Foundations of Dominant Strategy Mechanisms*

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Abstract

Wilson (1987) criticizes the existing literature of game theory as relying too much on common-knowledge assumptions. In reaction to Wilson's critique, the recent literature of mechanism design has started employing simpler mechanisms such as dominant strategy mechanisms. However, there has been little theory behind this approach. In particular, it has not been made clear why, when a mechanism designer is not willing to make strong common-knowledge assumptions, she would respond by using simpler mechanisms instead of even more complicated ones. This paper tries to fill the void and investigates some foundations for using simpler mechanisms such as dominant strategy mechanisms.

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1 Introduction

In the recent literature of mechanism design, there is a research agenda which is motivated by the so-called *Wilson Doctrine*. Roughly speaking, the Wilson Doctrine refers to the vision, articulated in Wilson (1987), that a good theory of mechanism design should not rely too heavily on assumptions of common knowledge:

> "Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one agent's probability assessment about another's preferences or information. [...] I foresee the progress of game theory as depending on successive reduction in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality."

Although there is no clear prescription from Wilson (1987) on how exactly to reduce the dependence on common knowledge assumptions, the methodology on which the literature has converged is to impose strong solution concepts which minimize the impact of any such assumption. For example, when Dasgupta and Maskin (2000) and Perry and Reny (2002) design efficient auctions in interdependent-value settings, they insist that their designs are ex post incentive compatible. Similarly, when Segal (2003) designs optimal auctions in private-value settings, he also insists that his designs are dominant strategy incentive compatible. Both ex post incentive compatibility and dominant strategy incentive compatibility are stronger solution concepts compared with Bayesian incentive compatibility, which in turn is the solution concept used in the traditional literature on mechanism design. Of course, requiring a stronger standard of incentive compatibility limits, in general, what can be achieved.

In this paper, we investigate the foundations of this methodology. We focus on private-value auctions, and ask whether or not there can be a rational basis for restricting attention to dominant strategy mechanisms.

A priori, it is not apparent at all why, when a mechanism designer is not willing to make strong common knowledge assumptions, she would respond by using *simpler* mechanisms such as dominant strategy mechanisms, as opposed to mechanisms that are even more complicated. In principle, a mechanism designer can ask her agents anything that she does not know, and she should do so if the answers are potentially useful. For example, if she is not sure whether a certain common knowledge assumption is true, she can (and probably should) add to her original mechanism an additional question concerning the validity of this common knowledge assumption. The fewer assumptions the mechanism designer is willing to make, the more questions she should ask, and hence the more complicated her mechanism should be. Pushing this logic to its extreme, if we were ever to achieve Wilson's ideal of "successive reduction" in the dependence of common knowledge assumptions, we would envision mechanisms that are so complicated that they ask agents to report *everything*. At the limit, mechanisms would become so complicated that they ask agents to report their whole infinite hierarchies of beliefs and higher-order beliefs, or in other words to report their universal types. It seems that the suggestion of using simpler mechanisms such as dominant strategy mechanisms is squarely at odds with this established intuition in the literature of mechanism design.

In this paper, we shall provide a rationale for using dominant strategy mechanisms which confronts this intuition head-on. Our theory is based on the following often-repeated informal motivation.[1] Imagine the mechanism designer as an auctioneer. She may have confidence in her estimate of the distribution ν of the bidders' valuations, perhaps based on data from similar auctions in the past. But she does not have reliable information about the bidders' beliefs (including their beliefs about one another's valuations, their beliefs about these beliefs, etc.), as these are arguably never observed. She can choose *any* selling mechanism. On the one hand, she could select a simple mechanism, asking the bidders to report only their valuations and ensuring that it is a dominant strategy to do so. Alternatively, she can choose to use some Bayesian incentive compatible mechanism that allows her to ask the bidders anything about their beliefs that might be relevant. It is well known that a dominant strategy mechanism secures a fixed expected revenue, independent of the bidders' actual beliefs. On the other hand, a Bayesian incentive compatible mechanism that performs well under certain common knowledge assumptions may perform badly if those assumptions turn out to be false. If the auctioneer is not sufficiently confident in any such assumption to stake the performance of the mechanism on it, she may optimally choose to use dominant strategies.

We call this story the *maxmin* foundation of dominant strategy mechanisms, because the auctioneer chooses among mechanisms according to their worst-case performance. Formally, the theorem we are seeking is illustrated in Figure 1. In Figure 1, we (heuristically) plot the performance of arbitrary Bayesian incentive compatible mechanisms against different assumptions about bidders' beliefs. The graph of any dominant strategy mechanism—and in particular the graph of the best one among all dominant strategy mechanisms—will be a horizontal line. To establish the maxmin foundation, we would need to show that the graph of any (potentially very complicated) Bayesian incentive compatible mechanism must dip below the graph of the best dominant strategy mechanism at some point.

Figure 1, although we believe captures the imagination of many advocates of dominant strategy mechanisms, turns out to be very difficult to prove in general. With no restriction on the environment, the set of all potentially useful mechanisms is quite rich, and it would be contrary to the spirit of our investigation to impose exogenous restrictions on the complexity of the mechanism.

Instead, in this paper, we introduce a sufficient condition on the distribution of bidders' *valuations* (recall that the auctioneer has confidence in the distribution of bidders' valuations although not in the distribution of bidders' beliefs). The condition generalizes to the case of an arbitrary (possibly correlated) ν what Myerson (1981) calls the "regular case" in his classical paper on optimal auctions with independent types. It is a familiar condition in the

[1]See, for example, Segal (2003) sec.VI, who conjectures a result similar to ours.

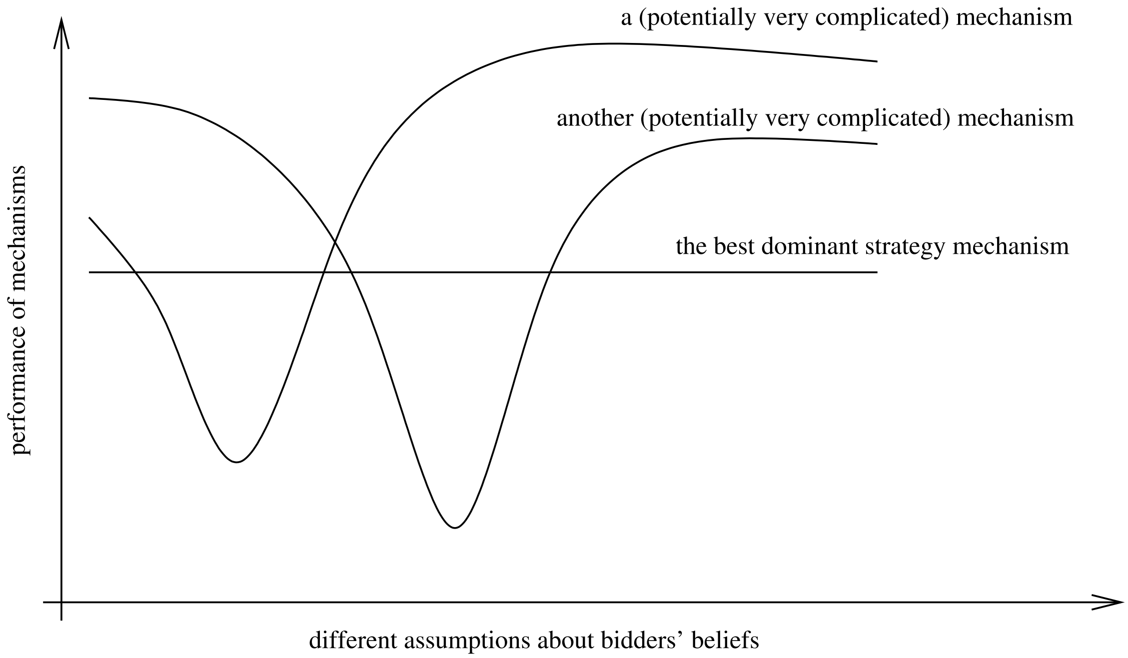

Figure 1: the graph of any mechanism dips below the graph of the best dominant strategy mechanism at some point.

literature of mechanism design and comfortably assumed in many applications.

In fact, under our condition, we are able to prove a stronger result (see Figure 2): there will be a particular distribution of bidders' beliefs, at which point the graph of *every* (potentially very complicated) Bayesian incentive compatible mechanism must dip below the graph of the best dominant strategy mechanism.

Clearly Figure 2 implies the maxmin foundation we seek. In addition, Figure 2 is significant in its own right. To expand on this, let us think about the auctioneer in a different, perhaps more standard, context.

Imagine the auctioneer as a Bayesian decision maker. When she needs to choose a mechanism under uncertainty about the bidders' beliefs, she forms a subjective belief about bidders' beliefs, and compares different mechanisms by calculating the expected performance with respect to that subjective belief. When we as outside observers observe that this auctioneer chooses a mechanism in dominant strategies, we can ask whether or not such a choice is consistent with Bayesian rationality; i.e., whether or not such a choice is optimal with respect to *some* subjective beliefs. If so, we say that there is a Bayesian foundation for dominant strategies. Our result (Figure 2) implies that, in the regular case, dominant strategy auctions have a Bayesian foundation.

If there exists a subjective belief for the auctioneer against which the optimal mechanism is in dominant strategies, we shall say that the belief *rationalizes* dominant strategies. Note that the existence of a rationalizing belief (Figure 2) is a stronger requirement than the maxmin foundation (Figure 1). We mentioned previously that we do not know whether the

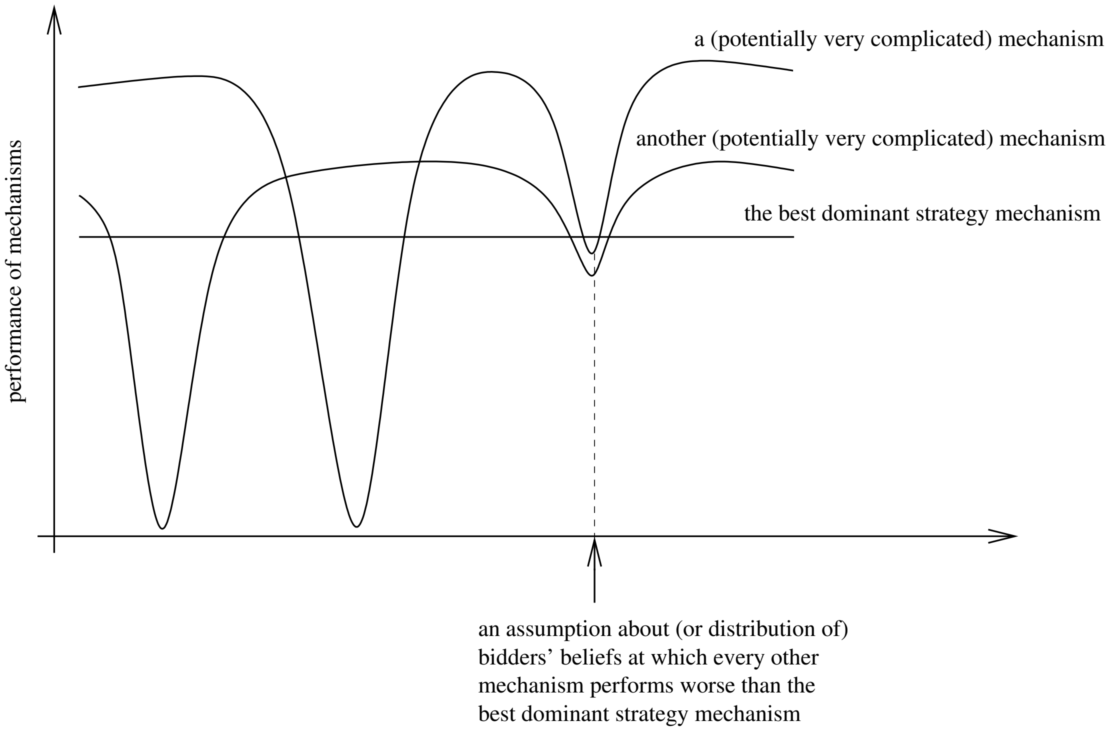

Figure 2: there is a particular point at which the graph of every mechanism dips below the graph of the best dominant strategy mechanism.

maxmin foundation is valid in general (beyond the regular case). However, we do show by example below that beyond the regular case, a Bayesian foundation need not exist. As a negative result about the rationality of imposing dominant strategies, we view this as particularly strong: for some distributions of valuations, no Bayesian expected-revenue maximizing auctioneer would optimally employ a dominant strategy mechanism, regardless of her beliefs.

