Generic Virtual Determinacy among Overlapping Generations

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Abstract: We reappraise the significance of indeterminacy in overlapping-generations models. In any of Gale's examples of locally-indeterminate (non-locally-unique) equilibrium, for instance, judiciously splitting each of Gale's goods into 2 close substitutes restricts that indeterminacy to each period's distribution of consumption between those substitutes. In particular, prices, interest rates, and utility levels are now determinate. Equilibrium is thus virtually determinate (virtually locally unique), and some forecasting and comparative-statics policy applications are now possible. Virtual determinacy is robust to further splitting goods, and is thus **generic**, holding for each economy in some open-dense set in an appropriate topological space.

Key words: Indeterminacy, overlapping generations, forecasting, comparative statics, locally unique.

JEL Classification Numbers: C62, D51, D90.

1. Introduction

This paper offers alternative analysis and conclusions to the literature appraising indeterminacy in overlapping-generations models. We focus on a common *thought experiment*: assume a forecaster or policy-maker observes an overlapping-generations economy with a stationary history of preference relations, endowments, and consumption consistent with pure-exchange perfect-foresight competitive equilibrium. Then unexpected sunspots occur. Assuming preferences and endowments continue at their historic levels and future consumption is again consistent with equilibrium after the sunspots, *will future consumption necessarily continue on the historic path? or might it follow some new, possibly non-stationary, path?* And if future consumption cannot be determined to follow the historic path, *is that indeterminacy significant?*

The literature's answers to the thought experiment depend on that particular economies forecasters or policy-makers observe. Some example economies have future consumption necessarily continuing on the historic path; the historic equilibrium is globally determinate. Other examples, beginning with Gale for the special case of one good per period and two periods per lifetime, have indeterminacy, with a continuum of alternative perfect-foresight equilibrium paths, each starting away from historic consumption just after the sunspots but monotonically converging back [5, 9]. And equilibrium utility levels vary between the alternative paths, so that indeterminacy is significant.

To motivate the literature, the indeterminacy in Gale's examples makes overlapping-generations models more useful for some applications but less useful for others. On the one hand, indeterminacy makes Gale's examples descriptive models of extrinsic or extraneous uncertainty, since unexpected sunspots affect equilibrium utility levels, which implies rationally-expected random sunspots also affect utility [1]. On the other hand, indeterminacy makes Gale's examples incapable of even the simplest forecasting or comparative-statics policy applications, since the sunspot effect (which is like the effect of an immaterial policy change that leaves equilibrium equations unaffected) on equilibrium utility levels cannot be determined, even if the forecasters and policy-makers *a priori* restrict attention to an arbitrarily-small neighborhood of historic consumption. Likewise, indeterminacy makes the perfect-foresight hypothesis internally inconsistent, since Gale's people cannot foresee from equilibrium equations whether prices and interest rates will continue on the historic path.

But is significant indeterminacy verifiable? Assuming preference relations are imprecisely observed by the forecasters and policy-makers in the thought experiment, significant indeterminacy in any observed economy is only verifiable if it is robust to any suitably-small perturbation of preference relations. The literature considers Gale's example economies to be robust, for instance, since any C^1-small perturbation of the bivariate utility functions representing preferences leaves a continuum of convergent equilibrium paths, each with a different path of equilibrium utility levels. But the literature's perturbations hold the number of goods per period fixed *a priori* [6, 8, 9], and the commodity-differentiation literature[1] justifies additional perturbations that vary the number of goods per period by *splitting* individual goods into 2 or more close substitutes. We find those additional perturbations make significant indeterminacy unverifiable.

Considering just the simplest possibility, compare any one of Gale's 1-good (1 good per period) economies (Example 1) to an equivalent 2-good version, where one of the goods is aggregate consumption over the first half of the period, the other good over the second half, and the two goods are perfect 1-for-1 substitutes. Assume historic consumption is positive. Evidently, significant local indeterminacy remains in the 2-good version; there remains a continuum of alternative perfect-foresight equilibrium paths, each starting away from historic consumption just after the sunspots but converging back, and equilibrium utility levels vary between the alternative paths. But that significant indeterminacy is not verifiable since it is not robust to a judicious C^1-small perturbation of the utility functions representing preferences in the 2-good economy. Specifically, there is a perturbation and a new economy (Example 2) were each alternative path of positive equilibrium consumption now generates utility constant at the historic level. Also constant are total young-age consumption (the sum of the two young-age goods), total old-age consumption, equilibrium prices, and equilibrium interest rates. Thus, the historic equilibrium of the perturbed economy is no longer significantly indeterminate; rather, it is **virtually determinate**.

[1] To prove the general existence of equilibrium in an economy with a finite number of consumers, Jones describes a topological space of economies whose preference relations can be defined over a continuum of commodities or over any finite subset of those commodities. His topology allows perturbations of preference relations that split a single good into 2 or more close substitutes [7, p. 518].

Such "virtual determinacy" in any overlapping-generations economy has the same practical implications as the standard sense of local determinacy. Specifically, virtual determinacy implies perfect foresight is internally consistent since any local indeterminacy in equilibrium consumption has no effect on equilibrium prices and interest rates, and so no effect on consumers' utility-maximization problems. Likewise, some forecasting and comparative-statics policy applications are possible since, locally, unexpected sunspots have virtually no effect on equilibrium.

The significant local indeterminacy in many other examples in the literature is also not verifiable and, instead, virtual determinacy is verifiable. To be precise, we form the potential observations by the forecasters and policy-makers in the thought experiment into a topological space of pure-exchange stationary single-consumer overlapping-generations economies with only two periods per lifetime but a variable number of goods per period. Hence, the historic equilibrium is virtually determinate for each economy in some dense set (Theorem 1), and is approximately virtually determinate for each economy in some open-dense set (Theorem 2). ("Approximate virtual determinacy" has practical implications like virtual determinacy.) Approximate virtual determinacy can thus be verified for an open-dense set of the economies potentially observed in the thought experiment (since that set is open), but virtually determinacy cannot be falsified for any economy (since the set of economies violating virtual determinacy has an empty interior). It is thus reasonable to *a priori* restrict the determinacy thought experiment, and forecasting and comparative-statics applications, to economies where the historic equilibrium is virtually determinate or approximately virtually determinate.

Here is the plan for the rest of the paper:

Sections 2 through 5 further specify the determinacy thought experiment with definitions and assumptions. Each time period is partitioned into a finite number of time subintervals, and goods are distinguished by both their physical type and by the time subinterval over which they are aggregated. Choosing finer partitions thus increases the number of goods per period and yields alternative versions of endowments, historic consumption, preference relations, and equilibria for the determinacy thought experiment.

Section 3 also exemplifies the thought experiment. Example 1 is one of Gale's economies with significant local indeterminacy. Example 2 splits the single good of Example 1 into two close substitutes. Although local indeterminacy remains in Example 2 , indeterminacy becomes insignificant because it does not affect equilibrium utility levels, prices, interest rates, or the aggregate consumption of the two close substitutes.

Sections 6 and 7 conclude the thought experiment by formulating Theorem 1 and Theorem 2 , which imply significant local indeterminacy is not verifiable for any economy, but virtual determinacy is verifiable for a generic set of economies. The formulations are carefully interpreted for realistic forecasters and policy-makers that have a fixed amount of endowment, consumption, and price data.

Sections 8 through 10 prove Theorem 1 and Theorem 2 .

Finally, about further work, our formulation of the determinacy thought experiment sacrificed generality for simplicity. We assumed pure-exchange, single-consumer generations, two-period lifetimes, and positive historic consumption. Generalizations have been completed, but they are complex and unremarkable [2]. Two other restrictions are harder to generalize. First, we only showed that, generically, the historic equilibrium is virtually determinate in the space of economies for which historic consumption is an equilibrium. We do not show that, generically, every equilibrium of every economy is virtually determinate (as, generically, every equilibrium of every Arrow-Debreu economy is locally determinate [3]). In defense, our restricted results are sufficient to restore the consistency of perfect foresight, and to restore forecasting and comparative-statics applications. Second, we restricted comparative statics to sunspots, which do not affect equilibrium equations. Our contribution is merely to overturn the common presumption that indeterminacy examples in the literature imply overlapping-generations economies are incapable of the simplest comparative-statics policy applications. We raise the possibility of further comparative-statics theorems.

2. Economies and equilibrium

2.1. Assumptions

This section defines a class of stationary pure-exchange overlapping-generations economies with 1 consumer per generation, 2 periods per lifetime, and a finite number of goods available per period. Time periods extend indefinitely into the past and future ($t = 0, \pm 1, ...$). For the thought experiment, interpret periods $t < 0$ as the past; period $t = 0$ as the present; and $t > 0$ as the future. Each period's commodity space is Euclidean space $\mathcal{L} := \Re^\ell$, with the usual structures, including the sup-norm $\| \cdot \|$ and vector orders $\geq$, $>$, and $\gg$.[2]

An **economy**

$$(\ (\mathbf{a}, \mathbf{b}) \ , \ (\bar{\mathbf{y}}, \bar{\mathbf{z}}) \ , \ \succ \ )$$

specifies the 2-period-lifetime commodity endowment $(\mathbf{a}, \mathbf{b})$ in $\mathcal{L} \times \mathcal{L}$ (with $\mathbf{a}$ in young-age and $\mathbf{b}$ in old-age) for each past, present, and future consumer; the historic consumption $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ in $\mathcal{L} \times \mathcal{L}$ (with $\bar{\mathbf{y}}$ in young-age and $\bar{\mathbf{z}}$ in old-age) for each past consumer; and the preference relation $\succ$ over each consumer's lifetime consumption set.

To simplify later analysis, assume endowments are non-negative and historic consumptions are positive:

A.1. $(\mathbf{a}, \mathbf{b}) \geq 0$ and $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \gg 0$.

Assume the lifetime consumption set of each consumer is the non-negative orthant, $\mathcal{L}_+ \times \mathcal{L}_+$. And assume

[2] The vector order $x > y$ means $x \geq y$ with strict inequality in at least one component of the vectors, and $x \gg y$ means strict inequality in every component of the vectors.

A.2. Preferences are represented by a utility function $u : \mathcal{L}_+ \times \mathcal{L}_+ \to \Re$ that is **concave** and **continuously differentiable** (C^1) and **increasing**[3] over its domain.

(Assuming utility functions are C^1 is weaker than the standard assumption of C^2 functions for determinacy theorems [8]. Assuming functions are concave and increasing, rather than quasi-concave and non-decreasing, merely simplifies analysis; generalizations are possible but are unremarkable.)

Finally, assume historic consumption is a steady state. That requires both material balance,

A.3. $\bar{\mathbf{z}} + \bar{\mathbf{y}} = \mathbf{b} + \mathbf{a}$

and budget-constrained utility maximization:

A.4. Consumption $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ solves

$$\max_{(\mathbf{y},\mathbf{z}) \in \mathcal{L}_+ \times \mathcal{L}_+} u(\mathbf{y}, \mathbf{z}) \quad \text{subject to} \quad \mathbf{p} \cdot (\mathbf{y} - \mathbf{a}) + \tfrac{1}{1+r} \mathbf{p} \cdot (\mathbf{z} - \mathbf{b}) = 0$$

for some price vector $\mathbf{p}$ in $\mathcal{L}$ and interest rate $r > -1$.

2.2. Non-stationary sunspot equilibrium

In the following definition, Consumer t lives in periods t and $t+1$, consuming $\mathbf{y}_t$ when young (period t) and $\mathbf{z}_t$ when old (period $t+1$).

Definition 1 *A consumption path*

$$\{(\mathbf{y}_t, \mathbf{z}_t)\} \in \prod_{t=0}^{\infty} \mathcal{L}_+ \times \mathcal{L}_+ \text{ with initial condition } \mathbf{y}_0 = \bar{\mathbf{y}} \tag{1}$$

is an **equilibrium** (*or competitive sunspot equilibrium with or without money*) *of the economy* ($(\mathbf{a}, \mathbf{b})$, $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$, u) *if, and only if, materials balance*

$$\mathbf{z}_{t-1} + \mathbf{y}_t = \mathbf{b} + \mathbf{a} \qquad (t = 1, 2, ...) \tag{2}$$

and, for some sequence of price vectors $\mathbf{p}_t \in \mathcal{L}$ *and interest rates* $r_t > -1$, *old-age consumption* $\mathbf{z}_0$ *solves Consumer* 0*'s optimization problem*

$$\text{Maximize}_{\mathbf{z} \in \mathcal{L}_+} \; u(\bar{\mathbf{y}}, \mathbf{z}) \quad \text{subject to} \quad \mathbf{p}_1 \cdot (\mathbf{z} - \mathbf{z}_0) = 0 \tag{3}$$

and lifetime consumption $(\mathbf{y}_t, \mathbf{z}_t)$, *for* $t > 0$, *solves Consumer* t*'s problem*

$$\text{Maximize}_{(\mathbf{y},\mathbf{z}) \in \mathcal{L}_+ \times \mathcal{L}_+} \; u(\mathbf{y}, \mathbf{z}) \quad \text{s.t.} \quad \mathbf{p}_t \cdot (\mathbf{y}_t - \mathbf{a}) + \tfrac{1}{1+r_t} \mathbf{p}_{t+1} \cdot (\mathbf{z}_t - \mathbf{b}) = 0 \tag{4}$$

[3] $(\mathbf{y}, \mathbf{z}) > (\mathbf{y}', \mathbf{z}')$ implies $u(\mathbf{y}, \mathbf{z}) > u(\mathbf{y}', \mathbf{z}')$

In particular, since historic consumption is a steady state (A.3,A.4), one equilibrium merely continues on the historic path, $(\mathbf{y}_t, \mathbf{z}_t) := (\bar{\mathbf{y}}, \bar{\mathbf{z}})$.

