The (S, s) Rule is an Optimal Trading Strategy in a Class of Commodity Price Speculation Problems

George Hall and John Rust[1]

Yale University

March 3, 2000

Abstract: This paper introduces a model of commodity price speculation and proves that the optimal trading strategy is of the (S, s) form. We consider a speculator who can purchase inventories of a durable commodity in a wholesale market at price p_t for subsequent resale to retail customers at price p^r_{t+j} on business day $t + j$, $j \geq 0$. A trading strategy is a rule for purchasing inventories of the commodity q^o_t that depends on information available to the speculator at the start of day t when purchase decisions are made. This information includes the current wholesale market "spot price" p_t, the level of inventories carried over from yesterday q_t, and a vector of other information x_t affecting the speculator's beliefs about retail demand, prices, and storage costs. We prove that the optimal trading strategy takes the form of an (S, s) rule in which the optimal order quantity $q^o(p, q, x)$ is given by $q^o(p, q, x) = S(p, x) - q$ if $q < s(p, x)$ and $q^o(p, q, x) = 0$ otherwise. Following Scarf (1960), the key to proving that the optimal trading strategy is of the (S, s) form is to demonstrate that the value function $V(p, q, x)$ is *K-concave* in its quantity argument q. A sufficient condition for V to be K-concave is that retail prices are set in such a way that a *no expected loss condition* holds. A strong form of this condition is that the retail price on day t always exceeds the expected discounted procurement price on day $t+1$, $p^r_t \geq \beta E\{p_{t+1}|p_t, x_t\}$, where $\beta \in (0, 1)$ is the speculator's discount factor. The optimal inventory level $S(p, x)$ is the level of q that equates the shadow value of an extra unit of inventory to its marginal cost, $\nabla_q V^n(p, S(p, x), x) = p$, where $V^n(p, q, x)$ is the value of not ordering any new inventory in state (p, q, x). The value of the speculator's position is given by $V = \max[V^o, V^n]$, where V^o is the value of ordering sufficient new inventory to attain the target inventory level $S(p, x)$. We show that the no expected loss condition implies that V^n is K-concave as a function of q, and the first point at which V^o and V^n intersect determines the purchase threshold $s(p, x)$. We show that a sufficient condition for $S(p, x)$ to be decreasing in p is that the shadow value of inventory does not increase faster than the wholesale price p at $q = S(p, x)$: $\nabla_p p^s(p, x) \equiv \nabla^2_{pq} V^n(p, q, x) < 1$.

Keywords: inventory investment, K-concavity, (S, s) policy
JEL classification: C13-15
URL: `ftp://gemini.econ.yale.edu/pub/johnrust/steel/ssproof.ps`

---

[1] We are extremely grateful to Herbert Scarf for pointing out an important error in a previous draft of this paper and for suggesting the key argument in a revised proof that fixed the problem. We also benefited from helpful feedback from William Brainard, Zvi Eckstein, Guy Laroque, Jean-Marc Robin, and participants of seminars at Yale, the Operations Research Center at MIT, and the Econometric Society Winter School at the Indian Statistical Institute, New Delhi. We welcome additional comments. Correspondence should be directed to George Hall at Department of Economics, Yale University, 37 Hillhouse Avenue, P.O. Box 208264 New Haven CT 06520-8264, phone: (203) 432-3566, fax: (203) 432-6323, e-mail: `george.hall@yale.edu`.

1 Introduction

This paper introduces a model of commodity price speculation and proves that the optimal trading strategy is of the (S, s) form. We consider a speculator who can purchase inventories of a durable commodity in a wholesale market at price p_t for subsequent resale to retail customers at price p^r_{t+j} on business day $t + j$, $j \geq 0$. A trading strategy is a rule for purchasing inventories of the commodity q^o_t that depends on the information available to the speculator at the start of business day t when purchase decisions are assumed to be updated. This information includes the current wholesale market "spot price" p_t, the level of inventories carried over from yesterday q_t, and a vector of other information x_t affecting retail demand, prices, or storage costs. We prove that the optimal trading strategy takes the form of a (S, s) rule in which the optimal order quantity $q^o(p, q, x)$ is given by $q^o(p, q, x) = S(p, x) - q$ if $q < s(p, x)$ and $q^o(p, q, x) = 0$ otherwise. Following Scarf (1960), the key to proving that the optimal trading strategy is of the (S, s) form is to show that the value function $V(p, q, x)$ is *K-concave* as a function of its quantity argument q. We show that a sufficient condition for the K-concavity of V is that a *no expected loss condition* holds:

$$p^r_t - \beta E\{p_{t+1}|p_t, x_t\} \geq 0, \tag{1}$$

where $\beta \in (0, 1)$ is the speculator's discount factor. This condition states that the retail price of the commodity is always at least as high as the expected discounted wholesale price on business day $t + 1$. This latter quantity represents the expected discounted replacement cost of the unit sold on business day t. This seems to be a mild restriction that would ordinarily be satisfied in practice: if it didn't hold, the speculator would be selling at a loss. If he has power to set his retail price, we would normally expect this condition to be satisfied. If the retail market is highly competitive, the speculator may have little control over p^r_t; however we would expect that if the no expected loss condition didn't hold, speculators would leave the market which would tend to drive up p^r_t and drive down p_{t+1} until the condition is satisfied.

We prove that the no expected loss condition implies that V is K-concave in q for any (p, x), and this in turn implies that an (S, s) trading rule is optimal. We show that $V = \max[V^o, V^n]$ where V^n is a K-concave concave function representing the value of not ordering any new inventory, and V^o is a piecewise linear concave function representing the value of placing an order for the optimal inventory level $S(p, x)$. The optimal inventory level $S(p, x)$ is the level of q that equates the shadow price of an extra unit of q to its marginal cost, $\nabla_q V^n(p, S(p, x), x) = p$. The purchase threshold $s(p,x)$ is the level of inventories at which V^n and V^o first intersect. We show that a sufficient condition for $S(p,x)$ to be decreasing in p is that the shadow value of inventory increases at a slower rate than its marginal cost at $q = S(p,x)$: $\nabla_p p^s(p,q,x) \equiv \nabla^2_{pq} V^n(p,q,x) < 1$.

Our work builds upon and helps to unify two previously separate literatures: the literature on optimal inventory investment (Arrow, Marschak and Harris 1951, Holt *et. al.* 1960, Scarf 1960), and the literature on the rational expectations commodity storage model (see, e.g. Deaton and Laroque, 1992, Miranda and Rui, 1999, and Williams and Wright, 1991, Working, 1949). The latter literature studies the role of commodity storage at an aggregate level, analyzing how the collective behavior of speculators affects prices in the commodity market. However this literature has not explicitly studied the decision problem faced by individual speculators in the commodity market. The literature on optimal inventory investment does focus on the decision problem faced by individual agents, but apparently the connection of this literature to the problem of commodity price speculation has gone unnoticed. All of the inventory-theoretic models that we are aware of focus on the role of inventory decisions in production problems, rather than on inventory management in commodity price speculation problems. Many previous authors[2] have computed (either numerically or using an analytical approximation) generalized (S,s) rules in which the S and s bands are functions of other state variables as an optimal inventory investment policy. However, we are not aware of any explicit proofs of this result — at least in a context that is sufficiently general to be applied to the class of problems we are studying here.