Section 2 of this paper presents the model and formalizes the problem. Section 3 then uses a two-bidder two-valuation example to illustrate our proof of Figure 2. In any two-bidder two-valuation example, the regular case holds as long as there is no unambiguous strong bidder; i.e., bidder 1's low valuation is lower than bidder 2's high valuation, and vice versa. Our main result will be presented and proved in Section 4. Section 5 presents an example to show that a Bayesian foundation for dominant strategies is in general impossible. In Section 6, we shall make some remarks on the common prior assumption. Section 7 then concludes the paper with an observation about the English auction.

1.1 Related Literature

This paper is not the first to offer a foundation for dominant strategy mechanisms. Bergemann and Morris (2003) offers an alternative foundation for ex post incentive compatible mechanisms, which in private-value settings are equivalent to dominant strategy mechanisms. The main difference between Bergemann and Morris (2003) and the present paper concerns the type of mechanism design problem considered. Bergemann and Morris (2003) focus exclusively on mechanisms in which the outcome can depend only on payoff-relevant data. These mechanisms are naturally suited to study *efficient* design. On the other hand, we are interested here in revenue maximization for a seller. The optimal mechanism for such a designer will almost always depend not just on the valuations, but also payoff-irrelevant data such as beliefs and higher-order beliefs.[2] This is why the results of Bergemann and Morris (2003) do not apply in our setting.

Neeman (2003) is similar in spirit in that he performs a worst-case assessment of the English auction (a dominant-strategy mechanism). He compares the revenue generated by the English auction to the benchmark of full-surplus extraction. The ratio of these two values he calls "effectiveness" and he shows that the effectiveness of the English auction can be fairly high, and in fact close to 1 for a wide variety of distributions of valuations. The benchmark of full-surplus extraction was used despite the fact that this benchmark may not be feasible even for the optimally chosen mechanism,[3] mainly because determining the optimal auction for an environment as general as he considers is a daunting task. One contribution of the present paper is to show how to derive the optimal auction *in the worst-case assumption* about bidder's beliefs. We are thus able to compare dominant strategy mechanisms with the optimal auction benchmark and show that the optimal dominant strategy auction performs at least as well in the worst-case. We discuss another connection with Neeman (2003) in footnote 11 after we introduce the regular case.

2 Preliminaries

2.1 Notation

If $\{X_i\}_{i=1}^N$ is a collection of sets, then X denotes the Cartesian product $\times_i X_i$, or the set of "profiles" of elements of $\{X_i\}$. We write $X_{-i} = \times_{j \neq i} X_j$ and if $x \in X$, then x_i refers to the ith co-ordinate, and we use x_{-i} to denote the element of X_{-i} obtained by removing x_i. If Y is a measurable set, then ΔY is the set of all probability measures on Y. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 Types

A single unit of an indivisible object is up for sale. There are N risk-neutral bidders with privately known valuations competing for the object. Each bidder has M possible valuations and for notational simplicity, we suppose that the set V_i of possible valuations is the same for each bidder i and that $V_i = \{v^1, v^2, \ldots, v^M\}$ where $v^m - v^{m-1} = \Delta$ for each m.[4] A bidder

[2] For instance, see the auctions depicted below in Figures 8, 9, 10, 11, 12, and 13.

[3] In fact, Neeman (2004) showed this.

[4] These notational conventions simplify the statements of results and notation, but are entirely innocuous. Assumptions of asymmetry in the bidders' valuation sets, or differing gaps between valuations would not

i with valuation v_i receives expected utility $p_i v_i - t_i$ if p_i is the probability with which he will be awarded the object and if his expected monetary payment is t_i. A typical element of V is v, and a typical element of V_{-i} is v_{-i}.

To characterize the (equilibrium) behavior of the bidders who compete in some given auction mechanism, it is not enough to specify the bidders' possible valuations or even the probability distribution from which they are drawn. In addition, we must also specify their beliefs about the valuations of their opponents (called the *first-order* beliefs), their beliefs about one another's' first-order beliefs (*second-order* beliefs), etc.

The standard approach to modeling the bidders' information is to use a type space. For each bidder i, there is a (measurable) set of *types* Ω_i. A typical element of Ω_i is a pair $\omega_i = (v_i, \rho_i)$, where $v_i \in V_i$ refers to the valuation of type ω_i, and ρ_i is an element of $\Delta\Omega_{-i}$, i.e. a belief about the types of the other bidders.

A type space is a parsimonious way to describe the beliefs and higher-order beliefs of the bidders. Observe that we can compute the first-order beliefs of a given type (v_i, ρ_i) from the marginal of ρ_i on V_{-i}. Once we have done this for each type of each bidder, we can then use bidders $-i$'s first-order beliefs to compute bidder i's second-order belief, and this can be repeated to compute all higher-order beliefs.

One simple kind of type space is the *naive type space*[5], which we shall denote by Ω^ν, generated from some distribution ν over the set of payoff-relevant types V. In the naive type space, each bidder believes that bidders' valuations are drawn from the distribution ν, and this is common-knowledge. In the formal notation of type spaces introduced above, this is modeled as follows. For each $v_i \in V_i$, there is a unique type $\omega^{v_i} = (v_i, \rho^{v_i})$. The belief ρ^{v_i} is defined in two steps: first the conditional probability $\nu(\cdot|v_i)$ over valuations of the other bidders is derived from ν, then this is transformed in the natural way into a belief over the other bidders' types: $\rho^{v_i}(\omega^{v_{-i}}) = \nu(v_{-i}|v_i)$.

The naive type space is used almost without exception in auction theory and mechanism design. The cost of this parsimonious model is that it implicitly embeds some strong assumptions about bidders' beliefs, and these assumptions are not innocuous.[6] These issues have been raised in Neeman (2004), Bergemann and Morris (2003), and Heifetz and Neeman (2004). The spirit of the Wilson doctrine is to avoid making such assumptions.

The now-widely adopted response to this is to diminish the impact of these assumptions by imposing stronger solution concepts which are not sensitive to the specifics of bidders' beliefs.[7] Our interest in this paper is to provide a foundation for this approach and so we will

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affect any of our results.

[5]This terminology originated in Bergemann and Morris (2003).

[6]For example, if the bidders' valuations are independent under ν, then in the naive type space, the bidders' beliefs are commonly known. On the other hand, for a generic ν, it is common-knowledge that there is a one-to-one correspondence between valuations and beliefs. Which of these cases holds makes a big difference for the structure and welfare properties of the optimal auction. See Myerson (1981) for the independent case and Crémer and McLean (1985) for the other case.

[7]For examples of this methodology, see Dasgupta and Maskin (2000), Lopomo (2000), Perry and Reny (2002), and Segal (2003), among others.

discuss it further below, but let us first consider the alternative approach. Rather than fixing the type space and strengthening the solution concept, we might instead enlarge the type space to include more and more possible beliefs. Larger and larger type spaces correspond to weaker and weaker assumptions about the bidders' beliefs.

Indeed, we can remove these assumptions altogether by considering the *universal* type space in which for every valuation and every conceivable (coherent) hierarchy of higher-order beliefs there is a representative type. Specifically, we construct the universal belief space from the basic payoff-relevant data as follows (the construction is standard, see Mertens and Zamir (1985) for the details and Brandenburger and Dekel (1993) for an alternative derivation).

The set of possible first-order beliefs for bidder i is

$$\mathcal{T}_i^1 := \Delta V_{-i},$$

and the set of all possible kth-order beliefs is

$$\mathcal{T}_i^k := \Delta(V_{-i} \times \mathcal{T}_{-i}^{k-1}).$$

Because the set ΔX is a compact metric space whenever X is, by induction each $\mathcal{T}_i^k$ is a compact metric space. The projections $\phi_i^k : \mathcal{T}_i^k \to \mathcal{T}_i^{k-1}$, defined inductively by $\phi_i^2(\tau_i^2)(v_{-i}) = \tau_i^2(\{v_{-i}\} \times \mathcal{T}_{-i}^1)$, and for each measurable subset $\{v_{-i}\} \times B \subset V_{-i} \times \mathcal{T}_{-i}^{k-2}$,

$$\phi_i^k(\tau_i^k)(\{v_{-i}\} \times B) = \tau_i^k(\{v_{-i}\} \times \left[\phi_{-i}^{k-1}\right]^{-1}(B)),$$

demonstrate that each kth-order belief for bidder i implicitly defines beliefs at lower orders as well.

A universal belief type for bidder i is a sequence (or *hierarchy*) $\tau_i = (\tau_i^1, \tau_i^2, \ldots)$ satisfying $\tau_i^k \in \mathcal{T}_i^k$ and the *coherency* condition that $\phi_i^k(\tau_i^k) = \tau_i^{k-1}$. The universal belief space for bidder i is then the set $\mathcal{T}_i^* \subset \prod_{k=1}^{\infty} \mathcal{T}_i^k$ of all such coherent hierarchies. This product space endowed with the product topology is compact. Since the set of coherent hierarchies is closed, the universal belief space is compact. By Mertens and Zamir (1985) and Brandenburger and Dekel (1993), there is a homeomorphism between $\mathcal{T}_i^*$ and $\Delta(V_{-i} \times \mathcal{T}_{-i}^*)$, so we may treat them interchangeably. Let $g_i : \mathcal{T}_i^* \to \Delta(V_{-i} \times \mathcal{T}_{-i}^*)$ be such a mapping.

A *type* is a pair $\omega_i = (v_i, \tau_i)$. Let $f_i(\omega_i) = v_i$ be the projection from bidder i's type to bidder i's valuation. A type space is a set $\Omega = \prod_{i=1}^N \Omega_i$, where $\Omega_i \subset V_i \times \mathcal{T}_i^*$. In this paper, we will mainly deal with two varieties of type spaces. The naive type space has already been introduced. The *universal type space* Ω^* is the type space where each $\Omega_i^* = V_i \times \mathcal{T}_i^*$. Let $\mathcal{T}^* = \prod_{i=1}^N \mathcal{T}_i^*$. For any $v \in V$, we shall write $\Omega^*(v)$ for the open subset $\{v\} \times \mathcal{T}^* \subset \Omega^*$.

Once the information of the bidders' has been specified through the choice of type space, the seller's problem is to design a selling procedure in order to maximize revenue. We turn to this in the next subsection.

2.3 Mechanisms

An auction mechanism consists of a set M_i of *messages* for each bidder i, an allocation rule $p : M \to [0,1]^N$, and a payment function $t : M \to \mathbf{R}^N$. Each bidder will select a message from his set M_i, and based on the resulting profile of messages m, the object is awarded according to $p(m)$ and payments are exacted according to $t(m)$. Player i receives the object with probability $p_i(m)$ and pays $t_i(m)$ to the seller.

