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Economic Theory

A note on the existence of a monetary equilibrium over an infinite horizon*

Gaetano Bloise

DEIR, University of Sassari, Sassari, ITALY
(e-mail: bloise@uniss.it)

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Summary. This note considers an economy with 'inside money' that extends over an infinite horizon in the case of certainty. It shows the existence of an equilibrium and the indeterminacy of the overall price level when the supply of balances is set exogenously.

Keywords and Phrases: Money, Equilibrium, Indeterminacy.

JEL Classification Numbers: D50, E40, E50.

1 Introduction

In this note, I show the existence of monetary equilibria over an infinite horizon in the case of certainty for a given supply of balances. In addition, I prove that there is an equilibrium for every overall price level. These findings confirm the theses advanced by Drèze and Polemarchakis [6] about economies extending over a finite horizon.

I here consider the monetary economy with 'inside money' of Drèze and Polemarchakis [6]. Balances, which are distinguished from bonds, are needed for transactions, as described by a traditional cash-in-advance constraint (Clower [5]) on trades in commodities. Money is supplied by a central bank, which trades balances against bonds and runs a balanced budget by redistributing its profit to shareholders.

A more canonical notion of equilibrium in monetary economies requires that balances be varied through lump-sum transfers to individuals. Remarkably, such

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Correspondence to: Gaetano Bloise, Dipartimento di Scienze Economiche, 12 Via Andrea Cesalpino, 00161 Roma, Italy

canonical equilibria are also equilibria as defined in this paper. Though I do not pursue this line of research here, perhaps with more structure on preferences and endowments, equilibria as defined in this paper would be canonical at all high enough overall price levels, provided that there are gains to trade. The former bear the advantage of analytical tractability.

My assumptions on preferences and endowments are only slightly more restrictive than those needed for the existence of a Walrasian equilibrium (*e.g.*, Bewley [2]). To the best of my knowledge, over an infinite horizon, no result of multiplicity of monetary equilibria has been established in the literature at such a level of generality.[1] In a cash-in-advance economy, where balances are the only tradable asset, Grandmont and Younès [11,12] proved the existence of a stationary monetary equilibrium under time additively separable preferences. Bewley [3] considered the case of uncertainty in a stationary economy where balances are the only store of value available for transferring wealth across periods. In both cases, there is no claim about the multiplicity of equilibria. Bloise, Drèze and Polemarchakis [4] treated the case of an interest rate pegging with uncertainty under a sequentially complete asset market.

A full understanding of conditions for the existence of a monetary equilibrium over an infinite horizon, to which this note contributes, is of help for addressing some issues that have recently been raised in the literature. First, Dubey and Geanakoplos [7,8] have submitted an interesting thesis, according to which the determination of interest rate cannot be properly understood without acknowledging the crucial interplay between inside and outside money. Their claim refers to a finite horizon and it is unclear whether it would meaningfully extend to the case of an infinite horizon. Second, price indeterminacy is well-known in cash-in-advance economies with a single representative individual. It is, however, not regarded as a serious issue as suitable (and, to some extent, reasonable) assumptions deliver a locally determinate steady state equilibrium and a continuum of equilibria exhibiting hyperinflation and converging to the autarchy.[2] Beyond generality, my minimal assumptions on preferences allow for the case of a non-uniform impatience across individuals. It is not obvious that the above described structure of equilibria is inherited by these more general settings.

The paper is organized as follows: in Section 2, I describe the monetary economy; in Section 3, I state the result on existence and multiplicity of equilibria; in Section 4, I comment on some complications arising in the proof because of cash-in-advance constraints; in Section 5, I present the formal proof.

[1] Of course, I do not include studies on overlapping generations economies.

[2] Interestingly, stricter assumptions deliver global uniqueness of equilibrium (Lucas and Stokey [13]). Such assumptions, however, imply that preferences are not continuous according to the definition in this paper.

2 Monetary equilibrium

2.1 Vector spaces

For a given countable set, $\mathcal{A}$, $\ell(\mathcal{A})$ denotes the ordered vector space of all real(-valued) maps on $\mathcal{A}$. A vector subspace is the space of all bounded (summable) real maps on $\mathcal{A}$, $\ell^\infty(\mathcal{A})$ ($\ell^1(\mathcal{A})$), that is, the space of all real maps, $x = (\dots, x_\alpha, \dots)$, on $\mathcal{A}$ such that, for some $\epsilon > 0$, $|x_\alpha| \leq \epsilon$ for every α in $\mathcal{A}$ (such that, for some $\epsilon > 0$, $\sum_\alpha |x_\alpha| \leq \epsilon$). A real map on $\mathcal{A}$ is positive if $x_\alpha \geq 0$ for every α in $\mathcal{A}$; it is strictly positive if $x_\alpha > 0$ for every α in $\mathcal{A}$; it is uniformly strictly positive if, for some $\epsilon > 0$, $x_\alpha \geq \epsilon$ for every α in $\mathcal{A}$. The positive cone $\ell_+(\mathcal{A})$ of $\ell(\mathcal{A})$ consists of all positive real maps on $\mathcal{A}$. For any map x in $\ell(\mathcal{A})$, x^+ and x^- are, respectively, its positive and its negative part, so that $x = x^+ - x^-$. Finally, for (z, x) in $\ell^1(\mathcal{A}) \times \ell^\infty(\mathcal{A})$, $z \cdot x = \sum_\alpha z_\alpha x_\alpha$ denotes the usual dual operation. Notice that positive is used to mean greater than or equal to zero throughout this paper.

2.2 Commodity space

Periods of trade are $\mathcal{T} = \{0, 1, \dots, t, \dots\}$. In every period of trade, physical commodities are $\mathcal{N} = \{\dots, n, \dots\}$, a finite set. The commodity space is the set of all bounded real maps on $\mathcal{T} \times \mathcal{N}$. The choice of such a commodity space, though disputable, is grounded on a well-established tradition, beginning with Bewley [2]. The consumption space is the positive cone of the commodity space. A typical element of the commodity space is denoted $x = (\dots, x_t, \dots)$, with each $x_t = (\dots, x_{tn}, \dots)$ being an element of $\mathbb{R}^{\mathcal{N}}$.

