Monetary Equilibria in a Cash-in-Advance Economy with Incomplete Financial Markets*

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Abstract

The general equilibrium model with incomplete financial markets (GEI) is extended by adding fiat money, fiscal and monetary policy and a cash-in-advance constraint. The central bank either pegs the interest rate or money supply while the fiscal authority sets a Ricardian or a non-Ricardian fiscal plan. We prove the existence of equilibria and characterize indeterminacy in all four scenarios. In Ricardian economies, the conditions required for existence are not more restrictive than in standard GEI. In non-Ricardian economies, the sufficient conditions for existence are more demanding. In the Ricardian economy, neither the price level

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nor the equivalent martingale measure are determinate.

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1 Introduction

In this paper we extend the standard general equilibrium model with incomplete financial markets by introducing fiat money and adding a public authority. The latter consists of a fiscal and a monetary authority. The fiscal authority sets a fiscal plan consisting of taxes, nominal transfers and a debt policy. The monetary authority, or the central bank, creates fiat money at zero costs and earns seignorage from its monetary policy. The actions of both authorities are linked by a common public budget constraint. The transactions technology is supposed to be a simple cash-in-advance constraint. If the nominal interest rates are positive, non-interest bearing fiat money is dominated as a store of value by an interest-bearing nominal bond. The demand for money comes from its role to facilitate trade by means of the cash-in-advance constraint within the states of the economy.

As argued in the Fiscal Theory of the Price Level (See, e.g., Woodford 1995), the introduction of a government which has to meet some budget constraint might add additional restrictions on the set of equilibria. It is well understood that this possibility depends on whether the fiscal policy is of the Ricardian or the non-Ricardian type. Following Woodford (2001), a fiscal policy is called Ricardian if the government budget is satisfied for every price vector. If the budget is valid only for some prices, it is called non-Ricardian. In the latter case, the government budget constraint adds additional restrictions on the equilibrium set.[1]

[1]The idea that a non-Ricardian policy might lead to a determinate equilibrium first ap-

We study four important combinations of fiscal and monetary policies by combining nominal interest rate peg and money supply policy of the central bank with a Ricardian and a non-Ricardian fiscal policy. For all these cases, we prove existence of an equilibrium and characterize its determinacy properties.

If the fiscal authority follows a Ricardian policy, there exist monetary competitive equilibria under assumptions which are close to the standard assumptions in GEI with financial assets. As in the standard GEI model without a central bank and a fiscal authority, the equilibrium in this Ricardian framework is not determinate. More precisely, there exists a monetary equilibrium under a Ricardian fiscal rule for every fixed positive price level and for every fixed equivalent martingale measure. This result is true for both interest rate peg and money supply policy. We argue that the indeterminacy of the price level in this Ricardian economy is purely nominal. However, the indeterminacy of the martingale measure can be expected to be real since markets are incomplete.

If the fiscal authority follows a non-Ricardian policy, existence of equilibrium requires more restrictive assumptions as compared to the Ricardian case. Loosely speaking, the existence of equilibrium requires either high enough gains to trade or positive tax returns. The intuition is that if the fiscal authority fixes nominal transfers at some predetermined and positive level, it must earn seignorage or tax returns to be able to balance its budget. If taxes are zero, then the gains to trade in the economy must be large enough to induce some positive seigniorage income for the government. We use the measure for the gains to trade introduced by Dubey and Geanakoplos (1992, 2003a) to prove existence for both interest rate peg and money supply policy even if taxes are zero.

Importantly, every obvious degree of indeterminacy we found in the Ricardian economy is lost under the assumption that the government only trades risk-

peared in Dubey and Geanakoplos (1992). They formally prove the generic local uniqueness under a particular non-Ricardian fiscal policy.

less bonds. Dubey and Geanakoplos (2006) provide a formal proof for generic local uniqueness of equilibria under such a fiscal policy. This result illustrates the role of fiscal policy for the determinacy of the equilibrium.

The main contributions of this paper to the recent literature are the following. First, we show existence and characterize indeterminacy in a cash-in-advance economy with incomplete financial market systems and a Ricardian fiscal policy.[2] Our results extend the previous findings on existence and indeterminacy in Dréze and Polemarchakis (2000), Bloise, Dréze and Polemarchakis (2005), Bloise (2006) and Nakajima and Polemarchakis (2005) under complete markets to incomplete markets. The incompleteness of the asset markets implies that the indeterminacy found in this paper can be expected to have real effects on the allocation. Second, we provide an alternative proof for existence in the non-Ricardian framework, which is first given in Dubey and Geanakoplos (2003(b)). The proof in the latter paper uses a strategic market game approach, while our proof adapts more traditional techniques in general equilibrium analysis.

The paper is organized as follows. In Section 2, we describe the monetary economy including the government and define the general equilibrium. In Section 3, we present our main results for a Ricardian economy, including both interest rate peg and money supply policy. In Section 4, we provide a parallel result for a non-Ricardian economy. In Section 5 we conclude the paper and give the proofs of all results in the Appendix.

[2] After completing the first draft of this paper, we learned that Gourdel and Triki (2005) independently studied a similar economy under interest rate peg. They obtain results similar to our Theorems 1 and 3. We will comment on this in Sections 3.3 and 4.3.

2 The model

2.1 The economy

We study an exchange economy which extends over two dates, the present time $t = 0$ and the future $t = 1$. The present is known with certainty, but at date 1 there are S possible states of nature which we index with $s \in S = \{1, \ldots, S\}$. Including the present, there are $S + 1$ states of nature lying in the set $S^* := \{0, 1, ..., S\}$. We add an accounting period at the end of each state $s \in S$ at $t = 1$. At every $s \in S^*$ there are L consumption goods indexed with $l = 1, \ldots, L$ traded at spot prices p_{sl}. We denote a consumption plan with $x = (x_0, x_1, ..., x_S) \in \mathbb{R}_+^{(S+1)L}$. All commodities are perishable.

At $t = 0$, there are asset markets for $J \leq S$ financial contracts indexed with $j = 1, \ldots, J$. Each asset is a promise to deliver $V_s^j \in \mathbb{R}_+$ units of money in every state $s \in S$ and is traded at price q_j in period zero. There is no default on such promises. The first asset is assumed to be a riskless government bond. Denote the $S \times J$-matrix of returns with V, the $S \times (J - 1)$-matrix of the returns of the risky assets with A, the $1 \times J$-vector of asset prices with q_V and the $1 \times (J - 1)$ -vector of asset prices excluding the price of the bond with q . In addition, there is fiat money, a government liability which can also be held as a store of value between the periods.

The state price of state s is as usual the present value at date 0 of one unit of income in state s. We denote these strictly positive state prices with $a = (a_0, a_1, ..., a_S) \in \mathbb{R}_{++}^{(S+1)}$, where $a_0 := 1$. The $1 \times S$-vector of state prices $(a_1, ..., a_S)$ is $\hat{a}$. It is well known that the absence of arbitrage is equivalent to asset prices satisfying $q_V = \hat{a}V$.

We define $\mu_s := a_s (1 + r_0) > 0$ for every $s \in S$. Since $\sum_{s \in S} \mu_s = 1$, μ defines a probability measure at $t = 0$. Following the terminology in the finance literature, μ is called the *equivalent martingale measure* at $t = 0$.

2.2 The households

The economy is populated by a finite set $I := \{1, ..., I\}$ of households. At $t = 0$, the asset markets open first. On this market, the household trades money $n_0^i \in \mathbb{R}_+$, riskfree government bonds $b_0^i \in \mathbb{R}$ and a portfolio of risky assets $\theta^i \in \mathbb{R}^{J-1}$. In addition, household i receives a (lump-sum) transfer $\delta^i H_0$ from the government, where $H_0 \in \mathbb{R}_+$ is the aggregate transfer from which every household i gets a share $\delta^i \in \mathbb{R}_{++}$. Therefore, household i faces the constraint

$$\frac{b_0^i}{1+r_0} + q \cdot \theta^i + n_0^i = \delta^i H_0, \tag{1}$$

where $\frac{1}{1+r_0}$ is the price of the nominal bond. In the goods markets, which open next, household i is subject to the following cash-in-advance constraint:[3]

$$p_0 \cdot (x_0^i - e_0^i)^+ \leq n_0^i. \tag{2}$$

The money at the end of $t = 0$, m_0^i, is

$$\begin{aligned} m_0^i &= \left(n_0^i - p_0 \cdot (x_0^i - e_0^i)^+\right) + p_0 \cdot (x_0^i - e_0^i)^- \\ &= n_0^i - p_0 \cdot x_0^i + p_0 \cdot e_0^i. \end{aligned} \tag{3}$$

Combining (1) and (3), we get

$$p_0 \cdot x_0^i + \frac{b_0^i}{1+r_0} + q \cdot \theta^i + m_0^i = \delta^i H_0 + p_0 \cdot e_0^i. \tag{4}$$

Equation (4) is the familiar flow budget constraint, which says that the total expenditure within one period cannot exceed the total wealth.

From (2) and (3), we get an equivalent formulation of the cash-in-advance constraint as

$$m_0^i \geq p_0 \cdot (x_0^i - e_0^i)^-. \tag{5}$$

We will use this formulation for the transactions technology because it turns out to be more convenient.

[3] We use the usual definition of the negative and the positive part of a vector: $x^+ := (\dots, \max\{x_i, 0\}, \dots)$ and $x^- := (\dots, \max\{-x_i, 0\}, \dots)$ so that $x = x^+ - x^-$.

The wealth tax of person i on endowments in $t = 0$, $p_0 \cdot \tau_0^i$ with $\tau_0^i \in \mathbb{R}_+^L$, is paid at the end of period zero or equivalently at the beginning of period one. In $t = 1$, in every state the asset markets open again to allow agents to borrow against their income which they receive at the end of this period. However, there is no uncertainty involved anymore, i.e. each state $s \in S$ has only one successor state which we call the end of period $t = 1$ and which serves for accounting purposes only. We assume that a riskfree bond can be traded at each $s \in S$. Denoting the quantity of bonds with b_s^i and the transfer in state $s \in S$ with $\delta^i H_s$, the flow budget constraint reads

$$p_s \cdot x_s^i + \frac{b_s^i}{1 + r_s} + m_s^i = b_0^i + A_s \cdot \theta^i + m_0^i + \delta^i H_s + p_s \cdot e_s^i - p_0 \cdot \tau_0^i, \qquad (6)$$

and the cash-in-advance constraint is

$$m_s^i \geq p_s \cdot (x_s^i - e_s^i)^-. \qquad (7)$$

At the end of $t = 1$, the only economic activity is the payment of the debt and of the income tax in state $s \in S$, $p_s \cdot \tau_s^i$. Therefore, the terminal condition is

$$0 = b_s^i + m_s^i - p_s \cdot \tau_s^i, \ \forall s \in S. \qquad (8)$$

Denote with e^i the vector $(e_s^i)_{s \in S^*}$ and with τ^i the vector $(\tau_s^i)_{s \in S^*}$. The budget set of every household i is the set [4]

$$B^i(p,q,r,H) := \left\{ (x^i, m^i, b^i, \theta^i) \in \mathbb{R}_+^{(S+1)L} \times \mathbb{R}_+^{S+1} \times \mathbb{R}^{S+1} \times \mathbb{R}^{J-1} \,\middle|\, (4) - (8) \text{ hold} \right\}$$

For the later analysis, it is convenient to derive an intertemporal version of the budget constraint by using the no-arbitrage conditions. First, combine (8) and (6) to get $p_s \cdot x_s^i + \frac{p_s \cdot \tau_s^i}{1+r_s} + \frac{r_s}{1+r_s} m_s^i = b_0^i + A_s \cdot \theta^i + m_0^i + \delta^i H_s + p_s \cdot e_s^i - p_0 \cdot \tau_0^i$. Multiply each such equation with its respective state price

[4] To save the notation, we suppress the parameters in the notation. The budget set should always be understood as $B^i(p,q,r,H) := B^i(p,q,r,H; e^i, \tau^i, \delta^i)$.

and add these equations up together with (4). Use the no-arbitrage condition $q_V = (\frac{1}{1+r_0}, q_2, \ldots, q_J) = \hat{a}V = \hat{a}(\mathbf{1}, V_2, \ldots, V_J)'$[5] in this sum to get

$$a \cdot \left(p \Box x^i\right) + \frac{r}{1+r} \cdot \left(a \Box m^i\right) = a \cdot \left(\delta^i H + p \Box \left(e^i - \frac{1}{1+r} \Box \tau^i\right)\right), \quad (9)$$

where $m \Box n := (m_s \cdot n_s)_{s \in S^*}$. The left hand side is the expenditure in terms of its date-0 value, while the right hand side is the discounted nominal wealth. By optimality, $\frac{r_s}{1+r_s} m_s^i = \frac{r_s}{1+r_s} p_s \cdot (x_s^i - e_s^i)^-$ for all $s \in S^*$. Using this in the intertemporal constraint, we can rewrite (9) as

$$a \cdot \left(p \Box (x^i - e^i)^+\right) = a \cdot \left(\delta^i H + \frac{1}{1+r} \Box \left(p \Box \left((x^i - e^i)^- - \tau^i\right)\right)\right). \quad (10)$$

Every household $i \in I$ gets utility from consuming in every node $s \in S^*$ according to a function $u^i : \mathbb{R}_+^{S^* L} \to \mathbb{R}$. We make the following general assumptions on asset markets and the household sector:

Assumption 1 *For each consumer i, the utility function u^i is continuous, quasi-concave and strictly increasing.*

Assumption 2 *Every household has some endowments after tax in every state, i.e., $\forall i \in I$, $\left(e_s^i - \tau_s^i\right) > 0$ for every $s \in S^*$. Household one has strictly positive endowments after tax at every node, i.e. $\left(e^1 - \tau^1\right) \gg 0$. Aggregate endowments are bounded, i.e. $\sum_i \left(e^i - \tau^i\right) \ll +\infty$.*[6]

Assumption 3 *$rank\,(V) = J \leq S$. There exists a riskfree asset b_s at each $s \in S^*$.*

2.3 The government

At each state $s \in S^*$, the government taxes the household and distributes transfers. We denote the total commodity tax by $\tau := (\tau_s)_{s \in S^*} \in \mathbb{R}_+^{(S+1)L}$, where

[5] The $S \times 1$-vector with 1 in every coordinate is $\mathbf{1}$.

