IDENTIFICATION, ESTIMATION, AND TESTING IN EMPIRICAL MODELS OF
SEQUENTIAL, ASCENDING-PRICE AUCTIONS WITH MULTI-UNIT DEMAND:
AN APPLICATION TO SIBERIAN TIMBER-EXPORT PERMITS*

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Abstract

Within the independent private-values paradigm, we construct a simple theoretical model of
participation and bidding at a sequential, oral, ascending-price, open-exit auction with multi-unit
demand. We characterize the strategic equilibrium, show that this equilibrium induces an efficient
allocation, and then describe a dominant-strategy implementation algorithm which allows us to
calculate the expected winning price for each lot sold. We then use the model to put structure on
data from a sample of sequential, oral, ascending-price, open-exit auctions of timber-export permits
held in the Krasnoyarsk Region of Siberia, Russia where (to a first approximation) the independent
private-values paradigm appears appropriate.

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1. Motivation and Introduction

During the past four decades, auctions at which only one object is sold to buyers demanding but
one object each have been studied extensively.\textsuperscript{1} In reality, however, many auctions involve the
sale of multiple units of the same good to buyers who may demand several units.\textsuperscript{2} Understanding
how such changes in the economic environment affect bidding behaviour at auctions and how these
behavioural changes in turn affect the process of price formation are open research questions.

Some recent research concerning multiple-object auction models suggests that such institutions
provide a host of additional economic issues that are typically absent in the analysis of single-object
auction models. For example, McAfee and Vincent (1993) have shown that the equilibrium at some
multiple-object auctions can involve mixed strategies, which can imply that equilibrium allocations
are inefficient. In addition, as the theoretical work of Weber (1983); McAfee and Vincent (1993);
and Laffont, Loisel, and Robert (1994) has shown, multiple-object, single-unit-demand auction
models can provide richer explanations of observed behaviour than can be obtained in single-object,
single-unit-demand auction models. For example, within the affiliated private-values paradigm and
assuming risk-averse single-unit buyers Laffont, Loisel, and Robert (1994) have been able to derive
inverse U-shaped expected price paths.

This paper differs from other studies on sequential auctions by allowing participants to desire
more than one unit. Specifically, within the independent private-values paradigm, we construct a
simple theoretical model of participation and bidding by potential bidders with multi-unit demand
at sequential, oral, ascending-price, open-exit auctions. We then use our model to put structure on
quite detailed data (to be discussed later) from a specific sequential, oral, ascending-price, open-exit
auction (the sale of timber-export permits in Siberia, Russia).

\textsuperscript{1} For a summary of the theoretical research, see Milgrom (1985,1987) as well as McAfee and McMillan

\textsuperscript{2} Weber (1983) has classified multiple-object auctions in three ways. The first category he describes is
\textit{simultaneous-dependent} auctions, at which bidders are required to take a single action that determines
both the allocation of the objects and the payments to the seller. Weekly auctions of U.S. Treasury bills
are an example of this type of multiple-object auction. A special case of the simultaneous-dependent
auction, viewed as distinct by Weber, is the \textit{simultaneous-independent} auction, at which the sale of one
object does not depend on the outcome of other sales. Sales of mineral rights such as oil and gas leases
are good examples of such auctions. The final category described by Weber is the \textit{sequential} auction,
at which one item (or lot containing several of the same item) is sold at a time. Art, stamps, and coins
as well as cattle, fish, flowers, timber, vegetables, and wine are often sold at sequential auctions.
Our main theoretical finding concerns the pattern of winning prices. When potential buyers desire at most one unit, the winning prices in a sequence of auctions are the same, equalling the highest infra-marginal valuation. On the other hand, when buyers desire more than one unit, our model predicts that winning prices will increase, on average. This prediction illustrates how incomplete our understanding of price determination in competitive markets is. For the lots on sale correspond, in all respects, to a homogenous good in the Arrow-Debreu sense. The auction's design insures competition among participants and generates considerable public information. We assume risk neutrality, so risk aversion does not limit attempts to exploit arbitrage possibilities. Yet, contrary to the conventional Walrasian wisdom, the price of each lot is not constant, either in actual terms or on average. This prediction is also supported by reduced-form econometric estimates obtained using data concerning Siberian timber-export permits. According to our reduced-form estimates, the average price increase from the first winning price to the last at a sequential auction where twenty lots are sold is nearly five percent. In our application, this difference corresponds to the monthly salary of an economics professor in Siberia. Apparently, not all arbitrage possibilities can be exploited.

The paper is in seven more parts. To provide the reader a context within which to appreciate the theoretical model that we construct, we describe in section 2 the economic environment from which the data used later in the paper were drawn; viz., the sale of timber-export permits in Siberia. In section 3, we construct a simple theoretical model of a sequential, oral, ascending-price, open-exit auction with multi-unit demand within the independent private-values paradigm. This model captures the essence of the particular bidding environment present in Siberia. The reader will notice, however, that ours is not specific to the Siberian case, and potentially could have wide applicability. In section 4, we characterize the strategic equilibrium and show that this equilibrium induces an efficient allocation. Subsequently, in section 5, we derive the implications of equilibrium strategic behaviour for the observed price process across lots of identical goods sold at the same auction. In section 6, we use our model to put structure on data from a sample of sequential, oral, ascending-price, open-exit auctions of timber-export permits held in Siberia between May of 1993 and May of 1994 where (to a first approximation) the independent private-values paradigm appears appropriate, first examining the reduced-form implications of the model and then developing and implementing a method for estimating the structural parameters of the model. We summarize
our results and conclude the paper in section 7. In an appendix to the paper, we document the development of the data set used, describing the source from which the data were taken. We also present in this appendix the proofs of results and theorems too detailed to be included in the text of the paper.

2. Sequential, Oral, Ascending-Price, Open-Exit Auctions of Siberian Timber-Export Permits

Siberia is one of the richest resource regions of Russia. While exploiting the vast wealth of minerals and furs is of considerable interest to the Russian government, exploiting the region’s timber is no less important. The bulk of the timber from Siberia is marketed either within Russia or in the other republics of the former Soviet Union, but some is also exported abroad. With the increasing importance of foreign hard currencies in the development of investment opportunities within Russia, timber exports abroad have taken on a special significance. In the Krasnoyarsk Region of Siberia, the Russian government regulates timber exports abroad by selling export permits at auction. This auction market, which is overseen by the Ministry of the Krasnoyarsk Region, is situated in the city of Krasnoyarsk, Siberia.

The potential bidders in the market for timber-export permits are the firms harvesting timber in the Krasnoyarsk Region. While the exact number of firms harvesting timber is reportedly unknown to the authorities of the Region, the total number of firms exporting timber is greater than twenty, but probably less than forty. Some of these firms are privately owned, but the bulk are state-owned enterprises. In either case, these firms have stands of timber, and the logs produced from harvesting this timber can potentially be exported. To export, however, these firms require a foreign buyer and an export permit. But without a buyer, an export permit has no value since such permits are non-transferable; i.e., no secondary market in timber-export permits exists. Moreover, an export permit will expire if not used within a fixed period of time, typically less than one year. Thus, the maximum value of an export permit to a potential bidder is the profit that firm could make by exporting timber from Siberia in the near future. To a first approximation, it would appear that such valuations are firm specific. Later, we shall assume that these valuations are independent draws from a common distribution of valuations.

As alluded to above, the supply of timber-export permits is set by the Ministry of the
Krasnoyarsk Region. By the Wednesday of the week prior to a particular auction (which is typically held on either a Tuesday or a Thursday), potential exporters of timber from the Krasnoyarsk Region are invited to submit requests for timber-export permits to the Ministry of the Region. While each potential exporter knows the volume of his request, he is ignorant of the total number of requests as well as the total volume of requests. On the following Tuesday or Thursday, the Ministry of the Region allocates for sale at auction a volume of timber-export permits in lots of between 180 and 5000 cubic metres of timber of a particular type. In Russia, one railway box-car holds between 90 and 100 cubic metres of timber. In the empirical part of this paper, we shall focus on the sale of lots of export permits for a product known as 4403, saw logs.

Potential bidders are required to pay a fee of 5000 R (about $3 US at the time the data we use below were collected) to attend the auction. In addition, for each lot a reserve price per cubic metre exported is announced. Reserve prices are relatively high, either 1500 R or 2500 R per cubic metre in the data available to us. The sale of a particular lot begins with the auctioneer calling out the lot number, describing the export permit, and then asking interested bidders to hold up the white cards they were issued when they paid the entry fee. By holding up a white card, a participant signifies his willingness to pay the reserve price of the export permit. As the price rises, bidders signify that they are no longer in the sale by dropping their white cards. The winner is the bidder who holds up his white card the longest; he pays the price at which his last opponent dropped his white card. Thus, the auction is a sequential, oral, ascending-price, open-exit auction with multi-unit demand because exit is perfectly observed by both the seller and the other participants and because many participants purchase more than one export permit.

3. Theoretical Model

Consider the case of a seller who wants to sell $T$ lots of a homogeneous good through a sequence of $T$ oral, ascending-price, open-exit sales. At each sale of the auction, the seller calls out a price that is progressively increased. Each potential buyer holds up a card (or presses a button or uses some alternative device) to signal his willingness to buy at the current price. As the price rises, some of the bidders lower their cards and forgo the possibility of buying the lot currently on sale. When all but one of the bidders have withdrawn, the price stops rising, and the lot is awarded to the last remaining bidder. The seller repeats the process until all $T$ lots are sold. Initially, we assume
that no reserve price exists and that the seller is committed to selling all \( T \) lots. Below, we shall
index the sales in descending order, letting sale \( t \) denote the sale at which \( t \) lots, including the one
currently on sale, remain to be sold.

