

Complementarities in Information Acquisition with Short-Term Trades

Christophe Chamley

Boston University, PSE and CREST

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Abstract

In a financial market where agents trade for prices in the short-term and where news can increase the uncertainty of the public belief, there are strategic complementarities in the acquisition of private information and a continuum of equilibrium strategies if the cost of information is sufficient small. Imperfect observation of the past prices reduces the continuum of Nash-equilibrium to a unique one which may be a Strongly Rational-Expectations Equilibrium. In that equilibrium, because of the strategic complementarity, there are two sharply different regimes for the evolution of the price, the volume of trade and the information acquisition which is either nil or at its maximum.

Keywords: financial markets, short-term, endogenous information, multiple equilibria, social learning, trading frenzies.

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1 Introduction

In a celebrated study, Grossman and Stiglitz (1980) show how strategic substitutability arises in the acquisition of private information about the fundamental value of an asset¹. In the model that is presented here, strategic complementarity arises when agents trade for the price of the asset in the near future before the revelation of the fundamental. The mechanism rests solely on the essential property of social learning in which current activity has an impact on the updating of the belief (the price) that is inversely related to the uncertainty of the belief from history: if more agents are informed today, the information provided by the market today may increase the uncertainty of the belief and thus reduce the weight of history tomorrow, in which case the trade tomorrow drives the price more strongly toward the fundamental. If agents today are short-term and care about the price tomorrow, then the payoff of information about the fundamental is increased. (The effect does not appear if agents hold until the realization of the fundamental). The complementarity in information gathering generates multiple equilibria with sudden changes in the volume of trade, in the information that is generated by the market, and in the volatility and the evolution of the asset price.

Grossman and Stiglitz present the argument for substitutability in a particular model, but the mechanism is general. Take a financial asset that is a claim to the realization of the fundamental θ after a one period market with the equilibrium price p . The expected profit from trading one unit of the asset (buying or selling short) is the absolute value $|E[\theta|s, p] - p|$, where $E[\theta|s, p]$ is the expected value of θ conditional on the private information of the agent s and the market price p . If more agents get information about θ , their private information is on average in the right direction and their trades move the price p in the direction of $E[\theta|s, p]$. (The price effect is the mechanism by which private information is conveyed, at least partially, through the market). The reduction of the margin $|E[\theta|s, p] - p|$, leads to a lower gross payoff of information and a lower incentive to invest in information².

¹In their second conjecture, the only one which will not be true here, “the more individuals who are informed, the lower the expected utility of the informed to the uninformed”.

²This argument applies if the information that agents get about θ is sufficiently strong, or if the inference problem from the price has “standard” properties, which apply in a Gaussian

The one-period model in which the fundamental is revealed after one period is a metaphor for long-term trades in which agents can, after an initial trade where they use their information, hold their position until the revelation of the fundamental. In this long-term setting, the strategic substitution is a general property because more private information has only one effect, the adverse movement of the value of the asset, (more expensive if the information is good, cheaper if it's bad).

When agents trade for the short-term, they plan to undo their position before the revelation of the fundamental. What matters now is the price at the end of their horizon. Assume for simplicity that agents hold their position for one period only, with an expected profit equal to $|E[p_{t+1}] - p_t|$, where $E[p_{t+1}]$ is the expectation for the informed agents. Agents want to predict the price in the next period $t + 1$, but they can get (at some cost) information on the fundamental θ . Information about θ is valuable if $E[p_{t+1} - p_t|\theta]$ is positive despite the positive relation between p_t and θ . There is strategic complementarity in the acquisition of private information if more informed agents lead to an increase of $E[p_{t+1} - p_t|\theta]$. The effect occurs only if the impact on $E[p_{t+1}]$ is stronger than the impact on p_t .

Why would the price difference $p_{t+1} - p_t$ be more correlated with the fundamental if more rational agents are informed in period t ? An example is given by Froot, Scharfstein and Stein (1992), where trade orders are executed randomly in the present and in the next period, presumably because of some frictions. Information about the fundamental is useful in predicting the information of others who boost the demand and the price in the next period when, by assumption, the other half of the orders is executed³.

Here, the analysis will rest only on a standard updating process in which new informed agents enter the market in each period and beliefs are updated from history as in standard models of social learning. Assume for simplicity that there is large mass of risk-neutral agents, and that the price p_t is equal to the expected value of θ , given the public information. Informed agents in period t have an

model. The argument does not apply in Barlevi and Veronesi (2000): strategic complementarity in information acquisition can arise in their one period model with risk-neutral agents because the structure of the noise is such that if the price moves toward the true value of θ (0 or 1), the price is less informative.

³In their model, agents can learn (at no cost) only one of the two independent components of the fundamental. The analysis remains to be done for the more natural framework of Grossman-Stiglitz with a costly information acquisition.

impact on the price p_t by their trade which is observed through a noise (for trade to be possible). The price in period t is the public belief about θ which is updated from the history's information using the market trade in period t . In the standard CARA-Gauss model, more observations always increase the weight of history and reduce the uncertainty of the public belief. If more agents are informed in period t , the history at the beginning of period $t + 1$ has more weight, and the informed trades have less impact on the price in period $t + 1$: more informed agents in period t imply a weaker relation between the price p_{t+1} and θ in the next period. There cannot be strategic complementarity⁴.

In order to generate the intuitive and important property that some market news may increase the uncertainty of the public belief, thereby reducing the weight of history, we need to depart from the Gaussian model. In the model that is presented in Section 2, the fundamental θ takes one of two possible values (an assumption that can be generalized)⁶. Uncertainty is small when the probability of $\theta = 1$ is near 0 or 1, and highest in the middle-range. In this context, a natural tool of analysis is the model of sequential trading by Glosten and Milgrom (1985). Two features are added: (i) agents who have private information hold the asset only for one period, (but the model could be extended to holdings for a few periods); (ii) some agents, information traders, can obtain information about the fundamental, at some cost, before entering the market, and their decision depends on the publicly available information⁷.

⁴We focus on the simplest mechanisms without risk-aversion. When a fixed number of informed and risk-averse agents trade for the short-term, there can be strategic complementarity in the amount of their trade: they trade more if the price is less uncertain in the next period, but the price is less uncertain when they trade more and hereby convey more information about the fundamental (Chamley, 2004).

⁶For previous studies that depart from the Gaussian framework to generate time-variable uncertainty see, among others, Detemple (1991), David (1997), Veronesi (1998).

⁷Dow and Gorton (1994) analyze the efficiency of financial markets with short-term trading and exogenous information. In a model of the Glosten-Milgrom type, they make the key assumption that the probability of an informed agent increases exogenously as the maturity of the asset goes to zero. There is a fixed cost of trading. When the maturity is long, the probability that the price moves in the right direction in the next period (because of the occurrence of an informed trader) is small and because of the fixed cost, agents do not trade. Trade begins only when the maturity is sufficiently short. Vives (1995) analyzes the informational content of prices with short-term traders in the CARA-Gauss model when private information is accrued over time and when the fundamental is revealed at the end of the N -period game.

The strategy of information traders is common knowledge in a rational-expectations equilibrium and if traders are more informed in some period, the price in that period moves more, on average, in the direction of the fundamental. If more agents get information in period $t + 1$, the payoff of information about the fundamental in period t increases. There is strategic complementarity from information investment in period $t + 1$ to information investment⁸ in period t . An equilibrium strategy in period t depends on the strategy in period $t + 1$.

Stationary strategies with an infinite number of periods are analyzed in Section 3: In each period, agents invest in information if the price is in some interval (p^{**}, p^*) which is strictly included in $(0, 1)$. The stationarity assumption fixes the expectation about the strategy in the next period. Given this expectation, if the degree of public uncertainty is not too large, there is a strategic complementarity between the information acquisitions of different agents within the same period.

To understand the key mechanism, suppose that the public belief is relatively high. A higher level of information investment in period t may induce a stronger fall of the belief in period $t + 1$ and a smaller “confidence” in that period. Should that happen, the weight of history will be smaller in period $t + 1$ and there will be a higher payoff of information investment in that period. Both effects enhance the variation of p_{t+1} toward the fundamental and thus the payoff of information in period t .

The strategic complementarity induces a continuum of equilibria in the class of these trigger strategies: the two threshold prices p^{**} and p^* define an equilibrium solution if each of them belong to some sub-interval of $(0, 1)$.

Some additional properties are presented in Section 4. First, the strategic complementarity is enhanced when hedging is endogenous. When informed traders are more active, the wider bid-ask spread crowds out some hedgers and thus enhances the information content of transactions. Second, it is shown that under some conditions, the only rationalizable strategies (*i.e.*, surviving the iterated elimination of dominated strategies), are those in the continuum. Non stationary strategies are also discussed.

⁸When agents trade for the short-term, the incentive to gather private information may be smaller than when they trade for the long-term, but this reduction affects the “first derivative” of the payoff while the strategic complementarity is about the “cross derivative”. (See Cooper and John, 1988).

The continuum of equilibria under common knowledge opens the issue of “equilibrium selection”, and the robustness of this property to a perturbation. The model is therefore extended in Section 5 with the very plausible assumption of an observation noise: agents have imperfect knowledge about the last transaction price before they decide whether to get information about the fundamental. This observation noise is bounded and that bound must be sufficiently small. In this extension to a “global game” (Carlsson and Van Damme, 1993), there are three types of properties: (i) the set of rationalizable strategies is reduced; (ii) under the behavioral assumption that in some distant period agents use a reasonable strategy (which will be defined), there is in the present period a unique rationalizable strategy and therefore a Strongly Rational-Expectations Equilibrium (SREE), which turns out to be a trigger strategy of the type described in Section 3; as a corollary, the continuum of trigger strategies found in Section 3 is reduced to that unique strategy; (iii) under the sufficient parametric conditions of Section 4 for which the continuum is the set of rationalizable strategies under common knowledge, that set is reduced to the SREE in (ii). The SREE property is important as it shows that the special trigger strategy is the unique stable strategy in a wide class of strategies.

The iterated elimination of dominated strategies cannot be applied period by period separately as in standard models. By the definition of an equilibrium which depends on the strategy in the following period, the iteration has to be implemented backwards between periods.

The extension shows that the existence of a continuum of equilibria depends on the common knowledge and is not robust to a perturbation. However, the essential property associated to multiple equilibria is a discontinuity in the behavior of agents. That property is strongly validated in the extension. In the unique equilibrium, when the price crosses a threshold value, the fraction of informed agents jumps between zero and its maximum, and the average amplitude of price changes between periods changes abruptly⁹.

