money

Fiat money is present in the economy: At time $t = 0$, each type i has a private endowment of outside money $m_i(0) \geq 0$ and of bonds $b_i > 0$ and a stock $M = M(0) > 0$ of money (inside money) is held by the Central Bank. Outside money is owned by households free and clear of debt. Inside money is always accompanied by debt when it comes into households' hands. The quantities $m = (m_i(0))_i, b = (b_i)_i$ and M are exogenously fixed, and $\overline{m} = \sum_i m_i(0)$ (resp. $\overline{b} = \sum_i b_i$) represents the aggregate stock of outside money (resp. bonds) held by the agents at the beginning of time. (Since b will remain constant across time, there is no need for indexing it with respect to time.) The monetary long-run economy is defined by $(\mathcal{E}, m, b, M)$ and the private sector is defined by $(\mathcal{E}, m) \equiv (u_i, \omega_i, m_i, b_i)_i$.

Let $p_c(t) > 0$ denote the price of good c in terms of money and $r(t) \geq 0$ denote the money rate of interest on the Bank loan. The vector $(p(t), r(t)) \in \mathbb{R}_+^C \times \mathbb{R}_+$ denotes the market price at time t.

The configuration space of our dynamics is the set of *states*, *i.e.* of feasible allocations in commodities and stocks of money $(x, m, M) \in \tau \times \mathbb{R}_+^{N+1}$ with $\sum_i m_i = \overline{m}$. Trades occur in continuous time. The stock of bonds $b = (b_i)_i$ is constant across time because we make the assumption that, at each time t, each individual i starts afresh with the same stock b_i (that will be solely used in order to borrow inside money in stage t_α, and is given back to household i as she repays her loan at the end of time t). At each time t, the profits of the Central Bank will be equal to $r(t)M(t)$ (where $r(t)$ is the current intra-period interest rate). It is distributed to its private shareholders according to some fixed ownership structure $\nu : i \mapsto \nu_i \in [0,1]$ such that $\sum_i \nu_i = 1$. However, shareholder i can use only at time $t + dt$, the cash received as dividend at time t. There is no loss of generality in postulating that, for every i, $\nu_i > 0$. Indeed, in our stripped-down model, myopic households make no expectations about the future, hence have no precautionary demand for money. As a consequence, they spend their whole cash at time t in order to repay their loans. Thus, an agent receiving no dividend from the Central Bank at time t, will enter the markets at time $t + dt$ with no outside money. Such an agent won't be able to trade any more, and could be disregarded as well.[7]

2.2 The short-run economy

At each time t, the state is $(x(t), m(t), M(t), b)$; the *short-run* economy $T_{x(t),m(t),M(t),b}\mathcal{E}$ is a monetary linear economy defined by the same characteristics as $\mathcal{E}$ except that:

[7] Actually, we could add a precautionary demand for money without altering our results.

▷ The set of infinitesimal trades of agent i in $T_{x(t),m(t)}\mathcal{E}$ is the shifted cone $-(x_i(t) + m_i(t)) + \mathbb{R}_+^{C+1}$

▷ Initial endowments of consumption goods (resp. outside money) are replaced by 0. Current allocations $(x_i(t))_i$ (resp. current endowment $(m_i(t))_i$) play the role of short-sale constraints.

▷ household's short-run preferences are given by the linear utilities over commodities induced by their current gradients.[8] In other words, agent's i short-run utility $v_i : \mathbb{R}^C \to \mathbb{R}$, is:

$$v_i(\dot{x}_i(t)) := \frac{\nabla u_i(x_i(t))}{\|\nabla u_i(x_i)\|} \cdot \dot{x}_i(t) := g_i(x_i(t)) \cdot \dot{x}_i(t). \tag{3}$$

At each state, the motion of $\mathcal{E}$ is dictated by some strategy-proof equilibrium of a strategic market game $G[T_{x(t),m(t),b(t),M(t)}\mathcal{E}]$ associated to the short-run economy $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$. This captures the myopia of consumers: They are not sophisticated enough to solve an intertemporal optimization programme (involving the heroic solution of, say, the associated Hamilton-Jacobi-Bellmann partial differential equation). Rather they try to trade in the direction of the steepest increase of their current utility.

2.3 The monetary short-run game

Each period $t \in \mathbb{R}_+$ is divided into three subperiods. At the first subperiod t_α, agents borrow money from the Bank by selling bonds ; at time t_β, they sell commodities for money and buy goods with money ; eventually, at time t_γ, Bank's loans are repaid with money. Agents enter the commodity market with the money they got from the loan market *plus* their initial holding in outside money.

In stage t_β, agents meet in a monetary version of the strategic market game induced by Merten's (2003) limit price mechanism. This mechanism itself can be viewed alternatively as the multi-item extension of double auctions, or as the extension of Shapley's windows model (see Sahi & Yao (1989)) to limit-price orders.[9] Money is denoted by m and bonds by b. The time period t is fixed.

Market orders

In order to describe the action of a player, let us begin with market orders. The vector q_i of i's market order has $2C + 1$ components:

- q_i^{bm} is the quantity of bonds sold by i to the Bank for money at time t_α;
- q_i^{mc} is the money spent by i to purchase c at time t_β, $c = 1, ..., C$;
- $q_i^{cc'}$ is the quantity of c sold by i for commodity c' at time t_β, $c \neq c' = 1, ..., C$.

Alternatively, a trader i's signal has two components: the first one consists in q_i^{bm} and is sent in stage t_α, the second is a $(C + 1) \times (C + 1)$ matrix whose k, ℓ-entry $q_{k\ell}^i$

[8] Money and bonds are of course worthless.

[9] See Giraud (2003) for an introduction to strategic market games, and a discussion of this point.

indicates the amount of item k s/he is offering in exchange for item ℓ. Prices are then computed according to the following set of equations:

$$r = \frac{\sum_i q^i_{bm}}{M}, \tag{4}$$

$$\sum_{k=1}^{C+1}\Big(\sum_{i=1}^{N} q^i_{k\ell}p_k\Big) = p_\ell \sum_{k=1}^{C+1}\Big(\sum_{i=1}^{N} q^i_{\ell k}\Big) \quad \ell = 1, ..., C+1 \tag{5}$$

where the price of money is normalized to 1. Finally, commodities are redistributed in such a way that i's final allocation is

$$x^i_\ell := \omega^i_\ell + \frac{1}{p_\ell}\sum_{k=1}^{C+1} p_k q^i_{k\ell} - \sum_{k=1}^{C+1} q^i_{\ell k}. \tag{6}$$

In other words, prices form so as to clear all markets; all of inside money M is disbursed to households in proportion to their bonds; the interest rate r forms as a market-clearing price on the market for bonds against inside money; and at each commodity-money market, all the money (or, commodity) received is disbursed to households in proportion to the commodity (or, money) sent by them.

