Nash Equilibrium Approximation in Games of Incomplete Information
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Abstract

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The most commonly used solution concept in game theory is that of Nash Equilibrium.[1] However, except under fairly restrictive assumptions whose empirical validity often is questionable, many games cannot be solved analytically for NE solutions. As an alternative to NE Armantier, Florens and Richard (2000) (hereafter AFR) introduce the concept of Constrained Strategic Equilibrium (hereafter CSE). Essentially, they propose to restrict attention to appropriate subsets of strategies, typically indexed by an auxiliary parameter vector, and to search for an equilibrium solution within such subsets. The authors show that CSE offer a major computational advantage, and they provide a powerful algorithm based upon Monte Carlo (hereafter MC) simulations to determine the CSE numerically. The concept of CSE appeared to be relevant under two scenarios: the first one is directly related to the general notion of 'bounded rationality' and more specifically to the concept of Rules of Thumb ; in the second scenario, one would use the computational advantage of the CSE with the intent to approximate an analytically untractable NE solution. The objective of the present essay is to establish conditions under which a sequence of CSE approximates a NE, in the context of games of incomplete information.

It appears natural to try to approximate a NE since the later might not be analytically tractable or might be too complex to calculate. This suggestion finds additional heuristic support in the common observation that in those few cases where NE strategies can be computed, their 'smooth' graphs clearly suggest that it ought to be possible to approximate them by simpler functional forms, such as low degree polynomials, piecewise linear and/or exponential functions. See e.g. some of the graphs of NE strategies found in Marshall et al. (1994) or AFR (2000).

Several approaches may be considered to approximate numerically the NE. If the NE were know one could use standard techniques to determine its best approximation with respect to a given distance on a constraint set of strategies. This approach is mostly irrelevant in practice since typically the NE cannot be calculated. One could also approximate the NE by finding the closest solution to the first order conditions (hereafter FOC) within a constraint set. For instance, in the context of auctions Bajari (1996) propose to solve the system of differential equation resulting from the FOC by finite elements techniques. However this approach requires to explicit the FOC in the extensive form game which might not be possible when the conditional individual expected utility functions are not

[1] Depending on the information available to players we shall consider Nash Equilibrium or Bayesian Nash Equilibrium. Hereafter both concepts in pure statregy are denoted NE.

differentiable or when actions are not continuous. Finally, one could approximate the NE by a CSE. This approach offers the advantages that it relies on "primitive" elements of the game (utility, distribution and strategy functions), it has also a game theoretic interpretation in finite distance and CSE can easily be computed with a flexible algorithm.

1. The general Model of incomplete Information

There are N players each of which is endowed with a privately known 'type' or 'signal' $\xi_i \in \Omega_i$ with $\Omega_i \subset \Re^p$ and $\Omega = \prod_{i=1}^{N} \Omega_i$. The types $\xi = (\xi_1, ..., \xi_N)$ are drawn from a joint distribution with cumulative distribution function (hereafter c.d.f.) $F(\xi)$ and density $f(\xi)$ (known to the players but not to the observer). Let $F_i(\xi_i)$ denote the marginal c.d.f. of ξ_i and $f_i(\xi_i)$ the corresponding density. This general framework includes as special cases of interest:

1. i.i.d. types,
$$f(\xi) = \prod_{i=1}^{N} f(\xi_i) \quad ; \tag{1.1}$$

2. exchangeable (affiliated) types,
$$f(\xi) = \int_{S_0} \prod_{i=1}^{N} f(\xi_i \mid \xi_0) \cdot f_0(\xi_0)\, d\xi_0 \quad , \tag{1.2}$$
where $\xi_0 \in \Omega_0$ denotes a 'linkage' random variable (e.g. the unknown value of an item being sold through a 'Common Value' auction) drawn from a distribution with p.d.f $f_0(.)$;

3. asymmetric independently distributed types,
$$f(\xi) = \prod_{i=1}^{N} f_i(\xi_i) \quad . \tag{1.3}$$

Unobserved signals are transformed into action by means of a transformation φ_i which depends upon $F(.)$,

$$\Omega_i \rightarrow X_i \tag{1.4}$$
$$\xi_i \rightarrow x_i = \varphi_i(\xi_i), \qquad i : 1 \rightarrow N \quad . \tag{1.5}$$

Where X_i is the set of possible actions, $\varphi_i(.) \in H_i$ and H_i is the set of admissible strategies for player i. H_i originally contains all possible functions, but, as we shall see in section 2, it can be reduced to a more amenable set of functions. Player i is endowed with an individual utility function $U_i(\varphi(\xi), \xi)$.[2] The number of players N (depending upon the situation, the decision to participate may be endogenous or exogenous), the joint distribution F, the utility functions $\{U_i\}_{i=1,...N}$ and the sets of admissible strategies $\{H_i\}_{i=1,...N}$ are common knowledge to all players. Symmetry assumes that the joint distribution F is exchangeable (i.e. F is invariant under a permutations of players), $(U_i, H_i, \Omega_i) = (U_j, H_j, \Omega_i)$ and the equilibrium strategies (subject to existence) are such that $\varphi_i = \varphi_j, \forall i \neq j$. Note that exchangeability reduces to the equality of marginal distributions when types are univariate or independent.

