Is intertemporal choice theory testable ?*

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Abstract

The recursive utility model is a widely used alternative to time-separability. However, we show in this paper that under certainty it does not impose any testable restrictions on a household's savings decisions or on choices in good markets over time. The additional assumption of a weakly separable aggregator is needed to ensure that the assumption of utility maximization restricts choices. Under this assumption, choices in spot markets are characterized by a strong axiom of revealed preferences.

Under uncertainty and stationarity, recursive utility imposes observable restrictions on portfolio-choice even without the assumption of a weakly separable aggregator.

Keywords: Intertemporal choice, non-parametric restrictions.

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1 Introduction

There is a large literature on testing individual demand data for consistency with utility maximization (see e.g. Afriat (1967), Varian (1982) or Chiappori and Rochet (1987)). In this literature it is assumed that one observes how an individual's choices vary as prices and his income vary. However, data of this sort can only be obtained through experiments. If one actually records an individual's actions in markets over time, these classical tests of demand theory might be useless because they neglect the fact that an agent's choices today may be affected by his choice set tomorrow or his savings from previous periods. Tests of demand theory which use market-data must be tests of intertemporal choice models. If one assumes that all agents maximize time-separable and time-invariant utility and only observes their choices in spot-markets (i.e. saving-decisions or incomes are unobservable) the analysis in Chiappori and Rochet (1987) remains valid and a strong version of the Strong Axiom of Revealed Preferences (SSARP, see Chiappori and Rochet (1987)) is necessary and sufficient for data to be consistent with utility maximization. However, time-separability is a very strong restriction on preferences which holds only if one assumes that preference orderings on consumption streams from $t = 1, ..., s$ are independent of an agent's expectations on his consumptions for the periods from $s + 1$ onwards.

While it seems intuitively reasonable to argue that history-independence together with some form of stationarity is enough to ensure that an agent's choice behavior is restricted by the assumption of utility maximization, we show that this intuition is wrong and that the assumption of recursive utility (see Koopmans (1960) or Becker and Boyd (1997)) does not impose any restriction on observed choices. It follows from our analysis that such widely used concepts as time-consistency or pay-off history independence are not testable if one does not use experimental data but is confined to data on individual behavior in markets.

Moreover, we derive a sufficient, additional, condition on the aggregator-function which ensures that the model is testable. If the aggregator function is weakly separable then choices on spot markets must satisfy SSARP. If interest rates are unobservable SSARP is also sufficient for the choices to be rationalizable by a time-separable utility function and the two specifications are therefore observationally equivalent.

Uncertainty adds an additional dimension to the agent's choice problem. Risk-aversion will generally impose restrictions on portfolio selection. The question then arises to what extend these restrictions are observable. We argue that in a model with a finite time-horizon observations on individual behavior can only consist of a single sample path. Potential choices off the sample path can be known only through questioning households. (Experimental data might be less restrictive - this paper focuses on market data). However, if we assume that asset's dividends, incomes and prices are stationary then the entire joint process can be estimated. In this case the assumption of recursive utility as in Kreps and Porteus (1978) imposes restrictions on portfolio-selection even when choices are only known along a sample path.

If off sample path prices and dividends cannot be estimated (or if portfolio-choices are not observable) then the conclusions which we derived under certainty remain valid. In particular for weakly separable recursive utility SSARP is necessary and sufficient for choices in spot markets to be rationalizable. Without assumptions on the aggregator-function there are no restrictions on choices.

The paper is organized as follows. In Section 2 we consider a choice problem under certainty. Section 3 analyses the problem under uncertainty.

2 Does recursive utility impose restrictions on choices?

2.1 The model

We consider a finite horizon choice problem. An individual lives for $T \geq 2$ periods $t = 1, ..., T$. In each period, the individual receives an exogenous income $I_t \in I\!R_+$(either from selling endowments or from transfers) and he is active in spot markets; he faces prices $p_t \in I\!R^L_{++}$ and chooses a consumption bundle $c_t \in I\!R^L_{++}$. (For simplicity we assume that choices do not lie on the boundary.) The individual also faces an interest rate $1/q_t$, $q_t > 0$ and chooses how much to save $s_t \in I\!R$. For simplicity we assume that he faces no borrowing constraints - however we impose the requirement that $s_T = 0$. An observation then consists of a vector $o = (o_t)_{t=1}^T = (I_t, p_t, q_t, c_t, s_t)_{t=1}^T$.