[6] A vector $x \in \mathbb{R}^n$ satisfies $x > 0$ if and only if $x_i \geq 0$, $\forall i = 1, \ldots, n$, and if there is a j such that $x_j > 0$. Accordingly, $x \gg 0$ if and only if $x_i > 0$, $\forall i = 1, \ldots, n$ and $x \geq 0$ if and only if $x_i \geq 0$, $\forall i = 1, \ldots, n$.

$\tau_s := \sum_{i=1}^{I} \tau_s^i$. The total lump-sum transfer is the vector $H := (H_s)_{s \in S^*} \in \mathbb{R}_+^{S+1}$. For simplicity, we assume throughout the paper that the transfer is distributed according to the shares $(\delta^i)_{i \in I}$, $\sum_i \delta^i = 1$.

The government trades riskfree bonds $B = (B_s)_{s \in S^*}$ and supplies balances $M = (M_s)_{s \in S^*}$. If $B_s > 0$ then the government sells bonds and hence the term represents new indebtedness against the private sector. If $B_s < 0$, it means the loan to the private sector.

Assumption 4 *The government only trades riskless bonds.*

This assumption can be justified by an appeal to realism. It has important consequences for the determinacy of equilibria. We will comment on this in Section 4.3. Combining all these elements, the sequential government budget constraint is

$$\frac{B_0}{1 + r_0} + M_0 = H_0 \tag{11}$$

in period zero and

$$\frac{B_s}{1 + r_s} + M_s + p_0 \cdot \tau_0 = B_0 + M_0 + H_s, \ \forall s \in S \tag{12}$$

in period one. Similar to the household, we have a terminal condition at the end of every node in period one:

$$B_s + M_s - p_s \cdot \tau_s = 0 \ , \forall s \in S. \tag{13}$$

For later analysis, we will study an equivalent formulation of the sequential constraint (11) – (13). Plug (13) as a function of B_s and (11) as a function of B_0 into equation (12) to get

$$\frac{r_s}{1 + r_s} M_s + r_0 M_0 + \frac{p_s \cdot \tau_s}{1 + r_s} + p_0 \cdot \tau_0 = (1 + r_0) H_0 + H_s, \ \forall s \in S. \tag{14}$$

The equation (14) is the intertemporal budget equation for each branch of the event tree. It is not difficult to see that the equation (14) is equivalent to the sequential budget constraint (11) – (13).

Multiply each equation (14) with its state price and add these equations over the states to get the government's intertemporal budget constraint at $t = 0$ for the entire event tree as

$$\frac{r}{1+r} \cdot (a \Box M) + a \cdot \left(\frac{1}{1+r} \Box p \Box \tau \right) = a \cdot H. \tag{15}$$

2.4 Competitive equilibria

The market clearing condition is specified in the usual way as

$$\sum_{i=1}^{I} e_s^i = \sum_{i=1}^{I} x_s^i, \quad \forall s \in S^*, \tag{16}$$

$$M_s = \sum_{i=1}^{I} m_s^i, \quad \forall s \in S^*, \tag{17}$$

$$B_s = \sum_{i=1}^{I} b_s^i, \quad \forall s \in S^*, \tag{18}$$

$$0 = \sum_{i=1}^{I} \theta_s^i, \quad \forall s \in S, \tag{19}$$

where the equation (16), (17), (18), and (19) are commodity, money and asset market clearing conditions, respectively. We write the bond and risky asset separately since we want to emphasize the difference of the market supply in two cases.

The primitives of the economy can be summarized by the vector

$$\mathcal{E} := \left\{ (u^i, e^i, \tau^i, \delta^i)_{i \in I}, V \right\}.$$

Definition 1 An *Equilibrium* for the economy $\mathcal{E}$ is a tuple

$$\left\{ (\overline{p}, \overline{q}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, (\overline{M}, \overline{B}, \overline{H}) \right\}$$

such that

(1) $(\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)$ maximizes $u^i(x)$ *subject to* $(\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i) \in B^i(\overline{p}, \overline{q}, \overline{r}, \overline{H})$.

(2) The actions of the monetary-fiscal authority $(\overline{M}, \overline{B}, \overline{H})$ satisfy (11)-(13).

(3) in every state, markets clear, i.e. (16)-(19) hold.

An equilibrium is said to be *monetary* if $\overline{p}_{sl} < +\infty$ for $\forall\ s \in S^*, l \in L$.

In the proof of the theorems given in the following sections, we indeed use another equivalent equilibrium concept. For household 1 define the complete markets budget set

$$\bar{B}^1\left(p, \mu, r, H\right) = \left\{ \left(x^1, m^1\right) \in \mathbb{R}_+^{(S+1)L} \times \mathbb{R}_+^{S+1} \middle| \, (5), (7), (9) \text{ hold.} \right\},$$

where we use μ because a is a function of μ and r. From the no-arbitrage conditions, the budget sets of agents $i \geq 2$ can also be expressed as depending on μ instead of q. Following Cass (1984) and Duffie and Shafer (1985), in Definition 2 we define a concept of effective monetary equilibrium.

Definition 2 An *effective equilibrium* for the economy $\mathcal{E}$ is a tuple

$$\left\{\left(\overline{p}, \overline{\mu}, \overline{r}\right), \left(\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i\right)_{i\in I}, \left(\overline{M}, \overline{B}, \overline{H}\right)\right\}$$

such that

(1) For $i \geq 2$ $(\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)$ maximizes $u^i(x)$ subject to $(\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i) \in B^i\left(\overline{p}, \overline{\mu}, \overline{r}, \overline{H}\right)$. For $i = 1$, $(\overline{x}^1, \overline{m}^1)$ maximizes $u^1(x)$ such that $(\overline{x}^1, \overline{m}^1) \in \bar{B}^1\left(\overline{p}, \overline{\mu}, \overline{r}, \overline{H}\right)$, and $\left(\overline{b}^1, \overline{\theta}^1\right) = \left(\overline{B} - \sum_{i=2}^{I} \overline{b}^i, -\sum_{i=2}^{I} \overline{\theta}^i\right)$.

(2) The actions of the monetary-fiscal authority $\left(\overline{M}, \overline{B}, \overline{H}\right)$ satisfy (11)-(13).

(3) In every state, commodity and money markets clear, i.e. (16)-(17) hold.

An effective equilibrium is said to be *monetary* if $\overline{p}_{sl} < +\infty$ for $\forall\ s \in S^*, l \in L$.

From Definitions 1 and 2, we can immediately see two differences. First, in the effective equilibrium, household 1 is only restricted by the intertemporal budget constraint and the cash-in-advance constraint. Second, household 1 does not choose $\left(\overline{b}^1, \overline{\theta}^1\right)$ directly. Instead he takes the residual asset to clear the asset market. Therefore, the asset market clears by construction.

It is immediate that every effective equilibrium is an equilibrium as defined in Definition 1. Indeed, it is easy to see that the no-arbitrage conditions determine $\overline{q}$ given $\overline{\mu}$ and $\overline{r}$. To show that a tuple $\left\{(\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i\in I}, \left(\overline{M}, \overline{B}, \overline{H}\right)\right\}$ as defined in the effective monetary equilibrium corresponds to a monetary equilibrium, we first need to check that the household 1 satisfies the budget equations (4)-(8) and second that his choice is still optimal in the sequential constraint. The first property follows directly from Walras law.[7] To see that household one still maximizes his utility, just notice that the sequential constraint is a subset of the intertemporal one. Hence, the old consumption vector must be optimal since it is still feasible under the sequential constraint and it was already optimal in the larger intertemporal constraint. These arguments are standard and not made explicit here.

2.5 Fiscal policy

In general, fiscal policy consists of a plan for taxes, transfers and bond market actions. However, in this paper we keep the taxes fixed and restrict attention to different transfer policies in combination with bond market actions. Subject to this restriction we will study four different combinations of fiscal and monetary policy of the government: the central bank might peg the interest rate or money supply, while the fiscal authority might run a Ricardian or a non-Ricardian kind of transfer policy.

We restrict attention to the following structure. In the Ricardian transfer policy we will consider here, the fiscal authority redistributes the seigniorage income and the tax returns at each state of the economy. The government bonds adjust accordingly to satisfy the sequential constraint of the government.

[7] We leave it an exercise to the reader to check these equations.

Bloise and Polemarchakis (2006) call such a policy a *balanced transfer rule.* We adopt their terminology and consequently define a balanced transfer fiscal policy as follows:

Assumption 5 The *balanced transfer fiscal policy* determines the vector (H, B) by the functions $H(p, M, r)$ and $B(p, M)$, where

$$\begin{aligned} H_s(p, M, r) &:= \frac{r_s}{1+r_s} M_s + \frac{p_s \cdot \tau_s}{1+r_s}, \ \forall s \in S^*, \\ B_s(p, M) &:= p_s \tau_s - M_s, \ \forall s \in S^*. \end{aligned}$$

Under Assumption 5, one can check that equations (11)-(13) always hold. Therefore, the fiscal policy is Ricardian.

A Fiscal policy which fixes transfers in every state of the world exogenously will be called *fixed transfer fiscal policy.* Formally,

Assumption 6 The *fixed transfer fiscal policy* determines the vector (H, B) by the functions $H(p, M, r)$ and $B(p, M, r)$, where

$$\begin{aligned} H_s(p, M, r) &:= \overline{H}_s, \forall s \in S^*, \ \textit{where} \ \overline{H}_0 > 0, \overline{H}_s \geq 0, \forall s \in S, \\ B_0(p, M, r) &:= (1+r_0)\left(\overline{H}_0 - M_0\right), \\ B_s(p, M, r) &:= (1+r_s)(B_0 + M_0 + \overline{H}_s - M_s - p_0 \cdot \tau_0), \ \forall s \in S. \end{aligned}$$

Using this rule, one can check that (11)-(12) hold, but the terminal condition (13) does not hold for some price and interest rate vector. Equivalently, (14) need not be true for some vector of prices and interest rates.

3 Monetary equilibria with balanced transfers

3.1 Interest rate peg

If the central bank pegs the nominal interest rate, then the vector $r := \{r_s\}_{s \in S^*}$ is fixed at a target value $\overline{r}$. To sustain $\overline{r}$ in the market, the central bank accommodates money demand. We impose the following assumption on monetary policy:

Assumption 7 *Interest rates are nonnegative and bounded above,* $0 \leq \overline{r}_s < +\infty$*,* $\forall s \in S^*$*, and the government accommodates money demand, i.e.* $M_s = \sum_i m_s^i$ *for each* $s \in S^*$*.*

A monetary equilibrium with interest rate peg and balanced transfers can now be defined as follows:

Definition 3 A *Monetary Equilibrium with interest rate peg and balanced transfers* is a monetary equilibrium according to Definition 2 with exogenously fixed r satisfying Assumption 7 and a fiscal policy rule which satisfies Assumption 5.

In the following theorem, we show that for every fixed price level and for every fixed martingale measure, there exists a monetary equilibrium which implements the interest rate target of the central bank.

Theorem 1 *Suppose Assumptions 1 - 5 and 7 hold. Fix* $0 < \overline{c} < +\infty$ *and* $\overline{\mu} \gg 0$*, then for every* $0 \leq \overline{r} \ll +\infty$ *there exists a Monetary Equilibrium with interest rate peg and balanced transfers* $\left\{(\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, \left(\overline{M}, \overline{B}, \overline{H}\right)\right\}$ *such that* $\overline{c} = \sum_l \overline{p}_{0l} + \sum_{l \in L, s \in S} \overline{\mu}_s \overline{p}_{sl}$*.*

3.2 Money supply control

Under money supply control, the central bank fixes the money supply process $M := (M_s)_{s \in S^*}$ at a target value $\overline{M}$. If this is the case, we impose

Assumption 8 *Under money supply policy,* $0 < \overline{M}_s < +\infty$*,* $\forall s \in S^*$*.*

Combining the balanced transfer policy with money supply control suggests the following definition:

Definition 4 A *Monetary Equilibrium with money supply control and balanced transfers* is a monetary equilibrium according to Definition 2 with exogenously fixed M satisfying Assumption 8 and a fiscal policy rule which satisfies Assumption 5.

In the next theorem we show that for every fixed price level and for every fixed martingale measure, there exists a monetary equilibrium which implements a money supply target $\overline{M}$ of the central bank. So the result parallels the result from the previous theorem under interest rate peg policy.

Theorem 2 *Suppose Assumptions 1 - 5 and 8 hold. Fix* $0 < \overline{c} < +\infty$ *and* $\overline{\mu} \gg 0$*, then for every* $0 \ll \overline{M} \ll +\infty$ *there exists a Monetary Equilibrium with money supply control and balanced transfers* $\left\{ (\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, (\overline{M}, \overline{B}, \overline{H}) \right\}$ *such that* $\sum_i \overline{m}^i = \overline{M}$ *and* $c = \sum_l \overline{p}_{0l} + \sum_{l \in L, s \in S} \overline{\mu}_s \overline{p}_{sl}$.

3.3 Interpretation and literature

We provide some intuition for the existence and the indeterminacy results in Theorems 1 and 2. To prove existence of an equilibrium we use similar assumptions as in the GEI-model with nominal assets. The balanced transfer rule always implies that the government balances its budget, so there are no additional assumptions necessary to achieve this. In addition, our equilibrium could be a no-trade equilibrium in which there is no seigniorage income for the central bank. In this case, the government just redistributes potential tax returns among the households according to their shares $(\delta^i)_{i \in I}$.

The intuition concerning indeterminacy can be given by counting equations and variables. The macro variables to be determined in the effective equilibrium are the $L(S+1)$ commodity spot prices, the $S-1$ dimensional equivalent martingale measure and the $S+1$ interest rates. In the interest rate peg case, all interest rates are exogenously fixed. There are $L(S+1)$ equilibrium restrictions coming from commodity market clearing. Finally, there is a single Walras law at work since household one only faces the intertemporal budget constraint. In total, there are S more variables than independent equations, so S is the number of total indeterminacy in this economy.

Among these S dimensions of indeterminacy there is one degree of homogeneity involved which allows us to fix the price level. Indeed, if agents react to a doubling of the commodity prices by doubling their portfolios and money demand, the transfers will also double by the balanced transfer rule and hence the allocation is unaffected. It can be expected that the remaining $S-1$ degrees of indeterminacy inherent in the measure are real. An argument which supports this conjecture is given in Nakajima and Polemarchakis (2001).

In the case of money supply control, we have $S+1$ more endogenous interest rates which have to be determined. However, there are $S+1$ additional equilibrium restrictions coming from money market clearing. So the intuition is exactly as in the case of interest rate peg.