We assume that there are \( N \) potential bidders who may bid for the lots for sale. Some of these
\( N \) potential bidders may not participate at a given auction. In fact, at any given auction, the seller
faces a random number of participants \( N \), each having some private values for the lots on sale.
Participants are assumed risk neutral, their objective being to maximize the sum of the valuations
for each unit purchased minus its price. The valuation for some participant \( i \) takes the form of a
vector

\[
W^i = \{W^i_1, W^i_2, \ldots, W^i_{m_i}, 0, 0, \ldots\}
\]

where \( W^j_i \) represents participants \( i \)'s valuation of his \( j \)th unit of the good, and \( m_i \) denotes the number
of positive valuations for participant \( i \). Thus, the number of elements in this vector could represent
the number of orders received by the firm for its product, and the valuations could represent the
amount that an export permit is worth to the firm. We assume decreasing marginal utility, so that
\( W^1_i \geq W^2_i \geq \cdots \geq W^{m_i}_i \). We also assume that orders arrive according to a Poisson process having
intensity parameter \( \lambda \); i.e., the distribution of waiting times between orders is memoryless being
distributed exponentially.\(^3\) Thus, the number of positive valuations drawn by each potential bidder
\( M \) is a random variable that is distributed Poisson with mean \( \lambda \) and probability mass function

\[
\Pr(M = m) = \frac{\lambda^m \exp(-\lambda)}{m!} \quad m = 0, 1, \ldots
\]  

(3.1)

When a potential bidder receives no orders, \( M \) equals zero, we assume that he does not attend the
auction, and hence signals that he will not bid. Conversely, we assume that all participants present
at the auction have at least one positive valuation. Consequently, the number of participants \( N \) is
distributed binomially with parameters \( N \) and \( [1 - \Pr(M = 0)] \) or \( [1 - \exp(-\lambda)] \). Thus,

\[
\Pr(N = n) = \binom{N}{n} [1 - \exp(-\lambda)]^n \exp(-\lambda)^{N-n}.  
\]  

(3.2)

We also assume that each valuation \( W \) is an independently and identically distributed draw
from the cumulative distribution function \( F(\cdot) \). Thus, for potential buyer \( i \), each of the \( m_i \) draws

\(^3\) Although this assumption may seem innocuous, it turns out to be crucial. Later, we shall demonstrate
that the assumption is necessary to maintain symmetry among bidders throughout the game and to
solve for the equilibrium of the auction game.
ranked in descending order represents another business opportunity, the value of which is the profit associated with an extra unit purchased. The valuations are ranked in descending order to reflect the fact that participants will exploit their most lucrative business opportunities first. The highest valuation $W_i^j$ corresponds to the utility accruing to participant $i$ from his first unit, while his $j^{th}$ valuation corresponds to the marginal utility of the $j^{th}$ unit if purchased. The vector $W^i$ and the number of positive valuation $m_i$ are assumed to be the private information of participant $i$. We assume that the actual number of participants $n$, the cumulative distribution function $F(\cdot)$, and the intensity parameter of the Poisson process $\lambda$ are common knowledge.

The state of nature or realization of types can be represented by a vector of descending valuations which ranks valuations of all $n$ participants for all lots

$$W = \{W_1; W_2, \ldots, W_t, \ldots, W_{\sum_{i=1}^n m_i}\}$$

where $W_1 \geq W_2 \geq \ldots \geq W_t \geq \ldots \geq W_{\sum_{i=1}^n m_i}$. Here, $W_t$ denotes the $t^{th}$ highest valuation among all $n$ participants. The expression $W_t = W_2$ means that participant $i$'s second highest valuation is the $t^{th}$ highest valuation overall. Similarly, we define $W^S$ as the vector of descending valuations which ranks valuations of all participants in coalition $S$, where $W_k^S$ denotes the $k^{th}$ highest valuation among all participants in $S$. Finally, we let $W_k = \{W^1_k, W^2_k, \ldots, W^n_k\}$ be the list for all participants of their individual $k^{th}$ valuations and $W_k^S$ be the list of the $k^{th}$ valuation of all participants in coalition $S$.

Given the above assumptions, the following properties hold:

A: For all $j < k$,

$$\Pr(W^j_k \leq y|W^j_k, W^j_{k-1}, \ldots, W^j_1) = \Pr(W^j_k \leq y|W^j_1).$$

B: For all $j < k$ and $i, \ell \in N = \{1, 2, \ldots, N\}$

$$\Pr(W^j_k \leq y|W^j_\ell = x) = \Pr(W^j_k \leq y|W^j_k = x) \quad \text{symmetry.}$$

C: For all $i \in N$ and for all $j < k$,

$$\Pr(W^j_k \leq y|W^j_j = x) = \Pr(W^j_k \leq y|W^j_{k-j+1} \leq y|W^j_1 = x).$$
Property A follows directly from the assumption of independent draws and the properties of order statistics. Property B follows directly from the assumption of symmetry. Property C is more involved: it states that the conditional distribution is invariant to a re-indexing of the order statistics. This is not a general property of order statistics, but it follows here from the assumption that the number of valuation draws follows a Poisson process. A formal proof of this result, like the proofs of all of our results, is contained in an appendix to the paper.

Property C proves to be very useful. The vector $W$ and the vectors $\hat{W}^i$'s represent the valuations at the beginning of the auction. When bidder $i$ wins one lot, his valuation for the next lot becomes $\hat{W}_2^i$ rather than $W_1^i$ as it was initially; so valuations need to be re-indexed to take into account the lots already purchased by each participant. Let $h(t)$ denote the history prior to the sale of lot $t$. A history includes all information available to the bidders. In particular, it includes the prices at which opponents withdrew, winning prices at previous sales, the identity of winners at previous sales, and the number of lots remaining. We denote $V^i[h(t)]$ as $i$'s re-indexed vector of types such that, if $i$ won $\ell_i$ lots prior to sale $t$, then $V^i_j[h(t)] = \hat{W}^i_{j+\ell_i}$. Similarly, we let $V^S[h(t)]$ be the re-indexed version of $W^S$, and $V^S_k[h(t)]$ be the re-indexed version of $\hat{W}^S_k$. When $t$ and $h(t)$ are unambiguous, we drop the argument $h(t)$ and simply use $V$, $V^S$, or $V^S_k$ for the re-indexed vectors. Given this notation, properties B and C imply

$$D: \text{For all } j < k, \ i, \ \ell \in \mathbb{N}, \text{ and after all histories } h(t),$$

$$\Pr \{V^S_k[h(t)] \leq y|V^S_j[h(t)] = x\} = \Pr \{V^S_k[h(t)] \leq y|V^S_j[h(t)] = x\} \quad \text{robust symmetry.}$$

Symmetry holds throughout the game, regardless of the history of the game. This assumption is instrumental in solving the game. The final property E reflects the correlation of order statistics: if the $j^{th}$ highest valuation of a bidder is high, then his $k^{th}$ highest value, $k > j$ is more likely to be high than low.

$$E: \text{For all } j < k, \ i \in \mathbb{N}, \text{ the function}$$

$$G(y|x) = \Pr(W^S_k \leq y|W^S_j = x)$$

is strictly decreasing in $x$ for all $y < x$ such that $F(y) > 0$. 

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4. Solving the Sequential, Oral, Ascending-Price, Open-Exit Auction Game

We wish to construct an equilibrium of the auction game induced by the sequence of $T$ sales. A bidding strategy at these sales specifies a stopping rule that indicates at what price a participant should withdraw from the current sale (lower his card). Such a stopping rule can be contingent on the entire history of previous as well as the one current sale; i.e., contingent on the prices at which other participants have withdrawn, previous winning prices, private information concerning the valuation attached to each lot purchased, the information shared by all participants, and so on. Hence, the strategies may be quite complex, not only because they will be history dependent, but also because each participant may want to manipulate his actions to influence the future bidding behaviour of others. Before describing the equilibrium for this complex game, we first analyze a simple case where only two units are for sale and only two buyers exist.

4.1. Two-Player, Two-Unit Model

In this subsection, we consider the simplest case of two bidders, Victoria and Walter, and two units for sale. Our objective is to provide the basic insight into understanding sequential sales with multi-unit demand.

Suppose that Victoria's willingness to pay, which is private information, is denoted by $\{V_1, V_2\}$, while Walter's willingness to pay, also private information, is denoted by $\{W_1, W_2\}$. Assume that units of the good are sold sequentially according to the rule specified above.

For this simple game, a continuum of possible equilibria exist. A list of equilibria exhibits "discouragement" behaviour: at the first sale, a bidder may learn he has no chance of winning the last sale. Therefore, he bids less at the second sale than what the extra units are worth to him. If bidding is not costly, then a "discouraged" player uses a weakly dominated strategy. We shall first focus our attention on equilibria where potential buyers bid up to their first valuation if they lose the first auction, and bid up to their second valuation, otherwise. Let $B(W_1, W_2)$ be Walter's bidding strategy at the first sale. The very best that Victoria can achieve is

$$\max[V_1 + V_2 - B(W_1, W_2) - W_1, V_1 - B(W_1, W_2), V_1 - W_2, 0].$$

If $B(W_1, W_2) < W_2$, then she can achieve this maximum by bidding up to $V_1$ at the first sale. She will win the first sale if $V_1 > B(W_1, W_2)$, and the second if $V_2 > W_1$. If $B(W_1, W_2) = W_2$, then
Victoria can achieve this maximum by offering any price in \( [V_2, V_1] \). Finally, if \( B(W_1, W_2) > W_2 \), then Victoria’s strategy is to bid up to a price \( p \) which solves

\[
V_2 = p + \mathbb{E}[W_1 | B(W_1, W_2) = p] - \mathbb{E}[W_2 | B(W_1, W_2) = p].
\]

That is, Victoria should drop out when she believes that

\[
V_1 + V_2 - B(W_1, W_2) - W_1 < V_1 - W_2.
\]

From the above discussion, we can already identify the following (unique) symmetric equilibrium:

*Equilibrium 1:* At the first sale, both bidders bid up to their second valuation. Next, they bid up to their first valuation if they lose the first sale and bid up to their second valuation, otherwise.

This equilibrium has several interesting features. First, it leads to an efficient allocation. Second, each bidder is indifferent between buying a unit at the first sale or at the second sale. Finally, the winning price is never decreasing. At the first sale, the winning price is given by \( \min[W_2, V_2] \). If Victoria wins the first sale (i.e., \( V_2 \geq W_2 \)), then the winning price of the second sale is given by \( \min[W_1, V_2] \geq W_2 \).

