Idiosyncratic risk and economic policy [1]

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Work in progress

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Abstract

In economies subject to aggregate as well as uninsurable idiosyncratic risks, competitive equilibrium allocations are constrained suboptimal: reallocations of assets support Pareto superior allocations. This is the case even if the asset market for the allocation of aggregate risks is complete.

Key words: uninsurable idiosyncratic risks; constrained suboptimaslity.

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Idiosyncratic risk does not affect the allocation of resources at Pareto optimal allocations.[1] At competitive equilibria, this is indeed the case if the asset market for the insurance of idiosyncratic risk is complete.

If realizations of idiosyncratic shocks are publicly unobservable or unverifiable, idiosyncratic risks may well not be insurable; indeed, this is a standard assumption in macroeconomic theory.[2]

We show that in economies subject to aggregate as well as uninsurable idiosyncratic risk, competitive equilibrium allocations are constrained suboptimal: reallocations of assets support Pareto superior allocations. This is the case even if the asset market for the allocation of aggregate risks is complete.

Our argument extends the argument for constrained suboptimality in economies with an incomplete asset market in Geanakoplos and Polemarchakis (1986). [3]

The identification of fundamentals from the equilibrium manifold is an issue.[4]

1 An example

Two types of individuals, $i = \alpha, \beta$, each consists of a continuum of individuals of unit mass.

Dates are 0 and 1.

One commodity is exchanged and consumed at date 0, and quantities of the commodity are x, while two commodities, $l = a, b$ are exchanged and consumed at date 1, and quantities of the commodities are x_a and x_b.

The intertemporal utility function of an individual of type β is

$$u^\beta = x + (1-\gamma)\ln x_a + \gamma \ln x_b, \quad 0 < \gamma < 1,$$

and his endowment at date 1 consists of b units of commodity b.

The intertemporal utility function of an individual of type α is

$$u^\alpha = x + \gamma \ln x_a + (1-\gamma)\ln x_b,$$

and his endowment at date 1 consists only of commodity a; but, importantly, it is subject to idiosyncratic shocks: it is $a \pm \varepsilon$, with equal probability.

At date 1, equal proportions of individuals of type α have endowments $a+\varepsilon$, and $a-\varepsilon$, and, as a consequence, there is no aggregate risk.

With quasi-linear preferences, it is not necessary to specify the endowments of individuals at date 0.

At date 0, the consumption good is numéraire, while q is a price of a risk-free bond of that matures at date 1.

At date 1, commodity a is numéraire, while the price of commodity b is p.

[1] Arrow and Lind (1970), Malinvaud (1973a and 1973b).
[2] Krebs (2003), Krussell and Smith (1999).
[3] Also, Citanna, Kajii and Villanacci (1998).
[4] Carvajal and Riascos (2004), Kübler, Chiappori, Ekeland and Polemarchakis (2002).

With holdings of the bond y for individuals of type α and $-y$ for individuals of type β, the equilibrium price at date 1 is

$$p(y) = \frac{(1-\gamma)a + (1-2\gamma)y}{(1-\gamma)b}.$$

The price at date 1 depends non-trivially on asset holdings as long as

$$\gamma \neq \frac{1}{2}.$$

At date 1, the marginal utility of revenue for individuals of type β is

$$\lambda^\beta = \frac{1}{pb - y},$$

while, for individuals of type α, it varies with the realization of the idiosyncratic shock or the personal state of an individual and it is

$$\lambda^\alpha(\varepsilon) = \frac{1}{a+\varepsilon+y}, \quad \text{or} \quad \lambda^\alpha(-\varepsilon) = \frac{1}{a+\varepsilon-y},$$

with equal probability.

The optimization of individuals of type β at date 0 requires that

$$q = \frac{1}{pb-y} = \frac{(1-\gamma)}{(1-\gamma)a - \gamma y},$$

while optimization of individuals of type α at date 0 requires that

$$q = (\frac{1}{2})\frac{1}{a+\varepsilon+y} + (\frac{1}{2})\frac{1}{a-\varepsilon+y} = \frac{a+y}{(a+y)^2-\varepsilon^2};$$

as a consequence, at equilibrium,

$$y^* = \frac{-a + \sqrt{a^2 + 4\varepsilon^2(1-\gamma)}}{2}.$$

Policy is a pair (dx, dy) of a transfer of revenue and bonds to individuals of type α.

The welfare effects of a policy are

$$du^\alpha = dx + qdy - \left((\frac{1}{2})\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + (\frac{1}{2})\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon)\right)p'dy,$$

and

$$du^\beta = -dx - qdy - \lambda^\beta(x_b^\beta - b)p'dy.$$

Pareto improving interventions exist if the matrix

$$\begin{pmatrix} 1 & q - \left((\frac{1}{2})\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + (\frac{1}{2})\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon)\right)p' \\ -1 & -q - \lambda^\beta(x_b^\beta - b)p' \end{pmatrix}$$

is nonsingular, which is the case, for $\varepsilon \neq 0$: singularity of the matrix would occur if and only if

$$\frac{1}{2}\lambda^{\alpha}(\varepsilon)x_b^{\alpha}(\varepsilon) + \frac{1}{2}\lambda^{\alpha}(-\varepsilon)x_b^{\alpha}(-\varepsilon) = -\lambda^{\beta}(x_b^{\beta} - b),$$

which is equivalent to

$$\frac{1-\gamma}{p} = -\frac{1}{pb-y}\left(\frac{\gamma\,(pb-y)}{p} - b\right),$$

or $y = 0$, which occurs only in the absence of idiosyncratic shocks, with $\varepsilon = 0$.

2 The economy

Individuals are of types $i = 1, \ldots, I$; within each type, there is a continuum of individuals of mass 1.

Dates are 0 and 1.

At date 1, aggregate risk is described by states of the world, $\sigma = 1, \ldots, \Sigma$; idiosyncratic risk is described by personal states, $s = 1, ..., S$.

Date-events are $0, \ldots, \sigma, \ldots$; personal date-events are $0, \ldots, (\sigma, s), \ldots$.

Commodities are $l = 1, \ldots L$; a bundle of commodities at a date-event is $x = (\ldots, x_l, \ldots)$. [5]

Individuals are heterogeneous: $I \geq 2$; neither aggregate nor individual risk vanish: $\Sigma \geq 2$, $S \geq 2$; and the set of commodities is sufficiently diverse: $L \geq IS$.

A consumption plan is $x = (x_0, \ldots, x_{\sigma,s}, \ldots) \gg 0$.

The distribution of individuals of type i across personal states at the aggregate state σ is $\pi^i_\sigma = (\ldots, \pi^i(s|\sigma), \ldots)$, which can be interpreted as a conditional probability measure over personal states; across states of aggregate risk, $\pi^i = (\ldots, \pi^i_\sigma, \ldots)$.

The preferences of an individual over consumption plans are described by the utility function[6]

$$u^i(x) = u^i_0(x_0) + \sum_{\sigma} \mathrm{E}_{\pi^i_\sigma} u^i_\sigma(x_\sigma).$$

The temporal, cardinal utility indices, belong to $\mathcal{U}$, the set of strictly monotonic, strongly concave, C^3 functions $v : \mathbb{R}^L_{++} \longrightarrow \mathbb{R}$, that satisfy the following interiority condition: for every sequence $(x_n)_{n=1}^{\infty}$ of strictly positive consumption bundles, if it converges to a bundle, x in the boundary of the consumption set, $x_n \longrightarrow x$, then $\|Dv\,(x_n)\|^{-1} Dv\,(x_n) \cdot x_n \longrightarrow 0$ and $\|Dv\,(x_n)\|^{-1} \longrightarrow \infty$. The space $\mathcal{U}$ is endowed with the (metrizable) topology of C^3, uniform convergence on compacta.

[5] It should be clear whether x denotes a consumption bundle, consumption plan or, later, an allocation of consumption plans.

[6] Given σ and i, for a personal random variable $\chi^i_\sigma = \ldots, \chi^i_{\sigma,s}, \ldots$, $\mathrm{E}_{\pi^i_\sigma} = \sum_s \pi^i\,(s|\,\sigma)\,\chi^i_{\sigma,s}$.

The endowment of an individual is $e^i = (e^i_0, \ldots, e^i_{\sigma,s}, \ldots) \gg 0$.

An economy is $(e, u, \pi) = (\ldots, (e^i, u^i, \pi^i), \ldots)$,endowed with the product topology.

An allocation is $x = (\ldots, x^i, \ldots)$; it is feasible if

$$\sum_i x^i_0 = \sum_i e^i_0,$$

$$\sum_i \mathrm{E}_{\pi^i} x^i_\sigma = \sum_i \mathrm{E}_{\pi^i} e^i_\sigma.$$

For simplicity, prices and gradients are denoted as row vectors, whereas quantities are denoted as column vectors.

3 Pareto Efficiency

A feasible allocation, x, is Pareto optimal if there does not exist another, feasible allocation, $_1$, such that $u^i(x^i_1) \geq u^i(x^i)$ for every type, with strict inequality for some.

Lemma 1. *An allocation, x, is Pareto optimal only if*

$$Du^i_\sigma \left(x^i_{\sigma,s}\right) = Du^i_\sigma \left(x^i_{\sigma,s'}\right),$$

and there exists $\gamma = (\ldots, \gamma^i, \ldots) \gg 0$, such that

$$\gamma^i Du^i_0 \left(x^i_0\right) = \gamma^{i'} Du^{i'}_0 \left(x^{i'}_0\right),$$

$$\gamma^i Du^i_\sigma \left(x^i_{\sigma,s}\right) = \gamma^{i'} Du^{i'}_\sigma \left(x^{i'}_{\sigma,s'}\right).$$

Proof. If the allocation is Pareto optimal, then, it solves the problem

$$\max_{\widehat{x}} u^1_0 \left(\widehat{x}^1_0\right) + \sum_\sigma \sum_s \pi^1 \left(s|\,\sigma\right) u^1_\sigma \left(\widehat{x}^1_{\sigma,s}\right),$$

$$\text{s.t.} \begin{cases} u^1_0 \left(\widehat{x}^1_0\right) + \sum_\sigma \sum_s \pi^1 (s|\sigma)\, u^i_\sigma \left(\widehat{x}^i_{\sigma,s}\right) \geq \underline{u}^i, \quad i = 2, \ldots, I, \\ \sum_i \left(\widehat{e}^i_0 - \widehat{x}^i_0\right) = 0, \\ \sum_i \sum_s \pi^i (s|\sigma) \left(\widehat{e}^i_{\sigma,s} - \widehat{x}^i_{\sigma,s}\right) = 0, \end{cases}$$

where

$$\underline{u}^i = u^i_0 \left(x^i_0\right) + \sum_\sigma \sum_s \pi^i (s|\sigma)\, u^i_\sigma \left(x^i_{\sigma,s}\right), \quad i = 2, \ldots, I.$$

First order necessary conditions yield the result. □

4 Competitive Equilibrium

At a date-event, prices of commodities are $p = (1, \ldots, p_l, \ldots) \in \mathcal{P}$. Across date-events, prices of commodities are $p = (p_0, \ldots, p_\sigma, \ldots) \in \mathcal{P}^{\Sigma+1}$. [7]

Elementary securities, $\sigma = 1, \ldots, \Sigma$, transfer wealth across states of the world. A portfolio of elementary securities is $y = (\ldots, y_\sigma, \ldots)$, and prices of elementary securities are $q = (\ldots, q_\sigma, \ldots)$.

The optimization problem of an individual is

$$\begin{aligned} \max_{x,y} \quad & u^i(x) = u_0^i(x_0) + \textstyle\sum_\sigma \mathrm{E}_{\pi_\sigma^i} u_\sigma^i(x_\sigma) \\ \text{s.t.} \quad & \Psi(p)\left(x - e^i\right) = R(q)\, y, \end{aligned}$$

where

$$\Psi(p) = \begin{bmatrix} p_0 & 0 & 0 & \ldots & 0 \\ 0 & \left(I_S \otimes p_1^\top\right)^\top & 0 & \ldots & 0 \\ 0 & 0 & \left(I_S \otimes p_2^\top\right)^\top & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \left(I_S \otimes p_\Sigma^\top\right)^\top \end{bmatrix}$$

is a matrix of dimension $(\Sigma S + 1) \times (\Sigma S + 1) L$, with

$$I_S \otimes p_\sigma^\top = \begin{bmatrix} p_\sigma^\top & 0 & \ldots & 0 \\ 0 & p_\sigma^\top & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_\sigma^\top \end{bmatrix}_{LS \times S},$$

the standard Kronecker product, and

$$R(q) = \begin{bmatrix} -q \\ I_\Sigma \otimes \mathbf{1}_S^\top \end{bmatrix},$$

a matrix of dimensions $(\Sigma S + 1) \times \Sigma$, with $\mathbf{1}_S$ denoting the (row) vector $(1, \ldots, 1, \ldots, 1)$ of dimension S.

[7] It should be clear whether p denotes prices at a date-event or across date-events.

First order necessary and sufficient conditions for a solution to the optimization problem of an individual are

$$Du^i\left(x^i\right) = \lambda^i \Psi\left(p\right),$$

$$\lambda^i R\left(q\right) = 0,$$

$$\Psi\left(p\right)\left(x^i - e^i\right) = R\left(q\right) y^i,$$

where $\lambda^i = (\lambda_0^i, \ldots, \lambda_\sigma^i, \ldots)$, and

$$Du^i\left(x^i\right)^\top = \begin{bmatrix} Du_0^i\left(x_0^i\right)^\top \\ \pi^i\left(1|\,1\right) Du_1^i\left(x_{1,1}^i\right)^\top \\ \pi^i\left(2|\,1\right) Du_1^i\left(x_{1,2}^i\right)^\top \\ \vdots \\ \pi^i\left(S^i\big|\,1\right) Du_1^i\left(x_{1,S^1}^i\right)^\top \\ \pi^i\left(1|\,2\right) Du_1^i\left(x_{2,1}^i\right)^\top \\ \vdots \\ \pi^i\left(S^i\big|\,\Sigma\right) Du_1^i\left(x_{\Sigma,S^i}^i\right)^\top \end{bmatrix},$$

a vector of dimensions $\left(\Sigma S^i + 1\right) L \times 1$.

For an economy, (e, u, π), an **equilibrium** is (x, y, p, q), allocations and prices of commodities and portfolios of assets, such that individuals optimize and markets clear.

We now maintain (u, π) fixed (and omit them as arguments). In later sections, we shall allow u to vary.

