Optimal Auctions with Ambiguity*

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Abstract

A crucial assumption in the optimal auction literature has been that each bidder's valuation is known to be drawn from a single unique distribution. In this paper we relax this assumption and study the optimal auction problem when there is ambiguity about the distribution from which these valuations are drawn and where the seller or the bidder may display ambiguity aversion. We model ambiguity aversion using the maxmin expected utility model where an agent evaluates an action on the basis of the minimum expected utility over the set of priors, and then chooses the best action amongst them. We first consider the case where the bidders are ambiguity averse (and the seller is ambiguity neutral). Our first result shows that the optimal incentive compatible and individually rational mechanism must be such that for each type of bidder the minimum expected utility is attained by using the seller's prior. Using this result we show that an auction that provides full insurance to all types of bidders is always in the set of optimal auctions. In particular, when the bidders' set of priors is the ϵ- contamination of the seller's prior the *unique* optimal auction provides full insurance to bidders of all types. We also show that in general, many classical auctions, including first and second price are not the optimal mechanism (even with suitably chosen reserve prices).

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We next consider the case when the seller is ambiguity averse (and the bidders are ambiguity neutral). Now, the optimal auction involves the seller being perfectly insured. Hence, as long as bidders are risk and ambiguity neutral, ambiguity aversion on the part of the seller seems to play a similar role to that of risk aversion.

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1 Introduction

Optimal auctions for an indivisible object with risk neutral buyers and independently distributed valuations have been studied by, among others, Vickrey [24], Myerson [20], Harris and Raviv [9], and Riley and Samuelson [22]. These papers show that the set of optimal mechanisms or auctions is quite large, and that the set contains both the first and second price auctions with reserve prices. One of the assumptions in this literature is that each bidder's valuation is drawn from a unique distribution. In this paper we relax this assumption and study how the design of the optimal auction is affected by the presence of ambiguity about the distribution from which the bidders' valuations are drawn.

The unique prior assumption is based on the subjective expected utility model, which has been criticized among others by Ellsberg [5]. Ellsberg shows that lack of knowledge about the distribution over states can effect choices in a fundamental way that can not be captured within the subjective expected utility framework. In one version of Ellsberg's experiment, a decision maker is offered two urns, one that has 50 black and 50 red balls, and one that has 100 black and red balls in unknown proportions. Faced with these two urns, most decision makers bet on drawing either color from the first urn, rather than on drawing the same color from the second urn. It is easy to show that such behavior is inconsistent with the expected utility model. Intuitively, decision makers do not like betting on the second urn because they do not have enough information or, put differently, there is too much ambiguity. Being averse to ambiguity, they prefer to bet on the first urn. Ellsberg and many subsequent studies have[1] demonstrated that ambiguity aversion is common and incompatible with the standard expected utility theory.

[1] See, for example, Camerer and Weber [3] for a survey.

Following Gilboa and Schmeidler [8], we model ambiguity aversion using the maxmin expected utility (MMEU) model. In this model agents have a set of priors (instead of a single prior), on the underlying state space, and compute their payoff as the minimum expected utility over the set of priors. The MMEU model is a generalization of the SEU model, and provides a natural and tractable framework to study ambiguity aversion.

When the buyers are more ambiguity averse than the seller, our main message is that auctions that provide perfect insurance to the buyers provide more revenue to the seller than other auctions. This result can explain some auction mechanisms that are observed in real life. In particular Goeree and Offerman (2004) observe that: "In Europe, sellers of houses, land,... regularly offer a premium to the highest losing bidder to promote competitive bidding. Many Dutch and Belgian towns have their own variant of premium auctions, some of which date back to the Middle Ages." Goeree and Offerman explain the existence of such auctions by asymmetries among bidders. They argue that even though premium auctions are not optimal in environments with asymmetries among bidders they may be "second best." In this paper we provide alternative explanation, since we show that even with symmetric bidders when there is ambiguity premium auctions may outperform standard auctions.

Under MMEU, when an ambiguity averse buyer is confronted with a selling mechanism, he evaluates each action on the basis of the minimum expected utility over the set of priors, and then chooses the best among them. An ambiguity averse seller on the other hand evaluates a mechanism on the basis of its minimum expected revenue over the set of priors and chooses the best mechanism. In order to better contrast our results with the risk case, we assume that the buyers and the seller are risk neutral (i.e. have linear utility functions).

We consider two cases, one where the bidders are ambiguity averse (and the seller is ambiguity neutral) and the other where the seller is ambiguity averse (and the buyers are ambiguity neutral)[2].

In the case where the bidders are ambiguity averse, our first result shows that the optimal mechanism is such that for each type of the buyer the minimum expected utility is attained by the seller's prior. We also show that an auction that provides complete insurance to the bidders (i.e., it keeps the

[2]See section 7 for a discussion of the case where both the buyers and the seller are ambiguity averse.

bidders' payoffs constant for all reports of the other bidders and consequently keeps them indifferent between winning or losing the object) is always in the set of optimal mechanisms. We use our first result to obtain further insights to the optimal auction problem. First we show that, unlike in the standard situation, when even a small amount of ambiguity is introduced, the complete insurance auction may be the *unique* optimal auction. To show this, we study an interesting example, ϵ-contamination. Suppose that the seller's prior is denoted by F, and the buyer's set of priors is given by $\Delta_B = \{G : G = (1-\epsilon) F + \epsilon H\}$ where H is any distribution and ϵ is a small positive number. That is, in the case of ϵ-contamination the buyers' set of beliefs contains all small perturbation of the seller's belief. The intuition is that the seller's prior is a focal point, but the buyers allow for an ϵ-order amount of noise. With ϵ-contamination we show that the complete insurance auction is the unique optimal auction. We also consider the often studied case of Choquet expected utility with convex capacity and show that the complete insurance auction is the unique optimal auction here as well. One practical interpretation of these results is that the complete insurance auction is the only auction within the traditional set of optimal auctions that is robust to the introduction of a small amount of ambiguity. We also show that in general, neither the first price nor the second price auction is optimal even with suitably chosen reserve prices.

To obtain some intuition for the first result, suppose that the optimal mechanism is such that the minimizing set of distributions for some type of the buyer does not include the seller's prior. In this case, it is clear that the particular type, say θ, of the buyer and the seller will be willing to bet against each other; essentially the seller recognizes that they have different beliefs about the underlying state space and offers "side bets" using transfers. However, the crucial issue is that the modified mechanism will have to maintain overall incentive compatibility constraint as well. An interesting technical point is that a mechanism, even if changed by a very small amount, could, as a result, be evaluated by an entirely different distribution than the one originally used, thus making comparisons often difficult. In our proof we address this issue by explicitly constructing the additional transfers that continue to satisfy incentive compatibility constraints while making the seller better off. Essentially, we show that these additional transfers (to the seller) can be chosen so that under the new mechanism, type θ under truth telling gets the minimum expected utility that he gets in the original mechanism in *every* state, and thus is completely insured against the ambiguity. Obviously, under truth telling, type θ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under both. More interestingly, no other type wants to imitate type θ in the new mechanism. This is because the additional transfers in the new mechanism are constructed so as to have zero expected value under the minimizing set of distributions of the original mechanism, but to have strictly positive expected value under any other distribution. Therefore, if type θ' imitates type θ in the new mechanism, he gets at best what he would get by imitating type θ in the original mechanism. In other words, for both types of bidders, - those whose transfers have been modified and those whose transfers haven't been - their payoff remain unchanged under truth telling, whereas if a type reports some other type his payoff is weakly lower under the new mechanism than under the original. Hence, since the original mechanism is incentive compatible, the new mechanism must also be incentive compatible. Moreover, since by assumption, the seller's distribution is not in the minimizing set for the original mechanism, the additional transfers (to the seller) must have strictly positive expected value under the seller's distribution, which means the seller is better off in the new mechanism. Note that the mechanism that is constructed to improve upon the original one gives complete insurance to the buyers and is always weakly preferred by the seller. Therefore, a complete insurance auction must always be in the set of optimal auctions. To see why in the case of ϵ-contamination the complete insurance auction is the *unique* optimal auction, suppose that the optimal auction does not provide complete insurance for some type of the buyer. Then that buyer must evaluate this auction by a distribution that moves the ϵ weight to the "unfavorable" states. But this contradicts our earlier result that the set of distributions that the buyer uses to evaluate the optimal auction must include the seller's distribution. A similar argument shows why the complete insurance auction is the uniquely optimal auction in the case of Choquet expected utility with convex capacity. Furthermore, for a wide class of situations, the first and the second price auctions are not optimal, since in both of these auctions the (minimizing) distributions used to evaluate the bidders' expected utilities will be different from the seller's prior.

The second case we consider is when the seller is ambiguity averse (and the bidders are ambiguity neutral). Within the risk framework, Eso and Futo [7] consider auctions (in IPV environments) with a risk averse seller and risk (and ambiguity) neutral buyers. They show that for every incentive compatible selling mechanism there exists a mechanism which provides deterministically the same (expected) revenue. From this it follows that the optimal selling mechanism must provide complete insurance to the seller. We show that the mechanism in Eso and Futo is the optimal mechanism in our case as well. Hence, as long as bidders are risk and ambiguity neutral, ambiguity aversion on the part of the seller plays a similar role to that of risk aversion.

1.1 Related Literature

Matthews [17] and Maskin and Riley [16] relax the assumption of risk neutrality replacing it with risk aversion. They show that the classic auctions (high bid, English) are no longer optimal. In order to contrast our results, where buyers are ambiguity averse, with the results obtained when buyers are risk averse, we will briefly summarize the main results of Maskin-Riley.

Maskin-Riley show that, in the setting with risk, the central problem is preventing high valuation buyers from bidding too low. Suppose that the seller devises an auction where bidders who bid low face risk, but high valuation buyers who bid low face greater risk. The seller would derive less revenue from the low valuation buyers than if he offered them complete insurance, but this loss would be compensated by the high valuation buyers' higher bids. Thus even though for a particular type of buyer, removal of risk can be done in such a way that the buyer type's utility remains unchanged while the seller's payoff increases, the usual screening condition dictates that risk should not be completely eliminated for types other than the one with the highest valuation.

Indeed Maskin-Riley show that the optimal way to confront a buyer with risk is by using the transfers. For example, low valuation buyers may be penalized if they lose and high valuation buyers may receive a subsidy if they win. Moreover, only the buyer with the highest valuation gets perfectly insured. In other words, all buyers except the most eager buyer are better off winning than losing.

These results contrast with ours in the ambiguity setting. As we argued in the previous section, providing full insurance to all types of buyers is an optimal mechanism, and in some cases, it is the uniquely optimal one. These results not only differ from those under pure risk, but they are also driven by very different considerations. To gain a better intuition on the difference between risk aversion and ambiguity aversion, let's consider the ϵ-contamination case where perfect insurance is provided to all types of buyers, and contrast that with the situation when buyers are risk averse but face no ambiguity. The problem faced by the seller in the risk aversion case is that if risk is reduced for a particular type of buyer then the expected utility of all types when they report this particular type goes up. Reduction of risk for a type thus affects the IC constraints adversely. With ambiguity averse, but risk-neutral, buyers that is not so. Starting from a situation where a buyer type faces variable ex post utility in the mechanism, the seller can modify the mechanism in such a way as to make that type's ex post utility constant, keep the expected utility of this type the same as before, and increase his own expected payoff. More importantly, this can be done in such a way that the *other* type's expected utility, when they report the type whose payment scheme is being changed, is either unaffected or actually goes down. In other words, unlike the risk aversion situation, here, the seller can provide full insurance to a type in a way that benefits the seller's expected payoff that does not create any adverse effect on the IC constraints.