We assume that the agent's choices are always in his budget set. We define the feasible set as $\mathcal{F}(o) = \{(c_t, s_t)_{t=1}^T : s_{t+1} = I_{t+1} + s_t/q_t - p_{t+1}c_{t+1} \text{ for } t = 1, ..., T-1, \ s_T = 0\}$. This is the set of all choices which satisfy the budget-constraint but which do not necessarily imply non-negative consumption (it turns out to be easier to work with this set than with the actual budget set).

To simplify the analysis we assume that the agent does not choose any consumption bundle twice, i.e. that for all $s \neq t = 1, ..., T$, $c_s \neq c_t$. From a data point of view this assumption seems perfectly reasonable - without this assumption we would need to treat the case $c_s = c_t$ separately. We define the set of all observations for which choices satisfy the budget constraint as

$$\mathcal{O} = \{(o_t)_{t=1}^T : o_t \in I\!R \times I\!R_{++}^L \times I\!R \times I\!R_{++}^L \times I\!R, (c_t, s_t)_{t=1}^T \in \mathcal{F}(o),\ c_s \neq c_t \text{ for all } s \neq t\}$$

2.2 Recursive utility

Following the definitions from Koopmans (1960) (see Kreps and Porteus (1978) for a finite horizon formulation under uncertainty) we say that an agent's utility function $u : I\!R_+^{LT} \to I\!R$ is recursive if $u((c_t)_{t=1}^T) = v_1$, where v_t, utility at time t, is recursively defined by

$$v_t = W(c_t, v_{t+1}) \text{ for } t = 1, ..., T,\ v_{T+1} = 0$$

We will assume throughout that the aggregator $W : I\!R_{++}^L \times I\!R_+ \to I\!R_+$ is twice continuously differentiable, strictly increasing and strictly concave. We assume that $W(.)$ is well defined at zero and normalize $W(0,0) = 0$.

It is well known that this utility function ensures time-consistency and pay-off history independence. However, the assumption of recursive utility does not restrict individual choice behavior unless one makes further assumptions on the aggregator $W(.)$. The intuitive reason for this is that changes in future consumption can change marginal rates of substitution for all goods in the current period.

In order to demonstrate this result formally we need the following definition.

Definition 1 *An observation $(\bar{o}_t)_{t=1}^T \in \mathcal{O}$ is said to be rationalizable by a utility function $u(.)$ if*

$$(\bar{c}_t, \bar{s}_t)_{t=1}^T \in \arg \max_{c \in I\!R_+^{LT}, s \in I\!R^T} u(c) \ \ s.t.\ \ (c_t, s_t)_{t=1}^T \in \mathcal{F}(o)$$

For our non-parametric analysis we need to derive Afriat-inequalities (Afriat (1967)). These non-linear inequalities completely characterize choices which are consistent with the maximization of a recursive utility function.

Lemma 1 *An observation* $(o_t)_{t=1}^T \in \mathcal{O}$ *is rationalizable by a recursive utility function if and only if there are* $U_t \in I\!R_+$ *and* $N(t) \in I\!R^2_{++}$ *for all* $t = 1, ..., T$ *such that*

1. *For all* $t = 1, ..., T-1$,

$$q_t N_1(t) = N_2(t) N_1(t+1) \tag{C1}$$

For all $s \neq t = 1, ..., T$,

$$U_s < U_t + N_t \cdot \left[\begin{pmatrix} p_t c_s \\ U_{s+1} \end{pmatrix} - \begin{pmatrix} p_t c_t \\ U_{t+1} \end{pmatrix} \right]. \tag{C2}$$

Proof.

Consider the agent's (necessary) first order conditions for optimality. If λ_t denotes the Lagrange-multiplier associated with period t, they can be written as:

$$\partial_c W(c_t, v_{t+1}) \prod_{s=1}^{t-1} \partial_v W(c_s, v_{s+1}) - \lambda_t p_t = 0 \text{ for } t = 1, ..., T$$

where we define $\prod_{s=1}^{t-1} \partial_v W(c_s, v_{s+1}) := 1$ for $t = 1$, and

$$q_t \lambda_t = \lambda_{t+1} \text{ for } t = 1, ..., T-1$$

The necessity part now follows if one defines $N_1(t) = \frac{\lambda_t}{\prod_{s=1}^{t-1} \partial_v W(c_s, v_{s+1})}$, $N_2(t) = \partial_v W(c_t, v_{t+1})$ and $U_t = v_t$. Inequality (C2) characterizes strict concavity of $W(.)$, where the first optimality-condition is used to substitute for $\partial_c W$.