The strategic form of the game is based upon the set of individual expected utility functions

$$\widetilde{U}_i(\varphi) = E_\xi \left[U_i(\varphi(\xi), \xi)\right] \quad . \tag{1.6}$$

The extensive form of the game is based upon the set of conditional individual expected utility functions

$$\widehat{U}_i(\varphi; \xi_i) = E_{\xi_{-i}|\xi_i} \left[U_i\left(\varphi_i(\xi_i), \varphi_{-i}\left(\xi_{-i}\right); \xi_i, \xi_{-i}\right)\right] \quad . \tag{1.7}$$

2. Unconstrained NE solutions

Subject to existence, a Bayesian Nash Equilibrium in pure strategy in the set of strategies $H = \prod_{i=1}^{N} H_i$ is defined by Harsanyi (1967) as a strategy profile $\varphi^{NE} = \left(\varphi_1^{NE}, ..., \varphi_N^{NE}\right)$ with $\varphi^{NE} \in \Phi_{NE}$[3] of mutually best responses strategies in the extensive form game:

$$\widehat{U}_i\left(\varphi_i^{NE}(\xi_i), \varphi_{-i}^{NE}; \xi_i\right) \geq \widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right),$$

$$\forall x_i \in X_i,\ \forall \xi_i \in \Omega_i \text{ and } \forall i : 1 \rightarrow N \quad . \tag{2.1}$$

Consider the following assumptions:

1) $H_i \subset \Theta_i$ where Θ_i is a measurable set and $\Theta = \prod_{i=1}^{N} \Theta_i$.

[2]For the ease of exposition we adopt the usual notation: $\xi = \left(\xi_i, \xi_{-i}\right) = (\xi_1, ..., \xi_N)$ and $\varphi(\xi) = \left(\varphi_i(\xi_i), \varphi_{-i}\left(\xi_{-i}\right)\right) = (\varphi_1(\xi_1), ..., \varphi_N(\xi_N))$.

[3]Note that we do not assume that the NE is unique.

2) $U(.,.)$, $\varphi(.)$ are measurable functions.
3) $\forall\varphi_{-i} \in H_{-i}$ and $\forall\Psi_i \in \Theta_i$ there exists $\varphi_i \in H_i$ such that

$$\widehat{U}_i\left(\varphi_i(\xi_i), \varphi_{-i}; \xi_i\right) \geq \widehat{U}_i\left(\Psi_i(\xi_i), \varphi_{-i}; \xi_i\right),$$
$$\forall\xi_i \in \Omega_i \text{ and } \forall i : 1 \to N \quad . \tag{2.2}$$

The following proposition provides conditions under which it is equivalent to consider the extensive or the strategic form of the game to derive the NE solution.

Proposition 2.1. *Consider φ^{NE} the NE in the extensive form game and $\widetilde{\varphi}^{NE}$ the NE derived from the strategic form game*

$$\widetilde{U}_i\left(\widetilde{\varphi}_i^{NE}, \widetilde{\varphi}_{-i}^{NE}\right) \geq \widetilde{U}_i\left(\varphi_i, \widetilde{\varphi}_{-i}^{NE}\right),$$
$$\forall\varphi_i \in H_i \text{ and } \forall i : 1 \to N \quad .$$

Under assumptions 1) to 3) $\varphi^{NE} = \widetilde{\varphi}^{NE}$.

Proof: TO BE COMPLETED

Assumption 3) implies that strategies are admissible in the set H if they are not strictly dominated. In the remainder conditions 1) to 3) are assumed to hold and we will consider either the extensive or the strategic form of the game to derive the NE solution.

Note that no general theorem insures the existence of a NE solution in a game of incomplete information with continuous types and actions. In practice, the problems of existence and uniqueness are solved by the direct determination of an analytical equilibrium solution. This solution obtains from the following optimization and fixed point problems,

$$\varphi_i^{NE}(\xi_i) \in \underset{x_i \in X_i}{ArgMax}\, \widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right), \ \forall\xi_i \in \Omega_i \text{ and } \forall i : 1 \to N \quad . \tag{2.3}$$

The corresponding First Order Conditions (FOCs) often are reformulated as

$$\frac{d}{dx_i}\widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right)|_{x_i=\varphi_i^{NE}(\xi_i)} = 0 \quad \forall\xi_i \in \Omega_i \text{ and } \forall i : 1 \to N, \tag{2.4}$$

which typically produce a set of equations (differential equations in the case of auctions) leading to the solution (provided that $\frac{d^2}{dx_i^2}\widehat{U}_i\left(x_i, \varphi_{-i}^{NE}; \xi_i\right)|_{x_i=\varphi_i^{NE}(\xi_i)} \leq 0$

$\forall i : 1 \rightarrow N$). Note that this approach requires $\widehat{U}_i(.)$ to be twice continuously differentiable in x_i. If we define the operator

$$A_i[\varphi](\xi_i) = \frac{d}{dx_i}\widehat{U}_i\left(x_i, \varphi_{-i}; \xi_i\right)_{|_{x_i=\varphi_i(\xi_i)}}, \qquad i : 1 \rightarrow N \quad ,$$

then a NE verifies $A_i\left[\varphi^{NE}\right](\xi_i) = 0 \ \forall \xi_i \in \Omega_i$ and $\forall i : 1 \rightarrow N$. Except under fairly restrictive assumptions (such as symmetry, risk neutrality, ...) it is often impossible to find an analytical or even numerical solution to such problems.

3. Constrained Strategic Equilibrium

Constrained sets of strategies are implicitly defined here as subsets $H_i^{(k)} \subset H_i$. The definition of CSE now parallels that of a NE in strategic form, except that strategies are now restricted to $H_i^{(k)}$:

Definition 3.1. *A CSE in the set of strategies* $H^{(k)} = \prod_{i=1}^{N} H_i^{(k)}$ *is a strategic implementation of the game* $\varphi_{CSE}^{(k)} = \left(\varphi_{1,CSE}^{(k)}, ..., \varphi_{N,CSE}^{(k)}\right)$ *with* $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$, *whereby the* $\varphi_{i,CSE}^{(k)}$*'s are mutually best responses in the strategic form game*

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \geq \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \quad ,$$

$$\forall \varphi_i^{(k)} \in H_i^{(k)}, \forall i : 1 \rightarrow N \quad .$$

AFR (2000) show that if

i) $H_i^{(k)}$ is compact and convex $\forall i = 1, ..., N$,

ii) the function $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is continuous in φ, $\forall i : 1 \rightarrow N$, $\forall \varphi \in H$,

iii) the function $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is quasi concave in φ_i, $\forall i : 1 \rightarrow N$, $\forall \varphi_i \in H_i$,

then there exists a CSE in $H^{(k)}$. AFR (2000) provide also primitive conditions on the utility function $U_i(\varphi(\xi), \xi)$ so that ii) and iii) are verified. In the remainder we assume that assumptions i) to iii) are verified.

