Private Monitoring, Likelihood Ratio Condition, and the Folk Theorem

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Abstract

This paper investigates infinitely repeated prisoner-dilemma games where the discount factor is less than but close to 1. We assume that players not only imperfectly but also privately monitor their opponents' choices of action. We provide a likelihood ratio condition under which there exist Nash equilibrium payoff vectors better than the one-shot Nash equilibrium payoff vector. We show that an efficient payoff vector is approximated by a Nash equilibrium payoff vector if the minimum likelihood ratio is zero. We show the full folk theorem when players' private signals are independent and a stronger version of this zero likelihood ratio condition is satisfied. In contrast with previous works, this paper assumes that monitoring is neither almost perfect nor almost public, and there exist no public signals, no announcement of publicly observed messages, and no public randomization devices.

We also investigate machine games, and we newly require that players always play a Nash equilibrium irrespective of their initial states of machine, i.e., they play according to a uniform equilibrium. We show that there exists the unique payoff vector sustained by a uniform equilibrium, i.e., the unique uniformly sustainable payoff vector, which Pareto-dominates all other uniformly sustainable payoff vectors. This is in contrast with the multiplicity of Pareto-undominated perfect equilibrium payoff vectors. We show also that this Pareto-dominant uniformly sustainable payoff is efficient if and only if the minimum likelihood ratio is zero.

Keywords: Repeated Prisoners' Dilemma, Private Monitoring, Likelihood Ratio Condition, the Folk Theorem, Uniform Sustainability.

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1. Introduction

This paper investigates infinitely repeated prisoner-dilemma games, where the discount factor is less than but close to 1. We assume that players not only imperfectly but also privately monitor their opponents' choices of action. Players cannot observe their opponents' choices of action directly, can observe no public signals, but can observe their own private signals which are drawn according to a probability density function conditional on their action profile.

The former part of this paper clarifies the possibility that implicit collusion can be sustained by Nash equilibria. We provide a sufficient condition for the existence of Nash equilibrium payoff vectors which are better than the one-shot Nash equilibrium payoff vector. The key condition of this sufficiency is a likelihood ratio condition which implies that the minimum of the likelihood ratio indicating whether the opponent has chosen the right action is sufficiently small. We show that an efficient payoff vector is approximated by a Nash equilibrium payoff vector if this minimum likelihood ratio is approximately zero, that is, if for each player there exists a private signal for this player which can discern whether her opponent has chosen the right action. We also show that the folk theorem holds, i.e., every feasible and individually rational payoff vector is approximated by a Nash equilibrium payoff vector, if players' private signals are independent and a stronger version of this zero likelihood ratio condition is satisfied.

The study of private monitoring is relatively a new literature in repeated games. Most previous works have assumed that monitoring is either perfect or public. As far as I know, Radner (1986) is the first paper on repeated games with private monitoring. Radner assumed that players' long-run payoffs are defined by the criterion of the limit of average with no discounting, and showed that every feasible and individually rational payoff vector may be sustained by a Nash equilibrium in the same way as the public monitoring case. Matsushima (1990a) firstly pointed out that there exist qualitative differences between discounting and no discounting in the private monitoring case, and qualitative differences between private monitoring and public monitoring in the discounting case. Matsushima showed that the anti-folk theorem holds, i.e., implicit collusion is impossible to be sustained by Nash equilibria in repeated games with discounting, when private signals are independent and these Nash equilibria are pure strategy profiles which are independent of payoff-irrelevant histories.

Several works on repeated games with private monitoring after these papers have provided more affirmative results even in the discounting case. Kandori and Matsushima (1998) and Compte (1998) showed that the folk theorem holds in the same way as the public monitoring case when players announce publicly observable messages through communication.[1] Sekiguchi (1997) investigated repeated prisoner-dilemma games on the assumption that private monitoring is almost perfect and players' private signals are independent. Sekiguchi firstly showed that an efficient payoff vector can be approximated by a Nash equilibrium payoff vector even though players cannot communicate. By using the public randomization devices, Bhaskar (1999) extends this Sekiguchi's result to more general cases in which monitoring is almost perfect but players' private signals are not necessarily independent. Ely and Valimaki (1999) also considered repeated prisoner-dilemma games with almost perfect monitoring, and provided the full folk theorem. Some parts of the proofs in the present paper are based on the idea of equilibrium construction addressed by Ely and Valimaki. Moreover, Mailath and Morris (1998) investigated the cases in which monitoring is truly imperfect, and clarified the robustness of perfect public equilibria in the public monitoring case with respect to small private noises for monitoring.