The only previous work that we are aware of that anticipates some of the results in this paper is some recent work in operations research on generalizations of optimal inventory policy with Markovian demands (Sethi and Cheng, 1997 and Cheng and Sethi, 1999) and work by Boizot, Robin, and Visser, 1997, (BRV) on inventory decisions with random sales. BRV derive an analytical expression for the value function and prove the optimality of a generalized version of the (S,s) rule in a model of food purchase and inventory decisions by consumers, where the price of a food can be either high or low. The articles by Sethi and Cheng use the traditional cost-minimization formulation of the inventory problem but introduce a finite-state Markov chain, whose current realized state affects the demand for and cost of acquisition of the commodity. They present a sufficient condition for the K-convexity of the value function and the optimality of the (S,s) rule that is remarkably similar to our no-expected loss condition. Their sufficient condition

[2]For example, see Abel and Eberly (1998).

requires that the marginal shortage cost exceed the expected unit ordering cost less an expected marginal inventory holding cost. We became aware of the Cheng and Sethi result after we wrote this paper.

Our formulation of the inventory management problem is significantly different from those studied by BRV and Sethi and Cheng. Our model was inspired by and was directly tailored to the problem of optimal commodity price speculation introduced in Hall and Rust (1999a). The problem of optimal commodity price speculation is more naturally specified as profit maximization rather than as a cost minimization problem. Our formulation contains a more general specification of the Markov process affecting the demand and acquisition cost of the commodity than the discrete Markov chain formulation considered by Cheng and Sethi. Finally, we allow general transition probabilities for the underlying forcing process $\{p_t, x_t\}$ that can accommodate continuous, discrete, or mixed discrete/continuous laws of motion for these variables.

2 Motivation and Notation

We work with a generalization of the (S, s) model of commodity price speculation introduced by Hall and Rust (1999a,b). This model characterizes the optimal trading policy of a commodity price speculator who is able to purchase bulk quantities of a durable commodity in a wholesale market at price p_t. Time is discrete and indexes successive business days. We assume that it is prohibitively costly for the speculator to resell his inventory in the wholesale market, but he can sell it in a retail market at a price p_t^r. Purchases in the wholesale market are also costly, requiring the speculator to incur a fixed transactions cost K to purchase any positive amount of the commodity, which we denote by $q^o > 0$. The transactions cost discourages the speculator from making frequent small purchases in the wholesale market. In all other respects the wholesale market is perfectly competitive, and the speculator has no ability to affect p_t.

We do allow the speculator to have a limited amount of market power in the retail market. Due to substantial informational frictions, the retail market can be conceptualized as a "telephone market" where sales result from private bilateral negotiations. The search frictions provide an opportunity for the speculator to charge his retail customers a potentially randomly varying markup over the wholesale market price p_t. While it seems reasonable to expect that the retail price p_t^r should satisfy $p_t^r \geq p_t$ with probability 1, if it is impossible or prohibitively costly for the speculator to re-sell the commodity in the wholesale market, then it is possible that under certain circumstances it could be optimal for the speculator to set $p_t^r < p_t$. Thus, we do not rule out the possibility that the speculator, even if behaving fully optimally, might incur *ex post* losses in his trading. This is why the trading problem we are modeling is best described as speculation rather than arbitrage.

We characterize the optimal trading strategy of a speculator who behaves strategically in the wholesale market by optimally choosing the level of new inventory purchases in the wholesale market, but behaves passively with respect to his sales decisions in the retail market. Let p_t^r denote the retail price at time t. This retail price could either represent the "going price" under the assumption that the retail market is perfectly competitive, or it could represent a price chosen by the speculator under the assumption that the retail market is imperfectly competitive, affording the speculator some control over retail prices. In either case we assume that p_t^r is a draw from a conditional distribution $\gamma(dp^r|p_t, x_t)$. The essence of a passive retail sales policy is that the speculator should be willing to sell his entire inventory $q_t + q_t^o$ to his retail customers at price p_t^r. To see why it might be reasonable to treat p_t^r as random variable with respect to the *ex ante* information available to the speculator at the beginning of business day t, note that the retail price p_t^r the speculator will ultimately charge his retail customers will generally depend on additional signals that the speculator receives about his customers and the state of the retail market during the course of business day t, which are random with respect to the information (p_t, q_t, x_t) available to the speculator at the start of the day. We assume that (p_t, x_t) is a sufficient statistic for these additional signals, so the speculator's beliefs about the retail price p_t^r he will subsequently charge during day t are given by a conditional probability distribution $\gamma(p_t^r|p_t, x_t)$ that depends on (p_t, x_t) but not q_t. The reason why we exclude q_t as an element of this conditioning set will be clear shortly.

Selling a unit of the commodity on day t yields an immediate revenue of p_t^r, but there is also an opportunity cost of having to replace that unit on business day $t+1$. Thus, a necessary condition for it to be optimal to sell a unit of the commodity on business day t is that the expected revenue, net of the expected discounted cost of replacing that unit via a purchase in the wholesale market on day $t+1$ be non-negative:

$$\int [p_t^r - \beta E\{p_{t+1}|p_t, x_t\}]\,[1 - \eta(p_t^r, p_t, x_t)]\,\gamma(dp_t^r|p_t, x_t) \geq 0. \tag{2}$$

This inequality states that the net expected benefit (conditional on the event that a retail sale actually occurs, which happens with probability $1 - \eta(p_t^r, p_t, x_t)$, is nonnegative, where the expectation is taken with respect to $\gamma(dp_t^r|p_t, x_t)$ which represents the conditional distribution of *ex post* retail prices p_t^r given the *ex ante* information at the start of the day (p_t, x_t). Under some relatively mild regularity conditions, we prove that the no expected loss condition is sufficient to prove the K-concavity of the speculator's value function $V(p, q, x)$ as a function of q for any (p, x). This implies that the (S, s) rule is an optimal trading strategy, i.e. an optimal policy for determining the optimal timing and size of purchases of new inventories of the commodity in the wholesale market.

We do not know whether it is possible to prove that the no expected loss condition holds when retail prices are chosen strategically. In such a case it might be optimal for the speculator to follow a strategic retail sales rule and deliberately refuse to sell the commodity to retail customers under certain situations. However it is hard for us to imagine realistic situations where this would actually be the case. It seems intuitive that if the speculator chooses the retail price strategically, he should be willing to sell to any retail customer willing to purchase at that price. This intuition suggests that that it may be possible to show that the no expected loss condition holds in a model where retail prices are chosen to maximize the speculator's expected discounted profits. We are currently working on this case, formulating more detailed bilateral bargaining models in which the speculator attempts to set an optimal limit price to maximize the amount of surplus he can extract from each customer. This can be viewed as a model of optimal price discrimination since the retail price charged each customer varies, depending on the speculator's beliefs about the customer's reservation value for the commodity.

In either case, the price p_t^r that the speculator sets during day t will generally depend on the quantity of inventory q_t at the start of the day since the latter quantity affects the shadow price of inventory, and thus the relevant opportunity cost for delaying and attempting to sell units in future periods. We conjecture that a necessary condition for any strategic pricing policy for p^r to be optimal is that the speculator should prefer selling at p^r now rather than delaying and selling units in some future period. This suggests that some version of our no expected loss condition will hold in models where retail prices are chosen strategically, although perhaps it will be necessary to include quantity q_t in addition to (p_t, x_t) in the information set. Initial computational experiments suggest that the (S, s) rule does indeed remain optimal even when the speculator is allowed to behave fully strategically in the retail market, charging an optimal limit price p^r that does depend on q. It may be possible to extend our proof of the optimality of the (S, s) rule to models that allow for endogenous optimal price determination in the retail market, but at the present time we do not have a complete proof of this result.