The function, $\mathcal{F}$, defined by

$$\mathcal{F}(x,\lambda,y,p,q,e) = \begin{bmatrix} \left(Du^1\left(x^1\right)-\lambda^1\Psi(p)\right)^\top \\ R(q)\,y^1-\Psi(p)\left(x^1-e^1\right) \\ \left(\lambda^1 R(q)\right)^\top \\ \vdots \\ \left(Du^I\left(x^I\right)-\lambda^I\Psi(p)\right)^\top \\ R(q)\,y^I-\Psi(p)\left(x^I-e^I\right) \\ \left(\lambda^I R(q)\right)^\top \\ \sum_i\left(\widetilde{e}^i_0-\widetilde{x}^i_0\right) \\ \sum_i\sum_s \pi^i\left(s|\,1\right)\left(\widetilde{e}^i_{1,s}-\widetilde{x}^i_{1,s}\right) \\ \vdots \\ \sum_i\sum_s \pi^i\left(s|\,\Sigma\right)\left(\widetilde{e}^i_{\Sigma,s}-\widetilde{x}^i_{\Sigma,s}\right) \\ \sum_i, \end{bmatrix},$$

where $\widetilde{e}$ and $\widetilde{x}$ exclude the numéraire commodity at all states, maps

$$\mathbb{R}^{I(\Sigma S+1)L}_{++}\times\mathbb{R}^{\Sigma I}\times\mathbb{R}^{I(\Sigma S+1)}_{++}\times\mathcal{P}^{\Sigma+1}\times\mathbb{R}^{\Sigma}\times\mathbb{R}^{I(\Sigma S+1)L}_{++}$$

into

$$\left(\mathbb{R}^{(\Sigma S+1)L}_{++}\times\mathbb{R}^{(\Sigma S+1)}_{++}\times\mathbb{R}^{\Sigma}\right)^I\times\mathbb{R}^{(\Sigma+1)(L-1)}_{++}\times\mathbb{R}^{\Sigma},$$

and it is straightforward that (x,y,p,q) is an equilibrium for endowments e if, and only if, there exists $\lambda=(\ldots,\lambda^i,\ldots)$, such that $\mathcal{F}(x,\lambda,y,p,q,e)=0$; we also refer to (x,λ,y,p,q) as an equilibrium for e.

Given (x,λ,y,p,q,e), define $D^2u^i=D^2u^i\left(x^i\right)$ by

$$\begin{bmatrix} D^2u^i_0\left(x^i_0\right) & 0 & \ldots & 0 \\ 0 & \pi^i\left(1|\,1\right)D^2u^i_1\left(x^i_{1,1}\right) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \pi^i\left(S|\,\Sigma\right)D^2u^i_1\left(x^i_{\Sigma,S}\right) \end{bmatrix};$$

also, denote $\widetilde{I}$the $(L-1)\times L$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and define, for each i, the $(\Sigma+1)(L-1)\times(\Sigma S+1)L$ matrix $\Phi^i = \Phi\left(\pi^i\right)$ by

$$\begin{bmatrix} \widetilde{I} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \pi^i(1|1)\widetilde{I} & \dots & \pi^i(S|1)\widetilde{I} & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & \pi^i(1|\Sigma)\widetilde{I} & \dots & \pi^i(S|\Sigma)\widetilde{I} \end{bmatrix},$$

the $(1+\Sigma S)L\times(1+\Sigma)(L-1)$ matrix $\Lambda^i = \Lambda\left(\lambda^i\right)$ by

$$\begin{bmatrix} \lambda_0^i\widetilde{I}^\top & 0 & 0 & \dots & 0 \\ 0 & \lambda_{1,1}^i\widetilde{I}^\top & 0 & \dots & 0 \\ 0 & \lambda_{1,2}^i\widetilde{I}^\top & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{1,S}^i\widetilde{I}^\top & 0 & \dots & 0 \\ 0 & 0 & \lambda_{2,1}^i\widetilde{I}^\top & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{\Sigma,S}^i\widetilde{I}^\top \end{bmatrix},$$

the $(1+\Sigma S)\times(1+\Sigma)(L-1)$ matrix $Z^i = Z\left(x^i, e^i\right)$ by

$$\begin{bmatrix} \left(\widetilde{e}_0^i-\widetilde{x}_0^i\right)^\top & 0 & 0 & \dots & 0 \\ 0 & \left(\widetilde{e}_{1,1}^i-\widetilde{x}_{1,1}^i\right)^\top & 0 & \dots & 0 \\ 0 & \left(\widetilde{e}_{1,2}^i-\widetilde{x}_{1,2}^i\right)^\top & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \left(\widetilde{e}_{1,S}^i-\widetilde{x}_{1,S}^i\right)^\top & 0 & \dots & 0 \\ 0 & 0 & \left(\widetilde{e}_{2,1}^i-\widetilde{x}_{2,1}^i\right)^\top & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \left(\widetilde{e}_{\Sigma,S}^i-\widetilde{x}_{\Sigma,S}^i\right)^\top \end{bmatrix}$$

and the $(1+\Sigma S)\times\Sigma$ matrix $Y^i = Y\left(y^i\right)$ by

$$\begin{bmatrix} y^{i\top} \\ 0 \end{bmatrix}.$$

With the arguments of $\mathcal{F}$ in the order

$$\left(\; x^1, \quad \lambda^1, \quad y^1, \quad \ldots \quad x^I, \quad \lambda^I, \quad y^I, \quad p, \quad q, \quad e^1, \quad \ldots \quad e^I \;\right),$$

$D\mathcal{F}\left(x, \lambda, y, p, q, e\right)$ is

$$\begin{bmatrix} D^2u^1 & -\Psi^\top & 0 & \ldots & 0 & 0 & 0 & \Lambda^1 & 0 & 0 & \ldots & 0 \\ -\Psi & 0 & R & \ldots & 0 & 0 & 0 & Z^1 & Y^1 & \Psi & \ldots & 0 \\ 0 & R^\top & 0 & \ldots & 0 & 0 & 0 & 0 & \lambda_0^1 I_\Sigma & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & D^2u^I & -\Psi^\top & 0 & \Lambda^I & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -\Psi & 0 & R & Z^I & Y^I & 0 & \ldots & \Psi \\ 0 & 0 & 0 & \ldots & 0 & R^\top & 0 & 0 & \lambda_0^I I_\Sigma & 0 & \ldots & 0 \\ \Phi^1 & 0 & 0 & \ldots & \Phi^I & 0 & 0 & 0 & 0 & -\Phi^1 & \ldots & -\Phi^I \\ 0 & 0 & I_\Sigma & \ldots & 0 & 0 & I_\Sigma & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

5 Existence of Equilibria

Let x^e be efficient an efficient allocation for e; by lemma 1, there exist $\gamma^e = (\ldots, \gamma^{e,i}, \ldots)$, such that

$$\gamma^{e,i} Du_0^i \left(x_0^{e,i}\right) = \gamma^{e,i'} Du_0^{i'} \left(x_0^{e,i'}\right)$$

and

$$\gamma^{e,i} Du_\sigma^i \left(x_{\sigma,s}^{e,i}\right) = \gamma^{e,i'} Du_\sigma^{i'} \left(x_{\sigma,s'}^{e,i'}\right).$$

Construct $(\lambda^e, y^e, p^e, q^e)$ as follows:

$$\begin{aligned} p_0^e &= \left(\frac{\partial u_0^1}{\partial x_{0,1}} \left(x_0^{e,1}\right)\right)^{-1} Du_0^1 \left(x_0^{e,1}\right), \\ p_\sigma^e &= \left(\frac{\partial u_\sigma^1}{\partial x_{\sigma,1,1}} \left(x_{\sigma,1}^{e,1}\right)\right)^{-1} Du_\sigma^1 \left(x_{\sigma,1}^{e,1}\right), \\ \lambda_0^{e,1} &= \frac{\partial u_0^1}{\partial x_{0,1}} \left(x_0^{e,1}\right), \\ \lambda_{\sigma,1}^{e,1} &= \frac{\partial u_\sigma^1}{\partial x_{\sigma,1,1}} \left(x_{\sigma,1}^{e,1}\right), \\ \lambda_0^{e,i} &= \frac{\gamma^{e,i}}{\gamma^{e,1}} \lambda_0^{e,1}, \\ \lambda_{\sigma,s}^{e,i} &= \frac{\gamma^{e,i} \pi^i \left(s|\, \sigma\right)}{\gamma^{e,1} \pi^1 \left(1|\, \sigma\right)} \lambda_0^{e,1}, \\ y^{e,i} &= 0, \\ q_\sigma^e &= \sum_s \frac{\lambda_{\sigma,s}^{e,1}}{\lambda_0^{e,1}}. \end{aligned}$$

Lemma 2. *$\mathcal{F}(x,\lambda,y,p,q,x^e)=0$ if, and only if, $(x,\lambda,y,p,q)=(x^e,\lambda^e,y^e,p^e,q^e)$.*

Proof. By construction, $\pi^1\left(s^1\middle|\,\sigma\right)\sum_s \lambda^{e,i}_{\sigma,s}=\frac{\gamma^{e,1}}{\gamma^{e,i}}\lambda^{e,1}_{\sigma,s^1}$, for every (i,σ,s^1), which implies that

$$\sum_s \lambda^{e,i}_{\sigma,s}=\frac{\gamma^{e,1}}{\gamma^{e,i}}\sum_{s^1}\lambda^{e,1}_{\sigma,s^1}=\frac{\gamma^{e,1}}{\gamma^{e,i}}\lambda^{e,1}_0 q^e_\sigma=\lambda^{e,i}_0 q^e_\sigma.$$

It is straightforward, then, that $\mathcal{F}(x^e,\lambda^e,y^e,p^e,q^e,x^e)=0$, by construction.

Now, fix (x,λ,y,p,q) such that $\mathcal{F}(x,\lambda,y,p,q,x^e)=0$. By strong concavity, since x^e is efficient given e, it follows that $x=x^e$. It is immediate, by construction, that $\mathcal{F}(x^e,\lambda,y,p,q,x^e)=0$ implies $(\lambda,y,p,q)=(\lambda^e,y^e,p^e,q^e)$. □

Lemma 3. *$D_{x,\lambda,y,p,q}\mathcal{F}(x^e,\lambda^e,y^e,p^e,q^e,x^e)$ has full row rank.*

Proof. Since $D_{x,\lambda,y,p,q}\mathcal{F}$ is square, it suffices to show that $D_{x,\lambda,y,p,q}\mathcal{F}(x^e,\lambda^e,y^e,p^e,q^e,x^e)$ has full column rank.

By definition, $D_{x,\lambda,y,p,q}\mathcal{F}(x^e,\lambda^e,y^e,p^e,q^e,x^e)$ is

$$\begin{bmatrix} D^2u^1 & -\Psi^\top & 0 & \dots & 0 & 0 & 0 & \Lambda^1 & 0\\ -\Psi & 0 & R & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & R^\top & 0 & \dots & 0 & 0 & 0 & 0 & \lambda^1_0 I_\Sigma\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & D^2u^I & -\Psi^\top & 0 & \Lambda^I & 0\\ 0 & 0 & 0 & \dots & -\Psi & 0 & R & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & R^\top & 0 & 0 & \lambda^I_0 I_\Sigma\\ \Phi^1 & 0 & 0 & \dots & \Phi^I & 0 & 0 & 0 & 0\\ 0 & 0 & I_\Sigma & \dots & 0 & 0 & I_\Sigma & 0 & 0 \end{bmatrix}.$$

Fix $\theta\neq 0$ such that $D_{x,\lambda,y,p,q}\mathcal{F}(x^e,\lambda^e,y^e,p^e,q^e,x^e)\,\theta=0$. Denote θ by its associated columns:

$$\theta^\top=\left[\begin{array}{ccccccccc}\theta^\top_{x^1}, & \theta^\top_{\lambda^1}, & \theta^\top_{y^1}, & \dots & \theta^\top_{x^I}, & \theta^\top_{\lambda^I}, & \theta^\top_{y^I}, & \theta^\top_{\widetilde{p}}, & \theta^\top_q\end{array}\right].$$

By definition,

1. $D^2u^i_0\left(x^{e,i}_0\right)\theta_{x^i_0}-p^{e\top}_0\theta_{\lambda^i_0}+\lambda^i_0\left[\begin{array}{cc}0 & \theta_{\widetilde{p}_0}\end{array}\right]^\top=0$;
2. $\pi^i\left(s\middle|\,\sigma\right)D^2u^i_\sigma\left(x^{e,i}_{\sigma,s}\right)\theta_{x^i_{\sigma,s}}-p^{e\top}_\sigma\theta_{\lambda^i_{\sigma,s}}+\lambda^i_{\sigma,s}\left[\begin{array}{cc}0 & \theta_{\widetilde{p}_\sigma}\end{array}\right]^\top=0$;
3. $-p^e_0\theta_{x^i_0}-q^e\theta_{y^i}=0$;
4. $-p^e_\sigma\theta_{x^i_{\sigma,s}}+\theta_{y^i_\sigma}=0$;
5. $-q_\sigma\theta_{\lambda^i_0}+\sum_\sigma\theta_{\lambda^i_{\sigma,s}}+\lambda^i_0\theta_{q_\sigma}=0$;
6. $\sum_i\theta_{\widetilde{x}^i_0}=0$;

7. $\sum_i \sum_s \pi\left(s|\,\sigma\right)\theta_{\widetilde{x}^i_{\sigma,s}} = 0$;

8. $\sum_i \theta_{y^i} = 0$.

Suppose that $\theta_{x^1} = \theta_{x^2} = ... = \theta_{x^I} = 0$. Then, by 1 and 2, $\theta_{\lambda^1} = \theta_{\lambda^2} = ... = \theta_{\lambda^I} = 0$ and $\theta_{p_0} = \theta_{p_1} = ... = \theta_{p_\Sigma} = 0$, which implies, by 4, that $\theta_{y^1} = \theta_{y^2} = ... = \theta_{y^I} = 0$, whereas by 5, $\theta_q = 0$. It follows that for some i, $\theta_{x^i} \neq 0$.

Since $\mathcal{F}\left(x^e, \lambda^e, y^e, p^e, q^e, x^e\right) = 0$, it follows that $Du^i_0\left(x^{e,i}_0\right) = \lambda^{e,i}_0 p^e_0$ and $\pi^i\left(s|\,\sigma\right) Du^i_\sigma\left(x^{e,i}_{\sigma,s}\right) = \lambda^{e,i}_{\sigma,s} p^e_\sigma$. Also, $\lambda^{e,i}_0 q^e_\sigma = \sum_s \lambda^{e,i}_{\sigma,s}$. So, by 3, $Du^i_0\left(x^{e,i}_0\right)\theta_{x^i_0} = -\lambda^{e,i}_0 q^e \theta_{y^i}$, whereas $\pi^i\left(s|\,\sigma\right) Du^i_\sigma\left(x^{e,i}_{\sigma,s}\right)\theta_{x^i_{\sigma,s}} = \lambda^{e,i}_{\sigma,s}\theta_{y^i_\sigma}$, by 4, and, hence,

$$\begin{aligned} Du^i\left(x^i\right)\theta_{x^i} &= -\lambda^{e,i}_0 q^e \theta_{y^i} + \sum_\sigma \sum_s \lambda^{e,i}_{\sigma,s}\theta_{y^i_\sigma} \\ &= \sum_\sigma \left(-\lambda^{e,i}_0 q^e_\sigma \theta_{y^i_\sigma} + \sum_s \lambda^{e,i}_{\sigma,s}\theta_{y^i_\sigma}\right) \\ &= \sum_\sigma \left(-\lambda^{e,i}_0 q^e_\sigma + \sum_s \lambda^{e,i}_{\sigma,s}\right)\theta_{y^i_\sigma} \\ &= 0. \end{aligned}$$

