Equilibrium Analysis of Dynamic Economies

Preliminary Version

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Abstract

We study equilibrium properties of discrete-time, infinite-horizon smooth economies whose equilibria can be represented as a sequence of interrelated equations, and provide sufficient conditions to guarantee existence, (generic) determinacy and (generic) robustness of equilibria. Our approach exploits the sequential nature of a dynamic economy. An equilibrium is represented by a sequence of endogenous variables which solve a sequence of smooth systems of equations. For every point in time, standard smooth analysis techniques are used. The task then is to link periods, in order to obtain properties of equilibria. This link is represented by a correspondence, whose parallel in the macro/capital theoretic literature is the law of motion for state variables. Regularity conditions for the systems of equations and continuity properties of the law of motion drive most of the results in the paper.

Keywords: Equilibrium correspondence, regular economy, locally unique equilibria, robust equilibria.

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1 Introduction

We consider infinite-horizon, discrete-time economies whose equilibria can be represented as a sequence of interrelated smooth equations, and study their equilibrium properties. Standard models in macroeconomics and capital theory usually have a sequential representation. A large class of such models consists of economies where the agents face an intertemporal problem that can be solved recursively, using dynamic programming tools (see [9]). The results found in this paper can be used, in particular, to analyze the equilibrium properties of recursive smooth economies.

The equilibrium properties analyzed in this paper are existence, determinacy and robustness.

An equilibrium is represented by a sequence of endogenous variables that solve a sequence of smooth systems of equations. Showing existence for each of the systems of equations and connecting the endogenous variables from period to period suffices to guarantee the existence of equilibria. We provide two sets of sufficient conditions, one based on applying degree theory, the other homotopy arguments.

Determinacy is studied in terms of two properties of equilibria, regularity and local uniqueness. An economy is said to be regular if the associated equilibria satisfy a rank condition in every period. The parallel of this definition in macro/capital theory is the smoothness of the policy function. References on this topic can be found in [1] and [7]. Local uniqueness has the typical interpretation given in GE. To obtain determinacy of equilibria we follow two steps. First, we make use of standard smooth analysis techniques (see [3] and [10]) at every point in time. Second, we link subsequent periods with a correspondence which plays the role of the law of motion for state variables in macro/capital theory. Rank conditions for the systems of equations in every period and continuity of the law of motion give us generic determinacy. We find that, for a generic set of economies, any economy is regular and the equilibria associated are locally unique. Moreover, for a given economy, at any point in time, the endogenous variable associated with that period can only take a finite number of values. In terms of the law of motion correspondence, it will consist of a finite collection of implicit smooth functions.

Finally, we study the behavior of equilibria after small perturbations of the sequence of systems of equations. We show that for a generic set of economies, equilibria are robust. In other words, for a given economy, we can perturb the systems of equations and still get the same number of equilibria and the same number of possible values for endogenous variables in every period.

The rest of the paper is organized as follows. Section 2 deals with preliminaries and notation. Section 3 presents the properties of the model for a particular point in time. Existence, determinacy, and robustness are studied in sections 4, 5, and 6, respectively. Section 7 applies the results to the smooth version of the neoclassical optimal growth model. An appendix contains some definitions and theorems.

2 Preliminaries and Notation

Let $t \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ denote the time index. For any t, consider the open sets $Y_t \subseteq \mathbb{R}^{s_t}$, $X_t \subseteq \mathbb{R}^{n_t}$, and $\Lambda \subseteq \mathbb{R}^m$, with typical elements y_t, x_t, and λ respectively. Define the sets $\Xi_t \equiv (Y_t \times X_{t+1})$ and $\Theta_t \equiv (X_t \times \Lambda)$ and the C^1 mapping $\Phi_t : (\Xi_t \times \Theta_t) \to \mathbb{R}^{(s_t+n_{t+1})}$ s.t. $(\xi_t, \theta_t) \mapsto \Phi_t(\xi_t, \theta_t)$.

For any point in time t, the vector θ_t contains exogenous variables while ξ_t endogenous ones. Given $\theta_t \in \Theta_t$, we want to solve $\Phi_t(\xi_t, \theta_t) = 0$ for the endogenous variable ξ_t. Starting at $t = 0$, given θ_0, we find (ξ_0, θ_0) such that $\Phi_0(\xi_0, \theta_0) = 0$. From (ξ_0, θ_0) we can obtain θ_1. For $t = 1$, considering the possible values found for θ_1 as exogenous, we solve $\Phi_1(\xi_1, \theta_1) = 0$ in the endogenous variable ξ_1. Applying the same logic for $t > 1$, we can construct sequences of endogenous variables $z \equiv \{\xi_t\}_{t\in\mathbb{N}_0} \in Z \equiv \times_{t\in\mathbb{N}_0}(\Xi_t)$ that satisfy the sequence of systems of smooth equations $\{\Phi_t(\xi_t, \theta_t) = 0\}_{t\in\mathbb{N}_0}$. Note that the only parameter for the economy is the exogenous vector θ_0. This analysis leads to the following definition.

Definition 2.1 (Equilibrium Correspondence). The equilibrium correspondence $\mathcal{E}$ is defined as $\mathcal{E} : \Theta_0 \rightrightarrows \mathcal{P}(Z)$ s.t. $\theta_0 \mapsto \{z \equiv \{\xi_t\}_{t\in\mathbb{N}_0} \in Z : \forall t \in \mathbb{N}_0 : \theta_t \in \Theta_t$, $\xi_t \in \Xi_t$ and $\Phi_t(\xi_t, \theta_t) = 0\}$. Any vector $\theta_0 \in \Theta_0$ is referred to as an economy, and any sequence $z \in \mathcal{E}(\theta_0)$ as an equilibrium.

The recursive representation of the model is clear from the definition of $\mathcal{E}$. At every point in time, ξ_t can be obtained by solving the smooth system $\Phi_t(\xi_t, \theta_t) = 0$, depending only on the vector of exogenous variables θ_t. This approach captures, in particular, models where the optimization problems of some of the agents can be solved using dynamic programming tools. In this context, we can think of $y_t, x_{t+1}, x_t, \lambda$ as being the vectors of control variables, endogenous state variables, exogenous state variables, and parameters respectively.

Define the set $M_t \equiv \{(\xi_t, \theta_t) \in (\Xi_t \times \Theta_t) : \Phi_t(\xi_t, \theta_t) = 0\}$ and the three natural projections (i) $\pi_t : M_t \to \Theta_t \; s.t. \; (\xi_t, \theta_t) \mapsto \theta_t$, (ii) $\rho_t : M_t \to \Theta_{t+1} \; s.t. \; (\xi_t, \theta_t) \mapsto \theta_{t+1}$, and (iii) $\omega_t : M_t \to \Xi_t \; s.t. \; (\xi_t, \theta_t) \mapsto \xi_t$. The mapping π_t is widely used in general equilibrium analysis. For any given vector $\tilde{\theta}_t$, the set $(\pi_t)^{-1}(\tilde{\theta}_t)$ consists of all the pairs $(\xi_t, \tilde{\theta}_t) \in M_t$. The second projection, ρ_t, turns out to be crucial in our analysis as well. For a given $\tilde{\theta}_t$, the set $(\rho_t \circ (\pi_t)^{-1})(\tilde{\theta}_t)$ contains all the possible values for θ_{t+1} that come after $\tilde{\theta}_t$, having satisfied $\Phi_t(\xi_t, \tilde{\theta}_t) = 0$. The correspondence $\rho_t \circ (\pi_t)^{-1}$ represents the **law of motion** for the exogenous variables in the model. The third projection, ω_t, takes a point in the solution set M_t and maps it into the endogenous variable ξ_t. For a given $\tilde{\theta}_t$, the set $(\omega_t \circ (\pi_t)^{-1})(\tilde{\theta}_t)$ gives all the possible values for ξ_t that solve $\Phi_t(\xi_t, \tilde{\theta}_t) = 0$.

As mentioned before, characteristics of M_t, π_t and ρ_t will determine the properties of the equilibrium correspondence $\mathcal{E}$.

3 Foundations of the Analysis: Properties of Solutions at a Point in time

In this section, we will study the properties of the set of solutions for a particular point in time, $M_t \equiv \{(\xi_t, \theta_t) \in (\Xi_t \times \Theta_t) : \Phi_t(\xi_t, \theta_t) = 0\}$. The results found are standard properties of finite-dimensional models. References for smooth analysis of finite economies can be found in [3] and [10].

Note that we will make extensive use of the definitions of manifolds, critical points and critical values of a mapping (see appendix). For any given mapping f, denote by $CP(f)$ and $C(f)$ the sets of critical points and critical values of f.

3.1 Existence at t

Proposition 3.1. *If* $0 \in R(\Phi_t)$, *then (i)* M_t *is an* (n_t+m)*-dimensional* C^1 *manifold, (ii)* $\tilde{\theta}_t \in R(\pi_t)$ *if and only if* $\forall(\tilde{\xi}_t, \tilde{\theta}_t) \in (\pi_t)^{-1}(\tilde{\theta}_t) : Rank[D_{\xi_t}\Phi_t(\tilde{\xi}_t, \tilde{\theta}_1] = (s_t+n_{t+1})$, *and (iii)* $\forall\tilde{\theta}_t \in R(\pi_t) : (\pi_t)^{-1}(\tilde{\theta}_t)$ *is either empty or a zero-dimensional* C^1 *manifold.*

Proof. We know that $M_t = (\Phi_t)^{-1}(0)$. Thus, applying the Regular Value Theorem (RVT; see appendix), we get (i). For (ii), see corollary 44, chapter 4, in [10]. Finally, (iii) is a straightforward application of the RVT. □

A zero-dimensional manifold is a set of isolated points. Therefore, for any given value of the exogenous variable $\tilde{\theta}_t \in R(\pi_t)$, we rule out a continuum of solutions.