There is a small but growing literature on auction theory with non-expected utility starting with Karni and Safra ([11], [13], [12]) and Karni [10]. The papers that look at auctions with ambiguity averse bidders, and thus are closer to this paper are by Salo and Weber [23], Lo [14], Volij [25] and Ozdenoren [21]. These papers look at specific auction mechanisms, such as the first and second price auctions, and not the optimal auction problem.

Billot, Chateauneuf, Gilboa and Tallon [2] analyze the question of when it is optimal to take bets for agents with MMEU preferences in a pure exchange economy. They show that if the intersection of the set of priors for all agents is non-empty, then any Pareto optimal allocation is a full insurance allocation. This result is in the same spirit as our results. Furthermore, even though a direct comparison of the two models are difficult, a possible implication of our result could be that Bilot et. al. [2] result may be robust to the introduction of incentive constraints. Another related paper is Mukerji [18] that shows that in the investment hold-up model ambiguity aversion can explain the existence of incomplete contracts. The incomplete or null contract is where the ex post surplus is split equally between the two parties and they thus agree on the ranking of the states. To implement more efficient investments, a contract has to introduce more variation in ex-post payoffs which would also result in disagreement among the two parties; and when ambiguity is sufficiently large any such contract would be dominated by the null contract.

2 Maxmin Expected Utility Model

In this section we introduce the MMEU model. Let Θ be the state space representing the agent's uncertainty. Let Σ be an algebra on Θ. Let $\mathcal{M}$ be the space of all probability measures on (Θ, Σ). Let X denote the set of outcomes. Suppose the decision maker's Von Neumann-Mongenstern utility function is given by $u : X \to \mathbb{R}$ and prior on Θ is given by a probability measure $\mu \in \mathcal{M}$. Let $\mathcal{A}$ be the set of all acts where an act is a Σ-measurable function $a : \Theta \to X$. In the standard expected utility model, utility of an act $a \in \mathcal{A}$ is,

$$U(a) = \int_{\Theta} u(a(\omega))\, d\mu(\omega).$$

In contrast, in the MMEU model, the decision maker's prior is given by a (weak*) closed and convex set of probability measures $\Delta^m \subseteq \mathcal{M}$, and the utility of an act $h \in \mathcal{F}$ is,

$$U(a) = \min_{\mu \in \Delta^m} \int_{\Theta} u(a(\omega))\, d\mu(\omega).$$

The interpretation of the set of priors is, even if the information of the decision maker is too vague to be represented by an additive prior, it may be represented by a set of priors.

In this paper, the state space will be the possible valuations of the other bidder: this is the domain of uncertainty. For simplicity we assume in the following that $\Theta = [0, 1]$ and Σ is the Borel algebra on Θ. We assume risk neutrality (linear utility function) throughout the paper.

3 The Optimal Auction Problem

There are two bidders and a seller. We assume that both the bidders and the seller have linear VNM utility functions. Bidders have one of a continuum of valuations $\theta \in \Theta$. Each bidder knows his true valuation but not that of the other. The set Δ_B is a set of distribution functions corresponding to a (weak*) closed, convex subset of the set of probability measures Δ_B^m over Θ, and this set represents each buyer's belief about the other bidder's valuation. Buyers believe that valuations are generated independently, but they are not confident about the probabilistic process that generates the valuations. This is reflected by the buyers having a set of priors rather than a single prior in this model.

The seller is also allowed to be uncertainty averse. The set Δ_S is a set of distribution functions corresponding to a (weak*) closed, convex subset of the set of probability measures Δ_S^m over Θ, and it represents the seller's belief about the bidders' valuations. That is, the seller believes that buyers' valuations are generated independently from some distribution in Δ_S.[3] We assume that $\Delta_S \subseteq \Delta_B$. Therefore the model covers two interesting cases. If Δ_S is a singleton set with the unique element F, then the seller is ambiguity neutral and believes that buyers' valuations are independently generated from the distribution F. On the other hand, if $\Delta = \Delta_S = \Delta_B$, then both the seller and the buyers are ambiguity averse with the common set of distributions Δ. We assume that there is a measure $\tilde{\mu}$ such that all measures in Δ_B^m and Δ_S^m are absolutely continuous with respect to $\tilde{\mu}$.

Each bidder's reservation utility is 0. As is standard, we assume that all of the above is common knowledge.

We focus on the direct revelation game. In the direct revelation game, each bidder is asked to report his type, where a report is some $\theta \in \Theta$. The mechanism stipulates a probability for assigning the item and a transfer rule as a function of reported types. Let $x(\theta, \theta')$ be the item assignment probability function and $t(\theta, \theta')$ the transfer rule. The convention is that the first entry is one's own report, the second entry is the report of the other bidder.[4] We consider mechanisms that are measurable with respect to $\Sigma \times \Sigma$ such that the transfers are uniformly bounded[5].

The seller's problem is to find a mechanism (x, t) that is incentive com-

[3]Formally the seller's belief is the set of product measures $\mu \times \mu$ on the product space $(\Theta \times \Theta, \Sigma \times \Sigma)$ where $\mu \in \Delta_S^m$.

[4]A word about notation before we proceed. Formally, one should have separate notation for reports, say $\widehat{\theta}$ as opposed to θ, and define the mechanism in terms of reports, not types. Since the optimal mechanisms we describe will all be incentive compatible, here and in several other places, we save on notation by describing the mechanisms directly in terms of θ. We hope this departure from convention, however, will cause no confusion.

[5]Note that the bound on transfers can be arbitrarily large, so we do not view this as an important restriction.

patible and individually rational and that maximizes:

$$\min_{F\in\Delta_S}\iint\left[t\left(\theta,\theta'\right)+t\left(\theta',\theta\right)\right]dF\left(\theta\right)dF\left(\theta'\right)$$

subject to

$$\text{(IC)}\quad \min_{G\in\Delta_B}\int\left(x\left(\theta,\theta'\right)\theta-t\left(\theta,\theta'\right)\right)dG\left(\theta'\right)$$

$$\geq\quad \min_{G\in\Delta_B}\int\left(x(\tilde{\theta},\theta')\theta-t(\tilde{\theta},\theta')\right)dG(\theta')\text{ for all }\theta,\tilde{\theta}\in\Theta$$

$$\text{(IR)}\quad \min_{G\in\Delta_B}\int\left(x\left(\theta,\theta'\right)\theta-t\left(\theta,\theta'\right)\right)dG\left(\theta'\right)\geq 0\text{ for all }\theta\in\Theta.$$

The first inequality gives the incentive compatibility (IC) constraints, and the second inequality gives the individual rationality (IR) or participation constraint. These are the usual constraints except that the bidders compute their utility in the mechanism using the MMEU rule. For example, the IC constraint requires that the minimum expected utility a bidder of type θ gets reporting his type truthfully is at least as much as the minimum expected utility that he gets under reporting any other type θ'.

In the appendix we show that the seller's problem can be written as above, in the sense that the minimums over the set of priors exist. We also show that an optimal mechanism exists.

4 Ambiguity averse buyers

The central result in this framework is that buyers and sellers "use the same distribution" to evaluate an optimal mechanism. That is, if a mechanism is optimal, then it must be the case that the intersection of the minimizing set of distributions for a given type of the buyer and the minimizing set of distributions for the seller must be non-empty for almost all types. Otherwise, the seller can offer a lottery to each type for which this intersection is empty, and be better off while keeping the resulting mechanism incentive compatible and individually rational.

Since more intuition can be obtained for the case where the seller is ambiguity neutral, that is $\Delta_S=\{F\}$, we focus on this case in the paper; however, in the appendix we give a proof of the result for the general case from which the proof of case $\Delta_S = \{F\}$ follows. Our main result says that the minimum payoff for the buyers will be attained by F as the distribution. This is because if for any type of buyer the expected utility is evaluated by a distribution other than F, then in effect the seller and that type of buyer agree to disagree about the value distribution of the other buyer. However, then the stipulated mechanism would not be optimal if the seller and that type of buyer can enter into side bets where the expected value of these side bets are zero for the type of buyer but positive for the seller. The reason why existence of such side bets are not obvious is that the transformed mechanism has to maintain overall incentive compatibility and individual rationality. In our proof we explicitly construct such side bets (or lotteries) to show how the seller can exploit such differences of opinion to make himself better off while maintaining the ICC and IR constrains. Thus, if given an ICC and IR mechanism, there is a positive measure of types θ for which $F \notin \Delta(\theta)$, then the given mechanism can not be optimal.

This result imposes quite a lot of structure on the set of optimal mechanisms, because many mechanisms will not be evaluated with the same distribution by the buyer and seller, and are hence not optimal. Next, we give the formal statement of the proposition. The formal proof also shows how to construct the lotteries described above.

Proposition 1 *Suppose that the seller is ambiguity neutral with distribution F and the buyers are ambiguity averse with the set of priors Δ_B. Let (x, t) be an arbitrary incentive compatible mechanism. For any $\theta \in \Theta$, let $\Delta(\theta) \subseteq \Delta_B$ be the set of minimizing distributions for θ under (x, t). That is,*

$$\Delta(\theta) = \arg \min_{G \in \Delta_B} \int [x(\theta, \theta')\theta - t(\theta, \theta')] \, dG(\theta').$$

If there exists some positive measure $\widetilde{\Theta} \subseteq \Theta$ such that, $F \notin \Delta(\tilde{\theta})$ for all $\tilde{\theta} \in \widetilde{\Theta}$ then (x, t) is not optimal and in fact the seller can strictly increase revenue using a mechanism that provides full insurance to all types in $\widetilde{\Theta}$.

To see in more detail, the intuition behind this result, suppose to the contrary that the optimal mechanism (x, t) is such that the minimizing set of distributions for some type of the buyer does not include the seller's prior. Let $\Delta(\theta)$ be the (possibly different) set of distributions that achieves the minimum expected utility for each type of buyer under truth telling Let the type, say θ, be such that $F \notin \Delta(\theta)$. In this case, the seller can offer additional transfers to θ, where these transfers have the following property: these additional transfers (to the seller) can be chosen so that type θ under truth telling gets the *same* minimum expected utility that he gets in the original mechanism in every state, and thus is completely insured against the ambiguity in the new mechanism. Furthermore, these transfers have zero expected value under $\Delta(\theta)$, the minimizing set of distributions in the original mechanism, and strictly positive expected value under any other distribution, i.e., for distributions in $\Delta_B - \Delta(\theta)$. Obviously, under truth telling, type θ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under both. More interestingly, no other type wants to imitate type θ in the new mechanism. This is because if a type, say θ' imitates type θ in the new mechanism, he gets at best what he would get by imitating type θ in the original mechanism. Since the original mechanism is incentive compatible, the new mechanism must also be incentive compatible. Moreover, by assumption, the seller's distribution is not in the minimizing set for the original mechanism, which means the additional transfers (to the seller) must have strictly positive expected value under the seller's distribution. Thus the seller is strictly better off in the new mechanism, contradicting the optimality of the original mechanism.