The recent literature in Ricardian economies can be summarized as follows. Dréze and Polemarchakis (2000) and Bloise, Dréze and Polemarchakis (2005) prove existence and indeterminacy under interest rate peg with complete asset markets under a finite and an infinite horizon, respectively. Bloise (2006) extends their results to money supply policy. We generalize this recent literature on Ricardian economies by proving existence and indeterminacy under both interest rate peg and money supply policy with incomplete markets and a finite time horizon.

Notice that our results do not rely on the number of assets. Hence, the same intuition as given above applies for the case of complete markets. This is why Bloise, Dréze and Polemarchakis (2005) and Dréze and Polemarchakis (2000) get basically the same results in terms of indeterminacy. Obviously, their indeterminacy is purely nominal.

Under interest rate peg policy, Gourdel and Triki (2005) independently studied a closely related economy. They obtained a result similar to our Theorem

1. Within the interest rate peg policy, there are two major differences between Gourdel and Triki (2005) and our model. First, in our model the asset markets open before the commodity markets, as in Woodford (1994) and Bloise, Dréze and Polemarchakis (2005). In Gourdel and Triki (2005), the bond market opens before the commodity market, but the latter opens before the markets for the risky assets. Second, we use different techniques to prove our results. In our proof we use a trick introduced by Cass (1984), while they use the method similar to Werner (1985). Our method leads us to characterize the indeterminacy in terms of the total price level and the equivalent martingale measure, while they use the price level within each state as the indeterminate variables.

4 Monetary equilibria with fixed transfers

In the fixed transfer case it is obvious from (15) that the government can only balance its budget if either taxes or seigniorage are strictly positive at some node. Under zero taxes, the gains to trade in the economy must be high enough to induce some positive seigniorage income for the government.

In the proof of the next two Theorems we will impose a gains to trade hypothesis which goes back to Dubey and Geanakoplos (1992, 2003(a), 2003(b)). Define the function $\bar{\zeta}_s : \mathbb{R}^L \times \mathbb{R}_+ \to \mathbb{R}^L$ by

$$\bar{\zeta}_s(\zeta_s, \gamma) := \begin{cases} \zeta_{sl} & \text{if } \zeta_{sl} < 0 \\ \frac{\zeta_{sl}}{1+\gamma} & \text{otherwise.} \end{cases}$$

A feasible allocation $(x_{-s}, e_s) := (x_0, x_1, \ldots, x_S)_{|x_s = e_s}$ is said to be γ-*Pareto optimal in state* $s \in S$ *at* e_s if there does not exist a trade vector $\zeta_s \in \mathbb{R}^{IL}$ in state s such that $\sum_i \zeta_s^i = 0$ and, $\forall i \in I$, $e_s^i + \zeta_s^i \geq 0$ and $u^i(x_0^i, x_1^i, \ldots, e_s^i + \bar{\zeta}_s(\zeta_s^i, \gamma), \ldots, x_S^i) \geq u^i(x_0^i, x_1^i, \ldots, e_s^i, \ldots, x_S^i)$ with at least one $i \in I$ where the strict inequality holds. If (x_{-s}, e_s) is γ-Pareto optimal in state $s \in S$ at e_s, then we equivalently say that there are *no gains to* γ*-diminished trade in* $s \in S$ at

(x_{-s}, e_s). Accordingly, the *gains to trade at* (x_{-s}, e_s) are defined by

$$\gamma_s(x_{-s}, e_s) := \min\{\gamma | \text{there are no gains to } \gamma\text{-diminished trade in } s \in S\}.$$

4.1 Interest rate peg

In the fixed transfer case, we assume that the interest rates are strictly positive.

Assumption 9 *Interest rates are strictly positive and bounded above,* $0 < \overline{r}_s < +\infty$, $\forall s \in S^*$. *The government accommodates money demand, i.e.* $M_s = \sum_i m_s^i$ *for each* $s \in S^*$.

Combining an interest rate peg policy of the central bank with the fixed transfer fiscal policy suggests the following definition:

Definition 5 A *Monetary Equilibrium with interest rate peg and fixed transfers* is a monetary equilibrium according to Definition 2 with exogenously fixed interest rates according to Assumption 9 and a fiscal policy rule which satisfies Assumption 6.

To rule out an exploding commodity price path, we need to impose either a strictly positive taxation or a gains to trade hypothesis. The following assumption says that if the tax in some state $s \in S$ is zero, then the gains to trade in this state exceed the interest rate. Intuitively, the friction caused by the transactions technology still allows for Pareto-improvements at the initial endowment allocation in the state $s \in S$.

Assumption 10 *For every* $s \in S$, *either* $\tau_s > 0$, *or* $\gamma_s(x_{-s}, e_s) > \overline{r}_s$ *for all feasible* (x_{-s}, e_s).

The following theorem states that every interest rate target of the central bank can be embedded in an equilibrium with fixed transfers. Note that we do not claim any indeterminacy result here.

Theorem 3 *Suppose that Assumptions 1 - 4, 6, 9 and 10 hold. For every* $0 \ll \overline{r} \ll +\infty$ *there exists a Monetary Equilibrium with interest rate peg and fixed transfers* $\left\{ (\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, \left(\overline{M}, \overline{B}, \overline{H}\right) \right\}$.

4.2 Money supply control

The definition of equilibrium is straightforward:

Definition 6 A *Monetary Equilibrium with money supply control and fixed transfers* is a monetary equilibrium according to Definition 2 with exogenously fixed money supply satisfying Assumption 8 and a fiscal policy rule which satisfies Assumption 6.

For the same reason as in the interest rate peg, we also need to impose a Gains-to-Trade hypothesis for money supply policy.

Assumption 11 *For every* $s \in S$*, either* $\tau_s > 0$*, or, for every feasible* (x_{-s}, e_s)*,* $\gamma_s(x_{-s}, e_s) > \frac{\overline{H}_0 + \overline{H}_s}{\overline{M}_s - \overline{H}_0 - \overline{H}_s}$ *together with* $\overline{M}_0 \geq \overline{H}_0$ *and* $\overline{M}_s > \overline{H}_0 + \overline{H}_s$.

The last theorem states the parallel result of Theorem 3 for the case of money supply control of the central bank.

Theorem 4 *Suppose Assumptions 1 - 4, 6, 8 and 11 hold. For every* $0 \ll \overline{M} \ll +\infty$*, there exists a Monetary Equilibrium with money supply control and fixed transfers* $\left\{ (\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, \left(\overline{M}, \overline{B}, \overline{H}\right) \right\}$ *such that* $\sum_i \overline{m}^i = \overline{M}$.

4.3 Interpretation and literature

We will now give an interpretation of Theorems 3 and 4. The economy with fixed transfers studied in Section 4 has only one important difference compared to the economy with balanced transfers in Section 3. In the balanced transfer case, the transfers always adjust to make the government budget hold. Therefore, (14) is an identity which does not impose restrictions on the equilibrium set. In the fixed transfer case, (14) imposes S additional restrictions on the set of equilibria. So all the S degrees of total indeterminacy we obtained in the balanced transfer case are lost here.

This conclusion follows from the assumption that the government only trades riskfree bonds and the fact that the transfers are fixed. Intuitively, the fixed transfers always impose some restrictions, only the number of restrictions depends on the set of assets the government trades. Our assumption that it only trades riskfree assets implies that it enters period one with state independent debt. To allow for budget balance, taxes and seigniorage must also be independent of the state. Since there are S states, this provides the intuition why there are S additional restrictions.

Now suppose there is a full set of Arrow securities and that the government trades every such security. Then there is only one additional restriction compared to the Ricardian case. Indeed, in this case the government only has to satisfy an intertemporal constraint. This imposes one additional restriction on the equilibrium set because of the exogenous transfers (see Bloise, Dréze and Polemarchakis (2005)).

The main contributions to the theoretical literature[8] in economies with non-Ricardian fiscal policies and an active monetary policy are Dubey and Geanakoplos (1992, 2003a, 2003b, 2006). Dubey and Geanakoplos (1992, 2003a) consider a one period model with a cash-in-advance constraint, inside and outside money. Dubey and Geanakoplos (2003b) extend this model to a stochastic economy with incomplete asset markets and a mixed asset structure. In all papers, they show, among several other results, existence of the equilibrium. They do so by using a strategic market game approach. Dubey and Geanakoplos (2006) formally prove generic local uniqueness in the stochastic economy with incomplete asset markets and nominal assets.

[8]As opposed to the quite huge macroeconomic literature on the Fiscal Theory of the Price Level.

We study a similar economy as Dubey and Geanakoplos (2003b, 2006), but to prove existence we basically follow the ideas in Bloise, Dréze and Polemarchakis (2005) by introducing a price determination mechanism in the fixed point mapping for every price object. This allows us to establish a unified framework to prove existence of equilibrium in all four cases we consider. In addition, by embedding each equilibrium object into the fixed point mapping, we provide a clear intuition for the mechanism which determines the equilibrium.

Gourdel and Triki (2005) provide a result similar to our Theorem 3 under interest rate peg policy. In addition to the differences mentioned in Section 3.3, there is one more major distinction in this case. While Gourdel and Triki (2005) need strictly positive taxes to establish the existence of a monetary equilibrium, our result also allows for the possibility of zero taxes provided that the economy has sufficiently high gains to trade.

5 Concluding remarks

To conclude the paper, we discuss some directions of future research. First, a different timing of transactions can be considered. One possibility is to use the cash-in-advance constraint as introduced by Svensson (1985), where the commodity markets open before the asset markets. This could be a suitable framework to study both the transaction and precautionary demand for money. However, different from our two-period model, the new timing needs an infinite horizon to support money's value. Second, it would be interesting to introduce a Baumol-Tobin structure in which households voluntarily hold money as a store of value even though other interest bearing bonds coexist. Both existence and determinacy in the Baumol-Tobin economy are open and difficult questions. Doing so probably requires more than two periods to enrich the potential transaction patterns. In particular, an infinite horizon model would be of interest. Finally, the model presented here delivers a unified framework for monetary and fiscal policy within a GEI-economy. Therefore, it would be of interest to study the general equilibrium effects of changing monetary policy parameters. Under incomplete financial market the effect can be expected to be real, an important feature for policy analysis. Such an analysis would contribute to the old but fundamental debate about the neutrality of money.

6 Appendix

In this appendix we give the proof for the theorems in the main text. The proofs are organized as follows. First, we define an abstract economy. Second, we show the properties of the household and aggregate demand. Then we prove the results under different monetary-fiscal policy combinations.

6.1 An abstract economy

Define the inverse price level as $c := \frac{1}{\sum_l p_{0l} + \sum_{s \in S, l} \mu_s p_{sl}}$ and the new prices by $\pi_{sl} := c\mu_s p_{sl}$ for all $s \in S$ and $\pi_{0l} := cp_{0l}$. By construction π lies in the unit simplex

$$\Delta := \left\{ \pi \in \mathbb{R}_+^{(S+1)L} \;\middle|\; \sum_{s,l} \pi_{sl} = 1 \right\}.$$

Multiply (4) by c, use the no-arbitrage equation $q = \widehat{a} \cdot A = \frac{1}{1+r_0} (\mu \cdot A)$, we get

$$\pi_0 \cdot x_0^i + \frac{1}{1+r_0} \left(\widetilde{b}_0^i + \mu \cdot A \cdot \widetilde{\theta}^i \right) + \widetilde{m}_0^i = \delta^i \widetilde{H}_0 + \pi_0 \cdot e_0^i, \tag{20}$$

where $\widetilde{b}_0^i := c\, b_0^i$, $\widetilde{\theta}^i := c\, \theta^i$, $\widetilde{m}_0^i := c\, m_0^i$ and $\widetilde{H}_0 := cH_0$. The cash-in-advance constraint in $t = 0$ is

$$\widetilde{m}_0^i \geq \pi_0 \cdot \left(x_0^i - e_0^i \right)^- . \tag{21}$$

Multiply (6) by $c\mu$ to get

$$\pi_s \cdot x_s^i + \frac{\widetilde{b}_s^i}{1+r_s} + \widetilde{m}_s^i = \mu_s \left(\widetilde{b}_0^i + A_s \cdot \widetilde{\theta}^i + \widetilde{m}_0^i - \pi_0 \cdot \tau_0^i \right) + \pi_s \cdot e_s^i + \delta^i \widetilde{H}_s, \tag{22}$$

where $\widetilde{b}_s^i := c\mu_s b_s^i$, $\widetilde{m}_s^i := c\mu_s m_s^i$ and $\widetilde{H}_s := c\,\mu_s H_s$. The cash-in-advance constraint at state $s \in S$ becomes

$$\widetilde{m}_s^i \geq \pi_s \cdot \left(x_s^i - e_s^i\right)^-, \tag{23}$$

and the terminal condition is

$$\widetilde{b}_s^i + \widetilde{m}_s^i - \pi_s \cdot \tau_s^i = 0. \tag{24}$$

We can now redefine household i's budget set by

$$B^i\left(\pi, \mu, r, \widetilde{H}\right) = \left\{(x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i) \in \mathbb{R}_+^{(S+1)L} \times \mathbb{R}_+^{S+1} \times \mathbb{R}^{S+1} \times \mathbb{R}^{J-1} \,\middle|\, (20) - (24) \text{ hold}\right\}.$$

Denote S-dimensional vectors for variables over the S states in $t = 1$ by $v_{\mathbf{1}} := (v_1, \ldots, v_S)$. In addition, $\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} := \left(\frac{r_s}{1+r_s}\right)_{s\in S}$ and $\frac{1}{1+r_{\mathbf{1}}} := \left(\frac{1}{1+r_s}\right)_{s\in S}$. By redefining variables, the intertemporal budget constraint (9) becomes[9]

$$\begin{aligned}&\pi_0 \cdot x_0^i + \frac{\pi_{\mathbf{1}} \cdot x_{\mathbf{1}}^i}{1+r_0} + \frac{r_0}{1+r_0}\widetilde{m}_0^i + \frac{1}{1+r_0}\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot \widetilde{m}_{\mathbf{1}}^i \\ &= \delta^i\left(\widetilde{H}_0 + \frac{\widetilde{H}_{\mathbf{1}} \cdot \mathbf{1}}{1+r_0}\right) + \pi_0 \cdot \left(e_0^i - \frac{\tau_0^i}{1+r_0}\right) + \frac{1}{1+r_0}\pi_{\mathbf{1}} \cdot \left(e_{\mathbf{1}}^i - \frac{1}{1+r_{\mathbf{1}}} \Box \tau_{\mathbf{1}}^i\right).\end{aligned} \tag{25}$$