For the sufficiency part one can construct piecewise linear utility functions as in Varian (1982):

Define $W(c, v) = \min_{t=1,...,T} \left\{ U_t + N_t \left[\begin{pmatrix} p_t c \\ v \end{pmatrix} - \begin{pmatrix} p_t c_t \\ U_{t+1} \end{pmatrix} \right] \right\}$

The resulting function is clearly concave and strictly increasing, $W(c_t, U_{t+1}) = U_t$ and the function rationalizes the observation o. Furthermore, the approach in Chiappori and Rochet (1987) can be used to construct a strictly concave and smooth aggregator function. Their argument goes through without any modification.□

We now use this characterization to show that the assumption of recursive utility is not testable since it imposes no restrictions on observed choices.

Theorem 1 *Any possible observation $O \in \mathcal{O}$ can be rationalized by some recursive utility function.*

Proof.

Solving for $(N_1(t))$ in Equation (C1) and writing M_t for $N_2(t)$ we obtain that an observation $o \in \mathcal{O}$ can be rationalized if and only if there exist $(U_t, M_t)_{t=1}^T \in I\!R_{++}^{2T}$ such that for all $s \neq t = 1, ..., T$

$$U_s < U_t + \left(\prod_{j=1}^{t-1} \frac{q_j}{M_j} \right) (p_t c_s - p_t c_t) + M_t (U_{s+1} - U_{t+1}).$$

In order to prove the theorem, it suffices to show that for small enough $\epsilon > 0$, there exist $(U_t, M_t)_{t=1}^T \in I\!R_{++}^{2T}$ with arbitrarily large M_1 and $M_t > 1$ for all t, such that for all $s, t = 1, ..., T$, $\epsilon < U_t - U_s + M_t(U_{s+1} - U_{t+1})$. This is true because for any $\delta > 0$ (in particular $\delta < \epsilon$) and any $o \in \mathcal{O}$, we can always choose $(M_t)_{t=1}^T$ to ensure that $\prod_{j=1}^{t-1} \frac{q_j}{M_j}(p_t c_s - p_t c_t) < \delta$.

But for any arbitrary choice of M_1, one can choose $(M_t)_{t=2}^T$ with $M_t > M_{t-1}$ and one can define $U_{T+1} = 0$, $U_T = 1$ and recursively $U_t = U_{t+1} + M_{t+1}(U_{t+1} - U_{t+2}) - \epsilon$ for $t = 1, ..., T-2$ with $\epsilon > 0$. The constant ϵ can always be chosen small enough to ensure that in fact $U_s < U_t + M_t(U_{s+1} - U_{t+1})$ for all s, t. □

2.3 Additional assumptions on the aggregator

We assume that $W(x, z)$ can be written as $F(w(x), z)$, where $F : I\!R_+ \times I\!R_+ \to I\!R$ is assumed to be increasing and where $w : I\!R_+^L \to I\!R_+$ is the concave and increasing utility function for spot consumption. We call this aggregator-function weakly separable. The assumption of weak separability is needed to ensure that marginal rates of substitution between different spot commodities are not affected by different future utilities. If there is only one good, i.e. $L = 1$ this assumption does not guarantee refutability. The assumption imbues the model with empirical content for $L > 1$ because it restricts possible choices on spot-markets. There are many utility functions satisfying this assumption - for example any nesting of concave CES-utility functions will give rise to a weakly separable aggregator.

The model is now testable. In fact, choices on spot-markets $(p_t, c_t)_{t=1}^T$ must satisfy the strong version of the strong axiom of revealed preferences.