Proposition 3.2 (Existence at t using Degree Theory). *Assume that* Θ_t *is connected. If (i)* $0 \in R(\Phi_t)$, *(ii)* M_t *is compact, and (iii)* $\exists\tilde{\theta}_t \in R(\pi_t)$ *such that*[1] $\#(\pi_t)^{-1}(\tilde{\theta}_t) = 1$, *then* $\forall\theta_t \in \Theta_t : (\pi_t)^{-1}(\theta_t) \neq \emptyset$. *In other words,* $\pi_t(M_t) = \Theta_t$.

Proof. Note that both, M_t and Θ_t, are $(n_t + m)$-dimensional manifolds. We know that $\#(\pi_t)^{-1}(\tilde{\theta}_t) = 1$ and $\tilde{\theta}_t \in R(\pi_t)$. Applying the Degree Theorem (see appendix), $\forall\theta_t \in R(\pi_t) : \#(\pi_t)^{-1}(\theta_t) = 1$. In other words, $\forall\theta_t \in R(\pi_t) : (\pi_t)^{-1}(\theta_t) \neq \emptyset$. Moreover, by the definition of critical value of π_t, if $\theta_t \in C(\pi_t)$, we have $(\pi_t)^{-1}(\theta_t) \neq \emptyset$. Since $\Theta_t = C(\pi_t) \cup R(\pi_t)$, the proof is complete. □

Proposition 3.3 (Existence at t using Homotopy). *Assume that* $0 \in R(\Phi_t)$, M_t *is compact, and* Θ_t *is connected. Suppose also that* $\exists\tilde{\theta}_t \in R(\pi_t)$, *such that* $\exists C^1$ *mapping* $\Psi_t : \Xi_t \to \mathbb{R}^{(s_t+n_{t+1})}$ *with two properties (i)* $0 \in R(\Psi_t)$, *and (ii)* $\#(\Psi_t)^{-1}(0) = 1$. *Moreover, suppose that there is a* C^0 *homotopy from* $\Phi_t|_{\tilde{\theta}_t}$ *to* Ψ_t, *say* $H_t : (\Xi_t \times [0,1]) \to \mathbb{R}^{(s_t+n_{t+1})}$, *such that* $(H_t)^{-1}(0)$ *is compact. Then,* $\forall\theta_t \in \Theta_t : (\pi_t)^{-1}(\theta_t) \neq \emptyset$. *In other words,* $\pi_t(M_t) = \Theta_t$.

[1] The symbol # stands for the *mod*2 degree.

Proof. We want to use the Existence using Homotopy Invariance Theorem (EHIT; see appendix). Note that Ξ_t and $\mathbb{R}^{(s_t+n_{t+1})}$ are C^∞ (s_t+n_{t+1})-dimensional manifolds. Identify (f,g) with $(\Phi_t|_{\tilde{\theta}_t}, \Psi_t)$. Since all the hypothesis for the EHIT are satisfied, we know that $\#(\Phi_t|_{\tilde{\theta}_t})^{-1}(0) = 1$. In other words, $\exists\tilde{\theta}_t \in R(\pi_t)$ *s.t.* $\#(\pi_t)^{-1}(\tilde{\theta}_t) = 1$. Applying proposition 3.2 we have the desired result. □

3.2 Determinacy at t

Proposition 3.4 (Closedness). *If π_t is proper, $C(\pi_t)$ is closed with respect to Θ_t.*

Proof. Applying the Closedness of Critical Points Theorem (see appendix), the set of critical points of the projection π_t, say $CP(\pi_t)$, is closed with respect to M_t. Using the fact that π_t is proper, we know that it is a closed mapping (i.e., maps closed sets into closed sets). Therefore, $C(\pi_t) = \pi_t(CP(\pi_t))$ is a closed set. □

Proposition 3.5 (Transversality). *If $0 \in R(\Phi_t)$, there exists a full measure subset of exogenous variables at t, say $\Theta_t^* \subseteq \Theta_t$, s.t. $\forall\tilde{\theta}_t \in \Theta_t^* : \tilde{\theta}_t \in R(\pi_t)$.*

Proof. We want to use the Transversality Theorem (see appendix). Let identify (M, Ω, N, h) with $(\Xi_t, \Theta_t, \mathbb{R}^{s_t+n_{t+1}}, \Phi_t)$. Thus, $h|_\omega$ is given by $\Phi_t|_{\tilde{\theta}_t} : \Xi_t \to \mathbb{R}^{s_t+n_{t+1}}$ s.t. $\xi_t \mapsto \Phi_t(\xi_t, \tilde{\theta}_t)$. If $0 \in R(\Phi_t)$, there is a full measure subset of exogenous variables, say $\Theta_t^* \subseteq \Theta_t$, such that $\forall\tilde{\theta}_t \in \Theta_t^* : 0 \in R(\Phi_t|_{\tilde{\theta}_t})$. This implies that $\forall\tilde{\xi}_t \in \omega_t((\pi_t)^{-1}(\tilde{\theta}_t)) : Rank[D_{\xi_t}\Phi_t|_{\tilde{\theta}_t}(\tilde{\xi}_t)] = (s_t + n_{t+1})$. Therefore, $Rank[D_{\xi_t}\Phi_t(\tilde{\xi}_t, \tilde{\theta}_t)] = (s_t + n_{t+1})$. Thus, $\forall(\tilde{\xi}_t, \tilde{\theta}_t) \in (\pi_t)^{-1}(\tilde{\theta}_t) : (\tilde{\xi}_t, \tilde{\theta}_t) \in RP(\pi_t)$. □

Definition 3.1 (Locally Unique Solutions at t). Consider any $\tilde{\theta}_t \in \pi_t(M_t)$. Define $(\pi_t)^{-1}(\tilde{\theta}_t) \equiv \{((\xi_t)_i, \tilde{\theta}_t), i \in I\}$, for some index set I. We say that $(\pi_t)^{-1}(\tilde{\theta}_t)$ consists of **locally unique solutions at** t if $\exists\tilde{\Theta}_t \subseteq \Theta_t$ *s.t.* $\tilde{\theta}_t \in \tilde{\Theta}_t$ and $\forall((\xi_t)_i, \tilde{\theta}_t) \in (\pi_t)^{-1}(\tilde{\theta}_t) : \exists(\tilde{M}_t)_i \subseteq M_t$ *s.t.* $((\xi_t)_i, \tilde{\theta}_t) \in (\tilde{M}_t)_i$, with the properties (i) $(\tilde{M}_t)_i \cap (\tilde{M}_t)_j = \emptyset$ iff $i \neq j$, (ii) $(\pi_t)^{-1}(\tilde{\Theta}_t) = \cup_{i\in I}(\tilde{M}_t)_i$, and (iii) $\pi_t|_{(\tilde{M}_t)_i}$ is a bijection.

Proposition 3.6 (Finite Local Uniqueness at t). *Let π_t be proper and $0 \in R(\Phi_t)$. For any $\tilde{\theta}_t \in R(\pi_t)$, such that $(\pi_t)^{-1}(\tilde{\theta}_t) \neq \emptyset$, the set $(\pi_t)^{-1}(\tilde{\theta}_t)$ consists of a* **finite** *number of* **locally unique solutions at** *t.*

Proof. Since $\Theta_t \subseteq \mathbb{R}^{n_t+m}$ is open and $0 \in R(\Phi_t)$, we know that M_t and Θ_t are $(n_t + m)$-dimensional manifolds. Thus, π_t is defined between manifolds of the same dimension. Applying the Stack of Records Theorem (SRT; see appendix), we get (1) $(\pi_t)^{-1}(\tilde{\theta}_t) = \{((\xi_t)_i, \tilde{\theta}_t),\ i \in \{1, 2, ..., q\}\}$, with $q < \infty$, (2) there are open neighborhood of $\tilde{\theta}_t$ in Θ_t, say Θ_t', open neighborhoods for each $((\xi_t)_i, \tilde{\theta}_t)$ in M_t, say $(M_t')_i$, s.t. (a) $\forall i \neq j : (M_t')_i \cap (M_t')_j = \emptyset$, (b) $(\pi_t)^{-1}(\Theta_t') = \cup_{i\in\{1,2,...,q\}}(M_t')_i$ and (c) $\forall i \in \{1, 2, ..., q\} : \pi_t|_{(M_t')_i}$ is a C^1 diffeomorphism (in particular, a bijection).

□

Proposition 3.7 (Generic Finite Local Uniqueness at t). *If π_t is proper and $0 \in R(\Phi_t)$, there exists a* **generic** *subset of* **regular** *exogenous variables at t, say $\Theta_t^* \subseteq R(\pi_t) \subseteq \Theta_t$, such that for every $\theta_t^* \in \Theta_t^*$, the set $(\pi_t)^{-1}(\theta_t^*)$ is either empty or consists of a* **finite** *number of* **locally unique solutions at** *t.*

Proof. Identify Θ_t^* with $R(\pi_t)$. Using proposition 3.4, $R(\pi_t) = \Theta_t \backslash C(\pi_t)$ is open. Moreover, by proposition 3.5, $R(\pi_t)$ has full measure. Finally, note that at every $\theta_t^* \in R(\pi_t)$, properness of π_t allows us to apply proposition 3.6. □

Proposition 3.8 (Locally Global Uniqueness at t). *Let π_t be proper and $0 \in R(\Phi_t)$. If $\exists \bar{\theta}_t \in R(\pi_t)$ s.t. $(\pi_t)^{-1}(\bar{\theta}_t) = \{(\bar{\xi}_t, \bar{\theta}_t)\}$, there is an open neighborhood of $\bar{\theta}_t$, say $\bar{\Theta}_t \subseteq \Theta_t$, such that $(\pi_t)^{-1}|_{\bar{\Theta}_t}$ is single-valued. Thus, $(\bar{\xi}_t, \bar{\theta}_t)$ is a* **unique, locally unique solution at** *t.*

Proof. Straightforward application of proposition 3.6, with $q = 1$. □

3.3 Robustness at t

Definition 3.2 (Robust Solutions at t). Consider any $\tilde{\theta}_t \in \Theta_t$, any particular mapping $\tilde{\Phi}_t$, and the solution set associated with the exogenous variable $\tilde{\theta}_t$, $(\tilde{\Phi}_t|_{\tilde{\theta}_t})^{-1}(0) = \{\xi_t \in \Xi_t \ s.t. \ \tilde{\Phi}_t(\xi_t, \tilde{\theta}_t) = 0\}$. Denote by $\tilde{\pi}_t$ the projection associated with $\tilde{\Phi}_t$. We say that $(\tilde{\pi}_t)^{-1}(\tilde{\theta}_t)$ consists of **robust solutions at t** if there exists a neighborhood for $\tilde{\Phi}_t|_{\tilde{\theta}_t}$, say $\mathcal{F} \subseteq \mathcal{C}^1(\Xi_t, \mathbb{R}^{s_t+n_{t+1}})$[2], such that $\forall \Phi_t|_{\tilde{\theta}_t} \in \mathcal{F}$: $Card((\Phi_t|_{\tilde{\theta}_t})^{-1}(0)) = Card((\tilde{\Phi}_t|_{\tilde{\theta}_t})^{-1}(0))$ [3]. In other words, the number of solutions associated with any system of equations in the set $\mathcal{F}$ is equal to the number of solutions for the original system $\tilde{\Phi}_t(\xi_t, \tilde{\theta}_t) = 0$.

