Coordination and Delay in Global Games*

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Abstract

What is the effect of offering agents an option to delay their choices in a global coordination game? We address this question by considering a canonical binary action global game, and allowing players to delay their irreversible decisions. Those that delay have access to accurate private information at the second stage, but receive lower payoffs. We show that, as noise vanishes, as long as the benefit to taking the risky action early is greater than the benefit of taking the risky action late, the introduction of the option to delay reduces the incidence of coordination failure in equilibrium relative to the standard case where all agents must choose their action at the same time. We outline the welfare implications of this finding, and probe the robustness of our results from a variety of angles.

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1 Introduction

Coordination problems arise naturally in many economic settings. These problems share the feature that for a given set of payoffs, agents may fail to take an action that would be in their collective interest, because they fear that others will not do so: a coordination failure. In this paper, we explore how providing participants in coordination problems with the option to delay their choices affects the extent of coordination failure.

Coordination games typically have multiple equilibria. This makes it hard to quantify the incidence of coordination failure. In order to begin with a well-specified measure of the extent of coordination failure, we focus on a well-known subclass of coordination problems called global games. Carlsson and van Damme (1993), Morris and Shin (2002), and Frankel, Morris, and Pauzner (2003) have identified a class of Bayesian coordination games each member of which is characterized by a unique dominance-solvable equilibrium. In this class of games, therefore, the extent of coordination failure can be easily quantified: it is the measure of states in which agents fail, in equilibrium, to select some action even though it is collectively in their interest to do so. We consider the effect of introducing an option to delay on the extent of coordination failure in global games. To be more specific about our theoretical motivation, consider the following canonical coordination problem.

A mass of players indexed by $[0, 1]$ choose between two actions, invest and not-invest. The payoff relevant state is indexed by θ. Not investing is "safe", with a cost and a benefit of 0. The cost of investing is given by $c > 0$, and benefits from investment depend on how the mass of players who choose to invest compares to some function $a(\theta)$, where $a' < 0$. In particular, if this mass is bigger than $a(\theta)$, then investment pays a gross benefit $b > c$. Otherwise, investment pays a gross amount of 0, leaving the investor with a loss of c. Suppose there exist values of the state, θ_l and θ_h, such that for $\theta \in (\theta_l, \theta_h)$, $a(\theta) \in (0, 1)$. If θ is common knowledge, there are multiple equilibria for $\theta \in (\theta_l, \theta_h)$: if everyone invests, it is optimal to invest; if nobody invests, it is optimal not to. This is, therefore, a coordination problem.

Papers in the literature on global games (referenced above) have shown that if θ is not common knowledge, and is instead observed privately by agents via signals $x_i = \theta + \sigma\epsilon_i$, where ϵ_i is independent of θ, and the support of θ is strictly bigger than (θ_l, θ_h), the induced Bayesian game has a unique rationalizable strategy profile. As the extent of private noise vanishes, i.e., as $\sigma \to 0$, when the induced Bayesian game becomes a perturbation of the original coordination problem, investment is undertaken in equilibrium only for some subset $(\theta^*, \theta_h) \subset (\theta_l, \theta_h)$. Thus, for a subset of states (θ_l, θ^*) there is coordination failure. The extent of coordination failure is given by the measure of this set.

We can now summarize the central question of this paper. Suppose we offered players in this coordination game the "option to delay" their decision to invest or not, while holding fixed all other features of the game. Delaying the decision endows a benefit: it generates more precise information. However, delaying the decision comes with a cost: the player who delays and then invests will receive a lower benefit than if he had invested early. Would the extent of coordination failure be reduced or increased?

In addition to being of theoretical interest, there is also a natural applied motivation for studying this question. Global games have recently been applied to a wide variety of economic settings.[1] Many such applications are inherently dynamic: players in these applications do have the option to delay their decision in order to garner more precise information. This makes it all the more important to examine the impact of the option to delay in global games. Before describing our theoretical results, we briefly digress to provide a leading example of such a situation: foreign direct investment (FDI) in an emerging market.

Consider an emerging market (e.g., India), which begins a liberalization program, giving foreigners access to positive net present value (NPV) domestic projects. The number of these projects is fixed. Eventual payoffs from such FDI projects depend both on the state of the Indian economy and on the number of FDI investors: for a given state of the economy, there must be sufficient numbers of foreign investors for the liberalization program to "take-off" and generate high returns for investors. Liberalization programs last several years. Foreign investors can invest early or late in the process, and entry involves a transaction cost. Early entrants have wide choice of positive NPV projects. Late entrants have less choice. Thus, there is a cost to delaying the entry decision. Late investors, however, have more information. For example, they could observe the choices of other potential FDI investors, which in turn informs them about the chances for success. Thus, there is also a benefit to delaying the entry decision. The information obtained by waiting in such a setting is typically not transparent or public (FDI figures are often late and unreliable). Investors have to individually obtain (private) information. This is an example of the class of settings we have in mind.

1.1 Summary of Results

We study a two-period binary action global game similar to the canonical coordination problem described above. A continuum $[0, 1]$ of players face an investment project and choose between investing (irreversibly) or not at time t_1, with the option to delay their decision until a later time t_2. There is a state variable θ which is observed privately with some initial noise at the beginning of the game. Investment succeeds as long as there are enough investors in the course of the game relative to the underlying state variable θ. Investment costs $c > 0$.

[1] Examples of applications of global games include the work of Morris and Shin (1998, 2004) to currency crises and debt pricing, Goldstein and Pauzner (2005) to bank runs, and Dasgupta (2004) and Goldstein and Pauzner (2004) to financial contagion.

Benefits to successful investment are b_1 for early investors and b_2 for late investors, where $b_1 > b_2 > c$, resulting in a cost to delaying the investment decision. Players who wait receive a private signal based on the aggregate number of early investors. This signal is informative about the state. We focus on games where such learning is precise, i.e., almost all uncertainty is resolved at the second stage of the game. Thus, there is an informational benefit to delay. In such games, as initial observation noise becomes small, we report the following results.

1. The provision of an option to delay always reduces the incidence of coordination failure relative to the static benchmark where choices are made only in one period. This is our main result.

2. An intermediate cost of delay minimizes the incidence of coordination failure.

3. We examine the robustness of our findings from a variety of angles, by varying both informational and payoff parameters.

   (a) We examine a larger class of possible payoffs by varying the cost of investment over time and show that the main limiting result continues to hold as long as the benefit to investing early is bigger than the benefit to investing late.

   (b) While our main focus is on limiting cases where the initial noise in observation is quite small, we numerically analyze the problem in the presence of substantial amounts of noise. In such cases, we show that initial levels of optimism may matter, and the main result can be reversed when agents are sufficiently optimistic ex ante.

4. Finally, we sketch the welfare implications of our results on coordination failure.

1.2 Related Literature

Within the global games literature, the paper that comes closest to ours is by Heidhues and Melissas (2003). Like us, they consider a global game with private learning and endogenous timing. However, both their model and the focus of their analysis is different from ours. In their game the payoff from taking the risky action varies continuously in the mass of players who take that action, and also depends on the time at which they take that action. They thus consider "cohort effects" which are absent in our model, and their main emphasis is on characterizing conditions under which the two-period global game will have a unique rationalizable strategy profile. In contrast, we focus on the effect of the option to delay on the incidence of coordination failure. Finally, in contrast to their work, our paper provides a microfoundation for how additional information can be generated later in the game.

A number of recent papers have considered endogenous public signals in global games. These include the work of Chamley (1999), Tarashev (2003), Angeletos and Werning (2005), and Hellwig, Mukherji, and Tsyvinsky (2005). These models can be more readily interpreted in terms of learning from prices. In contrast, learning in our model is private, and we do not consider the effects of public information in inducing multiplicity of equilibrium. Angeletos, Hellwig, and Pavan (2005) examine (public) information dynamics in global games. An early example of social learning in a two-stage global game can be found in Corsetti, Dasgupta, Morris, and Shin (2004).[2]

Beyond global games, the question of whether providing the option to delay participation in a risky project is socially beneficial or not has been debated extensively in the literature on coordination problems in general (e.g. Farrell and Saloner (1986), Gale (1995), Choi (1997), and Xue (2003)).[3] No unified conclusion emerges from this literature. Within this broader literature, the paper that is closest to ours is by Choi (1997). Choi studies a dynamic coordination game with social learning. He presents a two-player game in which the option to delay can be harmful for a range of parameter values: the fear of being stranded in a suboptimal technology induces excessive delay, thus hindering participation in the risky project. In contrast, in our model the option to delay increases participation in the risky project. There are at least two important distinctions between our model and Choi's. The first is that, unlike in Choi (1997), each of our players is "small", and thus individual investment does not produce social benefits or information. Second, we consider a finite-time setting where there is heterogeneity of beliefs amongst players, which creates strategic uncertainty and limits the set of possible equilibrium outcomes. Built into our model, therefore, are limits to the extent of delay and discouragement, making the models not directly comparable, and applicable to different settings.

The rest of the paper is organized as follows. In the next section we describe the investment problem. Section 3 analyzes the problem using the traditional static global games approach. In section 4 we extend the analysis to include the option to delay. Section 5 examines the effect of the option to delay on the incidence of coordination failure, while section 6 discusses welfare. Finally section 7 discusses limitations and generalizations.

[2] Our paper has a general connection to the literature on dynamics in global games. This includes the work of Morris and Shin (1999) and Angeletos, Hellwig, and Pavan (2002). Also related are the important equilibrium selection results of Burdzy, Frankel, and Pauzner (2001), Frankel and Pauzner (2000), and Frankel (2003). Finally, our work also has a general connection to the literature on endogenous timing in the absence of strategic complementarities, e.g. Chamley and Gale (1994) and Gul and Lundholm (1995).

[3] This debate has been mirrored in the literature on pure public goods provision (e.g. Admati and Perry (1991) and Marx and Matthews (2000)).

2 The Investment Project

The economy is populated by a continuum of risk neutral agents indexed by $[0,1]$. Each agent must choose whether to invest (irreversibly) in a risky project. Not investing (N) is a safe action with benefits and costs equal to zero. Economic fundamentals are summarized by a state variable θ which is distributed $N(\mu_\theta, \sigma_\theta^2)$ and is revealed at time T, when consumption occurs. There are two periods in which an agent might be able to invest in the risky project: $t \in \{t_1, t_2\}$, where $t_1 < t_2 < T$.

