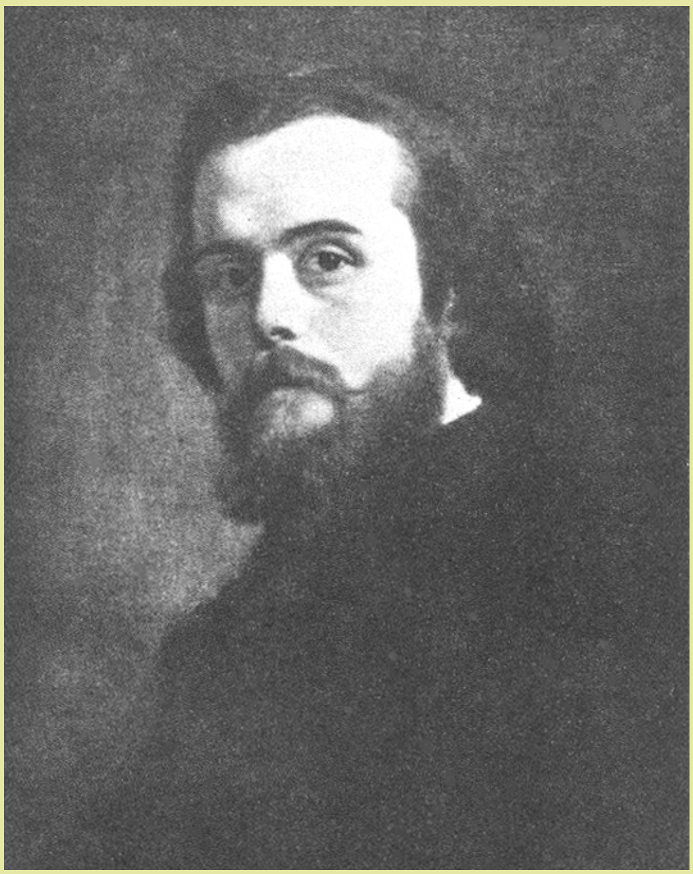
Indeterminacy, Nonparametric Calibration and Counterfactual Equilibria



Léon Walras
1834-1910

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The Walrasian Paradigm

Walras
Elements of Pure Economics (1874)

(a) Walrasian Hypothesis
(b) Tâtonnement Price Adjustment
(c) Comparative Statics