While (S, s) might appear to be a natural and robust trading rule, it is not hard to modify the model in ways that are likely to destroy the optimality of the (S, s) rule. For example, (S, s) is unlikely to remain optimal if the speculator faces significant quantity discounts, or other types of nonlinear pricing schedules in the wholesale market. Characterizing the form of optimal speculative trading strategies under these conditions remains a topic for future research.

Assumption 0: *(Timing of the speculator's information and actions)*

1. *At the start of day t the speculator knows his inventory level q_t, the current spot price p_t, and the values of the other state variables x_t.*
2. *Given (q_t, p_t, x_t) the speculator orders additional inventory q_t^o for immediate delivery.*
3. *Given (p_t, x_t) the speculator sets a retail price p_t^r that is modeled as a random draw from a conditional distribution $\gamma(p_t^r | p_t, x_t)$.*
4. *Given (p_t^r, p_t, x_t) the speculator observes a realized retail demand for the commodity, q_t^r, modeled as a draw from a distribution $H(q_t^r | p_t, p_t^r, x_t)$ with a point mass at $q_t^r = 0$, reflecting the possibility that if the retail or wholesale price are too high, there might not be any retail demand for the commodity on day t.*
5. *The speculator cannot sell more of the commodity than it has on hand, so the actual quantity sold satisfies*
$$q_t^s = \min\left[q_t + q_t^o, q_t^r\right]. \tag{3}$$
6. *The sales in period t determine the level of inventories on hand at the start of the next business day, $t+1$ by the standard inventory identity:*
$$q_{t+1} = q_t + q_t^o - q_t^s. \tag{4}$$
7. *New values of (p_{t+1}, x_{t+1}) are drawn from a Markov transition probability $g(p_{t+1}, x_{t+1} | p_t, x_t)$.*

Assumption 0 implies that the speculator does not face any delivery lags and cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand, the residual unfilled demand is lost. This implies that the amount of the commodity sold each period is the *minimum* of retail demand q_t^r and quantity on hand $q_t + q_t^o$ as given in equation (3).

For technical reasons, it is convenient to assume that the state space for the dynamic programming problem is compact.

Assumption 1: *The intermediary has a maximum storage capacity equal to $\overline{q} \leq \infty$. Further negative orders and inventories (representing backlogs) are not allowed, so q_t^o is restricted to the interval $[0, \overline{q} - q_t]$ and q_t must lie in the interval $[0, \overline{q}]$ with probability 1. The joint Markov process $\{p_t, x_t\}$ has support $P \times X$ where X is a compact subset of R^k and $P = [\underline{p}, \overline{p}]$ where $\underline{p} > 0$ and $\overline{p} < \infty$.*

To understand the implications of Assumptions 0 and 1, we need to describe the speculator's retail sales and revenue in a bit more detail. We assume that the speculator's retail sales on business day t is a random draw q_t^r from a conditional distribution $H(q_t^r | p_t^r, p_t, x_t)$ that depends on the retail price p_t^r, the current spot price p_t, and the values of the other observed state variables x_t. We assume that there is a positive probability $\eta(p^r, p, x) = H(0|p^r, p, x)$ that the speculator will not make any retail sales on a particular day. We assume that there are no other mass points so H can be represented as follows.

Assumption 2: *The conditional probability distribution for the speculator's retail sales on day t is given by:*

$$H(q^r|p^r, p, x) = \eta(p^r, p, x) + [1 - \eta(p^r, p, x)] \int_0^{q^r} h(dq|p^r, p, x), \tag{5}$$

where $\eta(p^r, p, x) \in [0, 1)$ and h is a continuous probability density function over the interval $[0, \infty)$ satisfying $h(q|p^r, p, x) \geq \epsilon > 0$ for all $q \in [0, \overline{q}]$, and all $(p^r, p, x) \in R^+ \times P \times X$.

Since the quantity demanded has support on the $[0, \infty)$ interval, equation (3) implies that there is always a positive probability of a *stockout* given by:

$$\delta(q, p, p^r, x) = 1 - H(q|p^r, p, x). \tag{6}$$

When a stockout occurs, the speculator may incur a per unit "goodwill cost" $g(p^r, p, x) \geq 0$ on the amount of unsatisfied demand. Assumption 3 introduces $g(p^r, q, x)$, the expected goodwill cost of lost sales in state (p, q, x).

Assumption 3: *The speculator faces an ex post per unit cost of* $g(p^r, p, x) \geq 0$ *for any unsatisfied demand due to a stockout.*

The speculator's *ex ante* expected total goodwill costs at the beginning of a business day with information (p, x) and inventory on hand of q is given by:

$$\begin{aligned} EG(p,q,x) &= E\{\tilde{g}(p^r,p,x)\max[(q^r-q),0]|p,q,x\} \\ &= \int_0^\infty g(p^r,p,x)[1-\eta(p^r,p,x)]\int_q^\infty (q^r-q)h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x). \end{aligned} \tag{7}$$

The key to the solution of the speculator's optimal trading is the expected per period retail sales revenue $ES(p, q, x)$. This is just the conditional expectation of realized sales revenue $p^r q^s$, the product of retail price quote p^r times the quantity actually sold $q^s = \min[q + q^o, q^r]$, given the current spot price p, quantity on hand q, and other information x:

$$\begin{aligned} ES(p,q,x) &= E\{\tilde{p}^r\tilde{q}^s|p,q,x\} \\ &= E\left\{\tilde{p}^r E\{\min[q,\tilde{q}^r]|\tilde{p}^r,p,q,x\}|p,q,x\right\} \\ &= \int_0^\infty p^r[1-\eta(p^r,p,x)]\left[\int_0^q q^r h(q^r|p^r,p,x)dq^r + q\int_q^\infty h(q^r|p^r,p,x)dq^r\right]\gamma(dp^r|p,x). \end{aligned} \tag{8}$$

Lemma 1: *If Assumptions 0-3 hold,* $ES(p, q, x)$ *is a strictly increasing and concave function of* q *for each* (p, x)*, and EG(p,q,x) is a strictly decreasing and convex function of* q *for each* (p, x)*.*
Proof: It is straightforward to verify via direct differentiation that

$$\nabla_q ES(p,q,x) = \int_0^\infty p^r[1-\eta(p^r,p,x)]\int_q^\infty h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x) > 0 \tag{9}$$

and

$$\nabla_q^2 ES(p,q,x) = -\int_0^\infty p^r[1-\eta(p^r,p,x)]h(q|p^r,p,x)\gamma(dp^r|p,x) < 0, \tag{10}$$

since $h(q|p^r, p, x) \geq \epsilon > 0$ and $\eta(p^r, p, x) < 1$ for all $(p^r, p, x) \in R^+ \times P \times X$. The properties of $EG(p, q, x)$ can be verified via a similar calculation.

Assumption 4: *The intermediary incurs a per period physical storage cost* $c^h(p, q, x)$ *of holding inventory level* q*, where* c^h *is nondecreasing, convex function of* q *for all* $(p, x) \in P \times X$*.*

We assume the intermediary incurs a cost of ordering q^o units of the commodity for inventory given by a function $c^o(p, q^o, x)$ that is linear in p, but discontinuous at $q^o = 0$ due to a fixed transactions cost of placing orders in the wholesale market.