2.3 Individuals

There is finite set of individuals, $\mathcal{I} = \{\dots, i, \dots\}$. An individual is described by preferences, $\succeq^i$, over the consumption space and an endowment of commodities, e^i, an element of the consumption space.

Assumption 1 (Preferences). *Preferences $\succeq^i$ of individual i are continuous in the relative Mackey topology, convex and strictly monotone.*

Assumption 2 (Endowment). *The endowment e^i of individual i is uniformly strictly positive.*

Assumption 3 (Bounds on marginal rates of substitution). *Preferences $\succeq^i$ of individual i are representable by a utility function $u^i : \ell^\infty_+(\mathcal{T} \times \mathcal{N}) \to \mathbb{R}$ that admits a strictly positive Gateaux derivative $\partial u^i(x^i)$ in $\ell^1_+(\mathcal{T} \times \mathcal{N})$ at every x^i in $\ell^\infty_+(\mathcal{T} \times \mathcal{N})$.*[3] *In addition, there is π in $\ell^1_+(\mathcal{T} \times \mathcal{N})$ such that, for all pairs of*

[3] Precisely, u^i is extended by a Gateaux differentiable function on a finitely open set containing $\ell^\infty_+(\mathcal{T} \times \mathcal{N})$, where a set is finitely open if its intersection with every finite-dimensional vector subspace of $\ell^\infty(\mathcal{T} \times \mathcal{N})$ is relatively open. Obviously, it would suffice to assume differentiability over some suitable order interval only.

individuals (i,j) *in* $\mathcal{I}\times\mathcal{I}$*, all* $\left(x^i,x^j\right)$ *in* $\ell_+^\infty(\mathcal{T}\times\mathcal{N})\times\ell_+^\infty(\mathcal{T}\times\mathcal{N})$ *and all* (x,z) *in* $\ell_+^\infty(\mathcal{T}\times\mathcal{N})\times\ell_+^\infty(\mathcal{T}\times\mathcal{N})$*,*

$$\left(\partial u^i\left(x^i\right)\cdot x_t\right)\partial u^j\left(x^j\right)\cdot z_t\leq\left(\partial u^j\left(x^j\right)\cdot x_t\right)\pi\cdot z_t$$

holds at every t.

Remark 1. By assumption 3, $\partial u^i\left(x^i\right)\leq\pi$ for every x^i in $\ell_+^\infty(\mathcal{T}\times\mathcal{N})$.

Assumptions 1-2 on fundamentals parallel those of Bewley [2] and are undisputable. Assumption 3 is perhaps more restrictive than necessary and it is used only to prove the indeterminacy of equilibrium. It is worth noticing that, for instance, recursive economies satisfy such a smoothness assumption under mild hypotheses on the aggregators.

For a better understanding of the nature of the restrictions arising from assumption 3, suppose that every individual is endowed with a time additively separable utility function,

$$u^i\left(x^i\right)=\sum_t\left(\frac{1}{1+\rho^i}\right)^t v^i\left(x_t^i\right),$$

where $\rho^i>0$ is the rate of impatience and $v^i:\mathbb{R}_+^{\mathcal{N}}\to\mathbb{R}$ is a smooth, smoothly strictly increasing and smoothly concave bounded felicity function. Hence (see, for instance, Shannon [15, Theorem 2.2]),

$$\partial u^i\left(x^i\right)=\left(\ldots,\left(\frac{1}{1+\rho^i}\right)^t\partial v^i\left(x_t^i\right),\ldots\right).$$

The requirement of assumption 3 then reduces to the existence of some π in $\ell_+^1(\mathcal{T}\times\mathcal{N})$ such that

$$\left(\frac{1}{1+\rho^i}\right)^t\left(\partial v^i\left(x_t^i\right)\cdot x_t\right)\partial v^j\left(x_t^j\right)\cdot z_t\leq\left(\partial v^j\left(x_t^i\right)\cdot x_t\right)\pi\cdot z_t;$$

that is, if $\partial v^j\left(x_t^i\right)\cdot x_t>0$,

$$\left(\frac{1}{1+\rho^i}\right)^t\left(\frac{\partial v^i\left(x_t^i\right)\cdot x_t}{\partial v^j\left(x_t^i\right)\cdot x_t}\right)\partial v^j\left(x_t^j\right)\cdot z_t\leq\pi\cdot z_t.$$

Therefore, it is satisfied if each v^i has bounded uniformly strictly positive derivatives.

2.4 Prices

Markets are sequentially open for commodities, (one-period) bonds and balances. Prices of commodities, p, are a positive map on $\mathcal{T} \times \mathcal{N}$. These are present value prices in terms of balances at the initial period of trade. Nominal rates of interest, r, are a positive real map on $\mathcal{T}$. (Nominal rates of interest are positive as, short sales of bonds being allowed, arbitrage opportunities would otherwise emerge.) They uniquely define state prices, a, a positive real map on $\mathcal{T}$, through $a_0 = 1$ and, for every t, $(1 + r_t)\, a_{t+1} = a_t$. Such a relation between nominal rates of interest and state prices is always implicitly assumed. Given nominal rates of interest, spot prices of commodities are obtained by

$$\left(\ldots, \frac{1}{a_t} p_t, \ldots \right).$$

The choice of expressing prices in terms of present values only reflects the need for a simplified notation.

2.5 Monetary policy

A central bank varies the supply of balances intertemporally by trading balances against bonds. A monetary policy consists of a given supply of balances, m, a positive real map on $\mathcal{T}$.

Assumption 4 (Monetary policy). *Monetary policy m is uniformly strictly positive.*

Transfers, h, are a positive real map on $\mathcal{T}$. The central bank runs a balanced budget by distributing its profit to individuals. This requires

$$\sum_t \left(\frac{r_t}{1 + r_t} \right) a_t m_t = \sum_t a_t h_t.$$

The left-hand side is the seignorage, that is, the profit of the central bank. The right-hand side is the overall intertemporal transfer to individuals. Transfers are distributed to individuals according to given shares,

$$(\ldots, \mu^i, \ldots) \gg 0,$$

with $\sum_i \mu^i = 1$. (Notice that, to simplify, I assume that shares are strictly positive.) The trade of the central bank in bonds needs not be made explicit for the purposes of this paper.