Proposition 3.9 (Robustness at t). *Consider any system $\tilde{\Phi}_t$ and any $\tilde{\theta}_t \in R(\tilde{\pi}_t)$, s.t. $(\tilde{\pi}_t)^{-1}(\tilde{\theta}_t) \neq \emptyset$. If Ξ_t is bounded, $(\tilde{\pi}_t)^{-1}(\tilde{\theta}_t)$ consists of* **robust solutions at** *t.*

Proof. Identify $(W, \mathbb{R}^n, F_0)$ with $(\Xi_t, \mathbb{R}^{s_t+n_{t+1}}, \tilde{\Phi}_t|_{\tilde{\theta}_t})$ and use the Robustness of the Solution Set Theorem (see appendix). □

Proposition 3.10 (Generic Robustness at t). *If Ξ_t is bounded and $0 \in R(\Phi_t)$, there is a* **generic** *set of* **regular** *exogenous variables at t, say $\Theta_t^* \subseteq R(\tilde{\pi}_t) \subseteq \Theta_t$, s.t. $\forall \theta_t^* \in \Theta_t^*$, the set $(\pi_t)^{-1}(\theta_t^*)$ is either empty or consists of* **locally unique** *and* **robust solutions at** *t.*

Proof. Use propositions 3.4 and 3.5 to get the generic set $\Theta_t^* = R(\pi_t)$. Finally, apply proposition 3.9 for every $\theta_t^* \in \Theta_t^*$. □

Remark 3.1. If, in addition, π_t is proper, $\forall \theta_t^* \in \Theta_t^* \subseteq R(\tilde{\pi}_t)$, the set $(\pi_t)^{-1}(\theta_t^*)$ consists of a **finite** number of **locally unique** and **robust solutions at** t.

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[2]See the appendix for the definition of the space of C^1 functions with the C^1-topology.

[3]Where $Card(A)$ stands for the cardinality of a set A.

4 Existence

We will give sufficient conditions in order to guarantee that $\mathcal{E}$ is nonempty-valued. Note that the equilibrium correspondence requires solving finite-dimensional systems of equations at every point in time. Therefore, it suffices to show, first, that $\forall t \in \mathbb{N}_0 : \pi_t(M_t) = \Theta_t$, and, second, that the values found for the vectors $(\xi_t, \theta_t) \in M_t$ are such that $\rho_t(\xi_t, \theta_t) \in \Theta_{t+1}$.

Theorem 4.1 (Existence using Degree Theory). *Assume that Θ_t is connected. If $\forall t \in \mathbb{N}_0$, (i) $0 \in R(\Phi_t)$, (ii) M_t is compact, (iii) $\rho_t(M_t) \subseteq \pi_{t+1}(M_{t+1})$, and (iv) $\exists \tilde{\theta}_t \in R(\pi_t) : \#(\pi_t)^{-1}(\tilde{\theta}_t) = 1$, then $\forall \theta_0 \in \Theta_0 : \mathcal{E}(\theta_0)$ is nonempty.*

Proof. From proposition 3.2, we know that $\forall t \in \mathbb{N}_0 : \pi_t(M_t) = \Theta_t$. Fix any $\hat{\theta}_0 \in \Theta_0$ and obtain $\hat{\xi}_0 \in \omega_0[(\pi_0)^{-1}(\hat{\theta}_0)]$. Use $(\hat{\xi}_0, \hat{\theta}_0)$ and the fact that $\rho_0(M_0) \subseteq \pi_1(M_1)$ to get $\hat{\theta}_1 \in \rho_0(\hat{\xi}_0, \hat{\theta}_0) \subseteq \pi_1(M_1) = \Theta_1$. Having found $\hat{\theta}_1$, proceed in the same way to get $\hat{\xi}_1 \in \omega_1[(\pi_1)^{-1}(\hat{\theta}_1)]$. Repeating this procedure, construct $\hat{z} = \{\hat{\xi}_t\}_{t \in \mathbb{N}_0}$ and note that $\hat{z} \in \mathcal{E}(\hat{\theta}_0)$. □

Theorem 4.2 (Existence using Homotopy). *Assume that $\forall t \in \mathbb{N}_0 : 0 \in R(\Phi_t)$, M_t is compact, Θ_t is connected, $\rho_t(M_t) \subseteq \pi_{t+1}(M_{t+1})$, and that $\exists \tilde{\theta}_t \in R(\pi_t)$, such that $\exists C^1$ mapping $\Psi_t : \Xi_t \to \mathbb{R}^{(s_t + n_{t+1})}$ with the properties (i) $0 \in R(\Psi_t)$, and (ii) $\#(\Psi_t)^{-1}(0) = 1$. Moreover, suppose that there is a C^0 homotopy from $\Phi_t|_{\tilde{\theta}_t}$ to Ψ_t, say $H_t : (\Xi_t \times [0,1]) \to \mathbb{R}^{(s_t+n_{t+1})}$, such that $(H_t)^{-1}(0)$ is compact.*
Then, $\forall \theta_0 \in \Theta_0 : \mathcal{E}(\theta_0)$ is nonempty.

Proof. By proposition 3.3, we know that $\forall t \in \mathbb{N}_0 : \pi_t(M_t) = \Theta_t$. For any given $\hat{\theta}_0 \in \Theta_0$, proceed as in the proof for theorem 4.1 to find $\hat{z} \in \mathcal{E}(\hat{\theta}_0)$. □

We have chosen to prove $\pi_t(M_t) = \Theta_t$ using degree theory and homotopy arguments. Applications of this method to GE models can be found in [10]. Any other technique that provides sufficient conditions to solve a system of nonlinear equations in Euclidean spaces can also be used (e.g., fixed point theorems).

5 Determinacy

In this section, we study two important properties of equilibria, regularity and local uniqueness. The purpose of the analysis is to find conditions under which these properties hold generically, that is, in a full measure and open subset of economies.

For any $\theta_0 \in \Theta_0$, let $\mathcal{G}_0(\theta_0) \equiv \{\theta_0\}$. For all $t \in \mathbb{N}$, define the correspondence $\mathcal{G}_t : \Theta_0 \rightrightarrows \Theta_t$ *s.t.* $\theta_0 \mapsto \{\theta_t \in \Theta_t$ *s.t.* $\forall \theta_{t-1} \in \mathcal{G}_{t-1}(\theta_0) : \theta_t \in (\rho_{t-1} \circ (\pi_{t-1})^{-1})(\theta_{t-1})\}$. The set $\mathcal{G}_t(\theta_0)$ contains all the possible values for the exogenous variable θ_t that might be reached by the optimal paths associated with the economy θ_0.

Definition 5.1 (Regular and Critical Economies). A pair $(\tilde{\theta_0}, \tilde{z}) \in Graph(\mathcal{E})$ is said to be a regular point if $\forall t \in \mathbb{N}_0 : \forall \tilde{\theta}_t \in \mathcal{G}_t(\tilde{\theta_0}) : \forall(\tilde{\xi_t}, \tilde{\theta_t}) \in (\pi_t)^{-1}(\tilde{\theta_t}) : Rank[D_{\xi_t}\Phi_t(\tilde{\xi_t}, \tilde{\theta_t})] = (s_t + n_{t+1})$. If $(\tilde{\theta_0}, \tilde{z}) \in Graph(\mathcal{E})$ is not regular, it is a critical point. A vector $\tilde{\theta_0} \in \Theta_0$ is a **critical economy** if $\exists \tilde{z} \in \mathcal{E}(\tilde{\theta_0})$ such that $(\tilde{\theta_0}, \tilde{z})$ is a critical point. If $\tilde{\theta_0} \in \Theta_0$ is not critical, it is a **regular economy**.

The definition of regular economies establishes that at every point in time, $\mathcal{G}_t(\tilde{\theta}_0) \subseteq R(\pi_t)$. Thus, $\forall t \in \mathbb{N}_0 : \forall \theta_t \in \mathcal{G}_t(\tilde{\theta}_0)$, the set $(\pi_t)^{-1}(\theta_t)$ consists of isolated points. Therefore, **no indeterminacy** is allowed for the equilibria associated with regular economies. Using terminology from macro/captital theory, the full rank of $D_{\xi_t}\Phi_t(\xi_t, \theta_t)$ can be interpreted as having a locally smooth policy function. It is important to mention that the regularity condition we work with does not guarantee the differentiability of optimal sequences (as elements of an infinite-dimensional space) with respect to the parameter θ_0. In addition to the full rank condition at t, some stability properties have to be satisfied (see [1]).

For any particular set $\mathcal{N} \subseteq \Theta_0$, define $\mathcal{L}_0(\mathcal{N}) \equiv \{\mathcal{N}\}$. For every $t \in \mathbb{N}$, let $\mathcal{L}_t : \mathcal{P}(\Theta_0) \to \mathcal{P}(\Theta_t)$ *s.t.* $\mathcal{N} \mapsto \{\bar{\Theta}_t \subseteq \Theta_t \ s.t. \ \bar{\Theta}_t = ((\rho_{t-1} \circ (\pi_{t-1})^{-1})(\mathcal{L}_{t-1}(\mathcal{N})))\}$. The set $\mathcal{L}_t(\mathcal{N})$ consists of all possible values for θ_t that can be reached by some optimal path associated with any economy $\theta_0 \in \mathcal{N}$. Note that when $\mathcal{N} = \{\theta_0\}$ (i.e., a singleton), the correspondences $\mathcal{G}_t$ and $\mathcal{L}_t$ coincide.