Now, by 1 and 3,

$$\theta^\top_{x^i_0} D^2 u^i_0\left(x^{e,i}_0\right)\theta_{x^i_0} = \theta_{\lambda^i_0}\theta^\top_{x^i_0} p^{e\top}_0 - \lambda^i_0 \theta^\top_{\widetilde{x}^i_0}\theta^\top_{\widetilde{p}_0} = -\theta_{\lambda^i_0} q^e \theta_{y^i} - \lambda^i_0 \theta^\top_{\widetilde{x}^i_0}\theta^\top_{\widetilde{p}_0},$$

whereas by 2 and 4,

$$\begin{aligned} \pi^i\left(s|\,\sigma\right)\theta^\top_{x^i_{\sigma,s}} D^2 u^i_\sigma\left(x^{e,i}_{\sigma,s}\right)\theta_{x^i_{\sigma,s}} &= \theta_{\lambda^i_{\sigma,s}}\theta^\top_{x^i_{\sigma,s}} p^{e\top}_\sigma - \lambda^i_{\sigma,s}\theta^\top_{\widetilde{x}^i_{\sigma,s}}\theta^\top_{\widetilde{p}_\sigma} \\ &= \theta_{\lambda^i_{\sigma,s}}\theta_{y^i_\sigma} - \lambda^i_{\sigma,s}\theta^\top_{\widetilde{x}^i_{\sigma,s}}\theta^\top_{\widetilde{p}_\sigma}, \end{aligned}$$

which implies, by 5, that

$$\begin{aligned} \sum_i \gamma^{e,i}\theta^\top_{x^i} D^2 u^i \theta_{x^i} &= \sum_{i,\sigma}\gamma^{e,i}\left(-\theta_{\lambda^i_0} q^e_\sigma + \sum_s \theta_{\lambda^i_{\sigma,s}}\right)\theta_{y^i_\sigma} \\ &\quad - \sum_i \gamma^{e,i}\left(\lambda^i_0 \theta^\top_{\widetilde{x}^i_0}\theta^\top_{\widetilde{p}_0} + \sum_{\sigma,s}\lambda^i_{\sigma,s}\theta^\top_{\widetilde{x}^i_{\sigma,s}}\theta^\top_{\widetilde{p}_\sigma}\right) \\ &= -\sum_{i,\sigma}\gamma^{e,i}\lambda^i_0 \theta_{q_\sigma}\theta_{y^i_\sigma} \\ &\quad - \sum_i \gamma^{e,i}\left(\lambda^i_0 \theta^\top_{\widetilde{x}^i_0}\theta^\top_{\widetilde{p}_0} + \sum_{\sigma,s}\lambda^i_{\sigma,s}\theta^\top_{\widetilde{x}^i_{\sigma,s}}\theta^\top_{\widetilde{p}_\sigma}\right). \end{aligned}$$

Since, $\gamma^{e,i}\lambda_0^{e,i} = \gamma^{e,1}\lambda_0^{e,1}$ and $\gamma^{e,i}\lambda_{\sigma,s}^{e,i} = \gamma^{e,1}\lambda_{\sigma,1}^{e,1}\pi^i\left(s|\,\sigma\right)/\pi^1\left(1|\,\sigma\right)$,

$$\begin{aligned}
\sum_i \gamma^{e,i}\theta_{x^i}^{\top}D^2u^i\theta_{x^i} &= -\sum_{i,\sigma}\gamma^{e,1}\lambda_0^{e,1}\theta_{q_\sigma}\theta_{y_\sigma^i} \\
&\quad -\sum_i \gamma^{e,1}\lambda_0^{e,1}\theta_{\widetilde{x}_0^i}^{\top}\theta_{\widetilde{p}_0}^{\top} - \sum_{i,\sigma,s}\gamma^{e,1}\lambda_{\sigma,1}^{e,1}\frac{\pi^i\left(s|\,\sigma\right)}{\pi^1\left(1|\,\sigma\right)}\theta_{\widetilde{x}_{\sigma,s}^i}^{\top}\theta_{\widetilde{p}_\sigma}^{\top} \\
&= -\sum_\sigma\left(\sum_i \theta_{y_\sigma^i}\right)\gamma^{e,1}\lambda_0^{e,1}\theta_{q_\sigma} - \gamma^{e,1}\lambda_0^{e,1}\left(\sum_i \theta_{\widetilde{x}_0^i}^{\top}\right)\theta_{\widetilde{p}_0}^{\top} \\
&\quad -\sum_\sigma \gamma^{e,1}\frac{\lambda_{\sigma,1}^{e,1}}{\pi^1\left(1|\,\sigma\right)}\left(\sum_{i,s}\pi^i\left(s|\,\sigma\right)\theta_{\widetilde{x}_{\sigma,s}^i}^{\top}\right)\theta_{\widetilde{p}_\sigma}^{\top} \\
&= 0,
\end{aligned}$$

by 6, 7 and 8.

The latter, however, is impossible by strong concavity, since for some i, $\theta_{x^i} \neq 0$ and $\gamma^e \gg 0$. □

Lemma 4. *If $(x_n, \lambda_n, y_n, p_n, q_n, \tau_n)_{n=1}^{\infty}$, with $(\tau_n)_{n=1}^{\infty}$ defined in $[0,1]$, satisfies that, for every n,*

$$\mathcal{F}\left(x_n, \lambda_n, y_n, p_n, q_n, \left(1-\tau_n\right)e + \tau_n x^e\right) = 0$$

then, there exists a subsequence $\left(x_{n(k)}, \lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}, \tau_{n(k)}\right)_{k=1}^{\infty}$ that converges to some $\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, \overline{\tau}\right)$, such that

$$\mathcal{F}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, , \left(1-\overline{\tau}\right)e + \overline{\tau}x^e\right) = 0$$

Proof. Since $(\tau_n)_{n=1}^{\infty}$ is defined in $[0,1]$, it has a convergent subsequence, which implies that $(x_n)_{n=1}^{\infty}$ has a convergent subsequence, because

$$\mathcal{F}\left(x_n, \lambda_n, y_n, p_n, q_n, \left(1-\tau_n\right)e + \tau_n x^e\right) = 0.$$

Let $\left(x_{n(k)}, \tau_{n(k)}\right)_{k=1}^{\infty} \longrightarrow \left(\overline{x}, \overline{\tau}\right)$. By construction,

$$\begin{aligned}
\lambda_{n(k),0}^i &= \frac{\partial u_0^i}{\partial x_{0,1}}\left(x_{n(k),0}^i\right) \longrightarrow \frac{\partial u_0^i}{\partial x_{0,1}}\left(\overline{x}_0^i\right), \\
\lambda_{n(k),\sigma,s}^i &= \pi^i\left(s|\,\sigma\right)\frac{\partial u_\sigma^i}{\partial x_{\sigma,s,1}}\left(x_{n(k),\sigma,s}^i\right) \longrightarrow \pi^i\left(s|\,\sigma\right)\frac{\partial u_\sigma^i}{\partial x_{\sigma,s,1}}\left(\overline{x}_{\sigma,s}^i\right).
\end{aligned}$$

Define $\overline{\lambda}_0^i = \frac{\partial u_0^i}{\partial x_{0,1}}\left(\overline{x}_0^i\right)$ and $\overline{\lambda}_{\sigma,s}^i = \pi^i\left(s|\,\sigma\right)\frac{\partial u_\sigma^i}{\partial x_{\sigma,s,1}}\left(\overline{x}_{\sigma,s}^i\right)$, so that $\lambda_{n(k)} \longrightarrow \overline{\lambda}$. Also,

$$\begin{aligned}
p_{n(k),0} &= \frac{1}{\lambda_{n(k),0}^1}Du_0^1\left(x_{n(k),0}^1\right) \longrightarrow \frac{1}{\overline{\lambda}_0^1}Du_0^1\left(\overline{x}_0^1\right), \\
p_{n(k),\sigma} &= \frac{1}{\lambda_{n(k),\sigma,1}^1}\pi^1\left(1|\,\sigma\right)Du_\sigma^1\left(x_{n(k),\sigma,s}^i\right) \longrightarrow \frac{1}{\overline{\lambda}_{\sigma,1}^1}\pi^1\left(1|\,\sigma\right)Du_\sigma^1\left(\overline{x}_{\sigma,s}^1\right),
\end{aligned}$$

and define $\overline{p}_0 = \frac{1}{\overline{\lambda}_0^1} Du_0^1\left(\overline{x}_0^1\right) \in S_{++}^{L-1}$ and $\overline{p}_\sigma = \frac{1}{\overline{\lambda}_{\sigma,1}^1}\pi^1\left(1|\,\sigma\right) Du_\sigma^1\left(\overline{x}_{\sigma,s}^1\right) \in S_{++}^{L-1}$, so $p_{n(k)} \longrightarrow \overline{p}$. Similarly,

$$q_{n(k),\sigma} = \sum_s \frac{\lambda^1_{n(k),\sigma,s}}{\lambda^1_{n(k),0}} \longrightarrow \sum_s \frac{\overline{\lambda}^1_{\sigma,s}}{\overline{\lambda}^1_0},$$

and we can define $\overline{q}_\sigma = \sum_{s=1}^{S^1} \frac{\overline{\lambda}^1_{\sigma,s}}{\overline{\lambda}^1_0} \in \mathbb{R}_{++}$ to get that $q_{n(k)} \longrightarrow \overline{q}$. Finally,

$$\begin{aligned} y^i_{n(k),\sigma} &= p_{n(k),\sigma}\left(x^i_{n(k),\sigma,1} - \left(1-\tau_{n(k)}\right)e^i_{\sigma,1} - \tau_{n(k)}x^{e,i}_{\sigma,1}\right) \\ &\longrightarrow \overline{p}_\sigma\left(\overline{x}^i_{\sigma,1} - \left(1-\overline{\tau}\right)e^i_{\sigma,1} - \overline{\tau}x^{e,i}_{\sigma,1}\right), \end{aligned}$$

so, if we define $\overline{y}^i_\sigma = \overline{p}_\sigma\left(\overline{x}^i_{\sigma,1} - \left(1-\overline{\tau}\right)e^i_{\sigma,1} - \overline{\tau}x^{e,i}_{\sigma,1}\right)$, we get that $y^i_{n(k)} \longrightarrow \overline{y}^i$.

It follows that $\left(x_{n(k)}, \lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}, \tau_{n(k)}\right)_{k=1}^\infty \longrightarrow \left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, \overline{\tau}\right)$, and then, by continuity, that $\mathcal{F}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, , \left(1-\overline{\tau}\right)e + \overline{\tau}x^e\right) = 0$. □

Theorem 1. *Competitive equilibria exist.*

Proof. Fix e and define the functions $\varphi(x,\lambda,y,p,q,\tau) = \mathcal{F}(x,\lambda,y,p,q,(1-\tau)e - \tau x^e)$ and $\gamma(x,\lambda,y,p,q) = \mathcal{F}(x,\lambda,y,p,q,x^e)$.[8] φ and γ, both, by $\varphi\left(x,\lambda,y,p,q,\tau\right) = \mathcal{F}(x,\lambda,y,p,q,(1-\tau)e- \tau x^e)$ and $\gamma\left(x,\lambda,y,p,q\right) = \mathcal{F}\left(x,\lambda,y,p,q,x^e\right)$.

It is immediate that $\varphi \in C^0$ and $\gamma \in C^1$. By lemmas 2 and 3, it follows that $\gamma \pitchfork 0$ and $\#\gamma^{-1}\left(0\right) = 1$. By construction, φ is a continuous homotopy from $\mathcal{F}_e = \mathcal{F}\left(\cdot, e\right)$ to γ, whereas from lemma 4, $\varphi^{-1}\left(0\right)$ is compact. It follows from theorem 57, in chapter 7 of Villanacci et al. that $\mathcal{F}_e^{-1}\left(0\right) \neq \varnothing$. □

6 Generic Determinacy of Equilibria

Maintain u fixed.

Equilibrium is **determinate** at e if all equilibria at e are locally unique.

A subset of a finite-dimentsional Euclidean space is **strongly generic** if it is open and its complement is null (zero Lebesgue measure), and that a subset of an abstract metric space is **generic** if it is open and dense.

Theorem 2. *There exists a strongly generic subset of endowments for which there is only a finite number of equilibria and, hence, equilibrium is determinate.*

Proof. Suppose that $\mathcal{F}\left(x,\lambda,y,p,q,e\right) = 0$.

[8] These functions map $\mathbb{R}_{++}^{I(\Sigma S+1)L} \times \mathbb{R}^{\Sigma I} \times \mathbb{R}_{++}^{I(\Sigma S+1)} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^\Sigma \times [0,1]$ and $\mathbb{R}_{++}^{I(\Sigma S+1)L} \times \mathbb{R}^{\Sigma I} \times \mathbb{R}_{++}^{I(\Sigma S+1)} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^\Sigma$, respectively, into $\mathbb{R}_{++}^{(\Sigma S+1)L} \times \mathbb{R}_{++}^{(\Sigma S+1)} \times \mathbb{R}^{\Sigma}{}^{I} \times \mathbb{R}_{++}^{(\Sigma+1)(L-1)} \times \mathbb{R}^\Sigma$.

By standard arguments, each square submatrix

$$\begin{bmatrix} D^2u^i\left(x^i\right) & -\Psi\left(p\right)^\top & 0 \\ -\Psi\left(p\right) & 0 & R\left(q\right) \\ 0 & R\left(q\right)^\top & 0 \end{bmatrix}$$

is invertible, so the submatrix constructed deleting the last two superrows and the last I supercolumns of $D\mathcal{F}\left(p,q,x,y,\lambda,e\right)$ has full rank.

For the full $D\mathcal{F}$, we add $\left(\Sigma+1\right)\left(L-1\right)+\Sigma$ rows and $I\left(S\Sigma+1\right)L$ columns, so, since $S\geq 2$, we are adding more columns than rows. So, for $D\mathcal{F}$ to have full row rank it suffices that for each v in the canonical basis of $\mathbb{R}^{(\Sigma+1)(L-1)+\Sigma}$, there exist α such that $D\mathcal{F}\left(x,y,\lambda,p,q,e\right)\alpha=\begin{bmatrix} 0 & v \end{bmatrix}^\top$.

For simplicity of notation, denote by $\alpha\left(\chi\right)$ the component of α corresponding to the argument χ of $\mathcal{F}$ (that is, the coefficient multiplying the χ column or supercolumn of $D\mathcal{F}$).

Let $l\in\{2,...,L\}$. Define α as follows: $\alpha\left(e^1_{0,l}\right)=1$, $\alpha\left(e^1_{0,1}\right)=-p_{0,l}$, and $\alpha\left(\chi\right)=0$ for every other χ. $D\mathcal{F}\left(p,q,x,y,\lambda,e\right)\alpha$ gives 0 everywhere except at the row corresponding to $\sum_i\left(\widetilde{e}^i_{0,l}-\widetilde{x}^i_{0,l}\right)$, where it gives 1.

Let $\sigma\in\{1,...,\Sigma\}$ and $l\in\{2,...,L\}$. Define α as follows: $\alpha\left(e^1_{\sigma,1,l}\right)=\pi^i\left(1|\,\sigma\right)^{-1}$, $\alpha\left(e^1_{\sigma,1,1}\right)=-p_{\sigma,l}\pi^i\left(1|\,\sigma\right)^{-1}$, and $\alpha\left(\chi\right)=0$ for every other χ. $D\mathcal{F}\left(p,q,x,y,\lambda,e\right)\alpha$ gives 0 everywhere except at the row corresponding to $\sum_{i,s}\pi^i\left(s|\,\sigma\right)\left(\widetilde{e}^i_{\sigma,s,l}-\widetilde{x}^i_{\sigma,s,l}\right)$, where it gives 1.

Let $\sigma\in\{1,...,\Sigma\}$. Define α as follows: $\alpha\left(y^1_\sigma\right)=1$, $\alpha\left(e^1_{0,1}\right)=q_\sigma$, for every s, $\alpha\left(e^1_{\sigma,s,1}\right)=-1$, and $\alpha\left(\chi\right)=0$ for every other χ. $D\mathcal{F}\left(p,q,x,y,\lambda,e\right)\alpha$ gives 0 everywhere except at the row corresponding to $\sum_i y^i_\sigma$, where it gives 1.