Proceeds to a particular investor depend on whether the project succeeds or not, and when the agent chooses to invest. The success of the project, in turn, depends on the actions of the agents and the realized value of θ. In particular, if p denotes the total mass of agents who invest at the times when opportunities are available, then investment succeeds if $p \geq 1 - \theta$. It costs an amount $c > 0$ to invest in the project. If the project succeeds, it pays b_1 to those who invested at time t_1, and b_2 to those who invested at time t_2. We impose the restriction that $b_1 > b_2 > c$. Thus there is a *cost to delay* in investment. Payoffs from the risky project can be summarized as follows, where I_j indicates the act of investing at time t_j:

$$u(I_1, p, \theta) = \begin{cases} b_1 - c & \text{if } p \geq 1 - \theta \\ -c & \text{otherwise} \end{cases} \tag{1}$$

$$u(I_2, p, \theta) = \begin{cases} b_2 - c & \text{if } p \geq 1 - \theta \\ -c & \text{otherwise} \end{cases} \tag{2}$$

$$u(N, p, \theta) = 0 \tag{3}$$

At the beginning of $t = t_1$ agents observe the state of fundamentals with idiosyncratic noise. In particular, each agent i receives the following signal at the beginning of the game:

$$x_i = \theta + \sigma \epsilon_i$$

where ϵ is distributed Standard Normal in the population and independent of θ.

In what follows, we normalize the prior mean of θ, $\mu_\theta = 0$, and the prior variance $\sigma_\theta^2 = 1$. This normalization is innocuous when considering limiting cases where θ is observed with vanishing noise ($\sigma \to 0$), which will be the main focus of the paper, and simplifies computations. However, when we discuss results away from the limit (in sections 5.2 and 6.2), we shall consider arbitrary prior parameters.

The unconstrained *efficient outcome* of this investment problem would have all agents investing at t_1 whenever $\theta \geq 0$ and not at all otherwise. We now present two games that can be used to study this investment problem in a decentralized context. We begin with the benchmark static global games analysis. We then extend by introducing the option to delay.

3 The Benchmark Static Game

To analyze this investment problem within the framework of static global games requires that we place a restriction on the actions of players: we insist that all players make their choices at $t = t_1$. The payoffs of this game are given by (1) and (3). We label this game Γ_{st}.

It is useful to begin with a preliminary definition. Note that in these games, agents' strategies map from their private information into their action spaces.

Definition 1 *An agent i is said to follow a* monotone *strategy if her chosen actions are increasing in her private information, i.e., if her strategy takes the form:*

$$\sigma_i(x_i) = \begin{cases} I & \text{when } x_i \geq x^* \\ N & \text{otherwise} \end{cases}$$

We shall call symmetric equilibria in monotone strategies *monotone equilibria.* Monotone equilibria can be given a natural economic interpretation: when an agent chooses to invest, she correctly believes (in equilibrium) that all agents who have more optimistic beliefs than her also choose to do so.

If a continuum of players follow monotone strategies, a threshold level emerges naturally in the underlying state variable of the game. Therefore, we look for monotone equilibria which take the form $(x^*_{st}, \theta^*_{st})$ where agent i invests iff $x_i \geq x^*_{st}$ and investment is successful iff $\theta \geq \theta^*_{st}$. Now we may state:

Proposition 2 [4] *If $\sigma < \sqrt{2\pi}$, there is a unique equilibrium in Γ_{st}. This equilibrium is in monotone strategies. In the limit as $\sigma \to 0$, it is given by the pair*

$$x^*_{st} \to \frac{c}{b_1} \quad \theta^*_{st} \to \frac{c}{b_1}$$

The proof is in the appendix. We note that this result does not rely on the specific mean and variance assumed for θ. For arbitrary μ_θ and σ^2_θ, uniqueness of monotone equilibria would have prevailed in the region $\frac{\sigma}{\sigma^2_\theta} < \sqrt{2\pi}$. Thus, the "small noise" condition of Proposition 2 should be read as a *relative* statement: uniqueness of monotone equilibria holds as long as private signals are sufficiently precise relative to the common prior. If the prior is diffuse ($\sigma^2_\theta \to \infty$), then there is always a unique monotone equilibrium. The same intuition is true for all of the results stated in the remainder of the paper. Morris and Shin (2002) discuss this issue further.

We now extend our analysis to introduce the option to delay.

[4] This result is a special case of Morris and Shin (2002): Proposition 3.1. It can be obtained by setting the precision of the public signal to 1. For an analysis of the role of public vs private information in inducing multiplicity of equilibria in global games, see Hellwig (2002).

4 The Dynamic Game with the Option to Delay

We now augment the original game to last two periods, and allow agents to choose both the action they take and the time at which they act. The payoffs of the game are given by (1-3).

The option to delay, when exercised, generates an informational benefit, which we model via Bayes social learning. Agents who choose to act at t_2 are able to observe a statistic based on the proportion of agents who chose to invest at t_1, which we denote by p_1. Agents observe this statistic with some idiosyncratic noise. Agents who delay receive an additional signal:

$$y_i = \Phi^{-1}(p_1) + \tau\eta_i$$

where η is Standard Normal in the population, and independent of ϵ and θ.

There are a number of points that should be noted regarding the second period signal. First, the precision of the signal is independent of the mass of agents who invest early. Thus, there is no informational externality in our model. Second, the signal is private. These two features make our model very different from the canonical social learning model.[5] Finally, while there are many ways of generating additional information (and thus a benefit of delay) at t_2, we have chosen a specific microfoundation (learning from the aggregate actions of others), and a specific technology (via the Φ^{-1}transformation). While we shall characterize equilibria for all (σ, τ), we shall draw economic conclusions only for the case where $\tau \to 0$, that is, when learning becomes very precise. In this limit, neither the specific learning technology (the Φ^{-1}transformation) nor the specific microfoundation for additional information (social learning) affect the results.[6]

At time t_1, agents have the choice to invest or not. If they invest, then their choice is final. If they choose not to invest, however, they get another opportunity at t_2 to make the same choice, based on the additional information they receive at that time. As we have noted earlier, the payoffs to the investment project given in (1-3) induce a cost to delay in investing. Agents will thus rationally trade off the possible excess gains to choosing early against the option value of waiting and choosing with more information at t_2. We call this game Γ_{en} and look for Bayes Nash equilibria.

[5] See Bikhchandani, Hirshleifer, and Welch (1992) and Banerjee (1992).

[6] In the limit as $\tau \to 0$, observing any monotone transformation of p_1 (for example, $\Phi^{-1}(p_1)$) is equivalent to observing p_1. Away from the $\tau \to 0$ limit, this transformation of p_1 is *not* without loss of generality.

Since the action of t_1 investors are monotone functions based on their informative signals, observing a statistic based on aggregate behaviour at $t_1$is informative about the state. In particular, with a continuum of players, observing the proportion without noise is equivalent to observing θ. As $\tau \to 0$, therefore, essentially all uncertainty is resolved, and the specific microfoundation for additional information (social learning) does not affect the results.

Players who wait until t_2 observe two noisy signals, x and y. Let $s(x,y)$ denote a sufficient statistic for (x,y).We look for equilibria in which agents choose monotone strategies with thresholds (x^*_{en}, s^*_{en}), such that:

1. Invest at $t = t_1$ iff $x_i \geq x^*_{en}$. Otherwise choose to wait.

2. Conditional on reaching $t = t_2$ with the option to invest, invest iff $s_i \geq s^*_{en}$

Note that $\theta|x$ is distributed $N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$. The mass of people who invest at t_1 in state θ is

$$p_1 = \Phi(\frac{\theta - x^*_{en}}{\sigma})$$

Substituting into the definition of the second period signal, y, we get:[7]

$$y_i = \frac{\theta - x^*_{en}}{\sigma} + \tau\eta_i$$

Now, using Bayes's Rule, we know that:

$$\theta|x_i, y_i \sim N\left[\frac{x_i + \frac{\sigma}{\tau^2}y_i + \frac{1}{\tau^2}x^*_{en}}{1 + \sigma^2 + \frac{1}{\tau^2}}, \frac{\sigma^2}{1 + \sigma^2 + \frac{1}{\tau^2}}\right]$$

Thus if we define:

$$s_i = \frac{x_i + \frac{\sigma}{\tau^2}y_i + \frac{1}{\tau^2}x^*_{en}}{1 + \sigma^2 + \frac{1}{\tau^2}}$$

then

$$\theta|x_i, y_i \equiv \theta|s_i \sim N\left[s_i, \frac{\sigma^2}{1 + \sigma^2 + \frac{1}{\tau^2}}\right]$$

In what follows, where there is no confusion, we drop the agent subscript i. In Γ_{st}, it was apparent that when agents followed monotone strategies there were corresponding equilibrium thresholds in the fundamentals above which investment would be successful, and below which it would fail. This characterization is not immediate in the current game (since the decisions to invest or not in the two periods are not independent) and requires closer examination.

When agents follow monotone strategies as outlined above, at any θ, a mass $Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta)$ will choose to invest. Thus, investment is successful at θ if and only if:

$$Pr(x \geq x^*_{en}|\theta) + Pr(x < x^*_{en}, s \geq s^*_{en}|\theta) \geq 1 - \theta$$

[7] It is clear that observing y_i is equivalent *in equilibrium* to observing an exogenous signal $z_i = \sigma y_i + x^*_{en} = \theta + \sigma\tau\eta_i$. The consequence of microfounding t_2 information via social learning is to make the precision of the second period signal increasing in the precision of the first period signal. However, as $\tau \to 0$, the precision of the first period signal becomes irrelevant, and thus our limiting results will also hold for any exogenous private second-period signal which becomes arbitrarily precise.

Lemma 8 (stated and proved in the appendix) shows that there exists a critical θ^* above which investment is successful and below which it is not.

Given Lemma 8, we can now look for monotone equilibria of the form $(x^*_{en}, s^*_{en}, \theta^*_{en})$ where x^*_{en} and s^*_{en} are defined as above, and investment is successful if and only if $\theta \geq \theta^*_{en}$.

Necessary conditions for such equilibria are as follows:

The indifference equation for those players who arrive at period t_2 with the option to invest:

$$Pr(\theta \geq \theta^*_{en} | s^*_{en}) = \frac{c}{b_2} \tag{4}$$

The critical mass condition:

$$Pr(x \geq x^*_{en} | \theta^*_{en}) + Pr(x < x^*_{en}, s \geq s^*_{en} | \theta^*_{en}) = 1 - \theta^*_{en} \tag{5}$$

Finally, the indifference condition of players in period t_1: At t_1, agents trade off the expected benefit of investing in early against the expected benefit of waiting and then acting optimally. Thus the marginal t_1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en} | x^*_{en}) b_1 - c = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en} | x^*_{en})[b_2 - c] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en} | x^*_{en})(-c) \tag{6}$$

Using (4), we can rewrite equation (5) as follows:

$$Pr(x \geq x^*_{en} | \theta^*_{en}) + Pr(x < x^*_{en}, s \geq \theta^*_{en} + M | \theta^*_{en}) = 1 - \theta^*_{en}$$

where M is a constant. This is an equation in x^*_{en} and θ^*_{en}. Lemma 9 (stated and proved in the appendix) shows that as long as σ is small enough, this equation implicitly defines θ^*_{en} as a smooth function of x^*_{en} with a bounded derivative. Using (4) and Lemma 9, we can express (6) purely in terms of x^*_{en}. In the appendix, we establish that there exists a unique solution to (6), which via, Lemma 9, implies that there is a unique solution to the system (4-6). We also show that agents who receive signals $x > (<) x^*_{en}$ prefer to invest early (wait). It is obvious that agents who wait until t_2 and receive signals $s > s^*_{en}$ at t_2 will invest at t_2, while those who receive signals $s < s^*_{en}$ will not. Thus, we can now state:[8]

[8] Readers familiar with the literature on global games will have noticed that the uniqueness result proved for the dynamic game is restricted to monotone strategy equilibria. For static global games Carlsson and van Damme (1993, later generalized by Frankel, Morris, and Pauzner 2003) prove a stronger result: the unique monotone equilibrium is also the unique strategy profile surviving the iterated deletion of dominated strategies. Existing arguments for this stronger result do not generalize to our dynamic game due to Bayesian learning. In Γ_{en} the type-space at t_2 depends on the strategies employed at t_1. Thus, starting from the unrestricted set of strategies at t_1 generates arbitrarily complex type-spaces at t_2, thereby vastly complicating the iterative deletion of dominated strategies. Nevertheless, as we shall show later, merely focussing on monotone equilibria is sufficient to generate interesting results. In particular, we shall show that there is a monotone equilibrium in which agents can coordinate more efficiently than in the unique rationalizable strategy profile of the static game.