We consider environments in which the seller cannot compel the bidders to participate in the auction, so we require that each M_i includes the *non-participation* message $\emptyset_i$. Selecting $\emptyset_i$ is equivalent to "opting-out" of the auction and so we assume that for any profile m in which $m_i = \emptyset_i$, the allocation and payments rules satisfy $p_i(m) = 0$ and $t_i(m) = 0$. A direct revelation mechanism for a given type space Ω is one in which $M_i = \Omega_i \cup \{\emptyset_i\}$.

The auction mechanism defines a game-form, which together with the type space constitutes a game of incomplete information. The auction design problem is to fix a solution concept and search for the auction mechanism that delivers the maximum revenue for the seller in some outcome consistent with the solution concept. The now-widely adopted approach to implement the Wilson-doctrine and minimize the role of assumptions built into the naive type space is to adopt a strong solution concept which does not rely on these assumptions. In our private-value setting the often-used solution concept for this purpose is dominant-strategy equilibrium.

By the revelation principle, the set of dominant-strategy equilibrium outcomes of auction mechanisms on the naive type space is equal to the set of truth-telling outcomes of dominant-strategy incentive compatible (dsIC) direct-revelation mechanisms.

Definition 1 *A direct-revelation mechanism Γ is dominant strategy incentive compatible with respect to the naive type space Ω^ν (or simply dsIC) if for each bidder i and type profile $\omega \in \Omega^\nu$,*

$$
\begin{aligned}
&p_i(\omega)v_i - t_i(\omega) \geq 0, \qquad \textit{and} \\
&p_i(\omega)v_i - t_i(\omega) \geq p_i(\hat{\omega}_i, \omega_{-i})v_i - t_i(\hat{\omega}_i, \omega_{-i}),
\end{aligned}
$$

for any alternative type $\hat{\omega}_i \in \Omega_i^\nu$.

Since $|\Omega_i^\nu| = |V_i|$, and since the incentive compatibility constraints for dsIC depend only on valuations, an auction mechanism is dsIC with respect to a naive type space Ω^ν if and only if it is dsIC with respect to any other naive type space $\Omega^{\nu'}$. So we can always discuss whether an auction mechanism is dsIC with respect to the naive type space without referring to the specific distribution ν from which the naive type space is generated.

To provide a foundation for the indirect approach based on strong solution concepts, we shall compare it to the direct route of completely eliminating assumptions about beliefs. We maintain the solution concept of Bayesian equilibrium but now we enlarge the type space all the way to the universal type space. The revelation principle implies that the set of resulting outcomes is equal to those that arise from truth-telling in Bayesian incentive compatible (BIC) direct-revelation mechanisms.

Definition 2 *A direct-revelation mechanism* Γ *is Bayesian incentive compatible with respect to the universal type space* Ω^* *(or simply BIC) if for each bidder* i *and type* $\omega_i \in \Omega_i^*$,

$$\int_{\Omega_{-i}^*} [p_i(\omega)v_i - t_i(\omega)] g_i(\tau_i)(d\omega_{-i}) \geq 0, \qquad \textit{and}$$

$$\int_{\Omega_{-i}^*} [p_i(\omega)v_i - t_i(\omega)]\, g_i(\tau_i)(d\omega_{-i}) \geq \int_{\Omega_{-i}^*} [p_i(\hat{\omega}_i, \omega_{-i})v_i - t_i(\hat{\omega}_i, \omega_{-i})]\, g_i(\tau_i)(d\omega_{-i}),$$

for any alternative type $\hat{\omega}_i \in \Omega_i^*$.

Note that any auction mechanism Γ that is dominant strategy incentive compatible with respect to the naive type space (i.e., dsIC) can be extended naturally into an auction mechanism that is Bayesian incentive compatible with respect to the universal type space (i.e., BIC) in a straightforward manner. We shall abuse notation and use Γ to denote this natural extension as well.

For either formulation of the problem, the revelation principle implies that there is no loss of generality in restricting attention to direct-revelation mechanisms, provided they satisfy the corresponding constraints. We shall do so in the remainder of the paper.

2.4 The Auctioneer as a Maxmin Decision Maker

When we start with the universal type space, we remove any implicit assumptions about the bidders' beliefs. We can now explicitly model any such assumption as a probability distribution over the bidders' universal types. Specifically, let μ be a distribution over Ω^*. For any BIC auction Γ, the performance of Γ under assumption μ, or the μ-expected revenue, is defined as $R_\mu(\Gamma) = \int_{\Omega^*} \bar{t}\, \mu(d\omega)$.

We take as given some full-support distribution ν over Ω^ν. This represents the auctioneer's estimate of the bidders' valuations. An assumption that is consistent with this estimate is a distribution μ on the universal type space Ω^* whose marginal on V is ν. Let $\mathcal{M}(\nu)$ denote the compact subset of such assumptions. Observe that there is a unique element ν^* in this subset that concentrates on the naive type space Ω^ν generated by ν. This represents the (common knowledge) assumption in the traditional literature that Wilson (1987) refers to. Unlike the standard formulation of the optimal auction design problem, we do not assume that the auctioneer has confidence in this particular assumption ν^*. Rather the auctioneer considers other assumptions within the set $\mathcal{M}(\nu)$ as possible as well.

A cautious auctioneer who chooses an auction that maximizes the worst-case performance is hence solving the *maxmin*[8]

$$\sup_{\Gamma \text{ is BIC}} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma). \tag{1}$$

Note that if an auction Γ is dominant strategy incentive compatible with respect to the naive type space (i.e., dsIC), then for any assumption $\mu \in \mathcal{M}(\nu)$, the μ-expected revenue of Γ—or, more precisely, Γ's natural extension into the universal type space—depends only on the distribution ν. Hence we can write $R_\mu(\Gamma)$ as $R_\nu(\Gamma)$ without confusion.

Given any distribution ν over V, the *optimal dsIC revenue* is defined as

$$\Pi^D(\nu) := \sup_{\Gamma \text{ is dsIC}} R_\nu(\Gamma).$$

The maxmin foundation of dominant strategy mechanisms refers to the following equation:

$$\Pi^D(\nu) = \sup_{\Gamma \text{ is BIC}} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma), \tag{2}$$

for every distribution ν over valuations.

In this paper, instead of proving that equation (2) holds for every ν, we shall prove that it hold for every ν satisfying a sufficient condition called regularity (to be defined in Section 4). Specifically, we shall prove that, whenever ν is regular, there will exist an assumption $\mu^* \in \mathcal{M}(\nu)$, under which we will have

$$\Pi^D(\nu) = \sup_{\Gamma \text{ is BIC}} R_{\mu^*}(\Gamma), \tag{3}$$

which implies

$$\begin{aligned}
\Pi^D(\nu) = \sup_{\Gamma \text{ is BIC}} R_{\mu^*}(\Gamma) &\geq \inf_{\mu \in \mathcal{M}(\nu)} \sup_{\Gamma \text{ is BIC}} R_\mu(\Gamma) \\
&\geq \sup_{\Gamma \text{ is BIC}} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma) \\
&\geq \sup_{\Gamma \text{ is dsIC}} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma) \geq \sup_{\Gamma \text{ is dsIC}} R_\nu(\Gamma) =: \Pi^D(\nu),
\end{aligned}$$

or simply

$$\Pi^D(\nu) = \sup_{\Gamma \text{ is BIC}} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma),$$

which delivers the maxmin foundation as promised.

[8] Another way to think about this formulation of the problem is to view the auctioneer as uncertainty averse. The beliefs of the bidders are ambiguous to the auctioneer and this ambiguity is modeled by supposing that the auctioneer holds all possible priors μ.

3 An Illustrative Example

In this section, we shall use a simple example to illustrate our main result as well as the strategy of proof.

Consider an auction example with two bidders, and each bidders have two possible valuations. Bidders' valuations are correlated according to the distribution ν depicted in Figure 3.

	$v_1 = 4$	$v_1 = 9$
$v_2 = 11$	$3/10$	$1/10$
$v_2 = 5$	$3/10$	$3/10$

Figure 3: The distribution ν of bidders' valuations.

The optimal dsIC auction is depicted in Figure 4. In Figure 4, "$\alpha = i$" is the shorthand for "allocating the object to bidder i" (i.e., $p_i = 1$ and $p_{-i} = 0$), and "$\alpha = 0$" means no sale.

	$v_1 = 4$	$v_1 = 9$
$v_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$
$v_2 = 5$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 1, t_1 = 9, t_2 = 0$

Figure 4: The optimal dsIC auction Γ.

In any two-bidder two-valuation example, the regular case holds as long as there is no unambiguous strong bidder; i.e., bidder 1's low valuation is lower than bidder 2's high valuation, and vice versa. Hence, according to Theorem 1 (to be proved in the next section), there exists an assumption μ^* consistent with the distribution ν such that equation (3) holds.

We construct one such assumption μ^* below, but shall keep our exposition informal. Let a_i (b_i) denote the first-order belief of a high-valuation (low-valuation) type of bidder i that bidder $-i$ has high valuation.

Consider an assumption μ^* which has a 4-point support: for every bidder i, every possible valuation is associated with only one possible belief type. The marginal distribution of μ^* over bidders' valuations and first-order beliefs is as depicted in Figure 5.

The bidders' higher-order beliefs are derived from Figure 5 by induction. For example, for a low-valuation type of bidder 1, his second-order belief assigns probability 2/5 (3/5) to bidder 2 having high (low) valuation and holding first-order belief $a_2 = 1/4$ ($b_2 = 2/5$), and a high-valuation (low-valuation) type of bidder 2 has a third-order belief that assigns probability 3/4 (3/5) to bidder 1 having low valuation and having such a second-order belief, and so on.

	$b_1 = 2/5$	$a_1 = 1/4$
$a_2 = 1/4$	3/10	1/10
$b_2 = 2/5$	3/10	3/10

Figure 5: The auctioneer's belief μ.

It is obvious that this assumption μ^* is consistent with the distribution ν.

Under this assumption μ^*, there are at least two possible ways to improve upon the optimal dominant strategy auction Γ in Figure 4. First, according to μ^*, conditional on bidder 1 having low valuation, the conditional probability that bidder 2 has high valuation is 1/2. This is different from the first-order belief of the low-valuation type of bidder 2, which is $b_1 = 2/5$. So one possible way to improve upon Γ is to bet against the low-valuation type of bidder 1 on bidder 2's types. Second, since high- and low-valuation types of bidder 1 hold different beliefs, another possible way to improve upon Γ is to separate these two types by introducing Crémer-McLean-kind of lotteries and relaxing incentive compatibility constraints. We shall see that neither of these can improve upon Γ.

First, consider introducing any bet (x, y) on bidder 2's type, where x and y are the amount bidder 1 pays the auctioneer in the events bidder 2 has low and high valuations respectively. If the bet is acceptable to both the auctioneer and the low-valuation type of bidder 1, we must have

$$\begin{aligned} (1/2)x + (1/2)y &\geq 0, \quad \text{and} \\ (3/5)(-x) + (2/5)(-y) &\geq 0, \end{aligned}$$

with at least one inequality strict unless $x = y = 0$. But then the high-valuation type of bidder 1 would find the bet acceptable as well, as

$$(3/4)(-x) + (1/4)(-y) = (5/2)[(3/5)(-x) + (2/5)(-y)] + (3/2)[(1/2)x + (1/2)y],$$

which is strictly bigger then the zero rent for the high-valuation type of bidder 1 under the auction Γ. With both high- and low-valuation types of bidder 1 accepting such a bet, such a bet turns sour for the auctioneer, as

$$(3/5)(-x) + (2/5)(-y) \leq 0,$$

and this explains why introducing the first kind of bets does not help.