Assumption 5: *The cost of purchasing q^o units of the commodity in the wholesale market is given by:*

$$c^o(p, q^o, x) = \begin{cases} K + pq^o & \text{if} \quad q^o > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

This specification can be easily modified to account for constant per unit shipping costs ρ, and to allow both ρ and K to depend on x. It will become obvious that all of the results below hold under this more general specification, but since the notation becomes more complex, we will initially ignore shipping costs and assume K is independent of x. The (S, s) rule may not be optimal for other specifications of the order cost function. For example if there are nonlinear pricing schedules such as quantity discounts that cause $\nabla_{q^o} c^o(p, q^o, x)$ to decline with q^o, then a generalized (S, s) policy may not be optimal. For notational simplicity we assume that any per unit shipping costs are already embodied in the spot price p, so that at least in this respect the simplified specification of order costs given in assumption 5 involves no loss of generality.

Under these assumptions, the intermediary's single-period profits π equals its sales revenues, less any goodwill costs due to unsatisfied demand, less the cost of new orders for inventory $c^o(p, q^o, x)$ and inventory holding costs $c^h(p, q + q^o, x)$:

$$\pi(p, p^r, q^r, q+q^o, x) = p^r \min[q^r, q+q^o] - g(p^r, p, x) \max[q^r - q - q^o, 0] - c^o(p, q^o, x) - c^h(p, q+q^o, x). \tag{12}$$

The intermediary's inventory investment behavior is governed by the decision rule:

$$q_t^o = q^o(p_t, q_t, x_t), \tag{13}$$

where the function q^o is the solution to:

$$V(p_t, q_t, x_t) = \max_{q^o} E\left\{ \sum_{j=t}^{\infty} \beta^{(j-t)} \pi(p_j, p_j^r, q_j^r, q_j^o + q_j, x_j) \Big| p_t, q_t, x_t \right\}. \tag{14}$$

The value function $V(p, q, x)$ is given by the unique solution to Bellman's equation:

$$V(p, q, x) = \max_{0 \leq q^o \leq \overline{q} - q} \Big[W(p, q + q^o, x) - c^o(p, q^o, x) \Big], \tag{15}$$

where:

$$W(p, q, x) \equiv \Big[ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) + \beta EV(p, q, x) \Big], \tag{16}$$

and EV denotes the conditional expectation of V given by:

$$
\begin{aligned}
& EV(p,q,x) = E\{V(\tilde{p}, \max[0, q-\tilde{q}^d], \tilde{x})|p,q,x) \\
& = \int_{p'}\int_{x'}\int_{p^r} \eta(p^r,p,x)V(p',q,x')\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
& + \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]V(p',0,x')\int_q^\infty h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
& + \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_0^q V(p',q-q^r,x')h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x)g(dp',dx'|p,x).
\end{aligned}
\tag{17}
$$

The optimal decision rule $q^o(p,q,x)$ is given by:

$$
q^o(p,q,x) = \inf \underset{0 \le q^o \le \overline{q}-q}{argmax} \Big[W(p,q+q^o,x) - c^o(p,q^o,x)\Big]. \tag{18}
$$

We invoke the inf operator in the definition of the optimal decision rule in equation (18) to handle the case where there are multiple maximizing values of q^o. This could arise if W is not strictly concave in q.

Definition 0: *A (S,s) rule is a trading strategy of the form:*

$$
q^o(p,q,x) = \begin{cases} 0 & \text{if} \quad q \ge s(p,x) \\ S(p,x) - q & \text{otherwise} \end{cases} \tag{19}
$$

where S and s are functions satisfying $S(p,x) \ge s(p,x)$ for all p and x.

Candidate functions for the upper and lower bands of the generalized (S,s) policy can be defined in terms of the optimal decision rule $q^o(p,q,x)$. The upper band $S(p,x)$ is defined as the optimal order quantity when the speculator has no inventory on hand:

$$
S(p,x) = q^o(p,0,x). \tag{20}
$$

The lower band $s(p,x)$ is the smallest value of q such that desired inventory investment is zero:

$$
s(p,x) = inf\Big\{ q \in [0,\overline{q}] \ \Big| \ q^o(p,q,x) = 0 \Big\}. \tag{21}
$$

It is not difficult to show that desired inventory investment at the upper $S(p,x)$ band is 0: $q^o(p,S(p,x),x) = 0$. Since $s(p,x)$ is the *smallest* value of q satisfying $q^o(p,q,x) = 0$ it follows that $s(p,x) \le S(p,x)$. We show that the speculator is indifferent between ordering and not ordering at $q = s(p,x)$ provided $s(p,x) > 0$. Defining q^o in terms of the functions S and s (which are defined in turn from q^o) may appear circular, but the (S,s) rule does amount to a real restriction on the trading strategy q^o. Substituting for $S(p,x)$ in equation (19) we have

$$
q^o(p,q,x) = q^o(p,0,x) - q, \tag{22}
$$

when $q^o(p,q,x) > 0$ and $q^o(p,q,x) = 0$ otherwise. The (S,s) rule further restricts the set of (p,q,x) for which $q^o(p,q,x) = 0$ to be a a set of the form $\{(p,q,x)|q \geq s(p,x)\}$ for some function $s(p,x)$. Thus, it should be clear that (S,s) rules are indeed a restricted subset of admissible trading strategies and our definition is not tautological.

The weakest known sufficient condition for the optimality of the (S,s) is the K-concavity condition introduced by Scarf (1960). For convenience, we re-state this condition below.

Definition 1: *A function $f : [0,\overline{q}] \to R$ is K-concave if and only if for all $q,z,b \in R$ satisfying $0 \leq q-b \leq q \leq q+z \leq \overline{q}$ we have:*

$$f(q+z) - K \leq f(q) + z\left[\frac{f(q)-f(q-b)}{b}\right]. \tag{23}$$

A function is K-concave if the secant approximation to $f(q+z)$ given on the right hand side of equation (23) exceeds $f(q+z)$ less the constant K. Clearly a concave function is 0-concave, and thus K-concave for all $K \geq 0$. A function $W(p,q,x)$ is K-concave in q if the inequality (23) also holds for W as a function of q for all (p,x). Scarf (1960) actually defined the property of K-convexity, but just as for ordinary convex functions, it is easy to show that f is K-concave if and only if $-f$ is K-convex.

The following lemma summarizes the key properties of K-concave functions. The proof is a trivial modification to the proof of an analogous result characterizing properties of K-convex functions (see Lemma 2.1 in Bertsekas, 1995) and is therefore omitted.

Lemma 2:

1. *A concave function is 0-concave and hence K-concave for all $K \geq 0$.*
2. *If $f_1(q)$ and $f_2(q)$ are K_1-concave and K_2-concave, respectively, for constants $K_1 \geq 0$ and $K_2 \geq 0$, then $\alpha f_1(q) + \beta f_2(q)$ is $(\alpha K_1 + \beta K_2)$-concave for any $\alpha > 0$ and $\beta > 0$.*
3. *If $\{f_n(q)\}$ is a sequence of K-concave functions and $f = \lim_{n\to\infty} f_n$ is the pointwise limit of these functions satisfying $|f(q)| < \infty$ for all $q \in R$, then f is K-concave.*
4. *If f is K-concave and $\tilde{w}$ is a random variable for which $E\{|f(q-\tilde{w})|\} < \infty$ for all q, then $g(q) = E\{f(q-\tilde{w})\}$ is K-concave.*
5. *If f is a continuous, K-concave function on the interval $[0,\overline{q}]$, then there exists scalars $0 \leq s \leq S \leq \overline{q}$ such that*

(a) $f(S) \geq f(q)$ *for all* $q \in [0, \overline{q}]$.