⁹In Veronesi (1999), agents are risk-averse and there is a non-linear relation between the asset price and the public belief about the fundamental. Volatility depends on the asset price. Because of the risk-aversion, bad news when agents are fairly confident about a high fundamental have a strong impact because they reduce this confidence; good news have a weak impact because they also increase the uncertainty. In the present model, agents are risk-neutral and the price is always equal to the expected value of the fundamental.

2 The model

There is a financial asset that is a claim on the “fundamental” value $\theta \in \{0, 1\}$, which is not directly observable. The public belief is defined as the probability that $\theta = 1$, conditional on the public information. At the beginning of the first period, the value of the public belief is defined by p_0 . The model builds on the standard setting of Glosten and Milgrom (1985). To introduce the short-term, the value of θ is revealed with probability δ in each period, conditional on no previous revelation. The value of δ will be small, in a sense that will be more precise later¹⁰.

When θ is not revealed at the beginning of the period (with probability $1 - \delta$), trading takes place: a new agent meets a risk-neutral profit maximizing market-maker and either trades one unit of the asset or does not trade. The new agent is of one of the following three types.

(i) With probability $\alpha > 0$, the agent has exogenous private information about the true state. To simplify, and without loss of generality¹¹, such an agent is perfectly informed about θ . In some special cases, which will be explicitly stated, α will be equal to zero.

(ii) With probability $\bar{\beta}$, the agent is an *information trader* who can “invest”, at a fixed¹² cost c , in information about the true state θ before trading. As for the agents of the previous type, this information is assumed to be perfect: if the agent pays the cost c , he is said to *invest* and he gets to know θ . The decision whether to invest in information is made at the beginning of the period, knowing the public belief about θ . The strategy of an agent is defined by the probability to invest, $\beta/\bar{\beta}$. An information trader who is informed knows the state and therefore trades according to that information, at any price: he buys (sells) when the state is good (bad). An information trader who is not informed will not trade, because of the spread between the ask and the bid, in equilibrium.

¹⁰One may also assume that $\delta = 0$ for a large number of periods T , and that the analysis of the paper applies to the periods before some $T_1 < T$. The introduction of a constant probability δ generates stationary solutions which may be more appealing.

¹¹Imperfectly informed traders may be crowded out in a more complicated equilibrium, without additional insight. Crowding out of some agents will appear in Section 4.2.

¹²Some heterogeneity in the cost could be introduced without changing the results because the strategic complementarity will induce jumps in the gross value of the information investment.

(iii) With probability $1 - \alpha - \bar{\beta}$, the agent trades for an exogenous “liquidity” or hedging purpose at any price. He sells, buys one unit of the asset or does not trade, each with probability $1/3$. Hedging will be endogenous in Section 4.2.

An agent who comes to the market for the first time is a “young” agent. Any young agent, whether he is motivated by liquidity or profit, undoes his position in the next period when he is “old”. For simplicity, old agents are identified as such. (One may also assume that agents contract the holding for one period with market-makers). Hence, the second trade with a market-maker is identified as an exogenous liquidity trade and does not convey any information. When a young informed agent buys or sells one unit, his action maximizes his expected profit that is determined by the difference between the expected price in the next trade and the current price. Without loss of generality, the interest rate is zero.

In each period, there is one “visit” by a new agent (who may or may not trade), followed by the undoing of the position, if any, of the old agent (who was young in the previous period). When the market-maker trades with a new agent, he does not know whether that agent is informed or not. The transaction price is established such that expected profit is driven to zero by perfect competition between market-makers. One may assume that market-makers trade for the short-term (for example in the second period with another market-maker), or that they hold the asset for the long-term until the revelation of θ . In the first case, we have

$$p_t = \delta E[\theta|h_{t+1}] + (1 - \delta)E[p_{t+1}|h_{t+1}], \quad (1)$$

with $h_{t+1} = \{h_t, x_t\}$. By iteration in $t + 1$, *ad infinitum*,

$$p_t = E[\theta|h_{t+1}]. \quad (2)$$

In the second case, when the market-maker prices the asset for the long-term, he uses equation (2). Since $E[p_{t+1}|h_{t+1}] = E[E[\theta|h_{t+1}, x_{t+1}]|h_{t+1}] = E[\theta|h_{t+1}]$, we have equation (1).

Since p_{t-1} summarizes the public information at the beginning of period t , the *strategies* of the information traders will be assumed to be Markov strategies that are defined by measurable functions $B_t(p_{t-1})$ from $(0, 1)$ to the closed interval $[0, \bar{\beta}]$. Without loss of generality, all information traders follow the same strategy. By assumption, the strategy B_{t+1} is common knowledge in period t .

The evolution of the price

Let $x_t \in \{-1, 0, 1\}$ describe the event when in period t , an agent sells the asset, does not trade or buys the asset. By definition of the model, the probabilities of transactions in the good state ($\theta = 1$) and the bad state ($\theta = 0$) are

$$\begin{aligned} P(x_t = -1|\theta = 1) &= P(x_t = 1|\theta = 0) = \frac{1 - \alpha - \bar{\beta}}{3} = \pi^O, \\ P(x_t = 1|\theta = 1) &= P(x_t = -1|\theta = 0) = \frac{1 - \alpha - \bar{\beta}}{3} + \alpha + \beta_t = \pi^O + \pi_t, \\ P(x_t = 0|\theta = 1) &= P(x_t = 0|\theta = 0) = 1 - (2\pi^O + \pi_t), \end{aligned} \quad (3)$$

with $\pi_t = \alpha + \beta_t$.

The parameter $\pi_t = \alpha + \beta_t$ measures the level of information (exogenous and endogenous) of agents who come to the market. It is also equal to the difference between the probabilities of a purchase and of a sale, conditional on the “good” fundamental $\theta = 1$. (The case $\theta = 0$ is symmetric). Since $\beta_t = B_t(p_{t-1})$ and the strategy B_t is common knowledge, π_t is common knowledge and is now used in the updating of p_{t-1} following the observation of the transaction x_t .

The probability of no trade is the same when $\theta = 1$ and $\theta = 0$. Hence, if there is no trade, the price which is equal to the public belief in equation (2) does not change. If there is trade, let $p^+(p_{t-1}, \pi_t)$ and $p^-(p_{t-1}, \pi_t)$ the values of the price in period t conditional on the events $x_t = 1$ and $x_t = -1$ in period t , given the beginning of period belief which is identical to p_{t-1} . Using Bayes’ rule and (2),

$$\begin{cases} p^+(p_{t-1}, \pi_t) = \frac{(\pi^O + \pi_t)p_{t-1}}{(\pi^O + \pi_t)p_{t-1} + \pi^O(1 - p_{t-1})}, \\ p^-(p_{t-1}, \pi_t) = \frac{\pi^O p_{t-1}}{\pi^O p_{t-1} + (\pi^O + \pi_t)(1 - p_{t-1})}. \end{cases} \quad (4)$$

Let the denominators in the Bayesian equations (4) be

$$\begin{aligned} D_1(p, \pi) &= (\pi^O + \pi)p + \pi^O(1 - p) = \pi^O + \pi p, \\ D_2(p, \pi) &= \pi^O p + (\pi^O + \pi)(1 - p) = \pi^O + \pi(1 - p). \end{aligned} \quad (5)$$

The difference between the ask p^+ , and the bid p^- , is the *spread*:

$$\Delta(p_{t-1}, \pi_t) = p^+(p_{t-1}, \pi_t) - p^-(p_{t-1}, \pi_t) = \frac{p_{t-1}(1 - p_{t-1})\pi_t(\pi_t + 2\pi^O)}{D_1(p_{t-1}, \pi_t)D_2(p_{t-1}, \pi_t)}. \quad (6)$$

The spread is an index about the transmission of private information to the market through transactions. A larger spread means that transactions have a stronger impact on the public belief. In any setting of social learning, the public belief evolves according to an averaging process between the previous belief, *i.e.*, p_{t-1} , which is derived from history, and the information from the currently observed action. When the history generates a high degree of confidence, its weight is larger in the Bayesian updating and the current observation has less impact on the evolution of the public belief. Here, when the price p_{t-1} is near 1 or 0, the transactions have a small effect on the price. When $p_{t-1} = 1/2$, history provides no information on θ and the value of private information is at its maximum. The properties of the spread are summarized in the next result which is proven easily¹³.

Lemma 1

In any period t , the bid-ask spread $\Delta(p_{t-1}, \pi_t)$ is decreasing in p_{t-1} if $p_{t-1} > 1/2$, and increasing if $p_{t-1} < 1/2$. It is increasing in π_t .

The value of the asset for an informed agent

Consider a young agent with a private belief μ (which may be derived from the public and the private information), who holds a position in the asset at the end of period t with the market price p_t , and who plans to undo his position in the next period. The price in the next period $t + 1$ is different from p_t if there is a transaction in period $t + 1$, in which case it depends on π_{t+1} according to (4). Since the value of $\pi_{t+1} = \alpha + \beta_{t+1}$ depends on the strategy $B_{t+1}(p_t)$ and the strategy is common knowledge, after the realization of the equilibrium price p_t , the value of π_{t+1} is common knowledge.

An appreciation $p^+(p_t, \pi_{t+1}) - p_t$ occurs with probability $\pi^O + \pi_{t+1}$ if the state is good ($\theta = 1$ with an *ex ante* probability equal to μ), and with probability π^0 if the state is bad. The value of holding one unit of the asset is the sum of the expected value, μ , if θ is revealed, the price p_t , if θ is not revealed, plus the

¹³Let $a = \pi/\pi^O$. The function $\Delta(p, \pi)$ is proportional to the inverse of $(1/p + a)(1/(1 - p) + a)$ which is increasing if and only if $p > 1/2$.

$$\frac{\partial D_1}{\partial \pi} = p = -\frac{\partial D_2}{\partial p}; \quad \frac{\partial D_1}{D_1} + \frac{\partial D_2}{D_2} = p \frac{D_2 - D_1}{D_1 D_2} = \frac{p\pi(1 - 2p)}{D_1 D_2} < 0.$$

expected value of a price change:

$$\begin{aligned}\omega(\mu, p_t; \pi_{t+1}) &= \delta\mu + (1 - \delta)p_t \\ &\quad + (1 - \delta)\mu \left((\pi^O + \pi_{t+1})(p^+(p_t, \pi_{t+1}) - p_t) + \pi^O(p^-(p_t, \pi_{t+1}) - p_t) \right) \\ &\quad + (1 - \delta)(1 - \mu) \left(\pi^O(p^+(p_t, \pi_{t+1}) - p_t) + (\pi^O + \pi_{t+1})(p^-(p_t, \pi_{t+1}) - p_t) \right).\end{aligned}$$

Since the price p_t is the price set by a market-maker with belief p_t , we can replace ω by p_t , and μ by p_t in the previous equation. Taking the difference between the two equations, we have the following result.