Second, consider introducing any Crémer-McLean-kind of lottery to separate the high- and low-valuation types of bidder 1. In a dominant-strategy mechanism, when the object is sold to the low-value type, the seller must leave "information rent" to the high-value type. By offering a bet (x, y) about the type of bidder 2, the seller can try to relax the downward incentive-compatibility constraint and sell to the low-valuation type of bidder 1 without leaving extra rent for the high-valuation type. If such a bet is successful then we must have

$$(3/5)(4-x)+(2/5)(-y)\geq 0, \quad \text{and}$$
$$(3/4)(9-x)+(1/4)(-y)\leq 0,$$

where the first inequality follows from the individual rationality constraint of the low-valuation of bidder 1, and the second from the incentive compatibility constraint of the high-valuation type. However, these together imply that any bet like this not profitable for the auctioneer, as

$$(1/2)x+(1/2)y=(2/3)[(3/4)(-x)+(1/4)(-y)]-(5/3)[(3/5)(-x)+(2/5)(-y)]\leq -1,$$

and this explains why introducing the second kind of bets does not help either.

In principle, there may still be other possible ways to improve upon the optimal dsIC auction Γ. But actually there are no more (this requires a proof, which will be the content of Theorem 1). Hence, under the assumption μ^* equation (3) holds, which in turn implies that there exists a maxmin foundation for dsIC in this example.

4 The Main Result

In this section, we shall first review the optimal dsIC auction design problem. We use a version of a standard argument to show that the dominant strategy incentive compatibility constraints can be replaced by a monotonicity constraint on the allocation rule. We then formally define the regularity condition, which in effect says the monotonicity constraint is not binding in the optimal dsIC auction design problem. Finally, we show that the maxmin foundation is valid in the regular case.

4.1 Review of Optimal dsIC Auctions

We can formulate the optimal dsIC auction design problem as follows:

$$\max_{p(\cdot),t(\cdot)} \quad \sum_{v_i\in V}\nu(v)\sum_{i=1}^{N}t_i(v) \tag{4}$$

$$\text{subject to:} \quad \forall\, i=1,\dots,N,\ \forall\, m,l=1,\dots,M,\ \forall\, v_{-i}\in V_{-i},$$

$$p_i(v^m,v_{-i})v^m-t_i(v^m,v_{-i})\geq 0, \qquad \langle DIR_i^m\rangle$$

$$p_i(v^m,v_{-i})v^m-t_i(v^m,v_{-i})\geq p_i(v^l,v_{-i})v^m-t_i(v^l,v_{-i}). \qquad \langle DIC_i^{m\to l}\rangle$$

By some standard manipulations, we shall eliminate some constraints and rewrite the problem in a form that will facilitate comparison with the optimal BIC auction. The following result is standard.

Lemma 1 *Say that an allocation rule p is dsIC if there exists a transfer rule t such that the auction mechanism (p, t) satisfies the constraints in (4). A necessary and sufficient condition for p to be dsIC is the following monotonicity condition:*

$$p_i(v^m, v_{-i}) \geq p_i(v^{m-1}, v_{-i}), \qquad \forall\ m = 2, \ldots, M, \qquad \forall\ v_{-i} \in V_{-i}. \qquad \langle M_i \rangle$$

It follows again from standard arguments that in an optimal dsIC auction, the constraints $\langle DIR_i^1 \rangle$ and $\langle DIC_i^{m \to m-1} \rangle$ are binding and (given that p is monotonic) all other constraints can be ignored. Combining the resulting two equalities, we see that when the other bidders report valuation profile v_{-i}, bidder i's net utility ("rent") will be

$$U_i(v^1, v_{-i}) = 0$$

for type v^1 and

$$U_i(v^m, v_{-i}) = p_i(v^{m-1}, v_{-i})(v^m - v^{m-1}) + U_i(v^{m-1}, v_{-i}) = \Delta \sum_{m'=1}^{m-1} p_i(v^{m'}, v_{-i})$$

for type v^m, $m > 1$. By definition, the total transfer received by the auctioneer is the total surplus generated by any sale of the object less the rent received by the bidders. Thus, an equivalent formulation of the problem is to choose a dsIC (i.e., monotonic) allocation rule to maximize the expected value of this difference:

$$\max_{p(\cdot)} \qquad \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{v_{-i} \in V_{-i}} \nu(v^m, v_{-i}) \left[p_i(v^m, v_{-i}) v^m - \Delta \sum_{m'=1}^{m-1} p_i(v^{m'}, v_{-i}) \right] \qquad (5)$$
$$\text{subject to } \langle M_i \rangle,\ i = 1, \ldots, N.$$

In accordance with Lemma 1, the monotonicity constraint appears as an equivalent expression for dsIC. This constraint may or may not bind at the solution. Below we will provide sufficient conditions on the distribution ν under which it will not bind. It turns out that these conditions also imply that dsIC can be rationalized.

For the moment, ignore the monotonicity constraint in (5), and consider the solution to the corresponding unconstrained problem. Fix a valuation profile v and differentiate the maximand with respect to $p_i(v)$ to obtain the dsIC-analogue of bidder i's "virtual valuation": $v_i - \Delta \sum_{\hat{v}_i > v_i} \nu(\hat{v}_i, v_{-i}) / \nu(v)$. It will be optimal at valuation profile v to award the object for sure to the bidder with the greatest non-negative virtual valuation, with the object going unsold if all virtual valuations are negative. Equivalently, if we use $F_i(v_i, v_{-i}) = \sum_{\hat{v}_i \leq v_i} \nu(\hat{v}_i, v_{-i})$ to denote the cumulative distribution function of i's valuation at profile v_{-i}, bidder i should receive the object if

$$\gamma_i(v) := v_i - \Delta \frac{1 - F_i(v)}{\nu(v)} > \max\{0, \max_{j \neq i} \gamma_j(v)\}$$

and in the event that two or more bidders tie for the greatest non-negative virtual valuation, the tie can be broken arbitrarily.[9]

If the resulting allocation rule satisfies $\langle M_i \rangle$, then the solution to the unconstrained problem also solves the constrained problem.[10] This is guaranteed to be the case if the virtual valuations satisfy the following version of the single-crossing condition.

Single-Crossing Condition *Let $\gamma_0(\cdot) \equiv 0$ denote the auctioneer's value for the object. The virtual valuations satisfy the single-crossing condition if for each v, $i \in \{1, \ldots, N\}$, and $j \in \{0, \ldots, N\}, j \neq i$,*

$$\gamma_i(v) \geq \gamma_j(v) \implies \gamma_i(\hat{v}_i, v_{-i}) > \gamma_j(\hat{v}_i, v_{-i})$$

for every $\hat{v}_i > v_i$.

Under the single-crossing condition, if bidder i is to receive the object at profile v, then for any higher true valuation $\hat{v}_i > v_i$, bidder i will continue to have the greatest virtual valuation and therefore receive the object. This ensures that the monotonicity constraint is satisfied.

When the induced virtual valuations satisfy the single-crossing condition, we say that ν is *regular.* Note that this condition generalizes the regularity condition in Myerson (1981) to the case of non-independent ν. This explains our choice of terminology.

The single-crossing condition is an assumption about the distribution ν. Our main result is that this assumption implies that dsIC is rationalizable. Unfortunately, because the assumption is stated in terms of virtual valuations, it is hard to give an intuitive interpretation. We will therefore also provide sufficient conditions on ν which are more familiar and easy to interpret.

The *monotone hazard rate condition* is satisfied if for each i and v_{-i}, the hazard rate: $h_i(\hat{v}_i | v_{-i}) = \frac{\nu(\hat{v}_i, v_{-i})}{1 - F_i(\hat{v}_i, v_{-i})}$ is an increasing function of $\hat{v}_i$. The valuations are *affiliated* if for each pair of profiles v, v', $\nu(v \vee v') \cdot \nu(v \wedge v') \geq \nu(v) \cdot \nu(v')$, where $v \vee v'$ is the component-wise maximum and $v \wedge v'$ the component-wise minimum of the two valuation vectors.[11]

We prove the following Proposition in Appendix C.

Proposition 1 *1. If the virtual valuations satisfy the single-crossing condition then any solution to the unconstrained problem (5) also satisfies the constraints $\langle M_i \rangle$.*

2. If ν satisfies both the monotone hazard rate condition and affiliation, then the virtual

[9] For related derivations, see Lopomo (2000) and Segal (2003).

[10] If not, then a version of the Myerson "ironing" procedure would have to be used.

[11] Affiliation is a strong form of positive correlation. In the worst-case analysis of Neeman (2003), the distribution of valuations itself was a free variable. He showed that the worst-case distribution of valuations involves *negative* correlation. It is thus not surprising that we use a condition such as affiliation. Furthermore, our counterexample in Section 5 also involves negative correlation. While the performance measure used in Neeman (2003) is not the same as ours, the similarity between this aspect of the two results suggests some deeper connection.

valuations satisfy the single-crossing condition.

4.2 The Possibility of Maxmin Foundation

Theorem 1 *If ν is regular, then dsIC has a maxmin foundation.*

Proof: The structure of the proof is as follows. We begin by supposing that ν is given such that the virtual valuations satisfy the single-crossing condition and an additional condition, called non-singularity. We show that a maxmin foundation exists for dsIC in this case. Next we show that we can find a sequence of such distributions to approach any ν satisfying the hypotheses of the theorem. We then apply a limiting argument to show that a maxmin foundation for dsIC exists for ν.

Given ν, write ν_i^m for the marginal probability of valuation $v_i = v^m$, and write $G_i(m) = \sum_{m'=m}^M \nu_i^{m'}$ for the associated de-cumulative distribution function. Let $\sigma_i^m = \nu(\cdot|v^m)$ be the conditional distribution over the valuations of bidders $j \neq i$ conditional on bidder i having valuation v^m. Say that ν is *non-singular* if the collection of vectors $\{\sigma_i^m\}_{m=1}^M$ is linearly independent.

Suppose ν is non-singular and the virtual valuations satisfy the single-crossing condition. We construct an assumption μ^* which concentrates on M possible types for each bidder. Let $\Omega = \times_i \Omega_i$ be the support of μ^*, with $\Omega_i = \{\omega_i^m\}_{m=1}^M = \{(v^m, \tau_i^m)\}_{m=1}^M$ representing the set of possible types of bidder i under assumption μ^*. The beliefs τ_i^m of these types will be specified next. For each $\omega_j \in \Omega_j$, let $f_j(\omega_j)$ be the valuation of ω_j. Note that f_j is a bijection for all j. For any belief τ over V_{-i}, define a corresponding belief $\pi_i(\tau)$ over Ω_{-i} in the straightforward way: $\pi_i(\tau)(\omega_{-i}) = \tau((f_j(\omega_j))_{j\neq i})$. In what follows, we shall occasionally use the notation τ interchangeably for $\pi_i(\tau)$, and the context will prevent any confusion.