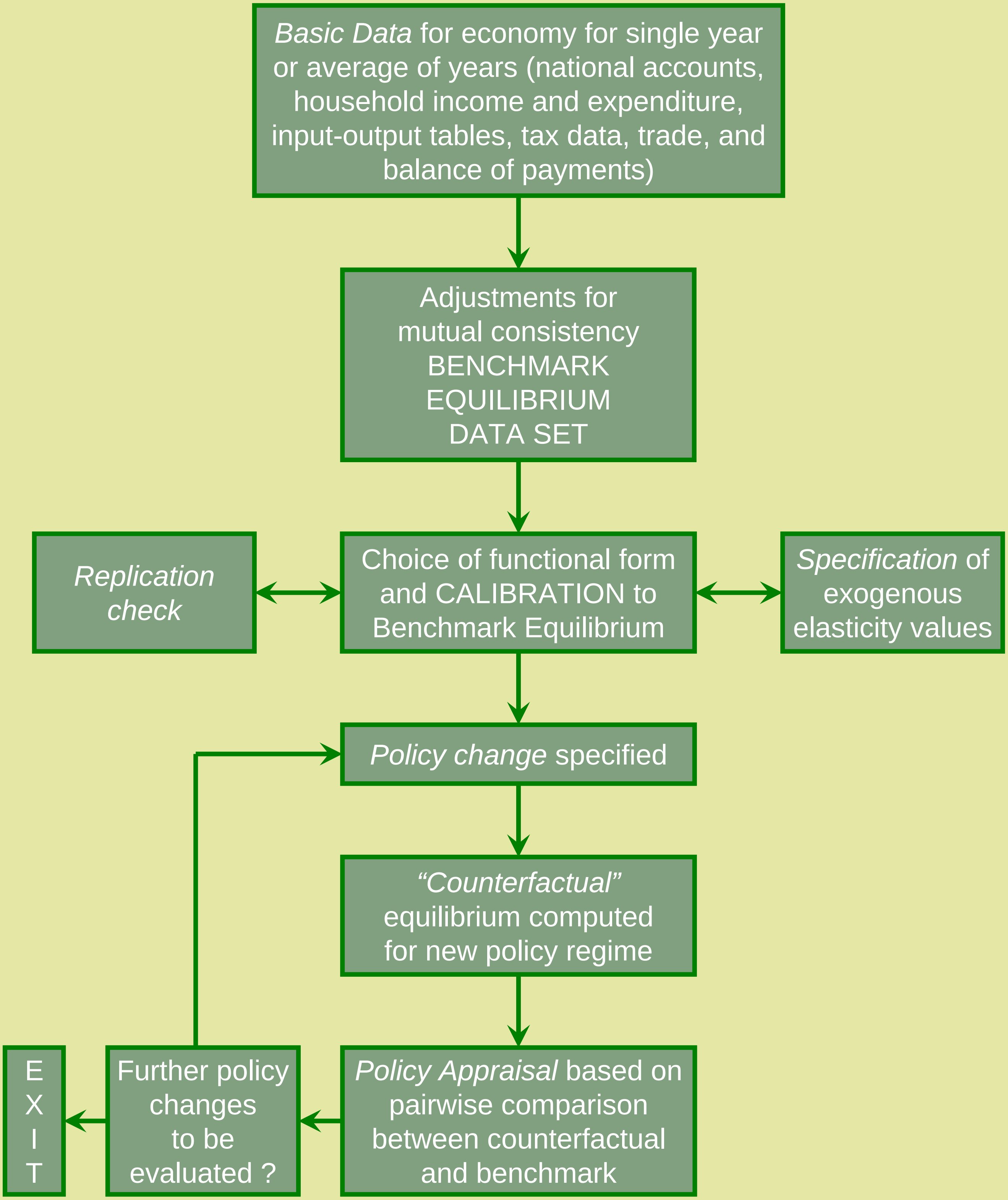
Shoven & Whalley
Applying General Equilibrium (1992)

(a) Indeterminacy
(b) Parametric Calibration
(c) Counterfactual Equilibria

Scarf
Computation of Economic Equilibrium (1973)

(a) Existence
(b) Numerical Specification
(c) Computation

Counterfactual Policy Analysis



From ***Applying General Equilibrium*** **by Shoven and Whalley**

Equilibrium Analysis on Finite Data Sets

Brown and Matzkin
***Econometrica*, 1996**
"Existence of Walrasian Equilibria Is Refutable"

Brown and Shannon
***Econometrica*, 2000**
"Local Tâtonnement Stability Is Not Refutable"

Brown and Calsamiglia
CFDP 1399, 2003
"Existence of Random Marshallian Equilibria Is Refutable"

Afriat's Theorem

Definition. A utility function $u(x)$ rationalizes a set of observations (p_i, x_i) for $i = 1, 2, \ldots, n$. If $u(x_i) \geq u(x)$ $\forall x$ s.t. $p_i x_i \geq p_i x$.

Theorem (Afriat, *International Economic Review*, 1967). The following conditions are equivalent:

(1) $\exists$ a nonsatiated utility function that rationalizes the data.

(2) the data satisfies "cyclical consistency," i.e., $p_r x_r \geq p_r x_s$, $p_s x_s \geq p_s x_t$, …, $p_q x_q \geq p_q x_r$ $\Rightarrow p_r x_r = p_r x_s = \ldots p_q x_r$.

(3) $\exists$ numbers U_i, $\lambda_i > 0$, $i = 1, \ldots, n$, s.t. $U_i \leq U_j + \lambda_j p_j (x_i - x_j)$ for $i, j = 1, \ldots, n$

(4) $\exists$ a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.

Condition (3) defines the Afriat Inequalities.