Proposition 3 *If $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$, there exists a unique monotone equilibrium in Γ_{en}.*

The proof is in the appendix.[9] We can now compare the equilibria of the static and dynamic games to understand the effects of providing the option to delay and learn.

5 The Effect of the Option to Delay

In this section, we compare the equilibria of the static and dynamic games to understand the effect of the option to delay. We focus exclusively on the case where t_2 information is very precise, that is $\tau \to 0$. This implies that we are examining the special case where (essentially) all uncertainty is resolved at time t_2. Our results provide a benchmark for a more general comparison for games for all (σ, τ).

As $\tau \to 0$, equation (5) reduces to:

$$\Phi(\frac{x^*_{en} - \theta^*_{en}}{\sigma})\frac{c}{b_2} = \theta^*_{en} \tag{7}$$

At the same time, equation (6) becomes:

$$\Phi\left(\frac{\frac{x^*_{en}}{1+\sigma^2} - \theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c}{b_1 - (b_2 - c)} \tag{8}$$

The formal derivations of (7) and (8) are given in the appendix, in section 8.1. Combining these two, we get:

$$\Phi\left(\frac{\Phi^{-1}\left(\frac{b_2}{c}\theta^*_{en}\right)}{\sqrt{1+\sigma^2}} - \frac{\sigma\theta^*_{en}}{\sqrt{1+\sigma^2}}\right) = \frac{c}{b_1 - (b_2 - c)} \tag{9}$$

In the limit as $\tau \to 0$, we can now use this characterization to compare equilibria for all $\sigma > 0$ which satisfy the uniqueness conditions in Γ_{st} and Γ_{en}. It is useful to divide this comparison into two parts. Our main emphasis will be on comparing equilibria in "small noise" games, that is, in games where agents receive very accurate signals at t_1, that is where $\sigma \to 0$. Following our examination of this limiting case, we proceed to examine the case of genuinely noisy signals at time t_1.

5.1 Comparing Equilibria in the Limiting Case

In order to compare equilibria in games where $\tau \to 0$ and t_1 signals are very precise, we let $\sigma \to 0$ in (9). This results in a characterization of the equilibria of Γ_{en} in the ordered limit

[9]This condition reduces to the familiar condition $\sigma < \sqrt{2\pi}$ as $\tau \to 0$.

where $\tau \to 0$ and $\sigma \to 0$.[10] In this limit, we can solve for the equilibrium thresholds of the endogenous order game in closed form. As $\sigma \to 0$, the unique solution to (9) is given by

$$\theta_{en}^* \to \frac{c^2}{b_2(b_1 - b_2 + c)}$$

Thus we can now summarize:

Proposition 4 *In the ordered limit as $\tau \to 0$ and $\sigma \to 0$, the unique equilibrium thresholds of Γ_{en} can be written as:*

$$x_{en}^* \to \frac{c^2}{b_2(b_1 - b_2 + c)} \quad s_{en}^* \to \frac{c^2}{b_2(b_1 - b_2 + c)} \quad \theta_{en}^* \to \frac{c^2}{b_2(b_1 - b_2 + c)}$$

Conveniently, when $\sigma \to 0$, we can also characterize the thresholds of the static game in closed form. Thus, in this case, we can compare thresholds explicitly. Recall from our earlier analysis that as $\sigma \to 0$,

$$x_{st}^* \to \frac{c}{b_1} \quad \theta_{st}^* \to \frac{c}{b_1}$$

Two important conclusions emerge immediately upon inspection of these thresholds. First, notice that

$$\frac{c^2}{b_2(b_1 - b_2 + c)} < \frac{c}{b_1}$$

If this were not so, then it must be the case that

$$\frac{c^2}{b_2(b_1 - b_2 + c)} \geq \frac{c}{b_1}$$

which implies that,

$$b_2(b_2 - c) \geq b_1(b_2 - c)$$

which is impossible because $b_1 > b_2 > c > 0$. Thus, we can now state our main result:

Corollary 5 *In the ordered limit as $\tau \to 0$ and $\sigma \to 0$, for all $b_1 > b_2 > c > 0$,*

$$\theta_{en}^* < \theta_{st}^*$$

Thus, successful coordinated investment becomes *more probable* when we let agents choose both how to act and when to act. A detailed discussion of this result is provided below.

A second result relates to the maximal probability of successful coordinated investment in the dynamic game. Given (b_1, c) what value of b_2 maximizes the probability of coordinated

[10] The interpretation of this ordered limit is that we are letting τ approach zero faster than σ, that is, $\sigma \to 0$ and $\frac{\tau}{\sigma} \to 0$. It would have been desirable to consider cases where $\sigma \to 0$ and $\frac{\tau}{\sigma} \to r$ for $r \geq 0$, but the problem proves analytically intractable.

investment? We want to choose: $b_2^* = \arg\min_{b_2} \frac{c^2}{b_2(b_1-b_2+c)}$. This is given by $b_2^* = \frac{1}{2}(b_1+c)$. Since $b_2 \in (c, b_1)$, this means that an *intermediate* cost of delay maximizes the probability of coordinated investment. We can thus summarize:

Corollary 6 *In the ordered limit as $\tau \to 0$ and $\sigma \to 0$, the probability of successful coordinated investment is maximized when $b_2 = \frac{1}{2}(b_1+c)$.*

In what follows, we provide a detailed discussion of these two limiting results.

5.1.1 Discussion of Results in the Limiting Case

We begin by discussing the main result: that the provision of an option to delay enhances the ability of agents to coordinate. Note that the dynamic critical threshold, θ_{en}^*, will be lower than the static critical threshold, θ_{st}^*, if and only if the mass of players who invest at the critical state in the dynamic game is greater than the mass of players who invest at the critical state of the static game. The mass of players who invest at the critical threshold in the dynamic game is the sum of the mass of players who invest at t_1 and those who invest at t_2. Let us consider these masses in turn.

Consider the static game first. Recalling our earlier analysis, we know that $x_{st}^* = (1+\sigma^2)\theta_{st}^* + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1})$, and thus the proportion of investors is given by

$$\Pr(x \geq x_{st}^*|\theta) = \Phi\left(\frac{\theta-\theta_{st}^*}{\sigma} - \sigma\theta_{st}^* - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1})\right)$$

As $\sigma \to 0$

$$\Pr(x \geq x_{st}^*|\theta) \to \begin{cases} 1 & \text{if } \theta > \theta_{st}^* \\ 1-\frac{c}{b_1} & \text{if } \theta = \theta_{st}^* \\ 0 & \text{if } \theta < \theta_{st}^* \end{cases}$$

Thus, the mass of agents who invest at t_1 at the critical state in the static game is given by $1-\frac{c}{b_1}$. In other words, the mass of investors at the critical state is "one minus the ratio of costs to benefits". This characterization will turn out to be useful in our analysis of the dynamic game. The mass of agents who invest at t_2 in the static game is 0.

Now consider the dynamic game. Consider late investors first. For any (σ, τ), the mass of players who invest at t_2 in state θ is given by

$$\Pr(x < x_{en}^*, s \geq s_{en}^*|\theta) = \int_{-\infty}^{x^*} Pr(s \geq s^*|\theta, x) f(x|\theta)dx$$

As $\tau \to 0$ we show in the appendix (see the derivation of equation 7) that:

$$\Pr(x < x_{en}^*, s \geq s_{en}^*|\theta) \to \begin{cases} \Pr(x < x_{en}^*|\theta) & \text{if } \theta > \theta_{en}^* \\ \Pr(x < x_{en}^*|\theta)\left(1-\frac{c}{b_2}\right) & \text{if } \theta = \theta_{en}^* \\ 0 & \text{if } \theta < \theta_{en}^* \end{cases}$$

The decomposition of the product terms arises because as $\tau \to 0$, $Cov(x, s|\theta) \to 0$, since $s \to \theta$.

Thus, the mass of late investors in the dynamic game can be expressed as the product of two terms. The first term is the mass of investors who chose to wait until t_2. Inspection of the second term shows that it is equal to the proportion of players who would have invested in the critical state of a *static* game played at t_2, with benefit b_2 and cost c.

In addition as $\tau \to 0$, we have shown that the agents t_1 indifference condition reduces to (8) which, in turn, implies that

$$x^*_{en} \to (1+\sigma^2)\theta^*_{en} + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1 - b_2 + c})$$

Since $x|\theta \sim N(\theta, \sigma^2)$, the mass of early investors can be written as follows:

$$\Pr(x \geq x^*_{en}|\theta) \to \Phi\left(\frac{\theta - \theta^*_{en}}{\sigma} - \sigma\theta^*_{en} - \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1 - b_2 + c})\right)$$

Now, as $\sigma \to 0$

$$\Pr(x \geq x^*_{en}|\theta) \to \begin{cases} 1 & \text{if } \theta > \theta^*_{en} \\ 1 - \frac{c}{b_1 - b_2 + c} & \text{if } \theta = \theta^*_{en} \\ 0 & \text{if } \theta < \theta^*_{en} \end{cases}$$

Thus, a mass $1 - \frac{c}{b_1 - b_2 + c}$ of agents invest early at the critical state of the dynamic game. Inspection of this term show that it is "as if" at t_1 agents were playing a static game with cost c, and benefit $b_1 - (b_2 - c)$, that is, the benefit from investing early, minus what they give up by not waiting.

The remaining agents, of mass $\frac{c}{b_1 - b_2 + c}$, enter the second period with their option to invest intact. Of these, a proportion $1 - \frac{c}{b_2}$ choose to invest at t_2 at the critical state, leading to a total mass of late investors of $\frac{c}{b_1 - b_2 + c}(1 - \frac{c}{b_2})$. Thus, the total mass of investors at the critical state is

$$1 - \frac{c}{b_1 - b_2 + c} + \frac{c}{b_1 - b_2 + c}(1 - \frac{c}{b_2}) = 1 - \frac{c}{b_1 - b_2 + c}\left(\frac{c}{b_2}\right)$$

It is no coincidence, then, that $\theta^*_{en} \to \frac{c^2}{b_2(b_1 - b_2 + c)}$.