The household 1's budget constraint is

$$B^1\left(\pi, r, \widetilde{H}\right) = \left\{(x^1, \widetilde{m}^1) \in \mathbb{R}_+^{(S+1)L} \times \mathbb{R}_+^{S+1} \,\middle|\, (21), (23) \text{ and } (25) \text{ hold}\right\}.$$

With the obvious definitions, equation (14) becomes

$$\frac{r_s}{1+r_s}\widetilde{M}_s + \mu_s r_0 \widetilde{M}_0 + \frac{\pi_s \cdot \tau_s}{1+r_s} + \mu_s \pi_0 \cdot \tau_0 = (1+r_0)\mu_s \widetilde{H}_0 + \widetilde{H}_s, \ \ \forall s \in S, \tag{26}$$

and (15) becomes

$$\frac{r_0}{1+r_0}\widetilde{M}_0 + \frac{1}{1+r_0}\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot \widetilde{M}_{\mathbf{1}} + \frac{\pi_0 \cdot \tau_0}{1+r_0} + \frac{1}{1+r_0}\pi_{\mathbf{1}} \cdot \left(\frac{1}{1+r_{\mathbf{1}}} \Box \tau_{\mathbf{1}}\right) = \widetilde{H}_0 + \frac{\widetilde{H}_{\mathbf{1}} \cdot \mathbf{1}}{1+r_0}. \tag{27}$$

A monetary equilibrium in this abstract economy is a vector $\Big\{ (\overline{\pi}, \overline{\mu}, \overline{r}, \overline{c})$, $(\overline{x}^i, \overline{\widetilde{m}}^i, \overline{\widetilde{b}}^i, \overline{\widetilde{\theta}}^i)_{i\in I}, \left(\overline{\widetilde{M}}, \overline{\widetilde{B}}, \overline{\widetilde{H}}\right) \Big\}$ such that markets clear, agents optimize, the

[9] We use the notation $\mathbf{1} := (\ldots, 1, \ldots)$.

government balances its budget, $\overline{c} > 0$ and $\overline{\mu} \gg 0$. Such an equilibrium corresponds to a monetary effective equilibrium $\left\{ (\overline{p}, \overline{\mu}, \overline{r}), (\overline{x}^i, \overline{m}^i, \overline{b}^i, \overline{\theta}^i)_{i \in I}, (\overline{M}, \overline{B}, \overline{H}) \right\}$ according to Definition 2. As argued earlier, the latter vector corresponds to a monetary equilibrium according to Definition 1. In the following proofs, we will therefore concentrate on equilibria in the abstract economy.

6.2 The household and market demand

μ is an element of the S-dimensional unit simplex, which we denote with Δ^{S-1}. The extended positive real line is as usual $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$. We start by deriving the properties of the budget sets in the following lemma:

Lemma 1 *Under Assumptions 2 and 3, the budget sets satisfy the following properties:*

(1.1) For $i \geq 2$, $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ is a non-empty and upper hemi-continuous correspondence for $\left(\pi, \mu, r, \widetilde{H}\right) \in \Delta \times \Delta^{S-1} \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.2) For $i \geq 2$, $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ is compact for $\left(\pi, \mu, r, \widetilde{H}\right) \in$ interior$(\Delta) \times$ interior$(\Delta^{S-1}) \times \mathbb{R}_{++}^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.3) For $i \geq 2$, $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ is lower hemi-continuous for $\left(\pi, \mu, r, \widetilde{H}\right) \in$ interior$(\Delta) \times \Delta^{S-1} \times \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.4) For $i \geq 2$, if $\widetilde{H}_0 > 0$ then $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ is lower hemi-continuous for $\left(\pi, \mu, r, \widetilde{H}\right) \in \Delta \times$ interior$(\Delta^{S-1}) \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.5) For $i \geq 2$, as long as $r_0 < +\infty$, $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ is lower hemi-continuous if $\left(\pi, \mu, r, \widetilde{H}\right) \in$ interior$(\Delta) \times$ interior$(\Delta^{S-1}) \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.6) $B^1\left(\pi, r, \widetilde{H}\right)$ is non-empty and upper hemi-continuous for $\left(\pi, r, \widetilde{H}\right) \in \Delta \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.7) $B^1\left(\pi, r, \widetilde{H}\right)$ is compact for $\left(\pi, r, \widetilde{H}\right) \in$ interior$(\Delta) \times \mathbb{R}_{++}^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.8) $B^1\left(\pi, r, \widetilde{H}\right)$ *is lower hemi-continuous for* $\left(\pi, r, \widetilde{H}\right) \in \Delta \times \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.9) *If* $\widetilde{H}_0 > 0$ *then* $B^1\left(\pi, r, \widetilde{H}\right)$ *is lower hemi-continuous for* $\left(\pi, r, \widetilde{H}\right) \in \Delta \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

(1.10) *As long as* $r_0 < +\infty$ *and* $\widetilde{H}_{\mathbf{1}} > 0$, $B^1\left(\pi, r, \widetilde{H}\right)$ *is lower hemi-continuous if* $\left(\pi, r, \widetilde{H}\right) \in \Delta \times \overline{\mathbb{R}}_+^{S+1} \times \mathbb{R}_+^{S+1}$.

Proof:

(1.1) To check non-emptiness, it is sufficient to notice that $\left(x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i\right) = (0, \pi \,\Box\, e^i, 0, 0)$ satisfies the equations $(20) - (24)$. Upper hemi-continuity is straightforward.

(1.2) Closedness is obvious. To show the boundedness of $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ under $(\pi, \mu, r) \gg 0$, note that an action $(x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i)$ in $B^i\left(\pi, \mu, r, \widetilde{H}\right)$ must satisfy (25), which implies $0 \leq (x^i, \widetilde{m}^i) \ll +\infty$. From (24) we know that $\widetilde{b}_s^i > -\infty$ for every $s \in S$. From the standard no-arbitrage argument, we have $-\infty \ll \left(\widetilde{b}_0^i, \widetilde{\theta}^i\right) \ll +\infty$. From (22) this further implies that $\widetilde{b}_s^i < +\infty$.

(1.3) To see that there is an interior point, take $\widetilde{\theta}^i = 0$ and for every $s \in S^*$ take $x_s^i = 0$, $\widetilde{b}_s^i = -2\,\epsilon_s^i$ and $\widetilde{m}_s^i = \pi_s \cdot e_s^i + \frac{\epsilon_s^i}{1+r_s}$ with $\epsilon_s^i > 0$. Using Assumption 2, $\pi \gg 0$ and $r \ll +\infty$, it is easy to see that (20) - (24) hold with a strict inequality for all ϵ_s^i small enough. Note that this is true even if $\mu_s = 0$ for some $s \in S$. This sequence shows that the interior of the budget set is nonempty. It is now easy to see that the interior is lower hemi-continuous. Since the closure of a lower hemi-continuous set is again lower hemi-continuous, the result follows.

(1.4) We only need to check that there is an interior point. Change the sequence defined in (1.3) by $\widetilde{b}_s^i = 0$ for every $s \in S^*$, $\widetilde{m}_0^i = \pi_0 \cdot e_0^i + \frac{\delta^i \widetilde{H}_0}{2}$ and $\widetilde{m}_s^i = \pi_s \cdot e_s^i + \mu_s \frac{\delta^i \widetilde{H}_0}{4}$ for every $s \in S$ to see that this is true.

(1.5) Again, use $x^i = 0$ and $\widetilde{\theta}^i = 0$. In period zero, take $\widetilde{b}_0^i = -2\,\epsilon_0^i$, $\widetilde{m}_0^i = \pi_0 \cdot e_0^i + \frac{\epsilon_0^i}{1+r_0}$ with $\epsilon_0^i > 0$ and in period one take $\widetilde{b}_s^i = 0$ and $\widetilde{m}_s^i = \pi_s \cdot e_s^i + \mu_s \frac{\widetilde{m}_0^i - \pi_0 \cdot \tau_0^i}{2}$ to see that the interior is nonempty for ϵ_0^i small enough.

(1.6) It holds that $\left(x^1, \widetilde{m}^1\right) = (0, \pi \,\Box\, e^1)$ is an element of $B^1\left(\pi, r, \widetilde{H}\right)$. Hence, $B^1\left(\pi, r, \widetilde{H}\right)$ is non-empty. The second part is straightforward.

(1.7) This property follows immediately.

(1.8) To see that the interior of $B^1\left(\pi, r, \widetilde{H}\right)$ is nonempty, take $\widetilde{m}_s^1 = \pi_s \cdot e_s^1 + \epsilon_s$ for every $s \in S^*$, $x^1 = 0$ and choose all $\epsilon_s > 0$ small enough. Note that this argument relies on Assumption 2 and $r \ll +\infty$.

(1.9) Under the assumption $\widetilde{H}_0 > 0$, the same sequence as in (1.8) is an interior point for ϵ_s^i small enough for every $s \in S^*$.

(1.10) Since $\widetilde{H}_{\mathbf{1}} > 0$ and $r_0 < +\infty$, the same argument as in (1.8) applies. ■

The demand correspondence for every consumer type $i \geq 2$ is defined to be

$$(x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i)(\pi, \mu, r, \widetilde{H}) := \left\{ (x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i) \in B^i(\pi, \mu, r, \widetilde{H}) \middle| (x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i) \in \arg\max u^i(x^i) \right\}$$

Let $\varphi^i(\pi, \mu, r, \widetilde{H})$ denote the projection of this demand set onto $(x^i, \widetilde{m}^i)$, $\varphi_x^i(\pi, \mu, r, \widetilde{H})$ the projection of the latter onto x^i and $\varphi_{\widetilde{m}}^i(\pi, \mu, r, \widetilde{H})$ the projection onto $\widetilde{m}^i$. Household $i = 1$ maximizes his utility by choosing $(x^1, \widetilde{m}^1)$ being an element of $B^1\left(\pi, r, \widetilde{H}\right)$. The demand correspondence is $\varphi^1\left(\pi, r, \widetilde{H}\right)$ and the projections are defined as above. We summarize the properties of individual demand in the following lemma:

Lemma 2 *Under Assumptions 1 - 3, household demand satisfies the following properties:*

(2.1) For $i \geq 2$, $\varphi^i\left(\pi, \mu, r, \widetilde{H}\right)$ is non-empty, compact and convex valued for $\left(\pi, \mu, r, \widetilde{H}\right) \in$ interior$(\Delta) \times$ interior$(\Delta^{S-1}) \times \mathbb{R}_{++}^{S+1} \times \mathbb{R}_{+}^{S+1}$.

(2.2) For $i \geq 2$, $\varphi^i\left(\pi, \mu, r, \widetilde{H}\right)$ is upper hemi-continuous under the conditions given in Lemma (1.3), (1.4) or (1.5).

(2.3) $\varphi^1\left(\pi, r, \widetilde{H}\right)$ is non-empty, compact and convex valued for $\left(\pi, r, \widetilde{H}\right) \in$ interior$(\Delta) \times \mathbb{R}^{S+1}_{++} \times \mathbb{R}^{S+1}_{+}$.

(2.4) $\varphi^1\left(\pi, r, \widetilde{H}\right)$ is upper hemi-continuous under the conditions given in Lemma (1.8), (1.9) or (1.10).

(2.5) Under the assumption of Lemma (1.8), (1.9) or (1.10), inf $\left\{ \|x\| \;\middle|\; x \in \varphi^1_x\left(\pi, r, \widetilde{H}\right)\right\} \to +\infty$ if $\pi_{sl} \to 0$ for some $s \in S^*$ and $l \in L$.

(2.6) $\forall s \in S^*$, if $r_s > 0$, then $\widetilde{m}^i_s \leq \pi_s \cdot e^i_s$ for all $(\dots, \widetilde{m}^i_s, \dots) \in \varphi^i_{\widetilde{m}}\left(\pi, \mu, r, \widetilde{H}\right)$ if $i \geq 2$ and for all $(\dots, \widetilde{m}^1_s, \dots) \in \varphi^1_{\widetilde{m}}\left(\pi, r, \widetilde{H}\right)$.

(2.7) For every $i \geq 2$, under the conditions of Lemma (1.4) it holds that if $r_0 \to +\infty$, then $\widetilde{m}^i_0 \to 0$ for all $(\widetilde{m}^i_0, \widetilde{m}^i_1 \dots, \widetilde{m}^i_S) \in \varphi^i_{\widetilde{m}}\left(\pi, \mu, r, \widetilde{H}\right)$. Under the conditions of Lemma (1.9), the same property is true for $i = 1$ for every $(\widetilde{m}^1_0, \widetilde{m}^1_1 \dots, \widetilde{m}^1_S) \in \varphi^1_{\widetilde{m}}\left(\pi, r, \widetilde{H}\right)$.

(2.8) For every $i \geq 2$, under the conditions of Lemma (1.5) it holds that if there is a $s' \in S$ with $r_{s'} \to +\infty$, then $\widetilde{m}^i_{s'} \to 0$ for all $(\widetilde{m}^i_0, \dots, \widetilde{m}^i_{s'}, \dots, \widetilde{m}^i_S) \in \varphi^i_{\widetilde{m}}\left(\pi, \mu, r, \widetilde{H}\right)$. Under the conditions of Lemma (1.10), the same property is true for $i = 1$ for every $(\widetilde{m}^1_0, \widetilde{m}^1_1 \dots, \widetilde{m}^1_S) \in \varphi^1_{\widetilde{m}}\left(\pi, r, \widetilde{H}\right)$.