We can also identify asymmetric equilibria for this simple game, one of which is described below:

*Equilibrium 2:* At the first sale, Walter bids up to \( W_1 \) and Victoria bids \( p(V_2) \) which solves \( V_2 = 2p - \mathbb{E}[W_2 | W_1 = p] \). Next, they bid up to their first valuation if they lose the first sale, and bid up to their second valuation, otherwise.

Here, Walter is very aggressive at the first sale, and Victoria offers a price less than her second valuation. Note that the above equilibrium does not lead to an efficient allocation. If \( V_2 > W_1 > p(V_2) \), then one unit will be allocated to Walter, while Victoria is willing to pay more for her second unit than Walter is willing to pay for his first unit.

Finally, note that if Victoria wins the first sale (i.e., \( W_1 < p(V_2) < V_2 \)), then she will necessarily win the second sale. Walter would not lose if he bids less than \( W_1 \) in the last sale. Hence, we can construct equilibria where at least one player is “discouraged”. The following forms an equilibrium for the game:
Equilibrium 3: At the first sale, Walter bids up to $W_1$ and Victoria bids $V_2$.
At the second sale, Walter bids $W_2$ whether he wins the first auction or not.
Victoria bids up to her first valuation if she loses the first sale, and bids up to
her second valuation, otherwise.

One can verify that the above equilibrium generates an efficient outcome. If the distribution of
valuations is symmetric, then the winning prices will, on average, be the same. Other equilibria
exist where Walter bids if he loses the first sale some function $C$ where $C(W_1, W_2) \in [W_2, W_1]$, and
Victoria offers at the first sale some price $p(V_2)$ which solves

$$V_2 = p + \mathcal{E}[C(W_1, W_2) | W_1 = p] - \mathcal{E}(W_2 | W_1 = p).$$

In the remainder of the paper, we shall focus our attention on a symmetric equilibrium for
the $n$-bidder, $T$-sale auction game. The equilibrium considered has features similar to those of
Equilibrium 1 above.

4.2. Strategic Equilibrium

In the theorem stated below, we characterize a (Perfect-Bayesian) strategic equilibrium of the $T$-
sale auction game.\footnote{R} Given all the information available to the participants of the game, a strategy
specifies an equilibrium stopping rule. Before stating the main result of this paper, let us introduce
some additional notation. Let $N_w$ be the list of participants who have won in previous lots. Let $S$
be the subset of participants who have withdrawn from the current auction, where $s = |S|$. Finally,
let $R$ be the subset of bidders still participating in the current auction, with $r = |R|$.

Theorem 4.1:

For all lots $t$ and all vectors $V^t[h(t)]$, the following characterizes a strategic equilibrium:

(i) Whenever $t \geq n$, each bidder $i$ remains active until all other bidders stop, or the price is
equal to or is above:

$$\mathcal{E}(V_{i - j}^t | V_j^t = V_j^t \forall j \in R, V_j^t, V_k^t \geq V_j^t \forall k \in N_w \cap S)$$

\footnotetext{\footnotemark[4]} Again, we do not claim that the equilibrium we construct is the only equilibrium, nor do we try to characterize all of the equilibria of the game if more than one exist.
where the $V_i$'s correspond to the bidders' second-highest valuation after re-indexing, and
$\hat{V}_i^S$ corresponds to the vector of second valuations for all bidders in $S$ which is consistent
with the prices at which they have withdrawn.

(ii) Whenever $t < n$, the participants first bid up to their first (highest) valuation, until $t = r$.
Afterward, each remaining bidder $i$ remains active until all other bidders stop, or the price
is equal to or is above:

$$E(V_i^j | V_i^j = V_j^j \forall j \in R, \hat{V}_i^S \setminus V_i^k \setminus S_1, V_i^k \geq V_j^j \forall k \in w \cap S_2) \quad (4.2)$$

where $S_1$ is the subset of $(n-t)$ bidders who first withdraw, and $S_2$ the subset of bidders
who withdraw later.

4.3. Properties of the Equilibrium

The Perfect-Bayesian equilibrium characterized above has two important properties. First, it is
efficient; i.e., the $T$ lots are allocated to the participants with the $T$ highest valuations overall.
Second, the expected price paid by buyer $i$ for his $(\ell_i + 1)^{th}$ lot is equal to the expected value of
$W_T^{-\ell_i}$; i.e., the $(T - \ell_i)^{th}$ highest valuations among all valuations of the other participants. Note
too that the only information from past sales that enters into the current bidding function is the set
of past winners. Hence, no one can influence future bidding by deviating, lowering or increasing his
bid, in ways that do not change the identity of the winner. Implicit in this solution is the property
that bid manipulation designed only to signal information to others is ignored.

The bidding strategy has two parts based on whether $t \geq n$ or $t < n$. When $t \geq n$, the
participants bid a function of their second valuations. This process identifies among all participants
those who, according to the efficiency rule, “deserve” more than one lot. As will be shown later,
the bidding function in (4.2) is monotone and symmetric, so the winners in this phase will be those
with the highest re-indexed second valuations.

If the number of lots remaining becomes less than $n$, then we enter a new phase of the game.
The equilibrium of this phase is constructed so that lots are allocated to the participants with
highest valuations including, of course, to those with highest first valuations. In this phase, everyone
first bids up to his highest valuation, until the price reaches the $(t - 1)^{th}$ highest individual first
valuation. Let $\hat{o}$ denote the price at which a $(n - t)^{th}$ participant drops out of the sale. If the
second valuations of all remaining participants are less than  θ, then according to the equilibrium condition they all drop out of the sale simultaneously. Indeed, the expected value of $V_t^{−i}$, given that $V_2^j = V_2^i$ for $(t-1)$ other bidders and $V_1^k ≤ θ$ for all $(n-t)$ others with equality for at least one, is given by $\min(θ, V_2^j)$. In this case, the lot will be allocated (randomly) among one of the last $t$ bidders. One can verify that in all of the $(t-1)$ remaining sales, the scenario will repeat itself and that all of these $t$ bidders will receive one lot and pay $θ$. If the second valuations of some of the remaining participants exceed $θ$, some will continue bidding until only one participant remains. Here, the winner of the sale will be the one with the highest second valuation.

At sale $t$, each participant continues to bid until the current price exceeds his best estimate of $V_t^{−i}[h(t)]$ given all available information and conditional on winning. Note that $V_t^{−i}[h(t)] = W_{T−l_t}^i$ if after $h(t)$, $i$ has already won $l_t$ lots. Given efficiency, this valuation corresponds to the price he would pay for his $(l_t+1)^{th}$ lot if he waits until the very last sale to win it. Implicit in the equilibrium is the property that a participant will pay up to but no more than this expected price in any of the sales. Note that, as in the affiliated private-values paradigm, bids depend on information revealed by the bids of others. The reason for this dependence is different, however. With affiliated values, the gains from winning depend on the private information of others. Here, the gains from winning depend on the willingness of others to buy. In all but the last sale, losing bidders can always win a later sale; the price they will need to pay in order to win this later sale will depend on the willingness of others to pay.

The results described above can be collected in the following theorem:

Theorem 4.2:

(i) If $t$ exceeds the number of active participants $r$ and at least two participants are still bidding, then their bids are strictly monotonic and symmetric functions of their second (re-indexed) valuations.

(ii) The allocation induced by the equilibrium is efficient; i.e., the $T$ lots are allocated to the buyers with the highest valuations.

(iii) If all other participants follow their equilibrium strategy, then in order to win sale $t$ participant $i$ must pay the price $\mathcal{E}(W_{T−l_t}^i, |Ω_t)$ where $l_t$ is the number of lots already won by participant $i$ and $Ω_t$ is the information available to participants at the end of lot $t$.  

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Moreover, participant $i$ needs to pay exactly $W_{T-\ell+i}$ in order to win when $n > t$ and the highest second valuation among all other participants is below their $t^{th}$ highest first valuation.

Before concluding this section, we would like to make comments on the result in Theorem 4.2(iii). This result is reminiscent of the dominant-strategy implementation of the efficient allocation (the generalized-Vickrey auction for multi-unit demand). Consider the following mechanism: First, each participant $i$ is asked to reveal his willingness to pay $W^i = \{W^i_1, W^i_2, \ldots, W^i_{m_i}, 0, 0, \ldots\}$; second, the $T$ lots are allocated to the participants with the $T$ highest (revealed) values; and third, each winner $i$ pays $W^i_T$ for his first unit, $W^i_{T-1}$ for his second unit, and $W^i_{T-\ell}$ for his $(\ell + 1)^{th}$ unit, etc. One can verify that it is a dominant strategy for each participant to reveal his actual valuations and that it implements the efficient allocation.\footnote{Let $w_{\ell}$ denote the $\ell^{th}$ highest value announced by some participant $i$. If $w_{\ell-1} < W_{T-\ell+2}$ then for all announced $w_{\ell} < w_{\ell-1}$, $i$ will not receive the $\ell^{th}$ lot, so $i$ cannot gain by mis-reporting $W^i_\ell$. Now suppose $w_{\ell-1} < W_{T-\ell+2}$, so that $i$ receives at least $(\ell - 1)$ lots. $i$ cannot gain by overstating his $\ell^{th}$ value because this will not affect the prices he will pay for any of his lots, and it could only make a difference if $w_{\ell} \geq W_{T-\ell+1} > W^i_\ell$ since he will be awarded the $\ell^{th}$ lot and he will pay a price above his valuation for this lot. Similarly, $i$ cannot gain by understating his $\ell^{th}$ value since $i$ may only lose the chance of purchasing the $\ell^{th}$ lot at a price below his valuation. This argument applies for all $\ell$, and all vectors $W^i$, so $i$ can never gain by mis-reporting his private information.} Theorem 4.2(iii) implies that the expected price paid by each winner in the sequential auction is equal to the price he would pay in the above mechanism. Hence, we can use the dominant-strategy implementation of the efficient allocation in conjunction with simulation methods to calculate the expected winning price for the sale of the $t^{th}$ lot.

5. Pattern of Winning Prices

Using the equilibrium constructed in the previous section, we can make predictions concerning the pattern of winning prices. In expected terms, the winning prices increase. The likelihood and the magnitude of these expected increases depend on the identity of the winners, specifically on the presence of repeated purchases by individual participants.