We construct the bidders' beliefs as follows:

$$\forall i, \forall m, \quad \tau_i^m = \frac{1}{G_i(m)} \sum_{m'=m}^{M} \nu_i^{m'} \sigma_i^{m'}.$$

Thus, conditional on having valuation v^m, bidder i's belief over opponents' valuations (and hence types) is a conditional expectation with respect to ν; in particular, it is the average of the auctioneer's beliefs conditional on i having valuation *at least* v^m.[12] Note that the collection $\{\tau_i^m\}_{m=1}^M$ is linearly independent by the non-singularity of ν. The following equivalent recursive definition of τ_i^m is useful:

$$\begin{aligned} \tau_i^M &= \sigma_i^M, \\ \tau_i^m &= \frac{1}{G_i(m)} \left(\nu_i^m \sigma_i^m + G_i(m+1) \tau_i^{m+1} \right), \quad \forall m < M. \end{aligned} \tag{6}$$

[12]Thus, each bidder type has beliefs which are a distortion of those that would be derived from ν, except for the highest valuation type, where there is "no distortion at the top."

Finally, we specify the assumption μ^* about types: $\mu^* = \pi(\nu)$; i.e., $\mu^*(\omega) = \nu(f_i(\omega_i)_{i=1}^N)$. Obviously $\mu^* \in \mathcal{M}(\nu)$. Under this assumption μ^*, the optimal BIC auction design problem is as follows:

$$\max_{p(\cdot),t(\cdot)} \quad \sum_{i=1}^{N} \sum_{\omega\in\Omega} \mu^*(\omega) t_i(\omega) \tag{7}$$

$$\text{subject to:} \quad \forall\, i = 1,\ldots,N,\ \forall\, m = 1,\ldots,M,\ \forall\, l = 1,\ldots,M,$$

$$\tau_i^m \cdot (p_i^m v^m - t_i^m) \geq 0, \qquad \langle IR_i^m \rangle$$

$$\tau_i^m \cdot (p_i^m v^m - t_i^m) \geq \tau_i^m \cdot \left(p_i^l v^m - t_i^l\right). \qquad \langle IC_i^{m\to l} \rangle$$

We have used the shorthand notation p_i^m and t_i^m to refer to the vectors $p_i(\omega_i^m, \cdot)$ and $t_i(\omega_i^m, \cdot)$ respectively in $\mathbf{R}^{M^{N-1}}$, and the inner product notation such as $\tau_i^m \cdot p_i^m$ for the expectations of these vectors with respect to the belief τ_i^m. Note that the IR and IC constraints for all types outside of the support of μ^* have been omitted. This is valid because (i) any BIC mechanism on the universal type space must satisfy the listed constraints, and (ii) any mechanism which satisfies the listed constraints can be extended to a revenue-equivalent mechanism that is BIC over the universal type space. Simply require every type to either opt-out or announce a type within the support of μ^*.

Say that an allocation rule p is BIC if there exists a transfer rule t such that the auction mechanism (p, t) satisfies the constraints in (7). Because the beliefs of the types of each bidder are linearly independent, every allocation rule is BIC. Indeed, by exploiting the differences in beliefs, the incentive compatibility and individual rationality constraints can be satisfied by building into the transfer rule lotteries which have positive expected value to the intended type and arbitrarily large negative expected values to the other types. This kind of construction is due to Crémer and McLean (1985), and we shall omit the details.

While the above argument shows that any allocation rule is implementable by some appropriate choice of transfer rule, we can further sharpen the conclusion and argue that certain constraints in (7) can be manipulated or even ignored without cost to the auctioneer. To begin with, each "upward" incentive constraint (i.e., $\langle IC_i^{m\to l} \rangle$ for $m < l$) can be ignored. Indeed, because bidder i's beliefs are linearly independent, there exists a lottery $\lambda \in \mathbf{R}^{M^{N-1}}$ such that $\tau_i^m \cdot \lambda = 0$ for all $m \geq l$ and $\tau_i^m \cdot \lambda < 0$ for all $m < l$. Since by (6) σ_i^l is a linear combination of τ_i^l and τ_i^{l+1}, we also have $\sigma_i^l \cdot \lambda = 0$. By adding (some sufficiently large scale of) λ to t_i^l, each $\langle IC_i^{m\to l} \rangle$ for $m < l$ can be relaxed. No other constraints are affected and the resulting change in the auctioneer's revenue is $\sigma_i^l \cdot \lambda = 0$.

We next show that for any auction mechanism (p, t) that satisfies the remaining constraints, there exists an auction mechanism (p', t') which satisfies the constraints $\langle IR_i^m \rangle$, for $m = 1,\ldots,M$, and $\langle IC_i^{m\to m-1} \rangle$, for $m = 2,\ldots,M$, with equality, and achieves at least as high an μ^*-expected revenue as (p, t) does.

To prove this, fix any auction mechanism (p, t) that satisfies the remaining constraints. Suppose $\langle IC_i^{m\to m-1} \rangle$ holds with strict inequality. Let $\boldsymbol{\tau}$ denote the matrix whose M rows are the vectors $\{\tau_i^m\}_{m=1}^M$, and let $(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1})$ be the matrix obtained by replacing the mth row of $\boldsymbol{\tau}$ with the vector σ_i^{m-1}. Note that the matrix $(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1})$ has rank M. We can thus solve the following equation for λ:

$$(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1}) \cdot \lambda = x^m,$$

where x^m denotes the mth elementary basis vector in $\mathbf{R}^M$. Note that because $\tau_i^{m-1} \cdot \lambda = 0 < \sigma_i^{m-1} \cdot \lambda$, and because τ_i^{m-1} is a convex combination of σ_i^{m-1} and τ_i^m according to (6), we have $\tau_i^m \cdot \lambda < 0$.

We will add the vector $\varepsilon\lambda$ to t_i^{m-1} for some scalar $\varepsilon > 0$. Because $\tau_i^{m'} \cdot \lambda = 0$ for $m' \neq m$, no constraints for types $\omega_i^{m'}$ are affected. As for type ω_i^m, the constraint $\langle IR_i^m \rangle$ is unaffected. The only incentive constraint of type ω_i^m that is affected is $\langle IC_i^{m \to m-1} \rangle$, and this constraint was slack by assumption. Let $S_i^m > 0$ be the slack in $\langle IC_i^{m \to m-1} \rangle$, and choose $\varepsilon = -S_i^m/(\tau_i^m \cdot \lambda) > 0$. Then, with the resulting transfer rule, $\langle IC_i^{m \to m-1} \rangle$ holds with equality. Finally, because $\varepsilon\sigma_i^{m-1} \cdot \lambda > 0$, the auctioneer profits from this modification.

We next show that each $\langle IR_i^m \rangle$ can be treated as an equality without loss of generality. Define $S_i^m = \tau_i^m \cdot (p_i^m v^m - t_i^m) \geq 0$ to be the slack in $\langle IR_i^m \rangle$. Construct a lottery λ that satisfies

$$\tau_i^m \cdot \lambda = S_i^m, \quad m = 1, \dots, M.$$

By the full-rank arguments such a lottery λ can be found. We will add λ to each t_i^m. No constraint of the form $\langle IC_i^{m \to l} \rangle$ will be affected, but now each constraint of the form $\langle IR_i^m \rangle$ holds with equality. Finally, we check that the auctioneer profits from this modification. Indeed, the auctioneer nets

$$\begin{aligned}
\sum_{m=1}^{M} \nu_i^m \left(\sigma_i^m \cdot \lambda\right) &= \sum_{m=1}^{M-1} \left(G_i(m)\tau_i^m - G_i(m+1)\tau_i^{m+1}\right) \cdot \lambda + \nu_i^M \tau_i^M \cdot \lambda \\
&= G_i(1)\tau_i^1 \cdot \lambda \\
&= G_i(1)S_i^1 \\
&\geq 0.
\end{aligned}$$

The proof for the non-singular case is now concluded as follows. Based on the preceding arguments, we consider the modified program in which the constraints $\langle IR_i^m \rangle$ and $\langle IC_i^{m \to m-1} \rangle$ are satisfied with equality. We will use these constraints to substitute out for the transfers in the objective function and reduce the problem to an *unconstrained* optimization with the only choice variable being the allocation rule (recall that any allocation rule is BIC). The resulting objective function will be identical to the objective function (4) for the dsIC problem. Thus the only difference between the two problems is the absence of any monotonicity constraint in the BIC case. It then follows that (*i*) the modified problem and hence the original problem (7) will have a solution, and (*ii*) this solution will be the same as the solution to the dsIC problem by Proposition 1 part 1.

We rewrite the objective function in (7) as below, and impose the constraints as equalities:

$$\begin{aligned}
\max_{p(\cdot),t(\cdot)} \quad & \sum_{i=1}^{N}\sum_{m=1}^{M}\nu_i^m\sigma_i^m\cdot t_i^m && (8)\\
\text{subject to:} \quad & \forall\ i=1,\ldots,N,\ \forall\ m=1,\ldots,M, \\
& \tau_i^m\cdot(p_i^m v^m - t_i^m) = 0, && \langle\overline{IR}_i^m\rangle\\
& \tau_i^m\cdot(p_i^m v^m - t_i^m) = \tau_i^m\cdot\left(p_i^{m-1}v^m - t_i^{m-1}\right). && \langle\overline{IC}_i^{m\to m-1}\rangle
\end{aligned}$$

We have substituted in the objective function using the definition $\mu^*(\omega)=\nu(f(\omega))=\nu_i(f_i(\omega_i))\nu(f_{-i}(\omega_{-i})|f_i(\omega_i))=\nu_i^m\sigma_i^m$ for the appropriate m.

By definition, $\sigma_i^M=\tau_i^M$, so $\langle\overline{IR}_i^M\rangle$ becomes $\sigma_i^M\cdot t_i^M = v^M\sigma_i^M\cdot p_i^M$. Now, for arbitrary $m<M$,

$$\begin{aligned}
\sigma_i^m\cdot t_i^m &= \frac{1}{\nu_i^m}\left[G_i(m)\tau_i^m - G_i(m+1)\tau_i^{m+1}\right]\cdot t_i^m\\
&= \frac{1}{\nu_i^m}\left\{G_i(m)v^m\tau_i^m\cdot p_i^m - G_i(m+1)\left[\tau_i^{m+1}\cdot(p_i^m - p_i^{m+1})v^{m+1} + \tau_i^{m+1}\cdot t_i^{m+1}\right]\right\}\\
&= \frac{1}{\nu_i^m}\left[G_i(m)v^m\tau_i^m\cdot p_i^m - G_i(m+1)v^{m+1}\tau_i^{m+1}\cdot p_i^m\right].
\end{aligned}$$

In the first line we used the recursive definition in (6), in the second line we used $\langle\overline{IR}_i^m\rangle$ and $\langle\overline{IC}_i^{m+1\to m}\rangle$, and in the third line we used $\langle\overline{IR}_i^{m+1}\rangle$.