Proof: Parts (2.1) - (2.5) follow from standard arguments using the results from Lemma 1. Since money is dominated as a store of value for a strictly positive interest rate, $\widetilde{m}^i_s = \pi_s \cdot (x^i_s - e^i_s)^-, \forall s \in S^*, \forall i \in I$. This implies (2.6). Concerning (2.7), we first argue for $i \geq 2$. For $r_0 \to +\infty$ we argue that the sequence of best responses converges to a $(x^i, \widetilde{m}^i, \widetilde{b}^i, \widetilde{\theta}^i)$ such that $\widetilde{m}^i_0 = 0$. From Lemma (2.2), the demand set is upper hemi-continuous along this sequence. We will argue that if $\widetilde{m}^i_0 > 0$ in the limit, then the household can increase his utility. Since the cash-in-advance constraint binds in the case of positive interest rates,

$\widetilde{m}_0^i > 0$ implies that he sells something of his endowment. If he deviates by selling nothing and consuming what he sold before, his utility increases. The revenue which he loses in period one from not selling the endowment in period zero can be taken from buying costless bonds. This deviation implies that $\widetilde{m}_0^i > 0$ cannot be the best response in the limit. For household $i = 1$, the same property is true. In fact, in the abstract economy, equation (10) becomes

$$\pi_0 \cdot \left(x_0^1 - e_0^1\right)^+ + \frac{\pi_{\mathbf{1}} \cdot \left(x_{\mathbf{1}}^1 - e_{\mathbf{1}}^1\right)^+}{1 + r_0} = \delta^1 \left(\widetilde{H}_0 + \frac{\widetilde{H}_{\mathbf{1}} \cdot \mathbf{1}}{1 + r_0} \right)$$
$$+ \frac{\pi_0 \cdot \left(\left(x_0^1 - e_0^1\right)^- - \tau_0^1 \right)}{1 + r_0} + \frac{1}{1 + r_0} \pi_{\mathbf{1}} \cdot \left(\left(\left(x_{\mathbf{1}}^1 - e_{\mathbf{1}}^1\right)^- - \tau_{\mathbf{1}}^1 \right) \square \frac{1}{1 + r_{\mathbf{1}}} \right).$$

This equation reveals that household one earns zero from selling his endowments in $t = 0$. Maximization thus implies selling no endowments. From the cash-in-advance it follows that money demand is zero. Part (2.8) follows from the same logic as part (2.7). ■

Define the market demand correspondence of the commodity and money as

$$Z\left(\pi, \mu, r, \widetilde{H}\right) := \varphi^1\left(\pi, r, \widetilde{H}\right) + \sum_{i \geq 2} \varphi^i\left(\pi, \mu, r, \widetilde{H}\right),$$

and the projections of this set onto commodity and money spaces by $Z_x\left(\pi, \mu, r, \widetilde{H}\right)$ and $Z_{\widetilde{m}}\left(\pi, \mu, r, \widetilde{H}\right)$, respectively.

Lemma 3 *Under Assumptions 1 - 3,* $Z\left(\pi, \mu, r, \widetilde{H}\right)$ *satisfies the following properties:*

(3.1) $Z\left(\pi, \mu, r, \widetilde{H}\right)$ *is non-empty, compact and convex-valued for* $\left(\pi, \mu, r, \widetilde{H}\right) \in$ *interior*$(\Delta) \times$ *interior*$(\Delta^{S-1}) \times \mathbb{R}_{++}^{S+1} \times \mathbb{R}_{+}^{S+1}$.

(3.2) $Z\left(\pi, \mu, r, \widetilde{H}\right)$ *is upper hemi-continuous for* $\left(\pi, \mu, r, \widetilde{H}\right) \in$ *interior*$(\Delta) \times \Delta^{S-1} \times \mathbb{R}_{+}^{S+1} \times \mathbb{R}_{+}^{S+1}$.

(3.3) If $(z_x, z_{\widetilde{m}}) \in Z\left(\pi, \mu, r, \widetilde{H}\right)$ *and if (27) holds, then* $(1+r_0)\, \pi_0 \cdot \left(z_{x_0} - \sum_i e_0^i\right) + \pi_{\mathbf{1}} \cdot \left(z_{x_{\mathbf{1}}} - \sum_i e_{\mathbf{1}}^i\right) + r_0 \left(z_{\widetilde{m}_0} - \widetilde{M}_0\right) + \frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot \left(z_{\widetilde{m}_{\mathbf{1}}} - \widetilde{M}_{\mathbf{1}}\right) = 0$.

(3.4) If $z_{\widetilde{m}} \in Z_{\widetilde{m}}(\pi, \mu, r, \widetilde{H})$ *then* $z_{\widetilde{m}_s} \leq \max_{s,l} \sum_i e^i_{sl}$ *for* $r \gg 0$ *and all* $s \in S^*$.

Proof: Lemma (3.1), (3.2) and (3.4) follow directly from individual demand (Lemma 2). Lemma (3.3) follows from adding up (25) over $i \in I$ and using (27).■

6.3 Proof of Theorem 1

We fix the martingale measure $\mu \gg 0$ and the inverse price level $c > 0$ at the outset. The transfers are determined endogenously according to the balanced transfer rule.

6.3.1 Preliminary definitions

From Assumption 5, we can define a government transfer function $\widetilde{H}\left(\pi, r, \widetilde{M}\right) := \left(\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_S\right)\left(\pi, r, \widetilde{M}\right)$, where

$$\widetilde{H}_s\left(\pi, r, \widetilde{M}\right) := \frac{r_s}{1+r_s}\widetilde{M}_s + \frac{\pi_s \cdot \tau_s}{1+r_s}, \forall s \in S^*.$$

We slightly abuse the notation by denoting both the function and the image with $\widetilde{H}$. By construction, $\widetilde{H}_s \geq 0$, $\forall s \in S^*$. Obviously, $\widetilde{H}\left(\pi, r, \widetilde{M}\right)$ is a bounded and continuous function for $\left(\pi, r, \widetilde{M}\right) \in \Delta \times \mathbb{R}^{S+1}_+ \times \mathbb{R}^{S+1}_+$.

6.3.2 Construction of a fixed point mapping

To make the proof compatible with a zero interest rate, we start by defining the modified interest rate process $r^n := (r^n_s)_{s \in S^*}$ by

$$r^n_s = \begin{cases} r_s & \text{if } r_s > 0 \\ \frac{1}{n} & \text{if } r_s = 0. \end{cases}$$

For $n > (S+1)L$ define

$$\Delta^n := \left\{\pi \in \Delta \middle| \pi_{sl} \geq \frac{1}{n}\right\}.$$

It is easy to see that $\bigcup\limits_{n>(S+1)L}^{\infty} \Delta^n = \text{interior}\,(\Delta)$. Let $K_{\widetilde{m}}$ be a compact and convex space such that

$$K_{\widetilde{m}} \supseteq Z^n_{\widetilde{m}}\left(\pi, \widetilde{H}\right)$$

for all $\pi \in \Delta$ and $\widetilde{H} \in \mathbb{R}^{S+1}_{+}$, where $Z^n_{\widetilde{m}}\left(\pi, \widetilde{H}\right) := Z_{\widetilde{m}}\left(\pi, \mu, r^n, \widetilde{H}\right)$. Since $r^n \gg 0$ for all finite n, such a compact set exists by Lemma (3.4).[10] Define a compact and convex set $K_{\widetilde{H}}$ such that

$$K_{\widetilde{H}} \supseteq \widetilde{H}^n\left(\pi, \widetilde{M}\right)$$

for all $\pi \in \Delta$ and $\widetilde{M} \in K_{\widetilde{m}}$, where $\widetilde{H}^n\left(\pi, \widetilde{M}\right) := \widetilde{H}\left(\pi, r^n, \widetilde{M}\right)$. Since $\widetilde{H}$ is a bounded function and $\widetilde{M} \in K_{\widetilde{m}}$, such a set $K_{\widetilde{H}}$ exists. Further define $Z^n_x\left(\pi, \widetilde{H}\right) := Z_x\left(\pi, \mu, r^n, \widetilde{H}\right)$ and a compact and convex set K^n_x such that

$$K^n_x \supseteq Z^n_x\left(\pi, \widetilde{H}\right)$$

for all $\pi \in \Delta^n$ and $H \in K_{\widetilde{H}}$. Denote the product set with $K^n := K^n_x \times K_{\widetilde{m}}$. Note that only K^n_x depends on n. Finally, define the mapping

$$f^n : \Delta^n \times K_{\widetilde{m}} \times K_{\widetilde{H}} \times K^n \rightrightarrows \Delta^n \times K_{\widetilde{m}} \times K_{\widetilde{H}} \times K^n$$

by

$$\left(\pi, \widetilde{M}, \widetilde{H}, z\right) \overset{f^n}{\mapsto} \left(f^n_\pi, f^n_{\widetilde{M}}, f^n_{\widetilde{H}}, f^n_z\right),$$

where

$$\begin{aligned}
f^n_\pi(\pi, \widetilde{M}, \widetilde{H}, z) &:= \underset{\{\pi\in\Delta^n\}}{\arg\max}\left\{(1+r_0)\,\pi_0 \cdot \left(z_{x_0} - \sum_i e^i_0\right) + \pi_{\mathbf{1}} \cdot \left(z_{x_{\mathbf{1}}} - \sum_i e^i_{\mathbf{1}}\right)\right\}, \\
f^n_{\widetilde{M}}(\pi, \widetilde{M}, \widetilde{H}, z) &:= z_{\widetilde{m}}, \\
f^n_{\widetilde{H}}(\pi, \widetilde{M}, \widetilde{H}, z) &:= \widetilde{H}^n\left(\pi, \widetilde{M}\right), \\
f^n_z(\pi, \widetilde{M}, \widetilde{H}, z) &:= Z^n\left(\pi, \widetilde{H}\right).
\end{aligned}$$

The first mapping is the price player's objective function, the second mapping says that the government accommodates money demand, the third mapping is the government transfer function and the last mapping is the market demand.

[10] Even though r depends on n, the set $K_{\widetilde{m}}$ does not depend on n by Lemma (3.4).

From Lemma 3 we infer that $f^n(\pi, \widetilde{M}, \widetilde{H}, z)$ is a non-empty, compact, convex-valued and upper hemi-continuous correspondence. Kakutani Fixed Point Theorem establishes the existence of a fixed point $\left(\pi^{*n}, \widetilde{M}^{*n}, \widetilde{H}^{*n}, z^{*n}\right)$.

6.3.3 The limit of the fixed points is an equilibrium

Since $\left(\pi^{*n}, \widetilde{M}^{*n}, \widetilde{H}^{*n}, z^{*n}\right)$ is bounded for each n, there exists a convergent subsequence with limit $\left(\pi^{*}, \widetilde{M}^{*}, \widetilde{H}^{*}, z^{*}\right)$, where π^* is trivially bounded. By Lemma (3.4), $z^*_{\widetilde{m}}$ is also finite since $z^n_{\widetilde{m}}$ is bounded above by the aggregate endowment for all n. By construction, $\widetilde{M}^* = z^*_{\widetilde{m}}$. Since $\widetilde{H}\left(\pi, \widetilde{M}\right)$ is continuous, $\widetilde{H}^* = \widetilde{H}\left(\pi^*, \widetilde{M}^*\right)$. This implies that $\widetilde{H}^*$ is finite. It only remains to show that $z^*_x = \sum_i e^i$ and $z^*_x \in Z_x\left(\pi^*, \widetilde{H}^*\right)$.

It follows from $\widetilde{M}^* = z^*_{\widetilde{m}}$ and Lemma (3.3) that for all n

$$(1+r_0)\,\pi_0^{*n} \cdot \left(z^{*n}_{x_0} - \sum_i e^i_0\right) + \pi^{*n}_{\mathbf{1}} \cdot \left(z^{*n}_{x_{\mathbf{1}}} - \sum_i e^i_{\mathbf{1}}\right) = 0,$$

which implies

$$(1+r_0)\,\pi_0^{*} \cdot \left(z^{*}_{x_0} - \sum_i e^i_0\right) + \pi^{*}_{\mathbf{1}} \cdot \left(z^{*}_{x_{\mathbf{1}}} - \sum_i e^i_{\mathbf{1}}\right) = 0$$

in the limit. Consequently, we have $z^*_x \leq \sum_i e^i$, and $z^*_x = \sum_i e^i$ if $\pi^* \gg 0$. However, from Lemma (2.5) we know that household one's demand goes to infinity if some $\pi^*_{sl} \to 0$. Since aggregate excess demand is bounded below, we get that $\|z^*_x\| \to +\infty$ if some $\pi^*_{sl} \to 0$. Therefore, $z^*_x \leq \sum_i e^i$ implies that $\pi^* \gg 0$ and $z^*_x = \sum_i e^i$. Since $Z_x\left(\pi, \widetilde{H}\right)$ is upper hemi-continuous for $\pi \gg 0$, we know that $z^*_x \in Z_x\left(\pi^*, \widetilde{H}^*\right)$.

It is straightforward to see that the vector $\left(\pi^{*}, \widetilde{M}^{*}, \widetilde{H}^{*}, z^{*}\right)$ corresponds to an equilibrium in the abstract economy under interest rate peg with balanced transfers.

6.4 Proof of Theorem 2

Similar to the proof of Theorem 1, we fix an arbitrary inverse price level $c > 0$ and an arbitrary martingale measure $\mu \gg 0$.

6.4.1 Preliminary definitions

In the abstract economy, the money supply vector is $\widetilde{M} = c \cdot \left(\overline{M}_0, \left(\mu_s \overline{M}_s \right)_{s \in S} \right) \gg 0$. Define the transfers $\widetilde{H}\left(\pi, r, \widetilde{M}\right)$ to individuals as in Section 6.3.1. Since $\widetilde{M}$ is fixed here, we write $\widetilde{H}(\pi, r)$.