Definition 2 (Chiappori and Rochet (1987)) *$(p_t, c_t)_{t=1}^T$ satisfies SSARP if for all sequences $\{i_1, ..., i_n\} \subset \{1, ..., T\}$*

$$p_{i_1} c_{i_1} \geq p_{i_1} c_{i_2},\ p_{i_2} c_{i_2} \geq p_{i_2} c_{i_3},\ \dots\ ,\ p_{i_{n-1}} c_{i_{n-1}} \geq p_{i_{n-1}} c_{i_n}$$

implies

$$c_{i_n} = c_{i_1},\ \text{or}\ p_{i_n}(c_{i_1} - c_{i_n}) > 0.$$

and if for all $s, t = 1, ..., T$, $p_s \neq p_t$ implies $c_s \neq c_t$.

Chiappori and Rochet (1987) show that in the context of static choice SSARP is necessary and sufficient for the data to be rationalizable by a smooth, strictly concave and strictly increasing utility function. In the intertemporal context, SSARP implies that choices are rationalizable by a separable (time-invariant) utility function if interest rates are unobservable.

We say that a utility function $u((c_t)_{t=1}^T)$ is time separable, if there is a $\beta \in (0, 1]$ and an increasing and strictly concave utility function $v : \mathbb{R}_+^L \to \mathbb{R}$ such that

$$u((c_t)_{t=1}^T) = \sum_{t=1}^T \beta^{t-1} v(c_t).$$

The following theorem is the main result of this section.

Theorem 2 *The following statements are equivalent.*

(a) An observation

$$\bar{o} = (\bar{I}_t, \bar{p}_t, \bar{q}_t, \bar{c}_t, \bar{s}_t)_{t=1}^T \in \mathcal{O}$$

is rationalizable by a recursive utility function with weakly separable aggregator.

(b) There are $V_t, U_t \in \mathbb{R}_+$, $M(t) \in \mathbb{R}_{++}^L$, $\lambda(t) \in \mathbb{R}_{++}$ and $N(t) \in \mathbb{R}_{++}^2$ for all $t = 1, ..., T$ such that

1. *For all $t = 1, ..., T-1$,*

$$\bar{q}_t \lambda(t) = N_2(t) \lambda(t+1) \tag{1}$$

2. For all $t = 1, ..., T$,

$$N_1(t)M(t) - \lambda(t)\bar{p}(t) = 0 \tag{2}$$

3. For all $s \neq t = 1, ..., T$,

$$U_s < U_t + N_t \cdot \left[\begin{pmatrix} V_s \\ U_{s+1} \end{pmatrix} - \begin{pmatrix} V_t \\ U_{t+1} \end{pmatrix} \right], \tag{3}$$

as well as

$$V_s < V_t + M_t \cdot (\bar{c}_s - \bar{c}_t). \tag{4}$$

(c) The prices and spot-market choices $(\bar{p}_t, \bar{c}_t)_{t=1}^T$ satisfy SSARP.

(d) There exist bond-prices $(q_t)_{t=1}^T$, incomes $(I_t)_{t=1}^T$ and saving-choices $(s_t)_{t=1}^T$ such that the observation $(I_t, \bar{p}_t, q_t, \bar{c}_t, s_t)_{t=1}^T$ can be rationalized by a time-separable utility function.

Proof.

The proof of Lemma 1 above implies that (a) is equivalent to (b). The additional requirement of weak separability introduces the $M_t = \partial_c W(c_t, v_{t+1})$ and gives rise to the set of inequalities (4).

The crucial part of the proof is to show that (b) is equivalent to (c): According to Afriat's Theorem (see Chiappori and Rochet (1987)) SSARP is necessary and sufficient for the existence of numbers $(V_t, \mu_t)_{t=1}^T$, $\mu_t > 0$ which satisfy

$$V_t - V_s < \mu_s p_s (c_t - c_s) \tag{5}$$

for all $s \neq t = 1, ..., T$.

For the necessity part, we define $\mu_t = \lambda_t / N_1(t)$. Inequality (4) then implies Inequality (5).

For sufficiency, assume that there exist numbers $(V_t, \mu_t)_{t=1}^T$, $\mu_t > 0$ which satisfy (5). We can then choose $(N_1(t))_{t=1}^T$ small enough to ensure that Inequality (4) has a solution (by the same argument as in the proof of Theorem 1). Inequality (3) can be satisfied because all these inequalities are homogeneous in $(\lambda_t)_{t=1}^T$.