Substituting the constraints into the objective function, it becomes:

$$\begin{aligned}
& \sum_{i=1}^{N}\left\{v^M\nu_i^M\sigma_i^M\cdot p_i^M + \sum_{m=1}^{M-1}\left[v^m G_i(m)\tau_i^m\cdot p_i^m - v^{m+1}G_i(m+1)\tau_i^{m+1}\cdot p_i^m\right]\right\}\\
= & \sum_{i=1}^{N}\left\{v^M\nu_i^M\sigma_i^M\cdot p_i^M + \sum_{m=1}^{M-1}\left[v^m\left(\nu_i^m\sigma_i^m + G_i(m+1)\tau_i^{m+1}\right)\cdot p_i^m - v^{m+1}G_i(m+1)\tau_i^{m+1}\cdot p_i^m\right]\right\}\\
= & \sum_{i=1}^{N}\left[\sum_{m=1}^{M}v^m\nu_i^m\sigma_i^m\cdot p_i^m - \sum_{m=2}^{M}(v^m - v^{m-1})G_i(m)\tau_i^m\cdot p_i^{m-1}\right].
\end{aligned}$$

Applying the definition of τ_i^m, the objective function becomes:

$$\begin{aligned}
& \sum_{i=1}^{N}\left[\sum_{m=1}^{M} v^m \nu_i^m \sigma_i^m \cdot p_i^m - \sum_{m=2}^{M} \Delta\left(\sum_{m'=m}^{M} \nu_i^{m'} \sigma_i^{m'}\right) \cdot p_i^{m-1}\right] \\
= & \sum_{i=1}^{N}\left[\sum_{m=1}^{M} v^m \nu_i^m \sigma_i^m \cdot p_i^m - \Delta \sum_{m=2}^{M} \sum_{m'=2}^{m} \nu_i^m \sigma_i^m \cdot p_i^{m'-1}\right] \\
= & \sum_{i=1}^{N} \sum_{m=1}^{M} \nu_i^m \sigma_i^m \cdot \left[v^m p_i^m - \Delta \sum_{m'=2}^{m} p_i^{m'-1}\right] \\
= & \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{v_{-i} \in V_{-i}} \nu_i(v^m, v_{-i}) \cdot \left[v^m p_i(v^m, v_{-i}) - \Delta \sum_{m'=1}^{m-1} p_i(v^{m'}, v_{-i})\right].
\end{aligned}$$

This is identical to the objective function in (5). This establishes equation (3), and hence also equation (2), for any non-singular ν under which the virtual valuations satisfy single-crossing.

Now consider an arbitrary regular ν, not necessarily non-singular. There exists a sequence ν_n converging to ν such that each ν_n is non-singular. Moreover, for ν_n close enough to ν, the strict inequalities in the definition of single-crossing will be preserved, and hence the virtual valuations derived from ν_n will also satisfy single crossing once n is large enough. For each such ν_n, construct the type space Ω^n exactly as in the first half of the proof. Let $\tau_i^m(n)$ denote the belief of type ω_i^m of bidder i in the type space Ω_i^n. Passing to a subsequence if necessary, take $\tau_i^m(n) \to \tau_i^m$ for each i and m. Let Ω be the limit type space with beliefs τ_i^m, and let $\mu^* = \pi(\nu)$. Write $\mu_n = \pi(\nu_n)$.

Note that for each of these type spaces (Ω^n or Ω) there is a one-to-one correspondence between types and valuations for each bidder i. Therefore, for any auction mechanism (p, t) defined over any of these type spaces, we can also think of it as mappings from V to probabilities and transfers. The following notations are hence defined regardless of which of these type spaces the auction mechanism (p, t) is defined over:

$$\begin{aligned}
\mathbf{E}_{n'}\bar{t} &:= \sum_{v \in V} \bar{t}(v) \nu^{n'}(v), \\
\mathbf{E}\bar{t} &:= \sum_{v \in V} \bar{t}(v) \nu(v).
\end{aligned}$$

For any one of these type spaces, say that an auction mechanism (p, t) is BIC for that type space if it satisfies the corresponding constraints in (7). Consider any (p, t) that is BIC for type space Ω.

Obviously $\mathbf{E}_n\bar{t} \to \mathbf{E}\bar{t}$. We will show that there exists a sequence of auction mechanisms $(p, t(n))$ such that each $(p, t(n))$ is BIC with respect to type space Ω^n, and such that $\mathbf{E}_n\bar{t}(n) - \mathbf{E}_n\bar{t} \to 0$.

For each i, m, and n, let

$$S_i^m(n) = \max\{0, \tau_i^m(n) \cdot (t_i^m - p_i^m \cdot v^m)\}$$

be the amount by which the $\langle IR_i^m \rangle$ constraint is violated by the auction mechanism (p,t) for type $\omega_i^m(n)$. Because (p,t) is BIC with respect to Ω, $S_i^m(n) \to 0$ for each i and m.

However, (p,t) may not be BIC with respect to Ω^n. To convert it into an auction mechanism that is, we first add the constant $-S_i^m(n)$ to t_i^m to restore all $\langle IR_i^m \rangle$ constraints. The cost of this to the auctioneer is the μ_n-expected value of $S_i^m(n)$ which is converging to zero. Let $\tilde{t}(n)$ be the transfer rule that results from this first step of modification.

Next, for each i, m, l, and n, let

$$L_i^{m \to l}(n) = \max\{0, \tau_i^l(n) \cdot (p_i^l v^m - \tilde{t}_i^l(n)) - \tau_i^m(n) \cdot (p_i^m v^m - \tilde{t}_i^m(n))\}$$

be the amount by which $\langle IC_i^{m \to l} \rangle$ is violated by the auction mechanism $(p, \tilde{t}(n))$. Note that $L_i^{m \to m}(n) = 0$. Again, because (p,t) is BIC with respect to Ω, and because $\tilde{t}(n) \to t$, we have $L_i^{m \to l}(n) \to 0$ for each i, m, and l. For each n, we construct $\lambda_i^l(n)$ to solve the system

$$\tau_i^m(n) \cdot \lambda_i^l(n) = L_i^{m \to l}(n), \qquad \forall\, i, m, l.$$

We will add $\lambda_i^l(n)$ to $\tilde{t}_i^l(n)$ to restore each $\langle IC_i^{m \to l} \rangle$ constraint without affecting the $\langle IR_i^m \rangle$ constraints. The resulting auction mechanism $(p, t(n))$ is now BIC with respect to Ω^n. Since $\tau_i^m(n) \cdot \lambda_i^l(n) \to 0$ for each m and l, so by (6) we have $\sigma_i^m(n) \cdot \lambda_i^m(n) \to 0$ for each m, and thus,

$$\mathbf{E}_n[\bar{t}(n) - \bar{t}] \to 0 \tag{9}$$

as promised.

Finally, recall that $\Pi^D(\nu)$ denotes the optimal dsIC ν-expected revenue. Because the constrained set in the optimal dsIC auction design problem (5) is compact, the maximum theorem implies

$$\Pi^D(\nu_n) \to \Pi^D(\nu). \tag{10}$$

We have already shown that $\Pi^D(\nu_n) \geq \mathbf{E}_n \bar{t}(n)$ because each ν_n is non-singular. This together with (9) and (10) delivers

$$\mathbf{E}(\bar{t}) = \lim_n \mathbf{E}_n \bar{t} \leq \Pi^D(\nu).$$

Since (p,t) was an arbitrary auction mechanism that is BIC with respect to Ω, this establishes equation (3), and hence also equation (2), for any regular ν. ■

5 The Bayesian Foundation for Dominant Strategy Mechanisms

In this section, we shall investigate another possible foundation of dominant strategy mechanisms, namely the Bayesian foundation. The Bayesian foundation can be loosely explained with the following story. Imagine the auctioneer as a Bayesian decision maker. When she needs to choose a mechanism under uncertainty of bidders' beliefs, she forms a subjective belief about bidders' beliefs, and compares different mechanisms by calculating the expected performance with respect to that subjective belief. When we as outside observers observe that this auctioneer chooses a particular mechanism, we can ask whether or not such a choice is consistent with Bayesian rationality; i.e., whether or not such a choice is optimal with respect to *some* subjective belief. If the answer is yes, then we say that such a choice is rationalizable. We can say that dominant strategy mechanisms are rationalizable if they are optimal with respect to *some* subjective beliefs. Given the predominant role of Bayesian rationality in the literature of mechanism design, it seems even more natural to pursue the Bayesian foundation.

To investigate the possibility of the Bayesian foundation, we only need minimal changes in our setting. Recall that we have already been modeling assumptions about bidders' beliefs as distributions over their types. So all we need to do now is to reinterpret an assumption as a subjective belief of the auctioneer. Similarly, if the auctioneer's estimate of the bidders' valuations is described by ν, then her subjective belief about bidders' beliefs must be a distribution μ over bidders' types that is consistent with ν.

It follows from the proof of Theorem 1 that there exists a Bayesian foundation for dominant strategy auctions when the distribution of valuations is regular. However, we show by example below that beyond the regular case, a Bayesian foundation need not exist. As a negative result about the rationality of imposing dominant strategies, we view this as particularly strong: for some distributions of valuations, no Bayesian expected-revenue maximizing auctioneer would optimally employ a dominant strategy mechanism, regardless of her beliefs.

In this example, there are two bidders and each has two possible valuations. The distribution of valuations ν is represented in Figure 6.[13]

The optimal dsIC auction is depicted in Figure 7, where we follow the convention in Section 3 and use "$\alpha = i$" as the shorthand for "allocating the object to bidder i.".

It is helpful to pay attention to a few noteworthy aspects of this environment and the optimal dsIC auction. Notice that the valuation of bidder 1 is always higher than that of bidder 2. Nevertheless, the auctioneer chooses to sell to bidder 2 when bidder 1 has low valuation. This is optimal because conditional on bidder 2 having low valuation, the probability that bidder 1 has high valuation is greater than 1/2. This means that it is

[13]The distribution ν in this example does not have full-support. This simplifies the exposition of the example, but the conclusion would be the same if the event $\{v_1 = 10, v_2 = 4\}$ had positive (but small) probability.

	$v_1 = 5$	$v_1 = 10$
$v_2 = 4$	1/6	0
$v_2 = 2$	1/3	1/2

Figure 6: The distribution ν.

	$v_1 = 5$	$v_1 = 10$
$v_2 = 4$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$v_2 = 2$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 7: The optimal dsIC auction Γ.

optimal to exclude the low valuation type of bidder 1 to relax the incentive constraint and sell to the high valuation type at his reservation price. Given this, the auctioneer may as well sell to bidder 2 when bidder 1 has a low valuation. If monotonicity were not a constraint, the auctioneer would choose to sell to bidder 1 when bidder 2 had high valuation. Thus, the monotonicity constraint binds here, and in order to satisfy it, the object is sold to bidder 2 in this case.

Proposition 2 *The optimal dsIC auction Γ depicted in Figure 7 cannot be rationalized by any subjective belief μ of the auctioneer that is consistent with the distribution ν depicted in Figure 6.*

In the remainder of this section we will present the proof of Proposition 2. In Appendix C we prove the following stronger result.

Proposition 3 *For the distribution ν depicted in Figure 6, the optimal BIC revenue is uniformly bounded away from the optimal dsIC revenue regardless of the auctioneer's subjective belief; i.e.,*

$$\inf_{\mu \in \mathcal{M}(\nu)} \sup_{\Gamma \text{ is BIC}} R_\mu(\Gamma) > V^D(\nu).$$

To prove Proposition 2, fix any subjective belief $\mu \in \mathcal{M}(\nu)$ that rationalizes the optimal dsIC auction Γ, we shall prove that there exists an BIC auction that generates higher μ-expected revenue than Γ does. This would contradict the assumption that μ rationalizes Γ and complete the proof.

The proof proceeds by a sequence of lemmas. In each we derive conditions that must be satisfied by a rationalizing subjective belief μ. Finally we show that no subjective belief μ can satisfy them all.

For the purpose of this proof, it suffices to work only with bidder 2's first-order beliefs in order to arrive at a contradiction. So, for notational convenience, we shall summarize bidder 2's belief type τ_2 by a single number: his first-order belief that bidder 1 has high valuation. Specifically, for any type $\omega_2 = (v_2, \tau_2)$ of bidder 2, if $v_2 = 4$, we shall use a to denote $g_2(\tau_2)(v_1 = 10)$; and if $v_2 = 2$, we shall use b to denote $g_2(\tau_2)(v_1 = 10)$. For any (measurable) subset $A \subset [0,1]$, we shall use "$a \in A$" to denote the event $\{\omega_2 = (v_2, \tau_2) : v_2 = 4, f_2(\tau_2)(v_1 = 10) \in A\}$; similarly for the notation "$b \in B \subset [0,1]$."