Incentive compatible mechanisms where the payoff of any bidder is constant for any report of the competing bidder are called *perfect (or full) insurance mechanisms*. Our next result shows that there is always a perfect insurance mechanism within the set of optimal mechanisms.

Proposition 2 *There exists an auction,* (x^*, t^*), *that maximizes the seller's revenue such that the payoff of any type of a bidder in his auction is constant as a function of the other bidder's report. That is for all* $\theta \in \Theta$, $x^*(\theta,\theta')\theta - t^*(\theta,\theta')$ *is constant in* θ'.

To see how (x^*, t^*) is constructed, suppose that (x,t) is an auction that maximizes the seller's revenue and let $q(\theta,\theta') = x(\theta,\theta')\theta - t(\theta,\theta')$. Let,

$$K(\theta) = \min_{G \in \Delta_B} \int q(\theta,\theta') dG(\theta')$$

so that $K(\theta)$ is buyer θ's expected payoff.

Define the function $\delta : \Theta \to \mathbb{R}$ as follows:

$$\delta(\theta,\theta') = [q(\theta,\theta') - K(\theta)], \text{ for all } \theta \in \Theta.$$

Now consider the mechanism (x^*, t^*) such that $x^*(\theta, \theta') = x(\theta, \theta')$ and $t^*(\theta, \theta') = t(\theta, \theta') + \delta(\theta, \theta')$ for all $\theta \in \Theta$. Now note that $x^*(\theta, \theta')\theta - t^*(\theta, \theta') = x(\theta, \theta')\theta - t(\theta, \theta') - \delta(\theta, \theta') = K(\theta)$, which does not depend on θ'.

We can prove using simple variations of the proofs of claims 1,2 and 3 in the proof of proposition 11 that (x^*, t^*) gives the seller at least as much revenue as (x, t) and satisfies IC and IR constraints for the buyers. Thus (x^*, t^*) must be in the set of optimal auctions as well.

For complete insurance mechanisms any distribution in Δ_B gives the minimum expected utility and proposition (1) is trivially satisfied. In general though there may be other selling mechanisms that are optimal. On the other hand, if the set Δ_B is sufficiently rich and if a bidder's ex post payoffs vary enough with the report of the other bidder, then typically the set of distributions that give the minimum expected utility will not include F. In section 5.1, we give an example where the set of priors include all perturbations of the seller's prior F and show that in fact in this case the *unique* optimal mechanism is the perfect insurance mechanism.

In the next section we provide some applications of these results.

5 Applications

For the examples in this section we again look at the case where the seller is ambiguity neutral with $\Delta_S = \{F\}$. The strength of proposition 1 is best seen through examples. We consider very natural Δ_B which result in perfect insurance for the buyers. We also establish that, under some broad conditions on Δ_B, first and second price auctions with reserve prices are not optimal.

5.1 Perfect insurance under ϵ-contamination

5.1.1 The optimal mechanism

Intuitively if the set Δ_B is sufficiently rich and if a bidder's ex post payoffs in a given mechanism vary enough with the report of the other bidder, then the set of distributions that give the infimum expected utility will not include F.. One such rich Δ_B is an ϵ-contamination of the seller's prior F. [6] In this setting, the seller's distribution F is a focal point, and buyers allow for an

[6]See Epstein and Wang [6].

ϵ-order amount of noise around this focal distribution. We assume F has a strictly positive density f, and, we construct the buyers' Δ_B as follows:

$$\Delta_B = \{G : G = (1-\epsilon) F + \epsilon H\}$$

where H is any distribution on Θ and $\epsilon \in (0,1]$. Note that in this case the absolute continuity assumption made earlier fails, and therefore the existence of a minimum over the set of priors and the existence of an optimal mechanism are not guaranteed. Nevertheless, we will show that an optimal mechanism can be constructed explicitly.

Wemake the standard assumption that

$$L(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$$

is strictly increasing in θ.

Next, we show that the optimal mechanism exists in this case by explicitly solving for the optimal mechanism. By proposition 12 we know that in our search for the optimal mechanism we can restrict attention to perfect insurance mechanisms. If we can find a mechanism that is optimal for the seller within this class, it must also be optimal overall. Fix a mechanism (x,t) such that $x(\theta,\theta')\theta - t(\theta,\theta')$ is constant for all θ'.

Next we define some useful notation. Let [7]

$$\begin{aligned} u(\theta) &= x(\theta,\theta')\theta - t(\theta,\theta'), \\ X(\theta) &= \int x(\theta,\theta')dF(\theta') \\ X^{\min}(\theta) &= \inf_{G\in\Delta_b} \int x(\theta,\theta')dG(\theta'), \\ X^{\max}(\theta) &= \sup_{G\in\Delta_b} \int x(\theta,\theta')dG(\theta'). \end{aligned}$$

Using the IC constraint we obtain,

$$\begin{aligned} u(\theta) &= \inf_{G\in\Delta_b} \int (x(\theta,\theta')\theta - t(\theta,\theta'))\, dG(\theta') \qquad (1) \\ &\geq \inf_{G\in\Delta_b} \int \left(x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') \right) dG(\theta') \\ &= u\left(\tilde{\theta}\right) + \inf_{G\in\Delta_b} \int \left(\theta - \tilde{\theta}\right) x(\tilde{\theta},\theta')dG(\theta'). \end{aligned}$$

[7] Since $x(\theta,\theta')\theta - t(\theta,\theta')$ is independent of θ', we write $u(\theta)$ instead of $u(\theta,\theta')$.

If $\theta > \tilde{\theta}$,we obtain,

$$u(\theta) \geq u\left(\tilde{\theta}\right) + \left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right). \tag{2}$$

Exchanging the roles of θ and $\tilde{\theta}$ in (1) we obtain.

$$u\left(\tilde{\theta}\right) \geq u(\theta) + \inf_{G \in \Delta_b} \int \left(\tilde{\theta} - \theta\right) x(\theta, \theta') dG(\theta')$$

Again for $\theta > \tilde{\theta}$,we obtain,

$$u\left(\tilde{\theta}\right) \geq u(\theta) + \left(\tilde{\theta} - \theta\right) X^{\max}(\theta). \tag{3}$$

Now observe that u is non-decreasing since, for $\theta > \tilde{\theta}$ by the IC constraint we have,

$$u(\theta) \geq u\left(\tilde{\theta}\right) + \left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \geq u\left(\tilde{\theta}\right).$$

Now, we are ready to prove a lemma that is useful in characterizing the optimal auction.

Lemma 3 *The function u is Lipschitz.*

Since u is Lipschitz, it is absolutely continuous and therefore is differentiable almost everywhere. For $\theta > \tilde{\theta}$ we use (2) and (3) to obtain,

$$X^{\max}(\theta) \geq \frac{u(\theta) - u\left(\tilde{\theta}\right)}{\theta - \tilde{\theta}} \geq X^{\min}\left(\tilde{\theta}\right).$$

We can take the limit as $\tilde{\theta}$ goes to θ to obtain for almost all θ that,

$$X^{\max}(\theta) \geq \frac{\partial u}{\partial \theta} \geq X^{\min}(\theta).$$

Since an absolutely continuous function is the definite integral of its derivative, we obtain,

$$\int_0^\theta X^{\max}(y)\, dy \geq u(\theta) - u(0) \geq \int_0^\theta X^{\min}(y)\, dy. \tag{4}$$

Equation (4) suggests that the auctioneer may set,

$$u(\theta) = \int_0^{\theta} X^{\min}(y)\, dy \tag{5}$$

and

$$t(\theta, \theta') = x(\theta, \theta')\,\theta - \int_0^{\theta} X^{\min}(y)\, dy, \tag{6}$$

since for a given allocation rule x, transfers as in (6) are the highest transfers the auctioneer can set without violating (4). Of course, (4) is only a necessary condition and for a given allocation rule x, the resulting mechanism (x, t) may not be incentive compatible. Fortunately, this difficulty does not arise if the allocation rule x is chosen optimally for transfers given as in (6). In other words, our strategy is to find the optimal allocation rule x, assuming that the transfers are given by (6), and then show that the resulting mechanism, (x, t) is incentive compatible.

For transfer function given by (6), we can rewrite the seller's revenue as,

$$R = 2\int_0^1 \int_0^1 \left(\theta x(\theta, \theta') - \int_0^{\theta} X^{\min}(y)\, dy \right) dF(\theta')\, dF(\theta).$$

Using integration by parts we obtain,

$$R = 2\int_0^1 \theta X(\theta)\, f(\theta)\, d\theta - \int_0^1 (1 - F(\theta))\, X^{\min}(\theta)\, d\theta. \tag{7}$$

Define,

$$L^{\epsilon}(\theta) = \theta - (1 - \epsilon)\,\frac{1 - F(\theta)}{f(\theta)},$$

and let $r \in (0, 1)$ be such that $L^{\epsilon}(r) = 0$.

The following proposition characterizes the optimal allocation when transfer function is given by (6).

Proposition 4 *For any θ and θ', let*

$$x(\theta, \theta') = \begin{cases} 1 & \text{if } \theta > \theta' \text{ and } \theta \geq r \\ 1/2 & \text{if } \theta = \theta' \text{ and } \theta \geq r \\ 0 & \text{otherwise} \end{cases}$$

and let t be given by (6). The mechanism (x, t) defined this way is optimal for the seller.

Proof. First note that L^ϵ is increasing in θ, if L is increasing in θ. To see this note that,

$$\begin{aligned}\theta-\frac{1-F(\theta)}{f(\theta)} &> \theta'-\frac{1-F(\theta')}{f(\theta')}\\ &\Rightarrow \theta-\theta'>\frac{1-F(\theta)}{f(\theta)}-\frac{1-F(\theta')}{f(\theta')}\\ &\Rightarrow \theta-\theta'>(1-\epsilon)\left(\frac{1-F(\theta)}{f(\theta)}-\frac{1-F(\theta')}{f(\theta')}\right)\\ &\Rightarrow \theta-(1-\epsilon)\frac{1-F(\theta)}{f(\theta)}>\theta'-(1-\epsilon)\frac{1-F(\theta')}{f(\theta')}.\end{aligned}$$

Note that $X^{\min}(\theta)\leq X(\theta)$. Therefore if $X(\theta)=0$, $X^{\min}(\theta)=0$ as well. Letting $\frac{X^{\min}(\theta)}{X(\theta)}=1$ whenever $X(\theta)=0$, we define $M(\theta)=\theta-\frac{X^{\min}(\theta)}{X(\theta)}\frac{1-F(\theta)}{f(\theta)}$. We can rewrite R as,

$$R=2\int_0^1\int_0^1 M(\theta)\,x(\theta,\theta')\,f(\theta')\,f(\theta)\,d\theta'd\theta. \tag{8}$$

Now we can show that the optimal allocation rule is given by setting $x(\theta,\theta')=1$ if $\theta>\theta'$ and $\theta\geq r$, $x(\theta,\theta')=\frac{1}{2}$ if $\theta=\theta'$ and $\theta\geq r$, and $x(\theta,\theta')=0$ otherwise. First note that, in the ϵ-contamination case, $X^{\min}(\theta)\geq(1-\epsilon)X(\theta)$ for all θ such that $X(\theta)<1$[8]. Under the above allocation rule $X^{\min}(\theta)=(1-\epsilon)X(\theta)$ for all θ such that $X(\theta)<1$. Therefore this allocation rule maximizes $M(\theta)$. By construction $x(\theta,\theta')=1$ if and only if $M(\theta)>M(\theta')$ and $M(\theta)\geq 0$ therefore maximizing (8).