Lemma 2

The profit from a transaction $x_t \in \{-1, 1\}$ by an agent with probability assessment μ of $\theta = 1$ is the product of x_t and of

$$\omega(\mu, p_t; \pi_{t+1}) - p_t = \left[(1 - \delta)\pi_{t+1}\Delta(p_t, \pi_{t+1}) + \delta \right] (\mu - p_t).$$

The absolute value of this expression represents the value of the optimal trade. Note the difference between the long-term and the short-term motive in this expression. If agents trade for the long-term, $\delta = 1$, as in the standard model. If δ is arbitrarily small, we have the short-term gain from trade which is proportional to the product of π_{t+1} and of the spread $\Delta(p_t, \pi_{t+1}) = p^+(p_t, \pi_{t+1}) - p^-(p_t, \pi_{t+1})$. This gain depends on the strategy in the next period $t + 1$, which is common knowledge in period t .

The value of private information

An information trader who comes to the market in period t knows that if he is informed, he will buy if $\theta = 1$ and sell if $\theta = 0$, (since information means knowing the true state). Before entering the market, he can compute the ask p_t^+ and the bid p_t^- (which depend only on the public information at the beginning of the period). Let π_{t+1}^+ and π_{t+1}^- be the value of π_{t+1} in period $t + 1$ after the observation of the two possible values p_t^+ and p_t^- in period t . No particular relation is assumed here between π_{t+1}^+ and π_{t+1}^- , but if $p_t > 1/2$, in general $\pi_{t+1}^- \geq \pi_{t+1}^+$ because the uncertainty is higher after p_t^- than after p_t^+ , and there is more incentive to invest. As we have seen, these two values are known at the beginning of period t .

An agent who learns θ gets from his transaction a profit which is given in

Lemma 2, where μ is replaced by 1 or 0. His profit in each of the two cases is

$$\begin{cases} G_t^+ = [(1 - \delta)\pi_{t+1}^+ \Delta(p_t^+, \pi_{t+1}^+) + \delta](1 - p_t^+), & \text{if } \theta = 1, \\ G_t^- = [(1 - \delta)\pi_{t+1}^- \Delta(p_t^-, \pi_{t+1}^-) + \delta]p_t^-, & \text{if } \theta = 0. \end{cases}$$

Before investing in information, the information trader has a belief equal to the public belief, $\mu = p_{t-1}$. Hence, the *ex ante* value of private information is

$$V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-) = p_{t-1}G_t^+ + (1 - p_{t-1})G_t^-.$$

It depends on the history h_t as summarized by the last transaction price p_{t-1} , on the strategy of others through $\pi_t = \alpha + \beta_t$, and on the strategy of others in period $t + 1$ through π_{t+1} . Using the previous expressions for G_t^+ and G_t^- ,

$$V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-) = (1 - \delta)\tilde{V}(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-) + \delta L(p_{t-1}, \pi_t), \quad (7)$$

$$\begin{aligned} \text{with } \tilde{V}(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-) = & p_{t-1}(1 - p_t^+(p_{t-1}, \pi_t))\pi_{t+1}^+ \Delta(p_t^+, \pi_{t+1}^+) \\ & + (1 - p_{t-1})p_t^-(p_{t-1}, \pi_t)\pi_{t+1}^- \Delta(p_t^-, \pi_{t+1}^-), \end{aligned} \quad (8)$$

$$\text{and } L(p_{t-1}, \pi_t) = [p_{t-1}(1 - p_t^+) + (1 - p_{t-1})p_t^-]. \quad (9)$$

The expression \tilde{V} represents the gross payoff of information investment from short-term trading, while L represents the payoff from long-term trading (for an agent who waits for the revelation of the fundamental).

Definition of an equilibrium

An equilibrium strategy is defined by a sequence of measurable functions $\{B_t(p)\}_{t \geq 1}$ from $(0, 1)$ to $[0, \bar{\beta}]$ such that for any $p \in (0, 1)$,

$$\text{if } B_t(p) = 0, \quad \text{then } V(p, B_t(p), \pi_{t+1}^+, \pi_{t+1}^-) \leq c,$$

$$\text{if } 0 < B_t(p) < 1, \quad \text{then } V(p, B_t(p), \pi_{t+1}^+, \pi_{t+1}^-) = c,$$

$$\text{if } B_t(p) = \bar{\beta}, \quad \text{then } V(p, B_t(p), \pi_{t+1}^+, \pi_{t+1}^-) \geq c,$$

where the functions π_{t+1}^+ and π_{t+1}^- are defined by

$$\pi_{t+1}^+ = \alpha + B_{t+1}(p^+(p, B_t(p))), \quad \pi_{t+1}^- = \alpha + B_{t+1}(p^-(p, B_t(p))),$$

Strategic complementarity and substitutability in information investment

A critical issue will be the strategic complementarity/substitutability in the information investment. When the endogenous investment β_t increases, the probability of an informed agent $\pi_t = \alpha + \beta_t$ increases. In the expression (7) of the payoff V , three effects can be identified.

(e-1) The market-maker raises the ask p_t^+ and lowers the bid p_t^- . This effect reduces the payoff of information for short-term trade in \tilde{V} and for long-term trade in L . It induces substitutability, as highlighted by Grossman and Stiglitz (1980).

(e-2) Assume that $1/2 < p_{t-1}^-$, (and therefore $1/2 < p_{t-1}^+$). The higher ask p_t^+ lowers the spread $\Delta(p_t^+, \pi_{t+1}^+)$, (Lemma 1). This effect reduces the payoff of information investment in period t , and is weaker when π_{t+1}^+ is smaller, (as can be seen in the first term of (8) or Lemma 2).

(e-3) The last effect takes place through the lower p_{t+1}^- . It operates in the opposite direction of the previous one. We will see that in general, it dominates the previous effect and that if the price p_{t-1} is sufficiently low or high, the net effect of (e-3) and (e-2) overrules the substitution effect in (e-1).

The impact of future investment on the value of current investment

A higher level of investment β_{t+1} in period $t + 1$ raises the probability π_{t+1} of an informed agent, and therefore the spread in that period. The larger spread increases the short-term payoff \tilde{V} of information in period t . The long-term payoff L is unaffected. Using simple algebra, one verifies the following property.

Proposition 1 (strategic complementarity from β_{t+1} to β_t)

The expected value of information investment in any period t , $V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$, is an increasing function of $\pi_{t+1}^+ = \alpha + \beta_{t+1}^+$ and of $\pi_{t+1}^- = \alpha + \beta_{t+1}^-$.

The impact of the belief from history on the value of information investment

In the present setting, the uncertainty about the fundamental is measured by its variance $p_{t-1}(1 - p_{t-1})$. When p_{t-1} is greater than $1/2$, this uncertainty is a decreasing function of p_{t-1} . There is a positive relation between uncertainty and the value of information. This intuitive property is formalized in the next result

which is proven in the appendix. Throughout the paper, increasing (decreasing), will mean strictly increasing (decreasing).

Lemma 3

For any $(\pi_t, \pi_{t+1}^+, \pi_{t+1}^-) \in [\alpha, \alpha + \bar{\beta}]^3$, the functions $V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$ and $\tilde{V}(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$, defined in (7) and (8), are decreasing in p_{t-1} if $p_{t-1} > p^+(1/2, \alpha + \bar{\beta})$, and increasing in p_{t-1} if $p_{t-1} < 1 - p^+(1/2, \alpha + \bar{\beta})$.

We have seen in the definition of an equilibrium that an equilibrium strategy B_t depends on the strategy B_{t+1} . We will mainly focus on stationary strategies where the function B_t does not depend on t : first, this class will be sufficiently rich for the existence of a continuum of equilibria; second, under imperfect information (Section 5), the unique equilibrium will be a stationary strategy. The computation of an equilibrium B_t for given B_{t+1} will be examined at the end of the next section.

3 Stationary equilibrium strategies

A stationary strategy is defined by the measurable function $B(p)$ from $(0, 1)$ to $[0, \bar{\beta}]$. We have seen in Lemma 3 that the information payoff is decreasing in p if p is high, and increasing if p is low. It is therefore natural to consider strategies in which an information trader invests if and only if the price is in some interval (p^{**}, p^*) . A particular strategy in that class is a trigger strategy.

Definition

*A (stationary) trigger strategy is defined by an interval (p^{**}, p^*) such that $B(p) = 1$ if $p \in (p^{**}, p^*)$, and $B(p) = 0$ if $p \notin (p^{**}, p^*)$.*

We will mainly look for equilibrium strategies that are trigger strategies. It will be shown that under imperfect information (Section 5), these are the only stable equilibrium stationary strategies if the cost of information is sufficiently small. The previous definition assumes that the “investment interval” (p^{**}, p^*) is open. Because of the externality of investment within a period, if $p_{t-1} = p^{**}$ or $p_{t-1} = p^*$, agents do not get the same payoff if all invest in information or if they do not. For any equilibrium strategy under the previous definition, it will be possible to include the boundaries p^* or p^{**} in the investment set. However, for the sake of the exposition and without loss of generality, these boundaries will be excluded: if

the initial price p_0 is determined according to an atomless distribution, the price is at one of the boundaries with zero probability and the strategy at p^* or p^{**} in one period has no impact on the payoff of investment in other periods.

3.1 Equilibria with strategic complementarity

For an information trader in period t , the short-term payoff of information investment, $\tilde{V}(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$ in (8), depends on the level of investment in the next period. Assume that p_{t-1} is in a neighborhood of the threshold p^* . If the information trader learns that $\theta = 1$, he trades at the ask $p_t^+ > p^*$ and, by definition of the stationary strategy, $\beta_{t+1}^+ = B(p_t^+) = 0$. If he learns that $\theta = 0$, he sells at p_t^- where $\beta_{t+1}^- = B(p_t^-) = \bar{\beta}$ if $p_t^- > 1/2 \in (p^{**}, p^*)$. We are therefore led to introduce the following *zero-one expectations* such that $\pi_{t+1}^+ = \alpha$, $\pi_{t+1}^- = \alpha + \bar{\beta}$: in the period that follows a transaction at the ask (at the bid), no information trader (any information trader) invests. Using (7), the payoff of investment under zero-one expectations is equal to

$$V(p, \pi, \alpha, \alpha + \bar{\beta}) = (1 - \delta)\tilde{V}(p, \pi, \alpha, \alpha + \bar{\beta}) + \delta L(p, \pi).$$

We omit the time subscript and replace p_{t-1} by p , and π_t by π . Recall that the long-term payoff L does not depend on the investment in period $t + 1$.