Provided that $\widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right)$ is twice continuously differentiable in φ_i, the CSE can be defined as a fixed point of the constrained best response correspondence

$$\varphi_{i,CSE}^{(k)} \in \underset{\varphi_i^{(k)} \in H_i^{(k)}}{ArgMax}\, \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \quad \forall i : 1 \rightarrow N \quad . \tag{3.1}$$

The determination of this fixed point is greatly simplified with a parametrization of the strategies in $H_i^{(k)}$ by a vector of $d_i^{(k)} \in \Re^k$. Such parametrization is always possible since $H_i^{(k)}$ is compact. This approach provides a major computational advantage since it requires to optimize over a finite set of parameters rather than an infinite set of functions as it is the case with NE (see AFR (2000) for numerical considerations).

4. General Approximation Theorem

In this section we assume that there exists a topology T such that $\underset{k\geq 1}{U} H^{(k)}$ is dense in H with respect to T and $H^{(k)} \subset H^{(k+1)}$ $\forall k \in N^*$. The first proposition show that a limit point of a sequence of CSE is a NE.

Proposition 4.1. *If the sequence* $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ *where* $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$ *has an accumulation point* $M \in H$ *with respect to* T *then there exists a NE in* H *and* $M \in \Phi_{NE}$

Proof:

Let us define $\widehat{H} = \prod_{k=1}^{\infty} H^{(k)}$ the set of sequences in H and

$\widehat{H}' = \left\{\left\{\varphi^{(k)}\right\}_{k=1\to\infty} \in \widehat{H}/\varphi^{(k)} \overset{k\to\infty}{\to} \varphi, \text{ with } \varphi \in H\right\}$ the set of convergent sequences in H. Since $\underset{k\geq 1}{U} H^{(k)}$ is dense in H, then $\forall \varphi \in H$ there exist $\left\{\varphi^{(k)}\right\}_{k=1\to\infty} \in \widehat{H}'$ such that $\varphi^{(k)} \overset{k\to\infty}{\to} \varphi$.

Consider $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ with $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$ $\forall k \in N^*$. Then $\forall k \in N^*$, $\forall \varphi^{(k)} \in H^{(k)}$ we have

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \geq \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \text{ and } \forall i : 1 \to N \tag{4.1}$$

or, equivalently, $\forall \left\{\varphi^{(k)}\right\}_{k=1\to\infty} \in \widehat{H}$

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \geq \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \; \forall i : 1 \to N \tag{4.2}$$

Since $\widehat{H}' \subseteq \widehat{H}$ the previous inequality holds also $\forall \left\{\varphi^{(k)}\right\}_{k=1\to\infty} \in \widehat{H}'$,

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \geq \widetilde{U}_i\left(\varphi_i^{(k)}, \varphi_{-i,CSE}^{(k)}\right) \; \forall i : 1 \to N \tag{4.3}$$

If the sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ has an accumulation point $M=(M_i,M_{-i})$ such that $M\in H$ then there exists $\left\{k^{(m)}\right\}_{m=1\to\infty}$ such that $\left(\varphi_{i,CSE}^{\left(k^{(m)}\right)},\varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right)\overset{m\to\infty}{\to}$ (M_i,M_{-i}) and we still have, $\forall\left\{\varphi^{\left(k^{(m)}\right)}\right\}_{m=1\to\infty}\in\widehat{H}'$

$$\widetilde{U}_i\left(\varphi_{i,CSE}^{\left(k^{(m)}\right)},\varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right)\geq\widetilde{U}_i\left(\varphi_i^{\left(k^{(m)}\right)},\varphi_{-i,CSE}^{\left(k^{(m)}\right)}\right)\ \forall i:1\to N \tag{4.4}$$

Note that each element of H is the limit of an element in $\widehat{H}'$. Then, since $\widetilde{U}_i\left(.\right)$ is continuous in φ $\forall i:1\to N$, we can write the previous equation as m tends toward infinity, $\forall\varphi\in H$

$$\widetilde{U}_i\left(M_i,M_{-i}\right)\geq\widetilde{U}_i\left(\varphi_i,M_{-i}\right)\ \forall i:1\to N \tag{4.5}$$

Therefore $M\in\Phi_{NE}$.■

Proposition 4.2. *if H is compact with respect to the topology T then there exists a NE in H and any sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ has a subsequence that converges towards a NE.*

Proof:

If H is compact any sequence $\left\{\varphi^l\right\}_{l=1\to\infty}\in\prod_{l=1}^{\infty}H$, has at least an accumulation point in H. Since $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}\in\prod_{k=1}^{\infty}H$ it has an accumulation point $M\in H$. Then, from the previous proposition M is a NE.■

The compactness of the strategy space is typically assumed in game of complete information to insure the existence of a NE. In games of incomplete information the exact structure of H is more delicate to determine and it may not be compact. However, in some instances one can use theoretic conditions to reduce the search of a NE to a compact set. For instance consider the independent private value auction model, where types are drawn from a continuous distribution define on a compact set Ω. It can be shown (see Athey 1998) that the equilibrium bid function has to be continuous and strictly increasing over Ω. We show in proposition 4.3 that such functions form a compact set. Therefore, the set of all possible functions H may be constrained to a compact set H' of continuous and strictly increasing functions over Ω.