The first lemma says that, conditional on any μ-non-null subset of low-valuation types of bidder 2, the μ-conditional-probability that bidder 1 having high valuation cannot be too low, otherwise the auctioneer can improve upon Γ by selling to some low-valuation types of bidder 1.[14]

Lemma 2 *For any $x \in (0,1]$ such that $\mu(b = x) = 0$, if $\mu(b < x) > 0$, then $\mu(v_1 = 10|b < x) \geq 3/8$.*

Proof: Suppose there exists $x \in (0,1]$ such that $\mu(b < x) = \mu(b \leq x) > 0$, and yet $\mu(v_1 = 10|b < x) < 3/8$. Consider the modified auction $\Gamma(x)$ as depicted in Figure 8.

	$v_1 = 5$	$v_1 = 10$
$a \in [0,1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 1, t_1 = 5, t_2 = 0$	$\alpha = 1, t_1 = 5, t_2 = 0$

Figure 8: The modified auction $\Gamma(x)$.

To see that $\Gamma(x)$ continues to be BIC, note that (*i*) truth-telling continues to be a dominant strategy of bidder 1, (*ii*) low-valuation types of bidder 2 always have zero rent regardless of what they announce, and (*iii*) high-valuation types of bidder 2 would not announce the (newly added) message "$b < x$" as that gives them zero rent.

The only difference between $\Gamma(x)$ and Γ is in the (μ-non-null) event of $b < x$, in which case $\Gamma(x)$ generates μ-expected revenue of $5\mu(v_1 = 5|b < x) + 5\mu(v_1 = 10|b < x) = 5$, whereas Γ only generates μ-expected revenue of $2\mu(v_1 = 5|b < x) + 10\mu(v_1 = 10|b < x) < 2(5/8) + 10(3/8) = 5$, contradicting the assumption that μ rationalizes Γ. ■

The second lemma says that for any low-valuation type of bidder 2 that the auctioneer subjectively perceives as possible, his first-order belief b also cannot be too low, otherwise his belief would be too different from the auctioneer's belief, so much so that the auctioneer can improve upon Γ by betting against him.

[14]In Lemma 2 (and similarly in Lemmas 3-5), the seemingly redundant requirement of $\mu(b = x) = 0$ is a null-boundary property used only in the proof of Proposition 3.

Lemma 3 $\mu(b < 3/13) = 0$.

Proof: Suppose not. Then pick $x < 3/13$ such that $\mu(b < x) > 0$ and $\mu(b = x) = 0$,[15] and consider the modified auction $\Gamma'(x)$ as depicted in Figure 9.

	$v_1 = 5$	$v_1 = 10$
$a \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 0, t_1 = 0, t_2 = -2$	$\alpha = 1, t_1 = 10, t_2 = 2(1 - x)/x$

Figure 9: The modified auction $\Gamma'(x)$.

To see that $\Gamma'(x)$ continues to be BIC, note that (*i*) truth-telling continues to be a dominant strategy of bidder 1, (*ii*) low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$b < x$" if and only if the resulting rent of $2(1 - b) - [2(1 - x)/x]b = 2(1 - b/x)$ is positive, or equivalently if and only if $b < x$, and (*iii*) high-valuation types of bidder 2 would not announce the (newly added) message "$b < x$" as that gives them rent of $2(1 - a) - [2(1 - x)/x]a = 2(1 - a/x)$, which is lower than the rent of $2(1 - a)$ if they tell the truth.

The only difference between $\Gamma'(x)$ and Γ is in the (μ-non-null) event of $b < x$, in which case $\Gamma'(x)$ collects from bidder 2 an μ-expected amount of

$$\begin{aligned}
& (-2)\mu(v_1 = 5|b < x) + [2(1 - x)/x]\mu(v_1 = 10|b < x) \\
\geq\ & (-2)(5/8) + [2(1 - x)/x](3/8) \\
=\ & 3/(4x) - 2 \\
>\ & [3/4(3/13)] - 2 \\
=\ & 5/4
\end{aligned}$$

(where the first inequality follows from Lemma 2), whereas Γ only collects from bidders 2 an μ-expected amount of $2\mu(v_1 = 5|b < x) \leq 2(5/8) = 5/4$, contradicting the assumption that μ rationalizes Γ. ■

The third lemma says that the first-order belief a of high-valuation types of bidder 2 cannot be too low. Otherwise beliefs held by high- and low-valuation types of bidder 2 would be too different, and this would enable the auctioneer to improve upon Γ by introducing Crémer-McLean-kind of bets to separate these types and relax incentive compatibility constraints.

Lemma 4 $\mu(a < 1/11) = 0$.

[15] It is always possible to pick such an x, as any distribution over $[0, 1]$ can have at most countably many mass points.

Proof: If not then let $y < 1/11$ such that $\mu(a = y) = 0$ and $\mu(a < y) > 0$. Notice that $y < 1/11$ implies $y < 3y/(2y+1) < 3/13$, and hence we can also choose x between $3y/(2y+1)$ and $3/13$ such that $\mu(b = x) = 0$. Consider the modified auction $\Gamma(x, y)$ as depicted in Figure 10.

	$v_1 = 5$	$v_1 = 10$
$a < y$	$\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$	$\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$
$a \geq y$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$	$\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 10: The modified auction $\Gamma(x, y)$.

To see that $\Gamma(x, y)$ continues to be BIC, note that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$b < x$" if and only if the resulting rent of $[2x(1-y)/(x-y)](1-b) - [2(1-x)(1-y)/(x-y)]b = 2(1-y)(x-b)/(x-y)$ is positive, or equivalently if and only if $b < x$, and (iii) high-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$a < y$" if and only if the resulting rent of $[2x(1-y)/(x-y)](1-a) - [2(1-x)(1-y)/(x-y)]a = 2(1-y)(x-a)/(x-y)$ is strictly higher than the truth-telling rent of $2(1-a)$, or equivalently if and only if $a < y$.

Since the event of $b < x$ is a μ-null event by Lemma 3, the only real difference between $\Gamma(x, y)$ and Γ is in the (μ-non-null) event of $a < y$, in which case $\Gamma(x, y)$ generates μ-expected revenue of

$$\begin{aligned}
& 5 - 2x(1-y)/(x-y) \\
= \; & 5 - 2(x-y+y)(1-y)/(x-y) \\
= \; & 5 - 2(1-y) - 2y(1-y)/(x-y) \\
> \; & 5 - 2(1-y) - 2y(1-y)(2y+1)/[3y - y(2y+1)] \\
= \; & 5 - 2(1-y) - 2y(1-y)(2y+1)/[2y(1-y)] \\
= \; & 2,
\end{aligned}$$

whereas Γ only generates μ-expected revenue of 2, contradicting the assumption that μ rationalizes Γ. ■

Finally, the fourth lemma says that the first-order belief a of high-valuation types of bidder 2 cannot be too high. Otherwise the beliefs of such types would be too different from the auctioneer's subjective belief, and this would enable the auctioneer to profit by offering an incentive compatible and individually rational bet. Obviously lemmas 4 and 5 deliver the contradiction and thus prove Proposition 2.

Lemma 5 $\mu(a < 1/11) > 0$.

Proof: Suppose $\mu(a < 1/11) = 0$. Consider the modified auction Γ' as depicted in Figure 11.

	$v_1 = 5$	$v_1 = 10$
$a \geq 1/12$	$\alpha = 2, t_1 = 0, t_2 = 123/61$	$\alpha = 2, t_1 = 0, t_2 = 233/61$
$a < 1/12$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 11: The modified auction Γ'.

To see that Γ' continues to be BIC, note that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 would not announce the (newly added) message "$a \geq 1/12$" as that gives them strictly negative rent regardless of what bidder 1 announces, and (iii) high-valuation types of bidder 2 would have weak incentive to announce the (newly added) message "$a \geq 1/12$" if and only if the resulting rent of $(4 - 123/61)(1 - a) + (4 - 233/61)a$ is weakly higher than their original rent of $2(1 - a)$, or equivalently if and only if $a \geq 1/12$.

Since the event $a < 1/12 < 1/11$ is a μ-null event by assumption, the only real difference between Γ' and Γ is in the (μ-non-null) event of $a \geq 1/12$, in which case Γ' generates μ-expected revenue of $123/61 > 2$, whereas Γ only generates μ-expected revenue of 2. This proves that μ does not rationalize Γ. ■

6 Remarks on the Common Prior Assumption

The validity, in the regular case, of the maxmin and Bayesian foundations for dominant strategies was shown by construction of a particular assumption about bidder's beliefs. It is noteworthy that the assumption constructed in the proof of Theorem 1 is inconsistent with the widely-adopted common prior assumption (CPA).

Loosely speaking, the CPA says that there is a common probability measure (the common prior) from which each bidder derives his belief by computing the conditional probability of opponents' types conditional on his own "signal" or "information." In our current setting, where any assumption about bidders' types is already modeled as a probability distribution over bidders' types, we can relate any assumption μ to the CPA as follows. For any subset $A \in \Omega_i^*$, we shall write $\mu(A)$ as a short hand for $\mu(A \times \Omega_{-i}^*)$. In other words, we abuse notation and use the same notation for a probability measure as well as its marginal distributions. Recall that $g_i : \mathcal{T}_i^* \to \Delta(V_{-i} \times \mathcal{T}_{-i}^*)$ is the homeomorphism between bidder i's belief types and distributions over his opponents' types.

Definition 3 *We say that an assumption* μ *is an CPA-assumption if for any measurable subsets* $A \subset \Omega_i^*$ *and* $B \subset \Omega_{-i}^*$*,*

$$\int_A g_i(\tau_i)(B)\,\mu(d\omega_i) = \mu(A \times B).$$

It is apparent that the particular assumption μ^* we used in the proof of Theorem 1 is not an CPA-assumption. Can we replace μ^* with some CPA-assumption μ in the proof? The answer is: sometimes, but not always. For some distribution ν over bidders' valuations, especially those that are close to being independent, it is indeed possible to use an CPA-assumption in the proof of Theorem 1.[16] But it is also not difficult to find an example of ν such that no such CPA-assumption can be constructed. We will give one such an example in the Appendix.

Do these observations cut back the appeal of Theorem 1? We believe the answer is: not at all, for two reasons. First, whether the CPA is an appropriate assumption to make is itself a subject of debate. Gul (1998) has explained why the CPA lacks appropriate motivations, and Morris (1995) has also explained why many defenses of the CPA are flawed, and why many interesting economic problems are better modeled without the CPA.

Second, recent studies on the CPA has uncovered the close relation between the CPA and common knowledge assumptions (see, for example, Lipman (2003)). In any study of the Wilson Doctrine, such as this paper, it seem inconsistent to pursue "successive reduction" in the dependence on common knowledge assumptions on one hand, but continue to insist on the CPA on the other.

7 Conclusion

We have identified a sufficient condition, a direct generalization of the regular case in Myerson (1981), under which the use of simple, dominant strategy auction mechanisms can be rationalized, either by appeal to maxmin or Bayesian optimality criteria. Let us conclude by pointing out one additional implication of this result. Suppose that in addition to the regularity assumption, the distribution of valuations ν is symmetric. This would be a natural assumption for a seller who does not know the identities or characteristics of the bidders. In this case, the English auction with a suitably chosen reserve price is an optimal dominant strategy auction.[17] We have thus shown that in symmetric, regular environments, the widespread use of the English auction as a selling mechanism can be justified as an optimal response to uncertainty about the bidders' beliefs.