Definition 5.2 (Locally Unique Equilibria). Consider any economy $\tilde{\theta}_0 \in \Theta_0$ such that $\mathcal{E}(\tilde{\theta}_0)$ is nonempty. We say that $\mathcal{E}(\tilde{\theta}_0)$ consists of **locally unique equilibria** if there is a sequence of open neighborhoods of $\tilde{\theta}_0$, say $\{\mathcal{N}_t\}_{t\in\mathbb{N}_0}$, $\tilde{\theta}_0 \in \mathcal{N}_t \subseteq \Theta_0$, such that $\forall t' \in \mathbb{N}_0$, (i) $(\pi_0)^{-1}(\tilde{\theta}_0)$ consists of locally unique solutions at $t = 0$, using $\mathcal{N}_{t'}$ as the corresponding set $\tilde{\Theta}_0$ (see definition 3.1), and (ii) $\forall t \in \mathbb{N}$ *s.t.* $t \le t'$: (a) $\forall \theta_t \in \mathcal{G}_t(\tilde{\theta}_0) : \exists! \ell_t(\theta_t) \subseteq \mathcal{L}_t(\mathcal{N}_{t'})$ s.t. $(\pi_t)^{-1}(\theta_t)$ consists of locally unique solutions at t, with the corresponding set $\ell_t(\theta_t)$, and (b) $\cup_{\theta_t \in \mathcal{G}_t(\tilde{\theta}_0)}(\ell_t(\theta_t)) = \mathcal{L}_t(\mathcal{N}_{t'})$.

Remark 5.1. If an economy $\tilde{\theta}_0 \in \Theta_0$ is s.t. at every $t \in \mathbb{N}_0$ and for any $\theta_t \in \mathcal{G}_t(\theta_0)$, we can apply the SRT, the optimal paths in $\mathcal{E}(\tilde{\theta}_0)$ are locally unique equilibria.

Remark 5.2. If there exists some $t' \in \mathbb{N}_0$ *s.t.* $\forall t \ge t' : \mathcal{N}_t \subseteq \mathcal{N}_{t+1}$, we can find an open set $\mathcal{N} = \cap_{t\in\mathbb{N}_0}(\mathcal{N}_t)$ and replace $\{\mathcal{N}_t\}_{t\in\mathbb{N}_0}$ by the constant sequence $\{\mathcal{N}\}_{t\in\mathbb{N}_0}$ in the previous definition. The condition $\mathcal{N}_t \subseteq \mathcal{N}_{t+1}$ obtains, for example, when $\tilde{\theta}_0$ is a local attractor.

Remark 5.3. For any $\tilde{\theta}_0 \in \Theta_0$, let $\mathcal{E}(\tilde{\theta}_0) \equiv \{z_i, i \in I\}$, for some index set I. Definition 5.2 implies that there is a neighborhood of $\tilde{\theta}_0$, say $\tilde{\mathcal{N}}$, and neighborhoods of every z_i[4], say $B(z_i) \subseteq Z$, such that (i) $\forall i, j \in I$ *s.t.* $i \ne j : B(z_i) \cap B(z_j) = \emptyset$, (ii) $\mathcal{E}(\tilde{\mathcal{N}}) = \cup_{i\in I}(B(z_i))$ and, (iii) $\forall i \in I : (\mathcal{E})^{-1}|_{B(z_i)}$ is a bijection.

[4]Assuming that Z is a space of bounded sequences with the *sup* norm.

Theorem 5.1 (Local Uniqueness). *Assume that $\mathcal{E}$ is nonempty-valued and that $\forall t, t' \in \mathbb{N}_0 : n_t = n_{t'} = n$. For all $t \in \mathbb{N}_0$, let the projections π_t and ρ_t be proper. Suppose that $\forall t \in \mathbb{N}_0 : \pi_{t+1}(M_{t+1}) \subseteq R(\rho_t)$ and $0 \in R(\Phi_t)$.*
Then, there exists a **full measure** *set of* **regular economies**, $\Theta_0^* \subseteq \Theta_0$, *such that $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ consists of* **locally unique equilibria**.

Proof. To show that Θ_0^* has full measure, consider the following five steps:
Step 1 *(Relevant space of economies):* We want the subset Θ_0^* to be such that $\forall \theta_0^* \in \Theta_0^* : \forall z^* \in \mathcal{E}(\theta_0^*) : (\theta_0^*, z^*)$ is a regular point. Thus, we have to get rid of all those economies $\tilde{\theta}_0$, where $\exists t \in \mathbb{N}_0$ *s.t.* $\exists \tilde{\theta}_t \in \mathcal{G}_t(\tilde{\theta}_0)$ *s.t.* $\tilde{\theta}_t \in C(\pi_t)$. Therefore, the relevant space of economies is: $\Theta_0^* = \Theta_0 \backslash \{[C(\pi_0)] \cup [\pi_0((\rho_0)^{-1}(C(\pi_1)))] \cup [\pi_0((\rho_0)^{-1}(\pi_1((\rho_1)^{-1}(C(\pi_2)))))] \cup ...\}$. Note that $\forall \theta_0^* \in \Theta_0^* : \mathcal{G}_t(\tilde{\theta}_0) \subseteq R(\pi_t)$. Thus, $\forall z^* \in \mathcal{E}(\theta_0^*) : (\theta_0^*, z^*)$ is a regular point and every θ_0^* in Θ_0^* is a regular economy.
Step 2 *(Covering $\tilde{C}(\pi_{t+1})$):* If for $\theta_{t+1} \in C(\pi_{t+1})$ we have $(\rho_t)^{-1}(\theta_{t+1}) = \emptyset$, we do not have to worry since this vector will not affect Θ_0. Thus, we want to delete $\tilde{C}(\pi_{t+1}) = \{\theta_{t+1} \in C(\pi_{t+1})$ *s.t* $(\rho_t)^{-1}(\theta_{t+1}) \neq \emptyset\}$ from Θ_{t+1}. Use the Countability Theorem (see appendix) to cover $\tilde{C}(\pi_{t+1})$ by a countable collection of arbitrarily small $(\epsilon/2)$-neighborhoods of elements of $C(\pi_{t+1})$, say $(\Theta'_{(t+1,\epsilon\backslash 2)})_j$, such that $\tilde{C}(\pi_{t+1}) \subseteq \cup_{j\in\mathbb{N}}((\Theta'_{(t+1,\epsilon\backslash 2)})_j)$. Since $\tilde{C}(\pi_{t+1}) \subseteq C(\pi_{t+1}) \subseteq R(\rho_t)$, we can choose ϵ small enough to have $\tilde{C}(\pi_{t+1}) \subseteq \cup_{j\in\mathbb{N}}\{(\Theta'_{(t+1,\epsilon\backslash 2)})_j\} \subset \cup_{j\in\mathbb{N}}\{(\Theta'_{(t+1,\epsilon)})_j\} \subseteq R(\rho_t) \subseteq \Theta_{t+1} \subseteq \mathbb{R}^{n+m}$. The purpose of covering $\tilde{C}(\pi_{t+1})$ with open neighborhoods, is to apply the SRT (going from open subsets of Θ_{t+1} to M_t, using $(\rho_t)^{-1}$). This is the reason why we have assumed that $\forall t \in \mathbb{N}_0 : \pi_{t+1}(M_{t+1}) \subseteq R(\rho_t)$ and ρ_t is proper. Therefore, the choice of ϵ can be done such that we are able to apply the SRT at every ϵ-neighborhood $(\Theta'_{(t+1,\epsilon)})_j$.
Step 3 *(Properties of $(\Theta'_{(t+1,\epsilon\backslash 2)})_j$ and the SRT)* Note that since $0 \in R(\Phi_{t+1})$, the set $\tilde{C}(\pi_{t+1}) \subseteq C(\pi_{t+1})$ has measure zero in Θ_{t+1} (see proposition 3.5). Thus, $\forall j \in \mathbb{N} : \{(\Theta'_{(t+1,\epsilon\backslash 2)})_j \cap \tilde{C}(\pi_{t+1})\}$ has measure zero in $(\Theta'_{(t+1,\epsilon)})_j$. We can use the SRT from $(\Theta'_{(t+1,\epsilon)})_j$ to M_t. Then, for any j, there is a finite collection of open sets in M_t, say $(M_t)_{(i,j)}, i \in \{1, 2, ..., q_j\}$, s.t. $\rho_t|_{(M_t)_{(i,j)}} : (M_t)_{(i,j)} \to (\Theta'_{(t+1,\epsilon)})_j$ is a C^1 diffeomorphism.
Step 4 *(The set $\pi_t[(\rho_t)^{-1}(C(\pi_{t+1}))]$ has measure zero in Θ_t):* For any j, i define the following C^1 mapping, $\varrho_{t(i,j)} \equiv (\rho_t|_{(M_t)_{(i,j)}})^{-1} : (\Theta'_{(t+1,\epsilon)})_j \to (M_t)_{(i,j)}$. Consider the composition, $\eta_{t(i,j)} \equiv \pi_t \circ \varrho_{t(i,j)} : (\Theta'_{(t+1,\epsilon)})_j \to \Theta_t$. Therefore, $\eta_{t(i,j)}$ is a C^1 function from an open subset of $\Theta_{t+1} \subseteq \mathbb{R}^{n+m}$ to $\Theta_t \subseteq \mathbb{R}^{n+m}$. Moreover, $\{(\Theta'_{(t+1,\epsilon\backslash 2)})_j \cap \tilde{C}(\pi_{t+1})\}$ is a measure zero subset of $(\Theta'_{(t+1,\epsilon)})_j$. Therefore, using the Image of Measure Zero Sets Theorem (see appendix), we have that $\eta_{t(i,j)}[(\Theta'_{(t+1,\epsilon\backslash 2)})_j \cap \tilde{C}(\pi_{t+1})]$ has measure zero in Θ_t. Therefore, $\cup_j \cup_i (\eta_{t(i,j)}[(\Theta'_{(t+1,\epsilon\backslash 2)})_j \cap \tilde{C}(\pi_{t+1})])$ has measure zero in Θ_t (the countable union of measure zero sets has measure zero). Since

$[\pi_t((\rho_t)^{-1}(C(\pi_{t+1})))] \subseteq \cup_j \cup_i \{\eta_{t(i,j)}[(\Theta'_{(t+1,\epsilon\backslash 2)})_j \cap \tilde{C}(\pi_{t+1})]\}$, we have that the set $\pi_t((\rho_t)^{-1}(C(\pi_{t+1})))$ has measure zero in Θ_t.