Proposition 4.3. *if H is the set of functions bounded at a ($|\varphi(a)| \leq \overline{a}\ \forall \varphi \in H$) and of uniformly bounded variation on $[a,b]$ then there exists a NE in H and any sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ has a subsequence that converges towards a NE.*

Proof:

It suffices to show that the set H defined in proposition 4.3 is compact. Consider $\varphi \in H$, the number

$$W_a^b(\varphi) = \sup_{a=\xi_1<\ldots.<\xi_{T+1}=b} \sum_{t=1}^{T} \left|\varphi(\xi_t) - \varphi(\xi_{t+1})\right| \tag{4.6}$$

is called the variation of the φ. Since H is a set of functions of uniformly bounded variation on $[a,b]$, there exists $W > 0$ such that $W_a^b(\varphi) \leq W\ \forall \varphi \in H$. It can be shown that $|\varphi(\xi)| \leq \overline{a} + W\ \forall \varphi \in H,\ \forall \xi \in [a,b]$. Consider $\overline{W} = \sup(\overline{a}, W)$ and a sequence in H $\{\varphi_l\}_{l=1\to\infty}$, then $|\varphi_l(\xi)| \leq 2\overline{W}$. By Jordan's canonical decomposition $\varphi_l = \overline{\varphi}_l - \underline{\varphi}_l$ where $\overline{\varphi}_l$ and $\underline{\varphi}_l$ are increasing and jointly bounded over $[a,b]$:

$$|\overline{\varphi}_l| = \frac{1}{2}\left|W_a^b(\varphi_l) + \varphi_l(\xi)\right| \leq \frac{3}{2}\overline{W} \tag{4.7}$$

$$\left|\underline{\varphi}_l\right| = \frac{1}{2}\left|W_a^b(\varphi_l) - \varphi_l(\xi)\right| \leq \frac{3}{2}\overline{W} \tag{4.8}$$

Helly's first theorem guarantees that any sequence of increasing and bounded functions over $[a,b]$ has a subsequence that converges over $[a,b]$. Now let us apply twice Helly's first theorem, first to the sequence $\left\{\underline{\varphi}_l\right\}_{l=1\to\infty}$ to obtain a convergent subsequence $\left\{\underline{\varphi}_{\beta(l)}\right\}_{\beta(l)=1\to\infty}$ and then to the sequence $\left\{\overline{\varphi}_{\beta(l)}\right\}_{\beta(l)=1\to\infty}$. This way we obtain two convergent subsequences:

$$\underline{\varphi}_{\alpha(l)}(\xi) \overset{l\to\infty}{\to} \underline{\varphi}(\xi) \text{ and } \overline{\varphi}_{\alpha(l)}(\xi) \overset{l\to\infty}{\to} \overline{\varphi}(\xi) \quad \forall \xi \in [a,b] \quad .$$

Hence,

$$\varphi_{\alpha(l)}(\xi) \overset{l\to\infty}{\to} \varphi(\xi) = \overline{\varphi}(\xi) - \underline{\varphi}(\xi) \quad \forall \xi \in [a,b] \quad .$$

Therefore, any sequence $\{\varphi_l\}_{l=1\to\infty}$ in H has a subsequence that converges to a function φ of bounded variation. Now, let us show that the variation of φ is smaller or equal to W. Consider any subdivision $a = \xi_1 < .... < \xi_{T+1} = b$ we know that

$$\sum_{t=1}^{T} \left|\varphi_l\left(\xi_t\right) - \varphi_l\left(\xi_{t+1}\right)\right| \leq W \ \forall l = 1 \to \infty \tag{4.9}$$

Now, take the limit as $l = 1 \to \infty$,

$$\sum_{t=1}^{T} \left|\varphi\left(\xi_t\right) - \varphi\left(\xi_{t+1}\right)\right| \leq W \tag{4.10}$$

Hence, taking the supremum of the left hand side, $W_a^b(\varphi) \leq W$.

Every sequence $\{\varphi_l\}_{l=1\to\infty}$ in H has a subsequence that converges in H therefore H is compact.■

The assumption in proposition 4.4 is less restrictive than the compactness of the strategy space. Indeed, functions of bounded variation include most well defined bounded functions such as the continuous monotonic functions on $[a, b]$, the bounded functions with a countable number of discontinuity points, and the differentiable bounded function with derivatives changing signs a countable number of time. Besides it has been shown (see Athey 1998) that in many games of incomplete information (such has the independent private value auction or Cournot competitions with private costs) the utility function verifies the single crossing property which insures that H is of bounded variation.

To apply proposition 4.3 actions and private signals need to be bounded. In most real life applications unbounded actions or private signals make little sense. Indeed, actions (such as bids, quantities, prices...), or privates signals (such as costs, level of effort, valuations...) are typically finite.

Note that one can use a bounded rationality argument to justify the assumptions of uniformly bounded variation strategies : agents have finite computational capacities that limit them to use a strategy with a variation lower than a certain W. Finally, if the equilibrium is not of bounded variation then two question arise: firstly, can we expect real agents to determine a NE which is a rather "wild" function? Secondly, will any method be able to approximate this NE?

5. Criteria of convergence

In many games the set of feasible strategies is not clearly defined and it is not always possible to apply theoretic restrictions to eliminate strictly dominated strategies. In these cases it might be impossible to verify the conditions of an approximation theorem. Moreover, even when the approximation theorem can be applied, it would be useful to know for any given constraint set $H^{(k)}$ how distant the CSE $\varphi_{CSE}^{(k)}$ is from the NE solution. In this section we say that a sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ approximate a NE φ^{NE} if $\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} = \varphi^{NE}$. We provide three criteria to document whether the CSE is a good approximation of the NE and how far the CSE is from a NE.

5.1. Convergence of the CSE sequence

Consider the criteria

$$C_1(k) = \left\| \varphi_{CSE}^{(k)} - \varphi_{CSE}^{(k-1)} \right\|.$$

where $\|.\|$ is a norm defined over H. Proposition 3.1 states that when a sequence of CSE converges it converges toward a NE. In other words, the sequence $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ is an approximation of the NE if and only if

$$\lim_{k\to\infty} C_1(k) = 0 \quad .$$

In practice, we want to monitor the convergence of a CSE sequence by verifying that the criteria $C_1(k)$ converges toward 0. However, there is no result regarding the rate of convergence. Therefore, even when the criteria $C_1(k)$ is close to 0 this does not provide any explicit information about the quality of the approximation.