2.6 Budget constraints

In every period of trade, t, individual i disposes of some accumulated wealth, w_t^i. Such an individual trades in balances, n_t^i, and bonds, b_t^i, subject to a sequential budget constraint,

$$n_t^i + \left(\frac{1}{1+r_t}\right) b_t^i \leq w_t^i + \mu^i h_t.$$

Commodities are then traded subject to a cash-in-advance constraint,

$$\frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i\right)^+ \leq n_t^i.$$

Individual i's holdings of balances, after trades in commodities occurred, is

$$m_t^i = n_t^i - \frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i\right)^+ + \frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i\right)^- = n_t^i - \frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i\right).$$

Available wealth in the following period of trade is, thus,

$$w_{t+1}^i = m_t^i + b_t^i.$$

An additional solvency constraint requires

$$-\frac{1}{a_{t+1}} \sum_{s \geq t+1} \left(\frac{1}{1+r_s}\right) p_s \cdot e_s^i - \mu^i \frac{1}{a_{t+1}} \sum_{s \geq t+1} a_s h_s \leq w_{t+1}^i,$$

so as to avoid Ponzi Games (Santos and Woodford [14]). Finally, at the initial period of trade, $w_0^i = 0$.

Notice that, under market clearing for commodities, the aggregate holding of balances, before and after transactions in commodities, are related by

$$\sum_i m_t^i = \sum_i n_t^i - \frac{1}{a_t} p_t \cdot \sum_i \left(x_t^i - e_t^i\right) = \sum_i n_t^i,$$

so that the demand for balances can be indifferently expressed as $\sum_i m_t^i$ or $\sum_i n_t^i$, with the former being preferred in the following discussion.

The above sequence of constraints can be equivalently consolidated into a single budget constraint of the form

$$\sum_t p_t \cdot \left(x_t^i - e_t^i\right)^+ \leq \mu^i \sum_t a_t h_t + \sum_t \left(\frac{1}{1+r_t}\right) p_t \cdot \left(x_t^i - e_t^i\right)^-,$$

with the underlying demand for balances, m^i, given by any positive real map on $\mathcal{T}$ that satisfies, for every t,

$$p_t \cdot \left(x_t^i - e_t^i\right)^- \leq a_t m_t^i,$$
$$\left(\frac{r_t}{1+r_t}\right) p_t \cdot \left(x_t^i - e_t^i\right)^- = \left(\frac{r_t}{1+r_t}\right) a_t m_t^i.$$

The demand for balances is indeed indeterminate when the interest rate vanishes, as balances and bonds become perfect substitutes for the purposes of intertemporal wealth transfers. The demand for bonds needs not be made explicit, as bond markets clear as a residual by a sequential application of Walras' Law.

Letting

$$q = \left(\dots , \left(\frac{1}{1+r_t} \right) p_t, \dots \right)$$

and adopting a standard vector notation, the single budget constraint is

$$p \cdot \left(x^i - e^i \right)^+ \leq \delta^i + q \cdot \left(x^i - e^i \right)^- ,$$

where δ^i is the present value of transfers to the individual. In addition, as nominal rates of interest are positive, $p \geq q$ and, hence, the budget set is convex.

These basic findings are discussed, among others, in Bloise, Drèze and Polemarchakis [4]. The way is which balances circulate is exactly that of Woodford [16], the difference with that work relying only in the hypothesis that the supply of balances is not varied by lump-sum transfers to individuals.

2.7 Equilibrium

Given a monetary policy, m, an equilibrium consists of prices, p, nominal rates of interest, r, and an allocation, $\left(\dots , x^i, \dots \right)$, such that the following conditions are satisfied.

(a) Consumption plans are individually optimal subject to the budget constraint, that is, for every individual i,

$$\sum_t p_t \cdot \left(x_t^i - e_t^i \right)^+ \leq \delta^i + \sum_t \left(\frac{1}{1+r_t} \right) p_t \cdot \left(x_t^i - e_t^i \right)^-$$

and

$$z^i \succ^i x^i \text{ implies } \sum_t p_t \cdot \left(z_t^i - e_t^i \right)^+ > \delta^i + \sum_t \left(\frac{1}{1+r_t} \right) p_t \cdot \left(z_t^i - e_t^i \right)^- ,$$

where

$$\delta^i = \mu^i \sum_t \left(\frac{r_t}{1+r_t} \right) p_t \cdot \sum_i \left(x_t^i - e_t^i \right)^- .$$

(b) Markets clear for balances and commodities, that is, in every period of trade t,

$$\sum_i x_t^i = \sum_i e_t^i$$

and

$$p_t \cdot \sum_i \left(x_t^i - e_t^i \right)^- \leq a_t m_t,$$

$$\left(\frac{r_t}{1+r_t} \right) p_t \cdot \sum_i \left(x_t^i - e_t^i \right)^- = \left(\frac{r_t}{1+r_t} \right) a_t m_t.$$

It is clear that, by the very definition on an equilibrium, jointly with the hypothesis of continuous preferences, $\sum_t (1 + r_t)^{-1} p_t \cdot e_t^i$ is finite for every individual. This, however, does not guarantee by itself that also $\sum_t p_t \cdot e_t^i$ is finite for every individual, unless nominal rates of interest are uniformly bounded.

2.8 Comments

Though the presentation is extremely concise, the notion of equilibrium is faithful to Drèze and Polemarchakis [6]. With further restrictions, it also precisely reproduces the canonical notion of equilibrium provided, for instance, by Woodford [16] for a cash-in-advance economy with a representative individual. A canonical equilibrium assumes that balances are varied by lump-sum transfers to individuals, thus requiring, in addition to conditions (a)-(b), that

$$\begin{aligned}\sum_t \left(\frac{r_t}{1+r_t}\right) p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- &= m_0 + \sum_t a_{t+1} \left(m_{t+1} - m_t\right) \\ &= \sum_t \left(\frac{r_t}{1+r_t}\right) a_t m_t + \lim_t a_t m_t.\end{aligned}$$

The notion of equilibrium of Dubey and Geanakoplos [7,8] distinguishes between inside and outside money, thus requiring, in addition to conditions (a)-(b), that

$$\sum_t \left(\frac{r_t}{1+r_t}\right) p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = \sum_t a_t h_t,$$

where $h \geq 0$ is interpreted as outside money, that is lump-sum transferred to individuals, and $m - h \geq 0$ as inside money, that is traded by the central bank against bonds. Importantly, both m and h are there considered as given exogenously. The case of Woodford [16], when the supply of balances never declines, can be reduced to a particular case of (my formulation of) Dubey and Geanakoplos [7,8] over an infinite horizon.