Before stating the formal result of this section, we wish to provide the basic intuition of why prices increase, on average. If each buyer desires at most one unit, then the winning price will always be constant. The participants will bid up to their first valuation until the $(n - T)^{th}$ participant
drops out. Then the \( T \) remaining participants will withdraw simultaneously. The winning price will correspond to the \((T+1)\) th highest valuation, \( W_{T+1} \). In the next auction, the bidding will stop as the price reaches the \( T \) th highest re-indexed valuation which is \( W_{T+1} \). Here, there is one fewer participant and one fewer unit for sale and these two factors exactly offset each other.

When participants desire one or more units, there is not necessarily one fewer participant each time a unit is allocated because the most recent winner may still desire an extra unit. Thus, in contrast to the case where participants desire at most one unit, one less unit for sale is not offset by one less buyer. One can object and claim that bidders may be more aggressive in the earlier auctions in order to keep the option of buying all subsequent units on sale. This does not turn out to be an issue. If valuations are such that one buyer ought to win all units, then he will not need to bid aggressively at the first sale in order to win. At an ascending-price auction, the bidding strategy is determined conditional on others offering a price similar to ones own. In a symmetric equilibrium, this implies that one bids conditional on not winning all the subsequent auctions.

The winning price at the first sale, conditional on the information set \( \Omega_1 \), is given by \( \mathcal{E}[W_T^i | \Omega_1] \) where participant \( i \) is the winner of the first sale. The winning price is participant \( i \)'s expected value of the \( T \) th highest value of the valuations for the other participants, conditional on his current information. The winning price at the last sale is given by \( W_{T+1} \), the highest sub-marginal valuation. If participant \( i \) has \( \ell_i \) values above \( W_{T+1} \), then \( W_T^i \leq W_{T+\ell_i} \). It follows that

\[
P_T = \mathcal{E}(W_T^i | \Omega_1) \leq \mathcal{E}(W_{T+\ell_i} | \Omega_1) \leq \mathcal{E}(W_{T+1} | \Omega_1) = \mathcal{E}(P_1 | \Omega_1)
\]

where the last winning price \( P_1 \) is greater than the first price \( P_T \) in conditional expectation. The upward drift in price is approximately equal to the difference between \( W_{T+1} \) and \( W_{T+\ell_i} \), where \( \ell_i \) is the expected number of lots won by \( i \), the winner of the first auction.

We explore here more precisely the difference between the winning price for some sale \( t \) and that of the next sale \((t - 1)\). Using the notation established above, \( P_t \) and \( P_{t-1} \) denote the winning prices for sales \( t \) and \((t - 1)\), respectively.

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Theorem 5.1:

(i) The winning prices form a sub-martingale

$$\mathbb{E}[P_{t-1} - P_t | P_t] > 0.$$ 

(ii) The expected difference $$\mathbb{E}[P_{t-1} - P_t | P_t]$$ is greater when the winner of sale $$t$$ wins more than two extra lots after sale $$t$$ and even greater when he wins sale $$(t-1)$$.

The first of the predictions stated in Theorem 5.1 can, in principle, be tested easily using data concerning the winning sales prices.

For a sample of 37 auctions held between May 1993 and May 1994 and involving 308 lots, we have the winning prices for each lot and the number of participants present. Thus, for example, for the lots sold at an auction held on April 5, 1994, we have the following information:

<table>
<thead>
<tr>
<th>Date</th>
<th>Lot No.</th>
<th>Product</th>
<th>Volume</th>
<th>Reserve Price</th>
<th>No. of Bidders</th>
<th>Winning Bid</th>
</tr>
</thead>
<tbody>
<tr>
<td>940905</td>
<td>1</td>
<td>4403</td>
<td>1000</td>
<td>2500</td>
<td>7</td>
<td>2500</td>
</tr>
<tr>
<td>940905</td>
<td>2</td>
<td>4403</td>
<td>1000</td>
<td>2500</td>
<td>7</td>
<td>2700</td>
</tr>
<tr>
<td>940905</td>
<td>3</td>
<td>4403</td>
<td>500</td>
<td>2500</td>
<td>7</td>
<td>2700</td>
</tr>
<tr>
<td>940905</td>
<td>4</td>
<td>4403</td>
<td>250</td>
<td>2500</td>
<td>7</td>
<td>2800</td>
</tr>
</tbody>
</table>

The development of this data set is described in an appendix to the paper where the sample descriptive statistics are also tabulated in table A.1.

6. Empirical Results

In this section, we use the model constructed in section 3 to put structure on data from a sample of sequential, oral, ascending-price, open-exit auctions of timber-export permits held in the Krasnoyarsk Region of Siberia, Russia. These auctions were described in section 2, while the data were described briefly at the end of section 5 and are described in complete detail in an appendix. We proceed in two steps. In the first, we examine the first reduced-form implications of the theory — Theorem 5.1 (i); we did not have enough information to investigate the second one — Theorem 5.1 (ii) — with any precision. In the second, we develop and implement a method which employs the dominant-strategy implementation of the equilibrium in conjunction with simulation methods.
to estimate the conditional means of the number of participants \( N_j \) and the winning price of the \( t^{th} \) lot sold at the \( j^{th} \) auction \( P_j^t \). These conditional means can then be used to recover estimates of the Poisson intensity parameter \( \lambda \) and the latent distribution of valuation heterogeneity \( F(\cdot) \). Such structural primitives are central to any exercise designed to evaluate alternative auction designs; \( e.g. \), evaluating the effect of changing the reserve price, \( etc. \).

6.1. Reduced-Form Estimates

The main reduced-form prediction of section 5 is that the price process across lots sold at the same auction should follow a sub-martingale. One parametric implementation of a sub-martingale is

\[
\mathcal{E}(P_{t-1}^j|P_t^j) = (1 + \phi)P_t^j.
\]

Thus, an empirical specification is

\[
(P_{t-1}^j - P_t^j) = \phi P_t^j + Z_t^j
\]

(6.1)

where \( Z_t^j \) is a mean zero, but potentially heteroskedastic error term.

Using the price data from 36 auctions of 307 lots, we obtained an estimate of \( \phi \) in (6.1) that equalled 0.00212 with a corresponding standard error of 0.00536.\(^6\) This suggests that \( \phi \) is not significantly different from zero, a fact consistent with the results of Weber (1983) who showed that at a multi-object auction with single-unit demand the price process should follow a martingale; \( i.e. \), \( \phi \) should equal zero. However, in our data many participants won more than one lot at a given auction. Thus, multi-unit demand appears relevant.

Note that because the standard error of the estimate of \( \phi \) is relatively large, the power of the test against interesting economic alternatives is relatively small. For example, if \( \phi \) actually equalled 0.010, which implies a 22 percent increase in the average winning price over 20 lots sold at auction, the power of this test would be only about 50 percent. In fact, the region of low power (as defined by Andrews [1991]) is quite wide, between 0.00212 and 0.0195. This latter value for \( \phi \) would imply a price increase over 20 lots sold of about 47.1 percent; see figure 1 for other alternatives and their related power.

\(^6\) One of the 37 auctions had but one lot for sale, so that auction could not be used.

Can a simple numerical example generate the sort of price paths which are consistent with these data? To investigate this question, consider the following: suppose the number of potential bidders \( N \) is twenty, the Poisson intensity parameters \( \lambda \) is five, and the latent distribution of unobserved heterogeneity is Weibull having cumulative distribution function:

\[
F(w; \alpha_1, \alpha_2) = \left[ 1 - \exp \left( -\alpha_1 w^{\alpha_2} \right) \right] \quad w > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0.
\]

The Weibull family has a flexible shape; see figure 2 for graphs of \( W \)'s probability density function conditional on different vectors \( \alpha \) where \( \alpha \) is defined as \( (\alpha_1, \alpha_2)^\top \). Assume further that the Weibull parameters \( (\alpha_1, \alpha_2) \) both equal one. In figure 3, we present the expected price paths from 1000 simulations for the cases of five, ten, and twenty lots for sale. For these parameter values, the price increases, from the first lot to the last lot sold, are quite modest: A total of 3.6 percent with an average price increase of 0.70 percent per lot for five lots; a total of 5.3 percent with an average price increase of 0.51 percent per lot for ten lots; and a total of 5.6 percent with an average price increase of 0.27 percent per lot for twenty lots. Such increases are of the same order of magnitude as those obtained from the reduced-form estimates presented in the previous section, suggesting that our model is at least broadly consistent with the data. Note too that when supply is relatively
FIGURE 2. — Weibull Probability Density Functions for Various Parameter Pairs.

Parameter Pairs
(a): (0.4000, 1.0)
(b): (0.3200, 1.5)
(c): (0.2048, 2.5)
(d): (0.1049, 4.0)

FIGURE 3. — Expected Winning Bid versus Lot Number: \( \lambda = 5; \alpha_1 = 1; \alpha_2 = 1; N = 20. \)

Low (only a few lots are for sale) prices are systematically higher than when supply is relatively high (many lots are for sale).
6.2. Structural Econometric Approach

In this subsection, we first outline a strategy for estimating the two main primitives of the structural-econometric auction model, the Poisson intensity parameter $\lambda$, which determines the number of positive valuations that a potential bidder receives, and the latent distribution of valuation heterogeneity $F(\cdot)$. We then implement this approach and present estimates of $\lambda$ and the distribution of valuations.

6.2.1. Estimation Strategy

Unlike some other work (see, for example, Elakime, Laffont, Loisel, and Vuong [1994]) in which non-parametric methods (such as those proposed by Guerre, Perrigne, and Vuong [1996]) are applied to obtain an estimate of $F(\cdot)$, here we assume that $F(\cdot)$ comes from a specific family of parametric distributions. We do this for two reasons. First, the distribution is unidentified non-parametrically because the bid function is defined in terms of an expectation and, without additional structure, it is typically impossible to recover a distribution function from only the expectation of the random variable. Second, for the $j^{th}$ auction, we only observe the truncated distribution of at best some $(T_j + 1)$ order statistics from an unknown number $\sum_{i=1}^{n_j} m_i$ of sample draws from $F(\cdot)$. Without additional structure, it is impossible to identify $F(\cdot)$ over its entire support given the data we have. Thus, we obtain identification by assuming that $W$ comes from the Weibull family of distributions described above.

