Examples of O-minimal Structures

- $\mathbb{R}_{\text{lin}}$, the semilinear context
- $\mathbb{R}_{\text{alg}}$, the semialgebraic context

As I shall discuss later, o-minimality implies a wealth of good analytic and topological properties. This provided ample motivation to seek out o-minimal structures that expand $\mathbb{R}_{\text{alg}}$ to include transcendental data.

We now survey some of the remarkable results that have been obtained beginning in the mid-1980's.

Expansions of $\mathbb{R}_{\text{alg}}$

1. **van den Dries 1986**

Consider the class of *restricted analytic functions*, **an**, where $g: \mathbb{R}^n \to \mathbb{R} \in$ **an** if there is some analytic $f: U^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}$ such that $[0,1]^n \subset U$, $g \restriction [0,1]^n = f \restriction [0,1]^n$, and $g(\bar{x}) = 0$ otherwise.

Let $\mathbb{R}_{\text{an}}$ be the expansion of $\mathbb{R}_{\text{alg}}$ by adding as basic functions all $g \in$ an. Then the structure $\mathbb{R}_{\text{an}}$ is admits elimination down to existential formulas and is o-minimal.

Depends on work of Łojasiewicz and Gabrielov (in the 1960's)

Polynomial growth: Let $f: (a, \infty) \to \mathbb{R}$ be definable in $\mathcal{R}_{\text{an}}$. Then there is some $N \in \mathbb{N}$ such that $|f(x)| < x^N$ for sufficiently large x.

2. Denef-van den Dries 1988

Adjoin to $\mathbb{R}_{\text{an}}$ the function $^{-1}$ given by $x \mapsto 1/x$ for $x \neq 0$ and $0^{-1} = 0$.

Theorem (Denef-van den Dries '88). $(\mathbb{R}_{\text{an}}, ^{-1})$ *admits elimination of quantifiers.*

Comment The languages in (1) and (2) are large, but nonetheless natural. Quantifier elimination always can be achieved by enlarging the language, but no advantage is gained: in general, the quantifier-free sets thus obained can be horribly badly behaved.

3. Wilkie 1991

Theorem (Wilkie '91). $\mathbb{R}_{\text{exp}}$ *admits elimination down to existential formulas.*

O-minimality then follows by a result of Khovanskii 1980 (which Wilkie also uses in his proof).

This theorem addresses a question posed originally by Tarski. He asked if his results on $\mathbb{R}_{\text{alg}}$ could be extended to $\mathbb{R}_{\text{exp}}$. Wilkie's result from the syntactic and topological points of view is best possible.

On Computational Decidability

Macintyre and Wilkie link decidability of the theory of the real exponential field to

Schanuel's Conjecture. *Let* $r_1, \ldots, r_n \in \mathbb{R}$ *be linearly independent over* $\mathbb{Q}$*. Then the transcendence degree over* $\mathbb{Q}$ *of*

$$\mathbb{Q}(r_1, \ldots, r_n, e^{r_1}, \ldots, e^{r_n})$$

is at least n*.*

Theorem (Macintyre-Wilkie 1993). *Schanuel's conjecture implies that the theory of the real exponential field is decidable.*

At present, Schanuel's conjecture is generally thought to be intractable.

An Aside

Using elimination down to existential formulas, some commutative algebra, and a functional theoretic version of Schanuel due to Ax 1971, Bianconi proves

Theorem (Bianconi '95). *No arc of the sine function is definable in* $\mathbb{R}_{\exp}$ *and conversely no restriction of the exponential function is definable in* $(\mathbb{R}_{\exp}, \sin \restriction [-\pi, \pi])$

4. van den Dries-Miller 1992

They adapt Wilkie's techniques to prove

Theorem (van den Dries-Miller '92). $\mathbb{R}_{\mathrm{an},\exp}$ *admits elimination down to existential formulas and (by Khovanskii) is o-minimal.*

5. **van den Dries-Macintyre-Marker 1992**

Inspired by work of Ressayre 1992, they analyze $\mathbb{R}_{\mathrm{an,exp}}$ further.