Finally we have to show that (c) is equivalent to (d): Following Chiappori and Rochet (1987) the SSARP is equivalent to the existence of numbers $(V_t, \lambda_t)_{t=1}^T$, $\lambda_t > 0$ such that $V_t - V_s <$

$\lambda_s p_s (c_t - c_s)$ for all $s \neq t = 1, ..., T$. The observation can be rationalized by a time-separable utility function if and only if in addition $\lambda_t q_t = \lambda_{t+1}$. Since we are free to choose the (q_t), this can always be satisfied. Saving choices (s_t) and incomes (I_t) must then be chosen to ensure that the budget constraint is satisfied. □

It is important to point out that weakly separable recursive utility is not observationally equivalent with time-separable utility if saving choices are observable. In this case, time-separability puts restrictions on saving-choices - weakly separable recursive utility does not. If the agent's income is growing (with constant prices) and the interest rate is zero, impatience will cause the agent to save if he has time-separable utility. With recursive utility, however, the degree of impatience might depend on future consumption and the agent might not save even when the interest rate is arbitrarily low.

3 Uncertainty

As mentioned in the introduction, Theorem 1 does not carry over to the case of uncertainty and Kreps-Porteus utility functions. Under the assumption that prices, dividends and income follow a stationary process which can be estimated, the model is testable even without the assumption of a weakly separable aggregator because the assumption of risk-aversion (which is implicit in the Kreps-Porteus formulation) imposes restrictions on agents' portfolio-choices even when the aggregator function is not weakly separable.

While under certainty it is conceivable that choices and prices are observable at every period, under uncertainty one can only observe one sample path of an underlying stochastic process. One has to make stationarity assumptions on the underlying stochastic processes for prices and incomes to imbue the model with empirical content.

Suppose that the uncertainty can be described by an event tree with a finite number of nodes. Under a stationarity assumption one can estimate the processes and one therefore knows prices, dividends and incomes at all nodes of the event tree. However, while prices, dividends and incomes[1] might be stationary, the life cycle aspect of the agent's finite horizon maximization

[1]Note that income is treated as an exogenous variable which is unrelated to a labor-supply choice. This is without loss of generality if one views this exogenous income as being generated by non-tradeable assets or simply as government transfers.

problem implies that choices are not stationary. We will argue below that it is implausible that all variables jointly follow a first order Markov chain. While it might be possible to construct a joint process for prices, dividends, incomes and choices which is stationary it is more appropriate the assume that individual choices can only be observed along a sample path.

We assume that the agent evaluates uncertain income streams according to the true (known) probabilities. While this might seem like a very strong assumption, it is standard in the applied literature and it is clear that without any assumption an agent's beliefs, Theorem 1 must hold. In this case the agent could put zero probability on all but one sample path. If this happens to be the observed sample path, the model is the same as under certainty.

With these assumptions on observability and expectation formation Theorem 1 no longer applies. Besides the trivial restriction that asset prices have to preclude arbitrage, the assumption of Kreps-Porteus utility imposes restrictions on individual portfolio-choices along the sample path. We provide an example to illustrate this and to prove that restriction in addition to the absence of arbitrage exist. Unfortunately, it is no longer possible to characterize optimal choices by a generalized version of the strong axiom as in Theorem 2.

Without the assumption of stationarity of the exogenous variables it is unlikely that observations can consist of dividends, prices and incomes at all nodes of the event tree. Note that given a finite data set, it is always possible to construct an event tree and a stationary process for prices, dividends and endowments such that the observed variables form a sample path. Therefore, the assumption of stationarity of the exogenous variables cannot be refuted.

The other extreme is to assume that both choices and dividends are only observable along a sample path (or equivalently that dividends are observable but that choices on asset market are not even observable along a sample path). Even under uncertainty, optimal choices in spot-markets are characterized by SSARP if agents' have time and state separable (and invariant) utility functions. In Theorem 4 we show that under recursive utility with weakly separable aggregator choices in spot markets have to satisfy SSARP as well - when dividends are only observable along a sample path SSARP fully characterizes the restrictions imposed by Kreps-Porteus utility.