6.4.2 Construction of a fixed point mapping

As before, we define $\Delta^n := \left\{ \pi \in \Delta | \pi_{sl} \geq \frac{1}{n} \right\}$ for $n > (S+1)L$. Define the set $\Omega^n := \left[\frac{1}{n}, n \right]^{S+1}$ carrying the interest rates r. Let $K_{\widetilde{m}}$ be a compact and convex space such that

$$K_{\widetilde{m}} \supseteq Z_{\widetilde{m}} \left(\pi, r, \widetilde{H} \right)$$

for all $\pi \in \Delta^n$, $r \in \Omega^n$ and $\widetilde{H} \in \mathbb{R}_+^{S+1}$. Define a set $K_{\widetilde{H}}$ such that

$$K_{\widetilde{H}} \supseteq \widetilde{H}(\pi, r)$$

for all $\pi \in \Delta^n$ and $r \in \Omega^n$. Define the compact and convex set K_x^n such that

$$K_x^n \supseteq Z_x^n \left(\pi, r, \widetilde{H} \right)$$

for all $\pi \in \Delta^n$, $r \in \Omega^n$ and $\widetilde{H} \in \mathbb{R}_+^{S+1}$. The product set is $K^n := K_x^n \times K_{\widetilde{m}}$. As before, define the mapping

$$f^n : \Delta^n \times \Omega^n \times K_{\widetilde{H}} \times K^n \rightrightarrows \Delta^n \times \Omega^n \times K_{\widetilde{H}} \times K^n$$

by

$$\left(\pi, r, \widetilde{H}, z \right) \overset{f^n}{\mapsto} \left(f_\pi^n, f_r^n, f_{\widetilde{H}}^n, f_z^n \right),$$

where

$$
\begin{aligned}
f_\pi^n\left(\pi, r, \widetilde{H}, z\right) &:= \arg\max_{\pi\in\Delta^n} \left\{(1+r_0)\,\pi_0\cdot\left(z_{x_0}-\sum_i e_0^i\right)+\pi_{\mathbf{1}}\cdot\left(z_{x_{\mathbf{1}}}-\sum_i e_{\mathbf{1}}^i\right)\right\},\\
f_r^n\left(\pi, r, \widetilde{H}, z\right) &:= \arg\max_{r\in\Omega^n} \left\{r_0\left(z_{\widetilde{m}_0}-\widetilde{M}_0\right)+\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}}\cdot\left(z_{\widetilde{m}_{\mathbf{1}}}-\widetilde{M}_{\mathbf{1}}\right)\right\},\\
f_{\widetilde{H}}^n\left(\pi, r, \widetilde{H}, z\right) &:= \widetilde{H}\left(\pi, r\right),\\
f_z^n\left(\pi, r, \widetilde{H}, z\right) &:= Z\left(\pi, r, \widetilde{H}\right).
\end{aligned}
$$

Again, all these mappings satisfy the assumptions required to apply Kakutani's Theorem, implying the existence of a fixed point $\left(\pi^{*n}, r^{*n}, \widetilde{H}^{*n}, z^{*n}\right)$.

6.4.3 The limit of the fixed points is an equilibrium

Since $\left(\pi^{*n}, r^{*n}, \widetilde{H}^{*n}, z^{*n}\right)$ is bounded for each n, there exists a convergent subsequence with limit $\left(\pi^*, r^*, \widetilde{H}^*, z^*\right)$, where $\left(\pi^*, \widetilde{H}^*, z^*_{\widetilde{m}}\right) \ll +\infty$. Since $\widetilde{H}(\pi, r)$ is continuous, we know that $\widetilde{H}^* = \widetilde{H}(\pi^*, r^*)$. Hence it remains to show that markets clear and $(r^*, z^*_x) \ll +\infty$.

Claim 1: *In period zero, the money market clears and* $r_0^* < +\infty$. To see this, we argue in three steps.

Step 1: We prove that $z^*_{\widetilde{m}_0} \leq \widetilde{M}_0$. Suppose not, i.e. $z^*_{\widetilde{m}_0} > \widetilde{M}_0$. From the construction of f_r^n, we must have $r_0^* = +\infty$. Then it follows from the definition of $\widetilde{H}(\pi^*, r^*)$ that $\widetilde{H}_0^*$ is strictly positive. Lemma (2.7) implies that $z^*_{\widetilde{m}_0} = 0$, a contradiction to $z^*_{\widetilde{m}_0} > \widetilde{M}_0^* > 0$. Hence $z^*_{\widetilde{m}_0} \leq \widetilde{M}_0$.

Step 2: We prove that $z^*_{\widetilde{m}_0} = \widetilde{M}_0$ if $r_0^* > 0$. Since the construction of f_r^n implies $r_0^*\left(z^*_{\widetilde{m}_0} - \widetilde{M}_0\right) \geq 0$ and $z^*_{\widetilde{m}_0} \leq \widetilde{M}_0$ implies $r_0^*\left(z^*_{\widetilde{m}_0} - \widetilde{M}_0\right) \leq 0$, we must have $r_0^*\left(z^*_{\widetilde{m}_0} - \widetilde{M}_0\right) = 0$. Therefore, $r_0^* > 0$ implies $z^*_{\widetilde{m}_0} - \widetilde{M}_0 = 0$, which means the money market clearing in period zero (with free disposal) in the limit.

Step 3: We prove $r_0^* < +\infty$. Suppose that $r_0^* = +\infty$. From the first step, we know that $z^*_{\widetilde{m}_0} = 0$; from the second step, we know $z^*_{\widetilde{m}_0} = \widetilde{M}_0$. These two facts imply $\widetilde{M}_0 = 0$, a contradiction. ■

Claim 2: *For every $s \in S$, the money market in state s clears and $r_s^* < +\infty$.* To see this, we argue in several steps.

Step 1: We show that $z_{x_0}^* \leq \sum_i e_0^i$ and $z_{x_{\mathbf{1}}}^* \leq \sum_i e_{\mathbf{1}}^i$. Indeed, using the definition of $\widetilde{H}(\pi, r)$ and Lemma (3.3), it follows $(1 + r_0^*)\, \pi_0^* \cdot \left(z_{x_0}^* - \sum_i e_0^i\right) + \pi_{\mathbf{1}}^* \cdot \left(z_{x_{\mathbf{1}}}^* - \sum_i e_{\mathbf{1}}^i\right) + r_0^* \left(z_{\widetilde{m}_0}^* - \widetilde{M}_0\right) + \frac{r_{\mathbf{1}}^*}{1+r_{\mathbf{1}}^*} \cdot \left(z_{\widetilde{m}_{\mathbf{1}}}^* - \widetilde{M}_{\mathbf{1}}\right) = 0$. In Claim 1 we established $r_0^* \left(z_{\widetilde{m}_0}^* - \widetilde{M}_0\right) = 0$. In addition, the definition of f_r^n implies that $\frac{r_{\mathbf{1}}^*}{1+r_{\mathbf{1}}^*} \cdot \left(z_{\widetilde{m}_{\mathbf{1}}}^* - \widetilde{M}_{\mathbf{1}}\right) \geq 0$. From this it follows that $(1 + r_0^*)\, \pi_0^* \cdot \left(z_{x_0}^* - \sum_i e_0^i\right) + \pi_{\mathbf{1}}^* \cdot \left(z_{x_{\mathbf{1}}}^* - \sum_i e_{\mathbf{1}}^i\right) \leq 0$. From the definition of f_π^n we get $z_{x_0}^* \leq \sum_i e_0^i$ and $z_{x_{\mathbf{1}}}^* \leq \sum_i e_{\mathbf{1}}^i$.

Step 2: The conditions of Lemma (2.5) apply. To see this, we argue that either the conditions of Lemma (1.8) or the conditions of Lemma (1.10) are satisfied. In fact, If $r_{\mathbf{1}}^* \ll +\infty$, then the conditions of Lemma (1.8) apply trivially. Alternatively, if there is some $s \in S$ with $r_s^* = +\infty$, then it follows from the definition of $\widetilde{H}(\pi, r)$ that $\widetilde{H}_{\mathbf{1}}^* > 0$. Then the conditions of Lemma (1.10) hold.

Step 3: We show that $\pi^* \gg 0$. In fact, we saw in the previous step that Lemma (2.5) can be applied. So if there is a $s \in S^*$ and a $l \in L$ such that $\pi_{sl}^* = 0$, then Lemma (2.5) implies a contradiction to Step 1.

Step 4: Since $r_0^* < +\infty$ and $\pi^* \gg 0$, Lemma (2.8) applies. With this in mind, it is easy to see that the Steps 1-3 of Claim 1 apply. Hence Claim 2 follows. ■

From money market clearing it follows now by the same arguments as in the proof of Theorem 1 that $z_x^* = \sum_i e^i$. From $\pi^* \gg 0$ and the upper hemi-continuity of the demand, $z^* \in Z(\pi^*, r^*, \widetilde{H}^*)$.

Finally, $\left(\pi^*, r^*, \widetilde{H}^*, z^*\right)$ corresponds to an equilibrium in the abstract economy with money supply control and balanced transfers.

6.5 Proof of Theorem 3

With fixed nominal transfers we just introduce a transfer mapping which transforms the original transfers into discounted real transfers. In addition, we now determine c and μ endogenously in the fixed point.

6.5.1 Preliminary definitions

Define an augmented taxation $\tau^n \in \mathbb{R}_+^{(S+1)L}$ as

$$\tau_s^n = \begin{cases} \tau_s & \text{if } \tau_s > 0 \\ \left(\frac{1}{n}, 0, .., 0\right) \in \mathbb{R}_+^L & \text{if } \tau_s = 0 \end{cases},$$

where $n \in \mathbb{N}$ and $n > \min_s \left(\frac{1}{e_{s1}^1}\right)$. Without loss of generality, if $\tau_s = 0$ we can assign the taxation to household 1, i.e. $\tau_s^{1n} = \left(\frac{1}{n}, 0, .., 0\right)$ and hence $\tau_s^{in} = 0$ for all $i \neq 1$. It is easy to see that household 1 will have a non-empty budget set for all $n > \min_s \left(\frac{1}{e_{s1}^1}\right)$.

Define a government transfer function $\widetilde{H}(c, \mu) := \left(\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_S\right)(c, \mu)$ by

$$\widetilde{H}_0(c, \mu) := c\,\overline{H}_0 \quad \text{and} \quad \widetilde{H}_s(c, \mu) := c\,\mu_s \overline{H}_s, \ \forall s \in S.$$

This function is obviously a bounded and continuous function for finite c. Next, define an inverse price level function c^n by

$$c^n\left(\pi, z_{\widetilde{m}}, \mu; r\right) := \frac{\frac{r_0}{1+r_0} z_{\widetilde{m}_0} + \frac{1}{1+r_0} \frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot z_{\widetilde{m}_{\mathbf{1}}} + \frac{\pi_0 \tau_0^n}{1+r_0} + \frac{1}{1+r_0} \pi_{\mathbf{1}} \cdot \left(\frac{1}{1+r_{\mathbf{1}}} \Box \tau_{\mathbf{1}}^n\right)}{\overline{H}_0 + \sum_{s \in S} \frac{\mu_s}{1+r_0} \overline{H}_s},$$

and use the shortcut $c^n\left(\pi, z_{\widetilde{m}}, \mu\right) := c^n\left(\pi, z_{\widetilde{m}}, \mu; r\right)$. This is a bounded and continuous function of $(\pi, z_{\widetilde{m}}, \mu)$ for each n as long as $\overline{H}_0 > 0$ and $z_{\widetilde{m}} \ll +\infty$. Under the latter condition, define the bounded and continuous martingale-measure function $\mu^n\left(\pi, z_{\widetilde{m}}, c, \mu\right) := \left(\mu_1^n, \ldots, \mu_S^n\right)(\pi, c, \mu, z_{\widetilde{m}}; r)$ by

$$\mu_s^n\left(\pi, c, \mu, z_{\widetilde{m}}; r\right) := \frac{\frac{r_s}{1+r_s} z_{\widetilde{m}_s} + \frac{1}{1+r_s} \pi_s \cdot \tau_s^n + \mu_s c \left(\max_{s' \in S} \overline{H}_{s'} - \overline{H}_s\right)}{\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot z_{\widetilde{m}_{\mathbf{1}}} + \pi_{\mathbf{1}} \cdot \left(\frac{1}{1+r_{\mathbf{1}}} \Box \tau_{\mathbf{1}}\right) + c \sum_{\sigma \in S} \mu_\sigma \left(\max_{s' \in S} \overline{H}_{s'} - \overline{H}_\sigma\right)}.$$

For fixed r, we just write $\mu_s^n\left(\pi, c, \mu, z_{\widetilde{m}}\right)$. By the construction of τ^n, as long as $\pi \gg 0$ we have $c^n > 0$, $\mu_s^n > 0$ and $\sum_{s=1}^S \mu_s^n = 1$ for all finite n.

6.5.2 Construction of a fixed point mapping

Denote aggregate demand with $Z^n\left(\pi, \mu, \widetilde{H}\right) := Z\left(\pi, r, \mu, \widetilde{H}, \tau^n\right)$. Lemma (3.4) allows us to define the compact and convex set $K_{\widetilde{m}}$ such that

$$K_{\widetilde{m}} \supseteq Z^n_{\widetilde{m}}\left(\pi, \mu, \widetilde{H}\right)$$

for all $\pi \in \Delta$, $\mu \in \Delta^{S-1}$ and $\widetilde{H} \in \mathbb{R}^{S+1}_{+}$. Notice that $K_{\widetilde{m}}$ does not depend on n. As argued above, for positive $\overline{H}_0$ we can define a compact and convex set K^n_c such that

$$K^n_c \supseteq c^n\left(\pi, z_{\widetilde{m}}, \mu\right)$$

for all $\pi \in \Delta$, $z_{\widetilde{m}} \in K_{\widetilde{m}}$ and $\mu \in \Delta^{S-1}$. $\mu^n\left(\pi, c, \mu, z_{\widetilde{m}}\right)$ lies in a compact and convex set

$$K^n_\mu \subset interior(\Delta^{S-1})$$

for $\pi \in \Delta^n$, $z_{\widetilde{m}} \in K_{\widetilde{m}}$ and $c \in K^n_c$. Introduce the set $K^n_{\widetilde{H}}$ such that

$$K^n_{\widetilde{H}} \supseteq \widetilde{H}(c, \mu)$$

for all $c \in K^n_c$ and $\mu \in K^n_\mu$. This set can be chosen to be compact and convex for every n since $c \in K^n_c$. Further define K^n_x such that

$$K^n_x \supseteq Z^n_x\left(\pi, \mu, \widetilde{H}\right)$$

for all $\pi \in \Delta^n$, $\mu \in K^n_\mu$ and $\widetilde{H} \in K^n_{\widetilde{H}}$. Finally, $K^n := K^n_x \times K_{\widetilde{m}}$. Define the mapping

$$f^n : \Delta^n \times K^n_c \times K^n_\mu \times K^n_{\widetilde{H}} \times K^n \rightrightarrows \Delta^n \times K^n_c \times K^n_\mu \times K^n_{\widetilde{H}} \times K^n$$

by

$$\left(\pi, c, \mu, \widetilde{H}, z\right) \overset{f^n}{\mapsto} \left(f^n_\pi, f^n_c, f^n_\mu, f^n_{\widetilde{H}}, f^n_z\right),$$

where

$$
\begin{aligned}
f_\pi^n\left(\pi,c,\mu,\widetilde{H},z\right) &:= \underset{\{\pi\in\Delta^n\}}{\arg\max}\left\{(1+r_0)\,\pi_0\cdot\left(z_{x_0}-\sum_i e_0^i\right)+\pi_{\mathbf{1}}\cdot\left(z_{x_{\mathbf{1}}}-\sum_i e_{\mathbf{1}}^i\right)\right\},\\
f_c^n\left(\pi,c,\mu,\widetilde{H},z\right) &:= c^n\left(\pi,z_{\widetilde{m}},\mu\right),\\
f_\mu^n\left(\pi,c,\mu,\widetilde{H},z\right) &:= \mu^n\left(\pi,c,\mu,z_{\widetilde{m}}\right),\\
f_{\widetilde{H}}^n\left(\pi,c,\mu,\widetilde{H},z\right) &:= \widetilde{H}(c,\mu),\\
f_z^n\left(\pi,c,\mu,\widetilde{H},z\right) &:= Z^n\left(\pi,\mu,\widetilde{H}\right).
\end{aligned}
$$

$f^n\left(\pi,c,\mu,\widetilde{H},z\right)$ is a non-empty, compact, convex-valued and upper hemi-continuous correspondence. Kakutani fixed point theorem establishes the existence of a fixed point $\left(\pi^{*n},c^{*n},\mu^{*n},\widetilde{H}^{*n},z^{*n}\right)$.