Theorem (D-M-M 1992). $\mathbb{R}_{\mathrm{an,exp,log}}$ *admits elimination of quantifiers.*

Their analysis further shows that every definable function in one variable is bounded by an iterated exponential.

Macintyre-Marker 1996 show that log is necessary for the quantifier elimination.

In a second paper (1995) D-M-M develop tools that enable them to obtain several further results.

Problem of Hardy

Let $f(x) = (\log x)(\log\log x)$ and let $g(x)$ be a compositional inverse to f defined on some interval (a, ∞). Hardy conjectured in 1912 that g is not asymptotic to a composition of $\exp, \log,$ and semialgebraic functions.

Theorem (D-M-M 1995). *Hardy's conjecture is true.*

Building on some remarkable ideas and results of Mourges-Ressayre 1993, D-M-M derive some "undefinability" results also.

Theorem (D-M-M 1995). *None of the following functions is definable in* $\mathbb{R}_{\mathrm{an,exp}}$.

i. *the restriction of the gamma function*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

to $(0, \infty)$

ii. *the error function* $\int_0^x e^{-t^2}\, dt$

iii. *the logarithmic integral* $\int_x^\infty t^{-1} e^{-t}\, dt$

iv. *the restriction of the Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to $(1, \infty)$

6. Miller 1994

For $r \in \mathbb{R}$ let x^r denote the real power function

$$x^r = \begin{cases} x^r & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ denote the expansion of $\mathbb{R}_{\text{an}}$ by all power functions x^r for $r \in \mathbb{R}$.

Theorem (Miller 1994). $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ *has elimination of quantifiers.*

An expansion $\mathfrak{R}$ of $\mathbb{R}_{\text{alg}}$ is *polynomially bounded* if for every definable $f\colon \mathbb{R} \to \mathbb{R}$ there is some $N \in \mathbb{N}$ so that

$$|f(x)| < x^N \quad \text{for sufficiently large } x.$$

Growth Dichotomy Theorem (Miller 1992). *Let $\mathfrak{R}$ be an o-minimal expansion of $\mathbb{R}_{\text{alg}}$. Then either the exponential function e^x is definable in $\mathfrak{R}$ or $\mathfrak{R}$ is polynomially bounded. In the second case, for every definable $f\colon \mathbb{R} \to \mathbb{R}$ in $\mathfrak{R}$ not ultimately identically zero, there are $c \in \mathbb{R} \setminus \{0\}$ and $r \in \mathbb{R}$ such that $f(x) = cx^r + o(x^r)$ as $x \to \infty$.*

7. van den Dries-Gabrielov 1993

Let $\partial\Phi$ be a collection of restricted analytic functions that is closed under differentiation.

Theorem. *The structure $(\mathbb{R}_{\text{alg}}, f)_{f \in \partial\Phi}$ has elimination down to existential formulas.*

8. van den Dries-Speissegger 1996

Using (delicate) generalized power series methods new expansions of $\mathbb{R}_{\text{alg}}$ are constructed.

There are two polynomially bounded versions that have elimination down to existential formulas.

Moreover, the exponential function can be added while preserving o-minimality. If, in addition the logarithmic function is adjoined as a basic function, these expansions admit quantifier elimination.

In one of these expansions, the gamma function on $(0, \infty)$ is definable, and in the second, the Riemann zeta function on $(1, \infty)$ is definable.

9. Wilkie 1996

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *Pfaffian* if there are functions $f_1, \ldots, f_k\colon \mathbb{R}^n \to \mathbb{R}$ and polynomials $p_{ij}\colon \mathbb{R}^{n+i} \to \mathbb{R}$ such that

$$\frac{\partial f_i}{\partial x_j}(\bar{x}) = p_{ij}(\bar{x}, f_1(\bar{x}), \ldots, f_i(\bar{x}))$$

for all $i = 1, \ldots, k$, $j = 1, \ldots n$, and $\bar{x} \in \mathbb{R}^n$.