For the thought experiment, interpret "equilibrium" so the economy initialy follows the historic path up to the end of period 0, leaving Consumer 0 with $\bar{\mathbf{y}}$ when young. Then unexpected **sunspots** occur, and consumers can reformulate their expectations away from historic consumption and onto an alternative equilibrium path.[4]

3. Perturbing Gale's indeterminacy examples

To exemplify the thought experiment, suppose a forecaster or policy-maker observes one of Gale's "Samuelson" economies [5, Theorem 4]. There is only one good available in each time period. (To interpret, there is only one physical type of commodity, and goods are measured as aggregates within each period.) Lifetime endowments $(\mathbf{a}, \mathbf{b}) := (2, 2)$, historic consumption $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) := (2, 2)$, and preference relations are represented by bivariate utility $u(\mathbf{y}, \mathbf{z}) := \mathbf{y} + 2\mathbf{z}$.

Example 1 *An interior consumption path* $\{(\mathbf{y}_t, \mathbf{z}_t)\} \in \prod_{t=0}^{\infty} \Re_{++} \times \Re_{++}$ *with initial condition* $\mathbf{y}_0 = 2$ *is an equilibrium* (2, 3, 4) *of Gale's economy* $((\mathbf{a}, \mathbf{b}) , (\bar{\mathbf{y}}, \bar{\mathbf{z}}) , u)$ *if, and only if,*

$$\mathbf{z}_0 = 2+x; \;\; (\mathbf{y}_1, \mathbf{z}_1) = (2-x, 2+2^{-1}x); \;\; (\mathbf{y}_2, \mathbf{z}_2) = (2-2^{-1}x, 2+2^{-2}x); \;\; ... \quad (5)$$

for some parameter x *between* -2 *and* $+2$. *In particular, at every interior equilibrium, utility* $u(\mathbf{y}_0, \mathbf{z}_0) = 6 + 2x$ *for Consumer* 0.

The parameter x indexes one dimension of local equilibrium indeterminacy around historic consumption (5) in Example 1 . And that indeterminacy is significant since it affects Consumer 0's equilibrium utility level, $u(\mathbf{y}_0, \mathbf{z}_0) = 6+2x$. Thus, even if the forecasters and policy-makers *a priori* restrict attention to equilibria in an arbitrarily-small neighborhood of historic consumption, the simplest forecasting or comparative-statics policy applications are impossible since the sunspot effect on equilibrium utility levels cannot be determined.

Contrary to the literature, we find the significant local indeterminacy in Gale's economy is **not verifiable**, assuming preference relations are imprecisely observed by forecasters and policy-makers. Specifically, the rest of this section shows indeterminacy is not significant in some alternative economy that we argue is an arbitrarily-close approximation of Gale's economy.

[4] Evidently, Consumer 0's old-age consumption is price suported (3) but not budget constrained; that is, $\mathbf{p}_1 \cdot \mathbf{z}_0$ is unconstrained. Thus Consumer 0 may hold money whose value may change as all consumers reformulate their expectations. In particular, the price of money can be zero on the stationary path of past consumption ($\mathbf{p} \cdot \bar{\mathbf{z}} = \mathbf{p} \cdot \mathbf{b}$) but non-zero on the new equilibrium path ($\mathbf{p}_1 \cdot \mathbf{z}_0 \neq \mathbf{p}_1 \cdot \mathbf{b}$). Such equilibria are excluded in Kehoe and Levine's indeterminacy examples [8], but we include them because increasing the set of potential equilibria strengthens our forthcoming determinacy conclusions.

The approximating economy has two goods per period, making the lifetime consumption of each consumer a vector $(\mathbf{y}, \mathbf{z}) = ((y_\alpha, y_\beta), (z_\alpha, z_\beta))$ in $\Re^2_+ \times \Re^2_+$. (To interpret, good α is the commodity aggregated within the first half of each time period; good β, within the second half.) For any positive perturbation parameter $\delta < 1/404$, Lemma 4 (Section 10) implies there is a concave and C^1 and increasing (A.2)[5] utility function u' over $\Re^2_+ \times \Re^2_+$ of the form

$$u'(\mathbf{y}, \mathbf{z}) \;=\; y_\alpha + y_\beta + 2(z_\alpha + z_\beta) - \delta(y_\alpha + y_\beta - 2)^2 \left(76 + (z_\alpha + 1)^2\right) \qquad (6)$$

over all consumption vectors $(\mathbf{y}, \mathbf{z}) = ((y_\alpha, y_\beta), (z_\alpha, z_\beta)) \geq 0$ such that $((y_\alpha + y_\beta), (z_\alpha + z_\beta)) \leq (4, 4)$.[6]

As the perturbation parameter $\delta \to 0$, the 4-variable utilty function (6) uniformly-C^1-converges to the linear function $(\mathbf{y}, \mathbf{z}) \mapsto y_\alpha + y_\beta + 2(z_\alpha + z_\beta)$. But that linear function is merely an equivalent, alternative version of Gale's original utility function, $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{y} + 2\mathbf{z}$, formed by splitting the single good in each period into perfect 1-for-1 substitutes. For any sufficiently small parameter δ, the 4-variable utility function (6) u' is thus an arbitrarily-close approximation of Gale's bivariate utility function.

It remains to show indeterminacy is not significant in any alternative economy with preferences represented by utility u'. Consider any 2-good-per-period endowment $(\mathbf{a}', \mathbf{b}')$ and historic consumption $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$ in $\Re^2 \times \Re^2$ that satisfy the positivity and material-balance assumptions $(A.1, A.3)$, and that specify the same aggregate young-age and aggregate old-age quantities as Gale's one-good-per-period versions $(\mathbf{a}, \mathbf{b}) = (2, 2)$ and $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) = (2, 2)$. Specifically,

$$\mathcal{A}(\mathbf{a}', \mathbf{b}') = (2, 2) \;\text{ and }\; \mathcal{A}(\bar{\mathbf{y}}', \bar{\mathbf{z}}') = (2, 2) \qquad (7)$$

for the aggregation map

$$\mathcal{A}((y_\alpha, y_\beta), (z_\alpha, z_\beta)) := ((y_\alpha + y_\beta), (z_\alpha + z_\beta)) \qquad (8)$$

(To interpret, the 4-dimensional vector $(\mathbf{a}', \mathbf{b}') = ((a_\alpha, a_\beta), (b_\alpha, b_\beta))$ specifies the endowment a_α aggregated within the first half of youth, a_β within the second half of youth, b_α within the first half of old-age, and b_β within the second half of old-age, so that total young-age consumption is $a_\alpha + a_\beta = 2$ and total old-age consumption is $b_\alpha + b_\beta = 2$. Interpret $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$ likewise.)

Example 2 *Historic consumption* $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$ *is a steady state* $(A.4)$ *of the approximating economy* $(\ (\mathbf{a}', \mathbf{b}')\ ,\ (\bar{\mathbf{y}}', \bar{\mathbf{z}}')\ ,\ u'\ )$.

[5] The parameter restriction $\delta < 1/404$ implies the utility function u' is increasing.

[6] The vector inequality $((y_\alpha + y_\beta), (z_\alpha + z_\beta)) \leq (4, 4)$ is satisfied by every equilibrium consumption in the alternative economy of Example 2 . Thus, the equilibrium of the alternative economy is only affected by utility over consumption satisfying that inequality.

Further, an interior consumption path $\{(\mathbf{y}_t, \mathbf{z}_t)\} \in \prod_{t=0}^{\infty}(\Re^2_{++} \times \Re^2_{++})$ *with initial condition* $\mathbf{y}_0 = \bar{\mathbf{y}}'$ *is an equilibrium* $(2, 3, 4)$ *of the approximating economy if, and only if,*

$$\mathbf{z}_0 = (1+x_0, 1-x_0); \;\; (\mathbf{y}_1, \mathbf{z}_1) = ((1-e-x_0, 1+e+x_0), (1+x_1, 1-x_1)); \;\; ... \quad (9)$$

for some sequence of parameters x_t *between* $-\min\{1, 1+e\}$ *and* $+\min\{1, 1-e\}$. *In particular, every interior equilibrium has the same utility levels, normalized gradients, and consumption aggregates*:

$$u'(\mathbf{y}_t, \mathbf{z}_t) = 6 \; ; \; \frac{\partial u'(\mathbf{y}_t, \mathbf{z}_t)}{\|\partial u'(\mathbf{y}_t, \mathbf{z}_t)\|} = \left((\frac{1}{2}, \frac{1}{2}), (1, 1) \right) \; ; \; \mathcal{A}(\mathbf{y}_t, \mathbf{z}_t) = (2, 2) \qquad (10)$$

for each consumer $t = 0, 1, ...$

To reduce notation in the equilibrium formula (9), the assumed consistency, $\mathcal{A}(\mathbf{a}', \mathbf{b}') = (2, 2)$, of the two versions of endowments implies the total endowment $\mathbf{a}' + \mathbf{b}' = (2 - e, 2 + e)$, for some scaler e between -2 and $+2$.

The parameter sequence x_t indexes *infinite* dimensions of equilibrium local indeterminacy around historic consumption (9) in Example 2 . But that indeterminacy of equilibrium consumption is **not** significant, as it was in Example 1 , because interior-equilibrium utility levels and normalized utility gradients (which determine prices and interest rates) are determinate (10) . Consumption aggregates (10) are also determinate. And since the perturbation parameter δ can be arbitrarily small, the 4-variable utilty function (6) can be arbitrarily close to the linear function $(\mathbf{y}, \mathbf{z}) \mapsto y_\alpha + y_\beta + 2(z_\alpha + z_\beta)$, so the two goods within each period can be arbitrarily close to perfect 1-for-1 substitutes. Hence, determining consumption aggregates (10) , $\mathcal{A}(\mathbf{y}_t, \mathbf{z}_t) = (2, 2)$, is almost as significant as determining individual consumption $(\mathbf{y}_t, \mathbf{z}_t)$. The historic equilibrium of the alternative economy is thus **virtually determinate**.

Proof of Example 1 . The only remarkable property to prove is that an interior consumption path $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ with initial condition $\mathbf{y}_0 = 2$ is an equilibrium $(2, 3, 4)$ of Gale's economy if, and only if, the path is of the required form (5) . The equilibrium condition that old-age consumption $\mathbf{z}_0$ solves Consumer 0's optimization problem (3) is vacuous since there is only one good per period. Hence, we show an interior consumption path $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ with initial condition $\mathbf{y}_0 = 2$ satisfies the remaining equilibrium conditions $(2, 4)$ if, and only if, the path is of the required form (5) .

For each index $t = 1, 2, ...$, material balance (2) at period t reads $\mathbf{z}_{t-1} + \mathbf{y}_t = 4$; and the condition that lifetime consumption $(\mathbf{y}_t, \mathbf{z}_t)$ solves Consumer t's problem (4) reads $(\mathbf{y}_t - 2) + 2(\mathbf{z}_t - 2) = 0$, since utility is linear and the gradient $\partial u(\mathbf{y}_t, \mathbf{z}_t) = (1, 2)$. Evidently, those equilibrium conditions $(2, 4)$

$$\mathbf{z}_{t-1} + \mathbf{y}_t = 4 \text{ and } (\mathbf{y}_t - 2) + 2(\mathbf{z}_t - 2) = 0 \qquad (t = 1, 2, ...)$$

are satisfied by an interior consumption path $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ with initial condition $\mathbf{y}_0 = 2$ if, and only if, the path is of the required form (5) . ▮

Proof of Example 2 . The only remarkable property to prove is that an interior consumption path $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ with initial condition $\mathbf{y}_0 = \bar{\mathbf{y}}'$ is an equilibrium $(2, 3, 4)$ of the approximating economy ($(\mathbf{a}', \mathbf{b}')$, $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$, u') if, and only if, the path is of the required form (9) .