The theorems in this paper are substantial departures from all these works. We assume that there exist no public signals, players announce no publicly observed messages, and there exist no public randomization devices. We do not require that monitoring is either almost perfect or almost public. We do not require that the probability that a player observes a private signal which can discern whether her opponent has chosen the right action is positive. Hence, the present paper can be regarded as the first work to provide affirmative answers to the possibility of implicit collusion in the discounting case when monitoring is truly imperfect and truly private.

The latter part of this paper considers machine games explored by Abreu and Rubinstein (1987) and others. A player behaves according to a finite automaton, or a machine, defined as a combination of a rule and an initial state of machine. A rule profile is called a uniform equilibrium if the machine profile associated with this rule profile is a Nash equilibrium irrespective of players' initial states. In contrast with related previous works, we require that players always play a Nash equilibrium irrespective of their initial states, that is, their rule profile is a uniform equilibrium.

Most previous works on repeated games with either perfect or imperfect public monitoring have confined their attentions to public perfect equilibria. Public perfect equilibrium requires that the past histories relevant to future play, or the states of machine, are always common knowledge among players in every sub-game. This common knowledge property makes perfect equilibrium analyses much tractable, because players' future play is always a Nash equilibrium in every sub-game. When monitoring is only private, however, it is inevitable

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[1] Matsushima (1990b) provided a conjecture that the folk theorem holds when players can communicate.

that an equilibrium sustaining implicit collusion depends on players' private histories, and therefore, the past histories relevant to future play, or players' states of machine, can hardly be common knowledge in every sub-game.

Uniform equilibrium is a refinement of perfect equilibrium which requires additionally that players' future play is always a Nash equilibrium in every sub-game even though players' states are not common knowledge. A payoff vector is called uniformly sustainable if there exists a uniform equilibrium which approximately induces this payoff vector irrespective of their initial states. We show that there exists the unique uniformly sustainable payoff vector which Pareto-dominates all other uniformly sustainable payoff vectors. This result is in contrast with the folk theorem in the study of Nash, or perfect Nash, equilibrium. The folk theorem implies that there exist multiple Pareto-undominated sustainable payoff vectors, whereas our result implies that there exists the unique Pareto-undominated uniformly sustainable payoff vector. We also show that this Pareto-dominant uniformly sustainable payoff vector is efficient if and only if the zero likelihood ratio condition is satisfied. This implies that the sufficient condition for efficiency in sustainability by Nash equilibrium provided in the former part of this paper is also necessary and sufficient for efficiency in uniform sustainability.

The organization of this paper is as follows. Section 2 defines the model. Section 3 provides a sufficient condition for implicit collusion (Theorem 1) and provides a sufficient condition for efficiency (Theorem 2). Section 4 presents the proof of Theorem 1. Section 5 considers independent private signals, provides a subset of sustainable payoff vectors by Nash equilibria (Theorem 3), and provides a sufficient condition for the full folk theorem (Theorem 4). Section 6 presents the proof of Theorem 3. Section 7 defines machine games and the notion of uniform equilibrium, and provides an upper-bound of the uniform equilibrium payoff vectors (Theorem 5). Section 8 defines the notion of uniform sustainability, provides a sufficient condition for uniform sustainability (Theorem 6), and also provides a necessary and sufficient condition for efficiency in uniform sustainability (Theorem 7).