Let $\mathcal{E}$ be the set of e such that $\mathcal{F}\left(\cdot,e\right)\pitchfork 0$. It follows immediately, from the transversality theorem, that $\mathcal{E}$ has full measure.

By the inverse function theorem, it follows that for every $e\in\mathcal{E}$, equilibrium is determinate.

Fix $e\in\mathcal{E}$ and suppose that there are infinitely many equilibria. Then, construct a sequence $\left(x_n,\lambda_n,y_n,p_n,q_n\right)_{n=1}^\infty$ such that $\mathcal{F}\left(x_n,\lambda_n,y_n,p_n,q_n,e\right)=0$ and $n\neq n'$ implies $\left(x_n,\lambda_n,y_n,p_n,q_n\right)\neq\left(x_{n'},\lambda_{n'},y_{n'},p_{n'},q_{n'}\right)$ for every n. By market clearing, there exists a subsequence $\left(x_{n(k)}\right)_{k=1}^\infty$ that converges to some $\overline{x}$. As before, $\lambda^i_{n(k),0}\longrightarrow\frac{\partial u^i_0}{\partial x_{0,1}}\left(\overline{x}^i_0\right)=\overline{\lambda}^i_0$, $\lambda^i_{n(k),\sigma,s}\longrightarrow\pi^i\left(s|\,\sigma\right)\frac{\partial u^i_\sigma}{\partial x_{\sigma,s,1}}\left(\overline{x}^i_{\sigma,s}\right)=\overline{\lambda}^i_{\sigma,s}$, $p_{n(k),0}\longrightarrow\frac{1}{\overline{\lambda}^1_0}Du^1_0\left(\overline{x}^1_0\right)=\overline{p}_0$, $p_{n(k),\sigma}\longrightarrow\frac{1}{\overline{\lambda}^1_{\sigma,1}}\pi^1\left(1|\,\sigma\right)Du^1_\sigma\left(\overline{x}^1_{\sigma,1}\right)=\overline{p}_\sigma$, $q_{n(k),\sigma}\longrightarrow\sum_s\frac{\overline{\lambda}^1_{\sigma,s}}{\overline{\lambda}^1_0}=\overline{q}_\sigma$ and $y^i_{n(k),\sigma}\longrightarrow\overline{p}_\sigma\left(\overline{x}^i_{\sigma,1}-e^i_{\sigma,1}\right)=\overline{y}^i_\sigma$.By continuity, that $\mathcal{F}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q},e\right)=0$, which contradicts, by the second property of the sequence, that equilibria are determinate.

To see that $\mathcal{E}$ is open, let $\left(e_n\right)_{n=1}^\infty$ be a sequence in $\mathcal{E}^c$, such that $e_n\longrightarrow e$. By definition, there exists $\left(x_n,\lambda_n,y_n,p_n,q_n\right)_{n=1}^\infty$ such that $\mathcal{F}\left(x_n,\lambda_n,y_n,p_n,q_n,e_n\right)=0$ and

$$\det\left(D_{x,\lambda,y,p,q}\mathcal{F}\left(x_n,\lambda_n,y_n,p_n,q_n,e_n\right)\right)=0.$$

Let $\left(x_{n(k)}\right)_{k=1}^{\infty}$ be a convergent subsequence, which exists, again, because of market clearing, and let x be the limit of that subsequence. Since, $\mathcal{F}(x_{n(k)}, \lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}, e_{n(k)}) = 0$, as before, for some (λ, y, p, q), $\left(\lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}\right) \longrightarrow (\lambda, y, p, q)$, and, by continuity, $\mathcal{F}(x, \lambda, y, p, q, e) = 0$ and $\det\left(D_{x,\lambda,y,p,q}\mathcal{F}(x, \lambda, y, p, q, e)\right) = 0$, so $e \in \mathcal{E}^c$. □

7 Generic Inefficiency of Equilibria

Theorem 3. *There exists a full measure subset of endowments for which, if (x, y, λ, p, q) is an equilibrium, then for every i, σ, s and s', with $s \neq s'$, $\frac{\lambda^i_{\sigma,s}}{\pi^i(s|\sigma)} \neq \frac{\lambda^i_{\sigma,s'}}{\pi^i(s'|\sigma)}$.*

Proof. For i, σ and $s \neq s'$, define the function[9]

$$\mathcal{G}^i_{\sigma,s,s'}(x, y, \lambda, p, q, e) = \begin{bmatrix} \mathcal{F}(p, q, x, y, \lambda, e) \\ \frac{\lambda^i_{\sigma,s}}{\pi^i(s|\sigma)} - \frac{\lambda^i_{\sigma,s'}}{\pi^i(s'|\sigma)} \end{bmatrix}.$$

For simplicity of exposition, consider $i = I$, $\sigma = 1$, $s = 1$ and $s' = S$. With (some of) the arguments in the order

$$\left(x^1 \quad \lambda^1 \quad y^1 \quad \dots \quad x^I \quad \lambda^I \quad y^I \quad e^1 \quad \dots \quad e^I \right),$$

$D\mathcal{G}^I_{1,1,S}(x, y, \lambda, p, q, e)$ is

$$\begin{bmatrix}
D^2u^1 & -\Psi^\top & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
-\Psi & 0 & R & \dots & 0 & 0 & 0 & \Psi^1 & \dots & 0 \\
0 & R^\top & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & D^2u^I & -\Psi^\top & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & -\Psi & 0 & R & 0 & \dots & \Psi^I \\
0 & 0 & 0 & \dots & 0 & R^\top & 0 & 0 & \dots & 0 \\
\Phi^1 & 0 & 0 & \dots & \Phi^I & 0 & 0 & -\Phi^1 & \dots & -\Phi^I \\
0 & 0 & I_\Sigma & \dots & 0 & 0 & I_\Sigma & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & \Pi & 0 & 0 & \dots & 0
\end{bmatrix},$$

where $\Pi = \begin{bmatrix} 0 & \pi^I(1|1)^{-1} & 0 & \cdots & 0 & -\pi^I(S|1)^{-1} & 0 & \cdots & 0 \end{bmatrix}$, with the nonzero components corresponding to the columns $\lambda^I_{1,1}$ and $\lambda^I_{1,S}$.

Without the last row and the e^I supercolumn, the matrix has full row rank. Adding the last supercolumn ads $\left(\Sigma S^I + 1\right) L \geq 1$ columns. Let

$$\beta = \begin{bmatrix} 0 & -\frac{\pi^I(1|1)}{2}p_1 & 0 & \cdots & 0 & \frac{\pi^I(S|1)}{2}p_1 & 0 & \cdots & 0 \end{bmatrix}^\top,$$

[9]Mapping $\mathbb{R}^{I(\Sigma S+1)L}_{++} \times \mathbb{R}^{\Sigma I} \times \mathbb{R}^{I(\Sigma S+1)}_{++} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{I(\Sigma S+1)L}_{++}$, into $\mathbb{R}^{I(\Sigma S+1)L}_{++} \times \mathbb{R}^{I(\Sigma S+1)}_{++} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{(\Sigma+1)(L-1)}_{++} \times \mathbb{R}^{\Sigma} \times \mathbb{R}$.

with the nonzero components on the second and $(S+1)$-th positions. Following the same notation as before, define $\alpha\left(x^I\right)=D^2u^I\left(x^I\right)^{-1}\beta$, $\alpha\left(\lambda^I_{1,1}\right)=\frac{\pi^I(1|1)}{2}$, $\alpha\left(\lambda^I_{1,S}\right)=-\frac{\pi^I(S^I|1)}{2}$, $\alpha\left(e^I\right)=\alpha\left(x^I\right)$, and $\alpha(\chi)=0$ for every other argument χ. Then, $D\mathcal{G}^I_{1,1,S^I}(x,y,\lambda,p,q,e)\,\alpha=\left[\begin{array}{cccc}0&\cdots&0&1\end{array}\right]^\top$, which implies that $D\mathcal{G}^I_{1,1,S^I}(x,y,\lambda,p,q,e)$ has full row rank.

It follows that $\mathcal{G}^I_{1,1,S^I}\pitchfork 0$, so the set $\mathcal{E}^I_{1,1,S^I}$ such that for every $e\in\mathcal{E}^I_{1,1,S^I}$, $\mathcal{G}^I_{1,1,S^I}(\cdot,e)\pitchfork 0$ has full measure . Since $D_{x,\lambda,y,p,q}\mathcal{G}^I_{1,1,S^I}$ has one fewer column than rows, $\mathcal{G}^I_{1,1,S^I}(\cdot,e)\pitchfork 0$ implies that

$$e\in\mathcal{E}^I_{1,1,S^I}\text{ and }\mathcal{F}(x,y,\lambda,p,q,x,y,\lambda,e)=0\Longrightarrow\frac{\lambda^I_{1,1}}{\pi^i(1|\,1)}-\frac{\lambda^I_{1,S^I}}{\pi^i(S^I|\,1)}\neq 0.$$

Doing the same for every i, every σ and every , $s\neq s'$, gives that the sets $\mathcal{E}^i_{\sigma,s,s'}$ such that

$$e\in\mathcal{E}^i_{\sigma,s,s'}\text{ and }\mathcal{F}(x,y,\lambda,p,q,x,y,\lambda,e)=0\Longrightarrow\frac{\lambda^i_{\sigma,s}}{\pi^i(s|\,\sigma)}-\frac{\lambda^i_{\sigma,s'}}{\pi^i(s'|\,\sigma)}\neq 0,$$

have full measure.

Set $\mathcal{E}=\bigcap_{i=1}^{I}\bigcap_{\sigma=1}^{\Sigma}\bigcap_{s,s'\in\{1,\ldots,S\},s\neq s'}\mathcal{E}^i_{\sigma,s,s'}$, which is a full measure set, has the desired property. □

Corollary 1. *There exists a strongly generic subset of endowments for which if (x,y,p,q) is equilibrium for endowments, then x is inefficient.*

Proof. Consider $\mathcal{E}=\mathcal{E}^1_{1,1,S}$ from the previous proof. To see that it is strongly generic, we only need to show that it is open. Let $(e^n)_{n=1}^\infty$ be a sequence in $\mathcal{E}^c$, so that $e_n\longrightarrow e$. By definition, for every $n\in\mathbb{N}$, there exists $(x_n,\lambda_n,y_n,p_n,q_n)$ such that $\mathcal{F}(x_n,\lambda_n,y_n,p_n,q_n,e_n)=0$, and $\lambda^1_{n,1,1}\pi^1(1|\,1)^{-1}=\lambda^1_{n,1,S^I}\pi^1\left(S^I\middle|\,1\right)^{-1}$ Let $\left(x_{n(k)}\right)_{k=1}^\infty$ be a convergent subsequence of $(x_n)_{k=1}^\infty$, which exists, again, because of market clearing, and let x be the limit of that subsequence. Since, $\mathcal{F}\left(x_{n(k)},\lambda_{n(k)},y_{n(k)},p_{n(k)},q_{n(k)},e_{n(k)}\right)=0$, for some (λ,y,p,q), $\left(\lambda_{n(k)},y_{n(k)},p_{n(k)},q_{n(k)}\right)\longrightarrow(\lambda,y,p,q)$. Since, by continuity, $\mathcal{F}(x,\lambda,y,p,q,e)=0$ and $\lambda^1_{1,1}\pi^1(1|\,1)^{-1}=\lambda^1_{1,S^I}\pi^1\left(S^I\middle|\,1\right)^{-1}$, it follows that $e\in\mathcal{E}^c$.

Now, if (x,y,p,q) is an equilibrium for endowments $e\in\mathcal{E}^1_{1,1,S}$, then $\pi^1(1|\,1)\,Du^1_1\left(x^1_{1,1}\right)=\lambda^1_{1,1}p_1$ and $\pi^1(S|\,1)\,Du^1_1\left(x^1_{S,1}\right)=\lambda^1_{S,1}p_1$, whereas if x is efficient $x^i_{1,1}=x^1_{S,1}$, which implies that $\frac{\lambda^1(1,1)}{\pi^1(1|1)}=\frac{\lambda^1\left(S^1,1\right)}{\pi^1(S^1|1)}$, which is impossible. □

Corollary 2. *There exists full measure subset of endowments for which, if (x,λ,y,p,q) is an equilibrium , then for every i, σ, s and s', with $s\neq s'$, $x^i_{\sigma,s}\neq x^i_{\sigma,s'}$.*

Proof. Fix $\mathcal{E}$ from the proof of theorem 3. Let $e \in \mathcal{E}$ and suppose that (x, y, λ, p, q) is an equilibrium for endowments e. For i, σ and $s \neq s'$, suppose that $x^i_{\sigma,s} = x^i_{\sigma,s'}$. Since $\pi^i\left(s|\,\sigma\right) Du^i_\sigma\left(x^i_{\sigma,s}\right) = \lambda^i_{\sigma,s} p_\sigma$ and $\pi^i\left(s'|\,\sigma\right) Du^i_\sigma\left(x^i_{\sigma,s'}\right) = \lambda^i_{\sigma,s'} p_\sigma$, it follows that $Du^i_\sigma\left(x^i_{\sigma,s}\right) = Du^i_\sigma\left(x^i_{\sigma,s'}\right)$ and, hence, $\frac{\lambda^i_{\sigma,s}}{\pi^i(s|\sigma)} = \frac{\lambda^i_{\sigma,s'}}{\pi^i(s'|\sigma)}$, contradicting the fact that $e \in \mathcal{E}$. □

8 Constrained inefficiency

We now consider u as a variable and use it as argument in the relavant functions.