In the analysis below, we consider an independent sample of $J$ auctions at which $\{T_j\}_{j=1}^{J}$ lots are sold to $\{n_j\}_{j=1}^{J}$ participants for the winning prices $\{p_{ij}^{j}\}_{i=1}^{T_j}$ assuming the number of potential bidders $N$ is thirty. We chose thirty because there seemed to be some consensus among officials in the Krasnoyarsk Region that this was a reasonable number.

Below, we first describe the sample objective function we should like to optimize, and then explain why that is impossible. Subsequently, we describe the method used to simulate the regression function for prices at the sequential auction, and then discuss the statistical properties of the estimator we propose to calculate the structural parameter vector $\theta$ which denotes $(\lambda, \alpha)^T$.

The sample objective function we should like to optimize with respect to the parameter vector
\( \theta \) is the sum of squared residuals from both the price and the participation regressions

\[
S(\theta) = \frac{1}{J} \left\{ \sum_{j=1}^{J} \sum_{i=1}^{T_j} \left[ p_{ij} - \mathcal{E}(P_i | n_j, T_j, r_j, N, \theta) \right]^2 + \sum_{j=1}^{J} [n_j - \mathcal{E}(N_j | r_j, N, \theta)]^2 \right\},
\]

We need the variation in participation to identify the Poisson intensity parameter \( \lambda \) and the variation in prices across lots and across auctions with different participation rates to identify the parameters of the Weibull distribution \( \alpha \). Unfortunately, while the regression function \( \mathcal{E}(N_j | r_j, N, \theta) \) does have a known functional form,\(^7\)

\[
\mathcal{E}(N_j; r_j, N, \theta) = N \left(1 - \exp \left\{ -\lambda [1 - F(r_j; \alpha)] \right\}\right),
\]

the regression function \( \mathcal{E}(P_i | n_j, T_j, r_j, N, \theta) \) does not. We can, however, estimate the regression function for \( P_i \) using simulation methods in conjunction with the algorithm for calculating the dominant-strategy implementation of the efficient allocation. We describe this next.

For notational parsimony, we let \( x_j \) denote the vector \((n_j, T_j, r_j, N)\). We first discuss what is done at each of the \( S \) simulations. Again, for notational parsimony, we omit the subscript \( s \) which indicates the simulation draw and, only after having discussed what is done at each draw, do we describe the way in which the draws are combined.

For the \( j \)th of the \( n_j \) participants at auction \( j \), we draw a number \( m_i \) of positive valuations, where \( m_i \) is at least one. To do this, we use the distribution of a conditionally positive Poisson random variable, noting that the parameter \( \lambda \) must be replaced by \( \lambda_j \) where

\[
\lambda_j = \lambda [1 - F(r_j; \alpha)]
\]

because only those potential buyers for whom valuations exceed the reserve price \( r_j \) will attend the auction. Thus, the probability mass function for a positive number of lots demanded for those who attend the auction for specific values of \( \lambda_j \) and \( \alpha \) denoted \((\bar{\lambda}_j, \alpha)\) is

\[
g(m; \bar{\lambda}_j) = \frac{\Pr(M = m | \bar{\lambda}_j)}{1 - \Pr(M = 0 | \bar{\lambda}_j)} = \frac{\bar{\lambda}_j^m \exp(-\bar{\lambda}_j)}{m! [1 - \exp(-\bar{\lambda}_j)]} \quad m = 1, 2, \ldots.
\]

\(^7\) The probability mass function of \( N \) in the presence of a reserve price \( r \) is

\[
\Pr(N = n; r, N, \theta) = \binom{N}{n} \left(1 - \exp \left\{ -\lambda [1 - F(r; \alpha)] \right\}\right)^n \exp \left\{ -\lambda [1 - F(r; \alpha)] \right\}^{N-n}.
\]
In the simulations described below, $\lambda$ and $\alpha$ are fixed and known numbers, $r_j$ is the observed reservation price, and $F$ is the Weibull cumulative distribution function. To summarize, for each simulation we have $m_i$ positive draws for each of the $n_j$ participants at auction $j$.

Next, given the draw $m_i$ for participant $i$ obtained above, $m_i$ independent, random variables, which are distributed Weibull conditioned on exceeding $r_j$, are drawn. Letting $F(w; \alpha)$ denote the cumulative distribution function for a Weibull random variable at assumed parameter values $\alpha$ and $f(w; \alpha)$ denote the corresponding probability density function, the draws of the $m_i$ valuations for individual $i$ have the following probability density function:

$$h(w; \alpha) = \frac{f(w; \alpha)}{[1 - F(r_j; \alpha)]}$$

which is non-zero over the range $[r_j, \infty)$. At this stage, for the $s^\text{th}$ simulation, we have generated $\sum_{i=1}^{n_j} m_i$ valuations for the $n_j$ participants at auction $j$. Using the $\sum_{i=1}^{n_j} m_i$ valuations and the rule for approximating the winning bid for the $t^\text{th}$ lot at auction $j$ given in Theorem 4.2, one can simulate the price for each of the $T_j$ objects at auction $j$. Now the regression function for the $t^\text{th}$ lot at auction $j$ can be written as

$$\mathbb{E}(P_j^t|\omega_j, \theta) = \mathbb{E}[\mathbb{E}(W_j^t|\Omega_t, \theta)|\omega_j, \theta].$$

Employing importance sampling, we estimate this regression function unbiasedly at a value $\theta$ using the sample average of the simulated prices for each lot

$$\frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} P_t^j h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r_j, n_j) \frac{h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r_j, n_j)}{h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r_j, n_j)}.$$

Here $P_t^j$ denotes the winning price of lot $t$ at auction $j$ based on the $s^{\text{th}}$ set of random draws of $m_i$ and the valuations and being determined by the rule discussed above.\(^8\) Note that $h_s(w_s; \alpha, m_s)$ denotes the joint density of all $\sum_{i=1}^{n_j} m_i$ valuations evaluated at the simulated valuations which

---

\(^8\) For example, consider the case where $T = 4$, $n = 3$ and where $W^1 = \{5, 4, 2\}$, $W^2 = \{6, 3, 1\}$ and $W^3 = \{4, 5, 2.5\}$. The list and order of winners is given by $\{1, 2, 3, 1\}$. For instance, in the third auction, the re-indexed values are given by: $V^1 = \{4, 2\}$, $V^2 = \{3, 1\}$ and $V^3 = \{4, 5, 2.5\}$; players bid up to their first valuation until player 2 drops out at price $p = 3$; then 3 and 1 drop out since their second valuation is less than 3 which will be the winning price for the last two sales. The simulated winning prices will be given by: $\{W_{5}^2, W_{4}^{3}, W_{4}^{3}, W_{3}^{2}\} = \{2.5, 2.5, 3, 3\}$. Note that this price does not correspond exactly to the price player 1 would pay according to the equilibrium but, according to Theorem 4.2(iii), on average, this price will equal the average equilibrium winning price.
at simulation draw $s$ are collected in $w_s$. Under the independence assumption, $h_s(w_s; \alpha, m_s)$ is just a product of the $\sum_{i=1}^{n_j} m_i$ density functions given by $h(w; \alpha)$. Also, $g_s(m_s; \theta, r_j, n_j)$ is the joint probability mass function of the $n_j$ conditional Poisson draws evaluated at the simulated $m_i$ values which are collected in $m_s$ at draw $s$. Again, by independence, this is just a product of the individual conditional Poisson probability mass functions each evaluated at the simulated $m_i$. In the denominator of the expression is the corresponding densities evaluated at the assumed parameter values $\bar{\theta}$. Note that the simulated price $P_i^{js}$ also depends on $\bar{\theta}$. However, since this value will be fixed, its dependence is omitted. Note too that only one set of simulated values need be drawn prior to estimation, which proceeds iteratively; this is a by-product of importance sampling.

Before discussing the distribution of the estimator obtained using the simulation method, we simplify notation. Let

$$p_i^{j*}(\theta) = p_i^{j*} h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r_j, n_j)$$

$$\bar{p}_i^{j*}(\theta) = \frac{\partial}{\partial \theta} p_i^{j*}(\theta)$$

with $\bar{p}_i^{j*}$ and $\bar{d}_i^{j*}$ denoting the averages across the $S$ simulations.

The proposed estimator $\hat{\theta}$ is obtained by minimizing the following objective function:

$$Q(\theta) = \frac{1}{J} \left\{ \sum_{j=1}^J \sum_{t=1}^{T_j} \left[ \bar{p}_i^{j*} - \bar{p}_i^{j*}(\theta) \right]^2 - \frac{1}{S(S-1)} \sum_{s=1}^S \left[ p_i^{j*}(\theta) - \bar{p}_i^{j*}(\theta) \right]^2 \right\} + \sum_{j=1}^J \left[ r_{ij} - E(N_j; \theta) \right]^2.$$

The first term in the braces is the simulated analogue to the first term in $S(\theta)$, while the second term in the braces is the correction necessary to account for the fact that $\bar{p}_i^{j*}$ differs from $E(P_i^{j*})$ due to simulation error. The last term of $Q(\theta)$ is identical to the last term of $S(\theta)$ since the mean of $N_j$ has a known functional form.