2. The Model

An infinitely repeated prisoner-dilemma game $\Gamma(d) = ((A_i, u_i, \Omega_i)_{i=1,2}, d, p)$ is defined as follows. In every period $t \geq 1$, players 1 and 2 play a prisoner-dilemma game $(A_i, u_i)_{i=1,2}$. Player $i's$ set of actions is given by $A_i = \{c_i, d_i\}$. Let $A \equiv A_1 \times A_2$. Player $i's$ instantaneous payoff function is given by $u_i : A \to R$. We assume that for every $i = 1,2$, $u_i(c) = 1$, $u_i(d) = 0$, $u_i(d / c_j) = 1 + x_i > 1$, and $u_i(c / d_j) = -y_i < 0$, where $c \equiv (c_1, c_1)$ and $d \equiv (d_1, d_2)$. The *feasible* set of payoff vectors $V \subset R^2$ is defined as the convex hull of the set $\{(1,1), (0,0), (1 + x_1, -y_2), (-y_1, 1 + x_2)\}$. The discount factor is denoted by $d \in [0,1)$.

At the end of every period, each player i observes her own *private* signal w_i. The set of player $i's$ private signals is defined as $\Omega_i \equiv [0,1]$. Let $\Omega \equiv \Omega_1 \times \Omega_2$. A signal profile $w \equiv (w_1, w_2) \in \Omega$ is determined according to a conditional density function $p(w|a)$. Let $p_i(w_i|a) \equiv \int_{w_j \in \Omega_j} p(w|a) dw_j$. We assume that $p_i(w_i|a)$ is continuous w. r. t. $w_i \in \Omega_i$, and $p_i(w_i|a) > 0$ for all $a \in A$ and for almost all $w \in \Omega$.

A private history for player i up to period $t \geq 1$ is denoted by $h_i^t = (a_i(t), w_i(t))_{t=1}^t$, where $a_i(t) \in A_i$ is the action chosen by player i and $w_i(t) \in \Omega_i$ is the private signal observed by player i in period t. The null history for player i is denoted by h_i^0. The set of all private histories for player i is denoted by H_i. A *strategy for player* i is defined as a function $s_i : H_i \to A_i$. The set of strategies for player i is denoted by S_i. Let $S \equiv S_1 \times S_2$. Player $i's$ normalized long-run payoff induced by a strategy profile $s \in S$ is given by $v_i(d, s) \equiv (1 - d) E[\sum_{t=1}^{\infty} d^{t-1} u_i(a(t)) | s]$. Let $v(d, s) \equiv (v_1(d, s), v_2(d, s))$. A strategy profile $s \in S$ is said to be a *Nash equilibrium* in $\Gamma(d)$ if for each $i = 1,2$ and every $s_i' \in S_i$, $v_i(d, s) \geq v_i(d, s / s_i')$.

Definition 1: A payoff vector $v = (v_1, v_2) \in R^2$ is *sustainable* if for every infinite sequence of discount factors $(d^{(m)})_{m=1}^{\infty}$ satisfying $\lim_{m \to +\infty} d^{(m)} = 1$, there exists an infinite sequence of strategy profiles $(s^{(m)})_{m=1}^{\infty}$ such that for every large enough $m = 1,2,\ldots$, s^m is a Nash equilibrium in $\Gamma(d^{(m)})$, and

$$\lim_{m \to +\infty} v(d^{(m)}, s^{(m)}) = v .$$

By definition, the set of sustainable payoff vectors is compact. We denote by $s_i|_{h_i^t}$ the strategy for player i induced by s_i after the private history $h_i^t \in H_i$ occurs.

3. Efficiency

The *likelihood ratio function* for player $i's$ private signals, $L_i : \Omega_i \times A^2 \to R$, is defined by

$$L_i(w_i, a, a') \equiv \begin{cases} \dfrac{p_i(w_i|a)}{p_i(w_i|a')} & \text{if } \; p_i(w_i|a') \neq 0 \\ \lim\limits_{w_i' \to w_i} \dfrac{p_i(w_i'|a)}{p_i(w_i'|a')} & \text{if } \; p_i(w_i|a') = 0 \end{cases}.$$