Finally we show that (x,t) is incentive compatible. To this end first we show that if $X^{\min}$ is non-decreasing selecting u as in (5) satisfies IC. We check two cases.

---

[8] This is true since:

$$\begin{aligned}X^{\min}(\theta) &= \min_{G\in\Delta_b}\int x(\theta,\theta')dG(\theta')\\ &= \min_{H\in\mathcal{D}}\int x(\theta,\theta')d\left((1-\epsilon)F+\epsilon H\right)(\theta')\\ &\geq (1-\epsilon)\int x(\theta,\theta')dF(\theta').\end{aligned}$$

If $\theta > \tilde{\theta}$,

$$\begin{aligned} u(\theta) - u\left(\tilde{\theta}\right) &= \int_{\tilde{\theta}}^{\theta} X^{\min}(y)\,dy \\ &\geq X^{\min}\left(\tilde{\theta}\right)\left(\theta - \tilde{\theta}\right) \end{aligned}$$

and if $\theta < \tilde{\theta}$,

$$\begin{aligned} u\left(\tilde{\theta}\right) - u(\theta) &= \int_{\theta}^{\tilde{\theta}} X^{\min}(y)\,dy \\ &\leq X^{\min}\left(\tilde{\theta}\right)\left(\tilde{\theta} - \theta\right) \end{aligned}$$

So in either case,

$$\begin{aligned} u(\theta) &\geq u\left(\tilde{\theta}\right) + \inf_{G \in \Delta_b} \int \left(\theta - \tilde{\theta}\right) x(\tilde{\theta}, \theta') dG(\theta') \\ &= \inf_{G \in \Delta_b} \int \left(x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta')\right) dG(\theta'). \end{aligned}$$

which is the IC constraint.

Now, note that for the allocation rule in proposition 4, $X^{\min}$ is non-decreasing, and thus the mechanism (x, t) is incentive compatible. ■

Next, we show that under ϵ-contamination perfect insurance auction is the unique optimal auction. The intuition is as follows; if there is any deviation from a constant payoff, then one of the elements of $\Delta_B \setminus \{F\}$ will yield a lower payoff, since an H can be found which weighs the discrepancies appropriately; namely inflates the low payoff states and understates the high payoff states relative to F. It follows that since the mechanism involving non-constant payoffs are being evaluated by a distribution other than F, the results of the previous section show that the mechanism cannot be the optimal one.

Next we provide a formal statement of this claim. The proof is in the appendix.

Proposition 5 *Suppose that (x, t) solves the seller's problem and let Δ_B be the ϵ-contamination of the seller's belief F:*

$$\Delta_B = \{G : G = (1 - \epsilon) F + \epsilon H\}$$

Then there is a measure 1 subset $\bar{\Theta} \subseteq \Theta$, for which, for each $\bar{\theta} \in \bar{\Theta}$, $q(\bar{\theta}, \theta)$ is constant for almost all $\theta \in \Theta$.

We conclude this section with a remark on how to implement the optimal auction.

Remark 6 *A natural question, at this stage, might be to ask how would one implement the optimal mechanism described above. There are actually several ways to implement the mechanism, and we will describe one such auction here. Consider an auction where bidders submit bids for the object and the allocation rule is the usual one, namely, the highest bidder who bids above the reservation value* r *obtains the object. The payment scheme is as follows: the winning bidder pays to the auctioneer an amount equal to his bid, and all bidders (regardless of winning or losing) who has bid above the reservation price receives a gift from the seller. For a bidder who bids, say,* b*, (where* b *is greater than* r*), the amount of the gift is given by* $S(b) = (1 - \varepsilon) \int_r^b F(y)\, dy$*. In this auction, the equilibrium strategy of a bidder with valuation* θ *is to bid his valuation. To see this note that the allocation rule is the same as the one in proposition 4. Moreover, a bidder who bids* θ *pays* $\theta - (1 - \varepsilon) \int_r^b F(y)\, dy$ *if he wins the auction and* $-(1 - \varepsilon) \int_r^b F(y)\, dy$ *if he loses the auction, and these transfers are also the ones in proposition 4. Since reporting one's true value is incentive compatible in the optimal mechanism, it is also optimal to bid one's true value in this auction as well.*

5.2 First and Second Price auctions are not optimal

It has been established in Lo [14] that the revenue-equivalence result of auction theory does not hold when bidders are ambiguity averse. In particular, it was shown that the second price auction is not optimal since the first price auction may generate more revenue. It was an open question, therefore, whether the first price auction with an optimally chosen reserve price is the optimal auction. Our result gives a negative answer to this question.[9] Under rather general conditions, the first and second price auctions, as well as many other natural auctions, will not be optimal in this setting. The intuition is as follows. In most auctions, sellers obtain more revenue the higher is the valuations of the buyers. Thus under ambiguity aversion, they consider *high valuations unlikely.* Similarly, buyers hope that their opponents' valuations are low; hence they consider *low valuations unlikely.* Quite naturally, then,

[9]When the type space is discrete, neither the first nor the second price auction is the optimal auction for reasons completely unrelated to the issues being studied in this paper.

one would expect that seller and buyer's beliefs would depart; and hence these auctions are not optimal. Even, as we assumed above, F is fixed, unless F is the lower envelope of Δ_B, these auctions will not be optimal.

We propose the following corollary which describes one possible restriction on Δ_B that is sufficient for the non-optimality of first- and second-price auctions.

Formally, consider the following corollary of the theorems presented in section 4:

Corollary 7 *Suppose there exists some distribution $G \in \Delta_B$ such that G first-order stochastically dominates F. Now, if under some mechanism (x, t), there exists a positive measure subset $\widetilde{\Theta} \subseteq \Theta$ such that for all $\tilde{\theta} \in \widetilde{\Theta}$,*

$$q(\tilde{\theta}, \theta') = x(\tilde{\theta}, \theta')\tilde{\theta} - t(\tilde{\theta}, \theta')$$

is weakly decreasing in θ' and over some interval strictly decreasing, then (x, t) is not optimal.

Proof. First-order stochastic dominance implies:

$$\int q(\tilde{\theta}, \theta')dG(\theta') < \int q(\tilde{\theta}, \theta')dF(\theta')$$

By Proposition 1, (x, t) is not optimal. ■

Now, to show that the first and second price auctions are not optimal, we need only establish that under first and second price auctions, there exists some $\widetilde{\Theta}$ which meets this criterion.

Consider a positive measure set $\widetilde{\Theta}$ of Θ such that, for all $\tilde{\theta} \in \widetilde{\Theta}$, $\tilde{\theta}$ is strictly above the reserve price (if there is one) and $\tilde{\theta}$ is strictly below the highest in Θ. We will show that the first and second price auctions are not optimal for any $\tilde{\theta} \in \widetilde{\Theta}$.

Now, consider an arbitrary $\tilde{\theta} \in \widetilde{\Theta}$. The following describes the first-price auction for type $\tilde{\theta}$:

$$x(\tilde{\theta}, \theta') = \begin{cases} 1, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases}$$

$$t(\tilde{\theta}, \theta') = \begin{cases} b(\tilde{\theta}), & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}b(\tilde{\theta}), & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases}$$

for a fixed optimal bid $b(\tilde{\theta}) < \tilde{\theta}$. Hence,

$$q(\tilde{\theta}, \theta') = \begin{cases} \tilde{\theta} - b(\tilde{\theta}) > 0, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}\left(\tilde{\theta} - b(\tilde{\theta})\right) > 0, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases}$$

So as θ' increases, $q(\tilde{\theta}, \theta')$ falls monotonically from $\left(\tilde{\theta} - b(\tilde{\theta})\right)$ to zero. So the first price auction is not optimal as long as there is some distribution in Δ_B which first-order stochastically dominates F.

Similarly, for second price auctions,

$$\begin{aligned} x(\tilde{\theta}, \theta') &= \begin{cases} 1, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \\ t(\tilde{\theta}, \theta') &= \begin{cases} \theta', & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}\theta', & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

So, for second price auctions,

$$q(\tilde{\theta}, \theta') = \begin{cases} \tilde{\theta} - \theta' > 0, & \text{if } \tilde{\theta} > \theta' \\ 0, & \text{otherwise} \end{cases}$$

So as θ' increases, $q(\tilde{\theta}, \theta')$ falls monotonically from $\tilde{\theta}$ to zero. So the second price auction is also not optimal as long as there is some distribution in Δ_B which first-order stochastically dominates F.

5.3 Choquet Expected Utility with Convex Capacity

Choquet expected utility (CEU) model is an alternative to MMEU model that is used to represent ambiguity averse preferences. In the CEU model an ambiguity averse agent's subjective belief is represented by a convex capacity μ satisfying the following properties: $0 \leq \mu(A) \leq 1$ for all $A \in \Sigma$, $\mu(\emptyset) = 0$, $\mu(\Theta) = 1$ and $\mu(A) + \mu(B) \leq \mu(A \cap B) + \mu(A \cup B)$ for $A, B \in \Sigma$. It is this last property that captures ambiguity aversion and says that the union of disjoint events may be assigned a larger weight than the sum of the weights assigned to each event, since these events may be more ambiguous than their union. The core of a convex capacity μ, denoted by $\Pi(\mu)$, is given by:

$$\Pi(\mu) = \{p \in \mathcal{M} : p(A) \geq \mu(A) \text{ for all } A \in \Sigma\}.$$

The Choquet expected utility (with respect to the convex capacity μ) of an agent who evaluates an act $f : \Theta \to \mathbb{R}$ is given by:

$$CE(f) = \int f d\mu = \min_{p \in \Pi(\mu)} \int f dp.$$

Here the first integral is the Choquet integral of f with respect to capacity μ, and the second integral is the usual Lebesgue integral of f with respect to probability measure p where the minimum is taken over all probability measures in the core of capacity μ.

In the case of CEU with convex capacity we can apply proposition 1 to show optimality of the complete insurance auction. To see this let's first introduce some additional notation. Let $\Delta(\mu)$ be the set of all distribution functions associated with measures in $\Pi(\mu)$. Now suppose that $\bar{p} \in \mathcal{M}$ is associated with the seller's distribution F. Suppose further that $\bar{p}(A) \geq \mu(A) + \varepsilon$ for all $A \in \Sigma$ and some $\varepsilon > 0$. In this case, the complete insurance auction is the unique optimal auction.

To see this, suppose that for some $\bar{\theta} \in \bar{\Theta}$, $q(\bar{\theta}, \theta)$ is a simple function that is not constant for a positive measure set of θ. [10] Therefore there exists $K_1 > \cdots > K_m$ and Θ_j for $j = 1, ..., m$ such that $q(\bar{\theta}, \theta) = K_j$ for all $\theta \in \Theta_j$ and $\cup_{j=1}^m \Theta_j = \Theta$.

Now,

$$\begin{aligned} CE\left(q(\bar{\theta}, \cdot)\right) &= \int q(\bar{\theta}, \cdot) d\mu \\ &= \sum_{i=1}^m (K_i - K_{i+1})\, \mu\left(\cup_{j=1}^i \Theta_j\right) \\ &< \int q(\bar{\theta}, \cdot) dF. \end{aligned}$$

The intuition behind the above result is in fact immediate upon observing that under our assumption on $\bar{p}$, the $\varepsilon-$ contamination set of F is a subset of the core, and hence by proposition 5, complete insurance auction is indeed the unique optimal auction.