A symmetric argument can be applied for low prices near p^{**} : from (8) and (9),

$$\tilde{V}(p, \pi, \alpha, \alpha + \bar{\beta}) = \tilde{V}(1 - p, \pi, \alpha + \bar{\beta}, \alpha), \tag{11}$$

$$L(p, \pi) = L(1 - p, \pi).$$

Hence, $V(p, \pi, \alpha, \alpha + \bar{\beta}) = V(1 - p, \pi, \alpha + \bar{\beta}, \alpha)$ and the analysis of the lower end of the investment set, p^{**} , is symmetrical to the one for p^* .

We now turn to the determination of p^* and introduce the function

$$W(p, \beta) = (1 - \delta)\tilde{W}(p, \beta) + \delta L(p, \alpha + \beta), \tag{12}$$

$$\text{with } \tilde{W}(p, \beta) = \tilde{V}(p, \alpha + \beta, \alpha, \alpha + \bar{\beta}),$$

or using (8),

$$\tilde{W}(p, \beta) = p(1 - p^+)\alpha\Delta(p^+, \alpha) + (1 - p)p^-(\alpha + \bar{\beta})\Delta(p^-, \alpha + \bar{\beta}), \tag{13}$$

$$p^+ = p^+(p, \alpha + \beta), \quad p^- = p^-(p, \alpha + \beta).$$

The expression $\tilde{W}(p, \beta)$ defines the short-term payoff of information as a function of the last transaction price p and the information investment β in the current period, under zero-one expectations about investment in the next period. For δ sufficiently small the relevant properties of the functions W and \tilde{W} will be the same. The next result provides a sufficient condition for the strategic complementarity of information investment within any period. This property will be critical for the results of the paper.

Proposition 2 (strategic complementarity in a period)

For given α and $\bar{\beta}$ with $\alpha < \bar{\beta}/3$, there exists \bar{p} such that if $p > \bar{p}$, then the function $\tilde{W}(p, \beta)$, defined in (12), is increasing in β for any $\beta \in [0, \bar{\beta}]$.

The result which is proven in the Appendix, holds only if the exogenous level of information α is not too large relative to the range of values of the endogenous information, $\bar{\beta}$. Such an assumption is not surprising: recall that the definition of the payoff function $\tilde{W}(p, \beta)$ rests on the assumption that there is no endogenous investment after a price rise. After that event, the probability of an informed agent in the next period is α and we have the effect that was described as (e-2) in the previous section, an effect that operates in the direction of strategic substitutability. When α is smaller, that effect is smaller. The properties of the function $\tilde{W}(p, \beta)$ in Lemma 3 and Proposition 2 are put together in the following result.

Corollary 1 (short-term payoff $\tilde{W}(p, \beta)$ with zero-one expectations)

If $\alpha < \bar{\beta}/3$, there exists $\bar{p} > p^+(1/2, \alpha + \bar{\beta})$ such that for any $(p, \beta) \in [\bar{p}, 1] \times [0, \bar{\beta}]$, the payoff of investment with zero-one expectations and pure short-term trading, $\tilde{W}(p, \beta)$, is decreasing in p and increasing in β .

When p tends to 0 or 1, the information payoff $\tilde{W}(p, 0)$ tends 0. Hence, there exists c^* such that if $c < c^*$, the equation $c = \tilde{W}(p_L, 0)$ has a unique solution p_L , and $p_L > \bar{p}$. In this case, from Corollary 1, the equation $c = \tilde{W}(p_H, \bar{\beta})$ has a unique solution with $p_H > p_L$. Furthermore, since $\tilde{W}(p, \beta)$ is continuous and strictly positive if $(p, \beta) \in [1/2, \tau] \times [0, \bar{\beta}]$ for any $\tau < 1$, one can choose c^* such that if $c \in (0, c^*]$, $\beta \in [0, \bar{\beta}]$, $p \in [1/2, p_L]$, then $\tilde{W}(p, \beta) > c$. We have the following lemma.

Lemma 4 (definition of p_L and p_H for the short-term payoff \tilde{W})

There exists c^* such that if $c < c^*$, then there are two values p_L and p_H such that with \bar{p} defined in Corollary 1, we have $\bar{p} < p_L < p_H$ and $\tilde{W}(p_L, 0) = \tilde{W}(p_H, \bar{\beta}) = c$. Furthermore, $\tilde{W}(p, \beta) > c$ for any $p \in [1/2, p_L)$, and $\beta \in [0, \bar{\beta}]$.

The information payoff with long-term trade, $L(p, \alpha + \beta)$ defined in (9), and its derivative L_β with respect to β are continuous for $p \in [0, 1]$ and $\beta \in [0, \bar{\beta}]$. Therefore, for any $\gamma > 0$, there exists $\hat{\delta} > 0$ such that if $\delta < \hat{\delta}$, then $|\delta L| < \gamma$ and $|\delta L_\beta| < \gamma$. We can extend the two previous results about the function \tilde{W} to the function W .

Corollary 2 (payoff W with zero-one expectations)

There exists $\hat{\delta} > 0$ such that for any $\delta < \hat{\delta}$:

(i) Corollary 1 applies to the payoff of investment with zero-one expectations, $W(p, \beta)$, with \bar{p} independent of δ ;

(ii) there exists c^* such that if $c < c^*$, there is a unique $\{p_L, p_H\}$ with $W(p_L, 0) = W(p_H, \bar{\beta}) = c$, $1/2 < \bar{p} < p_L < p_H$, and $W(p, \beta) > c$ for $(p, \beta) \in [1/2, p_L) \times [0, \bar{\beta}]$.

From this proposition, we have the Figure 1 where only the prices above $1/2$ are represented. Assume that agents follow a trigger strategy (p^{**}, p^*) with $p^* \in (p_L, p_H)$ and $p^{**} \in (1 - p_H, 1 - p_L)$. If $p_{t-1} > p^*$, no information trader invests and $\beta = 0$. It is shown in the Appendix that the payoff of a deviating agent who invests is bounded above by $W(p_{t-1}, 0)$, which is strictly smaller than c (Figure 1). Such a deviation is not profitable. Likewise, if $p_{t-1} < p^*$, $\beta_t = \bar{\beta}$ and the payoff of information investment is bounded below by $W(p_{t-1}, \bar{\beta}) > c$. We have seen previously that the case for p^{**} is symmetrical. The stationary trigger strategy $\{p^{**}, p^*\}$ defines a Nash-equilibrium.

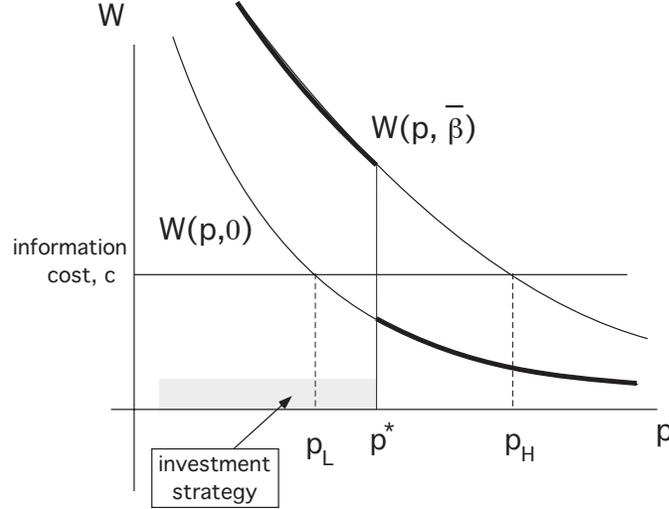
Proposition 3 (continuum of equilibria)

There exists c^* and $\hat{\delta}$ such that if $c < c^*$ and $\delta < \hat{\delta}$, $\alpha < \bar{\beta}/3$, there is a continuum of stationary equilibrium strategies which is defined by (p_L, p_H) as defined in Corollary 2: any pair $\{p^{**}, p^*\}$ such that $p^* \in [p_L, p_H]$, $p^{**} \in [1 - p_H, 1 - p_L]$ defines a trigger strategy that is an equilibrium strategy.

The sufficient conditions for the existence of a continuum of equilibrium strate-

gies are simple: the cost of information should be smaller than some value, traders should sufficiently care about the short-term profits, and the mass of exogenously informed agents should not be too large compared to the mass of traders for whom information is endogenous.

Throughout the subsequent sections, the assumptions of Proposition 3 will hold, and the thresholds p_L and p_H are defined.



$W(p, \beta) = V(p, \beta; \alpha, \alpha + \bar{\beta})$ is also the payoff of information investment when the price is p and in the future agents invest if and only if the price is in the interval $(1 - p, p)$.

FIGURE 1 The continuum of constant equilibria

3.2 An equilibrium with no investment

We have seen that the investment in period $t + 1$ contributed through the strategic complementarity to the payoff of investment in period t (Proposition 1). Can the strategy of no investment for any price in any period $t + 1$ reduce the payoff so that no investment is profitable in any period t ? The next result shows that the answer may be positive.

If no one invests in any period, $\pi_t = \alpha$ for any t , the short-term payoff of investment is $\tilde{V}(p, \alpha, \alpha, \alpha)$. A simple exercise shows that an upper-bound for this function of p is $\tilde{V}(1/2, \alpha, \alpha, \alpha)$. If this expression is smaller than the investment cost c and the long-term gain can be neglected because of low δ , there is an equilibrium with no information investment.

Proposition 4 (equilibrium with no endogenous information)

If $\tilde{V}(1/2, \alpha, \alpha, \alpha) < c$, then there exists $\hat{\delta}$ such that if $\delta < \hat{\delta}$, no investment is a stationary equilibrium strategy.

To show that a zero investment equilibrium can exist while there is also a continuum of equilibria with positive investment, fix $\bar{\beta}$ and take $\alpha \rightarrow 0$. From (13) and (4), for any p ,

$$\text{Lim}_{\alpha \rightarrow 0} \tilde{W}(p, 0) = v(p) = \bar{\beta}(1-p)p\Delta(p, \bar{\beta}).$$

One can find c such that the solution p_L of $c = v(p_L)$ is greater than the value \bar{p} in Proposition 3. Therefore, there is α sufficiently small such that $\alpha < \bar{\beta}/3$ and Proposition 3 applies, and there is a continuum of equilibria with positive investment. Since $\text{Lim}_{\alpha \rightarrow 0} \tilde{W}(1/2, \alpha, \alpha, \alpha) = 0$, for δ sufficiently small, there is also an equilibrium with no investment.