We assume that $L_i(w_i, a, a')$ is continuous w. r. t. $w_i \in \Omega_i$. We define

$$\underline{L}_i(a, a') \equiv \min_{w_i \in \Omega_i} L_i(w_i, a, a'),$$

$$\bar{v}_i \equiv 1 - \frac{\underline{L}_j(c, c/d_i)x_i}{1 - \underline{L}_j(c, c/d_i)},$$

and

$$\underline{v}_i \equiv \frac{\underline{L}_j(d, d/c_i)y_i}{1 - \underline{L}_j(d, d/c_i)},$$

Let $\bar{v} \equiv (\bar{v}_1, \bar{v}_2)$ and $\underline{v} \equiv (\underline{v}_1, \underline{v}_2)$. Note that if for each $i = 1,2$ and for $j \neq i$,

$$1 > \frac{\underline{L}_j(c, c/d_i)x_i}{1 - \underline{L}_j(c, c/d_i)} + \frac{\underline{L}_j(d, d/c_i)y_i}{1 - \underline{L}_j(d, d/c_i)}, \tag{1}$$

then $\bar{v} > \underline{v}$ holds. We define a subset $V^* \subset V$ by the convex hull of the set $\{(0,0), \bar{v}, (\bar{v}_1, \underline{v}_2), (\underline{v}_1, \bar{v}_2)\}$.

Theorem 1: *If* $\bar{v} > \underline{v}$, *then every* $v \in V^*$ *is sustainable.*

We provide the proof of Theorem 1 in the next section.

Theorem 2: *If for each* $i = 1,2$ *and for* $j \neq i$,

$$\underline{L}_i(c, c/d_j) = 0, \tag{2}$$

and

$$\underline{L}_i(d, d/c_i) < \frac{1}{1 + y_j}, \tag{3}$$

then, $(1,1)$ *is sustainable.*

Proof: Equalities (2) and inequalities (3) imply inequalities (1), and, therefore, $\bar{v} > \underline{v}$. Equalities (2) and the definition of $\bar{v}$ imply $\bar{v} = (1,1)$. Hence, Theorem 1 implies that $(1,1)$ is sustainable.

Q.E.D.

Suppose that $x_1 + x_2 \leq y_1 + y_2$. Then, (1,1) is efficient, and, therefore, one gets from Theorem 2 that efficiency can be sustained by Nash equilibria when the minimum likelihood ratio $\underline{L}_i(c, c/d_j)$ between c and c/d_j is zero and the minimum likelihood ratio $\underline{L}_i(d, d/c_i)$ between d and d/c_j is sufficiently small.

4. Proof of Theorem 1

The proof of Theorem 1 is divided into three steps.

Step 1: We show that for every $v^+ \in V^*$ and every $v^- \in V^*$, if $\bar{v} \geq v^+ > v^- \geq \underline{v}$, then, v^+, v^-, (v_1^+, v_2^-), and (v_1^-, v_2^+) are all sustainable.

Fix $i = 1,2$ arbitrarily. From the continuity of L_i, we can choose $\hat{w}_i \in \Omega_i$ which satisfies

$$v_j^+ = 1 - \frac{L_i(\hat{w}_i, d, c/d_j)x_j}{1 - L_i(\hat{w}_i, d, c/d_j)},$$

that is,

$$L_i(\hat{w}_i, c, c/d_j) = \frac{1 - v_j^+}{x_j + 1 - v_j^+}. \qquad (4)$$

Choose $e_i > 0$ close to 0. From equality (4), we can choose $\hat{v}_j = \hat{v}_j(e_i)$ which satisfies

$$\frac{\int_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c)dw_i}{\int_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c/d_j)dw_i} = \frac{1 - \hat{v}_j}{x_j + 1 - \hat{v}_j}. \qquad (5)$$

Note that $\hat{v}_j = \hat{v}_j(e_i)$ tends towards v_j^+ as e_i approaches 0. We define

$$a_j = a_j(e_i) \equiv \frac{\int_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c/d_j)dw_i}{x_j + 1 - \hat{v}_j}. \qquad (6)$$

Note that $a_j(e_i)$ tends towards 0 as e_i approaches 0.