(b) *Either* $s = 0$ *and* $f(S) - K \leq f(0)$ *or* $s > 0$ *and* $f(S) - K = f(s) > f(q)$ *for all* $q \in [0, s)$.

(c) f *is strictly increasing for* $q \in [0, s)$.

(d) $f(z) - K \geq f(q)$ *for all* z *and* q *satisfying* $s \leq q \leq z \leq \overline{q}$.

Scarf established the optimality of the (S, s) via an inductive proof that the value function W is K-convex in q.[3] We will prove an analogous result for K-concave functions, i.e., that the Bellman operator maps the class of K-concave functions, $\mathcal{F}_{KC}$, into itself. Then part (3) of Lemma 2 implies that V and W are uniform limits of sequences of K-concave functions, and must also be K-concave. Lemma 3 verifies the analog of Scarf's basic result for K-convex functions in our setting, namely that K-concavity of W implies the optimality of the (S, s) rule.

Lemma 3: *Suppose the function* $W(p, q, x)$ *is* K*-concave in* q *for all* (p, x). *Let* V *be given by:*

$$V(p, q, x) = \max_{0 \leq q^o \leq \overline{q} - q} [W(p, q + q^o, x) - c^o(p, q^o, x)], \tag{24}$$

and let the (S, s) *bands be given by:*

$$\begin{aligned} S(p, x) &= \inf \underset{0 \leq q^o \leq \overline{q}}{argmax} [W(p, q^o, x) - pq^o]. \\ s(p, x) &= \inf \{q \in [0, S(p, x)] \mid W(p, q, x) \geq W(p, S(p, x), x) - p[S(p, x) - q] - K\}. \end{aligned} \tag{25}$$

Then there is a solution $q^o(p, q, x)$ *to problem (24) that is of the* (S, s) *form with the functions* $S(p, x)$ *and* $s(p, x)$ *given above. The value function* V *can be expressed in terms of* W *and* (S, s) *as:*

$$V(p, q, x) = \begin{cases} W(p, S(p, x), x) - p[S(p, x) - q] - K & \text{if} \quad q \in [0, s(p, x)) \\ W(p, q, x) & \text{if} \quad q \in [s(p, x), \overline{q}]. \end{cases} \tag{26}$$

Proof: The function $S(p, x)$ exists and is well defined since the maximum of a continuous function over a compact set exists by the Theorem of the Maximum. The function $s(p, x)$ exists because the set of q satisfying $W(p, q, x) \geq W(p, S(p, x), x) - p[S(p, x) - q] - K$ is non-empty (for example $q = S(p, x)$ trivially satisfies this inequality). Now we wish to show that the (S, s) rule is indeed optimal. Suppose that $s(p, x) > 0$.

First, consider the case when $q < s(p, x)$. We must have $W(p, q, x) < W(p, S(p, x), x) - p[S(p, x) - q] - K$ otherwise we would have a contradiction of the definition of $s(p, x)$ as the

[3] More precisely, Scarf proves that $W + pq$ is K-concave.

smallest q satisfying $W(p,q,x) \geq W(p,S(p,x),x) - p[S(p,x)-q] - K$. But this implies that the speculator would prefer to order $q^o = S(p,x) - q$ units than to order none. By definition of $S(p,x)$ there is no other order quantity that would yield strictly higher expected discounted profits, so it follows that $q^o(p,q,x) = S(p,x) - q$ is indeed optimal when $s < s(p,x)$. By continuity this also holds at $q = s(p,x)$.

Second, consider the case when $s(p,x) < q < S(p,x)$. Lemma 2 implies that if W is K-concave, then so is the function $W(p,q,x) - pq$. By the definition of K-concavity we have:

$$
\begin{aligned}
&W(p,S(p,x),x) - pS(p,x) - K \\
&\leq W(p,q,x) - pq + \frac{S(p,x)-q}{q-s(p,x)}\left[W(p,q,x) - pq - W(p,s(p,x),x) + ps(p,x)\right]. \qquad (27)
\end{aligned}
$$

Rearranging terms, we have

$$
\begin{aligned}
&W(p,S(p,x),x) - pS(p,x) - K + \frac{S(p,x)-q}{q-s(p,x)}\left[W(p,s(p,x),x) - ps(p,x)\right] \\
&\leq \frac{S(p,x)-s(p,x)}{q-s(p,x)}\left[W(p,q,x) - pq\right]. \qquad (28)
\end{aligned}
$$

However, the definition of $s(p,x)$ implies that

$$
W(p,s(p,x),x) - ps(p,x) \geq W(p,S(p,x),x) - pS(p,x) - K, \qquad (29)
$$

so combining the above inequality with inequality (28) we conclude that

$$
\frac{S(p,x)-s(p,x)}{q-s(p,x)}\left[W(p,S(p,x),x) - pS(p,x) - K\right] \leq \frac{S(p,x)-s(p,x)}{q-s(p,x)}\left[W(p,q,x) - pq\right]. \qquad (30)
$$

The above inequality is algebraically equivalent to the inequality

$$
W(p,S(p,x),x) - p[S(p,x)-q] - K \leq W(p,q,x), \qquad (31)
$$

which says that it is not optimal for the speculator to order when $s(p,x) < q < S(p,x)$. Thus, the (S,s) rule yields the optimal decision $q^o(p,q,x) = 0$ in this case. It is easy to see that the optimal order quantity is also zero when $q = S(p,x)$.

Third and finally, consider the case when $q \in (S(p,x), \overline{q}]$. By K-concavity, for any $z \in [0, \overline{q} - q]$ we have

$$
\begin{aligned}
&W(p,q+z,x) - p(q+z) - K \\
&\leq W(p,q,x) - pq + \frac{z}{q-S(p,x)}\left[W(p,q,x) - pq - W(p,S(p,x),x) + pS(p,x)\right]. \qquad (32)
\end{aligned}
$$

By the definition of $S(p,x)$, the term on the right hand side of the above inequality is non-positive, so rearranging we have

$$W(p, q+z, x) - pz - K \leq W(p,q,x). \tag{33}$$

However this implies that $q^0(p,q,x) = 0$, so the (S,s) rule also yields the correct decision in this case. This concludes the proof.

Define the functions V^n and V^o as follows:

$$V^n(p,q,x) = W(p,q,x) \tag{34}$$

$$V^o(p,q,x) = \begin{cases} W(p, S(p,x), x) - p[S(p,x) - q] - K & \text{if } q \in [0, S(p,x)] \\ W(p,q,x) & \text{if } q \in (S(p,x), \overline{q}] \end{cases} \tag{35}$$

$V^n(p,q,x)$ represents the value of not ordering, whereas $V^o(p,q,x)$ represents the value of ordering the target inventory level $S(p,x)$ if $q \leq S(p,x)$ and not ordering otherwise.

Lemma 4: *Under the assumptions of Lemma 3, V can be represented as:*

$$V(p,q,x) = \max\left[V^n(p,q,x), V^o(p,q,x)\right]. \tag{36}$$

Note that even if both V^n and V^o were concave, since V is a maximum of two concave functions it is not necessarily concave. However we will now show by induction that both V and W are K-concave in q for all (p,x). To minimize the amount of notation required to carry out this induction argument, we introduce the following definition.