3.3 The case of high information cost

When c is sufficiently large (but not too large), the solution p_L of $W(p_L, 0) = c$ may be too low for the validity of Proposition 2. In this case, there is strategic substitutability near the threshold p^* and a unique equilibrium strategy. When the price enters the region with positive investment, endogenous investment increases gradually above zero. As there is more endogenous information, the market conveys more information and transactions generate larger price changes. If p moves toward $1/2$, the weight of history is reduced and this additional effect amplifies the price variations, but the increase of these variations operates through a gradual process without the jump that occurs when the information cost is small. The detailed analysis of this case is not the main focus in this paper and is left aside.

4 Properties**4.1 Convergence**

The public belief is equal to the price p_t and is a bounded martingale, hence it converges. If the probability of an exogenously informed agent α is sufficiently small, there may be a no investment equilibrium in which the price remains constant with a bid-ask spread equal to zero (Proposition 4).

Assume that agents coordinate on an equilibrium where they invest if the price is in the interval (p^{**}, p^*) . With probability one, after a finite number of periods the price enters the complement of that interval. In the regime $p \notin (p^{**}, p^*)$, if $\alpha = 0$ (which is compatible with a positive investment equilibrium), there is an informational cascade and the price stays constant¹⁴. If $\alpha > 0$, then $\pi_t = \alpha > 0$ and the model behaves like the standard Glosten-Milgrom model with exogenous information. There is no cascade and the convergence of the public belief is exponential (Chamley, 2004).

4.2 Endogenous hedging

In this section, the amount of liquidity trading is endogenous. In each period, the agent who comes to the market is randomly determined among four types. The first three types are the same as in the previous section: with probability α , the agent is exogenously informed (knows the state); with probability $\bar{\beta}$, he is an information trader who can learn the state at the cost c ; and with probability $3\gamma_0 \geq 0$, the agent is an exogenous liquidity trader who buys or sells the asset at any price, or does not trade, with probability 1/3 each.

With probability $2\gamma = 1 - \alpha - \bar{\beta} - 3\gamma_0 > 0$, the agent is an *endogenous hedger*. There are two types of these hedgers: with probability γ , the agent has a ratio between the marginal utilities of wealth in state $\theta = 1$ and state $\theta = 0$ that is $1 + \zeta$ with $\zeta > 0$. This ratio is generated by an idiosyncratic exogenous income that is correlated with θ in some way that does not need to be specified here. The value of ζ is randomly drawn, independently for each endogenous hedger, from a distribution with cumulative distribution function $F(\zeta)$. That agent can transfer wealth from state 0 to state 1 by purchasing the asset, and he is restricted (*e.g.*, by credit constraints as others agents), to trade at most one unit of the asset. Finally, with probability γ , the agent who meets the market-maker is symmetric to an agent of the previous kind: he has a higher marginal utility of wealth in state $\theta = 0$ and in state $\theta = 1$, and the ratio $1 + \zeta$ with ζ randomly determined by the same c.d.f. $F(\zeta)$. By assumption, these endogenous hedgers hold the asset for the long-term until the revelation of the fundamental. (The results are not qualitatively different if they are short-term traders).

Consider an agent with ratio $1 + \zeta$ between the marginal utilities of wealth in

¹⁴Cascades may take place in the Glosten-Milgrom model if agents are risk-averse (Decamps and Lovo, 2002), or if there are transaction costs (Romano, 2004).

state 1 and 0. His belief about state 1 at time t is the same as the public belief. If he buys one unit of the asset, he has to pay the ask p_t^+ (as he is not distinguishable from other agents), and his utility changes by $p_{t-1}(1+\zeta)(1-p_t^+) + (1-p_{t-1})(-p_t^+)$, which is positive if and only if

$$\frac{p_t^+}{1-p_t^+} \leq (1+\zeta) \frac{p_{t-1}}{1-p_{t-1}}. \quad (14)$$

The agent buys the asset only if his marginal utility $1+\zeta$ is sufficiently large with respect to the spread between the ask p_t^+ and the last price p_{t-1} . Likewise, an agent with higher marginal utility in state 0 sells the asset only if

$$\frac{p_t^-}{1-p_t^-} \geq \left(\frac{1}{1+\zeta}\right) \frac{p_{t-1}}{1-p_{t-1}}. \quad (15)$$

But the bid-ask spread is endogenous to the actions of hedgers: as shown by Dow (2004), more trade by hedgers reduces the signal to noise ratio and the bid-ask spread. Let z be the minimum ζ of hedgers who buy the asset. In equilibrium, the ask is determined by

$$\frac{p_t^+}{1-p_t^+} = \frac{\gamma_0 + \alpha + \beta + \gamma(1-F(z))}{\gamma_0 + \gamma(1-F(z))} \frac{p_{t-1}}{1-p_{t-1}}. \quad (16)$$

Likewise, if z' is the minimum ζ of hedgers who sell the asset,

$$\frac{p_t^-}{1-p_t^-} = \frac{\gamma_0 + \gamma(1-F(z'))}{\gamma_0 + \alpha + \beta + \gamma(1-F(z'))} \frac{p_{t-1}}{1-p_{t-1}}. \quad (17)$$

From (14), (15), (16) and (17), in an equilibrium, hedgers buy or sell the asset when their parameter ζ is greater than z^* which is solution of

$$z^* = H(z^*) = \frac{\alpha + \beta}{\gamma_0 + \gamma(1-F(z^*))}. \quad (18)$$

The function $H(z^*)$ is increasing and must have a slope smaller than 1 at a non-trivial stable equilibrium¹⁵. It follows immediately that if the investment β increases, the value of z^* is raised. As only the hedgers with $z > z^*$ trade, the noise trading is reduced and the spread increases. The endogeneity of hedging reinforces the property of strategic complementarity shown in the previous results and the main result, Proposition 3, holds.

¹⁵Dow (2004), in a slightly different specification, assumes that the distribution of ζ has two atoms. In this case, F is a step function and there can be multiple equilibria which are the intersection of the graph of H with the 45° line.

4.4 Rationalizable strategies

In this section, we show first that investment is not a rationalizable strategy if $p_{t-1} > p_H$, in the sense that it can be eliminated by the iteration of dominated strategies.

We will need the functions $q^+(p, \beta)$ and $q^-(p, \beta)$ which are defined as the inverse functions such that $p^+(q^+(p, \beta), \alpha + \beta) = p$ and $p^-(q^-(p, \beta), \alpha + \beta) = p$. Note that $q^+(p, \beta) < p < q^-(p, \beta)$. The next result provides the tool for the iterative elimination of dominated strategies.

Lemma 5 (backward induction of dominance)

Assume that in period $t + 1$, no agent invests in information if $p_t > p_{t+1}^$ for some $p_{t+1}^* > p_H$. Then conditional on that expectation, if $p_{t-1} > \text{Max}\{q^+(p_{t+1}^*, 0), p_H\}$, investment is strictly dominated in period t .*

The Lemma is proven by the method that was used in the proof of Proposition 3: if $p_{t-1} > \text{Min}\{q^+(p_{t+1}^*, 0), p_H\}$, the payoff of investment is bounded above by $W(p_{t-1}, \bar{\beta})$ which is strictly smaller than c , by definition of $p_H < p_{t-1}$.

Since the value of information tends to zero uniformly in $(\pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$ when p_{t-1} tends to 1, there exists $p(1)$ such that in any period t , if $p_{t-1} > p(1)$, investment is a dominated strategy. Applying Lemma 5, since no agent invests in any period $t + 1$ if $p_t > p(1)$, investment is a dominated strategy in any period t if $p_t > \text{Max}\{q^+(p(1), 0), p_H\}$.

If for $k \geq 1$, $q^+(p(k), 0) > p_H$, the argument can be repeated with $p(k + 1) = q^+(p(k), 0) < p(k)$. The distance $p(k + 1) - p(k)$ increases with k . For any $p > p_H$, after a finite number of steps, investment is iteratively dominated, and therefore not rationalizable. The symmetric argument can be applied for the low prices smaller than $1 - p_H$.

Proposition 5

Investment is not rationalizable if $p \notin [1 - p_H, p_H]$.

The iterative argument can be applied “on the left” of p_L only if there is a sufficiently wide “middle” interval contained in $(0, 1)$ in which investment is a

dominating strategy¹⁶. Such an interval is required to begin the iterative argument. Note that the middle region of $(0, 1)$ is precisely the one with the smallest weight of history and where information should be more valuable.

Assumption \mathcal{M} *The parameters of the model are such that $V(p, \alpha + \beta; \alpha, \alpha) > c$ for any $\beta \in [0, \bar{\beta}]$ and any $p \in (1 - p_m, p_m)$ where $p_m > 1/2$ is defined such that $p^-(p_m, \alpha + \bar{\beta}) > 1 - p_m$.*

Proposition 6

Under Assumption \mathcal{M} , any investment less than the maximum $\bar{\beta}$ is not rationalizable if $p \in (1 - p_L, p_L)$.

4.5 Non-stationary strategies

We now consider briefly non stationary strategies. We have seen that an equilibrium strategy in period t depends on the strategy in period $t + 1$. From Propositions 5 and 6, it is reasonable to assume that B_{t+1} is a trigger strategy (p^{**}, p^*) with $1 - p^{**}$ and p^* in (p_L, p_H) . Using the same arguments as in the proof of Lemma 5, we have the following result.

Proposition 7 (non stationary equilibria)

*When in period $t + 1$, agents use the trigger strategy $(1 - p^{**}, p^*)$ with $p^{**} \in (p_L, p_H)$ and $p^* \in (p_L, p_H)$, then in period t :*

- (i) *if $\text{Min}\{q^-(p^*, \bar{\beta}), p_H\} < p_{t-1}$, the only equilibrium strategy is $\beta_t = 0$;*
- (ii) *if $\text{Max}\{q^+(p^*, 0), p_L\} \leq p_{t-1} \leq \text{Min}\{q^-(p^*, \bar{\beta}), p_H\}$, the stable equilibrium strategies are $\beta_t = 0$ and $\beta_t = \bar{\beta}$;*
- (iii) *if $p_{t-1} \leq \text{Max}\{q^+(p^*, 0), p_L\}$, $\beta_t = 1$ is a stable equilibrium strategy.*

Symmetric properties occur for low prices. This proposition does not consider all possible cases, and the complete analysis would not add any insight¹⁷. From

¹⁶By an abuse of notation, investment is dominating if any $\beta < \bar{\beta}$ is dominated.

¹⁷One could not determine in general whether $q^-(p, \bar{\beta}) > p_H$, although this inequality was satisfied for all numerical parameters that were tried. The strategy $\beta_t = 0$ may be an equilibrium when $p < q^+(p, 0)$ for some parameter values, but this is a minor point.