3 Existence and indeterminacy

The result of this paper asserts the existence of an equilibrium provided that monetary policy is uniformly strictly positive. In addition, the overall price level is indeterminate.

Proposition 1 (Indeterminacy of equilibrium). *Under assumptions 1-4, there exists an equilibrium for every price normalization*

$$\sum_t p_t \cdot e_t = 1,$$

where e is any uniformly strictly positive bounded real map on $\mathcal{T} \times \mathcal{N}$.

This proposition does not establish that the multiplicity of overall price levels translates into a multiplicity of real equilibrium allocations. This, for instance, does not happen if the initial allocation is Pareto efficient, as nominal rate of interest vanishes and no exchange occurs in every period of trade. If there are gains to trade, instead, equilibrium prices and allocation vary with the overall price level, since market clearing for balances (condition (b)) are not homogeneous in price levels. To this purpose, it suffices to assume that the initial allocation is not Pareto efficient.

4 Remarks

4.1 Monetary versus *Walrasian equilibrium*

It is to be noticed that my existence proof would be trivial without the additional quantifier on the overall price level. Indeed, suppose that the economy admits a Walrasian equilibrium. This can be interpreted as an equilibrium with nominal rates of interest vanishing everywhere, provided that market clearing for balances is satisfied. As a Walrasian equilibrium obtains with a purely nominal multiplicity of prices, $\{\lambda p : \lambda > 0\}$, one might assume that, for some $\epsilon > 0$ small enough,

$$p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- \leq \epsilon$$

holds for every t. As monetary policy is uniformly strictly positive, this indeed shows that the Walrasian equilibrium is also an equilibrium as defined in this paper.

4.2 Canonical equilibrium

One might easily show that, if nominal rates of interest are strictly positive at equilibrium, then $\lim_t a_t m_t = 0$. Indeed, for every t, market clearing for balances implies

$$a_t m_t = p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^+,$$

where market clearing for commodities is used to obtain the second equality. Hence, $\sum_t a_t m_t$ is finite by intertemporal Walras' Law, so proving the claim. As a conclusion, if nominal rates of interest are strictly positive, an equilibrium is canonical.

4.3 Minimum income

As nominal rates of interest might not be bounded uniformly, a difficulty in the proof emerges because of a minimum income problem. Indeed, the budget constraint for an individual takes the form

$$\sum_t p_t \cdot \left(x_t^i - e_t^i\right)^+ \leq \delta^i + \sum_t \left(\frac{1}{1+r_t}\right) p_t \cdot \left(x_t^i - e_t^i\right)^-.$$

By varying nominal rates of interest, the right-hand side of the above inequality can be made arbitrarily small, even though the endowment is uniformly strictly positive and prices are strictly positive. One is then induced to exploit the intuition that, if nominal rates of interest grow unboundedly, then seignorage remains strictly positive, $\delta^i = \mu^i \sum_i \delta^i > 0$, which explains my requirement of a uniformly strictly positive monetary policy.[4]

5 Proof

5.1 Overview

The proof uses truncations with a limit argument. An equilibrium exists in any truncated economy over a finite horizon. The limit of such truncated equilibria is an equilibrium over the infinite horizon. Neither steps are trivial extensions of the corresponding ones for Walrasian economies.

5.2 Method of truncation

I here describe the method of truncation. Concerning notation, if x lies in $\ell(\mathcal{T} \times \mathcal{N})$, then $x\chi_t$ is its truncation at t, that is, $(x\chi_t)_s = x_s$, if $0 \leq s \leq t$, and $(x\chi_t)_s = 0$, otherwise. For a given s in $\mathcal{T}$, let X^{is} be the truncated consumption space for individual i, namely,

$$X^{is} = \left\{x^i \in X^i : x^i - x^i\chi_s = e^i - e^i\chi_s\right\},$$

where $X^i = \ell_+^\infty(\mathcal{T} \times \mathcal{N})$. Prices are

$$P^s = \{p \in P : p - p\chi_s = 0\},$$

where

$$P = \left\{p \in \ell_+(\mathcal{T} \times \mathcal{N}) : \sum_t p_t \cdot e_t = 1\right\}$$

is the space of normalized prices. (Notice that e is given in the statement of the proposition.) Let A be the space of positive real maps, a, on $\mathcal{T}$ such that $a_0 = 1$ and, for every t, $0 \leq a_{t+1} \leq a_t$. Finally, let V be the space of positive maps, v, on $\mathcal{T}$ such that, for every t, $0 \leq v_t \leq 1$. If $0 \leq v_t < 1$ for every t, then nominal rates of interest are obtained by $r_t = (1 + r_t)\, v_t$ for every t. Every v in V induces a unique a in A by setting $a_0 = 1$ and, for every t, $(1 - v_t)\, a_t = a_{t+1}$. Keeping in mind all these definitions, I omit the index for truncation in the following analysis of Section 5.3 and simply treat all the original vector spaces as if they were of finite dimension. Finally, as this does not involve any loss of generality, I assume that $e - \sum_i e^i$ is uniformly strictly positive.

[4] Alternatively, one could use the bounds on marginal rates of substitution arising from assumption 3. I prefer a limited use of those restrictions to point out that many of the results (in particular, over a finite horizon) are independent of them.

5.3 Truncated equilibria

The consumption space of every individual, i, is truncated by defining

$$\hat{X}^i = \left\{x^i \in X^i : x^i \leq e\right\},$$

which is a nonempty convex compact set. Let $Z = \cdots \times \hat{X}^i \cdots \times P \times V$, which is a nonempty convex compact set, with typical element $z = \left(\ldots, x^i, \ldots, p, v\right)$.