Definition 2: *Let $\mathcal{F}_{KC}$ denote the class of functions $V(p,q,x)$ which are K-concave in q for all $(p,x) \in P \times X$.*

Let B denote the Banach space of all continuous functions $W(p,q,x)$ mapping $P \times [0,\overline{q}] \times X$ into R under the usual sup-norm $\|\cdot\|$. Bellman's equation defines the value function $V(p,q,x)$ as the fixed point of a contraction mapping, hence it exists and is unique. We now characterize its properties, and in particular show that both V and W are K-concave in q. We do this by showing that the Bellman operator can be represented as the composition of two operators $\Gamma : B \to B$ and $\Lambda : B \to B$ given by:

$$\Gamma(W)(p,q,x) = \max_{0 \leq q^o \leq \overline{q} - q} \left[W(p, q+q^o, x) - c^o(p, q^o, x)\right]. \tag{37}$$

$$\Lambda(V)(p,q,x) = ES(p,q,x) - EG(p,q,x) - c^h(p,q,x) + \beta EV(p,q,x). \tag{38}$$

Lemma 5: *The value function V is the unique fixed point of the composition operator, $\Gamma \circ \Lambda : B \to B$ given by:*

$$V = \Gamma \Lambda V. \tag{39}$$

The function W is the unique fixed point to the composition operator $\Lambda \circ \Gamma : B \to B$ given by:

$$W = \Lambda \Gamma W. \tag{40}$$

The representation of the Bellman operator in Lemma 5 suggests that if we can prove that V and W are K-concave in q in two steps: 1) first we demonstrate that $\Gamma : \mathcal{F}_{KC} \to \mathcal{F}_{KC}$ and 2) we demonstrate that $\Lambda \circ \Gamma : \mathcal{F}_{KC} \to \mathcal{F}_{KC}$. This will enable us to establish the key induction step of our argument, which will imply that the fixed point V is a uniform limit of functions in $\mathcal{F}_{KC}$, and hence will also be a member of this class.

Lemma 6: *Assumptions $1-5$ imply that $\Gamma : \mathcal{F}_{KC} \to \mathcal{F}_{KC}$. That is, if W is K-concave in q, then $V = \Gamma(W)$ is K-concave in q.*

Proof: We need to show that for any $(p, x) \in P \times X$ and any points $q - b$, q and $q + z$ satisfying $0 \le q - b \le q \le q + z \le \overline{q}$ the function $V = \Gamma(W)$ satisfies the definition of K-concavity

$$V(p, q + z, x) - K \le V(p, q, x) + \frac{z}{b}\left[V(p, q, x) - V(p, q - b, x)\right]. \tag{41}$$

Let $S(p, x)$ and $s(p, x)$ be the (S, s) bands defined in Lemma 3. There are 3 cases to consider, depending on which side of the kink point at $s(p, x)$ the points $q - b$, q and $q + z$ lay on. Since V is linear in q for $q \le s(p, x)$, if $q + z \le s(p, x)$ then all these points lie in the interval $[0, s(p, x)]$ where V is linear, and thus K-concave. Similarly, if $s(p, x) \le q - b$ all of the points lie on the interval $[s(p, x), \overline{q}]$ where $V = W$, and since W is K-concave, then so is V. So the only remaining case to consider is where the points $q - b$, q and $q + z$ straddle the kink in V at $s(p, x)$. In this case we have $0 \le q - b < s(p, x)$ and $q + z > s(p, x)$. Equation (26) implies that $V(p, q + z, x) = W(p, q + z, x)$ and

$$\begin{aligned} V(p, q - b, x) &= W(p, S(p, x), x) - p\left[S(p, x) - (q - b)\right] - K \\ &= W(p, s(p, x), x) - p\left[s(p, x) - (q - b)\right]. \end{aligned} \tag{42}$$

Since Lemma 2, part 5(b) and the fact that $s(p, x) > 0$, we have

$$W(p, s(p, x), x) = W(p, S(p, x), x) - p\left[S(p, x) - s(p, x)\right] - K. \tag{43}$$

If $q < s(p,x)$ then $V(p,q,x) - V(p,q-b,x) = pb$, so that inequality (41) characterizing the K-concavity of V reduces to:

$$V(p,q+z,x) - K \leq W(p,S(p,x),x) - p(S(p,x)-q) - K + pz. \tag{44}$$

This expression can be rearranged into an equivalent inequality

$$V(p,q+z,x) - p(q+z) \leq V(p,S(p,x),x) - pS(p,x), \tag{45}$$

which necessarily holds via the definition of $S(p,x)$ in equation (25). Now if $q > s(p,x)$, then Lemma 3 implies that $V(p,q,x) = W(p,q,x)$. Suppose that q is such that

$$W(p,q,x) - pq \leq W(p,s(p,x),x) - ps(p,x). \tag{46}$$

Via some simple algebra, we see that this inequality is equivalent to the inequality

$$\frac{z}{q-s(p,x)}\left[W(p,q,x) - W(p,s(p,x),x)\right] \leq \frac{z}{b}\left[W(p,q,x) - W(p,s(p,x),x) + p(s(p,x)-q+b)\right], \tag{47}$$

which holds for any $z \geq 0$. Since W is K-concave, we have

$$W(p,q+z,x) - K \leq W(p,q,x) + \frac{z}{s(p,x)}\left[W(p,q,x) - W(p,s(p,x),x)\right]. \tag{48}$$

Using this inequality and inequality (47) we have

$$W(p,q+z,x) - K \leq W(p,q,x) + \frac{z}{b}\left[W(p,q,x) - W(p,s(p,x),x) + p(s(p,x)-q+b)\right]. \tag{49}$$

Using the identity (43) and the definition of V in (26) we see that the above inequality is equivalent to the inequality defining K-concavity of V in (41). The final case is where $q > s(p,x)$ and q satisfies

$$W(p,q,x) - pq > W(p,s(p,x),x) - ps(p,x). \tag{50}$$

Using this inequality and the definition of $S(p,x)$ in equation (25) we have

$$\begin{aligned} W(p,q+z,x) - p(q+z) - K &\leq W(p,S(p,x),x) - pS(p,x) - K = W(p,s(p,x),x) - ps(p,x) \\ &< W(p,q,x) - pq \\ &< W(p,q,x) - pq + \frac{z}{b}\left[W(p,q,x) - pq - W(p,s(p,x),x) + ps(p,x)\right]. \end{aligned}$$

Rearranging terms in the last inequality we obtain

$$W(p,q+z,x)-K<W(p,q,x)+\frac{z}{b}\left[W(p,q,x)-W(p,s(p,x))+p(s(p,x)-q+b)\right], \qquad (51)$$

which is equivalent to the inequality defining the K-concavity in (41).

The next key result, $\Lambda \circ \Gamma : \mathcal{F}_{KC} \to \mathcal{F}_{KC}$, is harder to establish and requires an additional condition. Although it is possible prove this using a weaker sufficient condition (which we will discuss following our proof of Lemma 7), we prefer to use the no expected loss condition below since it is easy to verify and has a simple economic interpretation.

Assumption 6: *(No Expected Loss Condition) With probability 1 the following inequality holds:*

$$\int_{p^r}\left[p^r-\beta\int_{p'}\int_{x'}p'g(dp',dx'|p,x)\right]\left[1-\eta(p^r,p,x)\right]\gamma(dp^r|p,x)\geq 0. \qquad (52)$$

Lemma 7: *Assumptions* $1-6$ *imply that* $\Lambda\circ\Gamma:\mathcal{F}_{KC}\to\mathcal{F}_{KC}$. *That is, if* U *is* K*-concave in* q, *then* $W=\Lambda(\Gamma(U))$ *is* K*-concave in* q.