Notice, then, that the mass of agents who invest at t_1 in the dynamic game is *lower* than the mass of agents who invest at t_1 in the static game (because $\frac{c}{b_1} < \frac{c}{b_1 - (b_2 - c)}$). In other words, the existence of the option to delay makes players *less aggressive* at t_1 in the dynamic game than in the static game. The mass of investors "lost" at t_1 ($L(t_1)$)can be expressed as follows:

$$L(t_1) = \left(1 - \frac{c}{b_1}\right) - \left(1 - \frac{c}{b_1 - (b_2 - c)}\right) = \frac{c}{b_1 - (b_2 - c)}\left(\frac{b_2 - c}{b_1}\right)$$

that is, the proportion of players who *do not* invest at t_1 times the ratio of the net gains from successful investment at t_2 to the gross benefits to successful investment at t_1.

However, each player who does not invest at t_1, gets the chance to invest at t_2 in the dynamic game. Hence, at the critical state of the dynamic game, the mass of investors "gained" at t_2 ($G(t_2)$) is given by

$$G(t_2) = \frac{c}{b_1 - (b_2 - c)}(1 - \frac{c}{b_2}) = \frac{c}{b_1 - (b_2 - c)} \left(\frac{b_2 - c}{b_2} \right)$$

that is, the proportion of players who *do not* invest at t_1 times the ratio of the net gains from successful investment at t_2 to the gross benefits to successful investment at t_2.

The total mass of investors at the critical state in the dynamic game will be higher than the total mass of investors at the critical state of the static game exactly when $G(t_2) > L(t_1)$. But this occurs exactly when $\frac{b_2 - c}{b_2} > \frac{b_2 - c}{b_1}$, that is, if and only if $b_2 < b_1$.

The second result, that an intermediate cost of delay maximizes the probability of coordinated investment, follows quite simply from the main result. The cost of delay, which is determined by the size of b_2 relative to b_1, has opposite impacts on the proportion of players who invest early and late. For a fixed (b_1, c), a low b_2 (high cost of delay) induces more people to invest at t_1, but, for those who choose to wait until t_2, acts as a deterrent to investment. In contrast, a high b_2 (low cost of delay) makes investment at t_2 attractive, but discourages investment at t_1, by inducing more players to wait. As $b_2 \to c$ essentially nobody waits, and the game reduces to Γ_{st} (indeed, $\lim_{b_2 \to c} \theta^*_{en} = \theta^*_{st}$). Similarly, as $b_2 \to b_1$ essentially everybody waits, and the game again reduces to Γ_{st} (indeed, $\lim_{b_2 \to b_1} \theta^*_{en} = \theta^*_{st}$). Thus, an interior extremum must exist. The main result, in turn, establishes that the interior extremum must be a maximum, since $1 - \theta^*_{en} > 1 - \theta^*_{st}$ for all $b_1 > b_2 > c$.

It is also instructive to consider the incentives of players at t_1 and t_2. At any stage of the game, players require a "high enough" probability of successful investment in order to invest. The higher this required probability, the lower will be the equilibrium mass of investors.[11] The required probability is defined by the beliefs of the marginal investor: the player who is indifferent between investing or not. At t_1, therefore, the minimum required probability of success is:

$$\Pr(\theta \geq \theta^*_{en} | x^*_{en}) = \frac{c}{b_1 - b_2 + c}$$

At t_2 the minimum required probability of success is:

$$\Pr(\theta \geq \theta^*_{en} | s^*_{en}) = \frac{c}{b_2}$$

Notice that $\Pr(\theta \geq \theta^*_{en} | x^*_{en})$ increases in b_2: that is, the mass of t_1 investors decreases in b_2, as already discussed earlier. Similarly, $\Pr(\theta \geq \theta^*_{en} | s^*_{en})$ decreases in b_2. Inspection of the

[11] Formally, it can be shown that as $\tau \to 0$, $\Pr(s \geq s^*_{en} | \theta^*_{en}) \to 1 - \Pr(\theta \geq \theta^*_{en} | s^*_{en})$, and as $\sigma \to 0$, $\Pr(x \geq x^*_{en} | \theta^*_{en}) \to 1 - \Pr(\theta \geq \theta^*_{en} | x^*_{en})$. Thus, there is a clear link between the incentives of the marginal investor and the proportion that invest at the critical state in the limit.

right-hand sides of the two equations above indicate that the (absolute value of the) impact of changing b_2 is highest on $\Pr(\theta \geq \theta^*_{en}|x^*_{en})$ and lowest on $\Pr(\theta \geq \theta^*_{en}|s^*_{en})$ when $b_2 = b_1$.[12] On the other hand, the (absolute value of the) impact of changing b_2 is lowest on $\Pr(\theta \geq \theta^*_{en}|x^*_{en})$ and highest on $\Pr(\theta \geq \theta^*_{en}|s^*_{en})$ when $b_2 = c$. Thus, when the cost of delay is increased from 0 (by reducing b_2 from b_1), the positive impact on the incentives of t_1 investors initially swamps the negative impact on incentives of t_2 investors. This leads to an increase in investment. However, when when b_2 is decreased far enough, bringing us closer to $b_2 = c$, the negative impact on the incentives of t_2 investors swamps the positive impact on the incentives of t_1 investors, reversing the earlier positive impact on investment.

5.2 Comparing Equilibria Away from the Limit

We now turn our attention to a comparison of equilibria of Γ_{st} and Γ_{en} when $\tau \to 0$ but $\sigma \nrightarrow 0$. In this case we cannot provide a closed form characterization of the equilibria in either Γ_{st} or Γ_{en}, and we must rely on numerical methods. Not surprisingly, therefore, the comparison away from the $\sigma \to 0$ limit is less clear cut. Nevertheless, a number of important conclusions can be reached, as we point out here.

First, we note that while the values of the parameters of the prior distribution of θ do not matter in the limit when $\sigma \to 0$, they may well have an effect away from the limit.[13] Varying the prior precision is not interesting, since the extent of noise in the game is determined by the ratio of the precision of the signal to the precision of the prior. Thus, we hold fixed $\sigma^2_\theta = 1$ but allow for a general prior mean μ_θ. We examine results for low ($\mu_\theta = -1$) and high ($\mu_\theta = 2$) prior means. Our choice of means is determined by their distance from the crucial region of θ, $\theta \in (0,1)$, in which the coordination problem is relevant. In all our simulations, normalize $b_1 = 1$, and let $b_2 = 1 - k$, thus denoting by k the cost of delay. Clearly k lies in the set $(0, 1-c)$. We set $c = 0.3$, and vary k in $(0, 0.7)$.[14] The plots show θ^*_{st} and $\theta^*_{en}(k)$ for selected parameters.

We present two sets of simulations. First, we illustrate, as a baseline case, that for small σ ($\sigma = 0.01$), the properties of the limiting case are preserved: $\theta^*_{en} < \theta^*_{st}$ for all k, and it is minimized for $k \approx \frac{1-c}{2} = 0.35$. This is shown in Figures 1 ($\mu_\theta = -1$) and 2 ($\mu_\theta = 2$).

INSERT FIGURES 1 AND 2 HERE

Next, we examine the case of substantial noise: $\sigma = 1$.[15] In Figure 3 and 4 we plot θ^*_{st}

[12] Formally, $\left|\frac{\partial}{\partial b_2}\frac{c}{b_1-b_2+c}\right|_{b_2=b_1} = \left|\frac{c}{(b_1-b_2+c)^2}\right|_{b_2=b_1} = \frac{1}{c} > \left|\frac{\partial}{\partial b_2}\frac{c}{b_2}\right|_{b_2=b_1} = \left|\frac{-c}{b_2^2}\right|_{b_2=b_1} = \frac{c}{b_1^2}$, since $b_1 > c > 0$.

[13] I thank a referee for urging me to explore this case.

[14] We have checked several high and low values of $c \in (0,1)$ and the qualitative properties discussed here are not affected by our choice of c.

[15] We have examined even larger values of σ, up to $\sqrt{2\pi}$, and the results are qualitatively similar.

and $\theta^*_{en}(k)$ for $\mu_\theta = -1$ and $\mu_\theta = 2$ respectively. It is evident that the results differ from the limiting case to varying degree. The main result of the limiting case, that $\theta^*_{en} < \theta^*_{st}$ for all k, is preserved for the $\mu_\theta = -1$ case, but is *reversed* for $\mu_\theta = 2$. We now discuss why this is so.

INSERT FIGURES 3 AND 4 HERE

The strategies of players at t_1 (x^*_{en}) are decreasing in μ_θ. It is easy to see that for any μ_θ, as $\tau \to 0$,

$$x^*_{en} \to (1+\sigma^2)\theta^*_{en} - \sigma^2\mu_\theta + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1 - b_2 + c})$$

Intuitively, this arises because the higher is μ_θ, the more optimistic players become, and the lower the incentive to wait. This intuition holds also for the static game, Γ_{st}, where

$$x^*_{st} = (1+\sigma^2)\theta^*_{st} - \sigma^2\mu_\theta + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1})$$

However, for those players who wait until t_2 in Γ_{en}, strategies are *independent* of the prior μ_θ, because $\tau \to 0$, and they thus observe θ with vanishing noise.

For low μ_θ, very few people invest in the static game, because of the pessimism generated by the prior. The same intuition applies to the first period of the dynamic game. In the dynamic game, however, the actions of those who wait until t_2 are unaffected by μ_θ, thus less pessimistic, generating more investment overall, and thus a lower θ^*_{en} (relative to θ^*_{st}). How much more investment (and thus how low $\theta^*_{en}(k)$ is relative to θ^*_{st}) depends on how many people wait until t_2, and thus play mean-independent strategies. The lower is k, the higher the proportion of agents who make choices at t_2, and thus the lower is $\theta^*_{en}(k)$ relative to θ^*_{st}.

For high μ_θ, many people invest in the static game, because of the optimism generated by the prior. The same intuition applies to the first period of the dynamic game. In the dynamic game, however, the actions of those who wait are unaffected by μ_θ, thus less optimistic, generating *less* investment overall, and thus a *higher* θ^*_{en} (relative to θ^*_{st}). How much less investment (and thus how high $\theta^*_{en}(k)$ is relative to θ^*_{st}) depends on how many people wait until t_2, and thus play mean-independent strategies. The lower is k, the higher the proportion of agents who make choices at t_2, and thus the higher is $\theta^*_{en}(k)$ relative to θ^*_{st}.

For an intermediate value of μ_θ the effect of the prior mean is not dominant, and the probability of coordinated investment can be non-monotonic in k, as in the limiting case. Figure 5 illustrates the case for $\mu_\theta = 0.5$, in the centre of the crucial $(0,1)$ area, and half way between the extreme means considered already.