We next state the asymptotic properties of the estimator $\hat{\theta}$. All asymptotics are done under the assumptions that $J$ goes to infinity and that the $T_j$s are bounded. Under conditions similar to those used in Laffont et al. (1995), one can show that $\hat{\theta}$ is parameter consistent for the true value $\theta^0$ and that

$$\sqrt{J}(\hat{\theta} - \theta^0) \overset{d}{\to} N[0, V(\theta^0)]$$

where

$$V(\theta) = (A_1 + A_2)^{-1}(B_1 + B_2)(A_1 + A_2)^{-1}$$

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We shall discuss $A_k$ and $B_\ell$ subsequently, but first we define some additional notation. Let

$$
\mathcal{E}(P_1^j|x_j;\theta) = \mu_1^j(\theta) \text{ and } \mathcal{E}(N_j;r_j;\theta) = \nu_j(\theta)
$$

and

$$
d\mu_1^j(\theta) = \frac{\partial}{\partial \theta} \mu_1^j(\theta) \text{ and } d\nu_j(\theta) = \frac{\partial}{\partial \theta} \nu_j(\theta).
$$

Define the residuals as well as the covariances between residuals at different lots and auctions and the variance of the residual in the participation equation by

$$
E_1^j = P_1^j - \mu_1^j(\theta^0) \text{ and } U_j = N_j^j - \nu_j(\theta^0),
$$

as well as

$$
\sigma_1^j = \mathcal{E}(E_1^j E_1^j|x_j) \text{ and } \delta_1^j = \mathcal{E}(U_j^2|r_j).
$$

Note that the covariance between residuals at different auctions is assumed to be zero, but there may be covariance between residuals for different lots at the same auction. Note too that $\mathcal{E}(U_j E_1^j|x_j)$ equals zero given the definition of these residuals, so that there are zero covariances across the two components of the objective function. We introduce the following final set of notation which is similar to that used in Laffont et al. (1995):

$$
\omega_1^j = \text{cov}[p_1^j(\theta^0), p_1^j(\theta^0)|x_j]
$$

$$
D_1^j = \text{cov}[d_1^j(\theta^0), d_1^j(\theta^0)|x_j]
$$

$$
C_1^j = \text{cov}[p_1^j(\theta^0), d_1^j(\theta^0)|x_j].
$$

Note that $\omega_1^j$ is a scalar, $D_1^j$ is a square matrix with dimension equal to the length of $\theta$, and $C_1^j$ is a vector with the same length as $\theta$.

Using this notation, we have

$$
A_1 = \lim_{j \to \infty} \frac{1}{J} \sum_{j=1}^{J} \mathcal{E} \left[ \sum_{i=1}^{T_j} d\mu_1^i(\theta^0) d\mu_1^i(\theta^0)^\top \right]
$$

$$
A_2 = \lim_{j \to \infty} \frac{1}{J} \sum_{j=1}^{J} d\nu_j(\theta^0) d\nu_j(\theta^0)^\top
$$

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as well as
\[
    B_1 = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \mathcal{E} \left( \sum_{t=1}^{T_j} \sum_{l=1}^{T_j} \left\{ \sigma_{t,l}^j \delta \mu_l^j(\theta^0) d\nu_l^j(\theta^0)^\top + \frac{1}{S} \left[ \omega_{t,l}^j D_{t,l}^j + \omega_{t,l}^j D_{t,l}^j(\theta^0) d\nu_l^j(\theta^0)^\top \right] + \frac{1}{S(S-1)} (\omega_{t,l}^j D_{t,l}^j + C_{t,l}^j C_{t,l}^j)^\top \right\} \right).
\]

\[
    B_2 = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \delta_{U,j}^2 d\nu_j(\theta^0) d\nu_j(\theta^0)^\top.
\]

Note that the last term in \( B_1 \) obtains because of the possible covariance across residuals for different lots at the same auction. Note too that no covariance exists across different auctions by the independent-sampling assumption.

In practice, these terms need to be estimated. The terms \( A_2 \) and \( B_2 \) can be estimated by substituting \( \hat{\theta} \) for \( \theta^0 \),

\[
    \hat{A}_2 = \frac{1}{J} \sum_{j=1}^{J} d\nu_j(\hat{\theta}) d\nu_j(\hat{\theta})^\top
\]

and

\[
    \hat{B}_2 = \frac{1}{J} \sum_{j=1}^{J} \delta_{U,j}^2 d\nu_j(\hat{\theta}) d\nu_j(\hat{\theta})^\top.
\]

with \( \delta_{U,j}^2 \) being estimated by the square of the \( j \)th fitted residual of the participation regression function. To estimate \( A_1 \) and \( B_1 \), we adapt the approach of Laffont et al. (1995) and use the following formula to estimate \( A_1 \):

\[
    \hat{A}_1 = \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T_j} \left\{ \tilde{d}_t^j(\hat{\theta}) \tilde{d}_t^j(\hat{\theta})^\top + \frac{1}{S(S-1)} \sum_{s=1}^{S} [\tilde{d}_t^{is}(\hat{\theta}) - \tilde{d}_t^j(\hat{\theta})] [\tilde{d}_t^{is}(\hat{\theta}) - \tilde{d}_t^j(\hat{\theta})]^\top \right\}.
\]

To estimate \( B_1 \), we introduce

\[
    \tilde{b}_j(\theta) = \sum_{t=1}^{T_j} [\tilde{p}_t - \tilde{p}_t(\theta)] \tilde{d}_t^j(\theta) + \frac{1}{S(S-1)} \sum_{s=1}^{S} [\tilde{p}_t^{is}(\theta) - \tilde{p}_t^j(\theta)] \tilde{d}_t^{is}(\theta)
\]

and then use

\[
    \hat{B}_1 = \frac{1}{J} \sum_{j=1}^{J} \tilde{b}_j(\hat{\theta}) \tilde{b}_j(\hat{\theta})^\top
\]

Together, these estimators provide us with a consistent means of performing inferences concerning \( \theta \).


6.2.2. Estimates of the Primitives

In table 1, we present estimates of the Poisson intensity parameter $\lambda$ and the parameters of the Weibull distribution $\alpha$ as well as their standard errors. In figure 4, we also present an estimate of the Weibull density function.

7. Summary and Conclusions

In this paper, we have developed a theoretically simple yet empirically tractable model of a sequential, oral, ascending-price, open-exit auction with multi-unit demand within the independent private-values paradigm. We have demonstrated that the outcomes at these auctions are efficient and that the price paths across consecutive lots sold at the same auction follow a sub-martingale. Using data from a sample of sequential, oral, ascending-price, open-exit auctions of timber-export
permits held in Siberia between May of 1993 and May of 1994, we have found that the broad predictions of the theoretical model are supported by reduced-form econometric estimates. We have also developed and estimated a parametric structural-econometric model which can provide information useful in evaluating the effects of alternative institutional design on the expected revenues of the seller.
A. Appendix

In this appendix, we document the development of the data set used and present the proofs of results and theorems stated in the text of the paper.

A.1. Winning Bid Data

The data used in this paper were obtained during an academic-exchange visit by the second author to the State University of Krasnoyarsk in Krasnoyarsk, Siberia, Russia. The collection, translation, and tabulation of the data from the records of an auction house called “The Birch” was directed by Professor Mihail I. Golovanov of the Faculty of Mathematics at the State University of Krasnoyarsk.

For each auction, we know how many lots of timber-export permits were sold as well as how many cubic metres each lot contained. From the records of “The Birch”, we have derived the number of participants who paid the entry fee, and subsequently received a white bidding card. For each lot, we observe the reserve price, the winning bid price, and an identification number of the winning bidder. The identification number of a winning bidder is unique to him at any auction. Unfortunately, these identification numbers may change across auctions. For example, identification number 1 at auction 1 could identify Firm A, which could also have identification number 81 at auction 37. This is because different people representing the same firm attended different auctions. Unfortunately, the way records were kept at “The Birch” did not include this distinction. Thus, although the raw data concerning identification numbers range from 1 to 88, suggesting that at least 88 potential bidders exist, there were, in fact, considerably fewer potential bidders, somewhere between 20 and 40. The important descriptive statistics are presented below in table A.1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>St.Dev.</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Lots</td>
<td>8.47</td>
<td>8.00</td>
<td>1</td>
<td>32</td>
<td>37</td>
</tr>
<tr>
<td>Number of Participants</td>
<td>6.64</td>
<td>3.52</td>
<td>2</td>
<td>15</td>
<td>37</td>
</tr>
<tr>
<td>Total Volume</td>
<td>8786.94</td>
<td>9801.60</td>
<td>180</td>
<td>35000</td>
<td>37</td>
</tr>
<tr>
<td>Reserve Price</td>
<td>1788.96</td>
<td>454.02</td>
<td>1500</td>
<td>2500</td>
<td>308</td>
</tr>
<tr>
<td>Winning Bid</td>
<td>3249.17</td>
<td>824.11</td>
<td>1666.67</td>
<td>5400.00</td>
<td>308</td>
</tr>
<tr>
<td>Volume</td>
<td>1036.79</td>
<td>761.42</td>
<td>180</td>
<td>5000</td>
<td>308</td>
</tr>
</tbody>
</table>
A.2. Proofs of Results and Theorems

In this subsection of the appendix, we collect the proofs of the results and theorems presented in the text of the paper.

Proof of Property C:

We need to show that \( \text{Pr}(W_k^j \leq y \mid W_j^x = x, m \geq j) \) depends only on \( \Delta \equiv (k - j), y, \) and \( x \) and not on \( j \) and \( k \) specifically. We have

\[
\text{Pr}(W_j^x = x, m \geq j) = \sum_{m=j}^{\infty} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j} [1 - F(x)]^{j-1} f(x) \\
= \frac{\lambda^j [1 - F(x)]^{j-1} f(x)}{(j-1)!} \sum_{m=j}^{\infty} \frac{\lambda^{m-j} F(x)^{m-j} \exp(-\lambda)}{(m-j)!} \\
= \frac{\lambda^j [1 - F(x)]^{j-1} f(x)}{(j-1)!} \exp(-\lambda)[1 - F(x)]
\]

and

\[
\text{Pr}(W_j^x = x, m \geq j, W_k^x \leq y) = \\
\sum_{m=j}^{k-1} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j} [1 - F(x)]^{j-1} f(x) + \\
\sum_{m=k}^{\infty} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j} [1 - F(x)]^{j-1} f(x) \\
\frac{(m-j)!}{(k-j)!(m-k-1)!} \int_{F(x)}^{F(x)} t^{m-k-1} (1-t)^{k-j} \, dt = \\
\frac{\lambda^j [1 - F(x)]^{j-1} f(x)}{(j-1)!} \left[ \sum_{m=j}^{k-1} \frac{\lambda^{m-j} \exp(-\lambda)}{(m-j)!} F(x)^{m-j} + \right. \\
\left. \sum_{m=k}^{\infty} \frac{\lambda^{m-j} \exp(-\lambda)}{(m-j)!} F(x)^{m-j} \frac{(m-j)!}{(k-j)!(m-k-1)!} \int_{F(x)}^{F(x)} t^{m-k-1} (1-t)^{k-j} \, dt \right]
\]

Dividing (A.2) by (A.1), we obtain after simplification and using \( \ell = (m - \Delta) \), as stated above, an expression independent of \( j \)

\[
\text{Pr}(W_k^j \leq y \mid W_j^x = x, m \geq j) = \sum_{\ell=0}^{\Delta-1} \frac{(\ell+1)!}{\ell!} \int_{F(x)}^{F(x)} t^{\ell} \Delta^{-\Delta} (1-t)^\Delta \, dt
\]

Proof of Theorem 4.1:

The proof proceeds by backwards induction. At the last sale, lot 1, the game corresponds to the usual single-unit, oral, ascending-price auction. The dominant strategy for each participant \( i \) is to
bid until the price reaches his valuation for the extra unit, if \( i \) has already won \( \ell_i \) units this is given by \( V_i[h(1)] = W_{i+1}^t \). Since, \( s_1 = (n - t) = (n - 1) \), the strategy prescribed by Theorem 4.1 forms the equilibrium when \( t = 1 \). We now show that if it is a best response for each player to follow the proposed equilibrium strategy for lot \( t - 1 \), then it is his best response to follow for lot \( t \).