[16] The details of such a construction are available from the authors upon request.
[17] This was shown for a slightly different environment by Lopomo (2000).

Appendix A: An Example for Section 6

In Section 6, we claim that there exists a distribution ν that satisfies Condition M such that there is no CPA-assumption μ for which equation 3 holds. We shall provide an example of such a distribution here.

As the proof below would make it clear, this example of ν is a robust to perturbations.

Consider the same example as in Section 3, where there are two bidders, and each bidder has two possible valuations. The joint distribution of valuations is as depicted in Figure 3, and the corresponding optimal dsIC auction is as depicted in Figure 4.

Suppose there exists an CPA-assumption $\mu \in \mathcal{M}(\nu)$ for which equation (3) holds. We shall prove that there exists an BIC auction that generates higher μ-expected revenue than Γ does. This would contradict the supposition that equation (3) holds.

It suffices to work only with bidder 2's first-order beliefs in order to complete this proof. So, following the convention in Section 3, we shall continue to use a (b) to denote the first-order belief of a high-valuation (low-valuation) type of bidder 2 that bidder 1 has high valuation. Let $\underline{b} = \sup\{x \in [0,1] : \mu(b < x) = 0\}$.

First, observe that $\underline{b} \geq 4/9$. Suppose, on the contrary, $\underline{b} < 4/9$. Then pick any number z between $\underline{b}$ and 4/9, and consider the modified auction $\Gamma(z)$ as depicted in Figure 12.

	$v_1 = 4$	$v_1 = 9$
$a \in [0,1]$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$
$b \geq z$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 1, t_1 = 9, t_2 = 0$
$b < z$	$\alpha = 1, t_1 = 4, t_2 = 0$	$\alpha = 1, t_1 = 4, t_2 = 0$

Figure 12: The modified auction $\Gamma(z)$.

It is obvious that $\Gamma(z)$ continues to be BIC. The only difference between $\Gamma(z)$ and Γ is in the (μ-non-null) event of $b < z$, in which case $\Gamma(z)$ generates μ-expected revenue of 4, whereas Γ only generates μ-expected revenue of $9\mu(v_1 = 9 | b < z) < 9z < 9(4/9) = 4$, where the first inequality comes from the fact that μ is an CPA-assumption. Since this would have contradicted the supposition that equation (3) holds, we must have $\underline{b} \geq 4/9$.

Then, consider the modified auction Γ'' as depicted in Figure 13.

To see that Γ'' continues to be BIC, it suffices to observe that, for low-valuation types of bidder 2 with $b \geq 4/9$, truth-telling gives them a non-negative rent of $(5-11)(1-b) + (15/2)b \geq (-6)(5/9) + (15/2)(4/9) = 0$.

Since $b < 4/9$ is a μ-null event, Γ'' generates μ-expected revenue of $9(4/10) + 11(6/10) - (15/2)(4/10) = 72/10$, whereas Γ only generates μ-expected revenue of $9(3/10) + 11(4/10) = 71/10$. This proves that equation (3) does not hold, a contradiction.

	$v_1 = 4$	$v_1 = 9$
$a \in [0,1]$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 1, t_1 = 9, t_2 = -15/2$
$b \geq 4/9$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 1, t_1 = 9, t_2 = -15/2$
$b < 4/9$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 0, t_1 = 0, t_2 = 0$

Figure 13: The modified auction Γ''.

Appendix B: Proof of Proposition 1

Proof of Proposition 1 For part 1, suppose that the virtual valuations satisfy the single-crossing condition, and let p be an allocation rule that solves the unconstrained maximization of (5). Then $p_i(v) > 0$ only if $\gamma_i(v) \geq \max_j \gamma_j(v)$, and $p_i(v) = 1$ if $\gamma_i(v) > \max_{j \neq i} \gamma_j(v)$. Fix v such that $p_i(v) > 0$, (so that $\gamma_i(v) \geq \max_j \gamma_j(v)$) and consider an increase in the valuation of bidder i to $\hat{v}_i > v_i$. By the single-crossing condition, $\gamma_i(v) > \max_{j \neq i} \gamma_j(v)$ and hence $p_i(v) = 1$. This shows that $\langle M_i \rangle$ is satisfied.

For part 2, suppose that both affiliation and the monotone hazard rate condition are satisfied and let v be a valuation profile at which $\gamma_i(v) \geq \gamma_j(v)$. Consider an increase in the valuation of bidder i to $\hat{v}_i > v_i$. Write $\hat{v} = (\hat{v}_i, v_{-i})$. It is well-known that affiliation implies that this "increases" the conditional distribution of other bidders' valuations in the sense of the monotone likelihood ratio ordering. That is, for any pair of valuations $v'_j > v_j$, $\frac{\nu(v'_j, \hat{v}_{-j})}{\nu(\hat{v})} \geq \frac{\nu(v'_j, v_{-j})}{\nu(v)}$.

The new virtual valuation for any bidder k is

$$\gamma_k(\hat{v}_i, v_{-i}) = \hat{v}_k - \Delta \frac{1 - F_k(\hat{v})}{\nu(\hat{v})}$$

By the monotone hazard rate condition $\gamma_i(\hat{v}) > \gamma_i(v)$. By affiliation, for each bidder $j \neq i$,

$$\begin{aligned} \frac{1 - F_j(\hat{v})}{\nu(\hat{v})} &= \sum_{v'_j > v_j} \frac{\nu(v'_j, \hat{v}_{-j})}{\nu(\hat{v})} \\ &\geq \sum_{v'_j > v_j} \frac{\nu(v'_j, v_{-j})}{\nu(v)} \\ &= \frac{1 - F_j(v)}{\nu(v)} \end{aligned}$$

and this implies $\gamma_j(\hat{v}) \leq \gamma_j(v)$. And for the seller ($j = 0$), the latter inequality holds by definition.

Combining these results we have $\gamma_i(\hat{v}) > \gamma_j(\hat{v})$. Since j was arbitrary, this proves that the single crossing condition holds. ■

Appendix C: Proof of Proposition 3

Lemma 6 *Suppose K is a compact topological space and that $\mathcal{F}$ is a family of real-valued functions on K such that, for each $x \in K$, there is some $f_x \in \mathcal{F}$ which is continuous at x and satisfies $f_x(x) > 0$. Then we have $\inf_{x\in K} \sup_{f\in\mathcal{F}} f(z) > 0$.*

Proof: For each $x \in K$, there exists an open neighborhood U_x such that, for each $y \in U_x$, we have $f_x(y) > f_x(x)/2$. The collection $\{U_x : x \in K\}$ forms an open covering of the compact space K, and hence there exists a finite sub-covering. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite sub-covering and let $\varepsilon = \min\{f_{x_1}(x_1), \ldots, f_{x_k}(x_n)\} > 0$. For each $x \in K$, we have $x \in U_{x_l}$ for some $l = 1, \ldots, n$ so that $\sup_{f\in\mathcal{F}} f(x) \geq f_{x_l}(x) > f_{x_l}(x_l)/2 \geq \varepsilon/2 > 0$. ■

Lemma 7 *Suppose $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are disjoint open subsets of Ω^* such that $\mu(\cup\mathcal{O}_l) = 1$, and $t : \Omega^* \to \mathbf{R}$ is a bounded real function that is constant on each $\mathcal{O}_l$. Then the mapping*

$$\mu' \to \int_{\Omega^*} t\, \mu'(d\omega)$$

is continuous at the point μ.

Proof: Fix any $\varepsilon > 0$. Let $\bar{t} > 0$ be an upper bound for $|t|$. The function $\mu' \to \mu'(\mathcal{O}_i)$ is lower semi-continuous (see Aliprantis and Border (1999)), hence we can set

$$\delta = \frac{\varepsilon}{\bar{t}n^2}$$

and find a neighborhood U of μ such that, for all $\mu' \in U$, $\mu'(\mathcal{O}_l) > \mu(\mathcal{O}_l) - \delta$ for $l = 1, \ldots, n$. Since $\mu(\cup\mathcal{O}_l) = 1$, it follows that $\mu'(\mathcal{O}_l) < \mu(\mathcal{O}_l) + (n-1)\delta$ and $\mu'(\Omega^* \setminus \cup\mathcal{O}_l) < \mu(\Omega^* \setminus \cup\mathcal{O}_l) + n\delta = n\delta$.

We can write

$$\int_{\Omega^*} t\, d\mu' = \sum_{l=1}^{n} \mu'(\mathcal{O}_l)t(\mathcal{O}_l) + \int_{\Omega^*\setminus\cup\mathcal{O}_l} t(\omega)\, d\mu',$$

so that

$$\begin{aligned}
& \sum_{l=1}^{n} \mu'(\mathcal{O}_l)t(\mathcal{O}_l) - \mu(\Omega^* \setminus \cup\mathcal{O}_l)\bar{t} \leq \int_{\Omega^*} t\, \mu'(d\omega) \leq \sum_{l=1}^{n} \mu'(\mathcal{O}_l)t(\mathcal{O}_l) + \mu'(\Omega^* \setminus \cup\mathcal{O}_l)\bar{t} \\
\Longrightarrow\quad & \sum_{l=1}^{n}[\mu(\mathcal{O}_l) - \delta]t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t\, \mu'(d\omega) < \sum_{l=1}^{n}[\mu(\mathcal{O}_l) + (n-1)\delta]t(\mathcal{O}_l) + n\delta\bar{t} \\
\Longrightarrow\quad & -\delta\sum_{l=1}^{n} t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t\, \mu'(d\omega) - \int_{\Omega^*} t\, \mu(d\omega) < (n-1)\delta\sum_{l=1}^{n} t(\mathcal{O}_l) + n\delta\bar{t} \\
\Longrightarrow\quad & -2n\delta\bar{t} < \int_{\Omega^*} t\, \mu'(d\omega) - \int_{\Omega^*} t\, \mu(d\omega) < n^2\delta\bar{t}.
\end{aligned}$$

This proves that $\left|\int_{\Omega^*} t\,\mu'(d\omega) - \int_{\Omega^*} t\,\mu(d\omega)\right| < \max\{2n\delta\bar{t}, n^2\delta\bar{t}\} = \varepsilon$. ■

Proof of Proposition 3 Notice that, for each of the mechanisms used in the proof of Proposition 2, the total transfer $(t_1+t_2)(\omega)$ satisfies the conditions of Lemma 7. For example, consider the mechanism $\Gamma(x)$ in Lemma 2. For any (v_1, v_2), the set of universal type profiles in which the valuation pair is (v_1, v_2) is open in the product topology with μ-null boundary. Moreover, since $\mu(b = x) = 0$, the event $b < x$ is also open in the product topology with μ-null boundary. Therefore, we can take $\mathcal{O}_1, \ldots, \mathcal{O}_6$ to be the interiors of the sets represented by the cells of the table in Figure 8. These open sets are disjoint, have μ-null boundaries, and have total μ-measure equal to 1 as required.

Thus, for any auctioneer's belief μ that is consistent with the distribution ν, there exists an BIC auction $\Gamma(\mu)$ such that $R_\mu\Gamma(\mu) - V^D(\nu) > 0$, and the mapping $\mu' \to R_{\mu'}\Gamma(\mu) - V^D(\nu)$ is continuous at the point $\mu' = \mu$. We can hence apply Lemma 6, taking $K = \mathcal{M}(\nu)$ and $\mathcal{F} = \{R_{(\cdot)}\Gamma - V^D(\nu) : \Gamma \text{ is BIC}\}$. ■

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