Proof: By Lemma 6 if $U\in\mathcal{F}_{KC}$, then $\Gamma(U)\in\mathcal{F}_{KC}$. By Lemma 3, there exist functions $S:P\times X\to R$ and $s:P\times X\to R$ satisfying $0\leq s(p,x)\leq S(p,x)\leq\overline{q}$ for which $\Gamma(U)$ can be represented as

$$\Gamma(U)(p,q,x)=\begin{cases}U(p,S(p,x),x)-p[S(p,x)-q]-K & \text{if}\quad q\in[0,s(p,x))\\ U(p,q,x) & \text{if}\quad q\in[s(p,x),\overline{q}].\end{cases} \qquad (53)$$

Although $\Gamma(U)$ is defined for $q\in[0,\overline{q}]$, it can be extended to a function V defined on $(-\infty,\overline{q}]$ by

$$V(p,q,x)=\begin{cases}\Gamma(U)(p,0,x)+pq & \text{if}\quad q\in(-\infty,0]\\ \Gamma(U)(p,q,x) & \text{otherwise.}\end{cases} \qquad (54)$$

It is not difficult to see that the proof of Lemma 6 implies that V is K-concave over the entire interval $(-\infty,\overline{q}]$. Now consider the function $\int_0^\infty V(p,q-q^r,x)h(q^r|p,x)dq^r$. Each translate $V(p,q-q^r,x)$ is K-concave in q over the interval $(-\infty,\overline{q}]$, and positive linear combinations and pointwise limits of K-concave functions are K-concave by Lemma 2; therefore $\int_0^\infty V(p,q-q^r,x)h(q^r|p,x)dq^r$ is K-concave in q on the interval $(-\infty,\overline{q}]$. Using equation (54) we have

$$\int_0^\infty V(p,q-q^r,x)h(q^r|p,x)dq^r=\int_0^q V(p,q-q^r,x)h(q^r|p,x)dq^r+\int_q^\infty V(p,q-q^r,x)h(q^r|p,x)dq^r \qquad (55)$$
$$=\int_0^q V(p,q-q^r,x)h(q^r|p,x)dq^r+V(p,0,x)\int_q^\infty h(q^r|p,x)dq^r+p\int_q^\infty(q-q^r)h(q^r|p,x)dq^r.$$

Using equations (18) and (55), we have

$$
\begin{aligned}
&\Lambda \circ \Gamma(U)(p,q,x) = \Lambda(V)(p,q,x) \\
&= ES(p,q,x) - EG(p,q,x) - c^h(p,q,x) + \beta EV(p,q,x) \\
&= ES(p,q,x) - EG(p,q,x) - c^h(p,q,x) \\
&+\beta \int_{p'}\int_{x'}\int_{p^r} \eta(p^r,p,x)V(p',q,x')\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
&+\beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]V(p',0,x')\int_q^\infty h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
&+\beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_0^q V(p',q-q^r,x')h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
&= ES(p,q,x) - EG(p,q,x) - c^h(p,q,x) \\
&-\beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_q^\infty p'(q-q^r)h(q^r|p,x)dq^r g(dp',dx'|p,x) \\
&+\beta \int_{p'}\int_{x'}\int_{p^r} \eta(p^r,p,x)V(p',q,x')\gamma(dp^r|p,x)g(dp',dx'|p,x) \\
&+\beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_0^\infty V(p',q-q^r,x')h(q^r|p^r,p,x)dq^r\gamma(dp^r|p,x)g(dp',dx'|p,x).
\end{aligned}
$$

The third and fourth terms in the last equation in (56) are K-concave since they are limits of convex combinations of K-concave functions. Since $EG(p,q,x)$ is a convex function of q by Lemma 1 and $c^h(p,q,x)$ is a convex function of q by Assumption 4, a sufficient condition for the K-concavity of $\Lambda \circ \Gamma(U)(p,q,x)$ is that the function

$$
ES(p,q,x) - \beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_q^\infty p'(q-q^r)h(q^r|p,x)dq^r g(dp',dx'|p,x), \tag{56}
$$

is concave in q. It is easy to see this function is continuously differentiable in q with second derivative given by:

$$
\begin{aligned}
&\nabla_{qq}\left[ES(p,q,x) - \beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_q^\infty p'(q-q^r)h(q^r|p,x)dq^r g(dp',dx'|p,x)\right] = \\
&-\int_{p^r}\left[p^r - \beta\int_{p'}\int_{x'} p'g(dp',dx'|p,x)\right][1-\eta(p^r,p,x)]\,h(q|p^r,p,x)\gamma(dp^r|p,x).
\end{aligned} \tag{57}
$$

However, $h(q|p^r,p,x) \geq \epsilon > 0$ by Assumption 2, so the no expected loss condition, Assumption 6, guarantees that expression on the right hand side of equation (57) is non-positive, and this enables us to conclude that $W = \Lambda(V) = \Lambda\Gamma U$ is K-concave in q for any $(p,x) \in P \times X$.

Note that we could have proven Lemma 7 under the weaker condition that the function

$$
\begin{aligned}
&ES(p,q,x) - EG(p,q,x) - c^h(p,q,x) - \\
&\beta \int_{p'}\int_{x'}\int_{p^r} [1-\eta(p^r,p,x)]\int_q^\infty p'(q-q^r)h(q^r|p,x)dq^r g(dp',dx'|p,x),
\end{aligned} \tag{58}
$$

is concave in q for all $(p,x) \in P \times X$. Assuming that c^h is twice continuously differentiable, a sufficient condition for this to hold is that the hessian of this function is negative. This leads to the following more general version of the no expected loss condition:

$$\int_{p^r} \left[p^r + g(p^r,p,x) - \beta \int_{p'} \int_{x'} p' g(dp',dx'|p,x)\right] [1 - \eta(p^r,p,x)]\, \gamma(dp^r|p,x) - \nabla_{qq} c^h(p,q,x) \geq 0. \tag{59}$$

By Assumption 3, $g(p^r,p,x) \geq 0$, and by Assumption 4, $\nabla_{qq} c^h(p,q,x) \leq 0$, so our formulation of the no expected loss condition in Assumption 6 is stronger than necessary to prove our result. Thus, we do not claim that we have found the weakest possible conditions under which our results can be proved; however it is easier to verify and provide a simple economic interpretation for the more restrictive form of the no expected loss condition in Assumption 6. Thus we have opted to stress this version since it is likely that this condition will be satisfied in many practical applications.

Lemma 8: *Under assumptions $1-6$, the functions V and W, the unique fixed points of the contraction mappings given in Lemma 5, are K-concave function of q for all $(p,x) \in P \times X$.*

Proof: We prove this by induction using Lemmas 6 and 7. Since $\Gamma \circ \Lambda$ is a contraction mapping, the fixed point $V = \Gamma\Lambda V$ can be uniformly approximated by the method of successive approximations starting from an initial guess, $V_0 = 0$. We have $\Lambda V_0(p,q,x) = ES(p,q,x) - c^h(p,q,x)$ is concave in q by Assumption 3 and Lemma 1. Since concave functions are automatically K-concave, we have that $\Lambda V_0 \in \mathcal{F}_{KC}$. Lemma 6 implies that $V_1 = \Gamma\Lambda V_0 \in \mathcal{F}_{KC}$. Lemma 7 implies that $\Lambda V_1 = (\Lambda \circ \Gamma)\Lambda V_0 \in \mathcal{F}_{KC}$. Continuing inductively, we see that for each $t \geq 0$ in the sequence of successive approximations, $V_t \in \mathcal{F}_{KC}$. Since the fixed point V is a uniform limit of functions in $\mathcal{F}_{KC}$ it follows that $V \in \mathcal{F}_{KC}$. Since $W = \Lambda\Gamma V$, Lemma 7 also implies that $W \in \mathcal{F}_{KC}$.