Consider the case of bidder \( i \) who has already won \( \ell_i \) units, where \( t < n \), and the highest (re-indexed) second valuation of all the bidders in \( N/\{i\} \) is lower than the \( e^t \) highest first valuation among these bidders. According to Theorem 4.2(iii), bidder \( i \) must pay exactly \( W_{T-\ell_i}^t \) in order to win an extra unit; he must also pay \( W_{T-\ell_i-1}^{-i} \) for an additional extra unit, etc. Given this, the best he could do is to buy \( q \) extra units if and only if \( W_{\ell_i+q}^t > W_{T-\ell_i-q}^t \). As we know from the efficiency argument, this is exactly what he will achieve if he follows his equilibrium strategy. Indeed if \( (\ell_i + q) \) is the number of units obtained by \( i \) in equilibrium, we have \( W_{\ell_i+q}^t = W_{T+1}^t = W_{T-\ell_i-q}^t \). Hence, he cannot do better by deviating.

Now consider the alternative case of bidder \( i \) who has already won \( \ell_i \) units, and where either \( t < n \), or the highest (re-indexed) second value of all the bidders in \( N/\{i\} \) exceeds the \( e^t \) highest first valuation among these bidders. Let \( j \) be the index of the bidder with the highest second valuation among all bidders but \( i \) after some history \( h(t) \). According to the equilibrium, no one but \( i \) might wish to out bid \( j \). We now show that \( i \) must be indifferent between the two following alternatives: (a) participate until he wins sale \( t \), withdraw from sale \( (t - 1) \), and let according to the equilibrium \( j \) win; or (b) let \( j \) win sale \( t \) and win sale \( (t - 1) \) instead. Because bidding strategies depend only on how many units each bidder has previously won, the choice between these two alternatives does not affect the outcome of sale \( (t - 2) \) to 1. The only thing that matters is the expected price that \( i \) will need to pay in order to win in sale \( t \) or \( (t - 1) \). In order to win sale \( t \) and out bid \( j \), \( i \) will need to pay: \( E(W_{T-\ell_i}^t | \Omega_t) \) where \( \ell_i \) is the number of units already won by \( i \). In order to win sale \( (t - 1) \), after \( j \) has won sale \( (t - 1) \), \( i \) will need to pay \( E(W_{T-\ell_i}^t | \Omega_{t-1}) \). Conditional on \( \Omega_t \), the expected value of this latter price is given by \( E[E(W_{T-\ell_i}^t | \Omega_{t-1}) | \Omega_t] = E(W_{T-\ell_i}^t | \Omega_t) \). It follows that if both expected price are equals, alternatives (a) and (b) yield the same expected payoff to bidder \( i \).

Now, suppose \( i \) were supposed to win sale \( t \), but he deviates and lets \( j \) win instead. Bidder \( i \) would win the next sale if he uses then his equilibrium strategy. But we know from above that deviating in sale \( t \) and then following the equilibrium strategy in sale \( (t - 1) \) is not better than following the equilibrium strategy, winning in sale \( t \), and withdrawing from sale \( (t - 1) \) to let \( j \) win. This strategy, in turn, under the assumption of equilibrium, cannot be better than following the equilibrium strategy in both sales. Hence, \( i \) has no incentive to bid less than prescribed by Theorem 4.1. Similarly, suppose that \( j \) were supposed to win sale \( t \), but that \( i \) deviates and out bids \( j \) in sale \( t \). \( j \) would win the next sale if \( i \) follows in sale \( (t - 1) \) the equilibrium. Here again, we know that to deviate in sale \( t \) and then to follow the equilibrium strategy in sale \( (t - 1) \) is not better than to follow the equilibrium strategy in sale \( t \) and to deviate in sale \( (t - 1) \). Hence, \( i \) has no incentive to bid more than prescribed by Theorem 4.1.

**Proof of Theorem 4.2:**

**Part (i)**

(Symmetry) Consider the bid function for bidder \( i \) in sale \( t \) after withdrawals by bidders in \( S_t \) and
$S_2$. We have

\[ b_i^j(x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w) \equiv \mathcal{E}(V_i^{-j}|V_2^k = x \forall k \in \mathbb{R}, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, V_i^k \geq x \forall k \in N_w \cap S_2) \]

\[ = \mathcal{E}(V_i^{-j}|V_2^k = x \forall k \in \mathbb{R}, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, V_i^k \geq x \forall k \in N_w \cap S_2) \]

\[ \equiv b_i^j(x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w). \]

From Property D, the conditional distribution of higher-order statistics given low-order statistics is the same across bidders and both expressions are expected value based on the same conditional information. This implies symmetry.

(Strict monotonicity) The function $b_i^j$ is an operator that evaluates the $d^{th}$ highest valuation among those all bidders but $i$ given some information. In all cases, there are at most $(n - |S_1| - 1) = \min[n - 1, t - 1] < t$ valuations in $V^{-i}$ which are above $x$. It follows that $V_i^{-i} \leq x$ in all cases. If $2(r - 1) + |S_2 \cap N_w| \leq t$, then there are at least $t$ values in $V^{-i}$ which are greater than or equal to $x$. In this case $b_i^j(x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w) = x$, and the function $b_i^j$ is clearly strictly increasing in $x$ and symmetric in $i$'s.

In general, we know by assumption that there are at least $(r - 1) + |S_2 \cap N_w|$ valuations in $V^{-i}$ which are greater than or equal to $x$ (these are the first valuations of bidders in $R/\{i\}$ and $S_2 \cap N_w$), and there are $(r - 1)$ valuations which exactly equal $x$ (these are the second valuation of all those in $R/\{i\}$). Let $x$ be the list of all the other valuations in $V^{-i}$, and let $H_k(x)$ denote the $k^{th}$ highest valuation in vector $x$. The $H_k$ is continuous and non-decreasing in all its arguments. We have that $V_i^{-i} = \min[x, H_{t-q}(x)]$ where $q = 2(r - 1) + |S_2 \cap N_w|$. Because not all valuations in $x$ are known, in particular the valuations $\{V_3^j, V_4^j, \ldots, V_t^j\}$ for all $j \in R/\{i\}$, the expression $H_{t-q}(x)$ is a random variable. We let $G(\cdot|x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1})$ denote the conditional distribution of $H_{t-q}(x)$. Given this, we have

\[ b_i^j(x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w) = x[1 - G(x|x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w)] + \int_0^x h \ dG(h|x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w). \]

If one differentiates $b_i^j$ with respect to $x$, then one obtains

\[ [1 - G(x|x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w)] - \int_0^x \frac{dG(h|x, \hat{V}_2^{S_2}, \hat{V}_1^{S_1}, N_w)}{dx} \ dh. \]

Because the valuations $\{V_3^j, V_4^j, \ldots, V_t^j\}$ for all $j \in R/\{i\}$ are positively correlated with $x$ (see Property E) and have support in $[0, x]$, the expression $\frac{dG(h|x, \hat{V}_2^{S_2})}{dx}$ is strictly negative for all $h \in [0, x]$. Hence, the function $b_i^j$ is strictly increasing in $x$.

Part (ii)

We need to show that each bidder $i$ receives $l$ units if and only if $W_i^l > W_{T+1}$. It suffices to show that in each successive round, the lots are allocated to a bidder whose highest re-indexed valuation exceeds $V_{i+1}[h(t)] = W_{T+1}$.

When $t \geq n$, bids are a monotonic and symmetric function of each bidder's second-highest valuations $V_2[h(t)]$. The winner after every history $h(t)$ of sale $t$ will then be the one with the
highest individual second-value. His highest valuation will necessarily be within the \( n \)th highest value overall and a fortiori within the \( t \)th highest valuation in \( V[h(t)] \).

When \( t < n \), everyone first bids up to his first valuation, until the price reaches the \((t + 1)\)th highest individual first valuation. After that, two thing may happen: first, if the second valuations of all remaining participants are less than the current price, they all drop out simultaneously and the lot is randomly allocated among these last bidders; second, if the second valuations of some of the remaining participants exceeds the current price, they will pursue bidding until the lot is awarded to the one with the highest second valuation. In either case, the lot is awarded to a bidder whose highest valuation exceeds the highest valuation of at least \((n - t)\) participants and the second highest valuation of the \((t - 1)\) others. Thus, the winner’s highest valuation must necessarily be included among the \( t \)th highest valuations in \( V[h(t)] \).