5.2. The best response to a CSE

Let us denote $\Phi_{BR}(\varphi) = \prod_{i=1}^{N} \Phi_{BR,i}(\varphi)$ where $\Phi_{BR,i}$ is player i best response correspondence defined as

$$\begin{aligned} \Phi_{BR,i} \quad &: \quad H \to H_i \\ \varphi \quad &\to \quad \Phi_{BR,i}(\varphi) = \left\{ \varphi_{BR,i} \in H_i / \widetilde{U}_i\left(\varphi_{BR,i}, \varphi_{-i}\right) \geq \widetilde{U}_i\left(\varphi_i, \varphi_{-i}\right) \right\} \end{aligned} \tag{5.1}$$

Note that $\Phi_{BR}(\varphi)$ is a subset of F. Let us assume that this best response correspondence is upper semicontinuous.

Consider a set $H^{(k)}$ and a constraint strategy profile $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$. $\Phi_{BR,i}\left(\varphi_{CSE}^{(k)}\right)$ represents the set of best response strategies of player i in H_i when his opponents play the CSE strategy $\varphi_{CSE,-i}^{(k)}$ in $H^{(k)}$. Then, we can measure the distance between the CSE and its best response which leads to a second criteria:

$$C_2(k) = \left\| \varphi_{CSE}^{(k)} - \Phi_{BR}\left(\varphi_{CSE}^{(k)}\right) \right\| \quad .$$

Proposition 5.1. *Every sequence of CSE* $\left\{\varphi_{CSE}^{(k)}\right\}_{k=1\to\infty}$ *that approximates a NE verifies*

$$\lim_{k\to\infty} C_2(k) = 0 \quad .$$

Proof:

The proof is trivial: If $\varphi_{CSE}^{(k)}$ is an approximation of a NE equilibrium then

$$\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} = \varphi^{NE} \quad .$$

This NE verifies, $\varphi^{NE} \in \Phi_{BR}\left(\varphi^{NE}\right)$. From the upper semicontinuity of $\Phi_{BR}(.)$ we have

$$\lim_{k=1\to\infty} \varphi_{CSE}^{(k)} \in \lim_{k=1\to\infty} \Phi_{BR}\left(\varphi_{CSE}^{(k)}\right) \quad .$$

which implies

$$\lim_{k=1\to\infty} C_2(k) = 0$$

■

One can monitor the quality of the approximation by looking at the distance between the CSE and its best response in H. When $C_2(k)$ is sufficiently close to 0 then the CSE is a good approximation of the NE.

Note that $\varphi_i^{BR(k)}$ the best responses in H to $\varphi_{CSE,i}^{(k)}$ is tremendously easier to calculate than the actual NE. Indeed, $\varphi^{BR(k)}$ is determined by N independent maximization problems, while the NE requires to solve N maximizations combined with a system of N differential equations associated with the fixed point problem. Besides, the strategies of player i opponents are known and fixed to $\varphi_{CSE,i}^{(k)}$ which eliminates the uncertainty about other players actions. This, typically reduces considerably the dimension of the integral in the derivation of the (conditional) expected utility.

Finally, it might not be possible in some games to determined explicitly the function $\varphi_i^{BR(k)}$. In this case the criteria $C_2(k)$ can be approximated by $\widehat{C}_1^L(k)$ defined as

$$\widehat{C}_2^L(k) = \sum_{i=1}^{N}\sum_{l=1}^{L} \left| \varphi_i^{BR(k)}\left(\xi_i^l\right) - \varphi_{CSE,i}^{(k)}\left(\xi_i^l\right) \right| .$$

where $\varphi_i^{BR(k)}\left(\xi_i^l\right)$ is the best response to $\varphi_{CSE,-i}^{(k)}$ when player i receives the private signal ξ_i^l. The points ξ^l $(\forall l : 1 \rightarrow L)$ are determined exogenously and we suggest to use some fractiles of the private signal distribution $f(.)$.

5.3. The CSE as NE of a neighboring game

Consider the functions

$$\begin{array}{ccc} \mathcal{F} & \rightarrow & H \\ f & \rightarrow & O(f) = \varphi_f^{NE} \end{array} \quad ,$$

where $\mathcal{F}$ is a set of distributions and φ_f^{NE} is the NE of the game where private signals are drawn from f. Note that the element of $\mathcal{F}$ are assumed to be defined almost everywhere. For the ease of exposition assume that $O(.)$ is an homeomorphism[4] with an invert function $O^{-1}(\varphi) = f_\varphi(.)$ where $f_\varphi(.)$ is a distribution such that if the private signals where drawn from $f_\varphi(.)$ then φ would be a NE. This assumption should be interpreted as : neighboring distributions define neighboring games and they should have neighboring NE solution. Conversely, neighboring strategies should be the NE solution of neighboring games. Note that this assumption is necessary to anyone conducting empirical work.

Then, a strategy φ is assumed to be admissible $(\varphi \in H)$ if there exists a distribution $f_\varphi(.)$ in $\mathcal{F}$ for which φ is a NE.