For every individual, i, define the correspondences $\beta^i : Z \mapsto \hat{X}^i$ and $\gamma^i : Z \mapsto \hat{X}^i$ by the rules

$$\beta^i\left(z\right) = \left\{\bar{x}^i \in \hat{X}^i : \sum_t p_t \cdot \left(\bar{x}^i_t - e^i_t\right)^+ < \delta^i + \sum_t \left(1 - v_t\right) p_t \cdot \left(\bar{x}^i_t - e^i_t\right)^-\right\}$$

and

$$\gamma^i\left(z\right) = \left\{\bar{x}^i \in \hat{X}^i : \sum_t p_t \cdot \left(\bar{x}^i_t - e^i_t\right)^+ \leq \delta^i + \sum_t \left(1 - v_t\right) p_t \cdot \left(\bar{x}^i_t - e^i_t\right)^-\right\},$$

where it is understood that

$$\delta^i = \mu^i \sum_t v_t p_t \cdot \sum_i \left(x^i_t - e^i_t\right)^- .$$

Notice that β^i is open, though its values might be empty. Now define a correspondence $\phi^i : Z \mapsto \hat{X}^i$ by

$$\phi^i\left(z\right) = \begin{cases} \beta^i\left(z\right), & \text{if } x^i \notin \gamma^i\left(z\right), \\ \left\{\bar{x}^i \in \hat{X}^i : \bar{x}^i \succ^i x^i\right\} \cap \beta^i\left(z\right), & \text{if } x^i \in \gamma^i\left(z\right). \end{cases}$$

This correspondence is open with convex values and, in addition, for every z in Z, x^i cannot be an element of $\phi^i\left(z\right)$. Also, define a correspondence $\phi^0 : Z \mapsto P \times V$ by

$$\phi^0\left(z\right) = \left\{\left(\bar{p}, \bar{v}\right) \in P \times V : f\left(z, \bar{p}, \bar{v}; m\right) > 0\right\},$$

where

$$\begin{aligned} f\left(z, \bar{p}, \bar{v}; m\right) = & \sum_t \left(\bar{p}_t - p_t\right) \cdot \sum_i \left(x^i_t - e^i_t\right) \\ & + \sum_t \left(\bar{v}_t - v_t\right) \left(p_t \cdot \sum_i \left(x^i_t - e^i_t\right)^- - a_t m_t\right). \end{aligned}$$

This correspondence is open with convex values and, for every z in Z, (p, v) is not an element of $\phi^0\left(z\right)$. Recall that a is inductively constructed by setting $a_0 = 1$ and, for every t, $a_{t+1} = \left(1 - v_t\right) a_t$. By Gale and Mas-Colell's [9, 10] Fixed Point Theorem, there is z in Z such that $\phi\left(z\right) = \cdots \times \phi^i\left(z\right) \times \cdots \times \phi^0\left(z\right) = \cdots \times \emptyset \times \cdots \times \emptyset$.

I now prove that the above fixed point, z, is indeed an equilibrium. The proof proceeds along the lines of Gale and Mas-Colell's [9, 10], provided that $\beta^i(z)$ is shown to be nonempty for every individual i. Suppose that, for some individual, i, $\beta^i(z)$ is empty. Since $\mu^i > 0$, it follows that

$$\sum_t v_t p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = 0.$$

Suppose that $a_t m_t > 0$. If $v_t = 1$, then

$$p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = 0$$

and, hence, setting $\bar{v}_t = 0$ for such a t would imply that $\phi^0(z)$ is nonempty, a contradiction. Hence, $0 \leq v_t < 1$, which implies $a_{t+1} m_{t+1} > 0$. Since $a_0 m_0 > 0$, this shows that $0 \leq v_t < 1$ for every t and, thus,

$$\sum_t (1 - v_t) p_t \cdot e_t^i \geq \sum_t \left(1 - \|v\|_\infty\right) p_t \cdot e_t^i > 0$$

because $p > 0$ and $e^i \gg 0$. One concludes that $\beta^i(z)$ is nonempty for every individual i, which implies that x^i is an element of $\gamma^i(z)$ as $\phi^i(z)$ is empty.

Summing budget constraints over individuals, one obtains

$$\sum_t p_t \cdot \sum_i \left(x_t^i - e_t^i\right) =$$
$$\sum_t p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^+ - \sum_t (1 - v_t) p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- - \sum_i \delta^i \leq 0.$$

Since $\phi^0(z)$ is empty, one concludes that $\sum_i x^i \leq \sum_i e^i$. In particular, this implies that x^i is in the relative interior of $\hat{X}^i$ for every individual i. As preferences are strictly monotone, it follows that $p \gg 0$ and that budget constraints hold with the equality. If $\sum_i x^i < \sum_i e^i$, since $p \gg 0$, one obtains a contradiction. Hence, $\sum_i x^i = \sum_i e^i$.

Suppose that there is $\bar{x}^i \succ^i x^i$ such that

$$\sum_t p_t \cdot \left(\bar{x}_t^i - e_t^i\right)^+ \leq \delta^i + \sum_t (1 - v_t) p_t \cdot \left(\bar{x}_t^i - e_t^i\right)^- .$$

By convexity of preferences and the balancedness of the allocation, $\sum_i x^i = \sum_i e^i$, one might assume that $\bar{x}^i$ is in the truncated consumption space $\hat{X}^i$ and, hence, in $\gamma^i(z)$. As $\beta^i(z)$ is nonempty and preferences are continuous, this contradicts the fact that $\phi^i(z)$ is empty.