The Walrasian Inequalities

Suppose there are 2 periods, 2 consumers and C commodities. Then the dual Walrasian equilibrium inequalities are:

(1) The strict dual Afriat inequalities for each agent:

(a) $V_1 > V_2 - \lambda_2 x_2 I_2 \left(\frac{p_1}{I_1} - \frac{p_2}{I_2} \right)$ (b) $W_1 > W_2 - \beta_2 y_2 J_2 \left(\frac{p_1}{I_1} - \frac{p_2}{I_2} \right)$

$$V_2 > V_1 - \lambda_1 x_1 I_1 \left(\frac{p_2}{I_2} - \frac{p_1}{I_1} \right) \qquad W_2 > W_1 - \beta_1 y_1 J_1 \left(\frac{p_2}{I_2} - \frac{p_1}{I_1} \right)$$

$$p_1 x_1 = I_1 \qquad p_1 y_1 = J_1$$

$$p_2 x_2 = I_2 \qquad p_2 y_2 = J_2$$

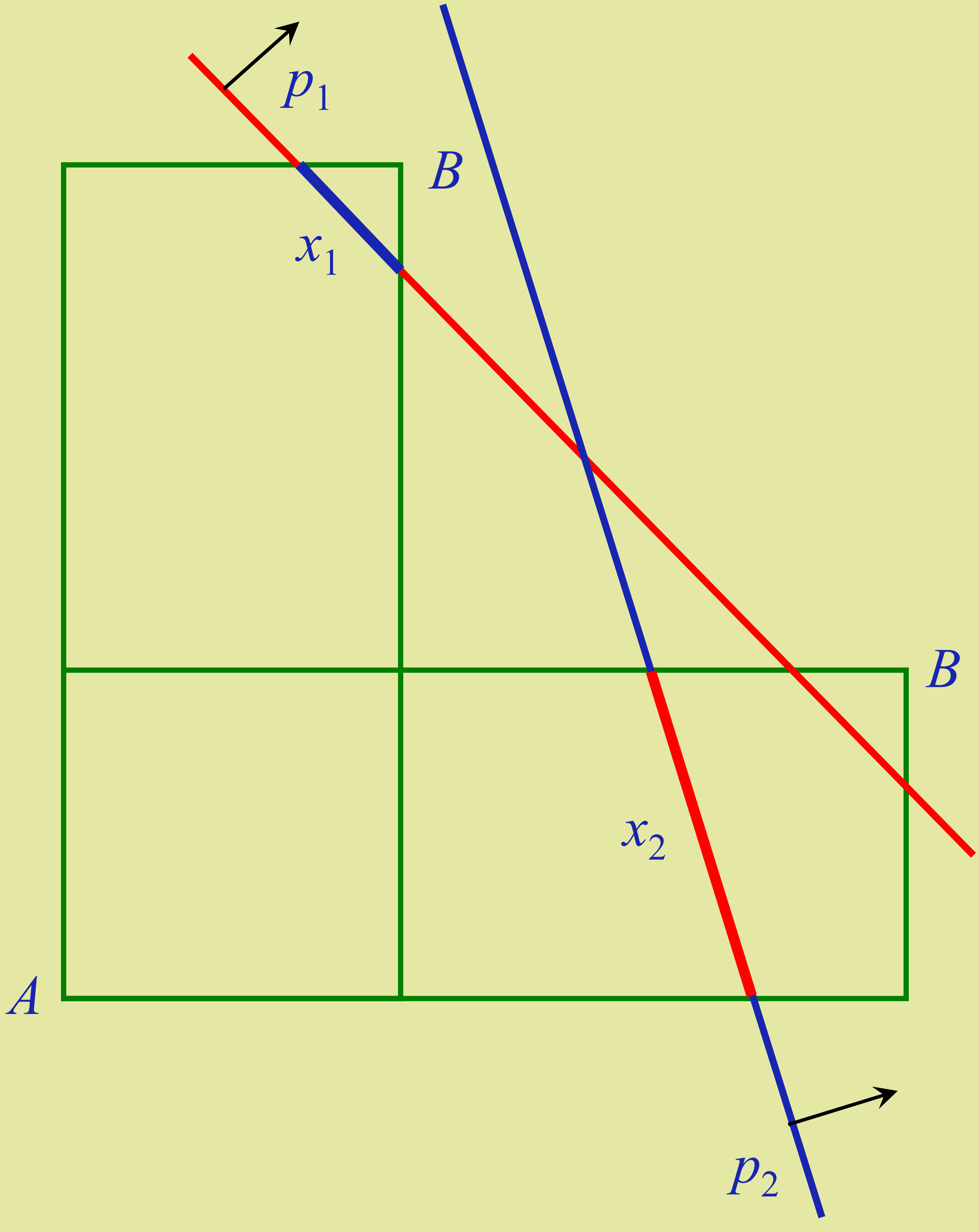
(2) The market clearing conditions in each period, i.e., supply = demand