Choose $x_j > 0$ close to 0. From the continuity of L_i, we can choose $\tilde{w}_i = \tilde{w}_i(e_i, x_j) \in \Omega_i$ which satisfies

$$\frac{v_j^- + x_j}{y_j + v_j^- + x_j} > L_i(\tilde{w}_i, d, d/c_j) > \frac{v_j^-}{y_j + v_j^-}. \qquad (7)$$

We define $\bar{l}_i = \bar{l}_i(e_i)$ and $\underline{l}_i = \underline{l}_i(e_i)$ by $\bar{l}_i > \underline{l}_i > 0$,

$$\int_{w_i \in (\tilde{w}_i - \bar{l}_i, \tilde{w}_i + \bar{l}_i]} p_i(w_i|d/c_j)dw_i = (y_j + v_j^- + x_j)a_j,$$

and

$$\int_{w_i \in (\tilde{w}_i - \underline{l}_i, \tilde{w}_i + \underline{l}_i]} p_i(w_i|d/c_j)dw_i = (y_j + v_j^-)a_j.$$

Note that both $\bar{l}_i(e_i)$ and $\underline{l}_i(e_i)$ tend towards 0 as e_i approaches 0, because $a_j = a_j(e_i)$ tends towards 0 as e_i approaches 0. Choose any continuous function $w_j = w_j(e_i, x_j) : [\underline{l}_i, \bar{l}_i] \to [v_j^-, v_j^- + x_j]$ which satisfies that $w_j(\bar{l}_i) = v_j^- + x_j$, $w_j(\underline{l}_i) = v_j^-$, and for every $l_i \in [\underline{l}_i, \bar{l}_i]$,

$$\int_{w_i \in (\tilde{w}_i - l_i, \tilde{w}_i + l_i]} p_i(w_i | d / c_j) dw_i = (y_j + w_j(l_i)) a_j . \quad (8)$$

Since $L_i(\tilde{w}_i, d, d / c_j)$ is approximated by $\dfrac{\int_{w_i \in (\tilde{w}_i - \bar{l}_i, \tilde{w}_i + \bar{l}_i]} p_i(w_i | d) dw_i}{\int_{w_i \in (\tilde{w}_i - \bar{l}_i, \tilde{w}_i + \bar{l}_i]} p_i(w_i | d / c_j) dw_i}$, one gets from inequalities (7) that

$$\frac{w_j(\bar{l}_i)}{y_j + w_j(\bar{l}_i)} = \frac{v_j^- + x_j}{y_j + v_j^- + x_j} > \frac{\int_{w_i \in (\tilde{w}_i - \bar{l}_i, \tilde{w}_i + \bar{l}_i]} p_i(w_i | d) dw_i}{\int_{w_i \in (\tilde{w}_i - \bar{l}_i, \tilde{w}_i + \bar{l}_i]} p_i(w_i | d / c_j) dw_i}$$

$$> \frac{v_j^-}{y_j + v_j^-} = \frac{w_j(\underline{l}_i)}{y_j + w_j(\underline{l}_i)} .$$

Hence, the continuity of $w_j(l_i)$ implies that there exists $\tilde{l}_i = \tilde{l}_i(e_i, x_j)$ such that

$$\frac{\int_{w_i \in (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]} p_i(w_i | d) dw_i}{\int_{w_i \in (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]} p_i(w_i | d / c_j) dw_i} = \frac{w_j(\tilde{l}_i)}{y_j + w_j(\tilde{l}_i)} . \quad (9)$$

We define

$$\tilde{v}_j = \tilde{v}_j(e_i, x_j) \equiv w_j(e_i, x_j)(\tilde{l}_i(e_i, x_j)) = w_j(\tilde{l}_i(e_i, x_j)) .$$

Note that $\tilde{v}_j = \tilde{v}_j(e_i, x_j)$ tends towards v_j^- as x_j approaches 0, because $\tilde{l}_i = \tilde{l}_i(e_i, x_j)$ tends towards 0 as x_j approaches 0. We define $d_j = d_j(e_i, x_j) \in (0,1)$ by

$$\frac{1 - d_j}{d_j} = (\hat{v}_j - \tilde{v}_j) a_j . \quad (10)$$

Note that d_j tends towards 1 as e_i approaches 0, because $a_j(e_i)$ tends towards 0 as e_i approaches 0. .