Note that the money market is always cleared since the central bank accommodates money demand. From the construction of $c^n\left(\pi,z_{\widetilde{m}},\mu\right)$, in the fixed point the equation (27) holds, i.e.

$$
r_0\widetilde{M}_0^{*n}+\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}}\cdot\widetilde{M}_{\mathbf{1}}^{*n}+\pi_0^{*n}\cdot\tau_0^n+\pi_{\mathbf{1}}^{*n}\cdot\left(\frac{1}{1+r_{\mathbf{1}}}\Box\tau_{\mathbf{1}}^n\right)=\widetilde{H}_0^{*n}(1+r_0)+\widetilde{H}_{\mathbf{1}}^{*n}\cdot\mathbf{1}. \quad (28)
$$

From the construction of $\mu^n\left(\pi,z_{\widetilde{m}},c,\mu\right)$ it follows

$$
\begin{aligned}
&\frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}}\cdot\widetilde{M}_{\mathbf{1}}^{*n}+\pi_{\mathbf{1}}^{*n}\cdot\left(\frac{1}{1+r_{\mathbf{1}}}\Box\tau_{\mathbf{1}}^n\right)+c^{*n}\sum_{\sigma\in S}\mu_\sigma^{*n}\left(\max_{s'\in S}\overline{H}_{s'}-\overline{H}_\sigma\right)\\
&=\frac{\frac{r_s}{1+r_s}\widetilde{M}_s^{*n}+\frac{1}{1+r_s}\pi_s^{*n}\cdot\tau_s^n+c^{*n}\mu_s^{*n}\left(\max_{s'\in S}\overline{H}_{s'}-\overline{H}_s\right)}{\mu_s^{*n}}.
\end{aligned}
$$

Use this equation and (28) to get

$$
\frac{r_s}{1+r_s}\widetilde{M}_s^{*n}+\frac{1}{1+r_s}\pi_s^{*n}\cdot\tau_s^n-\widetilde{H}_s^{*n}+\mu_s^{*n}r_0\widetilde{M}_0^{*n}+\mu_s^{*n}\pi_0^{*n}\cdot\tau_0^n=\mu_s^{*n}\widetilde{H}_0^{*n}(1+r_0),
$$

which proves that the government balances its budget (26).

6.5.3 The limit of the fixed points for $n\to\infty$

Since $\left(\pi^{*n},c^{*n},\mu^{*n},\widetilde{H}^{*n},z^{*n}\right)$ is bounded for each n, we can let $n\to\infty$ and denote the limit with $\left(\pi^*,c^*,\mu^*,\widetilde{H}^*,z^*\right)$, where $\left(\pi^*,\mu^*,z_{\widetilde{m}}^*\right)\ll+\infty$ and z_x^*

could be infinite. Clearly, $\tau^n \to \tau$. By continuity, $\widetilde{H}^* = H(c^*, \mu^*)$, $c^* = c(\pi^*, z^*_{\widetilde{m}}, \mu^*)$ and $\mu^* = \mu(\pi^*, c^*, \mu^*, z^*_{\widetilde{m}})$. From $z^*_{\widetilde{m}} \ll +\infty$ we know that $c^* < +\infty$. From this we infer $\widetilde{H}^* \ll +\infty$. It remains to argue that markets clear, the government balances its budget, $0 \ll (\pi^*, c^*, \mu^*)$, $z^*_x \ll +\infty$ and $z^* \in Z\left(\pi^*, \mu^*, \widetilde{H}^*\right)$.

Given the construction of $c^n(\pi, \widetilde{M}, \widetilde{H})$, Lemma (3.3) applies for every n. Together with money market clearing (the central bank accommodates money demand) it hence follows $(1 + r_0)\, \pi^{*n}_0 \cdot \left(z^{*n}_{x_0} - \sum_i e^i_0\right) + \pi^{*n}_{\mathbf{1}} \cdot \left(z^{*n}_{x_{\mathbf{1}}} - \sum_i e^i_{\mathbf{1}}\right) = 0$ for every n. Hence

$$(1 + r_0)\, \pi^*_0 \cdot \left(z^*_{x_0} - \sum_i e^i_0\right) + \pi^*_{\mathbf{1}} \cdot \left(z^*_{x_{\mathbf{1}}} - \sum_i e^i_{\mathbf{1}}\right) = 0.$$

Since the interest rates are always finite, Lemma (1.8) allows for the application of the Lemmas (2.4) and (2.5). Hence, $z^*_x = \sum_i e^i \ll +\infty$ and $\pi^* \gg 0$.

It is easy to see that the government budget is also balanced in the limit. $z^* \in Z\left(\pi^*, \mu^*, \widetilde{H}^*\right)$ follows from finite interest rates and $\pi^* \gg 0$ since Lemma (1.3) and Lemma (2.2) apply. Therefore, we need only to show that $c^* > 0$ and $\mu^* \gg 0$.

We first prove that $\mu^* \gg 0$. Suppose $\mu^*_s = 0$ for $s \in S$. From the definition of $\mu(\pi, \widetilde{m}, c, \mu)$ we get $z_{\widetilde{m}_s} = 0$ and $\tau_s = 0$. For every n along the sequence of fixed points, the consumer's budget in $s \in S$ is

$$\begin{aligned}
\pi^{*n}_s \cdot x^{i*n}_s + \frac{\widetilde{b}^{i*n}_s}{1 + r_s} + \widetilde{m}^{i*n}_s &\leq \pi^{*n}_s \cdot e^i_s + (\widetilde{b}^{i*n}_0 + A_s \widetilde{\theta}^{i*n} + \widetilde{m}^{i*n}_0)\mu^{*n}_s + \delta^i \widetilde{H}^*_s, \\
\widetilde{m}^{i*n}_s &\geq \pi^{*n}_s \cdot (x^{i*n}_s - e^i_s)^-, \\
\widetilde{b}^{i*n}_s + \widetilde{m}^{i*n}_s &= 0.
\end{aligned}$$

Since households optimize, we must have $\frac{r_s}{1+r_s} \widetilde{m}^{i*n}_s = \frac{r_s}{1+r_s} \pi^{*n}_s \cdot (x^{i*n}_s - e^i_s)^-$. We can use this to derives the equivalent formulation

$$\pi^{*n}_s \cdot (x^{i*n}_s - e^i_s)^+ = \frac{\pi^{*n}_s \cdot (x^{i*n}_s - e^i_s)^-}{1 + r_s} + (\widetilde{b}^{i*n}_0 + A_s \widetilde{\theta}^{i*n} + \widetilde{m}^{i*n}_0)\mu^{*n}_s + \delta^i \widetilde{H}^{*n}_s.$$

By the cash-in-advance constraint, $(x^{i*}_s - e^i_s)^- = 0$ for every $i \in I$. Since markets clear and nobody sells goods it follows that $x^{i*}_s = e^i_s$ for all $i \in I$. From $\widetilde{H}(c, \mu)$, we know that $\widetilde{H}_s^* = 0$. Hence we get $(\widetilde{b}_0^{i*} + A_s\widetilde{\theta}^{i*} + \widetilde{m}_0^{i*})\mu_s^* = 0$ from the budget constraint. For $n \to \infty$, we get from the continuity of the budget set and from what we said previously that

$$\pi_s^* \cdot (x_s^{i*} - e_s^i)^+ = \frac{\pi_s^* \cdot (x_s^{i*} - e_s^i)^-}{1 + r_s}.$$

Define, for every $i \in I$, the utility function $v^i : \mathbb{R}_+^L \to \mathbb{R}$ by $v^i(\zeta_s^i) := u^i(x_0^{i*}, x_1^{i*}, \ldots, e_s^i + \zeta_s^i, \ldots, x_S^{i*})$. From what we said before, it follows $0 = \arg\max \left\{ v^i(\zeta_s^i) \,\middle|\, \pi_s^* \cdot \zeta_s^{i+} \leq \frac{\pi_s^* \cdot \zeta_s^{i-}}{1+r_s} \right\}$, $\forall i \in I$. Define the function $\bar{\zeta}_s(\zeta_s^i, r_s)$ by

$$\bar{\zeta}_s(\zeta_s^i, r_s) := \begin{cases} \zeta_{sl}^i & \text{if } \zeta_{sl}^i < 0 \\ \frac{\zeta_{sl}^i}{1+r_s} & \text{otherwise} \end{cases}$$

and a utility function $\bar{v}_{r_s}^i(\zeta_s^i) := v^i(\bar{\zeta}_s(\zeta_s^i, r_s))$. As argued in Dubey and Geanakoplos (1992, pp. 418-419) we then get the equivalence that $0 = \arg\max\left\{ v^i(\zeta_s^i) \,\middle|\, \pi_s^* \cdot \zeta_s^{i+} \leq \frac{\pi_s^* \cdot \zeta_s^{i-}}{1+r_s} \right\}$ if and only if $0 = \arg\max \left\{ \bar{v}_{r_s}^i(\zeta_s^i) \,\middle|\, \pi_s^* \cdot \zeta_s^{i+} \leq \pi_s^* \cdot \zeta_s^{i-} \right\}$. If we consider an economy with I agents having concave utilities $\bar{v}_{r_s}^i(\zeta_s^i)$ and endowments e_s^i, then no-trade is a Walrasian equilibrium for this economy at prices π^*. By Lemma 2 in Dubey and Geanakoplos (2003(a)), at the initial endowment allocation (in state $s \in S$) there are no gains to r_s-diminished trade. Hence, $r_s \geq \gamma_s(x_{-s}^*, e_s)$ from the definition of $\gamma_s(x_{-s}^*, e_s)$ - a contradiction to the Gains-to-Trade Hypothesis in Assumption 10. Therefore, we must have $\mu_s^* > 0$ for every $s \in S$.

The definition of $c(\pi, z_{\widetilde{m}}, \widetilde{H})$ and $\mu(\pi, z_{\widetilde{m}}, c, \widetilde{H})$ now immediately imply $c^* > 0$.

It follows as before that the limit of the fixed point vectors correspond to an equilibrium in the abstract economy with interest rate peg and fixed transfers.

6.6 Proof of Theorem 4

This proof is a combination of the proofs of Theorems 2 and 3.

6.6.1 Preliminary Definitions

The augmented taxation $\tau^n \in \mathbb{R}_+^{(S+1)L}$, the government transfer function $\widetilde{H}(c,\mu) := \left(\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_S\right)(c,\mu)$, the inverse price level function $c^n(\pi, r, \mu, z_{\widetilde{m}})$ and the martingale-measure function $\mu^n(\pi, r, c, \mu, z_{\widetilde{m}}) := (\mu_1^n, \ldots, \mu_S^n)(\pi, r, c, \mu, z_{\widetilde{m}})$ are defined as in Theorem 3. $c^n(\pi, r, \mu, z_{\widetilde{m}})$ is a bounded and continuous function of $(\pi, r, \mu, z_{\widetilde{m}})$ for each n as long as $\overline{H}_0 > 0$ and $z_{\widetilde{m}} \ll +\infty$. Under the latter condition, $\mu^n(\pi, r, c, \mu, z_{\widetilde{m}})$ is also bounded and continuous. By the construction of τ^n, as long as $\pi \gg 0$ we have $c^n > 0$, $\mu_s^n > 0$ and $\sum_{s=1}^S \mu_s^n = 1$ for all finite n.