Proposition 7, if agents follow the strategy $(1-p^*, p^*)$ in period $t+1$, then in period t , an equilibrium strategy is found in any measurable function $B_t(p)$ that takes values in the set $\{0, 1\}$ on the interval $\mathcal{I} = (Max\{q^+(p^*, 0), p_L\}, Min\{q^-(p^*, \bar{\beta}), p_H\})$, that is equal to 0 on the right of \mathcal{I} , and to 1 on the left for $p \geq 1/2$, with a symmetric definition for $p \leq 1/2$. Trigger strategies are special cases of these strategies.

5 Imperfect information and equilibrium uniqueness

The property of multiple equilibria is indicative of potentially large changes of the evolution of the price, but it is not entirely satisfactory. In a trigger strategy, agents invest in information if and only if $p \in (p^{**}, p^*)$, and each agent knows that other agents behave likewise in any period (Proposition 3). It is not clear how agents coordinate on the strategy $\{p^{**}, p^*\}$ since there is a continuum of such values. Furthermore, one should check that the property is robust to a perturbation, for example a small observation noise. The introduction of an observation noise generates a global game of Carlsson and Van Damme (1993), which will be shown to display two types of properties: first, the continuum of Nash-equilibria in a wide class of strategies (which is more general than the stationary trigger strategies) is reduced to a unique Nash-equilibrium which is a stationary strategy; second, under some condition, that unique Nash-equilibrium strategy is the only one that survives the iterated elimination of dominated strategies and is therefore a Strongly Rational-Expectations Equilibrium (SREE). As is well known, the existence of a SREE shows that in the set of measurable functions from $(0, 1)$ to $[0, \bar{\beta}]$, there is no other strategy that is stable in the sense of a reaction function.

In the new information structure, the information trader who comes to the market in period t knows imperfectly the last transaction price p_{t-1} . The private information of an information trader in period t is the signal

$$s_t = p_{t-1} + \epsilon_t, \tag{19}$$

where ϵ_t is independently drawn from a distribution with support $[-\sigma, \sigma]$. The analysis holds for any non degenerate distribution of ϵ , but to simplify ϵ has a uniform distribution. The prior distribution on p_{t-1} is common knowledge and without loss of generality, it is assumed to be uniform¹⁸. Market-makers have

¹⁸When σ is arbitrarily small, the density of the prior p_{t-1} is nearly uniform for a given s_t , as emphasized by Carlsson and Van Damme. The important assumption is that the prior of an information trader has a support that includes an open interval that includes the interval

perfect information as befits their role. (A noisy observation on their part would probably not change the results). The other parameters of the model are the same as in Section 3 without observation noise, and are such that Proposition 3 holds.

If an information trader after observing the signal s_t , does not invest in information, he stays out of the market and does not trade because of the bid-ask spread, as in the case with no observation noise. If he invests at the cost c , he learns θ , and since his private information is sufficiently strong, he trades at whatever price. For any value of p_{t-1} , the market maker knows the distribution of private signals and the strategy of agents. He can therefore determine the bid and the ask as in the previous sections.

A strategy is now a measurable function of s and the value of the investment, β , depends on the strategy. We will not restrict the strategy to be a trigger strategy, but the payoff with a trigger strategy will be a useful tool. In a trigger strategy, an information trader invests if his signal is in the interval (\hat{s}', \hat{s}) . We will focus on behavior of agents near the value \hat{s} , which will be shown to be in the interval (p_L, p_H) . For σ sufficiently small, the level of investment is equal to

$$\beta(p, \hat{s}) = \bar{\beta} \text{Min}\left(\text{Max}\left(\frac{\hat{s} - p + \sigma}{2\sigma}, 0\right), 1\right). \quad (20)$$

In the model with perfect information, we have used the payoff function $W(p, \beta)$ with the zero-one expectations that in the period after a price rise (at the ask), there is no information investment, while investment is at the maximum $\bar{\beta}$ after a transaction at the bid. A similar function will play an important role here. Omitting the time subscript for the current period t , and proceeding as for the function $W(p, \beta)$ in (12), the payoff of information for an information trader with signal s and zero-one expectations about the next period is equal to the function

$$J(s, \hat{s}; \sigma) = (1 - \delta)\tilde{J}(s, \hat{s}; \sigma) + \delta K(s, \hat{s}; \sigma), \quad (21)$$

with

$$\begin{aligned} \tilde{J}(s, \hat{s}; \sigma) &= \mu(s) \int_{s-\sigma}^{s+\sigma} (1 - p^+) \pi^+ \Delta(p^+, \pi^+) \phi(p|s) dp \\ &\quad + (1 - \mu(s)) \int_{s-\sigma}^{s+\sigma} p^- \pi^- \Delta(p^-, \pi^-) \phi(p|s) dp, \\ K(s, \hat{s}; \sigma) &= \int_{s-\sigma}^{s+\sigma} \left(\mu(s)(1 - p^+) + (1 - \mu(s))p^- \right) \phi(p|s) dp, \end{aligned} \quad (22)$$

$[1 - p_H, p_H]$.

where $p^+ = p^+(p, \alpha + \beta)$, $p^- = p^-(p, \alpha + \beta)$, $\pi^+ = \alpha$, $\pi^- = \alpha + \bar{\beta}$, β is given by (20), $\mu(s)$ is the probability that $\theta = 1$ conditional on the signal s , and $\phi(p|s)$ is the density of p conditional on s .

Using the assumption about the distributions of p and of ϵ , the payoffs of information under a trigger strategy \hat{s} *cum* zero-one expectations, for short- and long-term trade are given by

$$\begin{aligned}\tilde{J}(s, \hat{s}; \sigma) &= s \int_{s-\sigma}^{s+\sigma} (1 - p^+) \pi^+ \Delta(p^+, \alpha) dp + (1 - s) \int_{s-\sigma}^{s+\sigma} p^- \pi^- \Delta(p^-, \alpha + \bar{\beta}) dp, \\ K(s, \hat{s}; \sigma) &= s \int_{s-\sigma}^{s+\sigma} (1 - p^+) dp + (1 - s) \int_{s-\sigma}^{s+\sigma} p^- dp,\end{aligned}\tag{23}$$

which are continuous functions of s and \hat{s} .

In Section 3 with no observation noise, agents have zero-one expectations when the price is near the threshold p^* , because a transaction lifts the price above p^* where no agent invest, or pushes it down below p^* where $\beta = \bar{\beta}$. The same property holds under imperfect information when the idiosyncratic heterogeneity σ is sufficiently small with respect to the spread between p^+ and p^- . But the spread has a minimum size because of the probability $\alpha > 0$ of exogenously informed agents. From the continuity of the bid $p^-(p, \alpha + \beta)$ and the ask $p^+(p, \alpha + \beta)$, as defined in the updating equations (4), we have the next result.

Lemma 7 (minimum size of the spread)

For any $\tilde{\eta} \in (1/2, 1)$, there exists η such that if $1 - \tilde{\eta} < p < \tilde{\eta}$, then for any $\beta \in [0, \bar{\beta}]$, we have $p^-(p, \beta) < p - \eta$ and $p + \eta < p^+(p, \beta)$.

In Section 3, investment was shown to be dominated if $p > p_H$ (Lemma 5). In that argument, the payoff of investment was bounded by the payoff under zero-one expectations. A similar technique is used here. First, under observation noise, the function J provides an upper-bound of the payoff of investment.

Lemma 8 (local boundedness of the payoff)

Assume $\sigma < \eta/4$ where η is defined in Lemma 7. If no agent invests in period $t + 1$ with a signal greater than \hat{s} , then the payoff of investment in period t for an agent with signal $s \in (\hat{s} - \eta/2, \hat{s} + \eta/2)$ is bounded above by $J(s, \hat{s}; \sigma)$.

The Lemma is proven in the appendix. Note that by assumption, agents do not invest with a signal greater than \hat{s} in period $t + 1$, and the ensuing property

applies to the function J with a trigger strategy at the same value \hat{s} , but in the period t .

In Section 3 under perfect information, the equilibrium strategies p^* where determined by the inequalities $W(p^*, 0) < c < W(p^*, \bar{\beta})$ where $W(p, \beta)$ is a decreasing function of p . A similar property will hold here when the function $W(p, \beta)$ is replaced by $J(s, s; \sigma)$. This function will have properties best related to those of $W(p, \beta)$ when the degree of heterogeneity σ is arbitrarily small. We therefore consider that case first.

Vanishingly small heterogeneity

In the expression (23) of \tilde{J} , taking $\hat{s} = s$ and using $p = s - \sigma(2\beta/\bar{\beta} - 1)$ from (20),

$$\tilde{J}(s, s; \sigma) = \frac{s}{\bar{\beta}} \int_0^{\bar{\beta}} (1 - p^+) \pi^+ \Delta(p^+, \pi^+) d\beta + \frac{1-s}{\bar{\beta}} \int_0^{\bar{\beta}} p^- \pi^- \Delta(p^-, \pi^-) d\beta,$$

with

$$p^+ = p^+(s - \sigma(2\frac{\beta}{\bar{\beta}} - 1), \beta), \quad p^- = p^-(s - \sigma(2\frac{\beta}{\bar{\beta}} - 1), \beta).$$

Recall that with perfect information, the short-term payoff of information (with $\delta \approx 0$) is given in (13):

$$\tilde{W}(p, \beta) = p(1 - p^+(p, \pi))\alpha\Delta(p^+; \alpha) + (1 - p)p^-(p, \pi)(\alpha + \bar{\beta})\Delta(p^-; \alpha + \bar{\beta}),$$

with $\pi = \alpha + \beta$.

Using these expressions, the expression of $K(s, s; \sigma)$ in (23) and the expression of J in (21),

$$\lim_{\sigma \rightarrow 0} J(s, s; \sigma) = \frac{1}{\bar{\beta}} \int_0^{\bar{\beta}} W(s, \beta) d\beta. \quad (24)$$

Because of the differentiability of the Bayesian functions p^+ and p^- on $[0, 1]$,

$$\lim_{\sigma \rightarrow 0} \frac{dJ(s, s; \sigma)}{ds} = \frac{1}{\bar{\beta}} \int_0^{\bar{\beta}} \frac{\partial W(s, \beta)}{\partial s} d\beta. \quad (25)$$

These equations show that for σ arbitrarily small, the function $J(s, s; \sigma)$ is approximated by an average of the functions $W(s, \beta)$. Using the properties of $W(p, \beta)$ in Corollary 2, and recalling that the assumptions of Proposition 3 hold throughout, the following property is shown in the appendix.