Consider now a game where private signals are drawn from a given distribution $f(.)$. Consider also a set $H^{(k)}$ and a constraint strategy profile $\varphi_{CSE}^{(k)} \in \Phi_{CSE}^{(k)}$. $f_{\varphi_{CSE}^{(k)}}(.) = O^{-1}\left(\varphi_{CSE}^{(k)}\right)$ is then such that

$$\widehat{U}_i\left(\varphi_{CSE,i}^{(k)}(\xi_i), \varphi_{CSE,-i}^{(k)}; \xi_i / f_{\varphi_{CSE}^{(k)}}\right) \geq \widehat{U}_i\left(\Psi_i(\xi_i), \varphi_{CSE,-i}^{(k)}; \xi_i / f_{\varphi_{CSE}^{(k)}}\right),$$
$$\forall \Psi_i \in H_i \ \forall \xi_i \in \Omega_i \quad , \tag{5.2}$$

or equivalently in the strategic form game,

$$\widetilde{U}_i\left(\varphi_{CSE,i}^{(k)}, \varphi_{CSE,-i}^{(k)} / f_{\varphi_{CSE}^{(k)}}\right) \geq \widetilde{U}_i\left(\Psi_i, \varphi_{CSE,-i}^{(k)} / f_{\varphi_{CSE}^{(k)}}\right)$$
$$\forall \Psi_i \in H_i \quad . \tag{5.3}$$

[4] Note that the result should generalized to the more realistic case where $O(.)$ associates a subset of strategies to a subset of distributions.

where the conditional and unconditional expected utility are calculated with the density $f_{\varphi^{(k)}_{CSE}}(.)$.

This leads to a different measure of the distance between the CSE and an the NE,

$$C_3(k) = \left\| f - f_{\varphi^{(k)}_{CSE}} \right\|.$$

Proposition 5.2. *The sequence of CSE* $\left\{\varphi^{(k)}_{CSE}\right\}_{k=1\to\infty}$ *approximates a NE if and only if*

$$\lim_{k\to\infty} C_3(k) = 0$$

Proof: The proof essentially relies on the continuity of $O(.)$ and its inverse. Consider a sequence of CSE $\left\{\varphi^{(k)}_{CSE}\right\}_{k=1\to\infty}$ such that

$$\lim_{k\to\infty} \varphi^{(k)}_{CSE} = \varphi^{NE} \qquad .$$

Then, since $O^{-1}(.)$ is continuous

$$\lim_{k\to\infty} O^{-1}\left(\varphi^{(k)}_{CSE}\right) = O^{-1}\left(\varphi^{NE}\right) \qquad ,$$

$$\Rightarrow \lim_{k\to\infty} f_{\varphi^{(k)}_{CSE}} = f \qquad ,$$

$$\Rightarrow \lim_{k\to\infty} C_3(k) = 0 \qquad .$$

Conversely,

$$\lim_{k\to\infty} C_3(k) = 0$$

implies

$$\lim_{k\to\infty} O\left(\varphi^{(k)}_{CSE}\right) = O\left(\varphi^{NE}\right) \qquad ,$$

which by continuity of $O(.)$ reduces to

$$\lim_{k\to\infty} \varphi^{(k)}_{CSE} = \varphi^{NE} \qquad .$$

■

If a sequence of CSE $\left\{\varphi^{(k)}_{CSE}\right\}_{k=1\to\infty}$ verifies proposition 5.2 then $\varphi^{(k)}_{CSE}$ can be interpreted as a NE in a slightly perturbed game where private signals are drawn from a distribution neighboring the original f. In other words, the CSEs are NE of games that become closer to the original game as k increases.

One can monitor the accuracy of the approximation by verifying that $C_3(k)$ is close enough to zero. In practice the determination of $f_{\varphi_{CSE}^{(k)}}$ requires to apply usual econometric techniques. For instance one can utilize, when available, the FOC associated with the determination of the NE as a moment condition.

$$\frac{d}{dx_i}\widehat{U}_i\left(x_i,\varphi_{-i}^{NE};\xi_i\right)|_{x_i=\varphi_i^{NE}(\xi_i)}=0 \quad \forall\xi_i\in\Omega_i \text{ and } \forall i:1\rightarrow N. \tag{5.4}$$

Provided identification we can apply the Method of Simulated Moment and estimate $f_{\varphi_{CSE}^{(k)}}$ by

$$\widehat{f}_{\varphi_{CSE}^{(k)}}=Arg\min_{f\in F}\sum_{l=1}^{L}\sum_{i=1}^{N}\left[\frac{d}{dx_i}\widehat{U}_i\left(x_i,\varphi_{CSE,-i}^{(k)};\xi_i^l/f\right)|_{x_i=\varphi_{CSE,i}^{(k)}\left(\xi_i^l\right)}\right]^2 \quad \forall i:1\rightarrow N. \tag{5.5}$$

where $\widehat{U}_i\left(x_i,\varphi_{CSE,-i}^{(k)};\xi_i^l/f\right)$ is the conditional expected utility calculated with the distribution $f(.)$ and ξ^l $(\forall l:1\rightarrow L)$ are private signals simulated from the distribution $f(.)$.

In most applications we assume that the private signals distribution belongs to a parametric family of distributions indexed by $\theta\in\Re^p$. In such cases, we have to estimate only the parameter $\widehat{\theta}$:

$$\widehat{\theta}=Arg\min_{\theta\in\Re^p}\sum_{l=1}^{L}\sum_{i=1}^{N}\left[\frac{d}{dx_i}\widehat{U}_i\left(x_i,\varphi_{-i}^{NE};\xi_i^l/\theta\right)|_{x_i=\varphi_i^{NE}\left(\xi_i^l\right)}\right]^2 \quad \forall i:1\rightarrow N. \tag{5.6}$$

where ξ^l $(\forall l:1\rightarrow L)$ are simulated from $f(./\theta)$.

To conclude this section let us remind the reader that in empirical applications, which is our primary interest, the distribution of private signals is not known and needs to be estimated. In other words the game is not perfectly defined and the actual NE strategy will vary slightly depending upon the estimation of the distribution. In this context is seems reasonable to consider the concept of CSE that can be interpreted as NE of a game with a slightly different distribution.

6. An analytical approach to the approximation of a NE solution

The present section focuses on the approximation of a class of games of incomplete information with continuous strategy functions. Such a class includes games such as auction, Cournot competition...We will assume that there exist a NE and that $\widetilde{U}_i(\varphi)$ and $\widehat{U}_i(\varphi;\xi_i)$ are such that it is possible to derive the FOC in (2.4). To simplify notations and w.l.o.g, players are assumed to be symmetric and we focus on a representative player i. A problem of auction with asymmetric bidders is addressed in section ().