6.6.2 Construction of a fixed point mapping

Define Δ^n and Ω^n as in Theorem 2 and denote aggregate demand with $Z^n\left(\pi, \mu, r, \widetilde{H}\right) := Z\left(\pi, \mu, r, \widetilde{H}, \tau^n\right)$. Lemma (3.4) allows us to define the compact and convex set $K_{\widetilde{m}}$ such that

$$K_{\widetilde{m}} \supseteq Z_{\widetilde{m}}^n\left(\pi, \mu, r, \widetilde{H}\right)$$

for all $\pi \in \Delta$, $\mu \in \Delta^{S-1}, r \in \Omega^n$ and $\widetilde{H} \in \mathbb{R}_+^{S+1}$. As argued above, for positive $\overline{H}_0$ we can define a compact and convex set K_c^n such that

$$K_c^n \supseteq c^n(\pi, r, \mu, z_{\widetilde{m}})$$

for all $\pi \in \Delta^n$, $r \in \Omega^n$, $\mu \in \Delta^{S-1}$ and $z_{\widetilde{m}} \in K_{\widetilde{m}}$. $\mu^n(\pi, r, c, \mu, z_{\widetilde{m}})$ lies in a compact and convex set

$$K_\mu^n \subset interior(\Delta^{S-1})$$

for $\pi \in \Delta^n$, $z_{\widetilde{m}} \in K_{\widetilde{m}}$, $\mu \in \Delta^{S-1}$, $c \in K_c^n$, and $z_{\widetilde{m}} \in K_{\widetilde{m}}$. Introduce the compact and convex set $K_{\widetilde{H}}^n$ such that

$$K_{\widetilde{H}}^n \supseteq \widetilde{H}(c,\mu)$$

for all $c \in K_c^n$ and $\mu \in \Delta$. Further define the compact and convex set K_x^n such that

$$K_x^n \supseteq Z_x^n\left(\pi, \mu, r, \widetilde{H}\right)$$

for all $\pi \in \Delta^n, \mu \in K_\mu^n$, $r \in \Omega^n$, and $\widetilde{H} \in K_{\widetilde{H}}^n$. Again, denote the product set by $K^n := K_x^n \times K_{\widetilde{m}}$. Define the mapping

$$f^n : \Delta^n \times \Omega^n \times K_c^n \times K_\mu^n \times K_{\widetilde{H}}^n \times K^n \rightrightarrows \Delta^n \times \Omega^n \times K_c^n \times K_\mu^n \times K_{\widetilde{H}}^n \times K^n$$

by

$$\left(\pi, r, c, \mu, \widetilde{H}, z\right) \overset{f^n}{\mapsto} \left(f_\pi^n, f_r^n,\ f_c^n,\ f_\mu^n, f_{\widetilde{H}}^n,\ f_z^n\right),$$

where

$$\begin{aligned}
f_\pi^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad \underset{\pi \in \Delta^n}{\arg\max} \left\{ (1 + r_0)\, \pi_0 \cdot \left(z_{x_0} - \sum_i e_0^i \right) + \pi_{\mathbf{1}} \cdot \left(z_{x_{\mathbf{1}}} - \sum_i e_{\mathbf{1}}^i \right) \right\}, \\
f_r^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad \underset{r \in \Omega^n}{\arg\max} \left\{ r_0 \left(\frac{z_{\widetilde{m}_0}}{c} - \overline{M}_0 \right) + \frac{r_{\mathbf{1}}}{1 + r_{\mathbf{1}}} \cdot \left(z_{\widetilde{m}_{\mathbf{1}}} \square \frac{1}{c\mu} - \overline{M}_{\mathbf{1}} \right) \right\}, \\
f_c^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad c^n\left(\pi, r, \mu, z_{\widetilde{m}}\right), \\
f_\mu^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad \mu^n\left(\pi, r, c, \mu, z_{\widetilde{m}}\right), \\
f_{\widetilde{H}}^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad \widetilde{H}(c, \mu), \\
f_z^n\left(\pi, r, c, \mu, \widetilde{H}, z\right) &:= \quad Z^n\left(\pi, r, \mu, \widetilde{H}\right),
\end{aligned}$$

where $\frac{1}{c\mu} := \left(\frac{1}{c\mu_s}\right)_{s \in S}$ in the second line. As before, there exists a fixed point $\left(\pi^{*n}, r^{*n}, c^{*n}, \mu^{*n}, \widetilde{H}^{*n}, z^{*n}\right)$ for every n.

6.6.3 The limit of the fixed points is an equilibrium

For $n \to \infty$, we get $\left(\pi^{*n}, r^{*n}, c^{*n}, \mu^{*n}, \widetilde{H}^{*n}, z^{*n}\right) \to \left(\pi^*, r^*, c^*, \mu^*, \widetilde{H}^*, z^*\right)$ and $\tau^n \to \tau$. We want to show that $\left(\pi^*, r^*, c^*, \mu^*, \widetilde{H}^*, z^*\right)$ is an equilibrium for the abstract economy with taxation τ.

By the the definitions of $c^n\left(\pi, r, \mu, z_{\widetilde{m}}\right)$ and $\mu^n\left(\pi, r, c, \mu, z_{\widetilde{m}}\right)$, we get for each n

$$\begin{aligned}
\frac{1}{c^{*n}\mu_s^{*n}} \frac{r_s^{*n}}{1 + r_s^{*n}} z_{\widetilde{m}_s}^{*n} + \frac{1}{c^{*n}\mu_s^{*n}} \frac{1}{1 + r_s^{*n}} \pi_s^{*n} \cdot \tau_s^n \ + \ \frac{1}{c^{*n}} \left(r_0^{*n} z_{\widetilde{m}_0}^{*n} + \pi_0^{*n} \cdot \tau_0^n \right) \\
= \overline{H}_0 (1 + r_0^{*n}) + \overline{H}_s,
\end{aligned}$$

or alternatively

$$\frac{1}{c^{*n}\mu_s^{*n}}\frac{r_s^{*n}}{1+r_s^{*n}}z_{\widetilde{m}_s}^{*n}+\frac{1}{c^{*n}\mu_s^{*n}}\frac{1}{1+r_s^{*n}}\pi_s^{*n}\cdot\tau_s^n \;+\; \frac{1}{c^{*n}}\left(r_0^{*n}z_{\widetilde{m}_0}^{*n}+\pi_0^{*n}\cdot\tau_0^n\right)-r_0^{*n}\overline{H}_0$$
$$=\overline{H}_0+\overline{H}_s.$$

From this,

$$\frac{r_s^{*n}}{1+r_s^{*n}}\left(\frac{1}{c^{*n}\mu_s^{*n}}z_{\widetilde{m}_s}^{*n}-\overline{M}_s\right)+r_0^{*n}\left(\frac{1}{c^{*n}}z_{\widetilde{m}_0}^{*n}-\overline{M}_0\right)\;+\;r_0^{*n}\left(\overline{M}_0-\overline{H}_0\right)$$
$$<\overline{H}_0+\overline{H}_s-\frac{r_s^{*n}}{1+r_s^{*n}}\overline{M}_s,$$

from which we infer

$$\lim_{n\to\infty}\frac{r_s^{*n}}{1+r_s^{*n}}\left(\frac{1}{c^{*n}\mu_s^{*n}}z_{\widetilde{m}_s}^{*n}-\overline{M}_s\right)+\lim_{n\to\infty}r_0^{*n}\left(\frac{1}{c^{*n}}z_{\widetilde{m}_0}^{*n}-\overline{M}_0\right)$$
$$+\lim_{n\to\infty}r_0^{*n}\left(\overline{M}_0-\overline{H}_0\right)\leq\overline{H}_0+\overline{H}_s-\frac{r_s^{*}}{1+r_s^{*}}\overline{M}_s.$$

By the construction of f_r^n and the fact that $\overline{M}_0>\overline{H}_0$, we get in the limit

$$\begin{aligned}\lim_{n\to\infty}r_0^{*n}\left(\frac{1}{c^{*n}}z_{\widetilde{m}_0}^{*n}-\overline{M}_0\right)&\geq 0,\\ \lim_{n\to\infty}\frac{1}{c^{*n}\mu_s^{*n}}\frac{r_s^{*n}}{1+r_s^{*n}}z_{\widetilde{m}_s}^{*n}&\geq\frac{r_s^{*}}{1+r_s^{*}}\overline{M}_s,\\ \lim_{n\to\infty}r_0^{*n}\left(\overline{M}_0-\overline{H}_0\right)&\geq 0.\end{aligned}$$

Therefore, we have

$$r_0^*<+\infty,$$

since otherwise $\lim_{n\to\infty}r_0^{*n}\left(\overline{M}_0-\overline{H}_0\right)=+\infty$, contradicting the inequality. In addition, since $\frac{r_s^*}{1+r_s^*}\overline{M}_s\leq\overline{H}_0+\overline{H}_s$, we know that

$$r_s^*<\frac{\overline{H}_0+\overline{H}_s}{\overline{M}_s-\overline{H}_0-\overline{H}_s}<+\infty. \tag{29}$$

From $r\ll+\infty$ and the construction of f_r^n we can infer that

$$\lim_{n\to\infty}\frac{1}{c^{*n}}z_{\widetilde{m}_0}^{*n}\leq\overline{M}_0, \tag{30}$$
$$\lim_{n\to\infty}\frac{1}{c^{*n}\mu_s^{*n}}z_{\widetilde{m}_s}^{*n}\leq\overline{M}_s, \tag{31}$$

which further implies

$$\lim_{n\to\infty} r_0^{*n}\left(\frac{1}{c^{*n}} z_{\widetilde{m}_0}^{*n} - \overline{M}_0\right) = 0, \quad (32)$$

$$\lim_{n\to\infty} \frac{r_s^{*n}}{1+r_s^{*n}}\left(\frac{1}{c^{*n}\mu_s^{*n}} z_{\widetilde{m}_s}^{*n} - \overline{M}_s\right) = 0. \quad (33)$$

From the definition of $c^n(\pi, r, \mu, z_{\widetilde{m}})$ and $\widetilde{H}(c, \mu)$ we get

$$r_0 z_{\widetilde{m}_0}^{*n} + \frac{r_{\mathbf{1}}}{1+r_{\mathbf{1}}} \cdot z_{\widetilde{m}_{\mathbf{1}}}^{*n} + \pi_0^{*n} \cdot \tau_0^n + \pi_{\mathbf{1}}^{*n} \cdot \left(\frac{1}{1+r_{\mathbf{1}}} \square \tau_{\mathbf{1}}^n\right) = \widetilde{H}_0^{*n}(1+r_0) + \widetilde{H}_{\mathbf{1}}^{*n} \cdot \mathbf{1}.$$

Adding up the intertemporal individual budget sets over all households and plugging in this equation gives for every n

$$(1+r_0^{*n})\,\pi_0^{*n} \cdot \left(z_{x_0}^{*n} - \sum_i e_0^i\right) + \pi_{\mathbf{1}}^{*n} \cdot \left(z_{x_{\mathbf{1}}}^{*n} - \sum_i e_{\mathbf{1}}^i\right) = 0.$$

The left hand side of this equation is just the commodity price players objective function. In the limit we get

$$(1+r_0^{*})\,\pi_0^{*} \cdot \left(z_{x_0}^{*} - \sum_i e_0^i\right) + \pi_{\mathbf{1}}^{*} \cdot \left(z_{x_{\mathbf{1}}}^{*} - \sum_i e_{\mathbf{1}}^i\right) = 0.$$

Given this, it is easy to see that the commodity markets clear. From Lemma (2.4) we get $\pi^* \gg 0$. Hence we have $z^* \in Z\left(\pi^*, r^*, \mu^*, \widetilde{H}^*\right)$.

From the construction of $c^n(\pi, r, \mu, z_{\widetilde{m}})$, we know that $c^* < +\infty$. Next, we show that $c^* > 0$ and $\mu^* \gg 0$.

For $\mu^* \gg 0$, the argument is quite similar to the one given in the Theorem 3. For every $s \in S$, if $\tau_s > 0$, it is obvious that $\mu_s^* > 0$. Suppose $\tau_s = 0$ for some $s \in S$ and $\mu_s^* = 0$. From the fact $c^* < +\infty$, we know that $\lim_{n\to\infty} c^{*n}\mu_s^{*n} = 0$. From the inequality $\lim_{n\to\infty} \frac{1}{c^{*n}\mu_s^{*n}} z_{\widetilde{m}_s}^{*n} \leq \overline{M}_s$, we know that $z_{\widetilde{m}_s}^{*} = 0$ (otherwise $\lim_{n\to\infty} \frac{1}{c^{*n}\mu_s^{*n}} z_{\widetilde{m}_s}^{*n} = +\infty > \overline{M}_s$). Therefore, the argument in the proof of Theorem 3 applies, which means $\gamma_s(x_{-s}^*, e_s) \leq r_s^*$. Hence, by (29), $\gamma_s(x_{-s}^*, e_s) < \frac{\overline{H}_0 + \overline{H}_s}{\overline{M}_s - \overline{H}_0 - \overline{H}_s}$, a contradiction to the Gains-to-Trade hypothesis in Assumption 11. Therefore, we must have $\mu_s^* > 0$ for all $s \in S$.

The result of $c^* > 0$ can be proved in a similar way as in Theorem 3.

Given $\mu^* \gg 0$ and $c^* > 0$ we can now infer from equations (30) - (33) that the money markets clear.

It follows that the limit of the fixed point vectors corresponds to an equilibrium in the abstract economy with money supply control and fixed transfers.

References

[1] Bloise, G., 2006. A note on the existence of monetary equilibrium over an infinite horizon, Economic Theory, 27, 59-77.

[2] Bloise, G., Dréze, J. H., Polemarchakis, H.M., 2005. Monetary equilibria over an infinite horizon. Economic Theory 25, 51-74.

[3] Bloise, G., Polemarchakis, H.M., 2006. Theory and practice of monetary policy. Economic Theory, 27, 1-23.

[4] Cass, D., 1984. Competitive equilibrium with incomplete financial markets. CARESS working paper 84-09 (University of Pennsylvania).

[5] Dréze, J.H., Polemarchakis, H.M., 2000. Monetary equilibria. in: G. Debreu, W. Neuefeind and W. Trockel, eds., Economic Essays: A Festschrift in Honor of W. Hildenbrand (Springer Verlag) p.p.83-108.

[6] Dubey, P., Geanakoplos, J.D., 1992. The value of money in a finite horizon Economy: A role for banks, in: P. Dasgupta, D. Gale, D. Hart, and E. Maskin, eds., Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn. Cambridge, MA: MIT Press, pp. 407–444.

[7] Dubey, P., Geanakoplos, J.D., 2003(a). Inside and outside fiat money, gains to trade, and IS-LM. Economic Theory 21, 347-397.

[8] Dubey, P., Geanakoplos, J.D., 2003(b). Monetary equilibrium with missing markets. Journal of Mathematical Economics 39, 585-618.

[9] Dubey, P., Geanakoplos, J.D., 2006. Determinacy with nominal assets and outside money. Economic Theory, 27,.79-106.

[10] Duffie, D., 1987. Stochastic equilibria with incomplete financial markets. Journal of Economic Theory 41, 405-416. Corrigendum 49, 384 (1989).

[11] Duffie, D., Shafer, W., 1985. Equilibrium in incomplete market I: Basic model of generic existence. Journal of Mathematical Economics 14, 285-300.

[12] Gourdel, P., Triki, L., 2005. Incomplete markets and monetary policy. working paper, CERMSEM.

[13] Nakajima, T., Polemarchakis, H.M., 2001. Money and prices under uncertainty. Working paper No. 01-32, Brown University.

[14] Nakajima, T., Polemarchakis, H.M., 2005. Money and prices under uncertainty. Review of Economic Studies, Volume 72: Issue 1, 223-246.

[15] Svensson, L., 1985. Money and asset prices in a cash-in-advance economy. Journal of Political Economy 93, 919-944.

[16] Werner, J., 1985. Equilibrium in economies with incomplete financial markets. Journal of Economic Theory 36, 110-119.

[17] Woodford, M., 1994. Monetary policy and price level determinacy in a cash-in-advance economy. Economic Theory 4, 345-380.

[18] Woodford, M., 1995. Price-level determinacy without control of a monetary aggregate, Carnegie-Rochester Conference Series on Public Policy 43, 1-46.

[19] Woodford, M., 2001. Fiscal requirements for price stability, Journal of Money, Credit and Banking, Vol. 33, No. 3, 669-728.