INSERT FIGURE 5 HERE

6 Welfare

As in the previous section, we discuss welfare only for games where essentially all uncertainty is resolved at t_2, i.e., as $\tau \to 0$. In the limit as $\tau \to 0$, we denote ex-ante social welfare in the static coordination game Γ_{st} by $W_{st}(b_1, c, \sigma)$. It is given by:

$$Pr(\theta \geq \theta^*_{st,1}, x \geq x^*_{st,1})(b_1 - c) + Pr(\theta < \theta^*_{st,1}, x \geq x^*_{st,1})(-c)$$

For the dynamic game, Γ_{en}, ex-ante social welfare $W_{en}(b_1, b_2, c, \sigma)$ is given by:

$$Pr(\theta \geq \theta^*_{en}, x \geq x^*_{en})(b_1 - c) + Pr(\theta < \theta^*_{en}, x \geq x^*_{en})(-c) + Pr(\theta > \theta^*_{en}, x < x^*_{en})(b_2 - c)$$

As before, it is useful to divide our discussion into two cases: first, we consider the limiting case where $\sigma \to 0$, and then we consider welfare away from this limit.

6.1 Welfare Comparison in the Limiting Case

As we let noise vanish in the games, i.e., as $\sigma \to 0$, the product probability terms simplify. Now, writing $W(b_1, b_2, c, \sigma \to 0)$ for $\lim_{\sigma \to 0} W(b_1, b_2, c, \sigma)$, we can state:

Remark 7 *In the ordered limit as $\tau \to 0$ and $\sigma \to 0$, for all $b_1 > b_2 > c > 0$*

$$W_{en}(b_1, b_2, c, \sigma \to 0) > W_{st}(b_1, c, \sigma \to 0)$$

In addition $W_{en}(b_1, b_2, c, \sigma \to 0)$ is maximized when $b_2 = \frac{1}{2}(b_1 + c)$.

As $\sigma \to 0$, ex-ante welfare in each game becomes a monotone decreasing function of its unique equilibrium fundamental threshold. The lower the threshold, the higher is ex-ante social welfare. Thus, Remark 7 follows immediately upon inspection of Corollaries 5 and 6. We now proceed to consider welfare comparisons away from the limit.

6.2 Welfare Comparison away from the Limit

Welfare in our games broadly depends on two factors. One factor is the probability of coordinated investment. It is welfare improving, *ceteris paribus*, to enhance the probability of coordinated investment.[16] The second factor is the measure of agents who take the incorrect

[16] It is worth being precise about what we mean by *ceteris paribus* here. Welfare in the dynamic game can be written as follows:

$$\Pr(\theta \geq \theta^*_{en}) \left[\Pr(x \geq x^*_{en} | \theta \geq \theta^*_{en})(b_1 - c) + \Pr(x < x^*_{en} | \theta \geq \theta^*_{en})(b_2 - c)\right] + \Pr(\theta < \theta^*_{en}) \Pr(x \geq x^*_{en} | \theta < \theta^*_{en})(-c)$$

By ceteris paribus, in this statement, we mean "holding the conditional probability terms constant". Clearly, $\Pr(\theta \geq \theta^*_{en})$ multiplies a positive number, while $\Pr(\theta < \theta^*_{en})$ multiplies a negative number. Thus, holding the conditional probability terms constant, it increases welfare to increase $\Pr(\theta \geq \theta^*_{en})$, the probability of coordinated investment.

action (that is, agents who receive signals $x > (<)x^*$ when $\theta < (>)\theta^*$) in equilibrium. It is welfare improving, holding fixed the probability of coordinated investment, to decrease the measure of agents who choose incorrectly in equilibrium. As $\sigma \to 0$ the second factor becomes irrelevant, and welfare is driven entirely by the probability of coordination. Away from this limit, both factors are relevant.

We first note, as before, that the limiting results also hold close to the limit. Figures 6 and 7 illustrate this for $\sigma = 0.01$. In this case, welfare is driven almost entirely by the probability of successful coordination. It is not surprising, then, that welfare in the dynamic game is always higher than welfare in the static game, and the former is maximized at an intermediate cost of delay.

INSERT FIGURES 6 AND 7 HERE

Further from the limit, for $\sigma = 1$, learning can make an important difference to welfare. When θ is observed with large amounts of noise at t_1, but perfectly (since $\tau \to 0$) at t_2, large welfare gains are to be had at small costs of delay because many agents wait and improve their information significantly. In addition, as before, the prior mean can have an impact.

We have seen that when the prior mean μ_θ is low, the probability of successful investment is higher in Γ_{en} than in Γ_{st}. Thus, for a low prior mean, the presence of the option to delay simultaneously enhances the probability of successful investment and reduces the probability of errors (via learning). Welfare should be unambiguously higher in Γ_{en} than in Γ_{st} when μ_θ is low. Figure 8 illustrates that this is the case. It is also intuitive that welfare should be decreasing in k, since both the probability of coordinated investment and the net benefit from learning are higher for lower costs of delay.

On the other hand, we have also seen that when the prior mean μ_θ is high, the probability of successful investment is lower in Γ_{en} than in Γ_{st}. Thus, for a high prior mean, the presence of the option to delay reduces the probability of successful investment and simultaneously reduces the probability of errors (via learning). The overall effect on welfare is ambiguous when μ_θ is high.

However, when μ_θ is sufficiently high, there is little loss from errors (because the probability of failed investment is very low), and thus learning becomes less important. Then, welfare should be lower in Γ_{en} than in Γ_{st}. Figure 9 illustrates that this is the case for $\mu_\theta = 2$.

INSERT FIGURES 8 AND 9 HERE

However, for intermediate values of μ_θ, welfare may be non-monotone in k, and no clear conclusions can be reached. Figures 10 illustrates the case for $\mu_\theta = 0.5$. As we have seen above, in this case the cost of delay has a non-monotone effect on the probability of coordination. The effect on welfare due to learning is, as always, decreasing in k. The overall effect is non-monotone, with highest welfare achieved at low k.

INSERT FIGURE 10 HERE

7 Limitations and Generalizations

In this section, we consider natural variations of the model, discuss when the results are robust and when they may fail. We focus on both the learning technology and the payoffs of the game.

First let us consider restrictions implicit in the learning technology. An important aspect of this model is that learning is private: signals observed at t_2 are observed with some idiosyncratic noise. Though we consider the limit as idiosyncratic noise vanishes ($\tau \to 0$), it is well understood from the higher order beliefs and global games literature that the limiting properties of a model in which $\tau \to 0$ are very different from those of one in which $\tau = 0$. For example, with $\tau = 0$, our model will have multiple equilibria. Thus, our model is not a natural candidate to analyze settings in which there is a public variable which aggregates information precisely, such as a market price. Instead, this set-up is better for analyzing instances where such publicly available variables are absent: foreign direct investment (where accurate figures are hard to come by, and often quite delayed), technology adoption, or club formation settings all share this property. For analyses of publicly observed variables in global games, see Chamley (1999), Tarashev (2003), Angeletos and Werning (2005), and Hellwig, Mukherji, and Tsyvinsky (2005).

Let us now turn to the payoffs of the game. The investment project modeled here is defined by the conditions for its success, and by the payoffs conditional on success. We have assumed an ostensibly specific success condition: Investment succeeds if the total mass of investors at t_1 and t_2 exceeds a given function of θ, $a(\theta) = 1 - \theta$. It is easy to see that our qualitative conclusions generalize to any decreasing function which preserves the dominance regions, i.e., all $a(\theta)$, such that $a'(\theta) < 0$ for all θ, $a(\theta) > 1$ for low enough θ, and $a(\theta) < 0$ for high enough θ. For any such $a(\cdot)$, an equilibrium in monotone strategies will be characterized by a critical threshold θ^*, which will be determined by: $p(\theta^*) = a(\theta^*)$, where $p(\theta)$ denotes the equilibrium mass of investors in state θ. We have shown above that when $\tau \to 0$ and $\sigma \to 0$ $p(\theta^*_{en}) > p(\theta^*_{st})$, and thus it will be the case that $a(\theta^*_{en}) > a(\theta^*_{st})$, implying that $\theta^*_{en} < \theta^*_{st}$, as in the baseline model. Note, however, that in our success condition, we do not differentiate between the mass of early and late investors. Such distinctions would lead to the creation of "cohort effects". For an analysis of global games with cohort effects, see Heidhues and

Melissas (2003).

Finally, we turn to the payoffs of the investment project, conditional on success. An obvious extension to the model would be to allow the cost of investing at t_2 to be different from the cost of investing at t_1: $c_1 \neq c_2$. Thus, our payoffs would be parameterized by four numbers: benefits (b_1, b_2) and costs (c_1, c_2), with $b_1 > c_1 > 0$ and $b_2 > c_2 > 0$. In addition, in order to ensure that our game has a genuine (payoff) cost of delay (without which, the problem becomes uninteresting), we require two crucial conditions. First, it must be the case that $\frac{b_1}{c_1} > \frac{b_2}{c_2}$: otherwise more players would invest in a static game played at t_2 than in a static game played at t_1, implying that we are simply improving payoffs over time. Second, it must be the case that $b_1 - c_1 > b_2 - c_2$: otherwise no player would ever be indifferent between investing at t_1 and waiting until t_2 as $\tau \to 0$. Let us call these the "cost of delay" conditions. Formally, note that the equilibrium characterization in Proposition 3 is valid, as the proof in the appendix shows, for all payoff parameters that satisfy the cost of delay conditions. In addition, we show in section 8.1, that the limiting characterization for all (b_1, b_2) and (c_1, c_2) that satisfy the cost of delay conditions is as follows: as $\tau \to 0$ and $\sigma \to 0$

$$\theta_{en}^* \to \frac{c_1}{(b_1 - b_2 + c_2)} \frac{c_2}{b_2}$$

while as $\sigma \to 0$, $\theta_{st}^* \to \frac{c_1}{b_1}$. It is clear that even in this more general set-up $\theta_{en}^* < \theta_{st}^*$ as long as $b_1 > b_2$.

The most natural interpretation of costs $\{c_t\}$ in this model (arising, for example, from the foreign direct investment application) is that the c_t is a transaction cost (physical cost) *paid at the time of investment t*. The payoff is realized later, at $T > t_2 > t_1$. Under this interpretation, the assumption at $c_2 = c_1 = c$, is equivalent the absence of discounting. If agents discounted, then for a given transaction cost, it is less costly to pay later, so that $c_2 < c_1$. Then, the cost of delay conditions would imply that $b_1 > b_2$, thus implying that $\theta_{en}^* < \theta_{st}^*$. Thus, our the main result is preserved in the most natural extension of the model.