Given prices (p, r), the (competitive) budget set $B(p, r, x_i(t), m_i)$ of household i is the set of all market orders and final allocations $(q_i, \dot{x}_i(t)) \in \mathbb{R}^{2C+1} \times \mathbb{R}^C_+$ that satisfy the constraints below for all $c \in C$ and all $t \geq 0$:

$$\sum_{c=1}^{C} q_i^{mc} \leq m_i + \frac{q_i^{bm}}{1+r} \tag{7}$$

$$q_i^{cm} \leq x_i^c(t) \tag{8}$$

$$q_i^{bm} \leq m_i + \frac{q_i^{bm}}{1+r} - \sum_{c=1}^{C} q_i^{mc} + \sum_{c=1}^{C} p_c q_i^{cm} \tag{9}$$

$$\dot{x}_i^c(t) \leq q_i^{cm} - x_i^c(t) + \frac{q_i^{mc}}{p_c} \tag{10}$$

(7) is the cash-in-advance constraint faced by each trader every period. A common criticism is that such cash-in-advance constraints are *ad hoc*, and do not adequately capture liquidity. However, in any strategic market game or in a monetary economy, they emerge naturally. Their main intuition is that the different instruments and commodities of the economy are not equally liquid. As long as there exist different liquidity parameters which are less than 1 for the endowment vector, money demand is positive to bridge the gap between expenditures and receipts. Otherwise, the budget constraints collapse to the standard Arrow-Debreu constraints.

(8) precludes commodity short sales, and (9) specifies loan repayment in the final subperiod from money carried over from the first subperiod and receipts from commodity sales. Finally, (10) describes the final allocations.

Limit-orders

We now supplement the preceding market structure by allowing traders to send limit-price orders to the market. For various reasons (that are spelled out in Mertens (2003)) only *selling* orders are allowed — but this implies no loss of generality: If a player wants to buy a commodity, s/he just has to sell some money !

Definition A **limit-order** to sell item ℓ in exchange for item c[10] gives a quantity $q_{\ell c}$ to be sold, and a relative price $\frac{p_\ell^+}{p_c^+}$. The order is to sell up to $q_{\ell c}$ units of item ℓ in exchange for item c if the relative price $\frac{p_\ell}{p_c}$ is greater than, or equal to, $\frac{p_\ell^+}{p_c^+}$. When $\frac{p_\ell^+}{p_c^+} = 0$, one gets a familiar *market order.*

> **Remark.** A limit-order to "sell" ℓ against m at relative prices $p_m^+ = 0, p_\ell^+ > 0$ is, in fact, an order *not* to buy money.

Every trader i in $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$ is allowed to send as many (market- and/or limit-)orders as s/he wants. Nevertheless, due to the fact that a short-run economy is linear, we shall see that every player has at her disposal a unique dominant strategy on the commodity market, which in addition involves at most $C+1$ limit-price orders (whose limit-prices will exactly coincide with this agent's current marginal rates of substitution among commodities and money).[11]

In order to prepare for the (next) definition of a *monetary order book*, observe that, given some collection of orders, one can define a *fictitious linear monetary* economy as follows:

Definition A **fictitious linear monetary** economy $\mathcal{L} = \langle I, \mathcal{I}, \mu, b, e, M\rangle$, is defined by a positive, bounded measure space $(I, \mathcal{I}, \mu)$ of traders, and measurable functions $b, e : I \to \mathbb{R}_+^{C+3}$, e being integrable.

In such a linear economy $\mathcal{L}$, there are $C+1$ objects of exchange: C consumption commodities and money. Every "trader's" $i \in I$ consumption set is $\mathbb{R}_+^{C+1}$. "Trader's" i utility for x^i is $b^i \cdot x^i$, and $e^i = (e_1^i, ..., e_{C+2}^i, e_{C+1}^i)$ is her/his initial endowment, with its last component being i's current holding of money. We designate the set of "agents" of a linear economy by an abstract measure space $(I, \mathcal{I}, \mu)$ because we will need later on to interpret it as a set of limit orders. Fortunately, I will rapidly turn out to be equal to $[0,1]^N$ (equipped with the product of Lebesgue measures) in most of the situations of interest for us.

- ▷ I is, now, the set of **orders**;
- ▷ For each fictitious "agent" i (i.e., for each order), its linear "utility" is given by $b^i := (p_1^+, ..., p_C^+, 0)$;
- ▷ Its "initial endowment" is defined as $e^i := (q_1^i, ..., q_C^i, q_m^i)$.

[10]Here, an *item* may designate a consumption commodity as well as money or a bond.
[11]See Giraud (2004a) for details.

Monetary order books

The timing of trades in each short-run economy is such that, in a third stage, households still have to deliver on their bonds. As shown by Dubey & Geanakoplos (2003a, Lemma 1), the multiple constraints of each stage t_α, t_β can be summarized in a unique (non-linear) constraint where revenues from sales are discounted by the interest rate. In a sense, the banking system extracts (inside) money every time a household i purchases beyond its outside financial wealth m_i.[12] We capture this property in our short-run game $G[T_{x,m,b,M}\mathcal{E}]$ as follows.

• Suppose first that $r = 0$. Any order on the commodity-money market of stage t_β, consisting of a linear "utility" u, together with an "endowment" e, is equivalent to a set of $C+1$ separate orders, the c^{th} of them selling an amount e^c (resp. e^m) of commodity c (resp. money) with the utility u. Therefore, we concentrate on sell-orders of a single item — say c_0. Since money is worthless, every truth-telling sell-order will involve a zero utility for money. But, being negligible, it is a dominant strategy for each player to "reveal the truth" when sending sell-orders to the market. Hence, a "utility function" will typically have the form:

$$v_i(\dot{x}) = \sum_{c=1}^{C} g_i^c(x_i)\dot{x}^c, \tag{11}$$

where $g_i^c(x_i)$ is the personal relative price of c for agent i (cf. (3)). This corresponds to a sell-order of commodity c_0 against any other commodity according to which one will yield the most value in terms of the personal relative price system $(g_i^1(x_i), ..., g_i^C(x_i))$. Formally, if p is the emergent price vector, this order will be executed as an order to sell e^{c_0} against $\dot{x}_{c^*} := \frac{p_{c^*}}{p_{c_0}} e^{c_0}$ for c^* in

$$\text{Argmax } \{v_i(\dot{x}_c)\ c = 1, ..., C\},$$

provided that there is at least some commodity c such that $\frac{p_{c_0}}{p_c} \geq \frac{g_i^{c_0}(x_i)}{g_i^c(x_i)}$. If this last condition is not satisfied, then the order is inexecuted (and automatically disappears from the order book). If this last condition is satisfied at most as an equality, then the order may be only partially executed (and the inexecuted part automatically disappears from the order book). If this condition is satisfied as a strict inequality, then the order is fully executed.

• Next, if $r > 0$, the same logic applies, except that, when a player is selling commodities, her revenue is rescaled at the ratio $\frac{1}{1+r}$ in order to take account of the dissipation of money in the system — which is but the cost to pay for the fact that inside money facilitates trades. (Remember, nevertheless, that the money dissipated at time t is immediately reinjected into the system at time $t + dt$ through the dividends.) This is equivalent, to requiring that, when a player announces (11) to the market and sells commodity c_0 (against money), then an outcome will be computed for the modified economy where (11) has been replaced by:

[12] This viewpoint is developed by Dubey & Geanakoplos (1992), and is consistent with Dubey & Shubik's (1978) seminal approach.

$$v_i(\dot{x}) = (1+r)g_i^{c_0}(x_i) + \sum_{c=1, c\neq c_0}^{C} g_i^c(x_i)\dot{x}^c. \tag{12}$$

On the other hand, when she is sending an order to sell money (i.e., to *buy* some commodity), this player's announcement is not modified. That such a modified order is equivalent to a standard one when due account is taken from the discount factor r can be best viewed as follows. As long as a player is using her own outside money in order to finance her purchase, she incurs no discount rate. This is reflected by the fact that an order to sell money (against commodity) stays unmodified. On the contrary, as soon as a player spends some inside money in order to finance additional purchases, then she has to incorporate this cost in her budget constraint. Everything then goes as if the price at which she will be ready to sell endowment in commodity c_0 was not p_{c_0} but $\frac{1}{1+r}p_{c_0} < p_{c_0}$. Thus, the corresponding order should remain inexecuted as long as

$$\frac{1}{1+r}\frac{p_{c_0}}{p_c} < \frac{g_i^{c_0}(x_i)}{g_i^c(x_i)}.$$

But this is equivalent to modifying the "utility function" associated with the corresponding limit-price order according to (12).