Wilkie proves (by quite different methods than used previously) that the expansion of $\mathbb{R}_{\mathrm{alg}}$ by all Pfaffian functions is o-minimal.

10. Speissegger 2000

Here Wilkie's methods are extended to obtain the "Pfaffian closure" of an o-minimal expansion of $\mathbb{R}_{\mathrm{alg}}$. In particular, such a structure is closed under integration (antidifferentiation) of functions in one-variable.

Finer Analytic and Topological Consequences of O-minimality

For this section, assume throughout that we work in some o-minimal expansion $\mathfrak{R}$ of $\mathbb{R}_{\text{alg}}$.

$\mathcal{C}^k$ Cell Decomposition Theorem. *For each definable set $X \subset R^m$ and $k = 1, 2, \ldots$, there is a decomposition of $\mathbb{R}^m$ that respects X and for which the data in the decomposition are $\mathcal{C}^k$.*

Triangulation Theorem. *Every definable set $X \subset R^m$ is definably homeomorphic to a semilinear set. More precisely, X is definably homeomorphic to a union of simplices of a finite simplicial complex in R^m.*

Number of Homeomorphism Types. *Let $S \subset R^{m+n}$ be definable, so that $\{S_{\bar{a}} \mid \bar{a} \in \mathbb{R}^m\}$ is a definable family of subsets of $\mathbb{R}^n$. Then there is a definable partition $\{B_1, \ldots, B_p\}$ of R^m such that for all $\bar{a}_1, \bar{a}_2 \in \mathbb{R}^m$, the sets $S_{\bar{a}_1}$ and $S_{\bar{a}_2}$ are homeomorphic if and only if there is some $j = 1, \ldots, p$ such that $\bar{a}_1, \bar{a}_2 \in B_j$.*

Uniform finiteness combined with Wilkie's theorem yields

Theorem (Khovanskii). *There exists a bound in terms of m and n for the number of connected components of a system of n polynomial inequalities with no more than m monomials.*

An o-minimal improvement: *there is a bound in terms of m and n for the number of homeomorphism types of the zero sets in $\mathbb{R}^n$ of polynomials $p(x_1, \ldots, x_n)$ over $\mathbb{R}$ with no more than m monomials.*

Theorem (Marker-CS 1994). *Let* $\mathfrak{R}$ *be an o-minimal expansion of* $\mathbb{R}_{\text{alg}}$*, and let*

$$g_{\bar{a}} \colon B \subseteq \mathbb{R}^m \to \mathbb{R} \quad \textit{for } \bar{a} \in A \subset \mathbb{R}^m$$

be an $\mathfrak{R}$*-definable family* $\mathcal{G}$ *of functions. Then every* $f \colon B \subseteq \mathbb{R}^m \to \mathbb{R}$ *which is in the closure of* $\mathcal{G}$ *is definable in* $\mathfrak{R}$*.*

Euler Characteristic

Let $S \subset \mathbb{R}^n$ be definable and $\mathcal{P}$ be a partition of S into cells. Let

$$n(\mathcal{P}, k) = \#\text{ cells of dimension } k \text{ in } \mathcal{P}$$

and define

$$E_{\mathcal{P}}(S) = \sum(-1)^k n(\mathcal{P}, k)$$

Proposition. *If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of S into cells, then $E_{\mathcal{P}}(S) = E_{\mathcal{P}'}(S)$.*

So we define $E(S) = E_{\mathcal{P}}(S)$ for any partition $\mathcal{P}$.

Some Properties of the Euler Characteristic

1. *Let A and B be disjoint definable subsets of $\mathbb{R}^n$. Then*

$$E(A \cup B) = E(A) + E(B).$$

2. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be definable. Then*

$$E(A \times B) = E(A)E(B).$$

3. *Let $f\colon A \subset \mathbb{R}^m \to \mathbb{R}^n$ be definable and injective. Then $E(A) = E(f(A))$.*

Some References

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