6.1. Piecewise strategy

The constrained set $H^{(k)}$ is made of the all the continuous piecewise functions defined as

$$\varphi^{(k)}(\xi_i) = \sum_{t=0}^{k} (\beta_t + \alpha_t \phi(\xi_i)) I_{]X_t,X_{t+1}]}(\xi_i) \tag{6.1}$$

where $\phi(.)$ is a continuous bounded function over an interval $[a,b]$, β_t and α_t belongs to compact euclidean spaces ($\forall t : 0 \rightarrow k$) and $I_{]X_t,X_{t+1}]}(\xi_i)$ is the characteristic function,

$$I_{]X_t,X_{t+1}]}(\xi_i) = \left\{ \begin{matrix} 1 & when\ \xi_i =]X_t, X_{t+1}] \\ 0 & otherwise \end{matrix} \right\} \tag{6.2}$$

$X_t \in [a,b]$ is a breaking point ($t = 1,...,k$ and $X_t < X_{t+1}$) exogenously determined and such that the distance between two consecutive breaking points in $H^{(k)}$ verifies $\lim_{k\to\infty} h_t = 0\ \forall t = 1,...,k$ where $h_t = \phi(X_{t+1}) - \phi(X_t)$. If Ω is not of the form $[a,b]$ we first select an interval $[a,b]$ and impose $X_0 = a$ and $X_{k+1} = b$. In this case a and b may be a certain percentile so that we approximate the NE over an interval where private signals are most likely to be drawn.

Since $\varphi^{(k)}$ is continuous we have,

$$\beta_t + \alpha_t \phi(X_{t+1}) = \beta_{t+1} + \alpha_{t+1}\phi(X_{t+1}), \qquad t : 0 \rightarrow k-1 \quad , \tag{6.3}$$

so,

$$\beta_t = \beta_{t+1} + (\alpha_{t+1} - \alpha_t)\phi(X_{t+1}), \qquad t : 0 \rightarrow k-1 \quad , \tag{6.4}$$

$$\beta_t = \beta_k + \sum_{i=t}^{k-1} (\alpha_{i+1} - \alpha_i)\phi(X_{i+1}), \qquad t : 0 \rightarrow k-1 \quad . \tag{6.5}$$

Therefore β_t $(t: 0 \to k-1)$ is a function of $(\alpha_t, ..., \alpha_k)$ and β_k. In other words the constrained strategy $\varphi^{(k)}$ is fully characterized by β_k and the vector of slopes $\alpha^{(k)} = (\alpha_0, ..., \alpha_k)$. In the remainder we assume that boundary conditions determine (α_k, β_k). If we insert 6.7 in 6.3, we can rewrite $\varphi^{(k)}$ as,

$$\varphi^{(k)}(\xi_i) = \alpha_k \phi(\xi_i) I_{]X_k, X_{k+1}]}(\xi_i) + \beta_k + \sum_{t=0}^{k-1} \left(\sum_{j=t}^{k-1} (\alpha_{j+1} - \alpha_j) \phi(X_{j+1}) + \alpha_t \phi(\xi_i) \right) I_{]X_t, X_{t+1}]}(\xi_i) \tag{6.6}$$

6.2. Piecewise CSE

The CSE obtains from the following optimization problem,

$$\underset{\alpha_i^{(k)}}{Max}\ E\left[U_i\left(\varphi_i^{(k)}, \varphi_{-i}^{(k)}, \xi\right)\right], \qquad i: 1 \to N \quad , \tag{6.7}$$

which leads to the FOC,

$$\frac{\partial E\left[U_i\left(\varphi_i^{(k)}, \varphi_{-i}^{(k)}, \xi\right)\right]}{\partial \alpha_{i,t}} = 0, \qquad i: 1 \to N, t: 0 \to k-1 \quad , \tag{6.8}$$

or, equivalently

$$\int_a^b \frac{\partial E\left[U_i\left(\varphi_i^{(k)}, \varphi_{-i}^{(k)}, \xi\right) / \xi_i\right]}{\partial \alpha_{i,t}} f(\xi_i) d\xi_i = 0, \qquad i: 1 \to N, t: 0 \to k-1 \quad , \tag{6.9}$$

$$\Rightarrow \int_a^b \frac{\partial \varphi^{(k)}(\xi_i)}{\partial \alpha_{i,t}} A_i\left[\varphi^{(k)}\right](\xi_i) f(\xi_i) d\xi_i = 0, \qquad i: 1 \to N, t: 0 \to k-1 \quad , \tag{6.10}$$

The partial derivative of $\varphi^{(k)}$ with respect to α_t $(t: 0 \to k-1)$ is

$$\frac{\partial \varphi^{(k)}(\xi_i)}{\partial \alpha_t} = (\phi(\xi_i) - \phi(X_{t+1})) I_{]X_t, X_{t+1}]}(\xi_i) + (\phi(X_t) - \phi(X_{t+1})) I_{]X_0, X_t]}(\xi_i) \tag{6.11}$$

and we can write the FOC as,

$$\int_{X_t}^{X_{t+1}} (\phi(X_{t+1}) - \phi(\xi_i)) A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i) f(\xi_i) \partial \xi_i + \tag{6.12}$$

$$\left(\phi\left(X_{t+1}\right)-\phi\left(X_t\right)\right)\int_{X_0}^{X_t} A_i\left[\varphi_{CSE}^{(k)}\right]\left(\xi_i\right) f\left(\xi_i\right) d\xi_i=0, \tag{6.13}$$

or equivalently if we divide by $h_t=\phi\left(X_{t+1}\right)-\phi\left(X_t\right)$,

$$\int_{X_t}^{X_{t+1}} \frac{\phi\left(X_{t+1}\right)-\phi\left(\xi_i\right)}{h_t} A_i\left[\varphi_{CSE}^{(k)}\right]\left(\xi_i\right) f\left(\xi_i\right) d\xi_i-\int_{X_0}^{X_t} A_i\left[\varphi_{CSE}^{(k)}\right]\left(\xi_i\right) f\left(\xi_i\right) d\xi_i=0, \tag{6.14}$$

where $\varphi_{CSE}^{(k)}$ is the symmetric constrained strategy equilibrium in $H^{(k)}$ that is fully characterized by $\alpha_{CSE}^{(k)}=\left(\alpha_0^{CSE},...,\alpha_k^{CSE}\right)$.