Let us call r-**monetary order** such a limit-price order where revenues from sales are discounted by the interest rate r according to (12). Observe that the discount imposed by the market makers is similar to some transaction cost that introduced a wedge between buying and selling prices. Even more, since it affects only revenues from *sales*, it works like a bid-ask spread.

Given some $r > 0$, a r-**monetary order book** in $G(T_{x,m,b,M}\mathcal{E})$ is a fictitious linear monetary economy $\mathcal{O} = \big(I, \mathcal{I}, \mu, \mathbf{b}, \mathbf{e}, M\big)$ such that each "agent's utility" verifies (**??**).

2.4 Monetary infinitesimal trades

We now define the short-term *outcome* that will be induced by a collection of r-monetary order books sent by players in the short-term economy $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$. Such an outcome will provide the direction in which the state of the underlying economy $\mathcal{E}$ moves (i.e., the infinitesimal trades, $(\dot{x}(t), \dot{m}(t), \dot{b}(t))$, occurring) at time t.

Consider a fictitious linear monetary economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e, M\big)$.

Definition A **pseudo-outcome** of $\mathcal{L}$ is a price system $p \in \mathbb{R}_+^{C+1} \setminus \{0\}$ and an infinitesimal trade $\dot{x}$ verifying $p_m = 1$,[13] and

(i) For every "agent" $i \in I$, $\quad p \cdot b^i = 0$ implies $\dot{x}^i = 0_{\mathbb{R}^{C+1}}$.

(ii) For every $i \in I$, $\quad \dot{x}^i$ maximizes $b^i \cdot \dot{x}$ subject to the (infinitesimal) budget constraints:

$$\dot{x} \geq -(x_i, m_i), \quad p \cdot \dot{x} \leq 0 \tag{13}$$

$$\text{and } \big(p^k = 0 \Rightarrow \dot{x}_k^i = 0\big).$$

[13] That is, the price of money is normalized to 1.

(iii) For every item c, $p^c = 0$ implies that, for μ-a.e. i, $\Big(p \cdot e^i > 0 \Rightarrow b^i(c) = 0\Big)$.

$P(\mathcal{L})$ will denote the set of pseudo-outcome prices, and for all $p \in P(\mathcal{L})$, $X_p(\mathcal{L})$ the corresponding set of pseudo-outcome allocations. We are now in a position to define monetary short-term outcomes.

Definition (Mertens (2003)) (i) A pseudo-outcome is **proportional** if all buyers who quoted the market price as limit price should get their orders executed in the same proportion, and similarly for sellers:

> For every pair of items $(c, c') \in \mathbb{N}_{C+1}$, with non-zero prices, there exists $m_{cc'} \geq 0$ s.t.
>
> a) $m_{cc'} + m_{c'c} > 0$;
>
> b) $m_{c_1c_2}m_{c_2c_3}m_{c_3c_4} = m_{c_1c_3}m_{c_3c_2}m_{c_2c_1}$ (consistency);
>
> c) all "agents" i with non-zero utility whose demand set
>
> $\delta_p^i \ni \{c, c'\}$ receive them in quantities proportional to $m_{cc'}$ and $m_{c'c}$,

where the **demand set** of i at price p is

$$\delta_p^i := \Big\{\ell \in \mathbb{N}_{C+1} \mid p_\ell \leq r(b_i, \ell, k)p_k, \quad \forall k \in \mathbb{N}_{C+1}\Big\},$$

with $r(b_i, \ell, k) := \frac{b_\ell^i}{b_k^i}$ denoting the marginal rate of substitution between ℓ and k (with the convention $\frac{b}{0} := 0$).

(ii) A **short-term outcome** of $\mathcal{L}$ is defined by the following algorithm: pick any proportional pseudo-outcome, settle the corresponding trades, and repeat the procedure with the linear sub-economy $\mathcal{L}'$ restricted to the commodities that had zero price. Until the algorithm ends.

2.5 Strategy-proof trade curves

Our short-run strategic market game is feasible: at the start of period t, the Bank holds $M(t)$ and households hold $\overline{m}(t)$ of money. Money market clearing (4) in stage t_α guarantees that the Bank stock $M(t)$ flows to traders at the end of t_α. When sending orders to the central clearing house in stage t_β, everything goes as if players would not use inside money, but only outside money. The use of outside money is implicit in the fact that they can send orders to sell commodities against commodities (but then with the specific cost (12) described in the preceding section). Thus, the commodities to be traded in stage t_β are the consumption commodities *plus* outside money. Commodity market clearing in stage t_β is guaranteed by the fact that Mertens' (2003) mechanism is balanced (cf. (6), see also Lemma 1 (a), p. 448 if needed). As a consequence, the total stock of commodities and outside money is conserved and redistributed among the households during the second stage. At the end of the two first stages, all of $(M + \overline{m})(t)$ is with households. The no-default constraint (9) is always satisfied because of (12), and implies that the total bonds sold by households do not exceed $(M + \overline{m})(t)$. At the end of stage t_γ, the Bank holds $(1 + r(t))M(t) \leq M(t) + \overline{m}(t)$, and households hold the balance $\overline{m}(t) - r(t)M(t)$. The profit of the Bank is therefore $r(t)M(t)$, and it is redistributed to its shareholders at the beginning of period $t+dt$ as a new endowment in outside money. Consequently, no money disappears from the system. The initial endowment, $m_i(t+dt) = m_i(t) + \dot{m}_i(t)$, in outside money of household i at the beginning of time $t+dt$ will be the amount of outside money she was left at the end of t_γ (i.e., the difference between the right-hand side and the left-hand side of (9)) *plus* her dividend:

$$m_i(t) + \dot{m}_i(t) = m_i(t) + \tfrac{q_i^{bm}(p(t))}{1+r} - \textstyle\sum_{c=1}^{C} q_i^{mc}(p(t))$$
$$+ \sum_{c=1}^{C} p_c(t) q_i^{cm}(p(t)) - q_i^{bm}(p(t)) + \nu_i r(t) M(t).$$

A **strategy** s_i of player i in the game $G(T_{x,m,b,M}\mathcal{E})$ consists in sending an order to buy inside money in subperiod t_α and a limit-price order for each pair (k, k') of items in period t_β. Players have no memory, and cannot condition their current behavior the past. Let us denote by $\varphi_i(\mathbf{s})$ the final allocation received by player i whenever the strategy profile $\mathbf{s} := (s_h)_h$ is played. Having defined the "rules of the game", it remains to characterize the players' behavior. We shall consider only weakly dominant strategies.

Definition A **strategy-proof profile** is a strategy profile $\underline{\mathbf{s}}$ such that a.e. player i plays a weakly dominant strategy in the short-run game, taking $m_i(t)$ as his/her current initial endowment in outside money, i.e., for a.e. player $\tau \in [0,1]^N$ one has:

$$g_i(x_i) \cdot \varphi_i(\underline{\mathbf{s}}) \geq g_i(x_i) \cdot \varphi_i(\underline{\mathbf{s}}^{-i})$$

where $\underline{\mathbf{s}}^{-i}$ is the strategy profile obtained by replacing $\underline{\mathbf{s}}^{i}$ with some arbitrary strategy.