Step 5 *(The set Θ_0^* has full measure):* We know that $[\pi_t((\rho_t)^{-1}(C(\pi_{t+1})))]$ has measure zero in Θ_t. Therefore, we can cover now $[\pi_t((\rho_t)^{-1}(C(\pi_{t+1})))]$ by a countable union of neighborhoods and proceed as we did in steps 2-4. Note that the assumption $\forall t \in \mathbb{N}_0 : \pi_{t+1}(M_{t+1}) \subseteq R(\rho_t)$ plays an important role, since it will allow us to apply SRT at every t, going from Θ_{t+1} to M_t. Therefore, $\forall t \in \mathbb{N}$: $[\pi_0((\rho_0)^{-1}(...(\pi_t((\rho_t)^{-1}(C(\pi_{t+1}))))))]$ has measure zero in Θ_0. Since $C(\pi_0)$ has also measure zero, we finally know that Θ_0^* has full measure in Θ_0.

To complete the proof, it remains to be shown that every equilibrium is locally unique. This is true since we can apply the SRT (from Θ_t to M_t) for every t. □

Remark 5.4. It is also true that $\forall \theta_0^* \in \Theta_0^* : \forall t \in \mathbb{N}_0$, the set $\mathcal{G}_t(\theta_0^*) \subseteq R(\pi_t)$ contains a finite number of elements. Since π_t is proper, $\forall \theta_t \in \mathcal{G}_t(\theta_0^*) : (\pi_t)^{-1}(\theta_t)$ is also finite.

Theorem 5.2 (Finite Local Uniqueness). *Suppose the hypothesis made for theorem 5.1 hold. If $\exists t^* \in \mathbb{N}_0$ s.t. $\forall t \geq t^* : (\pi_t)^{-1}|_{\mathcal{L}_t(\Theta_0^*)}$ is a bijection, the* **full measure** *subset of* **regular economies**, *$\Theta_0^* \subseteq \Theta_0$, is such that $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ contains a* **finite** *number of* **locally unique equilibria**.

Proof. Applying theorem 5.2, we get that (i) any $\theta_0^* \in \Theta_0^*$ is a regular economy, and (ii) any $z^* \in \mathcal{E}(\theta_0^*)$ is a locally unique equilibrium. It remains to be shown that $\mathcal{E}(\theta_0^*)$ is finite. For that purpose, fix any $\theta_0^* \in \Theta_0^*$. Using the fact that $\forall t \geq t^* : (\pi_t)^{-1}|_{\mathcal{L}_t(\Theta_0^*)}$ is a bijection, we know that after t^*, each element of the sequence z^* can take only one possible value. Thus, the number of optimal paths associated with θ_0^* is finite. □

Theorem 5.3 (Generic Local Uniqueness). *Suppose the assumptions made for theorem 5.2 hold. If $\exists t^* \in \mathbb{N}_0$ s.t. $\forall t \geq t^* : [\pi_t((\rho_t)^{-1}(C(\pi_{t+1})))] \subseteq C(\pi_t)$, the relevant subset of economies is* **generic**. *In other words, there is an* **open** *and* **full measure** *set of* **regular economies**, *$\Theta_0^* \subseteq \Theta_0$, s.t. $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ consists of* **locally unique equilibria**.

Proof. Note that for all $t \geq t^*$, we do not need to get rid of any other set $C(\pi_t)$. Thus, the relevant space of economies is $\Theta_0^* = \Theta_0 \backslash \{[C(\pi_0)] \cup [\pi_0((\rho_0)^{-1}(C(\pi_1)))] ... \cup \Delta\}$. Where $\Delta \equiv [\pi_0((\rho_0)^{-1}(...(\pi_{(t^*-1)}((\rho_{(t^*-1)})^{-1}(C(\pi_{t^*}))))))]$. Since $t^* \in \mathbb{N}_0$, the finite union of closed and measure zero sets $\{[C(\pi_0)] \cup [\pi_0((\rho_0)^{-1}(C(\pi_1)))] ... \cup \Delta\}$ is also closed and has measure zero. Thus, Θ_0^* is generic. □

Theorem 5.4 (Generic Finite Local Uniqueness). *If the assumptions made theorems 5.2 and 5.3 hold, there is a* **generic** *set of* **regular economies**, *$\Theta_0^* \subseteq \Theta_0$, s.t. $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ is a* **finite** *set of* **locally unique equilibria**.

Proof. Straightforward. □

Theorem 5.5 (Locally Global Uniqueness). *For every $t \in \mathbb{N}_0$, assume that $0 \in R(\Phi_t)$, π_t is proper, and $\exists \bar{\theta}_t \in R(\pi_t)$ s.t. $(\pi_t)^{-1}(\bar{\theta}_t) = \{(\bar{\xi}_t, \bar{\theta}_t)\}$. Denote by $\bar{\Theta}_t \subseteq \Theta_t$ the open set where $(\pi_t)^{-1}|_{\bar{\Theta}_t}$ is a bijection (see proposition 3.8). Assume also that $\rho_t[(\pi_t)^{-1}(\bar{\Theta}_t)] \subseteq \bar{\Theta}_{t+1}$. Then, $\forall \bar{\theta}_0 \in \bar{\Theta}_0$ the set $\mathcal{E}(\bar{\theta}_0)$ consists of a* **unique, locally unique equilibrium**.

Proof. Fix any $\bar{\theta}_0 \in \bar{\Theta}_0$. Since $(\pi_0)^{-1}(\bar{\theta}_0) = \{(\bar{\xi}_0, \bar{\theta}_0)\}$, we can get $\bar{\xi}_0 = \omega_0[(\pi_0)^{-1}(\bar{\theta}_0)]$. Since $\rho_t[(\pi_t)^{-1}(\bar{\Theta}_t)] \subseteq \bar{\Theta}_{t+1} \subseteq R(\pi_{t+1})$, we have $\rho_0[(\bar{\xi}_0, \bar{\theta}_0)] = \bar{\theta}_1 \in \bar{\Theta}_1$. Using $\bar{\theta}_1$, get $\bar{\xi}_1 = \omega_1[(\pi_1)^{-1}(\bar{\theta}_1)]$. Repeating this procedure, construct $\bar{z} = \{\bar{\xi}_t\}_{t \in \mathbb{N}_0}$. Since at each t, we have $(\pi_t)^{-1}(\bar{\theta}_t) = \{(\bar{\xi}_t, \bar{\theta}_t)\}$, we can conclude that $\exists! \bar{z} \in \mathcal{E}(\bar{\theta}_0)$. Finally, since we can apply the SRT at every $\bar{\theta}_t$, $\bar{z}$ is locally unique. □

Remark 5.5. To get global uniqueness, i.e., the correspondence $\mathcal{E}$ being single-valued, we need $\forall \tilde{\theta}_0 \in \Theta_0 : \forall t \in \mathbb{N}_0 : (\pi_t)^{-1}|_{\mathcal{G}_t(\tilde{\theta}_0)}$ to be single-valued.

6 Robustness

For any $\theta_0 \in \Theta_0$, recall $\mathcal{G}_0(\theta_0) \equiv \{\theta_0\}$ and $\mathcal{G}_t : \Theta_0 \rightrightarrows \Theta_t$ *s.t.* $\theta_0 \mapsto \{\theta_t \in \Theta_t$ *s.t.* $\forall \theta_{t-1} \in \mathcal{G}_{t-1}(\theta_0) : \theta_t \in (\rho_{t-1} \circ (\pi_{t-1})^{-1})(\theta_{t-1}))\}$.

Definition 6.1 (Robust Equilibria). Consider any sequence of systems $\{\tilde{\Phi}_t\}_{t \in \mathbb{N}_0}$. Denote by $\tilde{\mathcal{E}}$, $\tilde{\pi}_t$ and $\tilde{\mathcal{G}}_t$ the objects $\mathcal{E}$, π_t and $\mathcal{G}_t$ associated with the particular mapping $\tilde{\Phi}_t$. For any given $\tilde{\theta}_0 \in \Theta_0$, we say that $\tilde{\mathcal{E}}(\tilde{\theta}_0)$ consists of **robust equilibria** if there exists a sequence of sets $\{\mathcal{H}_t\}_{t \in \mathbb{N}_0}$, with $\{\tilde{\Phi}_t\}_{t \in \mathbb{N}_0} \in \{\mathcal{H}_t\}_{t \in \mathbb{N}_0}$, such that $\forall \{\Phi_t\}_{t \in \mathbb{N}_0} \in \{\mathcal{H}_t\}_{t \in \mathbb{N}_0}$, and every $t \in \mathbb{N}_0$, we have (i) $Card(\mathcal{G}_t(\tilde{\theta}_0)) = Card(\tilde{\mathcal{G}}_t(\tilde{\theta}_0))$, and (ii) $\forall \theta_t \in \mathcal{G}_t(\tilde{\theta}_0) : \exists! \tilde{\theta}_t \in \tilde{\mathcal{G}}_t(\tilde{\theta}_0)$ *s.t.* $Card((\pi_t)^{-1}(\theta_t)) = Card((\tilde{\pi}_t)^{-1}(\tilde{\theta}_t))$.