Note that the difference between two consecutive FOC,

$$\Delta_{t,t+1}=\frac{\partial E\left[U_i\left(\varphi_{CSE}^{(k)},\xi\right)\right]}{\partial\alpha_{i,t+1}}-\frac{\partial E\left[U_i\left(\varphi_{CSE}^{(k)},\xi\right)\right]}{\partial\alpha_{i,t}}= \tag{6.15}$$

$$\Delta_{t,t+1}=\int_{X_{t+1}}^{X_{t+2}}\left(\phi\left(X_{t+2}\right)-\phi\left(\xi_i\right)\right) A_i\left[\varphi_{CSE}^{(k)}\right]\left(\xi_i\right) f\left(\xi_i\right) d\xi_i- \tag{6.16}$$

$$\int_{X_t}^{X_{t+1}}\left(h_t+\phi\left(X_{t+1}\right)-\phi\left(\xi_i\right)\right) A_i\left[\varphi_{CSE}^{(k)}\right]\left(\xi_i\right) f\left(\xi_i\right) d\xi_i=0$$

Typically, one can express α_t^{CSE} as a function of $\left(\alpha_{t+1}^{CSE},...,\alpha_{k-1}^{CSE},\alpha_k\right)$ and the vector $\alpha_{CSE}^{(k)}$ obtains recursively as a function of the boundary conditions (α_k,β_k).

6.3. Approximation of a NE with a piecewise CSE

Consider $\varphi^{NE}(.)$ a NE and $\varphi_{CSE}^{(k)}(.)$ a CSE in $H^{(k)}$ defined as

$$\varphi_{CSE}^{(k)}\left(\xi_i\right)=\alpha_k\phi\left(\xi_i\right)_i I_{]X_k,X_{k+1}]}\left(\xi_i\right)+\beta_k+ \tag{6.17}$$

$$\sum_{t=0}^{k-1}\left(\sum_{j=t}^{k-1}\left(\alpha_{j+1}^{CSE}-\alpha_t^{CSE}\right)\phi\left(X_{j+1}\right)+\alpha_t^{CSE}\phi\left(\xi_i\right)\right) I_{]X_t,X_{t+1}]}\left(\xi_i\right)$$

Consider the following assumptions

xiii) The operator $A_i[\varphi](\xi_i)$ is continuous in ξ_i $\forall i:1\to N$, $\forall \xi_i\in\Omega_i$.

xiv) The operator $A_i[\varphi](\xi_i)$ does not change sign infinitely many times over the interval $[X_0,X_{k+1}]$

xv) (α_k,β_k) are given and such that $\lim\limits_{\xi_i\to X_k} A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)=0, \forall i:1\to N$ and $\forall k>0$.

Theorem 6.1. *under assumptions xiii) to xv)* $\lim\limits_{k\to\infty}\varphi_{CSE}^{(k)}(\xi)=\varphi^{NE}(\xi)$ $\forall\xi\in[a,b]^N$.

Proof: We know that $\forall k>1$ we have

$$0\le\left|\frac{\phi(X_{t+1})-\phi(\xi_i)}{h_t}I_{]X_t,X_{t+1}]}(\xi_i)\right|\le 1 \qquad t:0\to k-1 \quad , \tag{6.18}$$

If $A_i\left[\varphi_{CSE}^{(k)}\right](.)$ does not change sign infinitely many times, then there exists h_0 such that $\forall h<h_0$ the sign of $A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)$ is constant over $]X_t,X_{t+1}]$ $\forall t:0\to k$. Then, we have

$$0\le\left|\int\limits_{X_t}^{X_{t+1}}\frac{\phi(X_{t+1})-\phi(\xi_i)}{h_t}A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)f(\xi_i)\partial\xi_i\right|\le\left|\int\limits_{X_t}^{X_{t+1}}A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)f(\xi_i)\partial\xi_i\right| \quad , \tag{6.19}$$

if $A_i\left[\varphi_{CSE}^{(k)}\right](.)$ is a continuous function we have

$$\lim_{h\to 0}\left|\int\limits_{X_t}^{X_{t+1}}A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)f(\xi_i)\partial\xi_i\right|=0 \qquad , \tag{6.20}$$

and consequently

$$\lim_{h_t\to 0}\int\limits_{X_t}^{X_{t+1}}\frac{\phi(X_{t+1})-\phi(\xi_i)}{h_t}A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)f(\xi_i)\partial\xi=0 \qquad , \tag{6.21}$$

From equation (5.34) we have $\forall h<h_0$

$$\lim_{h_t\to 0}\int_{X_0}^{X_t} A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i)\, f(\xi_i)\,\partial\xi_i = 0, \qquad i:1\to N, t:0\to k-1 \quad , \tag{6.22}$$

Under assumption xv) this is possible only if

$$\lim_{h_t\to 0} A_i\left[\varphi_{CSE}^{(k)}\right](\xi_i) = 0, \qquad i:1\to N, \forall\xi_i\in[a,b] \quad , \tag{6.23}$$

Therefore, the constraint strategy $\varphi_{CSE}^{(k)}$ verifies the same F.O.C as the NE φ^{NE}, at any point $\xi_i\in[a,b]$ as k tend toward infinity.. $\varphi_{CSE}^{(k)}(.)$ is therefore an approximation of $\varphi_{CSE}^{(k)}(.)$ over the interval $[a,b]$. ■

7. Conclusion

[To be completed]

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