We are eventually ready to define the dynamics of the *Limit-Price exchange Process* (LPP). For every strategy profile $\mathbf{s}$, we denote by $\pi(\mathbf{s}) \in \mathbb{R}_{++}^{C+1}$ (resp. $\dot{x}(\mathbf{s})$) the set of short-term outcome prices (resp. trades) induced by $\mathbf{s}$ in $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$.

Definition. A **monetary strategy-proof trajectory** is a "solution" of the following differential inclusion:

$(x(0), m(0)) = (\omega, m(0))$ and

$$\dot{y}(t) = \varphi\Big(\mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]\Big) \text{ and } p(t) \in \Pi\Big(\mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]\Big). \tag{14}$$

where $\forall t \geq 0, \mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]$ is a strategy-proof profile of $G[T_{x,m,b,M}\mathcal{E}]$.

Here "solution" has to be understood as folllows. Let

$$\dot{x}(t) \in f(x(t)), \tag{15}$$

where $f : \mathbb{R}^m \hookrightarrow \mathbb{R}^m$ is a possibly discontinuous generalized vector field.

Definition (Filippov (1988))

A **Filippov solution** of (15), is an absolutely continuous trajectory $\phi : [a,b) \to \tau$ such that, for a.e. $t \in [a,b)$,

$$\dot{\phi}(t) \in G_F(\phi(t)) := \cap_{\varepsilon>0} \cap_{A \in \mathcal{N}} \overline{\text{co}}\big\{y \mid d(y, f(\phi(t)))) < \varepsilon, y \notin A\big\}. \tag{16}$$

where $\mathcal{N} :=$ family of sets $A \subset \mathbb{R}^m$ of (Lebesgue) measure zero.

3 Uniqueness and non-neutrality

3.1 Global nominal uniqueness

Definition. (Mertens (2003)) A market order to sell commodity k for commodity j is **inexecutable** if there exists a partition of $\mathbb{N}_L$ into $A \cup B$ such that $j \in A, k \in B$, and for every "agent" i,

(α) either $e^i_a = 0\ \forall a \in A$, ($\beta$) or $b^i_b = 0\ \forall b \in B$.

Definition (i) A linear economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e\big)$ is *"weakly reducible"* if there exists a partition $A \cup B = \mathbb{N}_L$ such that for each "agent" i, either $b^i_b = 0\ \forall b \in B$, or $e^i_a = 0\ \forall a \in A$, and there exists some triple (i_0, b, a) with $e^{i_0}_b > 0, b^{i_0}_b = 0$ and $b^{i_0}_a > 0$.

(ii) $\mathcal{L}$ is *weakly irreducible* if it is not weakly reducible, i.e., if it admits no inexecutable order.

We shall need the following, fairly weak assumption:[14]

Assumption (I) $\mathcal{E}$ is **dynamically weakly irreducible**, that is, for every $x \in \tau^*$, the short-term economy $T_{x,b,m,M}\mathcal{E}$ is weakly irreducible.

A *strict* trade $\dot{x}$ in some linear economy $\mathcal{L}$ is such that either $\dot{x}_i \geq 0$ or $b_i \cdot (\dot{x}_i - e_i) > 0$ for a.e. "agent" i. Mertens (2003) proves that every short-run outcome is Pareto optimal when optimality is checked only with respect to strict trades.

A feasible allocation (x, m) is infinitesimally Pareto-optimal if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$ and $\nabla u_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality. It is **infinitesimally optimal in strict trades** if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$, $\phi'(x)$ involves only strict trades in $T_{x,m,b,M}\mathcal{E}$, and $\nabla u_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality. We denote by $\overline{\theta}$ (resp. Θ) the set of such infinitesimally optimal allocations (resp. in strict trades). Clearly, $(x, m) \in \overline{\theta}$ (resp. Θ) iff (x, m) is Pareto-optimal (resp. Pareto-optimal when only strict trades are allowed) in $T_{x,m,b,M}\mathcal{E}$. Finally, θ is the relative interior of $\overline{\theta}$.

Finally, for a given $r \geq 0$, a trade $\dot{x}$ is **r-infinitesimally optimal in strict trades** if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$, $\phi'(x)$ involves only strict trades in $T_{x,b,m}\mathcal{E}$, and $\nabla \tilde{u}^r_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality, where $\tilde{u}^r_i$ is the "modified" utility function defined as follows.[15]

Let $z^i \in \mathbb{R}^C$ be a trade vector of i (with positive components representing purchases and negative ones representing sales). For any scalar $\gamma > -1$, define:

$$\tau^i_c(\gamma) := \ \min\big\{z^i_c, \frac{1}{1+\gamma} z^i_c\big\}. \tag{17}$$

$z^i_c(\gamma)$ entails a diminution of purchases in z^i by the fraction $\frac{1}{1+\gamma}$. The (continuous and concave) utility function $\tilde{u}^r_i(\cdot)$ is given by:

[14] See Giraud (2004) for a discussion of this assumption.

[15] See Dubey & Geanakoplos (2003a) for details.

$$\tilde{u}_i^r(x) = u_i(e^i + (x - e_i)(r)). \tag{18}$$

Needless to say, if $r = 0$, a feasible allocation x is r-infinitesimally optimal in strict trades if, and only if, it belongs to Θ. We denote by Θ_r the subset of r-infinitesimally optimal allocations in strict trades.

Of course, due to the transaction cost induced by r, the short-term outcome of a short-run economy need not be Pareto-optimal in the corresponding linear economy, even when optimality is checked with respect to strict allocations. (It would be so for sure if $M = +\infty$, i.e., $r = 0$.) Nevertheless, the next theorem says that the possibility of retrading in continuous time ensures that all r-gains to strict trades will be exhausted at the end of a monetary trade curve.

THEOREM 1.— *Under (C)(i), for $b > \overline{m} + M$, these three parameters being fixed,*

(i) every short-run economy $\mathcal{L} = T_{x,m,b,M}\mathcal{E}$ admits a unique short-term outcome $(\dot{x}, \dot{m}, p, r)$.

(ii) Moreover, $r = \frac{\overline{m}}{M}$. Except when $x \in \Theta_r$, the corresponding short-term price $\pi = (p, r)$ is unique, i.e. $P(\mathcal{L})$ reduces to a singleton.

(ii) Monetary trade curves exist.

(iii) Provided $M, m > 0$ and, if in addition, (C)(ii) and (I) hold, all monetary trade curves converge to $\overline{\theta}_r$.

Proof. (i) and (ii) If b is sufficiently large (e.g., $b(t) > \overline{m}(t) + M(t)$), we can be sure that the feasibility constraint on the market for bonds will never be binding: Indeed, each player i must return $q_b^i(t)$ to the Bank in stage t_γ, and there is at most $\overline{m}(t) + M(t)$ units of money in the economy. Since players are negligible and play a dominant strategy, there is no loss of generality in assuming that, at the end of the third stage t_γ, after repaying the Bank, no player will be left with unowed cash. Otherwise, she should have spent more money earlier in order to purchase commodities, or else curtailed her sale of commodities, improving her (strictly monotone) short-run utility. Hence, exactly $\overline{m}(t) + M(t)$ is owed to the Bank, so that $(1 + r(t))M(t) = M(t) + \overline{m}(t)$, i.e., $r(t) = \frac{\overline{m}(t)}{M(t)}$. As a consequence, exactly $r(t)M(t) = \overline{m}(t) = \overline{m}$ is redistributed to the households at the end of time t, so that i will start at time $t + dt$ with $\nu_i \overline{m}$ units of outside money in her pocket. From now on, we therefore consider the quantity $m_i(t)$ of outside money held by household i as a constant (independent from t for every $t > 0$).