(a) $x_1 + y_1 = e_1$ and (b) $x_2 + y_2 = e_2$

We observe price vectors p_i , incomes I_i and J_i , and total endowment vectors e_i , for $i = 1, 2$. These observations constitute the data set $\mathfrak{D}$.

The unknowns are V_i , W_i , λ_i , β_i , x_i and y_i for $i = 1, 2$.

Brown-Matzkin Example



WARP: $p_1 \cdot x_2 \leq p_1 \cdot x_1 \Rightarrow p_2 \cdot x_1 > p_2 \cdot x_2$

Methodology

A. *O*-Minimal Structures on the Real Field

1) Quantifier Elimination

2) Vapnik-Čveronkis (VC) Property

B. Random Algorithms

1) Monte Carlo

2) Markov Chain Monte Carlo

C. Applications

1) Specification tests for numerical general equilibrium models

2) Computation of an ε-net of solutions of the Walrasian inequalities, e.g., counterfactual equilibria

Definable Sets

An **ordered structure** on $\mathbb{R}$ is a sequence $S = (S_1, S_2, \ldots)$ where S_n is a Boolean algebra of subsets of $\mathbb{R}^n$ with the following properties

i) $\phi \in S_n$, $\mathbb{R}^n \in S_n$

ii) The set $\{(x,y) : x, y \in \mathbb{R}^n \text{ and } x = y\} \in S_{2n}$

iii) $a \in \mathbb{R}$, then $\{a\} \in S_1$

iv) The set $\{(x, y) : x, y \in \mathbb{R} \text{ and } x < y\} \in S_2$

v) If $A \in S_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A \in S_{n+1}$

vi) If $A \in S_{n+1}$ and $B \subseteq \mathbb{R}^n$ is the projection of A onto the first n-coordinates, then $B \in S_n$

If $A \in S_n$, we say that A is a **definable** set in the structure S.

O-Minimal Structures on $\mathbb{R}$, the Field of Real Numbers

An ordered structure S on $\mathbb{R}$ is o-minimal if every $A \in S_1$ is a finite union of points and intervals with endpoints in $\mathbb{R} \cup \{\pm\infty\}$.

EXAMPLES

1. S_{lin}, the semilinear sets in $\mathbb{R}^n$ for $n = 1, 2, \ldots$
2. S_{alg}, the semialgebraic sets in $\mathbb{R}^n$ for $n = 1, 2, \ldots$
3. S_{exp}, the smallest ordered structure on $\mathbb{R}$ containing S_{alg} and the graph of $x \to e^x$

Exponential Reals

Let $\mathbb{P}$ be a polynomial with real coefficients in $2n$ variables, then the set $\{x \in \mathbb{R}^n : P(x, e^x) = 0\}$ is called an **exponential set** in $\mathbb{R}^n$

A **subexponential set** in $\mathbb{R}^n$ is the image of an exponential set in $\mathbb{R}^{n+k}$, for some k, under the projection map $\mathbb{R}^{n+k} \to \mathbb{R}^n$.

The subexponential sets in $\mathbb{R}^n$ for $n = 1, 2, \ldots$ constitute the **exponential reals**.

Theorem (Wilke, 1991). S_{exp} = *the exponential reals*.