To prove the "if" part, consider any consumption path of the required form (9) . Material balance (2) is evident. For each consumer $t = 0, 1, ...,$ consumption $(\mathbf{y}_t, \mathbf{z}_t) = ((y_{t\alpha}, y_{t\beta}), (z_{t\alpha}, z_{t\beta}))$ evidently satisfies $y_{t\alpha} + y_{t\beta} = 2$ and $z_{t\alpha} + z_{t\beta} = 2$. In particular, the special form of utility (6) implies

$$\frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{y_\alpha} = \frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{y_\beta} = 1 - 2\delta(y_{t\alpha} + y_{t\beta} - 2)\left(13 + (z_{t\alpha} + 1)^2\right) = 1$$

and

$$\frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{z_\alpha} = 2 - 2\delta(y_{t\alpha} + y_{t\beta} - 2)^2(z_{t\alpha} + 1) = 2 \ ; \quad \frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{z_\beta} = 2$$

since $y_{t\alpha} + y_{t\beta} = 2$. That is, the young-age and old-age utility gradients

$$\partial_y u(\mathbf{y}_t, \mathbf{z}_t) = \mathbf{p}_t; \ \ \partial_z u(\mathbf{y}_t, \mathbf{z}_t) = \frac{1}{1 + r_t}\mathbf{p}_{t+1}$$

for the constant sequence of price vectors $\mathbf{p}_t := (1, 1)$ and interest rates $r_t := -1/2$. Old-age consumption $\mathbf{z}_0$ thus solves Consumer 0's optimization problem (3). And lifetime consumption $(\mathbf{y}_t, \mathbf{z}_t)$, for $t = 1, 2, ...,$ thus solves Consumer t's problem (4) once we prove the budget constraint

$$\mathbf{p}_t \cdot (\mathbf{y}_t - \mathbf{a}') + \frac{1}{1 + r_t}\mathbf{p}_{t+1} \cdot (\mathbf{z}_t - \mathbf{b}') = 0 \tag{11}$$

To that end, the definition $\mathbf{p}_t := (1, 1)$ and the assumed consistency (7) of endowment $(\mathbf{a}', \mathbf{b}')$ implies $\mathbf{p}_t \cdot \mathbf{a}' = 2$; likewise, $\mathbf{p}_{t+1} \cdot \mathbf{b}' = 2$. But $y_{t\alpha} + y_{t\beta} = 2$ implies $\mathbf{p}_t \cdot \mathbf{y}_t = 2$; likewise, $\mathbf{p}_{t+1} \cdot \mathbf{z}_t = 2$. The required budget constraint (11) follows. The consumption path (9) is thus an equilibrium $(2, 3, 4)$ of the approximating economy ($(\mathbf{a}', \mathbf{b}')$, $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$, u').

To prove the "only if" part, consider any interior equilibrium $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ of the approximating economy ($(\mathbf{a}', \mathbf{b}')$, $(\bar{\mathbf{y}}', \bar{\mathbf{z}}')$, u'). For each period $t = 1, 2, ...,$ the required form (6) of utility implies $\partial u(\mathbf{y}_{t+1}, \mathbf{z}_{t+1})/\partial y_\alpha = \partial u(\mathbf{y}_{t+1}, \mathbf{z}_{t+1})/\partial y_\beta$ for Consumer $t + 1$. But budget-constrained utility maximization (4) by Consumer $t + 1$ (the young) and by Consumer t (the old) imply their marginal rates of substitution between goods α and β must equal in period t. Hence, $\partial u(\mathbf{y}_t, \mathbf{z}_t)/\partial z_\alpha = \partial u(\mathbf{y}_t, \mathbf{z}_t)/\partial z_\beta$ for Consumer t. But the required form (6) of utility implies the latter partial derivative $\partial u(\mathbf{y}_t, \mathbf{z}_t)/\partial z_\beta = 2$; hence, the former derivative also equals 2; that is,

$$\frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{\partial z_\alpha} = 2 - 2\delta(y_{t\alpha} + y_{t\beta} - 2)^2(z_{t\alpha} + 1) = 2$$

Hence, $y_{t\alpha} + y_{t\beta} = 2$. But $y_{t\alpha} + y_{t\beta} = 2$, for each period $t = 1, 2, ...$, implies the path of young-age consumption is

$$\mathbf{y}_1 = (1 - x_0, 1 + x_0); \;\; \mathbf{y}_2 = (1 - x_1, 1 + x_1); \;\; ...$$

for some sequence of parameters x_t between -1 and $+1$. Hence, material balance $\mathbf{z}_{t-1} + \mathbf{y}_t = (2, 2)$ determines the path of old-age consumption

$$\mathbf{z}_0 = (1 + x_0, 1 - x_0); \;\; \mathbf{z}_1 = (1 + x_1, 1 - x_1); \;\; ...$$

Hence, the entire consumption path is of the required form (9) . ▮

4. The space of potential economies

4.1. Overview and comparison with the literature

The rest of the paper concludes the determinacy thought experiment by formulating then proving that the historic equilibrium is generically virtually determinate for the economies in Section 2, which includes each of Gale's examples and other familiar examples of significant indeterminacy from the literature. (As described in Section 3, virtual determinacy answers the thought experiment by asserting any local indeterminacy is insignificant.)

All of the alternative formulations of genericity used in the literature assume some observations of the economy are imprecise and so forecasters and policy-makers must consider a space of potential economies. The formulations differ according to which paratermers are observed imprecisely. The first and best-known formulation, Debreu [3], fixes preference orders and considers a space of alternative endowments. That formulation has the advantage that the space of economies is finite dimensional, and so Lebesque measure helps define genericity. But it assumes preferences are observed precisely and endowments imprecisely. We chose an alternative formulation, which may be more realistic, where preferences are observed imprecisely.

4.2. Endowments and historic consumption

For the rest of the paper, fix commodity index $\ell = 1, 2, ...$, non-negative endowment $(\mathbf{a}, \mathbf{b}) \in \Re^\ell_+ \times \Re^\ell_+$, and positive historic consumption $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \Re^\ell_{++} \times \Re^\ell_{++}$ satisfying material balance (A.3), $\bar{\mathbf{z}} + \bar{\mathbf{y}} = \mathbf{b} + \mathbf{a}$. Considering just the simplest interpretation, a forecaster or policy-maker perfectly observes endowments $(\mathbf{a}, \mathbf{b})$ and historic consumption $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ for ℓ physically different types of goods available in each period. Each good is measured as an aggregate within each time period.

To help forumulate futher assumptions, recursively define spaces $X^0 := \Re^\ell$ and

$$X^N := X^{N-1} \times X^{N-1} \qquad (N = 1, 2, ...)$$

Interpret X^N as each period's commodity space after subdividing the time period N times, forming 2^N subintervals, and distinguishing goods by both their physical characteristics and by the time subinterval over which they are aggregated. (Induction proves the space X^N has the required $\ell \times 2^N$ dimensions.) For an example with $\ell = 1$ physical type of good, $(3, 12)$ in X^1 specifies 3 units in the first half of the period and 12 in the second half. Likewise, $((1, 2), (4, 8))$ in X^2 specifies 1 unit in the first quarter of the period, 2 units in the second quarter, 4 units in the third, and 8 in the fourth.

Recursively define the **per-period-aggregation** map $A : \bigcup_{N=0}^{\infty} X^N \to X^0$. A is the identity over X^0. For each aggregation index $N = 1, 2, ...,$

$$A(y_\alpha, y_\beta) \;:=\; Ay_\alpha + Ay_\beta \tag{12}$$

for each (y_α, y_β) in $X^N = X^{N-1} \times X^{N-1}$. Interpret the aggregate (12) $A\mathbf{y}$ in X^0 as the aggregate version of the consumption vector $\mathbf{y}$ in the manifold commodity space $\bigcup_{N=0}^{\infty} X^N$. For an example with $\ell = 1$, $A(1, 2) = 3$ and $A(4, 8) = 12$ for vectors in X^1, and $A((1, 2), (4, 8)) = A(1, 2) + A(4, 8) = 15$ for vector $((1, 2), (4, 8))$ in X^2.

Assume forecasters and policy-makers have no observations of historic consumptions disaggregated within a period. For each aggregation index $N = 0, 1, ...,$ potential historic consumptions thus form the space

$$\Omega^N \;\;:=\;\; \left\{\; (\mathbf{y}, \mathbf{z}) \in X^N_+ \times X^N_+ \;\mid\; A\mathbf{y} = \bar{\mathbf{y}} \text{ and } A\mathbf{z} = \bar{\mathbf{z}} \;\right\} \tag{13}$$

where $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ is the historic consumption already fixed. In particular, $\Omega^0 := \{(\bar{\mathbf{y}}, \bar{\mathbf{z}})\}$.

To reduce notation in the rest of the paper, for each aggregation index $N = 0, 1, ...,$ since there are two periods per lifetime, we can also interpret $X^{N+1} = X^N \times X^N$ as each consumer's lifetime commodity space after subdividing the time period N times. Thus, "$(\mathbf{y}, \mathbf{z}) \in X^N_+ \times X^N_+$" in the definition (13) above can be written "$(\mathbf{y}, \mathbf{z}) \in X^{N+1}_+$".

4.3. Gradient notation

Let $\partial u \in X^0$ denote the gradient of any differentiable function u over $X^0 = \Re^\ell$. For each aggregation index $N = 0, 1, ...,$ recursively extend the gradient ∂u to each utility function $u : X^{N+1}_+ \to \Re$:

$$\partial u(\mathbf{y}, \mathbf{z}) := (\partial_y u(\mathbf{y}, \mathbf{z}), \partial_z u(\mathbf{y}, \mathbf{z})), \quad \text{for } (\mathbf{y}, \mathbf{z}) \in X^{N+1}_+ \tag{14}$$

where $\partial_y u(\mathbf{y}, \mathbf{z}) \in X^N_+$ is the gradient of the partial function $\mathbf{y} \mapsto u(\mathbf{y}, \mathbf{z})$ over X^N_+, and $\partial_z u(\mathbf{y}, \mathbf{z})$ is the gradient of $\mathbf{z} \mapsto u(\mathbf{y}, \mathbf{z})$. Interpret $\partial_y u(\mathbf{y}, \mathbf{z})$ as the gradient of marginal utility over young-age consumption; $\partial_z u(\mathbf{y}, \mathbf{z})$, over old-age consumption. Since utility is concave and increasing (A.3), gradients are positive, $\partial u(\mathbf{y}, \mathbf{z}) \in X^{N+1}_{++}$.

4.4. Utility functions

For the rest of the paper, fix a price vector $\bar{\mathbf{p}}$ in $\Re^{\ell}_{++}$ and an interest rate $r > -1$ satisfying the budget constraint for historic consumption, $\bar{\mathbf{p}} \cdot \bar{\mathbf{y}} + \frac{1}{1+r}\bar{\mathbf{p}} \cdot \bar{\mathbf{z}} = \bar{\mathbf{p}} \cdot \mathbf{a} + \frac{1}{1+r}\bar{\mathbf{p}} \cdot \mathbf{b}$. (Material balance (A.3) implies there is at least one such vector.) Interpret forecasters and policy-makers as perfectly observing prices and the interest rate for ℓ types of goods available in each period. But assume forecasters and policy-makers have no observations of prices disaggregated within a period, just as they have no observations of historic consumptions (13) disaggregated within a period.

Finally, assume forecasters and policy-makers can observe preferences over consumption disaggregated within a period (they can run experiments) but those observations are imperfect, so they must consider a space of potential utility functions representing preferences. The standard way in the literature to handle missing observations of equilibrium consumptions or prices disaggregated within a period is to *a priori* assume potential preferences or utility functions satisfy local substitution;[7] that is, for each type of good, the literature assumes consumption across all moments within each period is perfectly 1-for-1 substitutable. That way, the observed historic consumption aggregates are consistent with equilibrium no matter how historic consumption is actually distributed across all moments within each period. And historic prices are constant across all moments within each period. But that consistency of historic consumption and that constancy of historic prices can be deduced without assuming perfect 1-for-1 substitution; we offer a weaker sufficient condition, and so expand the class of potentially observed utility functions.

To be precise, recursively define price vectors $\mathbf{p}^0 := \bar{\mathbf{p}}$ in X^0 and

$$\mathbf{p}^N = (\mathbf{p}^{N-1}, \mathbf{p}^{N-1}) \in X^N = X^{N-1} \times X^{N-1} \tag{15}$$

Interpret $\mathbf{p}^N$ as the historic price vector after subdividing the time period N times, while keeping prices constant across the subintervals. Induction proves a useful price identity

$$\mathbf{p}^N \cdot \mathbf{x} = \bar{\mathbf{p}} \cdot A\mathbf{x} \tag{16}$$

for each aggregation index $N = 0, 1, ...$ and each vector $\mathbf{x}$ in X^N.