[10] Since any integrable function can be approximated to any arbitrary degree by a simple function, in what follows it is sufficient to consider q itself to be a simple function.

6 Ambiguity Averse Seller

In this section we first provide a counterpart of proposition 1 when the seller is ambiguity averse. The proof of the result is in the appendix.

Proposition 8 *Suppose that the seller is ambiguity averse, with a set of priors Δ_S and the buyers are ambiguity neutral with a prior $F \in \Delta_S$. Let (x,t) be an arbitrary incentive compatible mechanism. Let $\Delta_S^{\min} \subseteq \Delta_S$ be the set of minimizing distributions for the seller under (x,t). That is,*

$$\Delta_S^{\min} = \arg \min_{G \in \Delta_S} \iint t(\theta, \theta') + t(\theta', \theta) dG(\theta) dG(\theta').$$

If $F \notin \Delta_S^{\min}$, then (x,t) is not optimal.

The idea of the proof is based on Eso and Futo [7] who consider auctions with a risk averse seller in independent private values environments with risk (and ambiguity) neutral buyers. They show that for every incentive compatible and individually rational selling mechanism there exists an incentive compatible and individually rational mechanism which provides deterministically the same (expected) revenue. From this it follows that the optimal selling mechanism must provide complete insurance to the seller. We next show that the mechanism in Eso and Futo is the optimal mechanism in our case as well.

Proposition 9 *Suppose the conditions of proposition 8 holds. Then there exists an incentive compatible and individually rational selling mechanism, (x^*, t^*), that maximizes minimum expected revenue of the seller over the set of priors Δ_S and provides the same revenue to the seller no matter what the buyer's types are. That is $t(\theta, \theta') + t(\theta', \theta)$ is constant for all $\theta, \theta' \in \Theta$.*

The proof is omitted since it is a straightforward extension of the proof of proposition 8. The basic idea is very simple. For any individually rational and incentive compatible mechanism (x,t), one can define a new mechanism (x,t') with the same allocation rule, but with the following transfers:

$$t'(\theta, \theta') = T(\theta) - T(\theta') + \int T(i) dF(i).$$

Note that in the new mechanism $t'(\theta, \theta') + t'(\theta', \theta)$ is always $\int T(i) dF(i)$ which is constant. It is straightforward to check that this mechanism is incentive compatible as well. The reason this mechanism works in both risk and ambiguity settings is that, since the buyers are risk and ambiguity neutral (x, t') is incentive compatible in both settings and provides perfect insurance to the seller against both ambiguity and risk.

7 Conclusion

In this paper we analyzed selling mechanisms (or auctions) from the seller's point of view when either the buyers or the seller is ambiguity averse. We have showed that selling mechanisms that provide full insurance to the buyers when the buyers are ambiguity averse and to the seller when the seller is ambiguity averse are in the set of optimal mechanisms for the seller. We have also showed that a necessary condition for the optimality of a mechanism is that the buyers and the seller use the same subjective beliefs to evaluate the mechanism.

There are at least two directions these results may be extended. The first one is to allow the buyers and the seller both be ambiguity averse. Our results easily extend to the case when the buyers are more ambiguity averse (i.e., have a larger set of priors) than the seller. On the other hand, when the seller is more ambiguity averse than the buyers the above results may break down. This point is left for further research.

The second is to use the methods developed here in mechanism design problems with incomplete information where the agents are ambiguity averse. We believe that the results in this paper will naturally extend to these situations, especially in environments where the payoffs are quasilinear. For example in a bargaining problem (see Myerson [19]) we conjecture that the most efficient (from the mechanism designer's point of view) mechanism might require that some agent be completely insured against the ambiguity. In any case, and unlike the standard unique prior environment, the transfer and not just the allocation rule will play a crucial role in the design of the optimal mechanism in the presence of ambiguity. Furthermore, notice that in section 5.1, efficiency is helped by ambiguity since the cutoff type goes down with an increase in ε. In other contexts, and as shown in Mukerji [18], large amount of ambiguity may be a impediment to efficient trade. These are deep and interesting issue and we hope to explore these extensions in future research.

8 Appendix

8.1 Nonemptiness of the minimizing set of priors and the existence of an optimal mechanism

8.1.1 Mathematical Preliminaries

In this section we will prove that both the set minimizing priors and the set of optimal mechanisms is nonempty. In our proof we will use the following definitions and results.

Suppose that p and q are conjugate indices, i.e. $1/p + 1/q = 1$. If $p = 1$ then the conjugate is $q = \infty$. Suppose that $f^n \in L_p(\Theta, \Sigma, \tilde{\mu})$ for $n \in \{1, 2, ...\}$. (From now on we will write L_p instead of $L_p(\Theta, \Sigma, \tilde{\mu})$ for notational simplicity.) We say that f^n converges weakly to $f \in L_p$ if $\int g f^n d\tilde{\mu}$ converges to $\int g f d\tilde{\mu}$ for all $g \in L_q$.

Let $\text{ca}(\Sigma)$ be the set of countably additive measures on (Θ, Σ) having finite total variation. Set $\text{ca}(\tilde{\mu}) = \{\mu \in \text{ca}(\Sigma) : \mu << \tilde{\mu}\}$. We have the following result from Marinacci and Montrucchio [15].

Lemma 10 *If ■ is a ■-algebra, then a subset of ca(■) is weak*-compact if and only if it is weakly compact.*

Note that Δ_B^m and Δ_B^m are weak*-compact subsets of $\text{ca}(\tilde{\mu})$, therefore by the above lemma they are also weakly compact.

By the Radon-Nikodym Theorem, there is an isometric isomorphism between $\text{ca}(\tilde{\mu})$ and $L_1(\tilde{\mu})$ determined by the formula $\mu(A) = \int_A f d\tilde{\mu}$ (see Dunford and Schwartz [4], p. 306 and Marinacci and Montrucchio [15], Corollary 11). Hence, a subset is weakly compact in $\text{ca}(\tilde{\mu})$ if and only if it is in $L_1(\tilde{\mu})$ as well.

Let $\tilde{\Delta}_B$ be the set of Radon-Nikodym derivatives of measures in Δ_B^m with respect to $\tilde{\mu}$. Define $\tilde{\Delta}_S$ analogously. By the observation in the previous paragraph, since Δ_B^m and Δ_S^m are weak*-compact by assumption and both are subsets of $\text{ca}(\tilde{\mu})$, $\tilde{\Delta}_B$ and $\tilde{\Delta}_S$ are weakly compact.

Finally, let $\mathcal{B}_\infty^r = \{g \in L_\infty : \|g\|_\infty \leq r\}$. By theorem 19.4 in Billinsley [1], $\mathcal{B}_\infty^r$ is weakly compact.

8.1.2 Existence Results

Now, we turn our attention to showing that the minimizing set of priors is nonempty in (3). Let $g_{\tilde{\theta}\theta}(\theta') = x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta')$. Recall that we assume $|t(\theta, \theta')| \leq K$ for some $K > 0$. In other words transfers are uniformly bounded. Therefore by assumption $g_{\tilde{\theta}\theta} \in L_\infty$.

Now suppose that $f^n \in \tilde{\Delta}_B$ is such that $\int g_{\tilde{\theta}\theta} f^n d\tilde{\mu}$ converges to $\inf_{f \in \tilde{\Delta}_B} \int g_{\tilde{\theta}\theta} f d\tilde{\mu}$. Since $\tilde{\Delta}_B$ is weak compact, by passing to a subsequence we can find $\bar{f} \in \tilde{\Delta}_B$ such that f^n weakly converges to $\bar{f}$. Thus, $\bar{f} \in \arg\min_{f \in \tilde{\Delta}_B} \int g_{\tilde{\theta}\theta} f d\tilde{\mu}$. This proves that the minimizing set of priors is nonempty in the IC and IR constraints.

Now, we show that the minimizing set of priors in the seller's objective function is nonempty. Suppose $f^n \in \tilde{\Delta}_S$ is such that $\iint t(\theta, \theta') f^n(\theta) f^n(\theta') d\tilde{\mu}(\theta) d\tilde{\mu}(\theta')$ approaches to $\inf_{f \in \tilde{\Delta}_S} \iint t(\theta, \theta') f(\theta) f(\theta') d\tilde{\mu}(\theta) d\tilde{\mu}(\theta')$. Since $\tilde{\Delta}_B$ is weak compact, by passing to a subsequence we can find $\bar{f} \in \tilde{\Delta}_B$ such that f^n weakly converges to $\bar{f}$. Thus $\int t(\theta, \theta') f^n(\theta) d\tilde{\mu}(\theta)$ converges to $\int t(\theta, \theta') \bar{f}(\theta) d\tilde{\mu}(\theta)$. Let $g^n(\theta') = \int t(\theta, \theta') f^n(\theta) d\tilde{\mu}(\theta)$ and let $g(\theta') = \int t(\theta, \theta') \bar{f}(\theta) d\tilde{\mu}(\theta)$. Consider $\int g^n f^n d\tilde{\mu}$. Note that,

$$
\begin{aligned}
& \left| \int g^n f^n d\tilde{\mu} - \int g \bar{f} d\tilde{\mu} \right| \\
\leq & \left| \int g^n f^n d\tilde{\mu} - \int g f^n d\tilde{\mu} \right| + \left| \int g f^n d\tilde{\mu} - \int g \bar{f} d\tilde{\mu} \right| \\
\leq & \left| \int (g^n - g) f^n d\tilde{\mu} \right| + \left| \int g f^n d\tilde{\mu} - \int g \bar{f} d\tilde{\mu} \right| \\
\leq & \|g^n - g\|_\infty \|f^n\|_1 + \left| \int g f^n d\tilde{\mu} - \int g \bar{f} d\tilde{\mu} \right|.
\end{aligned}
$$

Note that $\|g^n - g\|_\infty \to 0$, since g^n converges to g pointwise therefore both terms at the end of the expression above go to zero. Thus,

$$
\bar{f} \in \arg\min_{f \in \tilde{\Delta}_S} \iint t(\theta, \theta') f(\theta) f(\theta') d\tilde{\mu}(\theta) d\tilde{\mu}(\theta').
$$

This proves that the minimizing set of priors in the seller's objective function is nonempty.

Next, we show that there exists a mechanism (x, t) that satisfies the IC and IR constraints and achieves the optimal revenue for the seller. Since transfers are bounded, the seller's revenue is bounded. Suppose that the value of the seller's problem (3) is R. This means that there exist a sequence of mechanisms $\{(x^n, t^n)\}$ such that (x^n, t^n) satisfies IC and IR constraints for each n, and if we let,

$$R^n = \min_{\mu \in \Delta_S^m} \iint \left[t^n(\theta, \theta') + t^n(\theta', \theta)\right] d\mu(\theta)\, d\mu(\theta'),$$

then $R^n \to R$.

Note that $x^n \in \mathcal{B}_\infty^1$ and $t^n \in \mathcal{B}_\infty^K$. Therefore passing to subsequences x^n converges weakly to x and and t^n converges weakly to t.