Homothetic Afriat Inequalities

$$\hat{u}_t^r < \hat{u}_t^s \frac{p^s \cdot x_t^r}{p^s \cdot x_t^s} \;\; (r \neq s = 1,\ldots,N;\; t = 1,\ldots,T) \quad (1)$$

$$\hat{u}_t^r > 0 \text{ and } x^r \geq 0 \;\; (r = 1,\ldots,N;\; t = 1,\ldots,T) \quad (2)$$

$$p^r \cdot x_t^r = I_t^r \;\; (r = 1,\ldots,N;\; t = 1,\ldots,T) \quad (3)$$

Following Brown and Lovasz in Brown (1995), let $\hat{u}_t^r = e^{\hat{z}_t^r}$ and rewrite (1) and (2):

$$e^{\hat{z}_t^r - \hat{z}_t^s} < \frac{p^s \cdot x_t^r}{I_t^s} \;\; (r \neq s = 1,\ldots,N;\; t = 1,\ldots,T) \quad (4)$$

$$\hat{z}_t^r \in \mathbb{R} \;\; (r = 1,\ldots,N;\; t = 1,\ldots,T) \quad (5)$$

(4) is a family of polynomial inequalities over the exponential reals in unobservable utility levels $\hat{z}_t^r$ and individual demands x_t^r.

Quantifier Elimination (QE)

1) **Semilinear Sets:** $\mathcal{A}$ is the collection of subsets of $\mathbb{R}^n$ (for each n) that are finite unions of sets of the form $\{x \in \mathbb{R}^n : f_i(x) = 0; g_j(x) > 0\}$ for $i = 1, ..., l; j = 1, ..., k$ where f_i and g_k are real polynomials in n variables of degree at most 1.

   Quantifier Elimination (Dines, Ann. Math., 1918). $\mathcal{A} = S_{\text{lin}}$.

2) **Semialgebraic Sets:** $\mathcal{A}$ is the collection of subsets of $\mathbb{R}^n$ (for each n) that are finite unions of sets of the form $\{x \in \mathbb{R}^n : f_i(x) = 0; g_j(x) > 0\}$ for $i = 1, ..., l; j = 1, ..., k$ where f_i and g_k are real polynomials in n variables.

   Quantifier Elimination (Tarski, *A Decision Method for Elemintary Algebra and Geometry*, 1951). $\mathcal{A} = S_{\text{alg}}$.

3) The exponential reals do **NOT** admit QE

The VC-Property in $\mathcal{O}$-Minimal Structures

For a collection of subsets $\mathcal{C}$ of a nonempty set X and points $x_1, \ldots, x_n \in X$, define

$$\Delta_n(\mathcal{C}; x_1, \ldots, x_n) = \#\{C \cap \{x_1, \ldots, x_n\} : C \in \mathcal{C}\}$$

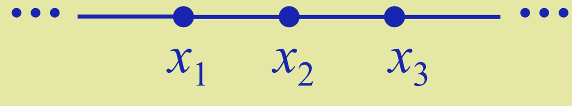
Set $S_{\mathcal{C}}(n) \equiv \max_{x_1,\ldots,x_n} \Delta_n(\mathcal{C}; x_1, \ldots, x_n)$ and define

$$V(\mathcal{C}) \equiv \inf\{n : S_{\mathcal{C}}(n) < 2^n\}$$

Where the infinum over the empty set is $+\infty$.

Definition. A collection $\mathcal{C}$ is called a VC-class if $V(\mathcal{C}) < \infty$, and $V(\mathcal{C})$ is called the VC-dimension of $\mathcal{C}$.

Example. $X = \mathbb{R}^d$, $\mathcal{C} = \{H_{u,c} : \|u\| = 1; c \in \mathbb{R}\}$, where $H_{u,c} = \{x \in \mathbb{R}^d : \langle x,u\rangle + c \geq 0\}$, then $V(\mathcal{C}) = d + 2$. Let $d = 1$ and consider



VC ≡ Vapnik-Chervonenkis

Laskowski's Theorem

Let $\Phi \subseteq \mathbb{R}^{j+n}$ be a definable relation in an O-minimal structure on $\mathbb{R}$, then Φ gives rise to a definable family (with parameter space $\mathbb{R}^n$) $\{\Phi_{\mathrm{b}}\}_{b\in\mathbb{R}^n}$, where $\Phi_b = \{a \in \mathbb{R}^j : (a,b) \in \Phi\}$. The collection of sets $\mathfrak{C} = \{\Phi_b : b \in \mathbb{R}^n\}$ is called a definable collection.