On the other hand, even if they are informed of the r-manipulation operated by the market-makers on their commodity-sell orders, negligible players have no interest to manipulate their preferences. Manipulating their announcements has no effect on the emerging sort-run price p, while the very definition of a short-term outcome implies that the induced outcome will solve:

$$\text{Max } \nabla u^i(x_i) \cdot \dot{x}^i(r),$$

under the constraints: $p_c = 0 \Rightarrow \dot{x}^i = 0$ and $p \cdot \dot{x} \leq 0$ and $\dot{x} \geq -(x_i, m_i)$, on the economy restricted to the commodities belonging to the support of p.

Thus, when analyzing a local game $G[T_{x,m,M,b}\mathcal{E}]$, one can restrict attention to the r-monetary linear economy $T_{x,m,r}\mathcal{E}$ obtained from $T_{x,m,M,b}\mathcal{E}$ by the same transformation as the one used to go from (11) to (12), and with $r := \frac{\overline{m}}{M}$.

Let us denote by $\big(\pi(T_{x,m,r}\mathcal{E}), \dot{x}(T_{x,m,r}\mathcal{E})\big)$ the unique short-term outcome associated to $T_{x,m,b,M}\mathcal{E})$. Existence and (ii) now follow from the existence and uniqueness results of "optimal allocations" (called short-term outcomes here) obtained in Mertens (2003) for general linear economies. There, uniqueness in price is understood up to a normalization. Here, as we impose that money's price be equal to 1, prices are automatically normalized (hence unique in nominal terms).

(iii) Existence of monetary trade curves then follows from standard arguments (see Giraud (2004) for details): We denote by $V : \tau \times T\tau \times \mathbb{R}^{C+1}$ the cone field associating to each state the set of infinitesimal trades in commodities and money $(\dot{x}, \dot{m})$ induced by our dynamics. Except on the Θ_r, this cone field reduces to a vector field. As is clear from Giraud (2004a), this vector field is discontinuous in general. Filippov (1988) then ensures that the differential equation with discontinuous right-hand associated to this vector field can be translated into a differential inclusion that is upper semi-continuous, non-empty-,convex- and compact-valued. Existence of monetary trade curves then follows from standard existence results for differential inclusions, see, e.g., Aubin & Cellina (1984) — see Giraud (2004, Theorem 4.1.1.).

(iv) We first remark that, under (C)(ii), every individually rational trade $\dot{x}$ in every short-run monetary economy $T_{x,b,m,M}\mathcal{E}$ at some state such that $x \in \tau^*$, must be strict. Indeed, $\nabla u_i(x_i) \cdot \dot{x}_i \geq \nabla u_i(x_i) \cdot x_i > 0$. Thus, Θ_r reduces to θ_r. The rest of the proof follows the standard Lyapounov argument, see Giraud (2004, Theorem 4.2.1).

□

The next result states that, given aggregate initial endowments $(\overline{\omega}, \overline{m}, M)$, and for a dense subclass of monetary economies of particular interest, namely finitely subanalytic (see Giraud (2004a,b)) economies, the vector field associated to our dynamics is smooth on an open and dense subset of the feasible set. Thanks to the Cauchy-Lipschitz theory of smooth differential equations, this implies that, when restricted to this generic subset, the Cauchy problem induced by our dynamics admits a (piecewise) unique solution path not only in real, but also in nominal terms.

PROPOSITION 1.— *For any finitely subanalytic economy $\mathcal{E}$ satisfying (C)(i) and (ii), then, for every fixed r, the feasible set τ can be partitioned as:*

$$\tau = \mathcal{R} \cup \mathcal{C}$$

where both $\mathcal{R}$ and $\mathcal{C}$ are finitely subanalytic subsets, the latter being closed, of dimension strictly less than $CN - C =$dim$\mathcal{R}$, and containing $\overline{\theta}_r$. Moreover, the restriction of V to the (open and dense) subset $\mathcal{R}$ is a real-analytic, hence smooth, vector field.

Proof. We first need to prove that θ_r is of measure zero in τ. It follows from the standard argument involving the strict quasi-concavity of preferences[16] and from Dubey & Geanakoplos (2003a, Lemma 2) that a point $x \in \tau_r$ belongs to θ_r iff it is

[16]See Giraud (2004, Lemma 3.1.1.).

Pareto-optimal (in the usual sense) for the auxiliary economy $\mathcal{E}_r$ defined as follows. Each household i's utility u_i is changed into $\tilde{u}_i^r$ as defined above by (18) (with the help of (17)). Since u_i is strictly quasi-concave and increasing, so is $\tilde{u}_i^r$. Thus, the set of Pareto points in $\mathcal{E}_r$ is homeomorphic to the $(N-1)$ unit simplex (cf. Mas-Colell (1985), Prop. 4.6.2, p. 155). So is therefore θ_r. As a consequence, it is negligible in τ.

Thus, we can perturb our generalized vector field in a way analogous to the one followed in Giraud (2004, Theorem 4.3.1) in order to be able to apply Filippov's theory. From there, the proof follows *verbatim* Theorem 4.3.1 in Giraud (2004).

□

Mathematically, the proof looks the same as in Giraud (2004a). From an economic viewpoint, there is a big difference however in the way prices have been normalized. In Giraud (2004a), prices are normalized *a priori* in the unit simplex, because the whole real dynamics is homogeneous of degree zero with respect to prices. Here, prices are endogenously normalized by equation (2). Of course, this generic global uniqueness result has to be contrasted with the generic local uniqueness of monetary equilibria obtained in Dubey & Geanakoplos (2003a, Theorem 3).

The set $\mathcal{C}$ of critical economies being finitely subanalytic, it is the finite, disjoint union of smooth submanifolds, all of them of dimension less than $CN - C$. The picture that can be derived from the previous theorem is therefore the following: τ can be partitioned into finitely many open, disjoint subsets, separated by smooth submanifolds, such that the union of these open subsets ($=\mathcal{R}$) is dense in the feasible set, and the restriction of our vector field to each open subset is smooth.

3.2 Long-run non-neutrality of money

Is money neutral in our model ? It is clear that if *both* m and M are multiplied by some constant $\lambda > 0$, then nothing changes in the analysis. This means that there is no money illusion. However, if either m or M is changed separately, then there will be, in general, a change in the long-run real variables characterizing the monetary trade curves of the economy. We show in this subsection how to characterize the short-run and long-run impact of such a monetary change on the real sector.[17] Unless otherwise specified, and in order to avoid trivialities, we assume throughout from now on that $\overline{m} > 0$.