Lemma 9 (threshold signal under imperfect information)

There exists $\hat{\sigma}$ such that if $\sigma < \hat{\sigma}$, the equation $J(s, s; \sigma) = c$ has a unique solution s^* on the interval $[p_L, 1]$. Furthermore, $s^* \in (p_L, p_H)$, $J(s, s; \sigma) < c$ for $s > s^*$, and if $\sigma \rightarrow 0$ then $s^* \rightarrow s^{**}$ which is defined by

$$\frac{1}{\bar{\beta}} \int_0^{\bar{\beta}} W(s^{**}, \beta) d\beta = c.$$

Iterative elimination

We use the previous tools to eliminate investment for high signal values s by an iterative procedure. Since the value of information tends to zero when the price tends to 1, there exists \hat{s}_1 such that investment is dominated for any $s > \hat{s}_1$. We can now start an iterated elimination of dominated strategies. Assume first that in any period, no agent invests with a signal greater than \hat{s}_k , $k \geq 1$.

According to Lemma 8, an agent in period t with signal $s \in (\hat{s}_k - \eta/2, \hat{s}_k + \eta/2)$ has a payoff of investment no greater than $J(s, \hat{s}_k; \sigma)$. Suppose that $\hat{s}_k > s^*$. By Lemma 9, $J(\hat{s}_k, \hat{s}_k; \sigma) < c$. The function $J(s, \hat{s}_k; \sigma)$ is continuous in s . Reduce now the first argument s in the function $J(s, \hat{s}_k; \sigma)$ while the second argument is kept constant at \hat{s}_k . Define \hat{s}_{k+1} such that

$$\hat{s}_{k+1} = \text{Min} \left\{ \hat{s} \geq s_k - \eta/2 \mid J(s, s_k) \leq 0 \text{ for all } s \in (\hat{s}, s_k) \right\}.$$

Conditional on no agent investing in any period with a signal greater than \hat{s}_k , investment is dominated in any period for an agent with signal $s > \hat{s}_{k+1}$.

The iterated elimination of strategies is illustrated in Figure 4. The decreasing sequence $\{\hat{s}_k\}$ which is bounded below by s^* , converges. Using an argument in Chamley (2004) (page 255 and an adaptation of Theorem 11.2), it converges to s^* . For any agent with $s > s^*$, investment is iteratively dominated. This proves the following result which extends the region of dominance in Proposition 5.

Proposition 8

There exists $\hat{\sigma}$ such that if $\sigma < \hat{\sigma}$, investment is iteratively dominated for any $s > s^*$ where $s^* \in (p_L, p_H)$ is defined by $J(s^*, s^*; \sigma) = c$. If $\sigma \rightarrow 0$, then $s^* \rightarrow s^{**}$ such that

$$\frac{1}{\bar{\beta}} \int_0^{\bar{\beta}} W(s^{**}, \beta) d\beta = c.$$

By symmetry, investment is iteratively dominated for any signal smaller than $1 - s^*$.

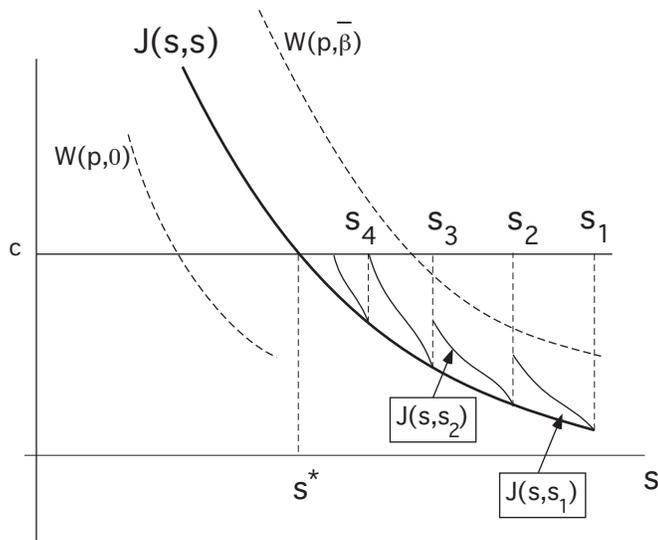


FIGURE 4 Iterated elimination of dominated strategies

Strongly Rational-Expectations Equilibrium (SREE)

The elimination of strategies with cutoff point smaller than s^* requires the existence of a price region in the middle of the interval $(0, 1)$ in which investment is a dominating strategy. Under perfect information, Proposition 4 showed that an equilibrium with zero investment may exist, and the condition for the iterative elimination of strategies in $(1 - p_L, p_L)$ is stronger than for the elimination of the zero equilibrium (Proposition 6). Here, we proceed in two parts. In the first, which is the main one, a restriction is introduced on the strategies that are played in some sufficiently distant period T . Since equilibrium strategies are determined backward, this restriction provides the first step for an iterative procedure that applies from period to period back to the present and which generate a SREE in the present period. As a consequence, the continuum of Nash-equilibria that was obtained under common knowledge is reduced to a unique equilibrium. In the second part, the restriction is introduced on the parameters of the model and the previous equilibrium is shown to be a SREE.

Suppose that in some period T , agents invest if the price is in the interval $(1 - p_L, p_L)$. Any investment rule is admissible if $p \notin (1 - p_L, p_L)$. The next result shows that if T is sufficiently large, in the present period, the only equilibrium strategy is $(1 - s^*, s^*)$.

Proposition 9 (SREE under a behavioral assumption)

There exists $\hat{\sigma}$ and an integer N such that if $\sigma < \hat{\sigma}$ and in period T an information trader invests if his signal is in $(1 - p_L + 2\hat{\sigma}, p_L - 2\hat{\sigma})$, then in any period $t \leq T - N$, the strategy $(1 - s^*, s^*)$ is a SREE.

The proof of the proposition proceeds as for Proposition 8: under the assumptions of Proposition 9, define $\hat{s}_1 \geq p_L - 2\hat{\sigma}$; using the same argument as in Lemma 8, $J(s, \hat{s}_1; \sigma)$ is a lower bound of the payoff of investment for an agent with signal s if s is in a neighborhood of \hat{s}_1 ; if $\hat{s}_1 < s^*$, one can find $\hat{s}_2 > \hat{s}_1$ such that all agents with $s \in [\hat{s}_1, \hat{s}_2)$ invest, and so on. The sequence $\{\hat{s}_k\}$ converges to s^* .

The SREE property shows that under the behavioral restriction of Proposition 9 for some distant period T , the strategy $(1 - s^*, s^*)$ is the only stable equilibrium when the set of strategies for the present period is the set of measurable functions from $(0, 1)$ to $[0, \bar{\beta}]$.

If there is a stationary Nash-equilibrium with a trigger strategy $(s^{*'}, s^*)$ such that $1 - s^{*'}$ and s^* are in the interval (p_L, p_H) , then it satisfies the conditions of Proposition 9. For any arbitrary period t , we can take N such that $T = t + N$ and apply Proposition 9 for T . We have then the following result.

Proposition 10 (uniqueness of a stationary Nash-equilibrium)

There exists $\hat{\sigma}$ such that if $\sigma < \hat{\sigma}$, the strategy $(1 - s^*, s^*)$ defines the unique Nash-equilibrium in the set of trigger strategies with $s^* \in (p_L, p_H)$. The value of s^* is determined in Proposition 8.

Proposition 10 shows that the continuum of equilibria which was found in Proposition 3 collapses to one point when an arbitrary small observation noise is introduced. (The result does not rule out other stationary strategies with threshold smaller than p_L). The next result will show that under some parametric restriction, any other equilibrium is indeed ruled out.

In the previous two propositions, we assumed some investment interval near $(1 - p_L, p_L)$ in order to start the iterations of the dominated strategies. Recall that under common knowledge, investment is not rationalizable (iteratively dominated) if $p \in (0, 1 - p_H) \cup (p_H, 1)$, and no investment is not rationalizable if $p \in (1 - p_L, p_L)$ provided that Assumption \mathcal{M} is added. When agents do not have common knowledge of history, Proposition 8 extended the non-rationalizability

interval $(0, 1 - p_H) \cup (p_H, 1)$ to $(0, 1 - s^*) \cup (s^*, 1)$ with $s^* \in (p_L, p_H)$. Under Assumption \mathcal{M} , the non-rationalizability of no investment in the interval $(1 - p_L, p_L)$ is extended to $(1 - s^*, s^*)$ by the next result which is shown as Proposition 9.

Proposition 11 (SREE under a parametric assumption)

Under Assumption \mathcal{M} , there exists $\hat{\sigma}$ such that if $\sigma < \hat{\sigma}$, the strategy to invest if and only $s \in (1 - s^, s^*)$ where s^* is defined in Proposition 8, is a SREE.*

6 Conclusion

We began by showing that the interaction of short-term trades and endogenous information generated strategic complementarities, and that these complementarities are sufficiently strong to generate a continuum of equilibria when agents have a common knowledge on the history. In my view, the property of multiple equilibria is important because it exhibits discontinuities in agents' behavior, here in the endogenous information investment. The apparent indeterminacy between different equilibria is not important. Indeed, this indeterminacy is not robust to perturbation with an observation noise. But the discontinuity in behavior is robust to the perturbation. The level of endogenous investment jumps between 0 and $\bar{\beta}$ when the price crosses the interval $s^* - \hat{\sigma}, s^* + \hat{\sigma}$ which is arbitrarily small. The model thus exhibits regime of “frenzies” of information gathering.

There is a linear relation in the model between the probability of a trade and the level of endogenous information. Hence, in the equilibrium, there is a positive relation between the volume of trade and the information that is generated by the market. Information frenzy is equivalent to trade frenzy.

A next step would be to consider small random changes or cycles of the fundamental²¹. One may anticipate that the present results will be extended and that in an equilibrium, there would be two regimes alternating randomly. In one of the two, the price should fluctuate by small amounts and the level of information conveyed by the market would be small. The other would be a regime of “frenzy” of information gathering and trade, and rapid changes of the price.

²¹David (1997) analyzes the learning process in a financial market when the state switches randomly between discrete values and agents have exogenous private information.