Theorem (Laskowski, *J. London Math. Soc.*, 1992). $\mathfrak{C}$ is a VC-class of sets.

Uniform Versions of Bernouli's Theorem

Let $\mathbb{P}$ be an arbitrary probability distribution on $\mathbb{R}^m$ and suppose $a_1, ..., a_m$ are drawn independently from $\mathbb{P}$. Then $\bar{a} = (a_1, ..., a_m)$ has probability distribution $\mathbb{P}^m$, the m-fold product measure, defined by $\mathbb{P}$.

$\mathbb{P}_{m;\bar{a}}$ is the empirical probability distribution defined by $\bar{a}$, i.e., $\mathbb{P}_{m;\bar{a}}(E)$ = fraction of a_i which lie in E, for each $\mathbb{P}$-measurable set E.

We assume that the function $\rho(\bar{a}) = \sup_{S\in\mathcal{C}}|\mathbb{P}_{m;\bar{a}}(S) - \mathbb{P}(S)|$ is $\mathbb{P}^m$-measurable.

Theorem (Blumer et al., 1989). For fixed δ, $\varepsilon > 0$ if $\mathcal{C}$ is a VC-class of sets on X and $m \geq \max\{(4/\varepsilon)\log 2(2/\delta), ((8d)/\varepsilon)\log_2(13/\varepsilon)\}$ then $\mathbb{P}^m(\{a_1, ..., a_m) : \exists S \in \mathcal{C}$ s.t. $\mathbb{P}(S) > \varepsilon$ and $a_1 \notin S, ..., a_m \notin S\}) < \delta$.

Corollary. With the notations of the Theorem, if $m \geq (8d/\varepsilon)\log_2(13/\varepsilon)$ then $\exists$ a set $\{x_1, ..., x_m\} \subseteq X$ (called an ε-net) such that $S \cap \{x_1, ..., x_m\} \neq \phi$ for every $S \in \mathcal{C}$ such that $\mathbb{P}(S) > \varepsilon$.

A Monte Carlo Algorithm for the Walrasian Decision Problem

Let Φ be the family of dual strict Walrasian inequalities, where the budget constraints and market clearing conditions have been substituted into the dual strict Afriat inequalities. Then Φ induces a definable relation in the *O*-minimal structure S_{alg}.

By Laskowski's Theorem, if a is the vector of unobservables: x_i, y_i, V_i and W_i and b is the vector of unobservables λ_i and β_i in Φ; then the family of nonempty solution sets for each a, $\Gamma(a)$ comprise a VC-class of sets. Moreover, each set in the class is open and has positive Lebesgue measure.

The Algorithm

(1) d, the VC-dimension of $\{\Gamma(a)\}_{a\in\mathbb{R}^J_{++}}$ is $s(J+2)$, where $J = TN$ and s is the number of Afriat inequalities

(2) $\inf_{a\in\mathbb{R}^J_{++},\Gamma(a)\neq\phi} \lambda(\Gamma(a)) \geq v$ where λ is Lebesgue measure on $\mathbb{R}^J$

(3) Choose δ, the confidence level in $(0,1)$

(4) Given v and δ, draw m samples from $[0,1]^J$ endowed with the uniform distribution, where $m \geq \max\left\{\frac{4}{v}\log_2\frac{2}{\delta}, \frac{8d}{v}\log_2\frac{13}{v}\right\}$.

(5) For each draw determine if the dual Walrasian inequalities have a solution, for the given family of marginal utilities of income, using a polynomial-time linear programming algorithm. If there is a solution, then stop.

(6) If after m draws, we have found no solution then stop and with confidence level $1-\delta$, decide that the dual Walrasian inequalities are not solvable for the given data set.