For the rest of this section, fix the aggregation index $N = 0, 1, ...$

Definition 2 *The universe $\mathcal{U}^N$ of utility functions consists of the concave-C^1-increasing (A.2) functions $u : X_+^{N+1} \to \Re$ for which*

[7] To prove that equilibrium with a continuum of goods can be approximated by equilibrium with only a finite number of goods, Jones assumes that commodities with nearly the same characteristics are good substitutes [7, p. 509]. In the extreme, equilibrium with a continuum of goods equals an equilibrium with only a finite number of goods when commodities with nearly the same characteristics are perfectly 1-for-1 substitutable.

$$\frac{\partial u(\mathbf{y},\mathbf{z})}{\|\partial u(\mathbf{y},\mathbf{z})\|} \;=\; (\mathbf{p}^N, \frac{1}{1+r}\mathbf{p}^N) \tag{17}$$

for each lifetime consumption vector $(\mathbf{y},\mathbf{z})$ *in* Ω^N*.*

Each utility function in that universe makes the observed historic consumption aggregates $(\bar{\mathbf{y}},\bar{\mathbf{z}})$ consistent with equilibrium no matter how historic consumption is actually distributed across all moments within each period. To be precise:

Observation 1 *For each utility function* $u \in \mathcal{U}^N$*, each potential historic* (13) *consumption* $(\mathbf{y},\mathbf{z})$ *in* Ω^N *maximizes utility* $u(\mathbf{y},\mathbf{z})$ *over* $(\mathbf{y},\mathbf{z}) \in X_+^{N+1}$ *subject to*

$$\mathbf{p}^N \cdot \mathbf{y} + \frac{1}{1+r}\mathbf{p}^N \cdot \mathbf{z} \;=\; \bar{\mathbf{p}} \cdot \bar{\mathbf{y}} + \frac{1}{1+r}\bar{\mathbf{p}} \cdot \bar{\mathbf{z}} \tag{18}$$

The price identity (16) implies that budget constraint (18) on consumption $(\mathbf{y},\mathbf{z})$ in X_+^{N+1} is equivalent to the constraint that

$$\mathbf{p}^N \cdot \mathbf{y} + \frac{1}{1+r}\mathbf{p}^N \cdot \mathbf{z} \;=\; \bar{\mathbf{p}} \cdot \mathbf{a}' + \frac{1}{1+r}\bar{\mathbf{p}} \cdot \mathbf{b}'$$

for each pair of endowments $(\mathbf{a}',\mathbf{b}')$ in X_+^{N+1} that are consistent with observed aggregate endowments, $A\mathbf{a}' = \mathbf{a}$ and $A\mathbf{b}' = \mathbf{b}$. Thus, each potential historic (13) consumption $(\mathbf{y},\mathbf{z})$ in Ω^N is a steady state (A.3,A.4) for each distribution of endowments $(\mathbf{a}',\mathbf{b}')$ in X_+^{N+1} that is consistent with the observed aggregates, $A\mathbf{a}' = \mathbf{a}$ and $A\mathbf{b}' = \mathbf{b}$.

Proof of Observation 1 . For each utility function $u \in \mathcal{U}^N$ and each potential (13) historic consumption $(\mathbf{y},\mathbf{z})$ in Ω^N, the first-order conditions for utility maximization follows from the definition (17) of the space $\mathcal{U}^N$. And the budget constraint (18) follows from the equalities $\mathbf{p}^N \cdot \mathbf{y} = \bar{\mathbf{p}} \cdot \bar{\mathbf{y}}$ and $\mathbf{p}^N \cdot \mathbf{z} = \bar{\mathbf{p}} \cdot \bar{\mathbf{z}}$, which follow from the price identity (16) since $(\mathbf{y},\mathbf{z})$ in Ω^N implies $A\mathbf{y} = \bar{\mathbf{y}}$ and $A\mathbf{z} = \bar{\mathbf{z}}$. ▮

5. The topology over utility functions

Since the space U^N of utility functions (17) is infinite dimensional, Lebesque measure is not available to define genericity. Instead we use the topological definition: we will prove virtual determinacy holds for a dense set of economies (Theorem 1), and an approximate form of virtual determinacy holds for an open-dense set of economies (Theorem 2). Density means virtual determinacy cannot be falsified (that is, significant indeterminacy cannot be verified), and openness means approximate virtual determinacy can be verified (that is, significant indeterminacy can be falsified).

5.1. A psuedometric over N-aggregated utility functions

Recursively extend the sup norm $\|\cdot\|$ from $X^0 := \Re^\ell$ to each space X^N with index $N = 1, 2, ...$:

$$\|(\mathbf{y}, \mathbf{z})\| := \max\{\|\mathbf{y}\|, \|\mathbf{z}\|\} \text{ over } (\mathbf{y}, \mathbf{z}) \in X^N = X^{N-1} \times X^{N-1} \tag{19}$$

For each aggregation index $N = 1, 2, ...$, material balance (equation 2 and A.3) implies the set

$$F^{N+1} := \left\{ (\mathbf{y}, \mathbf{z}) \in X_+^{N+1} \mid A\mathbf{y} \le \bar{\mathbf{y}} + \bar{\mathbf{z}} \text{ and } A\mathbf{z} \le \bar{\mathbf{y}} + \bar{\mathbf{z}} \right\} \tag{20}$$

contains each lifetime consumption vector $(\mathbf{y}_t, \mathbf{z}_t)$ that is part of some equilibrium (2,3,4) path. F^{N+1} is evidently compact.

Definition 3 *For each aggregation index $N = 0, 1, ...$, measure the distance $d^N(u, v)$ between two N-aggregated utility functions u and v in $\mathcal{U}^N$ by their uniform C^1-difference over the feasible set,*

$$\sup_{(\mathbf{y},\mathbf{z}) \in F^{N+1}} |\, u(\mathbf{y}, \mathbf{z}) \;-\; v(\mathbf{y}, \mathbf{z}) \,| \;+\; \sup_{(\mathbf{y},\mathbf{z}) \in F^{N+1}} \|\, \partial u(\mathbf{y}, \mathbf{z}) \;-\; \partial v(\mathbf{y}, \mathbf{z}) \,\| \tag{21}$$

Remember, $\|\cdot\|$ is the sup norm (19), and the gradients (14) $\partial u(\mathbf{y}, \mathbf{z})$ and $\partial v(\mathbf{y}, \mathbf{z})$ are vectors in X_{++}^{N+1}.

Lemma 1 *The distance function $d^N : (\mathcal{U}^N \times \mathcal{U}^N) \to \Re$ is a psuedometric, and $d^N(u, v) = 0$ for two elements if, and only if, the functions $u = v$ over the feasible set F^{N+1}.*

Lemma 1 follows from the properties of the uniform C^1-norm.

5.2. The manifold universe of utility functions

The aggregation map (8)

$$\mathcal{A}((y_\alpha, y_\beta), (z_\alpha, z_\beta)) := ((y_\alpha + y_\beta), (z_\alpha + z_\beta))$$

from Section 3 sends lifetime consumption vectors in $X^{N+2} = X^{N+1} \times X^{N+1}$ into their aggregated counterparts in $X^{N+1} = X^N \times X^N$. For an example with $\ell = 1$ physical type of good, $((1, 2), (4, 8))$ specifies 1 unit in the first half of youth, 2 units in the second half of youth, 4 units in the first half of old-age, and 8 units in the second half of old-age. The aggregate $\mathcal{A}((1, 2), (4, 8)) = (3, 12)$ specifies the counterpart of 3 units in youth and 12 units in old-age. Likewise, iterateted applications $\mathcal{A}^n$ of aggregation maps X^{N+n+1} into X^{N+1}, for $n = 1, 2, ...$

Step 3 in the proof of Lemma 4 in Section 10 proves a lemma that will help compare utility functions with different aggregation indicies.

Lemma 2 *For each aggregation index $N = 0, 1, ...$ and each N-aggregated utility function u in $\mathcal{U}^N$ and each index $M > N$, the composition $u \circ \mathcal{A}^{M-N}$ is an M-aggregated function in $\mathcal{U}^M$.*

Definition 4 *Define the* **manifold universe** *of utility functions $\mathcal{U}^\infty := \bigcup_{N=0}^{\infty} \mathcal{U}^N$. Call two functions $u \in \mathcal{U}^N$ and $v \in \mathcal{U}^M$ in the universe $\mathcal{U}^\infty$* **equivalent***, and write $u \equiv v$, if*

$$\begin{cases} u = (v \circ \mathcal{A}^{N-M}) \text{ over the feasible set } F^{N+1}, & \text{in case } \quad N \geq M \\ v = (u \circ \mathcal{A}^{M-N}) \text{ over the feasible set } F^{M+1}, & \text{in case } \quad M \geq N \end{cases} \tag{22}$$

And define the distance between two functions $u \in \mathcal{U}^N$ and $v \in \mathcal{U}^M$ in the manifold universe $\mathcal{U}^\infty$ by

$$d^\infty(u, v) \;=\; \begin{cases} d^N(u, v \circ \mathcal{A}^{N-M}) & \text{in case } \quad N \geq M \\ d^M(v, u \circ \mathcal{A}^{M-N}) & \text{in case } \quad M \geq N \end{cases} \tag{23}$$

To interpret equivalence, consider a single physical type of commodity ($\ell = 1$) and the bivariate utility $u(y, z) := y + 2z$ over $\Re^2_+$, from Section 3. The bivariate utility is equivalent to the 4-variable utility

$$v((y_\alpha, y_\beta), (z_\alpha, z_\beta)) \;\;:=\;\; y_\alpha + y_\beta + 2(z_\alpha + z_\beta)$$

because $v = (u \circ \mathcal{A})$ over $\Re^2_+ \times \Re^2_+$. Roughly, the 4-variable composition utility v is equivalent because it merely divides each of the goods in the bivariate utility u's domain into two perfect 1-for-1 substitutes. In general, the composition $u \circ \mathcal{A}^n$ is equivalent to u because it merely divides each of the goods in the bivariate utility u's domain into 2^n perfect 1-for-1 substitutes.

Lemma 3 *The distance function $d^\infty : (\mathcal{U}^\infty \times \mathcal{U}^\infty) \to \Re$ is a psuedometric, and $d^\infty(u, v) = 0$ for two elements if, and only if, $u \equiv v$.*

Section 10 proves Lemma 3 , which implies the distance function d^∞ (21,23) generates a topology. Endow the manifold universe $\mathcal{U}^\infty$ with that topology.

6. The indeterminacy index

Throughout this section, fix the aggregation index $N = 0, 1, ...$. We index the significance of local indeterminacy around historic consumption through the variation across interior equilibrium of consumption aggregates:

Definition 5 *For each utility function u in $\mathcal{U}^N$, let $\mathcal{E}(u)$ denote the set of allocations $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ in $\prod_{t=0}^{\infty} X_+^{N+1}$ satisfying the aggregate material balance*

$$A\mathbf{z}_{t-1} + A\mathbf{y}_t = \bar{\mathbf{z}} + \bar{\mathbf{y}} \qquad (t = 1, 2, ...) \tag{24}$$

and marginal-utility

$$\frac{\partial_z u(\mathbf{y}_{t-1}, \mathbf{z}_{t-1})}{\|\partial_z u(\mathbf{y}_{t-1}, \mathbf{z}_{t-1})\|} \quad = \quad \frac{\partial_y u(\mathbf{y}_t, \mathbf{z}_t)}{\|\partial_y u(\mathbf{y}_t, \mathbf{z}_t)\|} \qquad (t = 1, 2, ...) \tag{25}$$

conditions that are satisfied by every interior equilibrium $(2, 3, 4)$ *of an economy with utility* u*.*

Define the **indeterminacy** index

$$\mathcal{I}(u) \quad := \sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \; \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \; \sup_{t=0}^{\infty} \|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\| \tag{26}$$

That is, $\sup_{t=0}^{\infty} \|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\|$ is the variation of consumption aggregates, and index $\mathcal{I}(u)$ measures that variation across all pairs of interior equilibria (24,25) . Since only interior equilibria are considered, $\mathcal{I}(u)$ only measures local indeterminacy.

Observation 2 *For each utility function* u *in* $\mathcal{U}^N$ *such that* $\mathcal{I}(u) = 0$,

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \; \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \; \sup_{t=0}^{\infty} \left\| \frac{\partial u(\mathbf{y}_t, \mathbf{z}_t)}{\|\partial u(\mathbf{y}_t, \mathbf{z}_t)\|} - \frac{\partial u(\mathbf{y}'_t, \mathbf{z}'_t)}{\|\partial u(\mathbf{y}'_t, \mathbf{z}'_t)\|} \right\| = 0 \tag{27}$$

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \; \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \; \sup_{t=0}^{\infty} |u(\mathbf{y}_t, \mathbf{z}_t) - u(\mathbf{y}'_t, \mathbf{z}'_t)| = 0 \tag{28}$$

Observation 2 follows from the proof of Theorem 2 , in Section 9.

Observation 2 shows $\mathcal{I}(\cdot)$ better indexes the significance of indeterminacy than mearly counting the dimensions of the set of interior equilibrium. Specifically, Example 1 with utility function v has one dimension of indeterminacy (5) , and that indeterminacy is significant because it affects equilibrium utility levels. Hence, the variation in utility (28) is positive, and Observation 2 implies the indeterminacy index $\mathcal{I}(v) > 0$. In contrast, Example 2 with utility function u has infinite dimensions of indeterminacy (9) , but that indeterminacy is insignificant because it does not affect interior equilibrium utility levels, normalized gradients, or consumption aggregates (10) ; its index (26) $\mathcal{I}(u) = 0$.