7 Appendix: proofs

Lemma 3

The value function V_t is defined in (7) which is repeated here:

$$V_t = (1 - \delta)\tilde{V}_t + \delta[p_{t-1}(1 - p_t^+) + (1 - p_{t-1})p_t^-],$$

We will omit the time subscripts since there is no ambiguity. In the second term of this expression, p^+ and p^- are given by the Bayesian equations (4), and simple algebra shows that this term is a decreasing function of p if $p > 1/2$. Focusing now on the first term, from (8),

$$\tilde{V} = p(1 - p^+)\pi^+\Delta(p^+, \pi^+) + (1 - p)p^-\pi^-\Delta(p^-, \pi^-).$$

Taking the partial derivative with respect to p ,

$$\begin{aligned} \frac{\partial \tilde{V}(p, \pi)}{\partial p} &= \pi^+\Delta(p^+, \pi^+)\left((1 - p^+) - p\frac{\partial p^+}{\partial p}\right) \\ &\quad + p\pi^+(1 - p^+)\frac{\partial \Delta(p^+, \pi^+)}{\partial p} \\ &\quad + \pi^-\Delta(p^-, \pi^-)\left(-p^- + (1 - p)\frac{\partial p^-}{\partial p}\right) \\ &\quad + (1 - p)\pi^-p^-\frac{\partial \Delta(p^-, \pi^-)}{\partial p}. \end{aligned} \tag{31}$$

We show first that the signs of the terms on the second and the fourth line are negative. We have $\partial p^+/\partial p > 0$. Since $p^+ > p > 1/2$, from Lemma 1, the spread $\Delta(p^+, \pi^+)$ is decreasing in p^+ and

$$\frac{\partial \Delta(p^+, \pi^+)}{\partial p} < 0.$$

The same argument applies for the fourth line of (31) because $p > \hat{p}$, and from the definition of \hat{p} one can show that $p^-(\hat{p}, \alpha + \bar{\beta}) > 1/2$. Hence,

$$\frac{\partial \Delta(p^-, \pi^-)}{\partial p} < 0.$$

In the first line of (31), let $R = (1 - p^+) - p\frac{\partial p^+}{\partial p} = \frac{\partial(1 - p^+)p}{\partial p}$.

$$\text{From the expression of } p^+, \quad (1 - p^+)p = \frac{\pi^O(1 - p)p}{\pi^O + \pi p}.$$

For $p > 1/2$, this expression is decreasing in p . Hence, $R < 0$.

Likewise on the third line, let $H = -p^- + (1-p)\frac{\partial p^-}{\partial p} = \frac{\partial(1-p)p^-}{\partial p}$.

From the expression of p^- , $(1-p)p^- = \frac{\pi^O p(1-p)}{\pi^O + \pi(1-p)}$.

A straightforward computation shows that the inequality $H < 0$ is equivalent to $p > 1 - p^-$. One verifies that $p > p^+(1/2, \alpha + \bar{\beta})$ is sufficient for that inequality.

Proposition 2

Using (8),

$$\tilde{V}_t = p_{t-1} \frac{p_t^+(1-p_t^+)^2(\pi_{t+1}^+)(\pi_{t+1}^+ + 2\pi^O)}{D_1(p_t^+, \pi_{t+1}^+)D_2(p_t^+, \pi_{t+1}^+)} + (1-p_{t-1}) \frac{(p_t^-)^2(1-p_t^-)(\pi_{t+1}^-)(\pi_{t+1}^- + 2\pi^O)}{D_1(p_t^-, \pi_{t+1}^-)D_2(p_t^-, \pi_{t+1}^-)}.$$

Using the expressions of the ask and the bid (4) in period t ,

$$\tilde{V}_t = p_{t-1}^2(1-p_{t-1})^2(\pi^O)^2(\pi^O + \pi)A(\pi), \quad (32)$$

$$\begin{aligned} \text{with } A(\pi) &= \frac{\pi_{t+1}^+(\pi_{t+1}^+ + 2\pi^O)}{D_1^3(p_{t-1}, \pi)D_1(p_t^+(\pi), \pi_{t+1}^+)D_2(p_t^+(\pi), \pi_{t+1}^+)} \\ &+ \frac{\pi_{t+1}^-(\pi_{t+1}^- + 2\pi^O)}{D_2^3(p_{t-1}, \pi)D_1(p_t^-(\pi), \pi_{t+1}^-)D_2(p_t^-(\pi), \pi_{t+1}^-)}. \end{aligned} \quad (33)$$

The derivative with respect to β is the same as the derivative with respect to π . In the definition of $W(p, \beta)$, $\pi_{t+1}^+ = \alpha$ and $\pi_{t+1}^- = \alpha + \bar{\beta}$. For given past price p_{t-1} , \tilde{V}_t is a multiple of the function

$$\mathcal{G}(\pi) = \frac{(\pi^O + \pi)\alpha(\alpha + 2\pi^O)}{D_1^3(p, \pi)D_1(p^+, \alpha)D_2(p^+, \alpha)} + \frac{(\pi^O + \pi)(\alpha + \bar{\beta})(\alpha + \bar{\beta} + 2\pi^O)}{D_2^3(p, \pi)D_1(p^-, \alpha + \bar{\beta})D_2(p^-, \alpha + \bar{\beta})}. \quad (34)$$

The second term on the right-hand side is positively related to π while the first term is inversely related to π . This has an intuitive interpretation: if the exogenous level of informed agents is sufficiently high, the value of information decreases as a function of π .

When $p \rightarrow 1$, $D_1(p, \pi) \rightarrow \pi^O + \pi$, $D_2 \rightarrow \pi^O$. Let a be the first term in (34). Its derivative with respect to π is such that if $p \rightarrow 1$,

$$\begin{aligned} a'_\pi &\rightarrow a \left(\frac{1}{\pi^O + \pi} - \frac{3}{\pi^O + \pi} - \frac{\partial p^+}{\partial \pi} \left(\frac{\partial D_1}{\partial p^+} \frac{1}{D_1} - \frac{\partial D_2}{\partial p^+} \frac{1}{D_2} \right) \right) \\ &\rightarrow -\frac{2a}{\pi^O + \pi} = -\frac{2\alpha(\alpha + 2\pi^O)}{(\pi^O + \pi)^3(\pi^O + \alpha)\pi^O}. \end{aligned}$$

Likewise for the second term b in (34), $b'_\pi \rightarrow \frac{b}{\pi^O + \pi} \rightarrow \frac{(\alpha + \bar{\beta})(\alpha + \bar{\beta} + 2\pi^O)}{(\pi^O)^4(\pi^O + \alpha + \bar{\beta})}$.

Combining the two previous expressions, $\mathcal{G}'_\pi \rightarrow \lambda$ where for any π ,

$$\lambda > \frac{(\alpha + \bar{\beta})(\alpha + \bar{\beta} + 2\pi^O)}{(\pi^O)^4(\pi^O + \alpha + \bar{\beta})} - \frac{2\alpha(\alpha + 2\pi^O)}{(\pi^O + \alpha)^4\pi^O}.$$

To prove the proposition, it is sufficient to prove that $\lambda > 0$, for which a sufficient condition is

$$\frac{(\alpha + \bar{\beta})(\alpha + \bar{\beta} + 2\pi^O)}{(\pi^O + \alpha + \bar{\beta})} > \frac{2\alpha(\alpha + 2\pi^O)}{\pi^O}, \quad \text{which holds if } \alpha < \bar{\beta}/3.$$

Proposition 3

Suppose first $p_{t-1} > p^*$. By definition of the strategy p^* , no information trader invests in period t . Consider the payoff of a deviating agent who invests. If after paying the cost c , he learns that $\theta = 1$, then he buys at $p_t^+ > p_{t-1} > p^*$. By the definition of the strategy p^* , no agent invests in the next period and $\pi_{t+1}^+ = 0$. We do not need to be concerned by the outcome if he learns that $\theta = 0$ because of the strategic complementarity from period $t + 1$ to period t : from Proposition 1, the payoff of investment in period t is not greater than if $\beta_{t+1} = \bar{\beta}$. The payoff of investment in period t is therefore bounded above by the payoff under zero-one expectations, $W(p_{t-1}, \beta_t) = W(p_{t-1}, 0) < W(p_L, 0) = c$, using Corollary 2 and the definition of p_H .

Suppose now that $1/2 \leq p_{t-1} < p^*$: any information trader invests in period t . We consider again a deviating information trader who invests. If he learns that $\theta = 0$, he trades at the bid $p^-(p_{t-1}, \bar{\beta})$. Using $p_{t-1} \geq 1/2$ and the property of \bar{p} , we have $p^{**} < 1 - p_L < 1 - \bar{p} < p^-(p_{t-1}, \bar{\beta})$. Therefore, $\pi_{t+1}^- = \alpha + \bar{\beta}$. The payoff of investment in period t is now bounded *below* by the payoff under zero-one expectations, $W(p, \beta)$, which is strictly greater than c because of Corollary 2. The analysis of the case $p_{t-1} < 1/2$ follows immediately from the symmetry equation (11).

Lemma 5

If an information trader gets information and that information is $\theta = 1$, he buys at a price that must be greater than p_{t+1}^* by definition of $q^+(p_{t+1}^*, 0)$, and by definition of p_{t+1}^* , $\beta_{t+1} = 0$. To find an upper-bound of the payoff of investment in

period t , we can assume that β_{t+1} is at the maximum $\bar{\beta}$ if the information trader learns $\theta = 0$, (because of the strategic complementarity from period $t+1$ to period t in Proposition 1). That payoff is therefore not greater than $W(p_{t-1}, \bar{\beta})$.

Lemma 8

Using the backward complementarity (Proposition 1), we can assume that any agent in period $t+1$ with signal $s' < \hat{s}$ invests. For an agent with signal s in period t , the lower-bound of p_{t-1} is $s - \sigma$. If the price rises in period t , it rises by at least η (Lemma 7). Any signal s' in period $t+1$ is greater than $s + \eta - 2\sigma$ which is greater than \hat{s} if $\sigma < \eta/4$. Likewise after a transaction at the bid at p_t^- , no signal in period $t+1$ is greater than \hat{s} if $\sigma < \eta/4$, and all agents invest. The payoff of investment in period t for an agent with signal s is bounded above by the function \tilde{J} given in (23).

Lemma 9

Using the definition of $J(s, s; \sigma)$, equation (24), and since $W(p, \bar{\beta}) < c$ for all $p > p_H$, there exists $\hat{\sigma}_1$ such that if $\sigma < \hat{\sigma}_1$, then $J(s, s; \sigma) < c$ for all $s \geq p_H$.

Using Corollary 2, there exists $\nu < 0$ and an open interval \mathcal{O} that contains $[p_L, p_H]$ such that if $p \in \mathcal{O}$ and $\beta \in [0, \bar{\beta}]$, then $W(p, \beta) < \nu$. Using (25), there exists $\hat{\sigma}_2$ such that if $\sigma < \hat{\sigma}_2$, then $dJ(s, s)/ds < \nu$. The proof is concluded by noting that using the definition of p_L and (24), $J(p_L, p_L; \sigma) > c$, if σ sufficiently small.

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