It remains to show market clearing for balances. Suppose that $a_t m_t > 0$ and $v_t = 1$. By the optimality of consumption plans of individuals, it follows that $\left(x_t^i - e_t^i\right)^- = 0$ for every individual i and, hence, that

$$p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = 0.$$

Setting $\bar{v}_t = 0$ contradicts the fact that $\phi^0(z)$ is empty. Hence, $a_{t+1}m_{t+1} > 0$. Since $a_0 m_0 > 0$, this argument shows that $0 \le v_t < 1$ for every t, so that well-defined nominal rates of interest obtain from

$$v_t = \frac{r_t}{1 + r_t}.$$

Suppose that, for some t,

$$p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- > a_t m_t > 0.$$

Setting $\bar{v}_t = 1$, again, contradicts the fact that $\phi^0(z)$ is empty. Suppose that, for some t,

$$a_t m_t > p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- \ge 0.$$

If $r_t > 0$, then $v_t > 0$ and setting $\bar{v}_t = 0$ contradicts the fact that $\phi^0(z)$ is empty.

5.4 Limit

Consider a sequence of truncated equilibrium allocations, $\left(\ldots, x^{is}, \ldots\right)$, with associated prices p^s and

$$q^s = (\ldots, q_t^s \ldots) = \left(\ldots, \left(\frac{1}{1 + r_t^s}\right) p_t^s, \ldots\right).$$

For every s, p^s and q^s can be regarded as positive linear functionals on $\ell^\infty(\mathcal{T} \times \mathcal{N})$. By a standard argument dating back to Bewley [2], $p^s \to p \ge 0$ and $q^s \to q \ge 0$, where convergence is understood to mean that, for every x in $\ell^\infty(\mathcal{T} \times \mathcal{N})$, $p^s \cdot x \to p \cdot x$ and $q^s \cdot x \to q \cdot x$. Both p and q are norm continuous positive linear functionals on $\ell^\infty(\mathcal{T} \times \mathcal{N})$. In addition, by Yosida-Hewitt Decomposition Theorem, $p = p_f + p_b$ and $q = q_f + q_b$, where p_f (q_f) is a Mackey continuous positive linear functional on $\ell^\infty(\mathcal{T} \times \mathcal{N})$ and p_b (q_b) is a positive purely finitely additive measure on $\ell^\infty(\mathcal{T} \times \mathcal{N})$, vanishing on $\ell^0(\mathcal{T} \times \mathcal{N})$, the vector space of all real maps on $\mathcal{T} \times \mathcal{N}$ that are zero at all but finitely many t. In particular, this implies that, for every x in $\ell^\infty(\mathcal{T} \times \mathcal{N})$,

$$p_f \cdot x = \sum_t p_t \cdot x_t$$

and

$$q_f \cdot x = \sum_t q_t \cdot x_t,$$

as p_f and q_f can be identified with positive elements of $\ell^1(\mathcal{T} \times \mathcal{N})$. In addition, as $p^s \cdot e = 1$ for every s, $p \cdot e = 1$, which implies that $p > 0$. By Tychonoff Theorem, $a^s \to a$, $\left(\ldots, \delta^{is}, \ldots\right) \to \left(\ldots, \delta^i, \ldots\right)$ and $\left(\ldots, x^{is}, \ldots\right) \to \left(\ldots, x^i, \ldots\right)$ in the product topology. Notice that the Mackey and the product topology coincide on bounded subsets of $\ell^\infty(\mathcal{T}\times\mathcal{N})$ and, hence, $x^{is}\to x^i$ in the Mackey topology for every individual i. A useful reference for technical details on the Mackey topology and the mentioned decomposition theorem is Aliprantis and Border [1].

Lemma 1. $z^i \succeq^i x^i$ *implies*

$$p\cdot\left(z^i-e^i\right)^+\geq\delta^i+q\cdot\left(z^i-e^i\right)^-.$$

Proof. Let $0<v^i\in\ell^\infty(\mathcal{T}\times\mathcal{N})$. By continuity of preferences, $z^i\chi_s+v^i\chi_s+e^i-e^i\chi_s\succ^i x^{is}$ for all s large enough and, therefore,

$$p^s\cdot\left(z^i+v^i-e^i\right)^+>\delta^{is}+q^s\cdot\left(z^i+v^i-e^i\right)^-.$$

In the limit,

$$p\cdot\left(z^i+v^i-e^i\right)^+\geq\delta^i+q\cdot\left(z^i+v^i-e^i\right)^-.$$

Letting $v^i\to 0$ in the norm topology and noticing that lattice operations are norm continuous, one obtains the conclusion. □

Lemma 2. *If* $\delta^i+q\cdot e^i>0$*,* $z^i\succ^i x^i$ *implies*

$$p\cdot\left(z^i-e^i\right)^+>\delta^i+q\cdot\left(z^i-e^i\right)^-.$$

Proof. For some $1>\lambda>0$ small enough, $\lambda z^i\succ^i x^i$ and, so, by Lemma 1,

$$\lambda p\cdot\left(z^i-e^i\right)+(p-q)\cdot\left(\lambda z^i-e^i\right)^-\geq\delta^i+(1-\lambda)\,p\cdot e^i.$$

As $p\geq q$, $(p-q)\cdot x^-$ is convex in x, which leads to

$$\lambda p\cdot\left(z^i-e^i\right)+\lambda\,(p-q)\cdot\left(z^i-e^i\right)^-\geq\lambda\delta^i+(1-\lambda)\left(\delta^i+q\cdot e^i\right)>0,$$

so proving my claim. □

Lemma 3. $\sum_i\delta^i+q\cdot\sum_i e^i>0$ *and, hence,* $\delta^i+q\cdot e^i>0$ *for every individual* i*.*

Proof. Suppose not. Let $\mathcal{T}_+^s=\{t\in\mathcal{T}:r_t^s>0\}$ and $\mathcal{T}_-^s=\{t\in\mathcal{T}:r_t^s=0\}$, so that $\mathcal{T}=\mathcal{T}_+^s\cup\mathcal{T}_-^s$. (One might assume that $t>s$ implies $r_t^s=0$.) If $\mathcal{T}_+^s=\emptyset$ for every s, possibly extracting a subsequence, then $p=q>0$ and, hence, $q\cdot e^i=p\cdot e^i>0$. It follows that one might assume, without loss of generality, that $\mathcal{T}_+^s\neq\emptyset$ for every s. Observe that, by market clearing,

$$\sum_i\left(x^{is}-e^i\right)^+=\sum_i\left(x^{is}-e^i\right)^-;$$

$a_tm_t=p_t\cdot\sum_i\left(x^{is}-e^i\right)^-$ for every t in $\mathcal{T}_+^s$; $a_t=1$ for some t in $\mathcal{T}_+^s$. Aggregating budget constraints, for every s,

$$\sum_i\delta^{is}+q^s\cdot\sum_i e^i\geq p^s\cdot\sum_i\left(x^{is}-e^i\right)^+\geq\sum_{t\in\mathcal{T}_+^s}a_tm_t\geq\epsilon\sum_{t\in\mathcal{T}_+^s}a_t\geq\epsilon>0,$$

which delivers a contradiction. Hence, either $\sum_i\delta^i>0$ or $q\cdot\sum_i e^i>0$ or both. The conclusion follows from the observation that each e^i is uniformly strictly positive and that $\delta^i=\mu^i\sum_i\delta^i$ with $\mu^i>0$. □

