O-MINIMAL STRUCTURES

An $\mathcal{L}$-structure $\mathfrak{R}$ is *o-minimal* if every definable subset of $\mathbb{R}$ is the union of finitely many points and open intervals (a, b), where $a < b$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Examples (thus far)

- $\mathbb{R}_{\text{lin}}$, the semilinear context
- $\mathbb{R}_{\text{alg}}$, the semialgebraic context

Why "o-minimal"?

- o-minimal is short for ordered minimal.
- ordered minimal because the definable subsets of $\mathbb{R}$ are exactly those that must be there because of the presence of $<$.

Theme The hypothesis of o-minimality combined with the power of definability have remarkable consequences.

Monotonicity Theorem. *Let* $\mathfrak{R}$ *be an* $\mathcal{L}$-*structure that is o-minimal. Suppose that* $f\colon \mathbb{R} \to \mathbb{R}$ *is* $\mathfrak{R}$-*definable. Then there are*

$$-\infty = a_0 < a_1 < \cdots < a_{k-1} < a_k = \infty$$

in $\mathbb{R} \cup \{\pm\infty\}$ *such that for each* $j < k$ *either* $f \upharpoonright (a_j, a_{j+1})$ *is constant or is a strictly monotone bijection of (possibly unbounded) open intervals in* $\mathbb{R}$.

- In particular, all definable $f\colon \mathbb{R} \to \mathbb{R}$ are piecewise continuous.

CELLS

in $\mathbb{R}$

in $\mathbb{R}^2$

More formally, let $\mathfrak{R}$ be an $\mathcal{L}$-structure. The collection of *$\mathfrak{R}$-cells* is a subcollection $\mathcal{C} = \cup_{n=1}^{\infty} \mathcal{C}_n$ of the $\mathfrak{R}$-definable subsets of $\mathbb{R}^n$ for $n = 1, 2, 3, ...$ defined recursively as follows.

Cells in $\mathbb{R}$

The collection of cells $\mathcal{C}_1$ in $\mathbb{R}$ consists of all single point sets $\{r\} \subset \mathbb{R}$ and all open intervals $(a, b) \subseteq \mathbb{R}$, where $a < b$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Cells in $\mathbb{R}^{n+1}$

Assume the collection of cells $\mathcal{C}_n$ in $\mathbb{R}^n$ have been defined. The collection $\mathcal{C}_{n+1}$ of cells in $\mathbb{R}^{n+1}$ consist of two different kinds:

Graphs

Let $C \in \mathcal{C}_n$ and let $f\colon C \subseteq \mathbb{R} \to \mathbb{R}$ be $\mathfrak{R}$-definable and continuous. Then $\mathrm{Graph}(f) \subseteq \mathbb{R}^{n+1}$ is a cell;

Generalized Cylinders

Let $C \in \mathcal{C}_n$. Let $f, g\colon C \subseteq \mathbb{R} \to \mathbb{R}$ be $\mathfrak{R}$-definable and continuous such that $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in C$. Then the cylinder set $(f, g)_C \subset \mathbb{R}^{n+1}$ is a cell.

Some Elementary Properties of Cells

- Cells are $\mathfrak{R}$-definable.
- Cells are connected.
- *Dimension* for cells.

  For each cell $C \subseteq \mathbb{R}^n$ there is a largest $k \leq n$ and $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that if $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is the projection mapping given by

$$\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k}),$$

  then $\pi(C) \subseteq \mathbb{R}^k$ is an open cell in $\mathbb{R}^k$. This value of k we call the *dimension* of C.

Cell Decomposition

An *($\mathfrak{R}$-)decomposition* $\mathcal{D}$ of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many $\mathfrak{R}$-cells satisfying:

> If $n = 1$, then $\mathcal{D}$ consists of finitely many open intervals and points;
>
> If $n > 1$ and $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes projection onto the first $n - 1$ coordinates, then $\{\pi_n(C) : C \in \mathcal{D}\}$ is a decomposition of $\mathbb{R}^{n-1}$.

Cell Decomposition Theorem. *Let $\mathfrak{R}$ be o-minimal and let $S \subset \mathbb{R}^n$ be definable. Then there is a decomposition $\mathcal{D}$ of $\mathbb{R}^n$ every definable set $S \subset M^n$ can be partitioned definably into finitely many cells. In particular, if $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is definable, then there is a partition of A into cells such that the restriction of f to each cell is continuous.*

Some Obvious Consequences

- Using the dimension of a cell as defined above, we obtain a good geometric definition of the dimension of a definable set.
- Since cells are connected, it follows that every definable set has finitely many connected components.
- The topological closure of a definable set consists of finitely many connected components; same for the interior.

Definable families

Let $S \subset \mathbb{R}^{n+p}$ be a definable set in the o-minimal structure $\mathfrak{R}$. For each $\bar{b} \in \mathbb{R}^n$ define

$$S_{\bar{b}} := \{\bar{y} \in \mathbb{R}^p \mid (\bar{b}, \bar{y}) \in S\}.$$

Note Some $S_{\bar{b}}$ may be empty.

The family $\{S_{\bar{b}} \mid \bar{b} \in \mathbb{R}^n\}$ of subsets of $\mathbb{R}^p$ is called a definable family.

Uniform Bounds Theorem. *Let $\mathfrak{R}$ be o-minimal and let $S \subset \mathbb{R}^{n+1}$ be a definable set such that $S_{\bar{b}}$ is finite for all $\bar{b} \in \mathbb{R}^n$. Then there is a fixed $K \in \mathbb{N}$ satisfying*

$$|S_{\bar{b}}| \leq K \quad \textit{for all } \bar{b} \in M^n.$$

Note A definable subset of $\mathbb{R}$ in an o-minimal structure $\mathfrak{R}$ is infinite if and only if it contains an interval. So the theorem actually is stronger.

Quantifier elimination is not easy to come by

Theorem (van den Dries). *Let I be an index set and for each $i \in I$ let $f_i\colon \mathbb{R}^{n_i} \to \mathbb{R}$ be (total) analytic functions. Then the structure $(\mathbb{R}_{\text{alg}}, \{f_i : i \in I\}$ admits quantifier elimination if and only if each f_i is semialgebraic.*

So, e.g., $\mathbb{R}_{\text{exp}}$, the real exponential field does not have quantifier elimination.

Partial Elimination

Suppose that a structure $\mathfrak{R}$ has the property that every definable set is definable by an *existential formula*, that is, a formula having the form

$$\exists x_1 \, \exists x_2 \cdots \exists x_k \, \varphi$$

where φ is a quantifier-free $\mathcal{L}$-formula.

How can this help?

- Suppose that the $\mathfrak{R}$-definable sets that are definable using quantifier-free formulas can be analyzed, and that all such have finitely many connected components.
- The continuous image of a connected set is connected (elementary topology).
- Existential quantification corresponds to projection, and projection is a continuous map.
- Thus all $\mathfrak{R}$-definable subsets of $\mathbb{R}$ have finitely many connected components, that is, all such are the union of finitely many points and open intervals.
- Conclusion $\mathfrak{R}$ is o-minimal, and so all the geometric and topological properties available as consequences of o-minimality apply.

Some References

L. van den Dries, *Tame Topology and O-minimal Structures* (London Mathematical Society Lecture Note Series, vol. 248), Cambridge: Cambridge University Press, 1998.

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