Let us start by stressing that a short-term outcome of the short-run economy $T_{x,m,b,M}\mathcal{E}$ does not coincide, in general, with a monetary equilibrium (with rational expectations) in the sense given to this word by Dubey & Geanakoplos (2003a). For this purpose, we first recall the authors' definition. Given prices (p, r), the (competitive) budget set $B(p, r, x_i(t), m_i)$ of type i is the set of all market orders and final allocations $(q_i, \dot{x}_i(t)) \in \mathbb{R}^{2C+1} \times \mathbb{R}_+^C$ that satisfy the constraints below for all $c \in C$ and all $t \geq 0$:

$$\sum_{c=1}^{C} q_i^{mc} \leq m_i + \frac{q_i^{bm}}{1+r} \tag{19}$$

[17]The quantitative comparative dynamics of the impact of money in the long-run can be done, in principle, since our whole dynamics is computable (see Giraud (2004a). However, this point is left for further research.

$$q_i^{cm} \leq x_i^c(t) \tag{20}$$

$$q_i^{bm} \leq m_i + \frac{q_i^{bm}}{1+r} - \sum_{c=1}^{C} q_i^{mc} + \sum_{c=1}^{C} p_c q_i^{cm} \tag{21}$$

$$\dot{x}_i^c(t) \leq q_i^{cm} - x_i^c(t) + \frac{q_i^{mc}}{p_c} \tag{22}$$

Agents have zero endowment in bonds, and therefore no feasibility constraint is put on bond-selling market-orders. This is at variance with Mertens' (2003) non-monetary mechanism, but in accordance with the traditional modelling of financial assets in perfectly competitive general equilibrium theory and with, e.g., Peck & Shell (1991). Consequently, there is only a no-default constraint (21). This (infinitesimal budget) constraint can be written in a more compact way:

$$q_i^{bm} \leq \Delta(22) + \sum_c p_c q_i^{cm},$$

where $\Delta(22)$ is the difference between the right-and the left-hand side of (19). Due to this constraint, we do not end up with a full-blown game, on account of the fact that no player can default. But, as already remarked by Dubey & Geanakoplos (2003a, footnote 14), this is not a real issue. By adding sufficiently harsh default penalties, one could get a classical game, and still guarantee that, at least at (strategic) equilibrium, nobody goes bankrupt.

A vector $(p(t), r(t), (q_i, \dot{x}_i(t))_i) \in \mathbb{R}_{++}^C \times \mathbb{R}_+ \times (\mathbb{R}_+^{2C+1} \times \mathbb{R}_+^C)^I$ is a **monetary equilibrium** (in the sense of Dubey & Geanakoplos (2003a) adapted to our linear/short-run setting) of $T_{x(t),m(t),b,M(t)}\mathcal{E}$ if all agents' market orders are in their competitive budget sets:

$$(q_i, x_i(t)) \in B(p(t), r(t), x_i(t), m_i), \tag{23}$$

demand equals supply for the loan market and for all good markets:

$$\sum_i \frac{q_i^{bm}}{1+r(t)} = M(t) \tag{24}$$

$$\sum_i \frac{q_i^{mc}}{p_c(t)} = \sum_i q_i^{cm}, \quad c \in C \tag{25}$$

and each agent optimizes

$$v_i(\dot{x}_i(t)) \geq v_i(\underline{\dot{x}}_i(t)) \text{ for all } (\underline{q}_i, \underline{\dot{x}}_i(t)) \in B(p(t), r(t), x_i(t), m_i). \tag{26}$$

In order to compare short-term outcomes with monetary equilibria in this sense, we need to recall the definition of the measure of *gains to trade* $\gamma(x)$, as introduced by Dubey & Geanakoplos (1992). For any $\gamma \geq 0$, we say that $x = (x_i)_i \in \left(\mathbb{R}^C_{++}\right)^N$ is *not γ-Pareto-optimal* if there exist feasible trades $z = (z_i)_i \in \left(\mathbb{R}^C\right)^N$ such that $\sum_i z_i = 0; x_i + z_i >> 0$ and $u_i(x_i[\gamma z_i]) > u_i(x_i)$ for all i,[18] where $x_i^c[\gamma z_i] := x_i^c + \min\left\{z_i^c, \frac{z_i^c}{1+\gamma}\right\}$ for every $c = 1, ..., C$.

Thus, the feasible trades contemplated to γ-Pareto improve x involve a tax of $\frac{\gamma}{1+\gamma}$ on trades. If, at the allocation x, we can find some price vector $p \in \mathbb{R}^C_+$ such that

$$p \cdot z_i \leq 0 \Rightarrow u_i(x_i[\gamma z_i]) \leq u_i(x_i) \ \forall i,$$

then, x is γ-Pareto optimal. Of course, 0-Pareto optimality coincides with the standard notion of Pareto optimality. Finally, the gains to trade $\gamma(x)$ at $x \in \tau$ is defined as the supremum of all γ for which x is not γ-Pareto optimal.

Proposition 2.— *(i) Under (C), for b sufficiently large, if $t \mapsto M(t)$ grows sufficiently rapidly, so that*

$$M(t) > \frac{\overline{m}}{\gamma(x(t))}, \tag{27}$$

for every t, then every short-run outcome is a monetary equilibrium of the corresponding short-run economy. Provided that each u_i is strictly quasi-concave, every monetary trade curve converges to some point $x^* \in \Theta$.

(ii) On the contrary, if at some point x, $\gamma(x) < \frac{\overline{m}}{M}$, then the short-run outcome of $T_{x,m,b,M}\mathcal{E}$ coincides with no-trade, and x is a rest-point of the dynamics.

Proof.

(i) In view of Theorem 1 and Proposition 2, it suffices to show that, if

$$M(t) > \frac{\overline{m}}{\gamma(x(t))},$$

then $T_{x(t),m(t),b,M(t)}\mathcal{E}$ admits a monetary equilibrium which is also its unique short-run outcome. For this purpose, observe that all the assumptions of Theorem 2 in Dubey & Geanakoplos (2003a) are verified by $T_{x(t),m(t),b,M(t)}\mathcal{E}$. Take therefore a monetary equilibrium of $T_{x(t),m(t),b,M(t)}\mathcal{E}$. By Lemma 1 of Dubey & Geanakoplos (2003a), it is such that every agent is maximizing her short-run utility over the non-linear budget set induced by the condition (see also Figure 1 below):

[18] Since preferences are strictly increasing, there is no need to distinguish weak from strict γ-Pareto optimality.

$$C(r, \dot{x}_i - e_i) \leq m_i.$$

Equivalently, i is maximizing her modified short-run utility $\tilde{v}_i^r$ over her "true" competitive contraint $p \cdot \dot{x} \leq p \cdot 0$. Thus, the unique property of a the short-term outcome that this ME could fail to verify is proportionality. But the proportionality rule comes into play only if two orders are sent to the market with the same limit-price, i.e., when two players of the sort-run economy $T_{x,m,b,M}\mathcal{E}$ have marginal rates of substitution that converge to each other. In this last case, there is a continuum of possible infinitesimal trades that are compatible with the definition of a ME, and the proportionality rule just chooses one of them. In all the other cases (i.e., for the "heterogenous" "players"), the proportionality rule does not come into play, so that the monetary equilibrium is actually proportional, hence is the short-run outcome.

The rest follows from the general results obtained by Dubey & Geanakoplos (2003a) on ME (applied to each linear short-run economy). Thus, as time grows, $r(t)$ must decline as rapidly as $\gamma(x(t))$, and converge to 0 (as $x(t)$ converges to Θ).