An allocation x is **constrained-inefficient** if there exist prices $(p_\sigma)_{\sigma=0}^{\Sigma}$, an allocation $\widehat{x}$, date-zero revenue transfers $\left(\tau^i\right)_{i=1}^I$ and an asset allocation $\left(y^i\right)_{i=1}^I$ such that:

1. $\sum_i \tau^i = 0$;
2. $\sum_i y^i = 0$;
3. for every i,
$$\widehat{x}^i_0 \in \arg\max_x u^i_0\left(x\right) \text{ s.t. } p_0 x \leq p_0 e^i_0 + \tau^i$$
and
$$\widehat{x}^i_{\sigma,s} \in \arg\max_x u^i_\sigma\left(x\right) \text{ s.t. } p_\sigma x \leq p_\sigma e^i_{\sigma,s} + y^i_\sigma$$
for every σ and s;
4. $\sum_i \left(e^i_0 - \widehat{x}^i_0\right) = 0$ and for every σ, $\sum_{i,s} \pi^i\left(s|\,\sigma\right)\left(e^i_{\sigma,s} - \widehat{x}^i_{\sigma,s}\right) = 0$;
5. for every i,
$$u^i_0\left(\widehat{x}^i_0\right) + \sum_\sigma \mathrm{E}_{\pi^i_\sigma} u^i_\sigma\left(\widehat{x}^i_\sigma\right) > u^i_0\left(x^i_0\right) + \sum_\sigma \mathrm{E}_{\pi^i_\sigma} u^i_\sigma\left(x^i_\sigma\right).$$

Define function[10]

$$\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,e_{\mathbf{1}},u_{\mathbf{1}}) = \begin{bmatrix} \sum_\sigma \mathrm{E}_{\pi^1_\sigma} u^1_\sigma(x^1_\sigma) \\ \vdots \\ \sum_\sigma \mathrm{E}_{\pi^I_\sigma} u^I_\sigma(x^I_\sigma) \\ \left(Du^1_{\mathbf{1}}(x^1_{\mathbf{1}}) - \lambda^1_{\mathbf{1}}\Psi_{\mathbf{1}}(p_{\mathbf{1}})\right)^\top \\ \widetilde{R}y^1 - \Psi_{\mathbf{1}}(p_{\mathbf{1}})\left(x^1_{\mathbf{1}} - e^1_{\mathbf{1}}\right) \\ \vdots \\ \left(Du^I_{\mathbf{1}}(x^I_{\mathbf{1}}) - \lambda^I_{\mathbf{1}}\Psi_{\mathbf{1}}(p_{\mathbf{1}})\right)^\top \\ \widetilde{R}y^I = \Psi_{\mathbf{1}}(p_{\mathbf{1}})\left(x^I_{\mathbf{1}} - e^I_{\mathbf{1}}\right) \\ \sum_{i,s}\pi^i(s|1)\left(\widetilde{e}^i_{1,s} - \widetilde{x}^i_{1,s}\right) \\ \vdots \\ \sum_{i,s}\pi^i(s|\Sigma)\left(\widetilde{e}^i_{\Sigma,s} - \widetilde{x}^i_{\Sigma,s}\right) \\ \sum_i y^i \end{bmatrix},$$

where $\widetilde{R}^i_{\Sigma S^i\times\Sigma} = I_\Sigma \otimes \mathbf{1}^\top_{S^i}$, and the subindex $\mathbf{1}$ means that only the future components of the variable are considered: for example,

$$Du^i_{\mathbf{1}}(x^i_{\mathbf{1}})^\top_{\Sigma S^i L\times 1} = \begin{bmatrix} \pi^i(1|1)\,Du^i_1\left(x^i_{1,1}\right)^\top \\ \vdots \\ \pi^i\left(S^i\middle|1\right)Du^i_1\left(x^i_{1,S^i}\right)^\top \\ \vdots \\ \pi^i\left(S^i\middle|\Sigma\right)Du^i_\Sigma\left(x^i_{\Sigma,S^i}\right)^\top \end{bmatrix}$$

Theorem 4. *If (x,λ,y,p,q) is a financial markets equilibrium for (e,u) and*

$$D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,e_{\mathbf{1}},u_{\mathbf{1}})$$

has full (row) rank, then x is constrained inefficient.

Proof. Fix an equilibrium (x,λ,y,p,q).

By letting $\tau^i = qy^i$, $\widehat{p}_0 = p_0$ and $\left(\widehat{x}^i_0\right)^I_{i=1} = \left(x^i_0\right)^I_{i=1}$ it follows that x is constrained inefficient if there exist $\widehat{p}_{\mathbf{1}}$, $\widehat{x}_{\mathbf{1}}$ and $\widehat{y}$ such that

1. $\sum_{i=1}^{I}\widehat{y}^i = 0$;
2. for every i, σ and s, $\widehat{x}^i_{\sigma,s} \in \arg\max_x u^i_\sigma(x)$ s.t. $p_\sigma x \le p_\sigma e^i_{\sigma,s} + y^i_\sigma$;
3. for every σ, $\sum_{i,s}\pi^i(s|\sigma)\left(e^i_{\sigma,s} - \widehat{x}^i_{\sigma,s}\right) = 0$;

[10] Mapping $\mathbb{R}^{I\Sigma SL}_{++}\times\mathbb{R}^{I\Sigma S}_{++}\times\mathcal{P}^\Sigma\times\mathbb{R}^{\Sigma I}\times\mathbb{R}^{I\Sigma SL}_{++}\times\mathcal{U}^{I\Sigma}$ into $\mathbb{R}^I\times\mathbb{R}^{I\Sigma SL}_{++}\times\mathbb{R}^{I\Sigma S}_{++}\times\mathbb{R}^{(L-1)\Sigma}\times\mathbb{R}^\Sigma$.

4. for every i, $\sum_\sigma \mathrm{E}_{\pi^i_\sigma} u^i_\sigma\left(\widehat{x}^i_\sigma\right) > \sum_\sigma \mathrm{E}_{\pi^i_\sigma} u^i_\sigma\left(x^i_\sigma\right)$.

Hence, it suffices to show that there exist α and $\Delta \in \mathbb{R}^I_{++}$ such that

$$D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,e_{\mathbf{1}}\right)\alpha = \begin{bmatrix} \Delta \\ 0 \end{bmatrix},$$

which is the case whenever $D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,e_{\mathbf{1}}\right)$ has full rank. □

Lemma 5. *For every u, the subset of endowments such that,*

$$\mathcal{F}\left(x,\lambda,y,p,q,e\right)=0 \Longrightarrow \left(\forall\sigma\in\{1,...,\Sigma\}\right): rank\left(\mathbf{Z}_\sigma\left(\widetilde{x}_\sigma,\widetilde{e}_\sigma\right)\right) = IS-1,$$

where $\mathbf{Z}_\sigma\left(\widetilde{x}_\sigma,\widetilde{e}_\sigma\right)$ denotes the $(L-1)\times IS$ matrix

$$\begin{bmatrix} \widetilde{e}^1_{\sigma,1}-\widetilde{x}^1_{\sigma,1} & \cdots & \widetilde{e}^1_{\sigma,S^1}-\widetilde{x}^1_{\sigma,S^1} & \widetilde{e}^2_{\sigma,1}-\widetilde{x}^2_{\sigma,1} & \cdots & \widetilde{e}^I_{\sigma,S^I}-\widetilde{x}^I_{\sigma,S^I} \end{bmatrix},$$

has full measure.

Proof. Fix σ and define the function[11]

$$\mathcal{J}_\sigma\left(x,\lambda,y,p,q,e,\theta\right) = \begin{bmatrix} \mathcal{F}\left(x,\lambda,y,p,q,e,u\right) \\ \mathbf{Z}_\sigma\left(\widetilde{x}_\sigma,\widetilde{e}_\sigma\right)\theta \\ \pi_\sigma^\top\theta \\ \frac{1}{2}\left(\theta^\top\theta-1\right) \end{bmatrix},$$

where

$$\pi_\sigma = \begin{bmatrix} \pi^1\left(1|\,\sigma\right) & \cdots & \pi^1\left(S|\,\sigma\right) & \pi^2\left(1|\,\sigma\right) & \cdots & \pi^I\left(S|\,\sigma\right) \end{bmatrix}^\top,$$

and suppose that $\mathcal{J}_\sigma\left(x,\lambda,y,p,q,e,\theta\right)=0$.

With the arguments in the order

$$\begin{pmatrix} x^1 & \lambda^1 & y^1 & \ldots & x^I & \lambda^I & y^I & p & q & e^1 & \ldots & e^I & \theta \end{pmatrix},$$

$D\mathcal{J}_\sigma\left(x,\lambda,y,p,q,e,\theta\right)$ is

$$\begin{bmatrix}
D^2u^1 & -\Psi^\top & 0 & \ldots & 0 & 0 & 0 & \Lambda^1 & 0 & 0 & \ldots & 0 & 0 \\
-\Psi & 0 & R & \ldots & 0 & 0 & 0 & Z^1 & Y^1 & \Psi^1 & \ldots & 0 & 0 \\
0 & R^\top & 0 & \ldots & 0 & 0 & 0 & 0 & \lambda^1_0 I_\Sigma & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & D^2u^I & -\Psi^\top & 0 & \Lambda^I & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & -\Psi & 0 & R & Z^I & Y^I & 0 & \ldots & \Psi^I & 0 \\
0 & 0 & 0 & \ldots & 0 & R^\top & 0 & 0 & \lambda^I_0 I_\Sigma & 0 & \ldots & 0 & 0 \\
\Phi^1 & 0 & 0 & \ldots & \Phi^I & 0 & 0 & 0 & 0 & -\Phi^1 & \ldots & -\Phi^I & 0 \\
0 & 0 & I_\Sigma & \ldots & 0 & 0 & I_\Sigma & 0 & 0 & 0 & \ldots & 0 & 0 \\
\phi^1 & 0 & 0 & \ldots & \phi^I & 0 & 0 & 0 & 0 & -\phi^1 & \ldots & -\phi^I & \mathbf{Z}_\sigma \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \pi_\sigma^\top \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \theta^\top
\end{bmatrix}$$

[11] Mapping $\mathbb{R}^{I(\Sigma S+1)L}_{++}\times\mathbb{R}^{I(\Sigma S+1)}_{++}\times\mathbb{R}^{\Sigma I}\times\mathcal{P}^{\Sigma+1}\times\mathbb{R}^{\Sigma}\times\mathbb{R}^{I(\Sigma S+1)L}_{++}\times\mathbb{R}^{IS}$ into $\mathbb{R}^{I(\Sigma S^i+1)L}_{++}\times\mathbb{R}^{I(\Sigma S^i+1)}_{++}\times\mathbb{R}^{I\Sigma}\times\mathbb{R}^{(\Sigma+1)(L-1)}_{++}\times\mathbb{R}^{\Sigma}\times\mathbb{R}^{L-1}\times\mathbb{R}\times\mathbb{R}$.

where, for each i, $\phi^i = \phi^i\left(\theta^i\right)$ is the $(L-1)\times\Sigma S^i L$ matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & \theta_1^i \widetilde{I} & \theta_2^i \widetilde{I} & \dots & \theta_S^I \widetilde{I} & 0 & \dots & 0 \end{bmatrix},$$

with the nonzero components corresponding to the columns $x^i_{\sigma,1}, x^i_{\sigma,2}, ..., x^i_{\sigma,S}$.

By standard arguments, the matrix without the last three superrows and the supercolumns $e^2, ..., e^I, \theta$ has full row rank. When we add the third-to-last superrow and the $e^2, ..., e^I$ supercolumns, we add $(L-1)$ rows and $(I-1)(\Sigma S+1)L \geq L(I-1) \geq L-1$ columns. Fix $l \in \{2, ..., L\}$ and define α as follows: $\alpha\left(e^i_{\sigma,s,l}\right) = \theta^i_s$, $\alpha\left(e^i_{\sigma,s,1}\right) = -p_{\sigma,l}\theta^i_s$, and $\alpha(\chi) = 0$ for every other argument χ. Since $\pi_\sigma^\top \theta = 0$ and $\theta^\top\theta = 1$, it follows that if we post-multiply the matrix without the last two superrows and the last supercolumn by α, we get 0 everywhere except at the l-th component of its last superrow, where we get 1. It follows that that matrix has full row rank. If we now add the last two superrows and the last supercolumn, and since $\pi_\sigma \neq 0$, $\theta \neq 0$ and $\pi_\sigma^\top\theta = 0$, it follows that the whole matrix has full row rank.

The latter implies that $\mathcal{J}_\sigma \pitchfork 0$ and, hence, the set $\mathcal{E}_\sigma$ such that $\mathcal{J}_\sigma(\cdot, e) \pitchfork 0$ has full measure. Since $D_{x,\lambda,y,p,q,\theta}\mathcal{J}_\sigma$ has $I(\Sigma S+1)(L+1)+\Sigma I+(\Sigma+1)(L-1)+\Sigma+L+1$ rows and $I(\Sigma S+1)(L+1)+\Sigma I+(\Sigma+1)(L-1)+\Sigma+IS$ columns, and $L \geq IS$, it follows that $D_{x,\lambda,y,p,q,\theta}\mathcal{J}_\sigma(x,\lambda,y,p,q,e,\theta)$ cannot have full row rank. This implies that for every $e \in \mathcal{E}_\sigma$,

$$\begin{bmatrix} \mathcal{F}(x,\lambda,y,p,q,e) \\ \mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma)\theta \\ \pi_\sigma^\top\theta \end{bmatrix} = 0 \Longrightarrow \theta = 0.$$

Define, $\mathcal{E} = \bigcap_{\sigma=1}^{\Sigma} \mathcal{E}_\sigma$, which is a full measure set. Let $e \in \mathcal{E}^F$ and suppose that $\mathcal{F}(x,\lambda,y,p,q,e) = 0$ but for some $\sigma \in \{1, ..., \Sigma\}$, $rank(\mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma)) \neq IS-1$. Since $\mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma)\pi_\sigma$, it follows that $rank(\mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma)) \leq IS-2$, so there exists $\theta \in \mathbb{R}^{IS}$ such that $\mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma)\theta = 0$, $\pi_\sigma^\top\theta = 0$ and $\theta \neq 0$, contradicting the facts that $\theta \in \mathcal{E}_\sigma$ and $\mathcal{F}(x,\lambda,y,p,q,e) = 0$. □

Lemma 6. *For every u, define the function*[12]

$$\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}}) = \begin{bmatrix} \left(Du^1_{\mathbf{1}}\left(x^1_{\mathbf{1}}\right) - \lambda^1_{\mathbf{1}}\Psi_{\mathbf{1}}(p_{\mathbf{1}})\right)^\top \\ \widetilde{R}y^1 - \Psi_{\mathbf{1}}(p_{\mathbf{1}})\left(x^1_{\mathbf{1}} - e^1_{\mathbf{1}}\right) \\ \vdots \\ \left(Du^I_{\mathbf{1}}\left(x^I_{\mathbf{1}}\right) - \lambda^I_{\mathbf{1}}\Psi_{\mathbf{1}}(p_{\mathbf{1}})\right)^\top \\ \widetilde{R}y^I - \Psi_{\mathbf{1}}(p_{\mathbf{1}})\left(x^I_{\mathbf{1}} - e^I_{\mathbf{1}}\right) \\ \sum_{i,s}\pi^i(s|\,1)\left(\widetilde{e}^i_{1,s} - \widetilde{x}^i_{1,s}\right) \\ \vdots \\ \sum_{i,s}\pi^i(s|\,\Sigma)\left(\widetilde{e}^i_{\Sigma,s} - \widetilde{x}^i_{\Sigma,s}\right) \\ \sum_i y^i \end{bmatrix}.$$

[12]Mapping $\mathbb{R}^{I\Sigma SL}_{++} \times \mathbb{R}^{I\Sigma S}_{++} \times \mathcal{P}^\Sigma \times \mathbb{R}^{\Sigma I} \times \mathbb{R}^{I\Sigma SL}_{++}$ into $\mathbb{R}^{I\Sigma SL}_{++} \times \mathbb{R}^{I\Sigma S}_{++} \times \mathbb{R}^{(L-1)\Sigma} \times \mathbb{R}^{\Sigma}$.

The set of (date 1) endowments on which

$$\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}}) = 0 \Longrightarrow D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}}) \text{ has full row rank,}$$

has full measure.