7. The generic virtual determinacy statement

Theorem 1 *For a dense subset of utility functions* u *in* $\mathcal{U}^\infty$*, the indeterminacy index* $\mathcal{I}(u) = 0$ *and the function* $\mathcal{I}(\cdot)$ *is continous at* u*.*

Theorem 1 states a sense in which the historic equilibrium of a dense set of economies is virtually deterministic. In particular, Gale's examples of significant indeterminacy are fragile, as are all other examples in the economies of Section 4.

Theorem 2 *For any positive tolerance ε, an open-dense subset of utility functions u in $\mathcal{U}^\infty$ are ε-virtually deterministic in the sense that*

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \sup_{t=0}^{\infty} \|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\| < \varepsilon \tag{29}$$

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \sup_{t=0}^{\infty} \left\| \frac{\partial u(\mathbf{y}_t,\mathbf{z}_t)}{\|\partial u(\mathbf{y}_t,\mathbf{z}_t)\|} - \frac{\partial u(\mathbf{y}'_t,\mathbf{z}'_t)}{\|\partial u(\mathbf{y}'_t,\mathbf{z}'_t)\|} \right\| < \varepsilon \tag{30}$$

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u)} \sup_{t=0}^{\infty} |u(\mathbf{y}_t,\mathbf{z}_t) - u(\mathbf{y}'_t,\mathbf{z}'_t)| < \varepsilon \tag{31}$$

A common view of genericity in the literature is that any property can be imposed *a priori* if it can never be falsified (it holds on a dense set) and it can sometimes be verified (it holds on an open set). [8] Theorem 2 thus justifies the *a priori* restriction of potentially-observed economies in the determinacy thought experiement from all economies with utility in $\mathcal{U}^\infty$ to the subset with ε-virtually deterministic utility functions. For each of those utility functions, the determinacy thought experiement concludes any local indeterminacy is insignificant. In particular, an approximate version of perfect foresight is internally consistent since any local indeterminacy has, at most, a small effect on equilibrium prices and interest rates (30) , which are the essential variables defining consumers' budget constraints. Likewise, approximate versions of some forecasting and comparative-statics policy applications are possible since, locally, unexpected sunspots have virtually no effect on equilibrium (29,30,31) .

Finally, we justify restricting attention to local determinacy, rather than global determincy, for our study of overlapping generations for the same reasons that justify that restriction for other general-equilibrium economies: The conditions sufficient for global determinacy (like gross substitutes) are much more restrictive, and local determinacy has the same economic implications as global determinacy if the locally-determinate equilibrium serves as a focal point for consumers' expecations.

This concludes the determinacy thought experiment. The rest of the paper proves Theorem 1 and Theorem 2 , and some supporting Lemmas.

[8] Paraphrasing Mas-Colell's description of the generic point of view [10, pp. 318]: "in a world that is not precisely observed, only properties that hold for an open-dense set of admissible economies have a good chance to be observed."

8. Proof of Theorem 1

Recall, $\|\cdot\|$ denotes the sup norm (19) over each commodity space X^N. For convenience in this section, also consider the L^2 (the sum of squares) norm. Let $\|\cdot\|_2$ denote the L^2 norm over $X^0 := \Re^\ell$. Recursively extend the L^2 norm to each space X^N with index $N = 1, 2, \ldots$:

$$\|(\mathbf{y}, \mathbf{z})\|_2 := \|\mathbf{y}\|_2 + \|\mathbf{z}\|_2 \text{ over } (\mathbf{y}, \mathbf{z}) \in X^N = X^{N-1} \times X^{N-1} \tag{32}$$

One useful formula computes the derivative of the norm

$$\partial\|\mathbf{x}\|_2 := 2\mathbf{x} \tag{33}$$

That formula is evident for the sum-of-squares norm over $X^0 := \Re^\ell$, when elements of $\Re^\ell$ are written as row vectors, and follows by induction (32) for each space X^N with index $N = 1, 2, \ldots$.

To prove Theorem 1 , fix any utility function v in the manifold universe $\mathcal{U}^\infty = \bigcup_{N=0}^\infty \mathcal{U}^N$. For convenience, write v as an element of $\mathcal{U}^{N-1}$, for some index $N = 1, 2, \ldots$. Fix any positive vector ι in X_{++}^{N-1}. Section 10 proves

Lemma 4 *There exists a scaler θ such that, for each sufficiently small tollerence $\delta > 0$, there exists a utility function $u = u_\delta$ in $\mathcal{U}^N$ of the form*

$$u(\mathbf{y}, \mathbf{z}) \quad = \quad (v \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) - \delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) \tag{34}$$

over all consumption $(\mathbf{y}, \mathbf{z}) = ((\mathbf{y}_\alpha, \mathbf{y}_\beta), (\mathbf{z}_\alpha, \mathbf{z}_\beta))$ in the feasible set F^{N+1}.

Part of the proof of Lemma 4 and of the proof of Theorem 1 concern the gradient (14) of the utility function u described above (34) . At any vector $(\mathbf{y}, \mathbf{z})$ in the feasible set F^{N+1}, the partial gradient

$$\partial_z u(\mathbf{y}, \mathbf{z}) \;=\; (\; \partial_z v(\mathcal{A}(\mathbf{y}, \mathbf{z})) - 2\delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2(\mathbf{z}_\alpha + \iota) \;\;,\;\; \partial_z v(\mathcal{A}(\mathbf{y}, \mathbf{z})) \;) \tag{35}$$

To compute that gradient (35) , consider the auxiliary function

$$f(\mathbf{y}, \mathbf{z}) = \|\bar{\mathbf{y}} - A\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) \tag{36}$$

over all vectors $(\mathbf{y}, \mathbf{z}) = ((\mathbf{y}_\alpha, \mathbf{y}_\beta), (\mathbf{z}_\alpha, \mathbf{z}_\beta))$ in X^{N+1}. Since $u(\mathbf{y}, \mathbf{z}) = (v \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) - \delta f(\mathbf{y}, \mathbf{z})$, the required gradient formula (35) follows if the gradient of the auxiliary function (36) is

$$\partial_z f(\mathbf{y}, \mathbf{z}) \;=\; (\; 2\|\bar{\mathbf{y}} - A\mathbf{y}\|_2(\mathbf{z}_\alpha + \iota) \;\;,\;\; 0 \;)$$

But that formula follow directly from the formula (33) for the derivative of the L^2 norm $\|\cdot\|_2$.

Lemma 4 generates N-disaggregated utility functions $u = u_\delta$ in $\mathcal{U}^N$ (parameterized by scalers $\delta > 0$) such that, for sufficiently small $\delta > 0$, utility function u is of the particular form (34)

$$u(\mathbf{y}, \mathbf{z}) \quad = \quad (v \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) - \delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2 \left(\theta + \|\mathbf{z}_\alpha + \iota\|_2\right)$$

over the feasible set F^{N+1}, where $\mathbf{z}_\alpha$ is the first component of the vector $\mathbf{z} = (\mathbf{z}_\alpha, \mathbf{z}_\beta)$ in $X^N = X^{N-1} \times X^{N-1}$. In particular, as $\delta \to 0$, the uniform C^1-difference $d^N(u_\delta, v \circ A)$ (21) between functions u_δ and $v \circ \mathcal{A}$ in $\mathcal{U}^N$ converges to 0, which implies the d^∞-difference $d^\infty(u_\delta, v) \to 0$ (23) . Thus the set of utility functions of the particular form (34) over the feasible sets is a dense subset of the manifold universe $\mathcal{U}^\infty$. Thus, to prove Theorem 1 , fix any utility function u of the particular form (34) over the feasible set F^{N+1} and show the indeterminacy index $\mathcal{I}(u) = 0$, and show the index function $\mathcal{I}(\cdot)$ is continuous at utility u.

To that end, it is sufficient to consider any convergent $u^k \to u$ sequence of utility functions u^k in the manifold universe $\mathcal{U}^\infty = \bigcup_{N=0}^\infty \mathcal{U}^N$ and consider any positive tolerance $\varepsilon > 0$ then show

$$\limsup_k \mathcal{I}(u^k) \le \varepsilon \tag{37}$$

For each utility function u^k, let N_k ($N_k = 0, 1, ...$) be the aggregation index such that $u^k \in \mathcal{U}^{N_k}$.

Step 1. *Show $\mathcal{I}(u^k) \le \mathcal{I}(u^k \circ \mathcal{A}^n)$ for each utility function u^k and each index $n = 1, 2, ...$*

We will show $\mathcal{I}(u^k) \le \mathcal{I}(u^k \circ \mathcal{A})$; the more general inequality follows by induction for each index $n = 1, 2, ...$

Consider any pair $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ and $\{(\mathbf{y}'_t, \mathbf{z}'_t)\}$ of supported (24,25) allocations in $\mathcal{E}(u^k) \subset \prod_{t=0}^\infty X_+^{N_k+1}$. The pair $\{((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))\}$ and $\{((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))\}$ are, evidently, supported (24,25) allocations in $\mathcal{E}(u^k \circ \mathcal{A}) \subset \prod_{t=0}^\infty X_+^{N_k+2}$. Hence, the definition of the indeterminacy index (26) at the composition $u^k \circ \mathcal{A}$ implies

$$\sup_{t=0}^{\infty} \|(A(.5\mathbf{y}_t, .5\mathbf{y}_t), A(.5\mathbf{z}_t, .5\mathbf{z}_t)) - (A(.5\mathbf{y}'_t, .5\mathbf{y}'_t), A(.5\mathbf{z}'_t, .5\mathbf{z}'_t))\| \quad \le \quad \mathcal{I}(u^k \circ \mathcal{A})$$

Hence, the aggregation identity

$$\|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\| \;=\; \|(A(.5\mathbf{y}_t, .5\mathbf{y}_t), A(.5\mathbf{z}_t, .5\mathbf{z}_t)) - (A(.5\mathbf{y}'_t, .5\mathbf{y}'_t), A(.5\mathbf{z}'_t, .5\mathbf{z}'_t))\|$$

implies

$$\sup_{t=0}^{\infty} \|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\| \leq \mathcal{I}(u^k \circ \mathcal{A})$$

which holding for any pair of supported (24,25) allocations in $\mathcal{E}(u^k)$ implies the required inequality $\mathcal{I}(u^k) \leq \mathcal{I}(u^k \circ \mathcal{A})$ by the definition of the indeterminacy index (26) at u^k. ∎

Step 1 implies it is sufficient to prove the limit inequality $\limsup_k \mathcal{I}(u^k) \leq \varepsilon$ (37) in the special case that every aggregation index $N_k \geq N$. To see that, suppose the limit inequality (37) held for every utility seuqence $u^{k\prime}$ converging with u with indicies $\geq N$. For the original sequence u^k of utility functions, we can replace each utility function with index $N_k < N$ by the composition $u^{k\prime} := u \circ \mathcal{A}^{N-N_k}$, which has index N. That new utility sequence $u^{k\prime}$ also converges to u but now has indicies $\geq N$. Hence, the supposed limit inequality $\limsup_k \mathcal{I}(u^{k\prime}) \leq \varepsilon$ holds for that seqence. But Step 1 implies $\mathcal{I}(u^k) \leq \mathcal{I}(u^{k\prime})$; hence, the limit inequality $\limsup_k \mathcal{I}(u^k) \leq \varepsilon$ also holds for the original seqence u^k of utility functions. Thus, without loss of generality, for the rest of the proof of Theorem 1 , assume every aggregation index $N_k \geq N$.

For each utility function u^k, the definition of the indeterminacy index (26) implies

$$\left\|(A\mathbf{y}^k_{t_k}, A\mathbf{z}^k_{t_k}) - (A\mathbf{y}^{k\,\prime}_{t_k}, A\mathbf{z}^{k\,\prime}_{t_k})\right\| \geq .5\mathcal{I}(u^k)$$

for some consumer t_k $(t_k = 0, 1, ...)$ and some pair $\left\{(\mathbf{y}^k_t, \mathbf{z}^k_t)\right\}$ and $\left\{(\mathbf{y}^{k\prime}_t, \mathbf{z}^{k\prime}_t)\right\}$ of supported allocations in $\mathcal{E}(u^k)$. In particular, the limit inequality (37) (and so Theorem 1) follows once we fix any consumer t_k $(t_k = 0, 1, ...)$ for each index k and fix any pair $\left\{(\mathbf{y}^k_t, \mathbf{z}^k_t)\right\}$ and $\left\{(\mathbf{y}^{k\prime}_t, \mathbf{z}^{k\prime}_t)\right\}$ of supported allocations in $\mathcal{E}(u^k)$, then show the limit inequality

$$\limsup_k \left\|(A\mathbf{y}^k_{t_k}, A\mathbf{z}^k_{t_k}) - (A\mathbf{y}^{k\,\prime}_{t_k}, A\mathbf{z}^{k\,\prime}_{t_k})\right\| \leq .5\varepsilon \tag{38}$$

To that end, for each index k, applying the aggregation map $\mathcal{A}^{N_k-N}$ (8) to the two consumption vectors

$$(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k}), \quad (\mathbf{y}^k_{t_k+1}, \mathbf{z}^k_{t_k+1})$$

in the feasible set F^{N_k+1} (20) evidently yields two aggregate vectors

$$\mathcal{A}^{N_k-N}(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k}), \quad \mathcal{A}^{N_k-N}(\mathbf{y}^k_{t_k+1}, \mathbf{z}^k_{t_k+1})$$

in the feasible set F^{N+1}. Since F^{N+1} is evidently compact (20) , there exists a subsequence (without loss of generality, the entire sequence) of aggregate vectors that converge,

$$\mathcal{A}^{N_k-N}(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k}) \to (\mathbf{y}, \mathbf{z}) \ ; \ \mathcal{A}^{N_k-N}(\mathbf{y}^k_{t_k+1}, \mathbf{z}^k_{t_k+1}) \to (\mathbf{y}', \mathbf{z}') \tag{39}$$

for two limit vectors $(\mathbf{y},\mathbf{z})$ and $(\mathbf{y}',\mathbf{z}')$ in F^{N+1}.