First suppose there exists $\delta > 0$ such that $x(\theta, \theta') + x(\theta', \theta) \geq 1 + \delta$ for $(\theta, \theta') \in A$ where $\tilde{\mu} \times \tilde{\mu}(A) > 0$. Let 1_A be the indicator function of A. This is a function in L_1, and since x^n converges weakly to x we have,

$$\begin{aligned} (1+\delta)\, \tilde{\mu} \times \tilde{\mu}(A) &= \int\int (1+\delta)\, 1_A d\tilde{\mu} d\tilde{\mu} \\ &\leq \int\int \left(x(\theta, \theta') + x(\theta', \theta)\right) 1_A d\tilde{\mu}(\theta)\, d\tilde{\mu}(\theta') \\ &= \lim_{n\to\infty} \int\int \left(x^n(\theta, \theta') + x^n(\theta', \theta)\right) 1_A d\tilde{\mu}(\theta)\, d\tilde{\mu}(\theta') \\ &\leq \int\int 1_A d\tilde{\mu} d\tilde{\mu} < (1+\delta)\, \tilde{\mu} \times \tilde{\mu}(A) \end{aligned}$$

which leads to a contradiction. Therefore, $x(\theta, \theta') + x(\theta', \theta) \leq 1$ for almost all (θ, θ'). Thus (x, t) is a feasible mechanism.

Next, we will show that (x, t) satisfies IC and IR constraints. Note that it is sufficient to show that for any $\theta, \tilde{\theta} \in \Theta$,

$$\lim_{n\to\infty} \min_{\mu \in \Delta_B^m} \int \left(x^n(\tilde{\theta}, \theta')\theta - t^n(\tilde{\theta}, \theta')\right) d\mu(\theta') = \min_{\mu \in \Delta_B^m} \int \left(x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta')\right) d\mu(\theta'). \tag{9}$$

To simplify notation let $g_{\tilde{\theta}\theta}^n(\theta') = x^n(\tilde{\theta}, \theta')\theta - t^n(\tilde{\theta}, \theta')$ and $g_{\tilde{\theta}\theta}(\theta') = x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta')$ for all $\theta, \tilde{\theta} \in \Theta$. Observe that $g_{\tilde{\theta}\theta}^n$ and $g_{\tilde{\theta}\theta}$ are both bounded by $K+1$ and thus they are both in L_∞. Moreover since x^n and t^t converge weakly to x and t, $g_{\tilde{\theta}\theta}^n$ converges weakly to $g_{\tilde{\theta}\theta}$.

Now note that for all $\hat{\mu} \in \Delta_B^m$,,

$$\lim_{n\to\infty} \min_{\mu \in \Delta_B^m} \int g_{\tilde{\theta}\theta}^n(\theta') d\mu(\theta') \leq \lim_{n\to\infty} \int g_{\tilde{\theta}\theta}^n(\theta') d\hat{\mu}(\theta') = \int g_{\tilde{\theta}\theta}(\theta') d\hat{\mu}(\theta'),$$

where the equality follows $g^n_{\tilde{\theta}\theta}$ converges weakly to $g_{\tilde{\theta}\theta}(\theta')$. Thus,

$$\lim_{n\to\infty} \min_{\mu\in\Delta_B^m} \int g^n_{\tilde{\theta}\theta}(\theta')d\mu(\theta') \leq \min_{\mu\in\Delta_B^m} \int g_{\tilde{\theta}\theta}(\theta')d\mu(\theta'). \tag{10}$$

On the other hand, for a given n, and for any $\varepsilon > 0$ there exists $f^n \in \tilde{\Delta}_B$ such that,

$$\int g^n_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') < \min_{f\in\tilde{\Delta}_B} \int g^n_{\tilde{\theta}\theta}(\theta') f(\theta')\, d\tilde{\mu}(\theta') + \varepsilon. \tag{11}$$

Since $\tilde{\Delta}_B$ is weakly compact again by passing to a subsequence, f^n converges weakly to $\bar{f} \in \tilde{\Delta}_B$. Note that,

$$\begin{aligned}
& \left| \int g^n_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta') \bar{f}(\theta')\, d\tilde{\mu}(\theta') \right| \\
\leq\ & \left| \int g^n_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') \right| + \\
& \left| \int \left(x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') \right) f^n(\theta')\, d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta') \bar{f}(\theta')\, d\tilde{\mu}(\theta') \right| \\
\leq\ & \left| \int \left[g^n_{\tilde{\theta}\theta}(\theta') - g_{\tilde{\theta}\theta}(\theta') \right] f^n(\theta')\, d\tilde{\mu}(\theta') \right| + \\
& \left| \int g_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta') \bar{f}(\theta')\, d\tilde{\mu}(\theta') \right| \\
\leq\ & 2\,|K+1| \left| \int f^n(\theta')\, d\tilde{\mu}(\theta') \right| + \left| \int g_{\tilde{\theta}\theta}(\theta') f^n(\theta')\, d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta') \bar{f}(\theta')\, d\tilde{\mu}(\theta') \right|.
\end{aligned}$$

The last inequality follows from the fact that $\left| g^n_{\tilde{\theta}\theta}(\theta') \right| \leq K+1$ and $|g_{\tilde{\theta}\theta}(\theta')| \leq K+1$. Since f^n weakly converges to $f \in \tilde{\Delta}_B$ all the terms on the last line converge to 0. This implies by taking limits in equation (11) that,

$$\int g_{\tilde{\theta}\theta}(\theta') \bar{f}(\theta')\, d\tilde{\mu}(\theta') \leq \lim_{n\to\infty} \min_{f\in\tilde{\Delta}_B} \int \left(x^n(\tilde{\theta},\theta')\theta - t^n(\tilde{\theta},\theta') \right) f(\theta')\, d\tilde{\mu}(\theta') + \varepsilon,$$

which in turn implies that

$$\min_{f\in\tilde{\Delta}_B} \int g_{\tilde{\theta}\theta}(\theta') f(\theta')\, d\tilde{\mu}(\theta') \leq \lim_{n\to\infty} \min_{f\in\tilde{\Delta}_B} \int \left(x^n(\tilde{\theta},\theta')\theta - t^n(\tilde{\theta},\theta') \right) f(\theta')\, d\tilde{\mu}(\theta') + \varepsilon$$

Since this is true for all $\varepsilon > 0$, the previous inequality together with (10) implies (9) which concludes the proof.

8.2 Additional Results and Remaining Proofs

We will prove results that are more general than the ones in the main text of the paper. In particular we will drop the assumption that there exists a measure $\tilde{\mu}$ such that $\mu \ll \tilde{\mu}$ for all $\tilde{\mu} \in \Delta_S^m \cup \Delta_B^m$. Without this assumption the seller's problem becomes:

$$\begin{array}{rl} & \sup_{(x,t)} \left[\inf_{F\in\Delta_S} \iint \left[t\left(\theta,\theta'\right) + t\left(\theta',\theta\right)\right] dF\left(\theta\right) dF\left(\theta'\right) \right] \\ & \quad \text{subject to} \\ \text{(IC)} & \inf_{G\in\Delta_B} \int \left(x\left(\theta,\theta'\right)\theta - t\left(\theta,\theta'\right)\right) dG\left(\theta'\right) \\ \geq & \inf_{G\in\Delta_B} \int \left(x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta')\right) dG(\theta') \text{ for all } \theta, \tilde{\theta} \in \Theta \\ \text{(IR)} & \inf_{G\in\Delta_B} \int \left(x\left(\theta,\theta'\right)\theta - t\left(\theta,\theta'\right)\right) dG\left(\theta'\right) \geq 0 \text{ for all } \theta \in \Theta. \end{array}$$

The next proposition is more general than proposition 1 in the sense that both the buyers and the seller are allowed to be ambiguity averse as long as $\Delta_S \subseteq \Delta_B$. For the case where $\Delta_S = \{F\}$ and when we make the absolute continuity assumption mentioned above this proposition implies proposition 1 in the text.

For a given mechanism (x,t) let $\Delta_S^{\min}$ be the set of measures such that if $\tilde{\mu} \in \Delta_S^{\min}$ then there is a sequence of measures $\mu^n \in \Delta_S^m$ such that $\mu^n \stackrel{w^*}{\to} \tilde{\mu}$ and,

$$\lim_{n\to\infty} \iint 2t(\theta,\theta')d\mu^n\left(\theta\right) d\mu^n\left(\theta'\right) = \inf_{\mu\in\Delta_S^m} \iint 2t(\theta,\theta')d\tilde{\mu}\left(\theta\right) d\tilde{\mu}\left(\theta'\right). \qquad (12)$$

Note that $\Delta_S^{\min}$ is nonempty. To see this note that by definition there exists a sequence $\mu^n \in \Delta_S^m$ that approaches the infimum of the right hand side in (12). Since Δ_S^m is weak* compact, for any such sequence there exists a subsequence that converges to some $\mu \in \Delta_S^m$, which by definition is an element of $\Delta_S^{\min}$.

Similarly, let $\Delta(\theta)$ be the set of measures such that if $\tilde{\mu} \in \Delta(\theta)$ then there is a sequence of measures $\mu^n \in \Delta_B^m$ such that $\mu^n \stackrel{w^*}{\to} \tilde{\mu}$ and

$$\lim_{n\to\infty} \int \left[x(\theta,\theta')\theta - t(\theta,\theta')\right] d\mu^n(\theta') = \inf_{\mu\in\Delta_B^m} \int \left[x(\theta,\theta')\theta - t(\theta,\theta')\right] d\tilde{\mu}(\theta').$$

Similar reasoning as the one at the end of last paragraph indicates that $\Delta(\theta)$ is nonempty for each θ.

Proposition 11 *Suppose that the seller and the buyers are ambiguity averse. Suppose that the seller's set of priors is Δ_S and the buyers' set of priors is Δ_B. Suppose further that $\Delta_S \subseteq \Delta_B$. Let (x,t) be an arbitrary incentive compatible mechanism. Let $\Delta_S^{\min}$ and $\Delta(\theta)$ be defined as above. If there exists some positive measure $\widetilde{\Theta} \subseteq \Theta$ such that, for all $\tilde{\theta} \in \widetilde{\Theta}$,*

$$\Delta_S^{\min} \cap \Delta(\tilde{\theta}) = \emptyset$$

then the seller can strictly increase revenue using a full insurance mechanism.

Proof. Define,

$$q(\theta, \theta') = x(\theta, \theta')\theta - t(\theta, \theta').$$

Now let,

$$K(\theta) = \inf_{\mu \in \Delta_B^m} \int q(\theta, \theta') d\mu(\theta')$$

so that $K(\theta)$ is buyer θ's maxmin expected payoff.

Now we will find a set of transfers for types in $\widetilde{\Theta}$ which will make the seller better off and keep the mechanism incentive compatible. Define the function $\delta : \widetilde{\Theta} \to \mathbb{R}$ as follows:

$$\delta(\tilde{\theta}, \theta') = \left[q(\tilde{\theta}, \theta') - K(\tilde{\theta}) \right], \text{ for all } \tilde{\theta} \in \widetilde{\Theta}$$

Now consider the mechanism (x, t') such that:

$$t'(\theta, \theta') = \begin{cases} t(\theta, \theta') + \delta(\theta, \theta'), & \text{for all } \theta \in \widetilde{\Theta} \\ t(\theta, \theta'), & \text{otherwise} \end{cases}.$$

We show that the mechanism (x, t') makes the seller strictly better off, leaves the buyers' payoffs unchanged, and is incentive compatible.