Proof. Let $\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}}) = 0$. By definition, with the arguments of $\mathcal{G}_u$ in the order

$$\left(\begin{array}{ccccccccccc} x_{\mathbf{1}}^1 & \lambda_{\mathbf{1}}^1 & \dots & x_{\mathbf{1}}^I & \lambda_{\mathbf{1}}^I & p_{\mathbf{1}} & y^1 & \dots & y^I & e_{\mathbf{1}}^1 & \dots & e_{\mathbf{1}}^I \end{array} \right),$$

$D\mathcal{G}_u(\cdot)$ is

$$\begin{bmatrix} D^2u_{\mathbf{1}}^1 & -\Psi_{\mathbf{1}}^\top & 0 & 0 & \Lambda_{\mathbf{1}}^1 & 0 & \dots & 0 & 0 & \dots & 0 \\ -\Psi_{\mathbf{1}} & 0 & 0 & 0 & Z_{\mathbf{1}}^1 & \widetilde{R} & \dots & 0 & \Psi_{\mathbf{1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & D^2u_{\mathbf{1}}^I & -\Psi_{\mathbf{1}}^\top & \Lambda_{\mathbf{1}}^I & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & -\Psi_{\mathbf{1}} & 0 & Z_{\mathbf{1}}^I & 0 & \dots & \widetilde{R} & 0 & \dots & \Psi_{\mathbf{1}} \\ \Phi_{\mathbf{1}}^1 & 0 & \Phi_{\mathbf{1}}^I & 0 & 0 & 0 & \dots & 0 & -\Phi_{\mathbf{1}}^1 & \dots & -\Phi_{\mathbf{1}}^I \\ 0 & 0 & 0 & 0 & 0 & I_\Sigma & \dots & I_\Sigma & 0 & \dots & 0 \end{bmatrix}.$$

Without the last superrow and the $y^1, ..., y^I$ supercolumns, the matrix has full row rank, by standard arguments. Adding the last superrow and the y^1 supercolumn immediately gives that the $D\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}})$ has full row rank, so $\mathcal{G} \pitchfork 0$. The result follows, once again, from the transversality theorem. $\square$

8.1 A finite-dimensional, local space of economies.

Theorem 5. *There exists a generic subset of economies,* $\mathcal{D}_r \subseteq \mathbb{R}_{++}^{I(\Sigma S+1)L} \times \mathcal{U}^{I(\Sigma+1)}$*, such that for every* $(e, u) \in \mathcal{D}_r$ *there is a finite number of equilibria and, at all of them,*

1. $D_{x,\lambda,y,p,q}\mathcal{F}(x, \lambda, y, p, q, e, u)$ *has full rank;*
2. *for every* i, σ, s *and* s'*, with* $s \neq s'$, $x_{\sigma,s}^i \neq x_{\sigma,s'}^i$*;*
3. *for every* $\sigma \geq 1$, $(\mathbf{Z}_\sigma(\widetilde{x}_\sigma, \widetilde{e}_\sigma))$ *has rank* $IS - 1$*;*
4. $D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{G}_u(x_{\mathbf{1}}, \lambda_{\mathbf{1}}, p_{\mathbf{1}}, y, e_{\mathbf{1}})$ *has full row rank.*

Proof. Let $\mathcal{D}_r$ be the set of (e, u), such that $\mathcal{F}(x, \lambda, y, p, q, e, u) = 0$ implies the four properties.

Suppose that $\mathcal{D}_r$ is not dense. Then, there exist and economy $(\overline{e}, \overline{u})$ and an open set O, such that $(\overline{e}, \overline{u}) \in O \subseteq \mathcal{D}_r^c$. By definition of product topology, then, there exists $\epsilon > 0$ such that $B_\epsilon(\overline{e}) \times \{\overline{u}\} \subseteq O$ which is impossible by theorem 2, lemma 5 and corollaries 2 and 6 (because $\mathcal{F} = 0 \Longrightarrow \mathcal{G} = 0$), given that $I \in \mathbb{N}$, $\Sigma \in \mathbb{N}$ and $S \in \mathbb{N}$.

Opennes is argued as before, but since preferences are allowed to vary, we present the full argument. Consider a sequence $(e_n, u_n)_{n=1}^{\infty}$ in $\mathcal{D}_r^c$ that converges to some $(\overline{e}, \overline{u})$. Construct a sequence $(x_n, \lambda_n, y_n, p_n, q_n)_{n=1}^{\infty}$, such that

$$(\forall n \in \mathbb{N}) : \mathcal{F}(x_n, \lambda_n, y_n, p_n, q_n, e_n, u_n) = 0,$$

yet at least one in conditions 1 to 4 is violated. By market clearing, there exists a subsequence $\left(x_{n(k)}\right)_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that converges to some $\overline{x}$. As before, by construction, since each $u_{n,\sigma}^i$ converges to $\overline{u}_\sigma^i$ uniformly on any conpact set,

$$\begin{aligned}
\lambda_{n(k),0}^i &= \frac{\partial u_{n(k),0}^i}{\partial x_{0,1}}\left(x_{n(k),0}^i\right) \longrightarrow \frac{\partial \overline{u}_0^i}{\partial x_{0,1}}\left(\overline{x}_0^i\right), \\
\lambda_{n(k),\sigma,s}^i &= \pi^i\left(s|\,\sigma\right) \frac{\partial u_{n(k),\sigma}^i}{\partial x_{\sigma,s,1}}\left(x_{n(k),\sigma,s}^i\right) \longrightarrow \pi^i\left(s|\,\sigma\right) \frac{\partial \overline{u}_\sigma^i}{\partial x_{\sigma,s,1}}\left(\overline{x}_{\sigma,s}^i\right).
\end{aligned}$$

Define $\overline{\lambda}_0^i = \frac{\partial \overline{u}_0^i}{\partial x_{0,1}}\left(\overline{x}_0^i\right)$ and $\overline{\lambda}_{\sigma,s}^i = \pi^i\left(s|\,\sigma\right) \frac{\partial \overline{u}_\sigma^i}{\partial x_{\sigma,s,1}}\left(\overline{x}_{\sigma,s}^i\right)$, so $\lambda_{n(k)} \longrightarrow \overline{\lambda}$. Also,

$$\begin{aligned}
p_{n(k),0} &= \frac{1}{\lambda_{n(k),0}^1} Du_{n(k),0}^1\left(x_{n(k),0}^1\right) \longrightarrow \frac{1}{\overline{\lambda}_0^1} D\overline{u}_0^1\left(\overline{x}_0^1\right), \\
p_{n(k),\sigma} &= \frac{1}{\lambda_{n(k),\sigma,1}^1} \pi^1\left(1|\,\sigma\right) Du_{n(k),\sigma}^1\left(x_{n(k),\sigma,s}^i\right) \longrightarrow \frac{1}{\overline{\lambda}_{\sigma,1}^1} \pi^1\left(1|\,\sigma\right) D\overline{u}_\sigma^1\left(\overline{x}_{\sigma,s}^1\right),
\end{aligned}$$

and define $\overline{p}_0 = \frac{1}{\overline{\lambda}_0^1} D\overline{u}_0^1\left(\overline{x}_0^1\right)$ and $\overline{p}_\sigma = \frac{1}{\overline{\lambda}_{\sigma,1}^1}\pi^1\left(1|\,\sigma\right) D\overline{u}_\sigma^1\left(\overline{x}_{\sigma,s}^1\right)$, so $p_{n(k)} \longrightarrow \overline{p}$. As before, $q_{n(k),\sigma} \longrightarrow \sum_s \frac{\overline{\lambda}_{\sigma,s}^1}{\overline{\lambda}_0^1} = \overline{q}_\sigma$, and $y_{n(k),\sigma}^i \longrightarrow \overline{p}_\sigma\left(\overline{x}_{\sigma,1}^i - \overline{e}_{\sigma,1}^i\right) = \overline{y}_\sigma^i = \overline{p}_\sigma\left(\overline{x}_{\sigma,1}^i - \overline{e}_{\sigma,1}^i\right)$, so

$$\left(x_{n(k)}, \lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}\right)_{k=1}^{\infty} \longrightarrow \left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$$

and $\mathcal{F}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, \overline{e}, \overline{u}\right) = 0$.

Now, consider four cases: (i) Suppose that there exists a subsequence of the convergent subsequence along which $D_{x,\lambda,y,p,q}\mathcal{F}$ has less-than-full rank; then, $D_{x,\lambda,y,p,q}\mathcal{F}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, \overline{e}, \overline{u}\right)$ has rank less than full and $(\overline{e}, \overline{u}) \in \mathcal{D}_r^c$. (ii) Suppose that there exists a subsequence of the convergent subsequence along which, $x_{n'(k),\sigma,s}^i = x_{n'(k),\sigma,s'}^i$; then, $\overline{x}_{\sigma,s}^i = \overline{x}_{\sigma,s'}^i$ and $(\overline{e}, \overline{u}) \in \mathcal{D}_r^c$. (iii) Suppose that there exists a subsequence of the convergent subsequence along which $rank\left(\mathbf{Z}_\sigma\left(\widetilde{x}_{n'(k),\sigma}, \widetilde{e}_{n'(k),\sigma}\right)\right) \neq IS - 1$; by construction, this implies that, along this subsequence, $rank\left(\mathbf{Z}_\sigma\left(\widetilde{x}_{n'(k),\sigma}, \widetilde{e}_{n'(k),\sigma}\right)\right) \leq IS - 2$ and, by convergence, $rank\left(\mathbf{Z}_\sigma\left(\widetilde{\overline{x}}_\sigma, \widetilde{\overline{e}}_\sigma\right)\right) \leq IS - 2$, so $(\overline{e}, \overline{u}) \in \mathcal{D}_r^c$. (iv) Finally, since $I \in \mathbb{N}$, $\Sigma \in \mathbb{N}$ and $S \in \mathbb{N}$, if none of the previous cases holds true, then it must be that there exists a subsequence of the convergent subsequence along which $D_{x_\mathbf{1},\lambda_\mathbf{1},p_\mathbf{1},y}\mathcal{G}_{u_{n'(k)}}$ has less than full rank; then $D_{x_\mathbf{1},\lambda_\mathbf{1},p_\mathbf{1},y}\mathcal{G}_{\overline{u}}\left(\overline{x}_\mathbf{1}, \overline{\lambda}_\mathbf{1}, \overline{p}_\mathbf{1}, \overline{y}, \overline{e}_\mathbf{1}\right)$ has rank less than full and $(\overline{e}, \overline{u}) \in \mathcal{D}_r^c$.

That for every $(e, u) \in \mathcal{D}_r$ equilibria are finite follows, as in the proof of theorem 2, from property 1. □

Fix $(\overline{e}, \overline{u}) \in \mathcal{D}_r$ and let

$$W_{\overline{e},\overline{u}} = \left\{ \left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in \mathbb{R}_{++}^{I(\Sigma S+1)L} \times \mathbb{R}_{++}^{I(\Sigma S+1)} \times \mathbb{R}^{\Sigma I} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^{\Sigma} \middle| \mathcal{F}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}, \overline{e}, \overline{u}\right) = 0 \right\}$$

Fix $\overline{\epsilon} > 0$ such that for every $\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in W_{\overline{e},\overline{u}}$,

$$B_{2\overline{\epsilon}}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \cap W_{\overline{e},\overline{u}} = \left\{\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)\right\}$$

Theorem 6. *For each $\epsilon > 0$, there exists $\overline{\delta} > 0$ such that for every $(e, u) \in B_{\overline{\delta}}(e, u)$, $\mathcal{F}(x, \lambda, y, p, q, e, u) = 0$ implies that*

$$(x, \lambda, y, p, q) \in \bigcup_{\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in W_{\overline{e},\overline{u}}} B_{\epsilon}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$$

Proof. Fix $\epsilon > 0$ and suppose not: for every $\delta > 0$, there exist $(e, u) \in B_{\delta}(e, u)$ and $(x, \lambda, y, p, q) \notin \bigcup_{\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in W_{\overline{e},\overline{u}}} B_{\epsilon}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$, such that $\mathcal{F}(x, \lambda, y, p, q, e, u) = 0$. Then, one can construct a sequence $(x_n, \lambda_n, y_n, p_n, q_n, e_n, u_n)_{n=1}^{\infty}$ such that $(e_n, u_n) \longrightarrow (\overline{e}, \overline{u})$ and for every $n \in \mathbb{N}$, $(x_n, \lambda_n, y_n, p_n, q_n) \notin \bigcup_{\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in W_{\overline{e},\overline{u}}} B_{\epsilon}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$ and $\mathcal{F}(x_n, \lambda_n, y_n, p_n, q_n) = 0$. As in the proof of the previous theorem, there exists a subsequence $\left(x_{n(k)}, \lambda_{n(k)}, y_{n(k)}, p_{n(k)}, q_{n(k)}\right)_{k=1}^{\infty}$ that converges to some (x, λ, y, p, q). By construction, $\mathcal{F}(x, \lambda, y, p, q, \overline{e}, \overline{u}) = 0$ and $(x, \lambda, y, p, q) \notin \bigcup_{\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \in W_{\overline{e},\overline{u}}} B_{\epsilon}\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right) \supset W_{\overline{e},\overline{u}}$, which is a contradiction. □

For each $\epsilon > 0$, construct a C^{∞} (bump) function $\rho_{\epsilon} : \mathbb{R}^L \longrightarrow [0, 1]$ such that $\rho_{\epsilon}(\delta) = 1$ in $B_{\epsilon}(0)$ and $\rho_{\epsilon}(\delta) = 0$ outside $B_{2\epsilon}(0)$.

For each i and $\sigma \geq 1$, let $C_{\sigma}^i = \left\{ \overline{x} \in \mathbb{R}_{++}^L \middle| (\exists (x, \lambda, y, p, q, s) \in W_{\overline{e},\overline{u}}) : x_{\sigma,s}^i = \overline{x} \right\}$. By theorem 2 and construction, $S \leq \# C_{\sigma}^i \leq S \times \# W_{\overline{e},\overline{u}}$, so one can find $\epsilon_{\sigma}^i > 0$ such that for each $\overline{x} \in C_{\sigma}^i$, $B_{2\epsilon}(\overline{x}) \cap C_{\sigma}^i = \{\overline{x}\}$.

Let $(\Delta_{\overline{x}})_{\overline{x} \in C_{\sigma}^i}$ be a finite array of symmetric, $L \times L$ matrices, with norm less than $\delta > 0$ (i.e., vectors in the ball $B_{\delta}(0)$, defined in $\mathbb{R}^{L(L+1)/2}$). For δ small enough, the function $u : \mathbb{R}_{++}^L \longrightarrow \mathbb{R}$, defined as

$$u(x) = \overline{u}_{\sigma}^i(x) + \frac{1}{2} \sum_{\overline{x} \in C_{\sigma}^i} \rho_{\epsilon_{\sigma}^i}(x - \overline{x})(x - \overline{x})^{\top} \Delta_{\overline{x}}(x - \overline{x})$$

lies in $\mathcal{U}$.

Define the set $\mathbf{U}_{\delta} \subseteq \mathcal{U}^{I(\Sigma+1)}$ as follows: $u = \left(\left(u_{\sigma}^i\right)_{\sigma=0}^{\Sigma}\right)_{i=1}^{I} \in \mathbf{U}_{\delta}$ if, and only if, for every i, $u_0^i = \overline{u}_0^i$, and for every $\sigma \geq 1$, there exists $(\Delta_{\overline{x}})_{\overline{x} \in C_{\sigma}^i} \in B_{\delta}(0)^{\# C_{\sigma}^i}$ such that

$$u_{\sigma}^i(x) = \overline{u}_{\sigma}^i(x) + \frac{1}{2} \sum_{\overline{x} \in C_{\sigma}^i} \rho_{\epsilon_{\sigma}^i}(x - \overline{x})(x - \overline{x})^{\top} \Delta_{\overline{x}}(x - \overline{x}),$$

which is a finite-dimensional submanifold, parameterized by $\prod_{i,\sigma} B_{\delta}(0)^{\# C_{\sigma}^i}$.