3.1 Notation

We take as given an event tree Ξ with nodes $\xi \in \Xi$. Let ξ_0 be the root node, i.e. the unique node without a predecessor. For all other nodes, let ξ_- be the unique predecessor of node ξ, let $\wp(\xi)$ be the the set consisting of ξ itself and of all nodes leading up to ξ except for ξ_0, i.e. $\wp(\xi) = \{\xi\} \cup \{\xi_-\} \cup \{\xi_{--}\} \cup ....$

For all nodes $\xi \in \Xi$, let $\Im(\xi)$ be the set of its immediate successors. Nodes without successors, i.e. $\Im(\xi)$ is empty, are called terminal nodes.

For each terminal node ξ , there exists a sample path $\Sigma(\xi) = \wp(\xi) \cup \{\xi_0\}$.

At each node ξ there are J short-lived assets with asset j paying $d_j(\zeta) \in I\!R$ at all nodes node $\zeta \in \Im(\xi)$, its price being denoted by $q_j(\xi)$. The agent's consumption decisions must now be supported by portfolio-choices $(\theta(\xi))_{\xi\in\Xi}$, $\theta(\xi) \in I\!R^J$. All feasible consumptions and portfolio-choices $(c(\xi), \theta(\xi))_{\xi\in\Xi}$ must satisfy

$$p(\xi) \cdot c(\xi) + q(\xi) \cdot \theta(\xi) \leq I(\xi) + \sum_{j=1}^{J} \theta_j(\xi_-) d_j(\xi) \text{ for all } \xi \in \Xi$$

where $\theta(\xi_{0-}) := 0$.

In this context recursive utility means that the utility over consumptions can be represented as $u((c_\xi)_{\xi\in\Xi}) = v_{\xi_0}$, where v_ξ, the utility at node ξ, is recursively defined by

$$v_\xi = W(c_\xi, E_\xi v_{\Im(\xi)}) \text{ for all } \xi \in \Xi$$

where

$$E_\xi v_{\Im(\xi)} = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi) v(\zeta)$$

denotes the conditional expectation over next period utilities at node ξ. As mentioned above, it is assumed that agents have rational expectations and that therefore the expectation is taken under the objective probabilities $\pi(\zeta|\xi)$ which are known to the observer. We normalize $E_\xi v_{\Im(\xi)} = 0$ for all terminal nodes ξ. The aggregator $W(.,.)$ is assumed to satisfy all assumptions from Section 2.2.

We will call utility functions which can be represented in such a way 'Kreps-Porteus' utility in order to clarify that they are an extension of the recursive utility functions under certainty.

3.2 Restrictions on portfolio-choices

An observation o consists of prices, dividends and incomes at all nodes, $(p(\xi), q(\xi), I(\xi), d(\xi))_{\xi\in\Xi}$, choices at nodes in a subset of the event tree $\Gamma \subset \Xi$, $(c(\xi), \theta(\xi))_{\xi\in\Sigma}$ as well as conditional probabilities $\pi(\xi|\zeta)$ for all non-terminal $\zeta \in \Xi$ and all $\xi \in \Im(\zeta)$. We will assume below that the subset Γ is a sample path - however, in order to explain why choices cannot be stationary we state the lemma in a little more generality. We will assume throughout that for any observation o the asset prices preclude arbitrage, i.e. there exist strictly positive state prices $(\rho(\xi))_{\xi\in\Xi}$ such that $\rho(\xi)q_j(\xi) = \sum_{\zeta\in\Im(\xi)} \rho(\zeta)d_j(\zeta)$.

The following lemma is the analogue of Lemma 1 for the case of uncertainty.

Lemma 2 *An observation o is rationalizable by a Kreps-Porteus utility function if and only if there exist numbers $(\eta(\xi), \lambda(\xi), \mu(\xi), \gamma(\xi), W(\xi))_{\xi\in\Xi}$, with $\mu(\xi) = 0$ for all terminal $\xi \in \Xi$ and with*

$$\mu(\xi) = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi)W(\zeta)$$

for all non-terminal $\xi \in \Xi$, as well as $x(\xi) \in I\!R^L_{++}$ and $t(\xi) \in I\!R^J$ for all $\xi \in \Xi$ with $x(\xi) = c(\xi)$ and with $t(\xi) = \theta(\xi)$ for all $\xi \in \Gamma$ such that