Remark 2. The above lemma shows that, in the limit, the income of every individual is strictly positive. It uses the fact that monetary policy is uniformly strictly positive and, in fact, this assumption is not used anywhere else.

Lemma 4. *For every individual i,*

$$p_f \cdot \left(x^i - e^i\right)^+ \leq \delta^i + q_f \cdot \left(x^i - e^i\right)^-$$

and $z^i \succ^i x^i$ implies

$$p_f \cdot \left(z^i - e^i\right)^+ > \delta^i + q \cdot \left(z^i - e^i\right)^- .$$

Proof. Observe that, along the sequence of truncated equilibria,

$$(p^s - q^s) \cdot \sum_i \left(x^{is} - e^i\right)^- \chi_t \leq \sum_i \delta^{is}$$

holds for every given t. In the limit $s \to \infty$, one obtains

$$(p_f - q_f) \cdot \sum_i \left(x^i - e^i\right)^- \chi_t \leq \sum_i \delta^i$$

and, hence, letting $t \to \infty$,

$$(p_f - q_f) \cdot \sum_i \left(x^i - e^i\right)^- \leq \sum_i \delta^i. \tag{5.1}$$

If $z^i \succ^i x^i$, then $e^i + \left(z^i - e^i\right)^+ \chi_t - \left(z^i - e^i\right)^- \succ^i x^i$ for every t large enough. By Lemma 2, this implies

$$p_f \cdot \left(z^i - e^i\right)^+ \chi_t > \delta^i + q \cdot \left(z^i - e^i\right)^-$$

and, so, letting $t \to \infty$,

$$p_f \cdot \left(z^i - e^i\right)^+ > \delta^i + q \cdot \left(z^i - e^i\right)^- \geq \delta^i + q_f \cdot \left(z^i - e^i\right)^- .$$

As preferences and lattice operations are Mackey continuous, it follows that

$$p_f \cdot \left(x^i - e^i\right)^+ \geq \delta^i + q_f \cdot \left(x^i - e^i\right)^- . \tag{5.2}$$

Therefore, using inequality (5.1), one obtains the equality in (5.2), so proving the claim. □

Lemma 5. $q_b = 0$.

Proof. Suppose not and assume that $\left(x^i - e^i\right)^+ > 0$. (If not, then $x^i = e^i$ and the proof requires only minimal changes.) There is then $\alpha > 0$ such that $\alpha p_f \cdot \left(x^i - e^i\right)^+ \leq q_b \cdot e^i$. For t large enough, $x^i \chi_t + \alpha \left(x^i - e^i\right)^+ \chi_t \succ^i x^i$ and, hence,

$$\begin{aligned} p_f \cdot \left(x^i - e^i\right)^+ + \alpha p_f \cdot \left(x^i - e^i\right)^+ &\geq p_f \cdot \left(x^i - e^i\right)^+ \chi_t + \alpha p_f \cdot \left(x^i - e^i\right)^+ \chi_t \\ &> \delta^i + q_f \cdot \left(x^i - e^i\right)^- \chi_t + q \cdot \left(e^i - e^i \chi_t\right) \\ &\geq \delta^i + q_f \cdot \left(x^i - e^i\right)^- + q_b \cdot e^i, \end{aligned}$$

which is a contradiction. □

Remark 3. Without Lemma 6, which is the only place where assumption 3 is used, the existence of an equilibrium, with prices p_f, would follow from the conclusive argument in this proof. However, since $p_f \cdot e = 1 - p_b \cdot e$, the normalization of prices could be lost and, so, as far as the indeterminacy of equilibrium is concerned, proving that $p_b = 0$ becomes necessary.

Lemma 6. $p_b = 0$.

Proof. By a standard convex analysis technique, it is easy to prove that, for every s, there is $\lambda^{is} > 0$ such that, for every z^i in $\ell_+^\infty(\mathcal{S} \times \mathcal{N})$,

$$u^i\left(z^i\chi_s + e^i - e^i\chi_s\right) - u^i\left(x^{is}\right) \leq \lambda^{is}\left(p^s \cdot \left(z^i - e^i\right)^+ - \delta^{is} - q^s \cdot \left(z^i - e^i\right)^-\right).$$

In particular, observe that

$$\frac{u^i\left(e\chi_s + e^i\right) - u^i\left(x^{is}\right)}{p^s \cdot e - \delta^{is}} \leq \lambda^{is} \leq \frac{u^i\left(x^{is}\right) - u^i(0)}{\delta^{is} + q^s \cdot e^i}.$$

Hence, without loss of generality, I shall assume that $1 \leq \lambda^{is}$ for every individual i.