Theorem 6.1 (Robustness). *Assume the hypothesis made for theorem 5.1 hold. In addition, suppose that $\forall t \in \mathbb{N}_0$, the sets Ξ_t are bounded.*
Then, the relevant space of economies obtained from theorem 5.1, $\Theta_0^* \subseteq \Theta_0$, is such that $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ consists of **robust** *equilibria.*

Proof. Apply the SRT and proposition 3.9 for every $t \in \mathbb{N}_0$. □

Theorem 6.2 (Generic Robustness). *Assume the hypothesis made for theorems 5.3 and 6.1 hold.*
Then, the space of economies obtained from theorem 6.1 is a **generic** *set.*

Proof. Straightforward. □

Remark 6.1. Under the hypothesis made for theorems 5.2 and 6.2, the set of **regular economies**, $\Theta_0^* \subseteq \Theta_0$, is **generic**, and $\forall \theta_0^* \in \Theta_0^* : \mathcal{E}(\theta_0^*)$ consists of a **finite** number of **locally unique**, **robust equilibria**.

7 Neoclassical Optimal Growth Model

We will apply the techniques found in the previous sections to the standard neoclassical optimal growth model. Considering the social planner's problem (SPP), we will study existence, uniqueness and determinacy of equilibria.

Consider the standard optimization problem faced by a social planner: $Max_{z\in\mathcal{J}(k_0,\delta,\beta)}\sum_{t\in\mathbb{N}_0}\beta^t u(c_t,1-h_t)$, with:

(i) $z\equiv\{(c_t,k_{t+1},h_t)\}_{t\in\mathbb{N}_0}$,

(ii) $\mathcal{J}:(\mathbb{R}_{++}\times(0,1)\times(0,1))\rightrightarrows\mathcal{P}(Z)$, s.t. $(k_0,\delta,\beta)\mapsto\mathcal{J}(k_0,\delta,\beta)\equiv\{z\in Z\ s.t.\ \forall t\in\mathbb{N}_0:\ c_t=F(k_t,h_t)+(1-\delta)k_t-k_{t+1},h_t\in[0,1],and\ c_t,k_{t+1}\geq 0\}$.

Definition 7.1 (Equilibrium Correspondence for SPP). Define the following correspondence $\mathcal{E}:(\mathbb{R}_+\times(0,1)\times(0,1))\rightrightarrows\mathcal{P}(Z)$, s.t. $(k_0,\delta,\beta)\mapsto\{z^*\in Z\ s.t.\ z^*\in argmax_{z\in\mathcal{J}(k_0,\delta,\beta)}\sum_{t\in\mathbb{N}_0}\beta^t u(c_t,1-h_t)\}$. The sequence $z^*\equiv\{(c_t^*,k_{t+1}^*,h_t^*)\}_{t\in\mathbb{N}_0}$ is called an optimal path for the initial condition k_0 and the parameters (δ,β).

Definition 7.2 (Bellman Equation). Let $\Omega:(\mathbb{R}_+\times(0,1)\times(0,1))\rightrightarrows\mathcal{P}(\mathbb{R}_+\times[0,1])$, s.t. $(k_t,\delta,\beta)\mapsto\Omega(k_t,\delta,\beta)\equiv\{(k_{t+1},h_t)\in(\mathbb{R}_+\times[0,1])\ s.t.\ k_{t+1}\in[0,\bar{k}],h_t\in[0,1]\}$ [5]. The Bellman equation (BE) is defined as the following functional equation $V(k_t)=max_{(k_{t+1},h_t)\in\Omega(k_t,\delta,\beta)}[u(F(k_t,h_t)+(1-\delta)k_t-k_{t+1},1-h_t)+\beta V(k_{t+1})]$.

Definition 7.3 (Policy Correspondence). Define the correspondence $G:(\mathbb{R}_+\times(0,1)\times(0,1))\rightrightarrows\mathcal{P}(\mathbb{R}_+\times[0,1])\ s.t.\ (k_t,\delta,\beta)\mapsto\{(k_{t+1}^*,h_t^*)\in(\mathbb{R}_+\times[0,1])$ s.t. $(k_{t+1}^*,h_t^*)\in argmax_{(k_{t+1},h_t)\in\Omega(k_t,\delta,\beta))}[u(F(k_t,h_t)+(1-\delta)k_t-k_{t+1},1-h_t)+\beta V(k_{t+1})]\}$.

7.1 Results Using Standard Dynamic Programming Methods

Theorem 7.1 (Existence of Equilibria for SPP). *Assume that the instantaneous utility function,* $u:(\mathbb{R}_+\times[0,1])\to\mathbb{R}$, *and the production function,* $F:(\mathbb{R}_+\times[0,1])\to\mathbb{R}_+$, *are* C^0. *Suppose also that* $\beta\in(0,1)$ *and* $\forall h_t\in[0,1]:F(0,h_t)=0$.
Then, $\mathcal{E}$ *is nonempty-valued. In other words, for any initial condition* $k_0>0$ *and any parameters* $\delta\in(0,1),\beta\in(0,1)$, *we can always find an optimal path* $z^*\in\mathcal{E}(k_0,\delta,\beta)$.

Proof. Note that under the mentioned assumptions, the hypothesis of theorem 4.6 in [9] are satisfied. Thus, the BE has a unique solution. Moreover, the policy correspondence is nonempty-valued (uh-c and compact-valued). Thus, for a given vector (k_0,δ,β), we construct a sequence $z^*\equiv\{(c_t^*,k_{t+1}^*,h_t^*)\}_{t\in\mathbb{N}_0}$ such that $\forall t\in\mathbb{N}_0:(k_{t+1}^*,h_t^*)\in G(k_t^*,\delta,\beta)$ and $c_t^*=F(k_t^*,h_t^*)+(1-\delta)k_t^*-k_{t+1}^*$. Note now that theorem 4.5 in [9] applies; i.e., the sequence obtained iterating the right hand side (RHS) of the BE is an optimal path. Therefore, $z^*\in\mathcal{E}(k_0,\delta,\beta)$. □

[5] The value $\bar{k}$ is obtained as the upper bound for any k_{t+1}, when $c_t=0$ for all $t\in\mathbb{N}_0$.

Theorem 7.2 (Existence of Interior Equilibria for SPP). *In addition to the assumptions made for theorem 7.1, suppose that* u *and* F *are* C^1. *Assume also that (i)* $\forall h_t \in [0,1] : lim_{c_t \to 0} u_1(c_t, 1-h_t) = \infty$ *and* $lim_{c_t \to \infty} u_1(c_t, 1-h_t) = 0$, *(ii)* $\forall h_t \in [0,1] : lim_{k_t \to 0} F_k(k_t, h_t) = \infty$ *and* $lim_{k_t \to \infty} F_k(k_t, h_t) = 0$, *and (iii)* $\forall c_t \in \mathbb{R}_{++} : u_2(c_t, 0) > 0$ *and* $u_2(c_t, 1) < 0$.
Then, $\forall (k_0, \delta, \beta) \in (\mathbb{R}_{++} \times (0,1) \times (0,1)) : \forall z^* \equiv \{(c_t^*, k_{t+1}^*, h_t^*)\}_{t \in \mathbb{N}_0} \in \mathcal{E}(k_0, \delta, \beta) : \forall t \in \mathbb{N}_0 : (c_t^*, k_{t+1}^*, h_t^*) \in (\mathbb{R}_{++} \times (0, \bar{k}) \times (0,1))$. *In other words, the equilibria associated with any vector* (k_0, δ, β) *lie in the interior of the set of feasible paths.*

Proof. Fix any $(k_0, \delta, \beta) \in (\mathbb{R}_{++} \times (0,1) \times (0,1))$. We know that $\mathcal{E}(k_0, \delta, \beta)$ is nonempty. Fix any $z^* \equiv \{(c_t^*, k_{t+1}^*, h_t^*)\}_{t \in \mathbb{N}_0} \in \mathcal{E}(k_0, \delta, \beta)$. If $h_t^* = 0$, since $u_2(c_t, 1) < 0$, we can increase h_t and obtain higher utility without violating feasibility. Similarly; if $h_t^* = 1$, an increase in h_t will yield higher utility. Therefore, $h_t^* \in (0,1)$. For $c_t^*, k_{t+1}^* > 0$, use conditions (i) and (ii), as it is done in [9], chapter 5. □

Theorem 7.3 (Uniqueness of Interior Equilibria for SPP). *Let the hypothesis made for theorem 7.2 hold. Suppose* u *is strictly concave and* F *is strictly increasing. Then,* $\mathcal{E}$ *is single-valued. In other words, for any vector* $(k_0, \delta, \beta) \in (\mathbb{R}_{++} \times (0,1) \times (0,1))$, *there exists a unique optimal path associated with it.*

Proof. For any given $t \in \mathbb{N}_0$, the value function $V(k_t)$ is strictly concave and the policy correspondence G is a continuous function (see theorem 4.8 in [9]). Moreover, by theorems 4.4 and 4.5 in [9], z^* is an optimal path iff it is generated iterating over G. Therefore, given any $(k_0, \delta, \beta) \in (\mathbb{R}_{++} \times (0,1) \times (0,1))$, the unique solution to the SPP is found as follows: $\mathcal{E}(k_0, \delta, \beta) = z^* \equiv \{(c_t^*, k_{t+1}^*, h_t^*)\}_{t \in \mathbb{N}_0}$, where $\forall t \in \mathbb{N}_0 : (k_{t+1}^*, h_t^*) = G(k_t^*, \delta, \beta)$ *and* $c_t^* = F(k_t^*, h_t^*) + (1-\delta) k_t^* - k_{t+1}^*$. □

7.2 Results Using Smooth Analysis

We want to apply some of the tools developed in this paper to the SPP. Note that existence was shown before in theorem 7.2. We will focus here only on the determinacy of the solutions to the SPP. We want characterize the set of interior solutions by a sequence of smooth equations. This role is going to be played by the Euler equations for capital (EE), and by the first order conditions for labor (FOC). We know that the collection of EE and FOC represents a set of necessary conditions for any solution (see [9], section 4.5).