STEP 2. *Show the normalized-gradient equality*

$$\frac{\partial_z u(\mathbf{y},\mathbf{z})}{\|\partial_z u(\mathbf{y},\mathbf{z})\|} \quad = \quad \frac{\partial_y u(\mathbf{y}',\mathbf{z}')}{\|\partial_y u(\mathbf{y}',\mathbf{z}')\|} \tag{40}$$

By definition, each supported allocation $\{(\mathbf{y}_t^k,\mathbf{z}_t^k)\}$ in $\mathcal{E}(u^k)$ satisfies the gradient equality (25)

$$\left\| \frac{\partial_z u^k(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)}{\|\partial_z u^k(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)\|} - \frac{\partial_y u^k(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)}{\|\partial_y u^k(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)\|} \right\| = 0 \tag{41}$$

The convergence of utility $u^k \to u$ in the manifold universe $\mathcal{U}^\infty$ (23) implies $d^{N_k}(u^k, u \circ \mathcal{A}^{N_k-N}) \to 0$ (21) , and so

$$\left\| \frac{\partial_z u^k(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)}{\|\partial_z u^k(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)\|} - \frac{\partial_z (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)}{\|\partial_z (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)\|} \right\| \to 0$$

and

$$\left\| \frac{\partial_y u^k(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)}{\|\partial_y u^k(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)\|} - \frac{\partial_y (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)}{\|\partial_y (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)\|} \right\| \to 0$$

which with the gradient equality (41) and the triangle inequality implies

$$\left\| \frac{\partial_z (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)}{\|\partial_z (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k)\|} - \frac{\partial_y (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)}{\|\partial_y (u \circ \mathcal{A}^{N_k-N})(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k)\|} \right\| \to 0 \tag{42}$$

But the chain rule (as shown at the end of this section) shows the k'th term in the sequence (42) equals

$$\left\| \frac{\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))}{\|\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))\|} - \frac{\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))}{\|\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))\|} \right\|$$

so the latter terms converge,

$$\left\| \frac{\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))}{\|\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))\|} - \frac{\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))}{\|\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))\|} \right\| \to 0 \tag{43}$$

But consumption convergence $\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k) \to (\mathbf{y},\mathbf{z})$ and $\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k) \to (\mathbf{y}',\mathbf{z}')$ (39) and the continuity of gradients for functions in U^N (17) imply

$$\frac{\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))}{\|\partial_z u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k}^k,\mathbf{z}_{t_k}^k))\|} \to \frac{\partial_z u(\mathbf{y}',\mathbf{z}')}{\|\partial_z u(\mathbf{y}',\mathbf{z}')\|} \quad ; \quad \frac{\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))}{\|\partial_y u(\mathcal{A}^{N_k-N}(\mathbf{y}_{t_k+1}^k,\mathbf{z}_{t_k+1}^k))\|} \to \frac{\partial_y u(\mathbf{y}'',\mathbf{z}'')}{\|\partial_y u(\mathbf{y}'',\mathbf{z}'')\|}$$

Hence, convergence (43) implies

$$\left\| \frac{\partial_z u(\mathbf{y},\mathbf{z})}{\|\partial_z u(\mathbf{y},\mathbf{z})\|} - \frac{\partial_y u(\mathbf{y}',\mathbf{z}')}{\|\partial_y u(\mathbf{y}',\mathbf{z}')\|} \right\| = 0$$

and so the required normalized-gradient equality (40) . ▮

STEP 3. *Show* $A\mathbf{y} = \bar{\mathbf{y}}$.

Since the limit vector $(\mathbf{y}',\mathbf{z}')$ is in the feasible set F^{N+1}, utility function u is of the particular form (34)

$$u(\mathbf{y},\mathbf{z}) \quad = \quad (v \circ \mathcal{A})(\mathbf{y},\mathbf{z}) - \delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2)$$

at $(\mathbf{y}',\mathbf{z}')$. In particular, notice how young-age consumption $\mathbf{y}$ in that particular form (34) only affects utility $u(\mathbf{y},\mathbf{z})$ through $(v\circ\mathcal{A})(\mathbf{y},\mathbf{z}) = v(\mathbf{y}_\alpha+\mathbf{y}_\beta, \mathbf{z}_\alpha+\mathbf{z}_\beta)$ and through the aggregate $A\mathbf{y}$. In both cases, the effect of young-age consumption is symetric in the two components $\mathbf{y}_\alpha$ and $\mathbf{y}_\beta$ of the young-age vector $\mathbf{y} = (\mathbf{y}_\alpha,\mathbf{y}_\beta)$. In particular, the young-age gradient (14) of u must be symetric at $(\mathbf{y}',\mathbf{z}')$, meaning $\partial_y u(\mathbf{y}',\mathbf{z}') = (\mathbf{c},\mathbf{c})$ for some vector $\mathbf{c}$. Hence, the normalized-gradient equality (40) implies the old-age gradient $\partial_z u(\mathbf{y},\mathbf{z})$ is also symetric, $\partial_z u(\mathbf{y},\mathbf{z}) = (\mathbf{d},\mathbf{d})$ for some vector $\mathbf{d}$. But since the limit vector $(\mathbf{y},\mathbf{z})$ is in F^{N+1}, the formula (35) applies for the old-age gradient at $(\mathbf{y},\mathbf{z})$,

$$\partial_z u(\mathbf{y},\mathbf{z}) \;=\; (\;\partial_z v(\mathcal{A}(\mathbf{y},\mathbf{z})) - 2\delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2(\mathbf{z}_\alpha + \iota)\;\;,\;\;\partial_z v(\mathcal{A}(\mathbf{y},\mathbf{z}))\;)$$

In particular, symmetry $\partial_z u(\mathbf{y},\mathbf{z}) = (\mathbf{d},\mathbf{d})$ implies $\delta\|\bar{\mathbf{y}} - A\mathbf{y}\|_2(\mathbf{z}_\alpha + \iota) = 0$. But since the scaler $\delta > 0$, the vector $\mathbf{z}_\alpha \geq 0$, and the vector $\iota \gg 0$, the scaler $\|\bar{\mathbf{y}} - A\mathbf{y}\|_2$ must be 0, which implies $\bar{\mathbf{y}} = A\mathbf{y}$. ▮

The convergence (39) $\mathcal{A}^{N_k-N}(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k}) \to (\mathbf{y},\mathbf{z})$ of aggregate vectors and $\bar{\mathbf{y}} = A\mathbf{y}$ imply aggregate young-age consumption converges

$$A\mathbf{y}^k_{t_k} \to \bar{\mathbf{y}} \tag{44}$$

which with aggregate material balance (24) $A\mathbf{z}^k_{t_k-1} + A\mathbf{y}^k_{t_k} = \bar{\mathbf{z}} + \bar{\mathbf{y}}$ implies aggregate old-age consumption converges $A\mathbf{z}^k_{t_k-1} \to \bar{\mathbf{z}}$. Since the latter convergence holds for every selection t_k $(t_k = 0, 1, ...)$ of consumers, the convergence

$$A\mathbf{z}^k_{t_k} \to \bar{\mathbf{z}} \tag{45}$$

also holds for every selection of consumers. Likewise, since the convergence pair (44,45) holds for every supported allocation in $\mathcal{E}(u^k)$, the convergence

$$(A\mathbf{y}^k_{t_k}, A\mathbf{z}^k_{t_k}) \to (\bar{\mathbf{y}},\bar{\mathbf{z}}) \text{ and } (A\mathbf{y}^{k\,\prime}_{t_k}, A\mathbf{z}^{k\,\prime}_{t_k}) \to (\bar{\mathbf{y}},\bar{\mathbf{z}})$$

also holds for every pair $\{(\mathbf{y}_t^k, \mathbf{z}_t^k)\}$ and $\{(\mathbf{y}_t^{k\prime}, \mathbf{z}_t^{k\prime})\}$ of supported allocations in $\mathcal{E}(u^k)$. Hence, the limit inequality (38) $\limsup_k \left\|(A\mathbf{y}_{t_k}^k, A\mathbf{z}_{t_k}^k) - (A\mathbf{y}_{t_k}^k{}', A\mathbf{z}_{t_k}^k{}')\right\| \leq .5\varepsilon$ follows by the triangle inequality,

$$\left\|(A\mathbf{y}_{t_k}^k, A\mathbf{z}_{t_k}^k) - (A\mathbf{y}_{t_k}^k{}', A\mathbf{z}_{t_k}^k{}')\right\| \leq \left\|(A\mathbf{y}_{t_k}^k, A\mathbf{z}_{t_k}^k) - (\bar{\mathbf{y}}, \bar{\mathbf{z}})\right\| + \left\|((\bar{\mathbf{y}}, \bar{\mathbf{z}}) - (A\mathbf{y}_{t_k}^k{}', A\mathbf{z}_{t_k}^k{}')\right\|$$

Finally, recall that limit inequality (38) proves Theorem 1 . ▮

Step 2 required the following implication of the chain rule:

$$\left\| \frac{\partial_z(u \circ \mathcal{A}^n)(\mathbf{y}, \mathbf{z})}{\|\partial_z(u \circ \mathcal{A}^n)(\mathbf{y}, \mathbf{z})\|} - \frac{\partial_y(u \circ \mathcal{A}^n)(\mathbf{y}', \mathbf{z}')}{\|\partial_y(u \circ \mathcal{A}^n)(\mathbf{y}', \mathbf{z}')\|} \right\| = \left\| \frac{\partial_z u(\mathcal{A}^n(\mathbf{y}, \mathbf{z}))}{\|\partial_z u(\mathcal{A}^n(\mathbf{y}, \mathbf{z}))\|} - \frac{\partial_y u(\mathcal{A}^n(\mathbf{y}', \mathbf{z}'))}{\|\partial_y u(\mathcal{A}^n(\mathbf{y}', \mathbf{z}'))\|} \right\| \tag{46}$$

for each index $n = 1, 2, ...$ The general formula (46) follows by induction from the special case $n = 1$:

$$\left\| \frac{\partial_z(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z})}{\|\partial_z(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\|} - \frac{\partial_y(u \circ \mathcal{A})(\mathbf{y}', \mathbf{z}')}{\|\partial_y(u \circ \mathcal{A})(\mathbf{y}', \mathbf{z}')\|} \right\| = \left\| \frac{\partial_z u(\mathcal{A}(\mathbf{y}, \mathbf{z}))}{\|\partial_z u(\mathcal{A}(\mathbf{y}, \mathbf{z}))\|} - \frac{\partial_y u(\mathcal{A}(\mathbf{y}', \mathbf{z}'))}{\|\partial_y u(\mathcal{A}(\mathbf{y}', \mathbf{z}'))\|} \right\| \tag{47}$$

To see the special case (47) , for each pair of consumption vectors $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{y}', \mathbf{z}')$ in X^{N+1}, the chain rule relates the gradient of the composition $(u \circ \mathcal{A})$ at $(\mathbf{y}, \mathbf{z})$ to the gradient of u at the aggregate $\mathcal{A}(\mathbf{y}, \mathbf{z})$ (8) :

$$\partial_y(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) = (\mathbf{c}, \mathbf{c}) \quad \text{and} \quad \partial_z(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) = (\mathbf{d}, \mathbf{d}) \tag{48}$$

for vectors $\mathbf{c} := \partial_y u(\mathcal{A}(\mathbf{y}, \mathbf{z}))$ and $\mathbf{d} := \partial_z u(\mathcal{A}(\mathbf{y}, \mathbf{z}))$. Likewise,

$$\partial_y(u \circ A)(\mathbf{y}', \mathbf{z}') = (\mathbf{e}, \mathbf{e}) \quad \text{and} \quad \partial_z(u \circ A)(\mathbf{y}', \mathbf{z}') = (\mathbf{f}, \mathbf{f}) \tag{49}$$

for vectors $\mathbf{e} := \partial_y u(\mathcal{A}(\mathbf{y}', \mathbf{z}'))$ and $\mathbf{f} := \partial_z u(\mathcal{A}(\mathbf{y}', \mathbf{z}'))$. Hence, the required special case (47) reads

$$\left\| \frac{((\mathbf{c}, \mathbf{c}), (\mathbf{d}, \mathbf{d}))}{\|((\mathbf{c}, \mathbf{c}), (\mathbf{d}, \mathbf{d}))\|} - \frac{((\mathbf{e}, \mathbf{e}), (\mathbf{f}, \mathbf{f}))}{\|((\mathbf{e}, \mathbf{e}), (\mathbf{f}, \mathbf{f}))\|} \right\| = \left\| \frac{(\mathbf{c}, \mathbf{d})}{\|(\mathbf{c}, \mathbf{d})\|} - \frac{(\mathbf{e}, \mathbf{f})}{\|(\mathbf{e}, \mathbf{f})\|} \right\|$$

which is an identity for the sup norm (19) .