Using differentiability, it follows that, for z in $\ell_+^\infty(\mathcal{T} \times \mathcal{N})$,

$$\partial u^i\left(x^{is}\right) \cdot z\chi_s \leq \lambda^{is} p^s \cdot z; \tag{5.3}$$

for z in $\ell_+^\infty(\mathcal{T} \times \mathcal{N})$, with $\alpha z \leq x^{is}$ for some $\alpha > 0$,

$$\lambda^{is} q^s \cdot z \leq \partial u^i\left(x^{is}\right) \cdot z\chi_s; \tag{5.4}$$

for z in $\ell_+^\infty(\mathcal{T} \times \mathcal{N})$, with $\alpha z \leq \left(x^{is} - e^i\right)^+$ for some $\alpha > 0$,

$$\lambda^{is} p^s \cdot z \leq \partial u^i\left(x^{is}\right) \cdot z; \tag{5.5}$$

for z in $\ell_+^\infty(\mathcal{T} \times \mathcal{N})$, with $\alpha z \leq \left(x^{is} - e^i\right)^-$ for some $\alpha > 0$,

$$\partial u^i\left(x^{is}\right) \cdot z \leq \lambda^{is} q^s \cdot z. \tag{5.6}$$

By means of first-order characterization (5.3)-(5.6), for every z in $\ell_+^\infty(\mathcal{T} \times \mathcal{N})$, I shall prove that

$$p^s \cdot z = \sum_{t=0}^{s} p^s \cdot z_t \leq \sum_{t=0}^{s} \pi \cdot z_t \leq \pi \cdot z,$$

where π in $\ell_+^1(\mathcal{T} \times \mathcal{N})$ is given in assumption 3. In particular, I shall show that, for every $0 \leq t \leq s$, $p^s \cdot z_t \leq \pi \cdot z_t$ by considering two cases.

Case (i). If $r_t^s = 0$, then $p^s \cdot z_t = q^s \cdot z_t$. For some $\alpha > 0$, $\alpha z_t \leq \sum_i x^{is} = \sum_i e^i$ and, so, by Riesz Decomposition Theorem, there are $\left(\ldots, z_t^{is}, \ldots\right)$ with $\sum_i z_t^{is} = z_t$ and $0 \leq \alpha z_t^{is} \leq x^{is}$ for every individual i. By condition (5.4),

$$\begin{aligned} p^s \cdot z_t &= q^s \cdot z_t \\ &= \sum_i q^s \cdot z_t^{is} \\ &\leq \sum_i \frac{1}{\lambda^{is}} \partial u^i \left(x^{is}\right) \cdot z_t^{is} \\ &\leq \sum_i \pi \cdot z_t^{is} \\ &= \pi \cdot z_t, \end{aligned}$$

which proves the claim.

Case (ii). If $r_t^s > 0$, then

$$\sum_i \left(x_t^{is} - e_t^i\right)^+ = \sum_i \left(x_t^{is} - e_t^i\right)^- > 0$$

and, hence, there is a pair of individuals (i, j) such that

$$\left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^- > 0.$$

By conditions (5.5)-(5.6), it follows that

$$\begin{aligned} 1 + r_t^s &= \frac{p^s \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-}{q^s \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-} \\ &\leq \frac{\lambda^{js}}{\lambda^{is}} \frac{\partial u^i \left(x^{is}\right) \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-}{\partial u^j \left(x^{js}\right) \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-}. \end{aligned}$$

Decompose z_t as $z_t = v_t^s + w_t^s$, where $\alpha v_t^s \leq \sum_i \left(x^{is} - e^i\right)^+ = \sum_i \left(x^{is} - e^i\right)^-$, for some $\alpha > 0$, and $w_t^s \wedge \sum_i \left(x^{is} - e^i\right)^+ = w_t^s \wedge \sum_i \left(x^{is} - e^i\right)^- = 0$. By Riesz Decomposition Theorem, there are $\left(\ldots, v_t^{is}, \ldots\right)$ with $\sum_i v_t^{is} = v_t^s$ and $0 \leq \alpha v_t^{is} \leq \left(x^{is} - e^i\right)^+$ for every individual i. By condition (5.5), it follows that

$$\begin{aligned} p^s \cdot v_t^s &= \sum_i p^s \cdot v_t^{is} \\ &\leq \sum_i \frac{1}{\lambda^{is}} \partial u^i \left(x^{is}\right) \cdot v_t^{is} \\ &\leq \sum_i \pi \cdot v_t^{is} \\ &= \pi \cdot v_t^s. \end{aligned}$$

Also, since $\alpha w_t^s \leq x^{js}$ for some $\alpha > 0$ (indeed, e^j is strictly positive and, by construction, $w_t^s \wedge \left(x^{js} - e^j\right)^- = 0$ and $w_t^s \wedge \left(x^{js} - e^j\right)^+ = 0$), by condition (5.4),

$$\begin{aligned} p^s \cdot w_t^s &= \left(1 + r_t^s\right) q^s \cdot w_t^s \\ &\leq \left(1 + r_t^s\right) \frac{1}{\lambda^{js}} \partial u^j \left(x^{js}\right) \cdot w_t^s \\ &\leq \frac{1}{\lambda^{is}} \frac{\partial u^i \left(x^{is}\right) \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-}{\partial u^j \left(x^{js}\right) \cdot \left(x_t^{is} - e_t^i\right)^+ \wedge \left(x_t^{js} - e_t^j\right)^-} \partial u^j \left(x^{js}\right) \cdot w_t^s \\ &\leq \pi \cdot w_t^s, \end{aligned}$$

where the last inequality follows from assumption 3. This proves the claim since $p^s \cdot z_t = p^s \cdot v_t^s + p^s \cdot w_t^s \leq \pi \cdot v_t^s + \pi \cdot w_t^s = \pi \cdot z_t$.

It follows that, for every s, $p^s \leq \pi$ and, hence, $p \leq \pi$, which concludes the proof of the lemma. □

It only remains to observe that, by point-wise limits, for every t,

$$q_t = \lim_s \left(\frac{1}{1 + r_t^s}\right) p_t.$$

Hence, by induction, assume that $a_t = \lim_s a_t^s > 0$. If $r_t^s \to \infty$, then $q_t = 0$ and, thus, by the optimality of consumption plans,

$$\sum_i \left(x_t^i - e_t^i\right)^- = 0.$$

The latter implies that

$$a_t m_t > 0 = p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^-,$$

which is a contradiction as the market for balances clear along the sequence of truncated equilibria. Since $\lim_s r_t^s = r_t \geq 0$ is finite,

$$a_{t+1} = \lim_s a_{t+1}^s = \lim_s \left(\frac{1}{1 + r_t^s}\right) a_t^s = \left(\frac{1}{1 + r_t}\right) a_t > 0.$$

Noticing that $\lim_s a_0^s = a_0 = 1$, this concludes the proof of existence of an equilibrium.

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