- $$p(\xi) \cdot x(\xi) + q(\xi) \cdot t(\xi) \leq I(\xi) + \sum_{j=1}^{J} t_j(\xi_-)d_j(\xi) \text{ for all } \xi \in \Xi \quad \text{(U0)}$$

- *For all non-terminal $\xi \in \Xi$,*

$$\lambda(\xi)q_j(\xi) = \sum_{\zeta\in\Im(\xi)} \lambda(\zeta)d_j(\zeta) \text{ for } j = 1, ..., J \quad \text{(U1a)}$$

- *For all $\xi \in \Xi$, $\xi \neq \xi_0$,*

$$\eta(\xi) = \frac{\lambda(\xi)}{\prod_{\zeta\in\wp(\xi)} \pi(\zeta|\zeta_-)p(\zeta)\eta(\zeta)\gamma(\zeta_-)} \quad \text{(U1b)}$$

- *For all $\xi, \zeta \in \Xi$,*

$$W(\xi) - W(\zeta) \leq \eta(\zeta)p(\zeta)(x(\xi) - x(\zeta)) + \gamma(\zeta)(\mu(\xi) - \mu(\zeta)) \quad \text{(U2)}$$

  the inequality holds strict if $x(\xi) \neq x(\zeta)$ or $\mu(\xi) \neq \mu(\zeta)$.

Proof. For the necessity part, let $\mu(\zeta) = E_\zeta W_{\Im(\zeta)}$ for all non-terminal ζ, and let $\mu(\xi) = 0$ for all terminal ξ. Consider the agent's (necessary and sufficient) first order condition. At any node $\xi \in \Xi$,

$$\partial_c W(c(\xi), \mu(\xi)) - \eta(\xi)p(\xi) = 0$$

and

$$\lambda(\xi)q_j(\xi) = \sum_{\zeta \in \Im(\xi)} \lambda(\zeta)d_j(\zeta) \text{ for } j = 1, ..., J \text{ and for all non-terminal } \xi \in \Xi,$$

where for all $\xi \in \Xi$, $\xi \neq \xi_0$,

$$\eta(\xi) = \frac{\lambda(\xi)}{\prod_{\zeta \in \wp(\xi)} \partial_{v_\zeta}\mu(\zeta_-)\partial_\mu W(c(\zeta_-), \mu(\zeta_-))}$$

These first order condition, together with the assumption that $W(.,.)$ is concave and the usual characterization of concave functions proves necessity: (U0) stems from the budget constraint, (U1) is the analogue of (C1) and (U2) is the analogue of (C2) in Lemma 1.

The proof of sufficiency is identical to the proof of sufficiency in Lemma 1: One can a construct piecewise linear aggregator function W from (U2). (U1a) and (U1b) ensure that agents maximize expected future utility. $\square$

The lemma implies that it is possible that choices as well as all exogenous variables follow a first order Markov chain. However, this is a knife-edged case: individual incomes have to equal individual expenditures at all nodes - there cannot be any savings since it is never optimal for an agent to save at terminal node. In actual data one can be almost certain, however, to find positive savings. Therefore we will assume from now on that individual choices can only be observed along a sample path and that therefore $\Gamma = \Sigma$.

An example now shows that portfolio-choices are restricted by the assumption of Kreps-Porteus utility.

Example 1 *Consider a two-period model with two possible states in the second period. The states are numbered 0 (today), 1,2 and the probabilities are $\pi_1 = \pi_2 = 1/2$. Assume for simplicity that there is only one good and that the price of this good is one at each node. Assume that there are two Arrow securities, one paying one unit in state 1, the other paying one unit in state 2 and that $q_1 > q_2$. Suppose that $I_1 > I_2$ and that the portfolio-choice satisfies $\theta_1 > \theta_2$.*

The observed portfolio-choice is inconsistent with Lemma 1. By the budget constraint (U0) , we must have $c_1 > c_2$. However, by (U2) this is only possible if $\lambda_2 > \lambda_1$. However, by (U1a) this implies that $q_1 < q_2$ - a contradiction.

This example proves the following theorem.