Lemmas 1-8 constitute the proof of our main result:

Theorem 1: *Consider the function $W(p,q+q^o,x)$ defined in equation (16), where W is defined in terms of the unique solution V to Bellman's equation (15). Under Assumptions 0-6 for any $(p,x) \in P \times X$ the functions V and W are K-concave in q, and the speculator's optimal inventory investment policy $q^o(p,q,x)$ takes the form of a (S,s) rule. That is, there exist a pair of functions (S,s) satisfying $S(p,x) \geq s(p,x)$ where $S(p,x)$ is the target inventory level and $s(p,x)$ is the* *inventory order threshold, i.e.*

$$q^o(p,q,x) = \begin{cases} 0 & \text{if} \quad q \geq s(p,x) \\ S(p,x) - q & \text{otherwise} \end{cases} \tag{60}$$

where $S(p,x)$ is given by:

$$S(p,x) = argmax_{0 \leq q^o \leq \overline{q}-q} \Big[W(p,q^o,x) - c^o(p,q^o,x) \Big] \tag{61}$$

and the lower inventory order limit $s(p,x)$ is the value of q that makes the speculator indifferent between ordering and not ordering more inventory:

$$s(p,x) = \inf \{ q \in [0,\overline{q}] | W(p,q,x) \geq W(p,S(p,x),x) - p[S(p,x) - q] - K \} . \tag{62}$$

Corollary: *If fixed costs of placing orders are zero, $K = 0$, then the minimum order size is 0, i.e.*

$$S(p,x) = s(p,x). \tag{63}$$

Theorem 2: *$S(p,x)$ is a decreasing function of p iff the shadow price of inventory increases at a slower rate than the wholesale price p, i.e. iff*

$$1 > \nabla^2_{pq} W(p,q,x) \quad \text{at} \;\; q = S(p,x). \tag{64}$$

Proof: Totally differentiating the first-order (Euler) equation for $S(p,x)$ and solving for $\nabla_p S(p,x)$ we get

$$\nabla_p S(p,x) = \frac{1 - \nabla^2_{pq} W(p,q,x)}{\nabla^2_q W(p,q,x)}, \quad q = S(p,x). \tag{65}$$

The denominator is strictly negative since $q = S(p,x)$ is a global optimum of the W function. Thus the sign of $\nabla_p S(p,x)$ depends on the sign of $1 - \nabla^2_{pq} W(p,q,x)$.

3 Example: Scarf's Inventory Model

We conclude the paper by illustrating how our results apply to the model of optimal inventory investment studied by Scarf (1960). Scarf formulated the inventory problem as a cost minimization problem. However if it is recast as a profit maximization problem, then it is easily seen to be a special case of our framework where the vector of state variables x does not enter the model, $\eta = 0$, the wholesale price p is a non-random constant, and the retail price equals a nonrandom constant p^r. Thus the only state variable in Scarf's model is q.

Scarf considered the case where unfilled inventory can be backlogged, represented by negative inventory levels $q < 0$. We assume orders cannot be backlogged and impose the nonnegativity constraint $q \geq 0$. A (S, s) policy in this case is simply two scalars s and S satisfying $s \leq S$ where S is given by

$$S = \underset{0 \leq q \leq \overline{q}}{argmax} \left[W(q) - pq \right], \tag{66}$$

and s is given by

$$s = \inf \{ q \in [0, S] \mid W(q) \geq W(S) - p[S - q] - K \}. \tag{67}$$

It is easy to see that the expected sales function ES is given by

$$\begin{aligned} ES(q) &= p^r E \{ \min[q, q^r] \} \\ &= p^r \left[\int_0^q q^r h(q^r) dq^r + q \int_q^\infty h(q^r) dq^r \right]. \end{aligned} \tag{68}$$

Similarly, the expected value function EV is given by

$$\begin{aligned} EV(q) &= E \{ V(\max[0, q - q^r]) \} \\ &= V(0) \int_q^\infty h(q^r) dq^r + \int_0^q V(q - q^r) h(q^r) dq^r. \end{aligned} \tag{69}$$

The function W (the value of holding inventory q rather than purchasing up to the target quantity S) is given by:

$$W(q) = ES(q) - EG(q) - c^h(q) + \beta EV(q). \tag{70}$$

Bellman's equation is given by

$$V(q) = \max_{0 \leq q^o \leq \overline{q} - q} \left[W(q + q^o) - c^o(q^o) \right], \tag{71}$$

where the order cost function is given in Assumption 5. The no expected loss condition (Assumption 6) reduces to

$$p^r \geq \beta p. \tag{72}$$

If the other regularity conditions in Assumptions 1-6 hold, Theorem 1 guarantees that V and W will be K-concave in q and the (S, s) rule is an optimal inventory policy.

References

[1] Abel, A.B. and J.C. Eberly (1998) "The Mix and Scale of Factors with Irreversibility and Fixed Costs of Investment" *Carnegie-Rochester Conference Series on Public Policy* 48, 101–135.

[2] Arrow, K.J., Harris, T. and J. Marschak (1951) "Optimal Inventory Policy" *Econometrica* 19-3, 250–272.

[3] Bertsekas, D.P. (1995) *Dynamic Programming and Optimal Control* Athena Scientific, Belmont, MA.

[4] Boizot, C. J.-M. Robin, and M. Visser (1997) "The Demand for Food Products: An Analysis of Interpurchase Times and Purchased Quantities" working paper 9748, CREST-INSEE, Paris.

[5] Cheng, F. and S. Sethi (1999) "Optimality of State-Dependent (s, S) Policies in Inventory Models with Markov-Modulated Demand and Lost Sales" *Production and Operations Management* 8-2, 183–192.

[6] Deaton, A. and G. Laroque (1992) "On the Behavior of Commodity Prices" *Review of Economic Studies* 59, 1–23.

[7] Hall, G. and J. Rust (1999a) "A (S, s) Model of Commodity Price Speculation" manuscript, Yale University.

[8] Hall, G. and J. Rust (1999b) "Econometric Methods for Endogenously Sampled Time Series: The Case of Commodity Price Speculation in the Steel Market" `http://cowles.econ.yale.edu/`

[9] Holt,C. Modigliani, F. Muth, J. and H. Simon (1960) *Planning Production, Inventories and Work Force* Prentice-Hall, Englewood Cliffs, N.J.

[10] Miranda, M. and Rui, X. (1999) "An Empirical Reassessment of the Nonlinear Rational Expectations Commodity Storage Model" manuscript, Ohio State University, forthcoming, *Review of Economic Studies.*

[11] Scarf, H. (1960) "The Optimality of (S, s) Policies in the Dynamic Inventory Problem" In *Mathematical Methods in the Social Sciences* K. Arrow, S. Karlin and P. Suppes (ed.), Stanford, CA: Stanford University Press.

[12] Sethi, S. and F. Cheng (1997) "Optimality of (s, S) Policies in Inventory Models with Markovian Demand" *Operations Research* 45-6, 931–939.

[13] Williams, J.C. and B. Wright (1991) *Storage and Commodity Markets* Cambridge University Press, New York.

[14] Working, H. (1949) "Theory of Price and Storage" *American Economic Review* 39, 1254–62.