Notation described in section 2 of the paper also applies here. Assume that u and F are C^2 functions, and for any $t \in \mathbb{N}_0$, consider:

- $\Xi_t \equiv ([0,1] \times [0,1] \times \mathbb{R}_+)$, with typical element $\xi_t \equiv (h_t, h_{t-1}, k_{t+1})$,
- $\Theta_t \equiv (\mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \times [0,1])$, with typical element $\theta_t \equiv (k_t, k_{t-1}, \delta, \beta)$,

- $\Phi_t : (\Xi_t \times \Theta_t) \to \mathbb{R}^3$ s.t. $(\xi_t, \theta_t) \mapsto \Phi_t(\xi_t, \theta_t)$, Φ_t is C^1. Given $c_t = F(k_t, h_t) + (1-\delta)k_t - k_{t+1}$, Φ_t consists of the following three equations:

  (i) $\Phi_t^1 = u_1(c_{t-1}, 1-h_{t-1}) - \beta u_1(c_t, 1-h_t)\{F_k(k_t, h_t) + (1-\delta)\}$ $[EE_{t-1}]$,

  (ii) $\Phi_t^2 = u_1(c_t, 1-h_t)F_h(k_t, h_t) - u_2(c_t, 1-h_t)$ $[FOC_t]$,

  (iii) $\Phi_t^3 = u_1(c_{t-1}, 1-h_{t-1})F_h(k_{t-1}, h_{t-1}) - u_2(c_{t-1}, 1-h_{t-1})$ $[FOC_{t-1}]$,

Definition 7.4 (Equilibrium Correspondence for Smooth SPP). Consider the correspondence $\hat{\mathcal{E}} : (\mathbb{R}_{++} \times \mathbb{R}_{++} \times (0,1) \times (0,1)) \rightrightarrows \mathcal{P}(Z)$, s.t. $(k_0, k_{-1}, \delta, \beta) \mapsto \{z \equiv \{(h_t, h_{t-1}, k_{t+1})\}_{t\in\mathbb{N}_0} \in Z \ s.t. \ \forall t \in \mathbb{N}_0 : (\xi_t, \theta_t) \in M_t\}$.

The results for the smooth version of the SPP are presented below.

Proposition 7.1 (M_t is a Manifold). *Assume that $F_k \geq 0, F_h > 0, u_1 > 0, u_{11} \leq 0, u_{22} \leq 0, u_{12} > 0$, and $F_{hh} \leq 0$. Then, M_t is a 4-dimensional C^1 manifold.*

Proof. To apply the RVT, we have to check that $0 \in R(\Phi_t)$. In other words, we want $\forall(\xi_t, \theta_t) \in M_t : Rank[D\Phi_t] = 3$. Computing the Jacobian matrix $D\Phi_t$, we have:

$$D\Phi_t = \begin{bmatrix} D_{h_t}\Phi_t^1 & D_{h_{t-1}}\Phi_t^1 & D_{k_{t+1}}\Phi_t^1 & D_{k_t}\Phi_t^1 & D_{k_{t-1}}\Phi_t^1 & D_\delta\Phi_t^1 & D_\beta\Phi_t^1 \\ D_{h_t}\Phi_t^2 & 0 & D_{k_{t+1}}\Phi_t^2 & D_{k_t}\Phi_t^2 & 0 & D_\delta\Phi_t^2 & 0 \\ 0 & D_{h_{t-1}}\Phi_t^3 & 0 & D_{k_t}\Phi_t^3 & D_{k_{t-1}}\Phi_t^3 & D_\delta\Phi_t^3 & 0 \end{bmatrix}$$

If (i) $D_{h_t}\Phi_t^2 \neq 0$, (ii) $D_{h_{t-1}}\Phi_t^3 \neq 0$, and (iii) $D_\beta\Phi_t^1 \neq 0$, then $Rank[D\Phi_t] = 3$. For the first expression[6], $D_{h_t}\Phi_t^2 = F_h[u_{11}\ F_h - 2\ u_{12}] + u_1\ F_{hh} + u_{22} < 0$. The second one is similar to (i). Finally, $D_\beta\Phi_t^1 = -u_1\ \{F_k + (1-\delta)\} < 0$. Thus, $0 \in R(\Phi_t)$ and by the RVT, $(\Phi_t)^{-1}(0)$ is a 4-dimensional C^1 manifold. □

Proposition 7.2 ($R(\rho_t) = \Theta_{t+1}$). *In addition to the assumptions made for proposition 7.1, suppose that $F_{hk} > 0$ and that $|H| \geq 0$ (where $|H|$ stands for the determinant of the Hessian matrix of u). Then, $R(\rho_t) = \Theta_{t+1}$.*

Proof. $R(\rho_t) \equiv \{\tilde{\theta}_{t+1} \in \Theta_{t+1}$ *s.t. (i)* $(\rho_t)^{-1}(\tilde{\theta}_{t+1}) = \emptyset$, *or (ii)* $\forall(\tilde{\xi}_t, \tilde{\theta}_t) \in (\rho_t)^{-1}(\tilde{\theta}_{t+1}) :$ $Rank[D_{(h_t, h_{t-1}, k_{t-1})}\Phi_t] = 3\}$. The partial Jacobian matrix is:

$$D_{(h_t, h_{t-1}, k_{t-1})}\Phi_t = \begin{bmatrix} D_{h_t}\Phi_t^1 & D_{h_{t-1}}\Phi_t^1 & D_{k_{t-1}}\Phi_t^1 \\ D_{h_t}\Phi_t^2 & 0 & 0 \\ 0 & D_{h_{t-1}}\Phi_t^3 & D_{k_{t-1}}\Phi_t^3 \end{bmatrix}.$$

Since $D_{h_t}\Phi_t^2 < 0$, the matrix $D_{(h_t, h_{t-1}, k_{t-1})}\Phi_t$ will have full rank if the rank of the following submatrix is full:

$$D_{(h_{t-1}, k_{t-1})}\tilde{\Phi}_t = \begin{bmatrix} D_{h_{t-1}}\Phi_t^1 & D_{k_{t-1}}\Phi_t^1 \\ D_{h_{t-1}}\Phi_t^3 & D_{k_{t-1}}\Phi_t^3 \end{bmatrix}.$$

[6] For notational simplicity, we have not included the arguments of the different functions.

To show that $D_{(h_{t-1},k_{t-1})}\tilde{\Phi}_t$ has full rank, it suffices to show $|D_{(h_{t-1},k_{t-1})}\tilde{\Phi}_t| \neq 0$. After simplifications, $|D_{(h_{t-1},k_{t-1})}\tilde{\Phi}_t| = \{F_k + (1-\delta)\}\ [u_{12}^2 - u_{11}\ u_{22}] - u_{12}\ u_1\ F_{kh}$. Using the two assumptions, $|H| = (u_{11}\ u_{22} - u_{12}^2) \geq 0$ and $F_{kh} > 0$, we get $|D_{(h_t,k_t)}\tilde{\Phi}_t| < 0$. Thus, $Rank[D_{(h_t,h_{t-1},k_{t-1})}\Phi_t] = 3$, and $R(\rho_t) = \Theta_{t+1}$. □

Proposition 7.3 (Properness of π_t and ρ_t). *The natural projections π_t and ρ_t are proper mappings.*

Proof. To prove properness of π_t, it suffices to show that for every compact set $\bar{\Theta}_t \subseteq \Theta_t$, the set $Graph((\pi_t)^{-1}|_{(\bar{\Theta}_t)})$ is also compact. Fix any $\bar{\Theta}_t$. Consider an arbitrary sequence $\{(\theta_t^\nu, \xi_t^\nu, \theta_t^\nu)\}_{\nu \in \mathbb{N}}$ s.t. $\forall \nu \in \mathbb{N} : (\theta_t^\nu, \xi_t^\nu, \theta_t^\nu) \in Graph((\pi_t)^{-1}|_{(\bar{\Theta}_t)})$. We need to show that there is a subsequence, without loss of generality the sequence itself, such that $(\theta_t^\nu, \xi_t^\nu, \theta_t^\nu) \to (\bar{\theta}_t^\nu, \bar{\xi}_t^\nu, \bar{\theta}_t^\nu) \in Graph((\pi_t)^{-1}|_{(\bar{\Theta}_t)})$. By compactness of $\bar{\Theta}_t$, we have $\theta_t^\nu \to \bar{\theta}_t^\nu \in \bar{\Theta}_t$. Since $\xi_t^\nu \equiv (h_t^\nu, h_{t-1}^\nu, k_{t+1}^\nu) \in ([0,1] \times [0,1] \times [0,\bar{k}])$, we know that ξ_t^ν will converge to $\bar{\xi}_t$. Finally, since Φ_t is C^1, $\Phi_t(\bar{\xi}_t, \bar{\theta}_t) = 0$. Therefore, $(\bar{\xi}_t, \bar{\theta}_t) \in Graph((\pi_t)^{-1}|_{(\bar{\Theta}_t)})$ and π_t is proper. A similar argument shows that ρ_t is also proper. □

Proposition 7.4 (Generic Finite Local Uniqueness at t for Smooth SPP). *Suppose the hypothesis made for proposition 7.1 hold. Then, there is a* **generic** *set of exogenous variables at t, say $\Theta_t^* \subseteq \Theta_t$, s.t. $\forall \theta_t^* \in \Theta_t^* : (\pi_t)^{-1}(\theta_t^*)$ is either empty or consists of a* **finite** *number of* **locally unique solutions at** *t.*

Proof. Straightforward application of proposition 3.7. □

Theorem 7.4 (Local Uniqueness for Smooth SPP). *Assume the hypothesis made for the propositions 7.1 and 7.2 hold. Then, there is a* **full measure** *set of economies, say $\Theta_0^* \subseteq \Theta_0$, s.t. any vector in Θ_0^* is a* **regular economy**. *Moreover, $\forall \theta_0^* \in \Theta_0^* : \hat{\mathcal{E}}(\theta_0^*)$ is either empty or consists of* **locally unique** *optimal paths.*

Proof. Straightforward application of theorem 5.1. □

8 Appendix

We will state some definitions and theorems that were used in the paper. Notation and assumptions made in section 2 also apply here.