9. Proof of Theorem 2

Throughout the proof, fix any positive tolerance ε and fix any utility function u in $\mathcal{U}^\infty$ such that the indeterminacy index $\mathcal{I}(u) = 0$ and the index $\mathcal{I}(\cdot)$ is continous at u. Given Theorem 1 , it is sufficient for Theorem 2 to prove that u is in an open set of utility functions satisfying the required inequalities (29,30,31) . To that end, it is sufficient to consider any convergent $u^k \to u$

sequence of utility functions u^k in the manifold universe $\mathcal{U}^\infty = \bigcup_{N=0}^{\infty} \mathcal{U}^N$ then show the required inequalities (29,30,31)

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u^k)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u^k)} \sup_{t=0}^{\infty} \|(A\mathbf{y}_t, A\mathbf{z}_t) - (A\mathbf{y}'_t, A\mathbf{z}'_t)\| < \varepsilon \tag{50}$$

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u^k)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u^k)} \sup_{t=0}^{\infty} \left\| \frac{\partial u^k(\mathbf{y}_t,\mathbf{z}_t)}{\|\partial u^k(\mathbf{y}_t,\mathbf{z}_t)\|} - \frac{\partial u^k(\mathbf{y}'_t,\mathbf{z}'_t)}{\|\partial u^k(\mathbf{y}'_t,\mathbf{z}'_t)\|} \right\| < \varepsilon \tag{51}$$

$$\sup_{\{(\mathbf{y}_t,\mathbf{z}_t)\}\in\mathcal{E}(u^k)} \sup_{\{(\mathbf{y}'_t,\mathbf{z}'_t)\}\in\mathcal{E}(u^k)} \sup_{t=0}^{\infty} \left| u^k(\mathbf{y}_t,\mathbf{z}_t) - u^k(\mathbf{y}'_t,\mathbf{z}'_t) \right| < \varepsilon \tag{52}$$

for sufficiently large k.

The hypothesized continuity of the indeterminacy index (26) and utility convergence $u^k \to u$ imply index convergence $\mathcal{I}(u^k) \to \mathcal{I}(u)$, which implies the first required inequality (50) . To prove the other two inequalities, for each utility function u^k, let the index $\mathcal{I}'(u^k)$ denote the supremum in the second inequality (51) , and $\mathcal{I}''(u^k)$ the supremum in the third inequality (52) . For each utility function u^k, let N_k ($N_k = 0, 1, ...$) be the aggregation index such that $u^k \in \mathcal{U}^{N_k}$.

Step 1. *Show $\mathcal{I}'(u^k) \le \mathcal{I}'(u^k \circ \mathcal{A}^n)$ and $\mathcal{I}''(u^k) \le \mathcal{I}''(u^k \circ \mathcal{A}^n)$, for each utility function u^k and each index $n = 1, 2, ...$*

We will show $\mathcal{I}'(u^k) \le \mathcal{I}'(u^k \circ \mathcal{A})$; the more general inequality $\mathcal{I}'(u^k) \le \mathcal{I}'(u^k \circ \mathcal{A}^n)$ follows by induction for each index $n = 1, 2, ...$ Showing $\mathcal{I}''(u^k) \le \mathcal{I}''(u^k \circ \mathcal{A}^n)$ is similar, and is not presented.

Consider any pair $\{(\mathbf{y}_t, \mathbf{z}_t)\}$ and $\{(\mathbf{y}'_t, \mathbf{z}'_t)\}$ of supported (24,25) allocations in $\mathcal{E}(u^k) \subset \prod_{t=0}^{\infty} X_+^{N_k+1}$. The pair $\{((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))\}$ and $\{((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))\}$ are, evidently, supported (24,25) allocations in $\mathcal{E}(u^k \circ \mathcal{A}) \subset \prod_{t=0}^{\infty} X_+^{N_k+2}$. Hence, the definition of the index $\mathcal{I}'$ (51) at the composition $u^k \circ \mathcal{A}$ implies

$$\sup_{t=0}^{\infty} \left\| \frac{\partial(u^k \circ \mathcal{A})((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))}{\|\partial(u^k \circ \mathcal{A})((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))\|} - \frac{\partial(u^k \circ \mathcal{A})((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))}{\|\partial(u^k \circ \mathcal{A})((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))\|} \right\| \le \mathcal{I}'(u^k \circ \mathcal{A})$$

But the chain rule (as discussed in the proof of Theorem 2) implies

$$\left\| \frac{\partial(u^k \circ \mathcal{A})((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))}{\|\partial(u^k \circ \mathcal{A})((.5\mathbf{y}_t, .5\mathbf{y}_t), (.5\mathbf{z}_t, .5\mathbf{z}_t))\|} - \frac{\partial(u^k \circ \mathcal{A})((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))}{\|\partial(u^k \circ \mathcal{A})((.5\mathbf{y}'_t, .5\mathbf{y}'_t), (.5\mathbf{z}'_t, .5\mathbf{z}'_t))\|} \right\|$$

equals

$$\left\| \frac{\partial u^k(\mathbf{y}_t,\mathbf{z}_t)}{\|\partial u^k(\mathbf{y}_t,\mathbf{z}_t)\|} - \frac{\partial u^k(\mathbf{y}'_t,\mathbf{z}'_t)}{\|\partial u^k(\mathbf{y}'_t,\mathbf{z}'_t)\|} \right\|$$

and so

$$\sup_{t=0}^{\infty} \left\| \frac{\partial u^k(\mathbf{y}_t, \mathbf{z}_t)}{\|\partial u^k(\mathbf{y}_t, \mathbf{z}_t)\|} - \frac{\partial u^k(\mathbf{y}'_t, \mathbf{z}'_t)}{\|\partial u^k(\mathbf{y}'_t, \mathbf{z}'_t)\|} \right\| \;\leq\; \mathcal{I}'(u^k \circ \mathcal{A})$$

which holding for any pair of supported (24,25) allocations in $\mathcal{E}(u^k)$ implies the required inequality $\mathcal{I}'(u^k) \leq \mathcal{I}'(u^k \circ \mathcal{A})$ by the definition of the indeterminacy index $\mathcal{I}'$ (51) at u^k. ∎

Step 1 implies it is sufficient to prove the second (51) and third (52) inequalities in the special case that every index $N_k \geq N$, just like in Step 1 in the proof of Theorem 1 . Thus, without loss of generality, for the rest of the proof of Theorem 2 , assume every aggregation index $N_k \geq N$.

STEP 2. *Show the second required inequality* (51).

For each utility function u^k, fix any consumer t_k ($t_k = 0, 1, ...$) such that, for some pair $\{(\mathbf{y}^k_t, \mathbf{z}^k_t)\}$ and $\{(\mathbf{y}^{k\prime}_t, \mathbf{z}^{k\prime}_t)\}$ of supported allocations in $\mathcal{E}(u^k)$, the term

$$\left\| \frac{\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})}{\|\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})\|} - \frac{\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')}{\|\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')\|} \right\|$$

is greater than .5 times the supremum in (51) . In particular, the second inequality (51) follows once we show the limit inequality

$$\limsup_k \left\| \frac{\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})}{\|\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})\|} - \frac{\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')}{\|\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')\|} \right\| < .5\varepsilon \tag{53}$$

As in Steps 2 and 3 of the proof of Theorem 1 , for some subsequence of the original sequence (without loss of generality, the entire sequence), the consumption sequence $\mathcal{A}^{N_k - N}(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})$ converges to a limit $(\mathbf{y}, \mathbf{z})$ in Ω^N and the consumption sequence $\mathcal{A}^{N_k - N}(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')$ converges to a limit $(\mathbf{y}', \mathbf{z}')$ in Ω^N. And as in Step 2 of the proof of Theorem 1 , the convergence of utility $u^k \to u$ and the convergence of consumption $\mathcal{A}^{N_k - N}(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k}) \to (\mathbf{y}, \mathbf{z})$ and $\mathcal{A}^{N_k - N}(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}') \to (\mathbf{y}', \mathbf{z}')$ imply

$$\left\| \frac{\partial u(\mathbf{y}, \mathbf{z})}{\|\partial u(\mathbf{y}, \mathbf{z})\|} - \frac{\partial u(\mathbf{y}', \mathbf{z}')}{\|\partial u(\mathbf{y}', \mathbf{z}')\|} \right\| = \limsup_k \left\| \frac{\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})}{\|\partial u^k(\mathbf{y}^k_{t_k}, \mathbf{z}^k_{t_k})\|} - \frac{\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')}{\|\partial u^k(\mathbf{y}^k_{t_k}{}', \mathbf{z}^k_{t_k}{}')\|} \right\|$$

But the left-hand norm is zero since the definition (17) of utility space $\mathcal{U}^N$ implies both terms $\partial u(\mathbf{y}, \mathbf{z})/\|\partial u(\mathbf{y}, \mathbf{z})\|$ and $\partial u(\mathbf{y}', \mathbf{z}')/\|\partial u(\mathbf{y}', \mathbf{z}')\|$ equal $(\mathbf{p}^N, \frac{1}{1+r}\mathbf{p}^N)$ at the limits $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{y}', \mathbf{z}')$ in Ω^N. The limit inequality (53) and the second required inequality (51) follow. ∎

STEP 3. *Show the third required inequality* (52).

For each utility function u^k, fix any consumer t_k $(t_k = 0, 1, ...)$ such that, for some pair $\left\{(\mathbf{y}_t^k, \mathbf{z}_t^k)\right\}$ and $\left\{(\mathbf{y}_t^{k\prime}, \mathbf{z}_t^{k\prime})\right\}$ of supported allocations in $\mathcal{E}(u^k)$, the term

$$\left|u^k(\mathbf{y}_{t_k}^k, \mathbf{z}_{t_k}^k) - u^k(\mathbf{y}_{t_k}^k{}', \mathbf{z}_{t_k}^k{}')\right|$$

is greater than .5 times the supremum in (52) . In particular, the third inequality (52) follows once we show the limit inequality

$$\limsup_k \left|u^k(\mathbf{y}_{t_k}^k, \mathbf{z}_{t_k}^k) - u^k(\mathbf{y}_{t_k}^k{}', \mathbf{z}_{t_k}^k{}')\right| < .5\varepsilon \tag{54}$$

As in the proof of Theorem 1 and in Step 2 above, for some subsequence of the original sequence (without loss of generality, the entire sequence), the consumption sequence $\mathcal{A}^{N_k - N}(\mathbf{y}_{t_k}^k, \mathbf{z}_{t_k}^k)$ converges to a limit $(\mathbf{y}, \mathbf{z})$ in Ω^N and the consumption sequence $\mathcal{A}^{N_k - N}(\mathbf{y}_{t_k}^k{}', \mathbf{z}_{t_k}^k{}')$ converges to a limit $(\mathbf{y}', \mathbf{z}')$ in Ω^N (Step 3). And as above, the convergence of utility $u^k \to u$ and the convergence of consumption $\mathcal{A}^{N_k - N}(\mathbf{y}_{t_k}^k, \mathbf{z}_{t_k}^k) \to (\mathbf{y}, \mathbf{z})$ and $\mathcal{A}^{N_k - N}(\mathbf{y}_{t_k}^k{}', \mathbf{z}_{t_k}^k{}') \to (\mathbf{y}', \mathbf{z}')$ imply

$$|u(\mathbf{y}, \mathbf{z}) - u(\mathbf{y}', \mathbf{z}')| = \limsup_k \left|u^k(\mathbf{y}_{t_k}^k, \mathbf{z}_{t_k}^k) - u^k(\mathbf{y}_{t_k}^k{}', \mathbf{z}_{t_k}^k{}')\right|$$

But the left-hand value is zero since Observation 1 implies each consumption limit $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{y}', \mathbf{z}')$ in Ω^N solves the same utility maximization problem. The limit inequality (54) and the third required inequality (52) follow. ▮

10. Remaining proofs

10.1. Proof of Lemma 3

First prove the distance functions (21) satisfy

$$d^{N+n}(u \circ \mathcal{A}^n, v \circ \mathcal{A}^n) = d^N(u, v) \tag{55}$$

for all utility functions u and v in the N-aggregated space $\mathcal{U}^N$ and for all indices $n = 1, 2, ...$ By induction, it is sufficient to prove the equality (55) for the special case $n = 1$.