Can one conceive of scenarios where the main result fails? The only way to reverse the main result while maintaining the cost of delay assumptions is to choose $b_1 < b_2$; the cost of delay conditions then imply that it must be the case that $c_1 < c_2$. That is *both* costs and benefits increase over time. It is possible to choose such parameters, as following example, demonstrates, without violating the cost of delay conditions. For example, with $(b_1, c_1) = (1, \frac{3}{10})$ and $(b_2, c_2) = (\frac{12}{10}, \frac{6}{10})$, in the limit as $\tau \to 0$ and $\sigma \to 0$, $\theta_{en}^* \to \frac{3}{8}$, while as $\sigma \to 0$, $\theta_{st}^* \to \frac{3}{10} < \frac{3}{8}$. Thus, in this case, the option to delay and learn can actually *lower* the probability of successful investment. Such examples are characterized by the properties that the payoff benefit to investing *increases* with delay, and the transaction cost *increases* over time. These properties are not natural in the contexts we have discussed. The precise nature of payoffs, however, must necessarily follow from specific applications, and a microfounded approach is necessary to be substantive. A general analysis of such applications is clearly beyond the scope of the current exercise.

Finally, we note that in many of the applied settings that we have discussed, the cost of delay is not exogenous, as we have assumed in the model, but actually depends on the proportion of investors who invest early. Such a modification would vastly complicate the model, but remains an interesting area for future research.

8 Appendix

Proof of Proposition 2: The following are necessary for a monotone equilibrium:

The marginal agent, who receives signal x^*_{st} must be indifferent between investing or not, i.e.

$$Pr(\theta \geq \theta^*_{st}|x^*_{st}) = \frac{c}{b_1}$$

Since $\theta|x \sim N(\frac{x}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2})$, the indifference condition can be written as:

$$1 - Pr(\theta < \theta^*_{st}|x^*_{st}) = 1 - \Phi(\frac{\theta^*_{st} - \frac{x^*_{st}}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}) = \frac{c}{b_1}$$

Thus,

$$x^*_{st} = (1+\sigma^2)\theta^*_{st} + \sigma\sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1}) \tag{10}$$

The critical mass condition requires that:

$$Pr(x \geq x^*_{st}|\theta^*_{st}) = 1 - \theta^*_{st}$$

Substituting the indifference condition into the critical mass condition we get

$$\Phi(\sigma\theta^*_{st} + \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1})) = \theta^*_{st} \tag{11}$$

Consider the function

$$F(\theta^*_{st}) = \Phi(\sigma\theta^*_{st} + \sqrt{1+\sigma^2}\Phi^{-1}(\frac{c}{b_1})) - \theta^*_{st}$$

Clearly as $\theta^*_{st} \to 1$, $F(\cdot) < 0$, and as $\theta^*_{st} \to 0$, $F(\cdot) > 0$. Differentiating yields

$$F'(\theta^*_{st}) = \sigma\phi(\cdot) - 1$$

If $\sigma < \sqrt{2\pi}$, then $F'(\theta^*_{st}) < 0$ for all θ^*_{st}, which establishes that there is a unique $(x^*_{st}, \theta^*_{st})$ that solves the necessary conditions for the equilibrium. In addition, note that $Pr(\theta \geq \theta^*_{st}|x)$

is strictly increasing in x, so that agents who receive $x > x^*_{st}$ will choose to invest while those who receive $x < x^*_{st}$ will choose not to invest. Thus, there exists a unique monotone equilibrium. The nonexistence of monotone equilibria follows from the iterative deletion of dominated strategies, as is shown by Morris and Shin (2002) amongst others. This part of the proof is omitted for brevity. This establishes the first part of the result. Letting $\sigma \to 0$ in (10) establishes the second part. ■

Lemma 8 *Fix any* (x^*, s^*). *Let*

$$G(\theta) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq s^*|\theta) - 1 + \theta$$

Then $G(\theta)$ *is monotone and crosses zero exactly once.*

Proof: Since $s = \frac{\tau^2 x + \sigma y + x^*}{1+\tau^2+\sigma^2\tau^2}$, writing $x = \theta + \sigma\epsilon$, $y = \frac{\theta - x^*}{\sigma} + \tau\eta$, and substituting, we get $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\theta + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Then $s \geq s^* \Leftrightarrow \gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$, where $\gamma = \tau\epsilon + \eta$. Thus, we can rewrite:

$$G(\theta) = 1 - \Phi(A(\theta)) + \int_{-\infty}^{A(\theta)} \int_{B(\theta)}^{\infty} f(\epsilon, \gamma) d\gamma d\epsilon - 1 + \theta$$

where $A(\theta) = \frac{x^* - \theta}{\sigma}$ and $B(\theta) = \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}s^* - \frac{1+\tau^2}{\sigma\tau}\theta$. Differentiating under the double integral:

$$G'(\theta) = -A'(\theta)\phi(A(\theta)) + A'(\theta)\int_{B(\theta)}^{\infty} f(A(\theta, \gamma))d\gamma - B'(\theta)\int_{-\infty}^{A(\theta)} f(\epsilon, B(\theta))d\epsilon + 1$$

Writing the joint densities as products of conditionals and marginals:

$$f(\epsilon = A(\theta), \gamma) = \phi(A(\theta)) f(\gamma|\epsilon = A(\theta))$$

$$f(\epsilon, \gamma = B(\theta)) = \hat{\phi}(B(\theta)) f(\epsilon|\gamma = B(\theta))$$

writing $\phi(\cdot)$ to denote the standard normal PDF of ϵ, and $\hat{\phi}(\cdot)$ to denote the (non-standard) Normal PDF for γ. Finally,

$$A'(\theta) = -\frac{1}{\sigma}, B'(\theta) = -\frac{1+\tau^2}{\sigma\tau}$$

Now we can rewrite $G'(\theta)$ as:

$$\frac{1}{\sigma}\phi(A(\theta))\left[1 - \int_{B(\theta)}^{\infty} f(\gamma|\epsilon = A(\theta)d\gamma\right] + \frac{1+\tau^2}{\sigma\tau}\hat{\phi}(B(\theta))\int_{-\infty}^{A(\theta)} f(\epsilon|\gamma = B(\theta))d\epsilon + 1$$

i.e. $G'(\theta) > 0$. Note that $\lim_{\theta\to\infty} G(\theta) = \infty$, and $\lim_{\theta\to-\infty} G(\theta) = -\infty$. Thus there exists a unique solution to $G(\theta) = 0$. ■

Lemma 9 *Assume* $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$. *Then, for any* x^**there is a unique* $\hat{\theta}(x^*)$ *such that* $G(\hat{\theta}, x^*) = 0$ *where*

$$G(\theta, x^*) = Pr(x \geq x^*|\theta) + Pr(x < x^*, s \geq \theta + M|\theta) - 1 + \theta$$

Moreover, $\frac{d\hat{\theta}}{dx^*} \in (0, \frac{1}{1+\sigma^2})$

Proof: As above, we know that $s = \frac{1+\tau^2}{1+\tau^2+\sigma^2\tau^2}\hat{\theta} + \frac{\sigma\tau}{1+\tau^2+\sigma^2\tau^2}(\tau\epsilon + \eta)$. Since $s^* = \hat{\theta} + M$, $s \geq s^* \Leftrightarrow \gamma \geq \sigma\tau\hat{\theta} + \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}M$. Let

$$B(\hat{\theta}) = \sigma\tau\hat{\theta} + \frac{1+\tau^2+\sigma^2\tau^2}{\sigma\tau}M$$

Note that $B'(\hat{\theta}) = \sigma\tau$, and so, using the proof of Lemma 8,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial\hat{\theta}} = \frac{1}{\sigma}\phi(A(\hat{\theta}, x^*))\left[1 - \int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*))d\gamma\right] - \sigma\tau\hat{\phi}(B(\hat{\theta}))\int_{-\infty}^{A(\hat{\theta},x^*)} f(\epsilon|\gamma = B(\hat{\theta}))d\epsilon + 1$$

where $\hat{\phi}(\cdot)$ denotes the non-standard Normal pdf of γ. Let

$$P_1 = \int_{B(\hat{\theta})}^{\infty} f(\gamma|\epsilon = A(\hat{\theta}, x^*))d\gamma$$

$$P_2 = \int_{-\infty}^{A(\hat{\theta},x^*)} f(\epsilon|\gamma = B(\hat{\theta}))d\epsilon$$

Since the variance of γ is $1 + \tau^2$, $\hat{\phi}(\cdot) < \frac{1}{\sqrt{2\pi}\sqrt{1+\tau^2}}$, and $P_2 \leq 1$, clearly if $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, $\frac{\partial G(\hat{\theta},x^*)}{\partial\hat{\theta}} > 0$. Similarly,

$$\frac{\partial G(\hat{\theta}, x^*)}{\partial x^*} = -\frac{1}{\sigma}\phi(A(\hat{\theta}, x^*))\left[1 - P_1\right] < 0$$

By the implicit function theorem

$$\frac{d\hat{\theta}(x^*)}{dx^*} = -\frac{\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}}{\frac{\partial G(\hat{\theta},x^*)}{\partial\hat{\theta}}}$$

Let $Q = -\frac{\partial G(\hat{\theta},x^*)}{\partial x^*}$, where $Q > 0$. Then,

$$\frac{d\hat{\theta}(x^*)}{dx^*} = \frac{Q}{Q - \sigma\tau\hat{\phi}(\cdot)P_2 + 1}$$

It is easy to check, that when $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$

$$\frac{1}{1+\sigma^2} - \frac{d\hat{\theta}(x^*)}{dx^*} > 0$$

Since $\sigma < \frac{\sqrt{2\pi}}{1+\frac{\tau}{\sqrt{1+\tau^2}}}$ implies that $\sigma < \frac{\sqrt{2\pi}}{\frac{\tau}{\sqrt{1+\tau^2}}}$, we are done. ■

Proof of Proposition 3: For pedagogical purposes, it is worth writing this proof for a general set of payoffs (b_1, c_1) for t_1 and (b_2, c_2) for t_2 where $b_1 > c_1 > 0$, $b_2 > c_2 > 0$, $\frac{b_1}{c_1} > \frac{b_2}{c_2}$, and $b_1 - c_1 > b_2 - c_2$. Proposition 3 requires only the special case where $c_1 = c_2 = c$, which then implies that $b_1 > b_2$.