Part (iii)

We prove (iii) given efficiency and part (i). Suppose that either \( t \geq n \) or the highest (re-indexed) second value of all the bidders in \( N / \{ \ell \} \), say bidder \( j ’ s \), exceeds the \( t \)th highest first valuation among these bidders. According to the equilibrium, \( i \) in order to win will need to pay the following price:

\[
\begin{align*}
\mathcal{E}(V_i^{\cdot j} | V_2^j) &= \mathcal{E}(V_i^{\cdot j} | V_2^j, V_2^S, V_i^{S_1}, V_i^k \geq V_2^j \quad \forall k \in N \cap S_2) \\
&= \mathcal{E}(V_i^{\cdot j} | V_2^j, V_2^S, V_i^{S_1}, V_i^k \geq V_2^j \quad \forall k \in N \cap S_2) \\
&= \mathcal{E}(W_{T-\ell_1}^{\cdot i} | (W_2^k, \ldots, W_2^k)_{k \in S_1}, (W_1^k, \ldots, W_2^k)_{k \in S_1}) \\
&= \mathcal{E}(W_{T-\ell_1}^{\cdot i} | \Omega_t).
\end{align*}
\]

The first expression gives the price at which bidder \( j \) will drop out of the auction when in the end only he and \( i \) remain. The first equality follows from symmetry. The second equality is more involved. Let \( \ell_k \) be the number of lots won by some bidder \( k \) prior to sale \( t \). Since lots are allocated to those with the highest valuations, we have for all \( k \in N \cap S_2 \), \( W_2^k + 1 = V_2^k[h(t)] > W_2^j + 2 = V_2^j[h(t)] \). Also, since \( \sum_{k \in S_1} \ell_k = T - t - \ell_i \), we have \( W_{T-\ell_1}^{\cdot i} = V_2^{T-\ell_1} \). Further, since the distribution of \( W_{T-\ell_1}^{\cdot i} \) for \( q \geq 2 \) given \( W_{T-\ell_1}^{\cdot i} \) is independent of \( W_2^k, \ldots, W_2^k \), the second expression in (5.7) is equal to Equation (5.8). Finally, note that buyers reveal through bidding their re-indexed second valuation, which is given by \( W_{T-\ell_1}^{\cdot i} \). The first values are revealed only by those who are among the first \( s_1 \) bidders to withdraw when \( t < n \). Thus, \( \Omega_t \) is included in \( \{(W_2^k, \ldots, W_2^k)_{k \in S_1}, (W_1^k, \ldots, W_2^k)_{k \in S_1} \} \). Hence the last equality.

Finally, let \( \theta \) be the highest \( t \)th highest first valuation among all bidders but \( i \), and suppose that \( n > t \) and the highest second valuation among all bidders but \( i \) is below \( \theta \). If \( i \) desires to win auction \( t \), he will need to pay \( \theta = V_2^{T-\ell_1} = W_{T-\ell_1}^{\cdot i} \). According to the equilibrium, buyers will bid up to their first valuations until the bidder with first valuation \( \theta \) withdraws. Because the second valuation of all other bidders is less than \( \theta \), all remaining bidders but \( i \) will simultaneously withdraw so \( i \) wins at price \( \theta \).

Proof of Theorem 5.1:

Let \( i \) be the winner of sale \( t \) and let either some bidder \( j \) or \( i \) be the winner of the next sale
(t - 1). Following Theorem 4.2 (iii), the winning prices in equilibrium at sales \( t \) and \( (t - 1) \) are given respectively by
\[
\mathcal{E} \{ W_{T-t_i}^j | V_2^j[h(t)] = V_2^j[h(t)] , \Omega_i \} = \mathcal{E} \{ \mathcal{E}(W_{T-t_i}^j | \Omega_{t-1}) | V_2^j[h(t)] \geq V_2^j[h(t)], \Omega_i \}
\]
\[
= \mathcal{E} \{ W_{T-t_i}^j | V_2^j[h(t)] \geq V_2^j[h(t)], \Omega_i \}
\]
\[
\geq \mathcal{E} \{ W_{T-t_i}^j | V_2^j[h(t)] = V_2^j[h(t)], \Omega_i \} = P_t.
\]
This establishes (i). The former is the expected value of \( W_{T-t_i}^j \), given that \( V_2^j[h(t)] = V_2^j[h(t)] \), while the latter is its expected value given that \( V_2^j[h(t)] \geq V_2^j[h(t)] \). The inequality follows from the fact that the expected value of \( W_{T-t_i}^j \) is increasing in \( V_2^j[h(t)] \).

Now note that one of the new information revealed by the bidding in sale \( (t - 1) \) that was not revealed before is \( v_3^j[h(t)] \). Since the expression \( \mathcal{E} \{ W_{T-t_i}^j | V_3^j[h(t)], \Omega_i \} \) is non-decreasing in \( V_2^j[h(t)] \), we have
\[
\mathcal{E} \{ W_{T-t_i}^j | V_3^j[h(t)] = V_2^j[h(t)], \Omega_i \} \geq \mathcal{E} \{ W_{T-t_i}^j | V_2^j[h(t)] \geq V_3^j[h(t)] \geq W_{T+1}, \Omega_i \} \geq \mathcal{E} \{ W_{T-t_i}^j | W_{T+1} \geq V_3^j[h(t)], \Omega_i \}.
\]
The first expression is equal to the expected price paid by \( i \) if \( i \) wins auction \( (t - 1) \). The second expression is the expected winning price in sale \( (t - 1) \) conditional on \( i \)'s winning at least two other units after sale \( t \). The last expression is the expected value of \( P_{t-1} \) when \( i \) wins less than two extra units after \( t \). This establishes (ii).

A.3. Derivatives for Standard Errors

Below, we provide the formulae for the derivatives necessary to calculate the standard errors. For notational parsimony, we suppress the \( j \) superscript, which denotes the \( j \)th auction in the data set, so these formulae are valid for a representative observation in the data set.

Recall that
\[
\mathcal{E}(N) = N \left( 1 - \exp \left\{ -\lambda \left[ 1 - F(r; \alpha) \right] \right\} \right) = N \left\{ 1 - \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right] \right\} = \nu(\theta),
\]
so
\[
\frac{\partial \nu}{\partial \lambda} = -N \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right],
\]
\[
\frac{\partial \nu}{\partial \alpha_1} = -N \lambda r^{\alpha_2} \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right],
\]
\[
\frac{\partial \nu}{\partial \alpha_2} = -N \lambda \alpha_1 r^{\alpha_2} \log(r) \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right].
\]

Also,
\[
p_i^\theta = \frac{h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r, n)}{h_s(w_s; \alpha, m_s) g_s(m_s; \theta, r, n)}
\]
where

$$\log h_s(w_s; \alpha, m_s) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} \left[ \log \alpha_1 + \log \alpha_2 + (\alpha_2 - 1) \log(w_k) - \alpha_1 w_k^{\alpha_2} + \alpha_1 r^{\alpha_2} \right]$$

and

$$\log g_s(m_s; \theta, r, n) = \sum_{i=1}^{n} \left( m_i \log \left[ \lambda \exp(-\alpha_1 r^{\alpha_2}) \right] - \lambda \exp(-\alpha_1 r^{\alpha_2}) - \log(m_i) - \log \left\{ 1 - \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right] \right\} \right).$$

Now

$$\frac{\partial p_i}{\partial \lambda} = \frac{p_i \left[ \frac{\partial h_s(w_s; \alpha, m_s)}{\partial \lambda} g_s(m_s; \theta, r, n) + h_s(w_s; \alpha, m_s) \frac{\partial g_s(m_s; \theta, r, n)}{\partial \lambda} \right]}{h_s(w_s; \theta, \alpha, m_s) g_s(m_s; \theta, r, n)}$$

$$\frac{\partial p_i}{\partial \alpha_1} = \frac{p_i \left[ \frac{\partial h_s(w_s; \alpha, m_s)}{\partial \alpha_1} g_s(m_s; \theta, r, n) + h_s(w_s; \alpha, m_s) \frac{\partial g_s(m_s; \theta, r, n)}{\partial \alpha_1} \right]}{h_s(w_s; \theta, \alpha, m_s) g_s(m_s; \theta, r, n)}$$

$$\frac{\partial p_i}{\partial \alpha_2} = \frac{p_i \left[ \frac{\partial h_s(w_s; \alpha, m_s)}{\partial \alpha_2} g_s(m_s; \theta, r, n) + h_s(w_s; \alpha, m_s) \frac{\partial g_s(m_s; \theta, r, n)}{\partial \alpha_2} \right]}{h_s(w_s; \theta, \alpha, m_s) g_s(m_s; \theta, r, n)}.$$

Furthermore,

$$\frac{\partial \log h_s(w_s; \alpha, m_s)}{\partial \lambda} = 0$$

$$\frac{\partial \log h_s(w_s; \alpha, m_s)}{\partial \alpha_1} = \left[ \sum_{i=1}^{n} \frac{m_i}{\alpha_1} - \sum_{k=1}^{n} \sum_{i=1}^{m_i} w_k^{\alpha_2} + \sum_{i=1}^{n} \frac{m_i}{\alpha_1} \sum_{k=1}^{m_i} \right]$$

$$\frac{\partial \log h_s(w_s; \alpha, m_s)}{\partial \alpha_2} = \left[ \sum_{i=1}^{n} \frac{m_i}{\alpha_2} + \sum_{k=1}^{n} \log w_k - \sum_{k=1}^{n} \alpha_1 w_k^{\alpha_2} \log(w_k) + \alpha_1 r^{\alpha_2} \log(r) \sum_{i=1}^{n} m_i \right]$$

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and
\[
\frac{\partial \log g_s(m_s; \theta, r, n)}{\partial \lambda} = \left( \frac{\sum_{i=1}^n m_i}{\lambda} - n \exp(-\alpha_1 r^{\alpha_2}) - \frac{n \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right]}{1 - \exp(-\alpha_1 r^{\alpha_2})} \right)
\]
\[
\frac{\partial \log g_s(m_s; \theta, r, n)}{\partial \alpha_1} = \left( -\alpha_1 r^{\alpha_2} \sum_{i=1}^n m_i + n \lambda r^{\alpha_2} \exp(-\alpha_1 r^{\alpha_2}) + \frac{n \lambda r^{\alpha_2} \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right]}{1 - \exp(-\alpha_1 r^{\alpha_2})} \right)
\]
\[
\frac{\partial \log g_s(m_s; \theta, r, n)}{\partial \alpha_2} = \left( -\alpha_1 r^{\alpha_2} \log(r) \sum_{i=1}^n m_i + n \lambda r^{\alpha_2} \log(r) \exp(-\alpha_1 r^{\alpha_2}) + \frac{n \lambda r^{\alpha_2} \log(r) \exp(-\alpha_1 r^{\alpha_2}) \exp \left[ -\lambda \exp(-\alpha_1 r^{\alpha_2}) \right]}{1 - \exp(-\alpha_1 r^{\alpha_2})} \right).
\]
B. Bibliography