For future reference, notice that at any $x \in \mathbb{R}^L_{++}$, there exists at most one $\overline{x} \in C^i_\sigma$ such that $u^i_\sigma(x) = \overline{u}^i_\sigma(x) + r(x-\overline{x})^\top \Delta_{\overline{x}}(x-\overline{x})$ for $r>0$, whereas at every $(x,\lambda,y,p,q) \in W_{\overline{e},\overline{u}}$, for every i, σ and s, there exists $\overline{x}^i_{\sigma,s} \in C^i_\sigma$

$$\begin{aligned} u^i_\sigma\left(x^i_{\sigma,s}\right) &= \overline{u}^i_\sigma\left(x^i_{\sigma,s}\right), \\ Du^i_\sigma\left(x^i_{\sigma,s}\right) &= D\overline{u}^i_\sigma\left(x^i_{\sigma,s}\right), \\ D^2u^i_\sigma\left(x^i_{\sigma,s}\right) &= D^2\overline{u}^i_\sigma\left(x^i_{\sigma,s}\right) + \Delta_{\overline{x}^i_{\sigma,s}}, \end{aligned}$$

and, by theorem 2, $s \neq s'$ implies that $\overline{x}^i_{\sigma,s} \neq \overline{x}^i_{\sigma,s'}$.

Also, notice that taking F as a function of u, on the submanifold parameterized as before, F is smooth.

Theorem 7. *There exist $\widetilde{\delta} > 0$ and $\widetilde{\epsilon} > 0$ such that for every $u \in \mathbf{U}_\delta$ and every $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$*

$$\left(\mathcal{F}(x,\lambda,y,p,q,\overline{e},u) = 0 \text{ and } (x,\lambda,y,p,q) \in B_{\widetilde{\epsilon}}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)\right) \Longleftrightarrow (x,\lambda,y,p,q) = \left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right).$$

Proof. Since $(\overline{e},\overline{u}) \in \mathcal{D}_r$ and $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$, it follows that $D_{x,\lambda,y,p,q}\mathcal{F}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q},\overline{e},\overline{u}\right)$ is invertible and, hence, by the implicit function theorem, there exist $\delta_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)} > 0$, $\epsilon_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)} > 0$ and a C^1 function $g_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)} : \prod_{i,\sigma} B_{\delta_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}}(0)^{\#C^i_\sigma} \longrightarrow B_{\epsilon_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$ such that $\mathcal{F}(x,\lambda,y,p,q,\overline{e},\Delta) = 0$ with $(x,\lambda,y,p,q) \in B_{\epsilon_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$ if and only if $(x,\lambda,y,p,q) = g_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}(\Delta)$.

Let $\widetilde{\delta} = \min_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}} \left\{\delta_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}\right\} > 0$ and $\widetilde{\epsilon} = \min_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}} \left\{\epsilon_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}\right\} > 0$.

Fix $\Delta \in \prod_{i,\sigma} B_{\widetilde{\delta}}(0)^{\#C^i_\sigma}$ and $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$. By construction, $\mathcal{F}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q},\overline{e},u\right) = 0$ and $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in B_{\widetilde{\epsilon}}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$, so $g_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)}(\Delta) = \left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$, and, therefore, $\mathcal{F}(x,\lambda,y,p,q,\overline{e},u) = 0$ and $(x,\lambda,y,p,q) \in B_{\widetilde{\epsilon}}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$ occurs if and only if $(x,\lambda,y,p,q) = \left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$. ☐

Corollary 3. *There exist $\delta > 0$ such that for every $u \in \mathbf{U}_\delta$,*

$$\left\{(x,\lambda,y,p,q) \in \mathbb{R}^{I(\Sigma S+1)L}_{++} \times \mathbb{R}^{I(\Sigma S+1)}_{++} \times \mathbb{R}^{\Sigma I} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^{\Sigma} \,\middle|\, \mathcal{F}(x,\lambda,y,p,q,\overline{e},u) = 0\right\} = W_{\overline{e},\overline{u}}.$$

Proof. Let $\epsilon = \min\{\overline{\epsilon},\widetilde{\epsilon}\} > 0$.

By theorem 6, there exists $\overline{\delta} > 0$ such that for every $\Delta \in B_{\overline{\delta}}(0)$, $\mathcal{F}(x,\lambda,y,p,q,\overline{e},\Delta) = 0$ implies that $(x,\lambda,y,p,q) \in \bigcup_{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W} B_\epsilon\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$. Let $\delta = \min\left\{\overline{\delta},\widetilde{\delta}\right\}$, and fix $\Delta \in \prod_{i,\sigma} B_\delta(0)^{\#C^i_\sigma}$. By construction, $\mathcal{F}(x,\lambda,y,p,q,\overline{e},\Delta) = 0$ only if for some $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$, $(x,\lambda,y,p,q) \in B_\epsilon\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right)$, which implies, by theorem 7, since $\epsilon \leq \widetilde{\epsilon}$ that $(x,\lambda,y,p,q) = \left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$. Directly, $\mathcal{F}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q},\overline{e},\Delta\right) = 0$, so $\left\{\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \,\middle|\, \mathcal{F}\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q},\overline{e},u\right) = 0\right\} \supseteq W_{\overline{e},\overline{u}}$. ☐

8.2 Genericity of constrained suboptimality

Theorem 8. *There exists a generic subset of economies $\mathcal{D}$ such that for every $(e,u) \in \mathcal{D}$, if (x,λ,y,p,q) is an equilibrium for (e,u), then x is constrained inefficient.*

Proof. Fix $(\overline{e},\overline{u}) \in \mathcal{D}_r$.

Let $\delta > 0$ be such that $\{\overline{e}\} \times \mathbf{U}_\delta \subseteq \mathcal{D}_r$ and that for every $u \in \mathbf{U}_\delta$,

$$\{(x,\lambda,y,p,q) | \mathcal{F}(x,\lambda,y,p,q,\overline{e},u) = 0\} = W_{\overline{e},\overline{u}}.$$

Consider $\mathcal{F}$ and $\mathcal{H}$ as functions of u restricted to $\mathbf{U}_\delta$, parameterized by $\prod_{i,\sigma} B_\delta(0)^{\# C^i_\sigma}$.

Define the function[13]

$$\mathcal{M}(x,\lambda,y,p,q,\Delta,\theta) = \begin{bmatrix} \mathcal{F}(x,\lambda,y,p,q,\overline{e},\Delta) \\ D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}},\Delta)^\top \theta \\ \frac{1}{2}\left(\theta^\top\theta - 1\right) \end{bmatrix}$$

and suppose that $\mathcal{M}(x,\lambda,y,p,q,\Delta,\theta) = 0$.

By construction, for some $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) \in W_{\overline{e},\overline{u}}$, $\left(\overline{x},\overline{\lambda},\overline{y},\overline{p},\overline{q}\right) = (x,\lambda,y,p,q)$, so for every i, σ and s,

$$\begin{aligned} u^i_\sigma\left(x^i_{\sigma,s}\right) &= \overline{u}^i_\sigma\left(x^i_{\sigma,s}\right), \\ Du^i_\sigma\left(x^i_{\sigma,s}\right) &= D\overline{u}^i_\sigma\left(x^i_{\sigma,s}\right), \\ D^2u^i_\sigma\left(x^i_{\sigma,s}\right) &= D^2\overline{u}^i_\sigma\left(x^i_{\sigma,s}\right) + \Delta^i_{\sigma,s}, \end{aligned}$$

where, for simplicity, $\Delta^i_{\sigma,s} = \Delta_{\overline{x}^i_{\sigma,s}}$.

Name the rows of $D\mathcal{H}$ by

$$\left(\begin{array}{ccccccccccc} u1 & \cdots & uI & f1 & b1 & \cdots & fI & bI & c1 & \cdots & c\Sigma & a \end{array} \right),$$

so we can denote θ by

$$\theta^\top = \left[\begin{array}{cccccccccccc} \theta_{u1} & \cdots & \theta_{uI} & \theta^\top_{f1} & \theta^\top_{b1} & \cdots & \theta^\top_{fI} & \theta^\top_{bI} & \theta^\top_{c1} & \cdots & \theta^\top_{c\Sigma} & \theta^\top_a \end{array} \right],$$

where $\theta_{ui} \in \mathbb{R}$, $\theta_{fi} \in \mathbb{R}^{\Sigma SL}$, $\theta_{bi} \in \mathbb{R}^{\Sigma S}$, $\theta_{c\sigma} \in \mathbb{R}^{L-1}$ and $\theta_a \in \mathbb{R}^\Sigma$.

$D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,e_{\mathbf{1}},\Delta)^\top \theta = 0$ is the system:

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[13] Mapping

$$\mathbb{R}^{I(\Sigma S+1)L}_{++} \times \mathbb{R}^{\Sigma I} \times \mathbb{R}^{I(\Sigma S+1)}_{++} \times \mathcal{P}^{\Sigma+1} \times \mathbb{R}^\Sigma \times \prod_{i=1}^{I}\prod_{\sigma=1}^{\Sigma} B_{\widetilde{\delta}}(0)^{K^i_\sigma} \times \mathbb{R}^{I+I\Sigma S(L+1)+\Sigma(L+1)+\Sigma}$$

into

$$\mathbb{R}^{I(\Sigma S+1)L} \times \mathbb{R}^{I(\Sigma S+1)} \times \mathbb{R}^{I\Sigma} \times \mathbb{R}^{(\Sigma+1)(L-1)} \times \mathbb{R}^\Sigma \times \mathbb{R}^{I\Sigma SL} \times \mathbb{R}^{I\Sigma S} \times \mathbb{R}^{\Sigma(L-1)} \times \mathbb{R}^{\Sigma I} \times \mathbb{R}$$

1. $\theta_{ui}\pi^i\left(s|\,\sigma\right)D\overline{u}_\sigma^i\left(x_{\sigma,s}^i\right)^\top + \pi^i\left(s|\,\sigma\right)\left(D^2\overline{u}_\sigma^i\left(x_{\sigma,s}^i\right)+\Delta_{\sigma,s}^i\right)\theta_{fi}^{\sigma,s} - \theta_{bi}^{\sigma,s}p_\sigma^\top + \pi^i\left(s|\,\sigma\right)\widetilde{I}^\top\theta_{c\sigma}=0;$
2. $-p_\sigma\theta_{fi}^{\sigma,s}=0;$
3. $\sum_{i,s}\lambda_{\sigma,s}^i\widetilde{I}\theta_{fi}^{\sigma,s}+\sum_{i,s}\theta_{bi}^{\sigma,s}\left(\widetilde{e}_{\sigma,s}^i-\widetilde{x}_{\sigma,s}^i\right)=0;$
4. $\sum_s\theta_{bi}^{\sigma,s}+\theta_a^\sigma=0.$

Claim 1. *For some i, $\theta_{ui}\neq 0$.*

Proof of the claim. Suppose that for every i, $\theta_{ui}=0$. This implies that

$$D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{G}_u\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}}\right)^\top\tilde{\theta}=0,$$

where

$$\widetilde{\theta}^\top=\left[\begin{array}{cccccccccc}\theta_{f1}^\top & \theta_{b1}^\top & \cdots & \theta_{fI}^\top & \theta_{bI}^\top & \theta_{c1}^\top & \cdots & \theta_{c\Sigma}^\top & \theta_a^\top\end{array}\right].$$

Since $\mathcal{G}_u\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}}\right)=0$ and $(\overline{e},u)\in\mathcal{D}_r$, it follows that $D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{G}_u\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}}\right)$ has full row rank and, therefore, $\widetilde{\theta}=0$ and $\theta=0$, which contradicts the fact that $\theta^\top\theta-1=0$. □

Claim 2. *For every σ, i and s, $\theta_{fi}^{\sigma,s}\neq 0$.*

Proof of the claim. Suppose that for some σ, i^* and $s^*\in\left\{1,...,S^i\right\}$, $\theta_{fi^*}^{\sigma,s^*}=0$. Then, from 1

$$\theta_{ui^*}\pi^{i^*}\left(s^*|\,\sigma\right)D\overline{u}_\sigma^i\left(x_{\sigma,s^*}^{i^*}\right)^\top-\theta_{bi^*}^{\sigma,s^*}p_\sigma^\top+\pi^{i^*}\left(s^*|\,\sigma\right)\widetilde{I}^\top\theta_{c\sigma}=0.$$

Since $\mathcal{F}\left(x,\lambda,y,p,q,\overline{e},\Delta\right)=0$, $\pi^{i^*}\left(s^*|\,\sigma\right)D\overline{u}_\sigma^{i^*}\left(x_{\sigma,s^*}^{i^*}\right)^\top=\lambda_{\sigma,s^*}^{i^*}p_\sigma^\top$, so

$$\theta_{ui^*}\lambda_{\sigma,s^*}^{i^*}p_\sigma^\top-\theta_{bi^*}^{\sigma,s^*}p_\sigma^\top+\pi^{i^*}\left(s^*|\,\sigma\right)\widetilde{I}^\top\theta_{c\sigma}=0,$$

which implies, from the first component, that $\theta_{ui^*}\lambda_{\sigma,s^*}^{i^*}=\theta_{bi^*}^{\sigma,s^*}$ and, then, that $\pi^{i^*}\left(s^*|\,\sigma\right)\widetilde{I}^\top\theta_{c\sigma}=0$, which means that $\theta_{c\sigma}=0$. For every other i and every other s, from 1,

$$\theta_{ui}\pi^i\left(s|\,\sigma\right)D\overline{u}_\sigma^i\left(x_{\sigma,s}^i\right)^\top+\pi^i\left(s|\,\sigma\right)\left(D^2\overline{u}_\sigma^i\left(x_{\sigma,s}^i\right)+\Delta_{\sigma,s}^i\right)\theta_{fi}^{\sigma,s}-\theta_{bi}^{\sigma,s}p_\sigma^\top=0.$$

As before, this means that

$$\theta_{ui}\lambda_{\sigma,s}^ip_\sigma^\top+\pi^i\left(s|\,\sigma\right)\left(D^2\overline{u}_\sigma^i\left(x_{\sigma,s}^i\right)+\Delta_{\sigma,s}^i\right)\theta_{fi}^{\sigma,s}-\theta_{bi}^{\sigma,s}p_\sigma^\top=0,$$

so, premultiplying by $\left(\theta_{fi}^{\sigma,s}\right)^\top$, by 2, we have that $\left(\theta_{fi}^{\sigma,s}\right)^\top\left(D^2u_\sigma^i\left(x_{\sigma,s}^i\right)+\Delta_{\sigma,s}^i\right)\theta_{fi}^{\sigma,s}=0$, which now implies that $\theta_{fi}^{\sigma,s}=0$. This immediately implies that $\theta_{ui}\lambda_{\sigma,s}^i=\theta_{bi}^{\sigma,s}$, whereas, from 3, given that $\left(\left(\theta_{fi}^{\sigma,s}\right)_{s=1}^S\right)_{i=1}^I=0$, we have that $\sum_{i,s}\theta_{bi}^{\sigma,s}\left(\widetilde{e}_{\sigma,s}^i-\widetilde{x}_{\sigma,s}^i\right)=$

0. Since $rank\left(\mathbf{Z}_{\sigma}\left(\widetilde{x}_{\sigma}, \widetilde{e}_{\sigma}\right)\right)=IS-1$ and $\sum_{i,s}\pi^{i}\left(s|\,\sigma\right)\left(\widetilde{e}^{i}_{\sigma,s}-\widetilde{x}^{i}_{\sigma,s}\right)=0$, it follows that for some $k\in\mathbb{R}$, $\theta^{\sigma,s}_{bi}=k\pi^{i}\left(s|\,\sigma\right)$ and, then, $\theta_{ui}\lambda^{i}_{\sigma,s}=k\pi^{i}\left(s|\,\sigma\right)$, for every i and s, which implies that $k\neq 0$, by the previous claim. It then follows that for some i, $\frac{\lambda^{i}_{\sigma,1}}{\pi^{i}(1|\sigma)}=\frac{\lambda^{i}_{\sigma,S}}{\pi^{i}(S|\sigma)}$, and, therefore, $x^{i}_{\sigma,1}\neq x^{i}_{\sigma,S}$, contradicting the fact that $(\overline{e},u)\in\mathcal{D}_{r}$ and $\mathcal{F}\left(x,\lambda,y,p,q,\overline{e},\Delta\right)=0$. $\square$

Now, with the arguments in the order

$$\left(\left(x,\lambda,y,p,q\right),\left(\left(\left(\Delta^{i}_{\sigma,s}\right)_{s=1}^{S}\right)_{\sigma=1}^{\Sigma}\right)_{i=1}^{I},\theta\right),$$

we have that

$$D\mathcal{M}\left(x,\lambda,y,p,q,\Delta,\theta\right)=\left[\begin{array}{ccc} D_{x,\lambda,y,p,q}\mathcal{F}\left(x,\lambda,y,p,q,\overline{e},\Delta\right) & 0 & 0 \\ M & N\left(\theta\right) & D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}},\Delta\right)^{\top} \\ 0 & 0 & \theta^{\top} \end{array}\right].$$

Since $D_{x,\lambda,y,p,q}\mathcal{F}\left(x,\lambda,y,p,q,\overline{e},\Delta\right)$has full row rank, because $(\overline{e},u)\in\mathcal{D}_{r}$, it suffices that

$$\mathbf{M}=\left[\begin{array}{cc} N\left(\theta\right) & D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}\left(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\overline{e}_{\mathbf{1}},\Delta\right)^{\top} \\ 0 & \theta^{\top} \end{array}\right]$$

have full row rank for $D\mathcal{M}\left(x,\lambda,y,p,q,\Delta,\theta\right)$ to have full row rank.