Initially, agents trade off the expected benefit of investing in period 1 against the expected benefit of retaining the option value to wait. Thus the marginal period 1 investor who receives signal x^*_{en} must satisfy:

$$Pr(\theta \geq \theta^*_{en}|x^*_{en})b_1 - c_1 = Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x^*_{en})[b_2 - c_2] + Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x^*_{en})(-c_2) \tag{12}$$

We can rewrite the indifference condition for t_2 players as:

$$s^*_{en} = \theta^*_{en} + \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}}\Phi^{-1}(\frac{c_2}{b_2}) \tag{13}$$

By Lemma 9, we can write $\theta^*_{en} = g(x^*_{en})$, and thus rewrite equation (13) as:

$$s^*_{en} = g(x^*_{en}) + M \tag{14}$$

where where $M = \frac{\sigma}{\sqrt{1+\sigma^2+\frac{1}{\tau^2}}}\Phi^{-1}(\frac{c_2}{b_2})$. Write x for x^*_{en} and let

$$G(x) = Pr(\theta \geq \theta^*_{en}|x)b_1 - c_1 - (b_2 - c_2)Pr(\theta \geq \theta^*_{en}, s \geq s^*_{en}|x) + c_2 Pr(\theta < \theta^*_{en}, s \geq s^*_{en}|x)$$

Note that

$$Pr(\theta \geq \theta^*_{en}|x) = 1 - \Phi(\frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}})$$

Let $A(x) = \frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$. Given x,

$$s = \frac{\tau^2 x + \theta + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2}$$

Rearranging terms, we can write this as

$$s = \frac{x}{1+\sigma^2} + \frac{\sigma}{1+\tau^2+\sigma^2\tau^2}\left[\frac{z}{\sqrt{1+\sigma^2}} + \tau\eta\right]$$

where $z = \frac{\theta - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$ is distributed $N(0,1)$ conditional on x. Let $\gamma = \frac{z}{\sqrt{1+\sigma^2}} + \tau\eta$. Then, $s \geq s^*$ is equivalent to

$$\gamma \geq \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}} A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c_2}{b_2})$$

Let

$$B(x) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}} A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c_2}{b_2})$$

Now, we may rewrite:

$$G(x) = b_1(1-\Phi(A(x))) - c_1 - (b_2-c_2)Pr(z \geq A(x), \gamma \geq B(x)) + c_2 Pr(z < A(x), \gamma \geq B(x)) \tag{15}$$

Differentiating under the double integral and rearranging we get:

$$G'(x) = -\phi(A(x))A'(x)\left[b_1 - b_2 P_1\right] + B'(x)\hat{\phi}(B(x))\left[b_2 P_2 - c_2\right]$$

where by $\hat{\phi}(\cdot)$ we denote the non-standard normal density of γ, and P_1 and P_2 are defined as follows:

$$P_1 = \int_{B(x)}^{\infty} f(\gamma | z = A(x)) d\gamma$$

$$P_2 = \int_{A(x)}^{\infty} f(z | \gamma = B(x)) dz$$

Using standard formulae for computing conditional distributions of Normal random variables (see, for example, Greene 1996), we know that:

$$z|\gamma = B(x) \sim N(A(x) + \frac{\tau\sqrt{1+\sigma^2}}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c_2}{b_2}), \frac{\tau^2(1+\sigma^2)}{1+\tau^2+\sigma^2\tau^2})$$

Thus,

$$P_2 = \int_{A(x)}^{\infty} f(z|\gamma = B(x))dz = \frac{c_2}{b_2}$$

and therefore

$$G'(x) = -\phi(A(x))A'(x)[b_1 - b_2 P_1]$$

Under the conditions of the theorem $A'(x) < 0$, so $G'(x) > 0$. In addition, note that as $x \to -\infty$, $G(x) \to -c_1 < 0$, and as $x \to \infty$, $G(x) \to (b_1 - c_1) - (b_2 - c_2) > 0$. Thus, there exists a unique $(x^*_{en}, s^*_{en}, \theta^*_{en})$ that satisfies the three necessary conditions for monotone equilibrium in Γ_{en}.

Finally, fixing θ^*_{en}, note that inspection of (15) shows that the indifference condition for t_1 players depends on x only via the functions $A(x) = \frac{\theta^*_{en} - \frac{x}{1+\sigma^2}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}$, and $B(x) = \frac{1+\tau^2+\sigma^2\tau^2}{\sqrt{1+\sigma^2}}A(x) + \tau\sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c_2}{b_2})$. Fixing θ^*_{en}, it is clear that $A(x, \theta^*_{en})$ is always strictly decreasing in x (for all $\sigma > 0$), and thus agents who receive signals $x > x^*_{en}$ will choose to invest at t_1, and agents who receive signals $x < x^*_{en}$ will choose to wait. Therefore the proof is complete. ■

8.1 Detailed derivations of equations (7) and (8):[17]

Again, for pedagogical purposes, we write this derivation for a general set of payoffs (b_1, c_1) for t_1 and (b_2, c_2) for t_2 where $b_1 > c_1 > 0$, $b_2 > c_2 > 0$, $\frac{b_1}{c_1} > \frac{b_2}{c_2}$, and $b_1 - c_1 > b_2 - c_2$. The equations to be derived require only the special case where $c_1 = c_2 = c$, which then implies that $b_1 > b_2$. First consider the derivation of (7). By Lebesgue dominated convergence:

$$Pr(x < x^*, s \geq s^*|\theta^*) = \int_{-\infty}^{x^*} Pr(s \geq s^*|\theta^*, x) f(x|\theta^*) dx$$

By definition $s = \frac{\tau^2 x + \sigma y + x^*}{1+\tau^2+\sigma^2\tau^2}$. Given x and θ^*, and substituting in $y = \frac{\theta^* - x^*}{\sigma} + \tau\eta$, we get $s = \frac{\tau^2 x + \theta^* + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2}$. We know that $s^* = \theta^* + \frac{\sigma\tau}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c_2}{b_2})$. Thus,

$$s \geq s^*|x, \theta^* \Leftrightarrow \frac{\tau^2 x + \theta^* + \sigma\tau\eta}{1 + \tau^2 + \sigma^2\tau^2} \geq \theta^* + \frac{\sigma\tau}{\sqrt{1 + \tau^2 + \sigma^2\tau^2}}\Phi^{-1}(\frac{c_2}{b_2})$$

After some algebra, this reduces to:

$$\eta \geq \frac{\tau}{\sigma}[\theta^*(1 + \sigma^2) - x] + \sqrt{1 + \tau^2 + \sigma^2\tau^2}\Phi^{-1}(\frac{c_2}{b_2})$$

As $\tau \to 0$, the RHS converges pointwise to $\Phi^{-1}(\frac{c_2}{b_2})$. Thus

$$Pr(s \geq s^*|\theta^*, x) \to 1 - \Phi(\Phi^{-1}(\frac{c_2}{b_2})) = 1 - \frac{c_2}{b_2}$$

Thus,

$$Pr(x < x^*, s \geq s^*|\theta^*) \to Pr(x \leq x^*|\theta^*)[1 - \frac{c_2}{b_2}] = \Phi(\frac{x^* - \theta^*}{\sigma})[1 - \frac{c_2}{b_2}]$$

Thus, equation (5) reduces to $1 - \Phi(\frac{x^*-\theta^*}{\sigma}) + \Phi(\frac{x^*-\theta^*}{\sigma})[1 - \frac{c_2}{b_2}] = 1 - \theta^*$, or in other words: $\Phi(\frac{x^*-\theta^*}{\sigma})\frac{c_2}{b_2} = \theta^*$, which, setting, $c_2 = c$, is (7).

Now consider the derivation of (8).

$$Pr(\theta \geq \theta^*, s \geq s^*|x^*) = \int_{\theta^*}^{\infty} Pr(s \geq s^*|\theta, x^*) f(\theta|x^*) d\theta$$

$$Pr(\theta < \theta^*, s \geq s^*|x^*) = \int_{-\infty}^{\theta^*} Pr(s \geq s^*|\theta, x^*) f(\theta|x^*) d\theta$$

[17] I am particularly grateful to an anonymous referee for proposing this elegant shortening of my original proof.

Given x^* and θ, it is easy to see that $s = \frac{\tau^2 x^* + \theta + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2}$. Thus,

$$s \geq s^* \Leftrightarrow \frac{\tau^2 x^* + \theta + \sigma\tau\eta}{1+\tau^2+\sigma^2\tau^2} \geq \theta^* + \frac{\sigma\tau}{\sqrt{1+\tau^2+\sigma^2\tau^2}}\Phi^{-1}(\frac{c_2}{b_2})$$

which reduces to

$$\eta \geq \frac{\theta^* - \theta}{\sigma\tau} + \frac{\tau}{\sigma}[(1+\sigma^2)\theta^* - x^*] + \sqrt{1+\tau^2+\sigma^2\tau^2}\Phi^{-1}(\frac{c_2}{b_2})$$

As $\tau \to 0$, the RHS tends to $-\infty$ or ∞ depending on whether $\theta > \theta^*$ or $\theta < \theta^*$. Thus

$$Pr(\theta \geq \theta^*, s \geq s^* | x^*) \to Pr(\theta \geq \theta^* | x^*)$$

$$Pr(\theta < \theta^*, s \geq s^* | x^*) \to 0$$

Thus, (6) reduces to

$$Pr(\theta \geq \theta^* | x^*) b_1 - c_1 = Pr(\theta \geq \theta^* | x^*)(b_2 - c_2)$$

In other words,

$$\Phi\left(\frac{\frac{x^*_{en}}{1+\sigma^2} - \theta^*_{en}}{\frac{\sigma}{\sqrt{1+\sigma^2}}}\right) = \frac{c_1}{b_1 - (b_2 - c_2)}$$

which, setting $c_2 = c_1 = c$, is (8).

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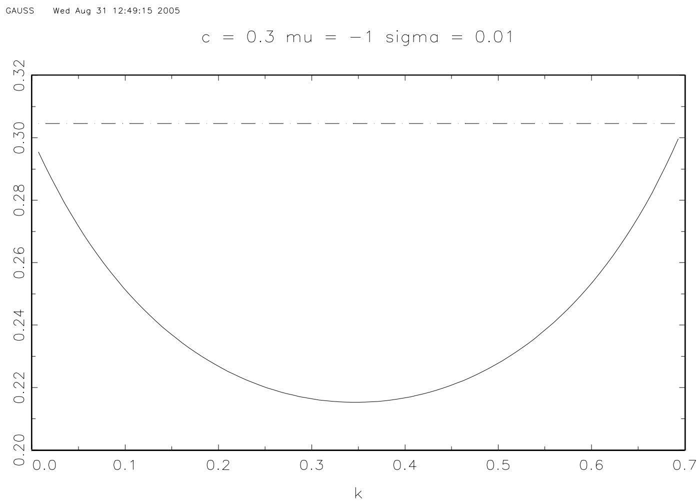

Figure 1: $\sigma = 0.01$: Thresholds with Low Prior ($\mu_\theta = -1$). (Γ_{en} : solid line, Γ_{st} : dashed line)

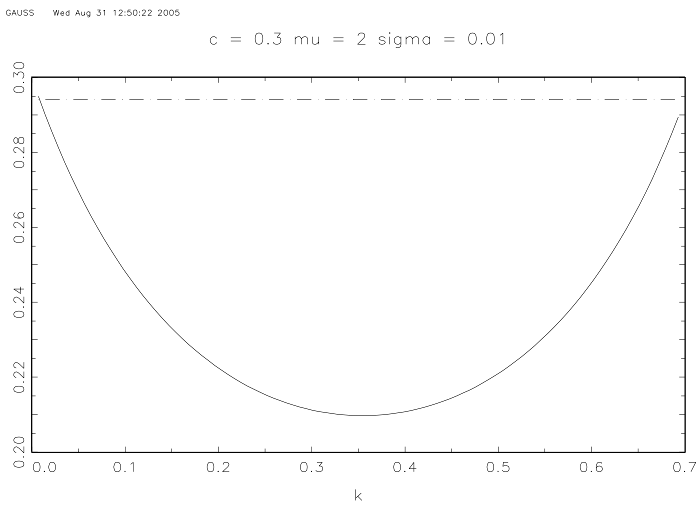

Figure 2: $\sigma = 0.01$: Thresholds with High Prior ($\mu_\theta = 2$). (Γ_{en} : solid line, Γ_{st} : dashed line)