Claim 1: (x, t') makes the seller strictly better off
The seller's payoff for the mechanism (x, t') is:

$$\inf_{\mu\in\Delta_S^m} \iint \left[t'(\theta,\theta')+t'(\theta',\theta)\right] d\mu(\theta)d\mu(\theta') \tag{13}$$
$$= \inf_{\mu\in\Delta_S^m} \iint 2t'(\theta,\theta')d\mu(\theta)d\mu(\theta')$$
$$= \inf_{\mu\in\Delta_S^m} \left[\iint 2t(\theta,\theta')d\mu(\theta)d\mu(\theta') + \int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu(\theta')d\mu(\tilde{\theta})\right]$$

Note that for any $\mu \in \Delta_S^m$,

$$\int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu(\theta')d\mu(\tilde{\theta}) = 2\int_{\widetilde{\Theta}}\left[\int_{\Theta} q(\tilde{\theta},\theta')d\mu(\theta') - K(\tilde{\theta})\right] d\mu(\tilde{\theta})$$
$$= 2\int_{\widetilde{\Theta}}\left[\int_{\Theta} q(\tilde{\theta},\theta')d\mu(\theta') - \inf_{\mu'\in\Delta_B^m}\int q(\tilde{\theta},\theta')d\mu'(\theta')\right] d\mu(\tilde{\theta}) \geq 0.$$

Note that by definition of $\Delta(\tilde{\theta})$, if $\mu \notin \Delta(\tilde{\theta})$ then for any sequence $\mu^n \in \Delta_B^m$ with $\mu^n \xrightarrow{w^*} \mu$,

$$\lim_{n\to\infty}\inf \left[\int_{\Theta} q(\tilde{\theta},\theta')d\mu^n(\theta') - \inf_{\mu'\in\Delta_B^m}\int q(\tilde{\theta},\theta')d\mu'(\theta')\right] > 0 \tag{14}$$

Now suppose that $\mu^n \in \Delta_S^m$ is such that,

$$\inf_{\mu\in\Delta_S^m} \left[\iint 2t(\theta,\theta')d\mu(\theta)d\mu(\theta') + \int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu(\theta')d\mu(\tilde{\theta})\right] \tag{15}$$
$$= \lim_{n\to\infty} \left[\iint 2t(\theta,\theta')d\mu^n(\theta)d\mu^n(\theta') + \int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu^n(\theta')d\mu^n(\tilde{\theta})\right]$$
$$\geq \lim_{n\to\infty}\inf \iint 2t(\theta,\theta')d\mu^n(\theta)d\mu^n(\theta') + \lim_{n\to\infty}\inf \int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu^n(\theta')d\mu^n(\tilde{\theta}).$$

Since Δ_S^m is weak* compact there exists $\tilde{\mu} \in \Delta_S^m$ such that (by going to a subsequence if necessary) $\mu^n \xrightarrow{w^*} \tilde{\mu}$.

If $\tilde{\mu} \in \Delta_s^{\min}$, then for all $\tilde{\theta} \in \widetilde{\Theta}$ we have $\tilde{\mu} \notin \Delta(\tilde{\theta})$ (since $\Delta(\tilde{\theta}) \cap \Delta_s^{\min} = \emptyset$.) This implies that the difference in (14) is strictly positive for all $\tilde{\theta} \in \widetilde{\Theta}$. This in turn implies that,

$$\lim_{n\to\infty}\inf \int_{\widetilde{\Theta}}\int_{\Theta} 2\delta(\tilde{\theta},\theta')d\mu^n(\theta')d\mu^n(\tilde{\theta})$$
$$= 2\lim_{n\to\infty}\inf \int_{\widetilde{\Theta}}\left[\int_{\Theta} q(\tilde{\theta},\theta')d\mu^n(\theta') - \inf_{\mu'\in\Delta_B^m}\int q(\tilde{\theta},\theta')d\mu'(\theta')\right] d\mu^n(\tilde{\theta}) > 0.$$

To see this suppose that μ^{n_k} is a subsequence such that the above limit along this subsequence is 0, that is,

$$\begin{aligned} & \lim_{k\to\infty} \int_{\widetilde{\Theta}} \left[\int_{\Theta} q(\tilde{\theta},\theta') d\mu^{n_k}(\theta') - \inf_{\mu'\in\Delta_B^m} \int q(\tilde{\theta},\theta') d\mu'(\theta') \right] d\mu^{n_k}(\tilde{\theta}) \\ = & \lim_{k\to\infty} \int_{\widetilde{\Theta}} f^{n_k}\left(\tilde{\theta}\right) d\mu^{n_k}(\tilde{\theta}) = 0 \end{aligned}$$

where f^{n_k} denotes the function in the square parantheses. Now note that it must be that $f^{n_k}\left(\tilde{\theta}\right) \to 0$ for almost all $\tilde{\theta}$ (since μ^{n_k} and $\tilde{\mu}$ are all equivalent measures.) But this contradicts (14).

On the other hand, if $\tilde{\mu} \notin \Delta_s^{\min}$ then by definition of $\Delta_s^{\min}$,

$$\lim_{n\to\infty} \inf \iint 2t(\theta,\theta') d\mu^n(\theta) d\mu^n(\theta') > \inf_{\mu\in\Delta_S^m} \iint 2t(\theta,\theta') d\mu(\theta) d\mu(\theta').$$

In either case,

$$\begin{aligned} & \lim_{n\to\infty} \inf \iint 2t(\theta,\theta') d\mu^n(\theta) d\mu^n(\theta') + \lim_{n\to\infty} \inf \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta') d\mu^n(\theta') d\mu^n(\tilde{\theta}) \\ > & \inf_{\mu\in\Delta_S^m} \iint 2t(\theta,\theta') d\mu(\theta) d\mu(\theta'). \end{aligned}$$

This completes the proof of the claim.

Claim 2: (x, t') **leaves the buyers' payoffs unchanged under truth-telling**

For all buyers $\theta \notin \widetilde{\Theta}$, the mechanisms (x, t) and (x, t') are identical under truth-telling. So their payoffs are trivially identical. Now, consider an arbitrary buyer $\tilde{\theta} \in \widetilde{\Theta}$. Under truth-telling, the payoff for this buyer is:

$$\begin{aligned} & \inf_{\mu\in\Delta_B^m} \int \left[x(\tilde{\theta},\theta')\tilde{\theta} - t'(\tilde{\theta},\theta') \right] d\mu(\theta') \\ = & \inf_{\mu\in\Delta_B^m} \int \left[x(\tilde{\theta},\theta')\tilde{\theta} - t(\tilde{\theta},\theta') - \delta(\tilde{\theta},\theta') \right] d\mu(\theta') \\ = & \inf_{\mu\in\Delta_B^m} \int \left[q(\tilde{\theta},\theta') - q(\tilde{\theta},\theta') + K(\tilde{\theta}) \right] d\mu(\theta') = K(\tilde{\theta}) \end{aligned}$$

where $K(\tilde{\theta})$ is buyer $\tilde{\theta}$'s original payoff.

Claim 3: (x, t') **is incentive compatible**

For $\theta \notin \widetilde{\Theta}$, $(x(\theta,\theta'), t'(\theta,\theta'))$ is identical to $(x(\theta,\theta'), t(\theta,\theta'))$. Furthermore, truth-telling payoffs for all types are unchanged. This immediately implies that, since (x,t) was incentive compatible, no θ can profitably deviate to any $\hat{\theta} \notin \widetilde{\Theta}$. Hence we only check if any type has an incentive to deviate to some $\tilde{\theta} \in \widetilde{\Theta}$.

The payoff for $\theta \in \Theta$ to deviate to an arbitrary $\tilde{\theta} \in \widetilde{\Theta}$, $\theta \neq \tilde{\theta}$, is:

$$\begin{aligned}
& \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t'(\tilde{\theta},\theta') \right] d\mu(\theta') \\
= & \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') - \delta(\tilde{\theta},\theta') \right] d\mu(\theta') \\
\leq & \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') \right] d\mu(\theta') - \inf_{\mu \in \Delta_B^m} \int \delta(\tilde{\theta},\theta') d\mu(\theta')
\end{aligned}$$

Because

$$\begin{aligned}
& \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') - \delta(\tilde{\theta},\theta') + \delta(\tilde{\theta},\theta') \right] d\mu(\theta') \\
\geq & \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') - \delta(\tilde{\theta},\theta') \right] d\mu(\theta') + \inf_{\mu \in \Delta_B^m} \int \delta(\tilde{\theta},\theta') d\mu(\theta')
\end{aligned}$$

But note that

$$\begin{aligned}
\inf_{\mu \in \Delta_B^m} \int \delta(\tilde{\theta},\theta') d\mu(\theta') &= \inf_{\mu \in \Delta_B^m} \int \left[q(\tilde{\theta},\theta') - K(\tilde{\theta}) \right] d\mu(\theta') \\
&= \inf_{\mu \in \Delta_B^m} \int q(\tilde{\theta},\theta') d\mu(\theta') - K(\tilde{\theta}) = 0
\end{aligned}$$

This implies that

$$\begin{aligned}
& \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t'(\tilde{\theta},\theta') \right] d\mu(\theta') \\
\leq & \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') \right] d\mu(\theta') - \inf_{\mu \in \Delta_B^m} \int \delta(\tilde{\theta},\theta') d\mu(\theta') \\
= & \inf_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta') \right] d\mu(\theta').
\end{aligned}$$

Now the payoff for type θ to truth-telling must be weakly larger than the last expression, because the mechanism (x,t) was assumed to be incentive compatible. Hence the payoff to truth-telling is weakly larger than deviating to any type $\tilde{\theta} \in \widetilde{\Theta}$. Hence the mechanism (x, t') is incentive compatible.

Since the mechanism (x, t') makes the seller strictly better off, leaves buyers' payoffs unchanged, and remains incentive compatible, the mechanism (x, t) must not have been optimal. This completes the proof. ■

The next result shows that there is always a perfect insurance mechanism within the set of optimal mechanisms whenever this set is nonempty.

Proposition 12 *When an optimal mechanism exists there exists an auction, (x^*, t^*), that maximizes the seller's revenue such that the payoff of any type of a bidder in his auction is constant as a function of the other bidder's report. That is for all $\theta \in \Theta$, $x^*(\theta, \theta')\theta - t^*(\theta, \theta')$ is constant in θ'.*

The proof is omitted since it is essentially identical to the proof of proposition 11.