We can rewrite matrix $\mathbf{M}$, with the columns in the order

$$\left(\begin{array}{cccccccccccccc} f1 & b1 & \cdots & fI & bI & c1 & \cdots & c\Sigma & a & \Delta^{1} & \cdots & \Delta^{I} & u1 & \cdots & uI \end{array}\right)$$

to get

$$\left[\begin{array}{ccccccccccccc}
D^{2}\overline{u}^{1}_{\mathbf{1}}+\Delta^{1} & -\Psi^{\top}_{\mathbf{1}} & \ldots & 0 & 0 & \left(\Phi^{1}_{\mathbf{1}}\right)^{\top} & 0 & \mathbf{N}^{1} & \ldots & 0 & \left(Du^{1}_{\mathbf{1}}\right)^{\top} & \ldots & 0 \\
-\Psi_{\mathbf{1}} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & D^{2}\overline{u}^{I}_{\mathbf{1}}+\Delta^{I} & -\Psi^{\top}_{\mathbf{1}} & \left(\Phi^{I}_{\mathbf{1}}\right)^{\top} & 0 & 0 & \ldots & \mathbf{N}^{I} & 0 & \ldots & \left(Du^{I}_{\mathbf{1}}\right)^{\top} \\
0 & 0 & \ldots & -\Psi_{\mathbf{1}} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\left(\Lambda^{1}_{\mathbf{1}}\right)^{\top} & \left(Z^{1}_{\mathbf{1}}\right)^{\top} & \ldots & \left(\Lambda^{I}_{\mathbf{1}}\right)^{\top} & \left(Z^{I}_{\mathbf{1}}\right)^{\top} & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \widetilde{R}^{\top} & \ldots & 0 & 0 & 0 & I_{\Sigma} & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \widetilde{R}^{\top} & 0 & I_{\Sigma} & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\theta^{\top}_{f1} & \theta^{\top}_{b1} & \ldots & \theta^{\top}_{fI} & \theta^{\top}_{bI} & \theta^{\top}_{c} & \theta^{\top}_{a} & 0 & \ldots & 0 & \theta_{u1} & \ldots & \theta_{uI}
\end{array}\right],$$

where $\mathbf{N}^i = \mathbf{N}^i(\theta_{fi})$ is the $\Sigma SL \times \Sigma SL(L+1)/2$ matrix

$$\begin{bmatrix} N\left(\theta_{fi}^{1,1}\right) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & N\left(\theta_{fi}^{1,2}\right) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N\left(\theta_{fi}^{1,S^i}\right) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & N\left(\theta_{fi}^{2,1}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & N\left(\theta_{fi}^{\Sigma,S^i}\right) \end{bmatrix},$$

where for any $t \in \mathbb{R}^L$,

$$N(t)_{L\times L(L+1)/2} = \begin{bmatrix} t_1 & t_2 & \cdots & \cdots & t_L & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & t_1 & \cdots & \cdots & 0 & t_2 & t_3 & \cdots & t_L & \cdots & 0 \\ 0 & 0 & \ddots & & 0 & 0 & t_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & t_1 & 0 & 0 & \cdots & t_2 & \cdots & t_L \end{bmatrix},$$

a matrix that has full rank if (and only if) $t \neq 0$.

Consider the matrix resulting from the $x_{\mathbf{1}}^1, \lambda_{\mathbf{1}}^1, ..., x_{\mathbf{1}}^I, \lambda_{\mathbf{1}}^I$ superrows and the $\theta_{f1}, \theta_{b1}, ..., \theta_{fI}, \theta_{bI}$ supercolumns on $\mathbf{M}$. It is straightforward that this matrix has full rank. The remaining rows and columns add $\Sigma(L-1) + \Sigma I + 1$ rows and $\Sigma(L-1) + \Sigma + I\Sigma SL(L+1)/2 + I$ columns. Since $\frac{1}{2} \geq \frac{1}{S} \geq \frac{(\Sigma-1)(I-1)}{\Sigma IS} = \frac{\Sigma I+1-I-\Sigma}{\Sigma IS}$, it follows that $\Sigma I + 1 - I - \Sigma \leq \frac{\Sigma IS}{2} \leq \frac{L(L-1)}{2}\Sigma IS$, or, equivalently, that $\frac{L(L-1)}{2}\Sigma IS + I + \Sigma \geq \Sigma I + 1$, showing that we add more columns than rows.

Assume, without loss of generality, that $\theta_{u1} \neq 0$.

Fix σ and $l \geq 2$, and define α as follows: $\alpha\left(\theta_{f1}^{\sigma,1,l}\right) = \frac{1}{\lambda_{\sigma,1}^1}$, $\alpha\left(\theta_{f1}^{\sigma,1,1}\right) = -\frac{p_{\sigma,l}}{\lambda_{\sigma,1}^1}$,

$$\alpha(\theta_{u1}) = -\frac{\frac{\theta_{f1}^{\sigma,1,l}}{\lambda_{\sigma,1}^1} - \frac{p_{\sigma,l}\theta_{f1}^{\sigma,1,1}}{\lambda_{\sigma,1}^1}}{\theta_{u1}}$$

and $\alpha(\chi) = 0$ for every other column χ, except for Δ^1, where $\alpha(\Delta^1)$ is fixed such that

$$\mathbf{N}^1(\theta_{f1})\,\alpha\left(\Delta^1\right) + \left(D^2\overline{u}_{\mathbf{1}}^1\left(x_{\mathbf{1}}^1\right) + \Delta^1\right)\alpha(\theta_{f1}) + \alpha(\theta_{u1})\,Du_{\mathbf{1}}^1\left(x_{\mathbf{1}}^1\right)^\top = 0,$$

which we can do, because each $\theta_{f1}^{\sigma,s} \neq 0$, so each $N\left(\theta_{f1}^{\sigma,s}\right)$ has full rank and hence $\mathbf{N}^i(\theta_{fi})$ has full rank. By construction, $\mathbf{M}\alpha$ gives 0 at every component, except at the l-th component for the superrow corresponding to $\widetilde{p}_\sigma$, where it gives 1.

Fix σ and define α by $\alpha\left(\theta_{b1}^{\sigma,1}\right)=1$,

$$\begin{aligned}(\forall l\in\{2,...,L\}):\alpha\left(\theta_{f1}^{\sigma,1,l}\right)&=-\frac{\left(e_{\sigma,1,l}^{1}-x_{\sigma,1,l}^{1}\right)}{\lambda_{\sigma,1}^{1}},\\ \alpha\left(\theta_{f1}^{\sigma,1,1}\right)&=\sum_{l=2}^{L}p_{\sigma,l}\frac{\left(e_{\sigma,1,l}^{1}-x_{\sigma,1,l}^{1}\right)}{\lambda_{\sigma,1}^{1}},\\ \alpha\left(\theta_{u1}\right)&=-\frac{\alpha\left(\theta_{b1}^{\sigma,1}\right)\theta_{b1}^{\sigma,1}+\alpha\left(\theta_{f1}^{\sigma,1}\right)\theta_{f1}^{\sigma,1}}{\theta_{u1}},\end{aligned}$$

and $\alpha(\chi)=0$ for every other column χ, except for Δ^1, where $\alpha\left(\Delta^1\right)$ is fixed such that

$$N^1\left(\theta_{f1}^{\sigma,1}\right)\alpha\left(\Delta_{\sigma,1}^{1}\right)+\pi^1\left(1|\,\sigma\right)\left(D^2\overline{u}_{\sigma}^{1}\left(x_{\sigma,1}^{1}\right)+\Delta^1\right)\alpha\left(\theta_{f1}^{\sigma,1}\right)-\alpha\left(\theta_{b1}^{\sigma,1}\right)p_{\sigma}^{\top}+\alpha\left(\theta_{u1}\right)Du_{\mathbf{1}}^{1}\left(x_{\mathbf{1}}^{1}\right)^{\top}=0,$$

which we can do because each $\theta_{f1}^{\sigma,s}\neq0$, so each $N\left(\theta_{f1}^{\sigma,s}\right)$ and, hence, $\mathbf{N}^1\left(\theta_{f1}\right)$ have full rank. By construction, $\mathbf{M}\alpha$ gives 0 at every component, except at the σ-th component for the superrow corresponding to y^1, where it gives 1.

Now fix $i\geq2$ and σ and define α by $\alpha\left(\theta_{bi}^{\sigma,1}\right)=1$

$$\begin{aligned}(\forall l\in\{2,...,L\}):\alpha\left(\theta_{fi}^{\sigma,1,l}\right)&=-\frac{\left(e_{\sigma,1,l}^{i}-x_{\sigma,1,l}^{i}\right)}{\lambda_{\sigma,1}^{i}},\\ \alpha\left(\theta_{fi}^{\sigma,1,1}\right)&=\sum_{l=2}^{L}p_{\sigma,l}\frac{\left(e_{\sigma,1,l}^{i}-x_{\sigma,1,l}^{i}\right)}{\lambda_{\sigma,1}^{i}},\\ \alpha\left(\theta_{u1}\right)&=-\frac{\alpha\left(\theta_{bi}^{\sigma,1}\right)\theta_{bi}^{\sigma,1}+\alpha\left(\theta_{fi}^{\sigma,1}\right)\theta_{fi}^{\sigma,1}}{\theta_{u1}},\end{aligned}$$

and $\alpha(\chi)=0$ for every other column χ, except for $\Delta_{\sigma,1}^{i}$ and Δ^1, where $\alpha\left(\Delta_{\sigma,1}^{i}\right)$ is fixed such that

$$N^i\left(\theta_{fi}^{\sigma,1}\right)\alpha\left(\Delta_{\sigma,1}^{1}\right)+\pi^i\left(1|\,\sigma\right)\left(D^2u_{\sigma}^{1}\left(x_{\sigma,1}^{1}\right)+\Delta^1\right)\alpha\left(\theta_{fi}^{\sigma,1}\right)-\alpha\left(\theta_{bi}^{\sigma,1}\right)p_{\sigma}^{\top}=0,$$

and $\alpha\left(\Delta^1\right)$ is fixed such that $\mathbf{N}^1\left(\theta_{f1}\right)\alpha\left(\Delta^1\right)+\alpha\left(\theta_{u1}\right)Du_{\mathbf{1}}^{1}\left(x_{\mathbf{1}}^{1}\right)^{\top}=0$, which we can do, because $\theta_{fi}^{\sigma,1}\neq0$ and each $\theta_{f1}^{\sigma,s}\neq0$, so $N^i\left(\theta_{fi}^{\sigma,1}\right)$, each $N\left(\theta_{f1}^{\sigma,s}\right)$ and, hence, $\mathbf{N}^1\left(\theta_{f1}\right)$ have full rank. By construction, $\mathbf{M}\alpha$ gives 0 at every component, except at the σ-th component for the superrow corresponding to y^i, where it gives 1.

Also, let $\alpha\left(\theta_{u1}\right)=\frac{1}{\theta_{u1}}$ and $\alpha(\chi)=0$ for every other column χ, except for Δ^1, where $\alpha\left(\Delta^1\right)$ is fixed such that $\mathbf{N}^1\left(\theta_{f1}\right)\alpha\left(\Delta^1\right)+\alpha\left(\theta_{u1}\right)Du_{\mathbf{1}}^{1}\left(x_{\mathbf{1}}^{1}\right)=0$, which we can do, because each $\theta_{f1}^{\sigma,s} \neq 0$, so each $N\left(\theta_{f1}^{\sigma,s}\right)$ has full rank and hence $\mathbf{N}^i\left(\theta_{fi}\right)$ has full rank. By construction, $\mathbf{M}\alpha$ gives 0 at every component, except at the last component where it gives 1.

It follows that $\mathbf{M}$ has full row rank, and hence $D\mathcal{M}$ has full row rank, implying that $\mathcal{M} \pitchfork 0$.

Then, there exists full measure $\mathbf{B} \subseteq \prod_{i,\sigma} B_\delta(0)^{\# C_\sigma^i}$ such that for every $\Delta \in \mathbf{B}$, $\mathcal{M}(\cdot,\Delta) \pitchfork 0$. Now, fix $\Delta \in \mathbf{B}$ and suppose that $\mathcal{M}(x,\lambda,y,p,q,\Delta,\theta) = 0$. Then, $D_{x,\lambda,y,p,q,\theta}\mathcal{M}(x,\lambda,y,p,q,\Delta,\theta)$ has full row rank, but has

$$I(\Sigma S+1)(L+1)+\Sigma I+(1+\Sigma)(L-1)+\Sigma+I\Sigma S(L+1)+\Sigma(L-1)+\Sigma I+1$$

rows and

$$I(\Sigma S+1)(L+1)+\Sigma I+(1+\Sigma)(L-1)+\Sigma+I+I\Sigma S(L+1)+\Sigma(L-1)+\Sigma$$

columns. Since $\Sigma \geq 2$ and $I \geq 2$, it follows that $\Sigma I \geq \Sigma + I$ and hence that $\Sigma I + 1 > \Sigma + I$, which means that the matrix has more rows than columns and hence cannot have full row rank. Then, it follows that for every $\Delta \in \mathbf{B}$,

$$\begin{array}{c} \mathcal{F}(x,\lambda,y,p,q,\bar{e},\Delta) = 0 \\ D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\bar{e}_{\mathbf{1}},\Delta)^\top \theta = 0 \end{array} \Longrightarrow \theta = 0,$$

meaning that $D_{x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y}\mathcal{H}(x_{\mathbf{1}},\lambda_{\mathbf{1}},p_{\mathbf{1}},y,\bar{e}_{\mathbf{1}},\Delta)$ has full row rank. It follows from proposition 4 that x is constrained inefficient.

All this implies that for any neighborhood O of $(\bar{e},\bar{u})$, there exists $\Delta \in \prod_{i,\sigma} B_\delta(0)^{\# C_\sigma^i}$ such that $(\bar{e},\Delta) \in (\{\bar{e}\} \times \mathbf{B}) \cap O$, which implies denseness.

Openness can be argued as in the proof of theorem 5. □

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