To that end, the left side $d^{N+1}(u \circ \mathcal{A}, v \circ \mathcal{A})$ of the required equation (55) is defined (21) to equal

$$\sup_{(\mathbf{y},\mathbf{z}) \in F^{N+2}} |\ (u \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ -\ (v \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ | + \sup_{(\mathbf{y},\mathbf{z}) \in F^{N+2}} \|\ \partial(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ -\ \partial(v \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ \|$$

But for each $(\mathbf{y}, \mathbf{z})$ in F^{N+2}, the chain rule implies

$$\|\ \partial(u \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ -\ \partial(v \circ \mathcal{A})(\mathbf{y}, \mathbf{z})\ \| = \|\ \partial u(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ -\ \partial v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ \|$$

so

$$d^{N+1}(u\circ\mathcal{A},v\circ\mathcal{A}) = \sup_{(\mathbf{y},\mathbf{z})\in F^{N+2}} \mid u(\mathcal{A}(\mathbf{y},\mathbf{z})) \;-\; v(\mathcal{A}(\mathbf{y},\mathbf{z})) \mid + \sup_{(\mathbf{y},\mathbf{z})\in F^{N+2}} \parallel \partial u(\mathcal{A}(\mathbf{y},\mathbf{z})) \;-\; \partial v(\mathcal{A}(\mathbf{y},\mathbf{z})) \parallel$$

Hence, since the aggregation map $\mathcal{A}$ (8) evidently sends the feasible set F^{N+2} into F^{N+1} (20) ,

$$d^{N+1}(u\circ\mathcal{A},v\circ\mathcal{A}) = \sup_{(\mathbf{y},\mathbf{z})\in F^{N+1}} \mid u(\mathbf{y},\mathbf{z}) \;-\; v(\mathbf{y},\mathbf{z}) \mid + \sup_{(\mathbf{y},\mathbf{z})\in F^{N+1}} \parallel \partial u(\mathbf{y},\mathbf{z}) \;-\; \partial v(\mathbf{y},\mathbf{z}) \parallel$$

But the right side of that equation is defined (21) to equal $d^N(u,v)$, which is the right side of the required equation (55) . ▮

The only remarkable property of the distance function d^∞ (23) to prove in Lemma 3 is triangle inequality,

$$d^\infty(u,w) \leq d^\infty(u,v) + d^\infty(v,w) \tag{56}$$

for any three elements $u \in \mathcal{U}^L$, $v \in \mathcal{U}^M$, and $w \in \mathcal{U}^N$. But the distance equation (55) implies

$$d^\infty(u,w) = d^{L+M+N}((u\circ\mathcal{A}^{M+N}),(w\circ\mathcal{A}^{L+M})) \tag{57}$$

To see that equation if $L \geq N$, the definition of d^∞ (23) implies $d^\infty(u,w) = d^L(u,(w\circ\mathcal{A}^{L-N}))$, and the distance equation (55) for $n = M+N$ implies $d^L(u,(w\circ\mathcal{A}^{L-N})) = d^{L+M+N}((u\circ\mathcal{A}^{M+N}),(w\circ\mathcal{A}^{L+M}))$. Likewise, the distance equation (55) implies the distance equation (57) if $L < N$, as well as

$$d^\infty(u,v) = d^{L+M+N}((u\circ\mathcal{A}^{M+N}),(v\circ\mathcal{A}^{L+N})); \quad d^\infty(v,w) = d^{L+M+N}((v\circ\mathcal{A}^{L+N}),(w\circ\mathcal{A}^{L+M}))$$

The required triangle inequality (56) for d^∞ thus follows from the known triangle inequality

$$d^{L+M+N}((u\circ A^{M+N}),(w\circ A^{L+M})) \leq d^{L+M+N}((u\circ A^{M+N}),(v\circ A^{L+N})) + d^{L+M+N}((v\circ A^{L+N}),(w\circ A^{L+M}))$$

for the pseudometric d^{L+M+N} (Lemma 1). ▮

10.2. Proof of Lemma 4

Fix any utility function v in $\mathcal{U}^{N-1}$, for some index $N = 1,2,...$; fix any positive vector ι in X^{N-1}_{++}; and fix the scaler

$$\theta \;:=\; \max\left\{\, 3\|\mathbf{z}_\alpha + \iota\|_2 \;:\; \mathbf{z}_\alpha \in 2F^{N-1} \,\right\} \tag{58}$$

To prove Lemma 4 , show, for each sufficiently small tollerence $\delta > 0$, there exists a utility function $u = u_\delta$ in $\mathcal{U}^N$ of the required form (34) .

STEP 1. *Show the auxiliary function* (36)

$$f(\mathbf{y},\mathbf{z}) = \|\bar{\mathbf{y}} - A\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2)$$

is convex over vectors $(\mathbf{y},\mathbf{z}) = ((\mathbf{y}_\alpha,\mathbf{y}_\beta),(\mathbf{z}_\alpha,\mathbf{z}_\beta))$ *in* $2F^{N+1}$.[9]

For each vector $(\mathbf{y},\mathbf{z}) = ((\mathbf{y}_\alpha,\mathbf{y}_\beta),(\mathbf{z}_\alpha,\mathbf{z}_\beta))$ in $2F^{N+1}$, the component $\mathbf{y} \in 2F^N$ (since $F^{N+1} \subset F^N \times F^N$ (20)) and the component $\mathbf{z}_\alpha \in 2F^{N-1}$ (since $F^N \subset F^{N-1} \times F^{N-1}$ (20)). Hence, since the mapping

$$\mathbf{y} \in (2F^N) \mapsto (\bar{\mathbf{y}} - A\mathbf{y}) \in X^0$$

is evidently affine, the required convexity of f (36) over $2F^{N+1}$ follows from the convexity of the function

$$(\mathbf{y},\mathbf{z}_\alpha) \in X^0 \times (2F^{N-1}) \quad\mapsto\quad \|\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) \tag{59}$$

To show that convexity (59) , write each Euclidean vector $\mathbf{y} \in X^0 = \Re^\ell$ as $\mathbf{y} = \{y_i\}$. Hence, the function values (59) are the sum

$$\|\mathbf{y}\|_2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) = \sum_i (y_i)^2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2)$$

Since the sum of convex functions is convex, the required convexity of function values (59) over $X^0 \times (2F^{N-1})$ follows from the convexity of the function

$$(y,\mathbf{z}_\alpha) \in \Re \times (2F^{N-1}) \quad\mapsto\quad y^2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) \tag{60}$$

To show that convexity (60) , write function values (60)

$$y^2\,(\theta + \|\mathbf{z}_\alpha + \iota\|_2) = g(y, r(\mathbf{z}_\alpha))$$

for the bivariate function

$$g(y,z) := y^2(\theta + z^2) \tag{61}$$

and the square-root function $r(\mathbf{z}_\alpha) := \sqrt{\|\mathbf{z}_\alpha + \iota\|_2}$. The square-root function is evidently convex over $(2F^{N-1})$. Hence, the required convexity of function values (60) over $\Re \times (2F^{N-1})$ follows if the bivariate function $g(y,z)$ (61) over (y,z) in $\Re \times r(2F^{N-1})$ is non-decreasing in z, and is convex.

[9] $2F^{N+1} := \left\{2(\mathbf{y},\mathbf{z}) \mid (\mathbf{y},\mathbf{z}) \in F^{N+1}\right\}$.

To prove the bivariate function g (61) is non-decreasing in z, for pairs (y, z) in $\Re \times r(2F^{N-2})$, the partial derivative $\partial g(y, z)/\partial z = 2zy^2 \geq 0$ since, evidently, $r(2F^{N-2}) \subset \Re_+$.

To prove convexity, use the determinant test to show the matrix

$$\partial^2 g(y, z) \;=\; \begin{bmatrix} 2(\theta + z^2) & 4yz \\ 4yz & 2y^2 \end{bmatrix}$$

of second derivatives of the bivariate function g (61) is positive semidefinite for pairs $(y, z) \in \Re \times r(2F^{N-1})$. The two diagonal terms are evidently non-negative, as required. The determinant

$$\det \partial^2 g(y, z) = 4(\theta + z^2)y^2 - 16y^2z^2 = 4y^2(\theta - 3z^2)$$

is also non-negative since the scaler $\theta = \max\left\{3z^2 \;:\; z \in r(2F^{N-1})\right\}$ (58) . The bivariate function g (61) is thus convex, and so the auxillary function (36) is convex over $2F^{N+1}$. ∎

Step 2. *Restrict the tollerence $\delta > 0$, and define the utility function u.*

For any $\delta > 0$, define the auxillary function

$$h(\mathbf{y}, \mathbf{z}) \quad := \quad (v \circ \mathcal{A})(\mathbf{y}, \mathbf{z}) - \delta f(\mathbf{y}, \mathbf{z}) \tag{62}$$

over $2F^{N+1}$. Since utility v is concave and C^1 and increasing (17) and since the auxiliary function f (36) over $2F^{N+1}$ is convex (Step 1) and is evidently C^1, for each sufficiently small tollerence $\delta > 0$, the auxillary function h (62) over $2F^{N+1}$ is concave and C^1 and increasing.

Fix any auxillary map $\psi : X_+^{N+1} \to 2F^{N+1}$ such that

M.1. ψ is concave and C^1 and increasing, meaning $\mathbf{y} > \mathbf{z}$ implies $\psi(\mathbf{y}) > \psi(\mathbf{z})$.

M.2. ψ is the identity over the feasible set F^{N+1}.

Hence, define utility

$$u(\mathbf{y}, \mathbf{z}) := h(\psi(\mathbf{y})) \tag{63}$$

Since ψ maps X_+^{N+1} into $2F^{N+1}$ and the function h (62) is defined over $2F^{N+1}$, the composition $u(\mathbf{y}, \mathbf{z}) := h(\psi(\mathbf{y}, \mathbf{z}))$ is a well-defined function over X_+^{N+1}.

Step 3. *Show the required properties of the utility function u.*

The composition $u(\mathbf{y}, \mathbf{z}) := h(\psi(\mathbf{y}))$ (63) is of the required form (34) over the feasible set F^{N+1} since the definition (36) of the function f implies the function h (62) is of that form (34) , and since the map ψ is the identity over F^{N+1} (M.2). It only remains to show the utility function u is in the space $\mathcal{U}^N$.

To that end, the composition $u(\mathbf{y}, \mathbf{z}) := h(\psi(\mathbf{y}))$ (63) is concave and C^1 and increasing over its domain X_+^{N+1} since the auxillary function h (62) is concave and C^1 and increasing over $2F^{N+1}$, and since the auxillary mapping ψ is concave and C^1 and increasing. Finally, for each $(\mathbf{y}, \mathbf{z})$ in Ω^N, it remains to show (17)

$$\frac{\partial u(\mathbf{y}, \mathbf{z})}{\|\partial u(\mathbf{y}, \mathbf{z})\|} = (\mathbf{p}^N, \frac{1}{1+r}\mathbf{p}^N)$$

To begin, $(\mathbf{y}, \mathbf{z}) \in \Omega^N$ (13) implies $\bar{\mathbf{y}} = A\mathbf{y}$, which with the gradient formula (35) implies

$$\partial_y u(\mathbf{y}, \mathbf{z}) = (\ \partial_y v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ ,\ \partial_y v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ )\ ;\ \partial_z u(\mathbf{y}, \mathbf{z}) = (\ \partial_z v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ ,\ \partial_z v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\ ) \tag{64}$$

But $(\mathbf{y}, \mathbf{z}) \in \Omega^N$ also implies $\mathcal{A}(\mathbf{y}, \mathbf{z}) \in \Omega^{N-1}$, which with $v \in \mathcal{U}^{N-1}$ implies

$$\frac{\partial v(\mathcal{A}(\mathbf{y}, \mathbf{z}))}{\|\partial v(\mathcal{A}(\mathbf{y}, \mathbf{z}))\|} = (\mathbf{p}^{N-1}, \frac{1}{1+r}\mathbf{p}^{N-1})$$

or $\partial v(\mathcal{A}(\mathbf{y}, \mathbf{z})) = \lambda(\mathbf{p}^{N-1}, \frac{1}{1+r}\mathbf{p}^{N-1})$, for some scaler λ. Hence, the gradient equality (64) implies

$$\partial u(\mathbf{y}, \mathbf{z}) = \lambda\left((\mathbf{p}^{N-1}, \mathbf{p}^{N-1}), \frac{1}{1+r}(\mathbf{p}^{N-1}, \mathbf{p}^{N-1})\right)$$

which implies

$$\frac{\partial u(\mathbf{y}, \mathbf{z})}{\|\partial u(\mathbf{y}, \mathbf{z})\|} = \left((\mathbf{p}^{N-1}, \mathbf{p}^{N-1}), \frac{1}{1+r}(\mathbf{p}^{N-1}, \mathbf{p}^{N-1})\right)$$

which implies the required gradient equality (17) by the definition (15) of price, $\mathbf{p}^N := (\mathbf{p}^{N-1}, \mathbf{p}^{N-1})$. ▮

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