Proof of Lemma 3

Proof. We need to show that there exists $M > 0$ such that

$$\left| u(\theta) - u\left(\tilde{\theta}\right) \right| \leq M \left| \theta - \tilde{\theta} \right|.$$

We know that,

$$\left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \leq u(\theta) - u\left(\tilde{\theta}\right) \leq \left(\theta - \tilde{\theta}\right) X^{\max}\left(\tilde{\theta}\right).$$

So if $\theta > \tilde{\theta}$, using the fact that u is increasing we can conclude that,

$$u(\theta) - u\left(\tilde{\theta}\right) \leq \left(\theta - \tilde{\theta}\right) X^{\max}\left(\tilde{\theta}\right) \leq \left|\theta - \tilde{\theta}\right|.$$

Similarly if $\theta < \tilde{\theta}$, then

$$-\left(u(\theta) - u\left(\tilde{\theta}\right)\right) \leq -\left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \leq \left|\theta - \tilde{\theta}\right|.$$

Together these imply that Lipschitz condition holds with $M = 1$. ■

Proof of Proposition 5

Proof. For any $\theta \in \bar{\Theta}$ we have

$$\inf_{G \in \Delta_B^m} \int q(\theta, \theta') dG(\theta') = K(\theta).$$

We know by Proposition **??** that there is a measure 1 set $\bar{\Theta} \subseteq \Theta$ such that for all $\theta \in \bar{\Theta}$ we have $F \in \Delta(\theta)$. This means that there exists a sequence $G^n \to F$ such that

$$\int q(\theta, \theta') dG^n(\theta') \to K(\theta).$$

Towards a contradiction, suppose that for some $\bar{\theta} \in \bar{\Theta}$, $q(\bar{\theta}, \theta)$ is not constant for a positive measure set of θ. Let Θ_+ be the set of all $\theta' \in \Theta$ such that $q(\bar{\theta}, \theta') > K(\bar{\theta}) + \delta$ and let Θ_- be the set of all $\theta' \in \Theta$ such that $q(\bar{\theta}, \theta') < K(\bar{\theta}) - \delta$ for some $\delta > 0$. There must exist some $\delta > 0$ such that both Θ_+ and Θ_- have positive measure. Let $\Theta_+ \bigcup \Theta_- = \hat{\Theta}$ and note that $\Theta_+ \bigcap \Theta_- = \emptyset$.

Now construct the following sequence of measures dH^n (corresponding to the distribution H^n) which increases the probability weight on the low-payoff states:

$$dH^n(\theta') = \begin{cases} = \frac{1}{2} dG^n(\theta'), & \text{if } \theta' \in \Theta_+ \\ = (1 + \frac{1}{2}\gamma^n) dG^n(\theta'), & \text{if } \theta' \in \Theta_- \\ = dG^n(\theta'), & \text{if } \theta' \in \Theta - \hat{\Theta} \end{cases}$$

where $\gamma^n = \frac{\int_{\Theta_+} dG^n}{\int_{\Theta_-} dG^n}$[11]. Note that $\Theta - \hat{\Theta}$ could be the empty set. This would not influence our results.

Claim: H^n is a well-defined distribution, in that $H^n(\theta)$ is increasing in θ, and $\int dH^n(\theta') = 1$.

Proof: $H^n(\theta)$ is increasing since dG^n is a measure and dH^n is constructed as a positive multiple of dG^n.

[11]Since Θ_- and Θ_+ are of positive measure, $0 < \gamma < \infty$. If one of these sets were not of positive measure, then there would not be positive probability mass that could be moved relative to F, so no such H could be constructed.

Now

$$\begin{aligned}
\int dH^n(\theta') &= \frac{1}{2}\int_{\Theta_+} dG^n(\theta') + (1+\frac{1}{2}\gamma^n)\int_{\Theta_-} dG^n(\theta') + \int_{\Theta-\hat{\Theta}} dG^n(\theta') \\
&= \frac{1}{2}\int_{\Theta_+} dG^n(\theta') + \int_{\Theta_-} dG^n(\theta') + \frac{1}{2}\frac{\int_{\Theta_+} dG^n}{\int_{\Theta_-} dG^n}\int_{\Theta_-} dG^n(\theta') + \int_{\Theta-\hat{\Theta}} dG^n(\theta') \\
&= \int_{\Theta_+} dG^n(\theta') + \int_{\Theta_-} dG^n(\theta') + \int_{\Theta-\hat{\Theta}} dG^n(\theta') = \int dG^n(\theta') = 1.
\end{aligned}$$

So H^n is indeed a well-defined distribution.

Note that by definition of the ϵ-contamination set for n large enough $(1-\frac{\epsilon}{2})dG^n+\frac{\epsilon}{2}dH^n \in \Delta_B^m$. Since Δ_B^m is weak* compact (by going to a subsequence if necessary) $(1-\frac{\epsilon}{2})dG^n + \frac{\epsilon}{2}dH^n \to d\hat{F} \in \Delta_B^m$, and by construction $\hat{F} \neq F$. Now, we will show that, if $q(\bar{\theta},\theta)$ is not constant for a positive measure set of θ, then the existence of $\left(1-\frac{\epsilon}{2}\right)G^n + \frac{\epsilon}{2}H^n$ in Δ_B^m implies that $F \notin \Delta(\theta)$, which leads to a contradiction. First note that,

$$\int_{\Theta_+} q(\bar{\theta},\theta')dG^n(\theta') > \left(K(\bar{\theta})+\delta\right)\int_{\Theta_+} dG^n,$$

since for all $\theta' \in \Theta_+$, $q(\bar{\theta},\theta') > K(\bar{\theta})+\delta$.Similarly,

$$\int_{\Theta_-} q(\bar{\theta},\theta')dG^n(\theta') < \left(K(\bar{\theta})-\delta\right)\int_{\Theta_-} dG^n,$$

since for all $\theta' \in \Theta_-$, $q(\bar{\theta},\theta') < K(\bar{\theta})+\delta$. Putting these two inequalities together we obtain,

$$\begin{aligned}
\frac{\int_{\Theta_-} q(\bar{\theta},\theta')dG^n(\theta')}{\int_{\Theta_-} dG^n} + \delta &< \frac{\int_{\Theta_+} q(\bar{\theta},\theta')dG^n(\theta')}{\int_{\Theta_+} dG^n} - \delta \Leftrightarrow \\
\frac{\int_{\Theta_+} dG^n}{\int_{\Theta_-} dG^n}\int_{\Theta_-} q(\bar{\theta},\theta')dG^n(\theta') &< \int_{\Theta_+} q(\bar{\theta},\theta')dG^n(\theta') - 2\delta \Leftrightarrow \\
\gamma^n\int_{\Theta_-} q(\bar{\theta},\theta')dG^n(\theta') &< \int_{\Theta_+} q(\bar{\theta},\theta')dG^n(\theta') - 2\delta \qquad (16)
\end{aligned}$$

Now we show that,

$$\begin{aligned}
&\liminf_{n\to\infty} \int q(\bar{\theta},\theta')\left[(1-\frac{\epsilon}{2})dG^n(\theta') + \frac{\epsilon}{2}dH^n(\theta')\right] \\
\leq\ & (1-\frac{\epsilon}{2})K(\bar{\theta}) + \frac{\epsilon}{2}\liminf \int q(\bar{\theta},\theta')dH^n(\theta') < K(\bar{\theta}),
\end{aligned}$$

which proves the claim that $F \notin \Delta(\theta)$. To prove this we need to show that,

$$\lim_{n\to\infty} \inf \int q(\bar{\theta},\theta')dH^n(\theta') < K(\bar{\theta})$$

or

$$\lim_{n\to\infty} \inf \int q(\bar{\theta},\theta')dH^n(\theta') < \lim_{n\to\infty} \inf \int q(\bar{\theta},\theta')dG^n(\theta').$$

But,

$$\begin{aligned} & \int q(\bar{\theta},\theta')dH^n(\theta') - \int q(\bar{\theta},\theta')dG^n(\theta') \\ = & \frac{1}{2}\gamma^n \int_{\Theta_-} q(\bar{\theta},\theta')dG^n(\theta') - \frac{1}{2}\int_{\Theta_+} q(\bar{\theta},\theta')dG^n(\theta') \\ < & -\delta. \end{aligned}$$

where the last inequality follows from 16. We can rewrite this as:

$$\int q(\bar{\theta},\theta')dH^n(\theta') + \delta < \int q(\bar{\theta},\theta')dG^n(\theta').$$

Taking limits,

$$\lim_{n\to\infty} \inf \int q(\bar{\theta},\theta')dH^n(\theta') + \delta \leq \lim_{n\to\infty} \inf \int q(\bar{\theta},\theta')dG^n(\theta').$$

This proves the claim that $F \notin \Delta(\theta)$ which leads to a contradiction. Therefore for almost all θ, $q(\theta,\theta')$ must be constant for almost all θ' which consludes the proof. ■

Proof of Proposition 8

Proof. Define $T(\theta)$ as buyer θ's expected transfer under F, that is,

$$T(\theta) = \int t(\theta,\theta')dF(\theta') \tag{17}$$

Now let

$$\tilde{t}(\theta,\theta') = T(\theta) - T(\theta') + \int T(i)dF(i) \tag{18}$$

We show that the mechanism $(x,\tilde{t})$ makes the seller strictly better off, leaves the buyers' payoffs unchanged, and is incentive compatible.

Claim 1: $(x, \tilde{t})$ **makes the seller strictly better off**

TThe seller's payoff for the mechanism (x, t') is:

$$
\begin{aligned}
& \min_{G \in \Delta_S} \iint \left[\tilde{t}(\theta, \theta') + \tilde{t}(\theta', \theta)\right] dG(\theta) dG(\theta') \\
= & \min_{G \in \Delta_S} \iint [T(\theta) - T(\theta') + \int T(i) dF(i) + T(\theta') - T(\theta) + \int T(j) dF(j)] dG(\theta) dG(\theta') \\
= & \min_{G \in \Delta_S} \iint \left[2 \int T(i) dF(i)\right] dG(\theta) dG(\theta') \\
= & \ 2 \int T(i) dF(i) \\
= & \ 2 \iint t(\theta, \theta') dF(\theta) dF(\theta')
\end{aligned}
$$

However, $F \notin \Delta_S^{\min}$ implies:

$$
\min_{G \in \Delta_S} \iint [t(\theta, \theta') + t(\theta', \theta)] dG(\theta) dG(\theta') < 2 \iint t(\theta, \theta') dF(\theta) dF(\theta')
$$

Hence the seller is made strictly better off by this change.

Claim 2: $(x, \tilde{t})$ **leaves the buyers' payoffs unchanged under truth-telling**

By construction:

$$
\begin{aligned}
\int \tilde{t}(\theta, \theta') dF(\theta') &= \int \left[T(\theta) - T(\theta') + \int T(i) dF(i)\right] dF(\theta') \\
&= T(\theta) - \int T(\theta') dF(\theta') + \int T(i) dF(i) \\
&= T(\theta)
\end{aligned}
$$

Since $T(\theta) = \int t(\theta, \theta') dF(\theta')$, we are finished.

Claim 3: (x, t') **is incentive compatible**

The payoff for type θ to pretend to be $\tilde{\theta}$ remains unchanged. Namely, θ's deviation payoff under (x, t) is:

$$
\begin{aligned}
& \int \left[\theta x(\tilde{\theta}, \theta') - t(\tilde{\theta}, \theta')\right] dF(\theta') \\
= & \int \theta x(\tilde{\theta}, \theta') dF(\theta') - \int t(\tilde{\theta}, \theta') dF(\theta') \\
= & \int \theta x(\tilde{\theta}, \theta') dF(\theta') - T(\tilde{\theta}) \\
= & \int \theta x(\tilde{\theta}, \theta') dF(\theta') - \int \tilde{t}(\tilde{\theta}, \theta') dF(\theta')
\end{aligned}
$$

as per the above proof. Hence the payoff to deviation is unchanged for all types $\theta \in \Theta$, so, since (x, t) was incentive compatible, $(x, \tilde{t})$ must be as well.

Since the mechanism (x, t') makes the seller strictly better off, leaves buyers' payoffs unchanged, and remains incentive compatible, the mechanism (x, t) must not have been optimal. This completes the proof. ■

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