Theorem 8.1 (Implicit Function Theorem). *Consider any pair* $(\hat{\xi}_t, \hat{\theta}_t) \in (\Xi_t \times \Theta_t)$ *s.t.* $\Phi_t(\hat{\xi}_t, \hat{\theta}_t) = 0$. *If* $Rank[D_{\xi_t}\Phi_t(\hat{\xi}_t, \hat{\theta}_t)] = (n_t + s_t)$, *there exists an open neighborhood of* $\hat{\theta}_t$, *say* $\Theta'_t \subseteq \Theta_t$, *an open neighborhood of* $\hat{\xi}_t$, *say* $\Xi'_t \subseteq \Xi_t$, *and a* C^1 *mapping* $\phi_t : \Theta'_t \to \Xi'_t$ *such that:* $\forall \theta_t \in \Theta'_t : [\xi_t = \phi_t(\theta_t)] \Leftrightarrow [\Phi_t(\xi_t, \theta_t) = 0]$.

Proof. See [4], pages 214-215. □

Definition 8.1 (Manifold). A set $M \subseteq \mathbb{R}^l$ is an n-dimensional C^r manifold if for every $x \in M$, there exists an open neighborhood of x in M, say V, such that V is locally C^r diffeomorphic to a subset of $\mathbb{R}^n$, say $U \subseteq \mathbb{R}^n$. In other words, $\exists m : U \to V$ *s.t.* m is a C^r diffeomorphism. The set (m, U, V) is called a local parametrization of M around x.

From now on, let $M \subseteq \mathbb{R}^l$ and $N \subseteq \mathbb{R}^p$ be C^r m and n-dimensional manifolds respectively. Define $f : M \to N$ *s.t.* $x \mapsto y = f(x)$ and $g : M \to N$ *s.t.* $x \mapsto y = g(x)$. Assume that f and g are C^1 mappings.

Definition 8.2 (Regular and Critical Points of f). An element $x \in M$ is a regular point of f if df_x is surjective and is a critical point otherwise. We denote by $RP(f)$ and $CP(f)$ the sets of regular and critical points of f respectively.

Remark 8.1. In definition 8.2, df_x stands for the differential of f at $x \in M$. Differentiation of functions between manifolds can be found in [5], chapter 1.

Definition 8.3 (Regular and Critical Values of f). An element $y \in N$ is a regular value of f if every $x \in f^{-1}(y) \subseteq M$ is a regular point of f. If not, we say that y is a Critical value of f. We denote by $R(f)$ and $C(f)$ the sets of regular and critical values of f respectively.

Theorem 8.2 (Robustness of the Solution Set). *Let* $W \subseteq \mathbb{R}^n$ *be a bounded set. Denote by* $C^1(\bar{W}, \mathbb{R}^n)$ *the set of* C^1 *mappings from* $\bar{W}$ *to* $\mathbb{R}^n$, *endowed with the* C^1*-topology*[7]. *For a given* $F_0 : \bar{W} \to \mathbb{R}^n \in C^1(\bar{W}, \mathbb{R}^n)$, *if* $0 \in R(F_0)$ *and* $F_0(x) \neq 0$ *for all* $x \in \delta W \equiv \bar{W} \backslash W$, *there exists an open neighborhood of* F_0, *say* $\mathcal{F}$, *and a finite number of* C^0 *mappings* $s_k : \mathcal{F} \to W$, $k = 1, 2, ..., q$, $q < \infty$, *such that* $\forall F \in \mathcal{F} : \cup_{k=1,2,...,q}\{s_k(F)\} = \{x \in W \text{ s.t. } F(x) = 0\}$.

Proof. See [2], mathematical appendix. □

[7] For any $F, G \in C^1(\bar{W}, \mathbb{R}^n)$, define $d(F, G) = sup_{x \in \bar{W}}||F(x) - G(x)|| + sup_{x \in \bar{W}}||DF(x) - DG(x)||$.

Theorem 8.3 (Regular Value Theorem). *If $y \in f(M)$ is a regular value of f, then $f^{-1}(y) \subseteq M$ is a C^r $(n-m)$-dimensional manifold.*

Proof. See [4], page 22. □

The previous theorem plays an important role in GE analysis since it generalizes the Implicit Function Theorem. In practice, if we check that $\forall x \in f^{-1}(y) : df_x$ is surjective, the set $f^{-1}(y)$ is locally diffeomorphic to a subset of $\mathbb{R}^{(n-m)}$.

Definition 8.4 (Proper Function). We say that f is proper if it is C^0 and for every compact subset of N, say $N' \subseteq N$, the set $f^{-1}(N')$ is a compact subset of M.

Theorem 8.4 (Closedness of Critical Points of f). *The set of critical points of f, $CP(f) \subseteq M$, is closed with respect to M.*

Proof. In [10], page 83, it is shown that the set $RP(f)$ is open. Thus, $CP(f) = M \backslash RP(f)$ is closed. □

Theorem 8.5 (Closedness of Critical Vales of f). *If f is proper, $C(f) \subseteq N$ is closed with respect to N.*

Proof. By theorem 8.4, we know that $CP(f)$ is closed. If f is proper, it is a closed mapping (i.e., maps closed sets into closed sets). Moreover, $f(CP(f)) = C(f)$. Therefore, $C(f)$ is the image of a closed set under a proper map. □

Theorem 8.6 (Stack of Records Theorem). *Assume that $m = n$ and f is proper. Let $y \in R(f)$ be s.t. $f^{-1}(y) \neq \emptyset$. Then,*
(1) $f^{-1}(y)$ is finite; i.e., $f^{-1}(y) = \{x_i, i \in \{1, 2, ..., q\}\}$ with $q < \infty$,
(2) $\exists$ open neighborhood of y in N, say N', open neighborhoods for each x_i in M, say M'_i, s.t. (a) $\forall i \neq j : M'_i \cap M'_j = \emptyset$, (b) $f^{-1}(N') = \cup_{i \in \{1,2,...,q\}} M'_i$ and (c) $\forall i \in \{1, 2, ..., q\} : f|_{M'_i}$ is a C^1 diffeomorphism.

Proof. See [10], page 94. □

Theorem 8.7 (Transversality Theorem). *Assume $\Omega \subseteq \mathbb{R}^w$ is a C^r u-dimensional manifold. Let $h : M \times \Omega \to N$ be a C^1 function. If $y \in R(h)$, then $\exists$ a full measure subset of Ω, say $\Omega^* \subseteq \Omega$, s.t. $\forall \omega \in \Omega^* : y \in R(h|_\omega)$.*

Proof. See [10], page 152. □

Theorem 8.8 (Degree Theorem). *Assume that $m = n$, M is compact and N is connected. Then, $\forall y_1, y_2 \in R(f) : \#f^{-1}(y_1) = \#f^{-1}(y_2)$ (where $\#$ stands for the mod2 degree).*

Proof. See [5], page 24. □

Definition 8.5 (Homotopy). Assume f and g are C^r. We say that f is homotopic to g if $\exists H : (M \times [0,1]) \to N$ s.t. (i) H is C^r and (ii) $\forall x \in M : H(x,0) = f(x)$ *and* $H(x,1) = g(x)$. The function H is called a C^r homotopy from f to g.

Theorem 8.9 (Existence using Homotopy Invariance). *Let M and N be C^2 m-dimensional manifolds. Assume that f and g are C^0 and C^1 respectively. Let $y \in R(g)$. If (i) $\#g^{-1}(y) = 1$ and (ii) $\exists$ a C^0 homotopy from f to g, say H, s.t. $H^{-1}(y)$ is compact; then $\#f^{-1}(y) = 1$. So in particular, $f^{-1}(y) \neq \emptyset$.*

Proof. See [10], page 200. □

Theorem 8.10 (Countability). *Let A be a countable set, B_n be the set of all n-tuples $(b_1, b_2, ..., b_n)$ s.t. $\forall k \in \{1, 2, ..., n\} : b_k \in A$. Then, B_n is a countable set.*

Proof. See [6], page 29. □

Theorem 8.11 (Image of Measure Zero Sets). *Let Θ_{t+1} and Θ_t be open subsets of $\mathbb{R}^n$. Assume that $\eta_t : \Theta_{t+1} \to \Theta_t$ is a C^1 mapping. Then, for every measure zero subset of Θ_{t+1}, say $\tilde{\Theta}_{t+1} \subseteq \Theta_{t+1}$, the set $\eta_t(\tilde{\Theta}_{t+1})$ is a measure zero subset of Θ_t.*

Proof. See [10], page 139. □

References

[1] Araujo, A, and J.A. Scheinkman. 1977. *Smoothness, Comparative Dynamics, and the Turnpike Property*. Econometrica, 45, No.3, 601-620.

[2] Balasko, Yves. 1997. *Equilibrium Analysis of the Infinite Horizon Model with Smooth Discounted Utility Functions*. Journal of Economic Dynamics and Control, 21, 783-829.

[3] Balasko, Yves. 1988. *Foundations for the Theory of General Equilibrium*. Academic Press, Boston.

[4] Hirsch, Morris W. 1976. *Differential Topology (Graduate Texts in Mathematics; 33)*. Springer-Verlag, New York.

[5] Milnor, John W. 1965. *Topology from the Differential Viewpoint*. Princeton University Press, Princeton, New Jersey.

[6] Rudin, Walter. 1976. *Principles of Mathematical Analysis*. Third Edition. McGraw-Hill, New York, NY.

[7] Santos, M. 1991. *Smoothness of the Policy Function in Discrete Time Economic Models*. Econometrica, 59, No.5, 1365-1382.

[8] Shannon, Chris. 1999. *Determinacy of Competitive Equilibria in Economies with Many Commodities*. Economic Theory, 14, 29-87.

[9] Stokey, Nancy L., Robert E. Lucas, with Edward C. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, Massachusetts.

[10] Villanacci, Antonio, Laura Carosi, Pierluigi Beneveri, and Andrea Battinelli. 2002. *Differential Topology and General Equilibrium with Complete and Incomplete Markets*. Kluwer Academic Publishers, Boston.