Theorem 3 *Under the assumption that prices, dividends and incomes follow a stationary process which can be estimated from existing time-series data, there exist observations on individual choices which cannot be rationalized by a Kreps-Porteus utility function.*

While the example only shows that there are restrictions on portfolio-choices at time $T-1$, there also exist restrictions at other nodes. Consider for example an economy with identical prices, dividends and income at all last period nodes. This implies that μ_ξ has to be identical for all ξ in the second to last period. Given a node ξ at $T-2$, the consumption at all nodes $\zeta \in \Im(\xi)$ is then determined by the budget constraint and (U2) determines $\lambda(\zeta)$. This puts additional restrictions on $\theta(\xi)$, just like in Example 1. The assumption of a single good is without loss of generality as well. With several goods, example 1 goes through as well, as long as all goods have the same price.

3.3 Consumption choices with weakly separable aggregator

The assumption of stationary dividends and asset prices might seem overly restrictive. In order to investigate how this assumption affects our results, we now assume that dividends are only observable along a sample path, i.e. an observation o consists of prices and incomes at all nodes, $(p(\xi), q(\xi), I(\xi), d(\xi))_{\xi\in\Xi}$, choices and dividends along a sample path $\Sigma \subset \Xi$, $(c(\xi), \theta(\xi), d(\xi))_{\xi\in\Sigma}$ as well as conditional probabilities $\pi(\xi|\zeta)$ for all non-terminal $\zeta \in \Xi$ and all $\xi \in \Im(\zeta)$.

If dividends are not observable there are no restrictions (other than positivity) on $(\lambda(\xi))_{\xi\in\Xi}$ since they are determined by (U1a). Therefore, by (U1b), there are no restrictions on $(\eta(\xi))_{\xi\in\Xi}$ either and the conditions of Lemma 1 are identical to those for the certainty case. The additional assumption of a weakly separable aggregator is needed to endow the model with empirical content.

With this assumption an analogue of Theorem 2 holds - choices in spot markets are rationalizable if and only if they satisfy SSARP. The additional requirement of weak separability imposes Inequality (4) in addition to (U1) and (U2) and SSARP must hold. On the other hand, if SSARP holds (U2) can always be satisfied and without knowledge of $d(\zeta)$ for all $\zeta \in \Im(\xi)$ (U1) is trivially satisfied. We therefore have the following theorem.

Theorem 4 *Under the assumption that prices and incomes follow a stationary process but that choices and dividends can only be observed along a sample path the following statements are equivalent.*

(a) The observation can be rationalized by a Kreps Porteus utility function with weakly separable aggregator.

(b) Consumptions and spot-prices along the sample path, $(c(\xi), p(\xi))_{\xi \in \Sigma}$, satisfy SSARP.

(c) The observation can be rationalized by a time-separable von-Neumann-Morgenstern utility function.

When dividends are observable, the SSARP remains necessary but it is no longer sufficient. As Theorem 3 shows (even without a weakly separable aggregator) Kreps-Porteus utility restricts portfolio-choices.

Note that instead of assuming that dividends are only observable along a sample path, we could have assumed in the theorem that portfolio-choices θ are not observable at all. In this case the only existing observable restriction on choices is the SSARP.

Furthermore, when dividends are only observable along a sample path, it is irrelevant whether or not asset prices and portfolio choices are observable. A different interpretation of Theorems 3 and 4 is that with a weakly separable aggregator function assumptions on the evaluation of future uncertain utilities impose restrictions on asset prices, but commodity prices and choices in commodity markets are only restricted by SSARP.

4 Conclusion

Assuming the existence of utility functions to explain the behavior of consumers is standard in economics. In order to imbue models which use utility functions with empirical content one would hope that by watching the behavior of individuals throughout their life, one can test the hypothesis that these individuals maximize utility. However, we show in this paper that this is only possible under additional assumptions on the utility function. Recursive utility with a weakly separable aggregator is one class of utility functions which imposes restrictions on individual behavior. These restrictions can be formulated in a tractable way - one can test a large data set for consistency with utility maximization (see Varian (1982) for such tests).

The situation is more complicated when there is no data on individual choices and when one has examine restrictions on aggregate data. Brown and Matzkin (1996) show that there exist observable restrictions for the case where one can observe how aggregate consumption varies as prices and the income-distribution vary. The criticism in this paper against traditional tests of utility maximization which use individual data applies to the analysis in Brown and Matzkin (which uses aggregate data) as well. In Kubler (1999), we extend their analysis to a multi-period model where the observations consist of a time-series on aggregate data.

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