(ii) Now, if $\gamma(x) = \gamma(e) < \frac{\overline{m}}{M} = r$, then the linear short-run economy at x admits *no* monetary equilibrium. Indeed, according to Theorem 6 in Dubey & Geanakoplos (2003a), if

$$\frac{\overline{m}}{M} > \Gamma(x),$$

then no monetary equilibrium exists — where $\Gamma(x) := \sup_y \gamma(y)$, the supremum being taken over all the y that are feasible and individually rational with respect to x. But, in a short-term *linear* economy, $\Gamma(e) = \gamma(e)$. Hence, if $\gamma(x) = \gamma(e) < \frac{\overline{m}}{M} = r$, then obviously our short-run outcome cannot coincide with any monetary equilibrium, because the later fails to exist. This implies that the short-run price must arise from a combination of several pseudo-equilibrium prices. Suppose that the short-run outcome does not reduce to no-trade. In view of (i) in this proof, there must be some non-trivial partition of the set of commodities and outside money, $\{1, ..., C+1\} = \cup_k C_k$, such that, when restricted to the economy with C_k as commodities, the short-run price is a monetary equilibrium price. Let us denote by $T_{x,b,m,M}\mathcal{E}_k$ the linear economy obtained by restricting oneself to the commodities in C_k, and let $x_i \in \left(\mathbb{R}^k\right)^N$ be the corresponding truncated allocation. Obviously,

$$\gamma(e_k) = \gamma(x_k) < \gamma(x) = \gamma(e),$$

so that $T_{x,b,m,M}\mathcal{E}_k$ also fails to admit a monetary equilibrium. Thus the unique possibility is that our short-run outcome induces no-trade.

□

When a Pareto-point is reached, there are no more gains to trade, hence money becomes a veil. Interest rate is then zero. If there is not enough inside money along a monetary trade curve, then the economy still converges to some rest-point (by compactness of the feasible set) but not necessarily to some infinitesimal Pareto-point. According to Theorem 1, it converges to some point x^* in Θ_r. This means that, in order to go away from x^*, every Pareto-improving path $\phi : [a, b) \to \tau$ with $\phi(a) = x^*$ in strict trades requires more inside money than there actually is. Thus, the economy can remain stuck at some inefficient state due to the lack of liquidity. Of course, such monetary trade curves are incomplete in the sense of Smale (1976).

To summarize, our theory provides a minimal growth rate of inside money for the dynamic analogue of the First Welfare Theorem to hold true. This minimal growth is given by (27), and can be rewritten as:

$$\dot{M}(t) > \frac{\overline{m}}{\gamma(x(t))} - M(t), \quad \forall t$$

or, equivalently,

$$M(t) > \frac{\overline{m}}{\gamma(\omega)} e^{\int_0^t \frac{r(s)-\gamma(x(s))}{\gamma(x(s))} ds} \quad \forall t$$

On the other hand, when the economy stops at some point in $x^* \in \Theta_r$, then it falls in a **liquidity trap**. Indeed, a small change in r will not suffice to move the economy out of x^* provided the change is sufficiently small so as to still verify:

$$\gamma(x^*) < r. \tag{28}$$

In this case, a small monetary change has no real effect because real trades still collapse, and the state of the economy remains constant.

Most textbooks devoted to monetary theories with rational expectations conclude that money is non-neutral in the short-run, but neutral in the long-run. Here, we get the opposite conclusion. This paradox can be explained as follows: in the short-run, if $\gamma(x) \neq r$, a sufficiently small change in r will not affect the direction in which the state of the long-run economy moves. Indeed, either $\gamma(x) < r$, in which case there is no trade; or $\gamma(x) > r$, in which case the long-run economy still moves in the direction of a Walrasian equilibrium of its linear short-run approximation. In this narrow sense, money can be said to be **locally neutral in the short-run** — “locally” because the preceding argument holds only for “small” changes in the monetary variables. Observe, nevertheless, that if $\gamma(x) = r$ (a non-generic event), then the slightest change of r will have a real effect, *even in the short-run.*

Now, in the long-run, different amounts of inside or outside money will induce different trade curves in real terms. Indeed, if r is fixed, then the trajectory followed by the long-run economy will stop at some point $x \in \tau$ where $\gamma(x) = r$. If $r \neq r'$, then $x \neq x'$. As a consequence, money is non-neutral in the long-run.

3.3 An example

A simple example will clarify the picture. In order to facilitate comparisons, we adopt a linearized version of Dubey & Geanakoplos (2003a)'s example (section 6). Suppose $N = C = 2, e^1 = e^2 = (50, 50), m^1 = m^2 = 5, M = 90$, and $v^1(\dot{x}_1^1, \dot{x}_2^1) = \frac{10}{75}\dot{x}_1^1 + \frac{3}{25}\dot{x}_2^1, v^2(\dot{x}_1^2, \dot{x}_2^2) = \frac{3}{25}\dot{x}_1^2 + \frac{10}{75}\dot{x}_2^2$. In this (very exceptional) situation, at the unique short-run outcome, household 2 sells part of its endowment of commodity 1, and buys commodity 2; household 1 sells part of its endowment of commodity 2 and buys 1; both agents borrow money from the central Bank. In short-run outcome, $p_1 = p_2 = 2, p_m = 1, r = \frac{1}{9}, \dot{x}^1 + e^1 = (75, 25), \dot{x}^2 + e^2 = (25, 75)$. Agent 1 spends his €5 and buys 2.5 units of good 1. She also borrows €; 45 from the Bank, promising to repay €; 50. This loan is spent to buy 22.5 additional units of good 1. Finally, agent 1 sells 25 units of good 2 to agent 2, and is able to repay the Bank. Traders' final gradients are not parallel, because:

$$\frac{\frac{\partial v^1}{\partial x_1^1}}{p_1} = (1+r)\frac{\frac{\partial v^1}{\partial x_2^1}}{p_2}$$

$$\frac{\frac{\partial v^2}{\partial x_2^2}}{p_2} = (1+r)\frac{\frac{\partial v^2}{\partial x_1^2}}{p_1}.$$

This misalignment confirms that a short-run outcome may fail to be Pareto-optimal in the short-run economy, and is clearly due to the transaction cost r.

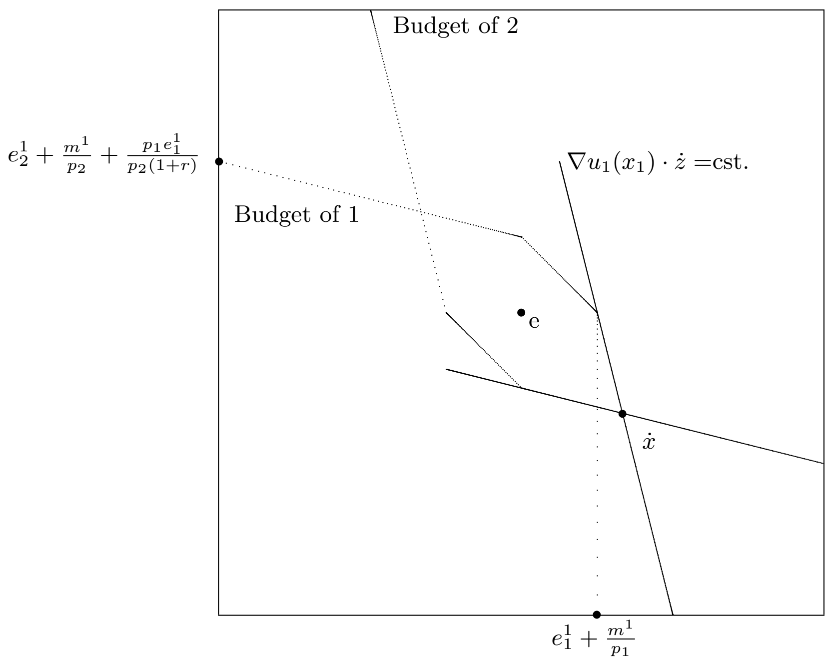

Figure 1. An interior short-run monetary equilibrium

What happens if, everything else being kept fixed, M decreases ? Then r increases above $\gamma(x) = \gamma(e) = \frac{1}{9}$, and the unique short-run outcome is no-trade. On the contrary, when M increases, r decreases below $\gamma(e)$. If the resulting short-term outcome $\dot{x}$ was still an interior monetary equilibrium, then we should have $0 < \gamma(\dot{x}) \leq r < \frac{1}{9}$ (Theorem 4 in Dubey & Geanakoplos (2003a)). But due to the linearity of short-term preferences, $\gamma(\dot{x})$ is constant in the interior of the Edgeworth box. Hence, the short-term outcome must lie on the boundary $\partial\tau$. Thus, the unique short-term outcome then coincides with the unique Walras equilibrium of this economy (which is also the unique short-term outcome of this linear economy when there is no money at all): $\dot{x}_1 = (100, 0)$ and $\dot{x}_2 = (0, 100)$. How do prices evolve as M increases ? For a given $r < \frac{1}{9}$, one gets:

$$p_1 = p_2 = \frac{1+r}{10r}. \tag{29}$$

Therefore, as soon as there is enough grease that turns the wheels of commerce, i.e., as soon as $M > \frac{\gamma(e)}{m}$, then the "classical dichotomy" holds in the short-run: an increase of inside money just increases prices proportionally and decreases the interest rate without affecting real trades. Notice that, in this example, when M increases from $\frac{\gamma(e)}{\overline{m}} = \frac{1}{90}$ to any higher value, the resulting direction in which the economy moves is the same (namely the left-bottom angle of the Edgeworth box). Only the speed at which the economy moves is modified: the state moves more slowly when $M = \frac{\gamma(e)}{\overline{m}}$, than when $M > \frac{\gamma(e)}{\overline{m}}$. Thus, above a certain threshold, an increase of inside money has no impact but nominal inflation.

Suppose, now, that M is fixed. What happens as m varies (proportionally for each household) ? As long as $m > M\gamma(e)$, no-trade is the unique outcome and prices are indeterminate. Whenever $m = M\gamma(e)$, the economy starts moving; it is actually driven by the unique interior monetary equilibrium of its short-run economy. When m further decreases, the economy moves slightly more rapidly in the same direction, r decreases and prices decrease as well. As $m \to 0^+$, the interest rate r goes to zero, and at the limit, the unique short-term outcome converges again to the unique Walras equilibrium of the short-run economy with prices equal to $p = (\frac{1}{10}, \frac{1}{10})$.

Thus, one can summarize the short-run effects of (i) monetary policy (M varies) and (ii) non-discriminatory fiscal policy (m varies proportionally for each household) by means of the following two diagrams:

Figure 2. m fixed

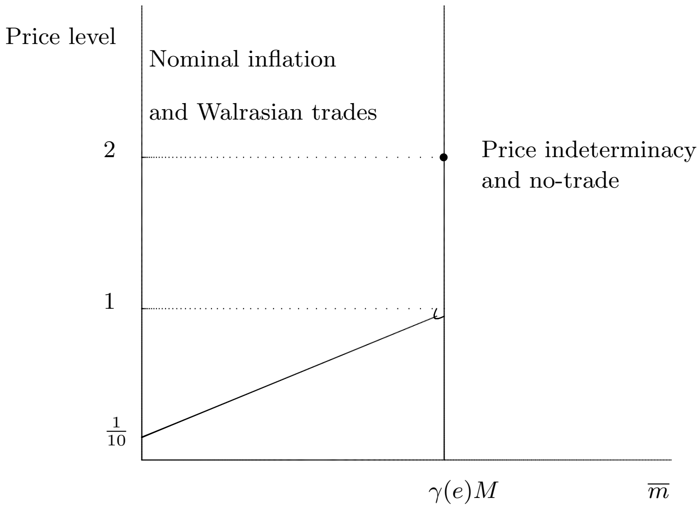

Figure 3. M fixed

When compared with the Figure 6 in Dubey & Geanakoplos (2003a), notice that, here, there is no "hyperinflation phenomenon": as M decreases to $\frac{\overline{m}}{\gamma(e)}$ ($\overline{m}$ being fixed), prices converge to 1. At the moment where $M = \frac{\overline{m}}{\gamma(e)}$, prices jump to 2. Similarly, when M is fixed, as $\overline{m}$ increases towards $\gamma(e)M$, prices converge, and then jump.[19] Looking now at the dynamic picture, one sees that the trade curve followed by our long-run economy depends upon the quantity of circulating money in the following way:

a) either $\overline{m} > 0$ and there is enough inside money throughout, in which case the economy follows a unique trade curve ϕ (which coincides with the non-monetary "Walrasian" trade curve as studied in Giraud (2004a)); in particular, it converges to some Pareto-optimal point, $r(t) \to 0^+$ and prices remain bounded ;

(b) or $\overline{m} = 0$, in which case, whatever being the amount $M(t) > 0$ of inside money, the economy follows the same trade curve ϕ;

(c) or $\overline{m} > 0$ and, at some point t, there is not enough inside money, i.e., $0 < M(t) < \frac{\overline{m}}{\gamma(x)}$, in which case the economy stops at x (even though $x \notin \Theta$), with $r(t) > 0$.

4 Concluding Remarks

In order to focus on the essentials, we restricted ourselves to a finite-dimensional economy populated by finitely many types of agents. Using the same trick as in Giraud (2004a), one could partially drop this restriction by assuming that there are only finitely many types of preferences but that the endowment map $i \mapsto \omega_i$ can be any integrable map. Apart from this, a couple of issues arise:

A. A quantitative analysis of the long-run impact of money will be performed later, taking advantage of the global nominal uniqueness of trade curves in our dynamics, and of the fact that this dynamics is computable (see Giraud (2004a)).

[19]Of course, our linear economy can be approximated by a strictly concave one by replacing each linear short-run preference v_i by $v_i + \varepsilon \sum_c \sqrt{x^i_c}$. One then sees that our diagrams are degenerate limits of figures 6 and 7 of Dubey & Geanakoplos (2003a).

B. Everyday experience on the interbank market suggests that (at least in Europe) this is usually a highly imperfectly competitive market, where a few "big" atomic players interact strategically. Thus, this first study calls for an analogous analysis within an imperfectly framework. This implies studying Mertens' limit-price mechanism with finitely many players. A first step in this direction has been made by Weyers (2003).
C. The presence of outside money in our model, although it plays a crucial role in order to get (global) uniqueness results, remains questionable from an economic point of view. We plan to explore in a subsequent work the impact of allowing for a certain amount of default along trade curves on the determinacy of such curves, taking inspiration from Tsomocos (2003). Default, indeed, is known to be able to play a role analogous to outside money in the analysis of money in a general equilibrium setting. Default and different lending and deposit rates as in Goodhart, Sunirand & Tsomocos (2003) allow for analyzing credit spreads.
D. In a companion paper, Giraud & Rochon (2004) have extended the basic framework underlying the present work to economies with (possibly non-convex) production. One difficulty that was not overcomed there was to model production sectors with fixed costs. An artifact of the methodology developed there is indeed that such production sectors never produce. By combining production and money, we could allow firms with fixed costs to borrow money in order to be able to survive during the production period involving fixed costs (as they actually do in "real life"). Such a promising "wedding" would also enable to explore a dynamic version of IS-LM within our general equilibrium set-up. Finally, since our monetary setting enables to endogenously normalize prices (or, equivalently, to endogenously fix the nominal level of prices), it should be instrumental in order to solve the normalization problem when defining a firm's objective.

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