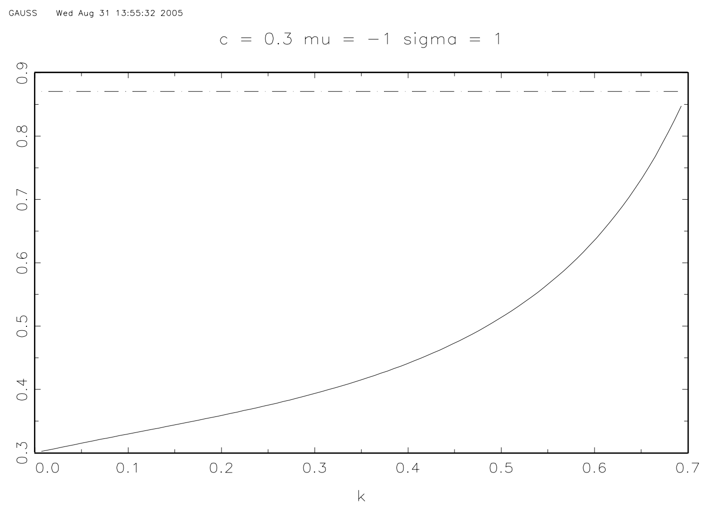

Figure 3: $\sigma = 1:$ Thresholds with Low Prior ($\mu_\theta = -1$). (Γ_{en} : solid line, Γ_{st} : dashed line)

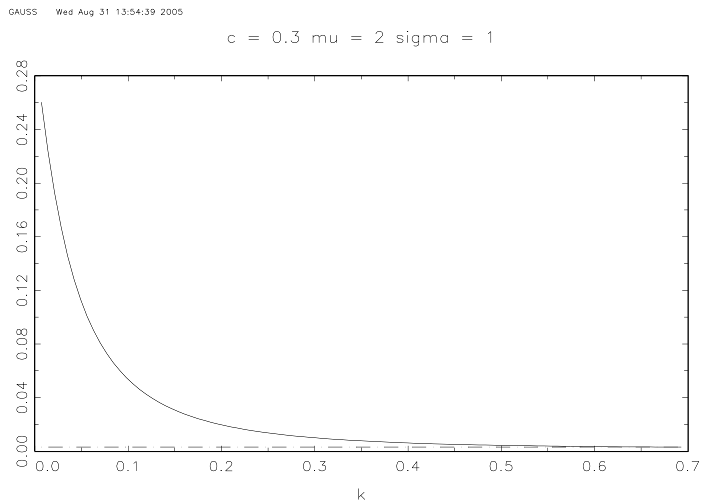

Figure 4: $\sigma = 1$: Thresholds with High Prior ($\mu_\theta = 2$). (Γ_{en} : solid line, Γ_{st} : dashed line)

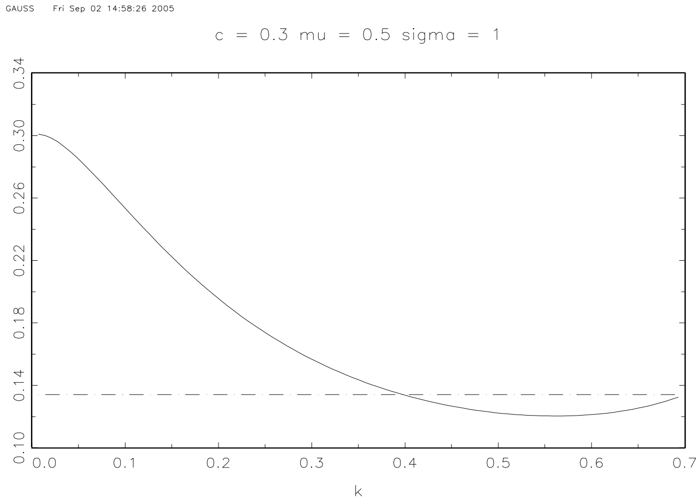

Figure 5: $\sigma = 1$: Thresholds with Intermediate Prior ($\mu_\theta = 0.5$). (Γ_{en} : solid line, Γ_{st} : dashed line)

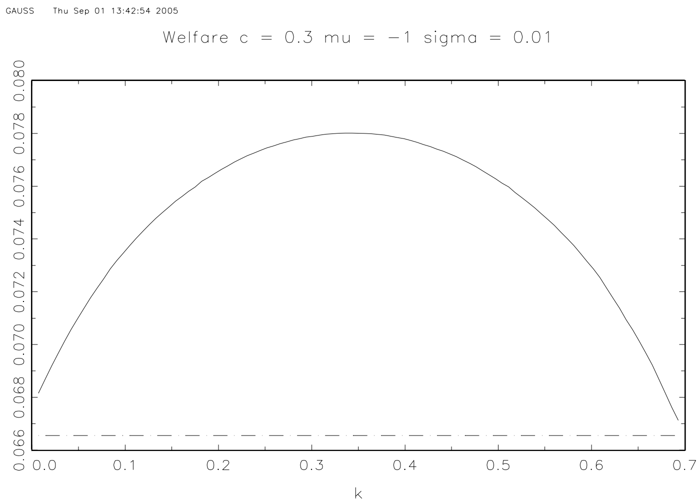

Figure 6: $\sigma = 0.01$: Welfare with Low Prior ($\mu_\theta = -1$). (Γ_{en} : solid line, Γ_{st} : dashed line)

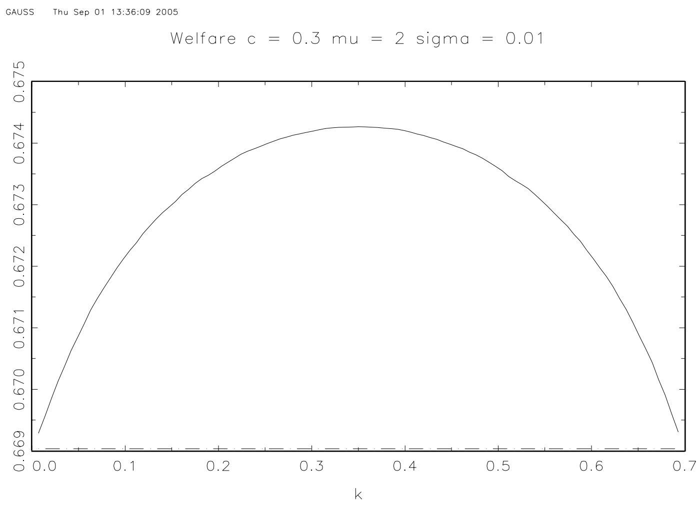

Figure 7: $\sigma = 0.01$: Welfare with High Prior ($\mu_\theta = 2$). (Γ_{en} : solid line, Γ_{st} : dashed line)

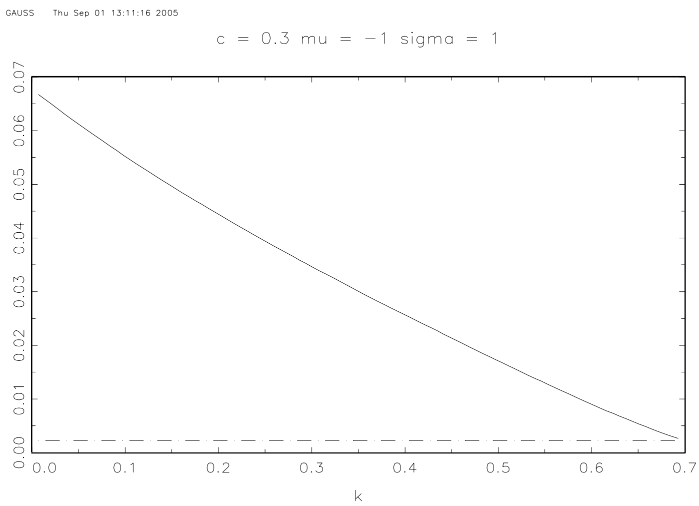

Figure 8: $\sigma = 1$: Welfare with Low Prior ($\mu_\theta = -1$). (Γ_{en} : solid line, Γ_{st} : dashed line)

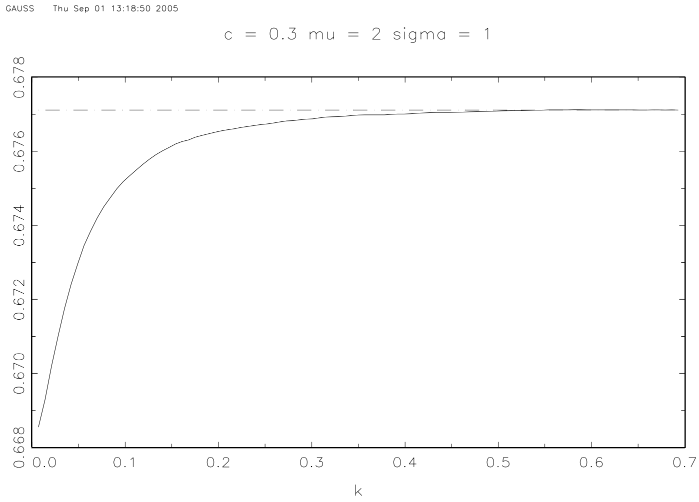

Figure 9: $\sigma = 1$: Welfare with High Prior ($\mu_\theta = 2$). (Γ_{en} : solid line, Γ_{st} : dashed line)

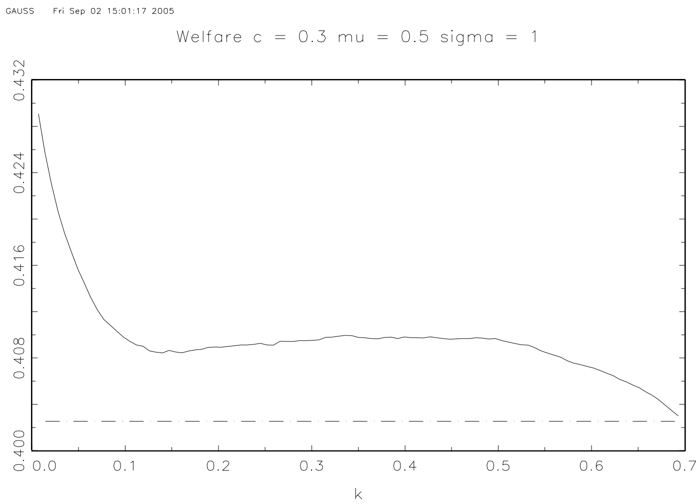

Figure 10: $\sigma = 1$: Welfare with Intermediate Prior ($\mu_\theta = 0.5$). (Γ_{en} : solid line, Γ_{st} : dashed line)