Fix $(d^{(m)})_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m \to +\infty} d^{(m)} = 1$. The above arguments imply that there exists $(e_1^{(m)}, x_1^{(m)}, e_2^{(m)}, x_2^{(m)})_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} (e_1^{(m)}, x_1^{(m)}, e_2^{(m)}, x_2^{(m)}) = (0,0,0,0) ,$$

and for every large enough m,

$$d^{(m)} = d_1(e_2^{(m)}, x_1^{(m)}) = d_2(e_1^{(m)}, x_2^{(m)}) .$$

Choose $(\tilde{w}_i^{(m)}, \tilde{l}_i^{(m)}, \hat{v}_i^{(m)}, \tilde{v}_i^{(m)})_{m=1}^{\infty}$ satisfying that for every large enough m, $\tilde{w}_i^{(m)} \equiv \tilde{w}_i(e_i^{(m)}, x_j^{(m)})$, $\tilde{l}_i^{(m)} \equiv \tilde{l}_i(e_i^{(m)}, x_j^{(m)})$, $\hat{v}_i^{(m)} \equiv \hat{v}_i(e_j^{(m)})$, and $\tilde{v}_i^{(m)} \equiv \tilde{v}_i(e_j^{(m)}, x_i^{(m)})$. From equalities (5), (6) and (10), one gets

$$\int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j | c) dw_j = \left(\frac{1 - d^{(m)}}{d^{(m)}}\right)\left(\frac{1 - \hat{v}_i^{(m)}}{\hat{v}_i^{(m)} - \tilde{v}_i^{(m)}}\right),$$

and, therefore,

$$\hat{v}_i^{(m)} = 1 - d^{(m)} + d^{(m)}\{\hat{v}_i^{(m)}(1 - \int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j|c)dw_j)$$
$$+ \tilde{v}_i^{(m)} \int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j|c)dw_j\}. \quad (11)$$

From equalities (6) and (10), one gets

$$\int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j|c/d_i)dw_j = (\frac{1-d^{(m)}}{d^{(m)}})(\frac{x_i + 1 - \hat{v}_i^{(m)}}{\hat{v}_i^{(m)} - \tilde{v}_i^{(m)}}),$$

and, therefore,

$$\hat{v}_i^{(m)} = (1 - d^{(m)})(1 + x_i) + d^{(m)}\{\hat{v}_i^{(m)}(1 - \int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j|c/d_i)dw_j)$$
$$+ \tilde{v}_i^{(m)} \int_{w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]} p_j(w_j|c/d_i)dw_j\}. \quad (12)$$

From equalities (8) and (10), one gets

$$\int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d/c_i)dw_j = (\frac{1-d^{(m)}}{d^{(m)}})(\frac{y_i + \tilde{v}_i^{(m)}}{\hat{v}_i^{(m)} - \tilde{v}_i^{(m)}}),$$

and, therefore,

$$\tilde{v}_i^{(m)} = (1 - d^{(m)})(-y_i) + d^{(m)}\{\hat{v}_i^{(m)}(1 - \int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d/c_i)dw_j)$$
$$+ \tilde{v}_i^{(m)} \int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d/c_i)dw_j\}. \quad (13)$$

From equalities (8), (9) and (10), one gets

$$\int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d)dw_j = (\frac{1-d^{(m)}}{d^{(m)}})(\frac{\tilde{v}_i^{(m)}}{\hat{v}_i^{(m)} - \tilde{v}_i^{(m)}}),$$

and, therefore,

$$\tilde{v}_i^{(m)} = d^{(m)}\{\hat{v}_i^{(m)}(1 - \int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d)dw_j)$$
$$+ \tilde{v}_i^{(m)} \int_{w_j \in (\tilde{w}_j^{(m)} - \tilde{l}_j^{(m)}, \tilde{w}_j^{(m)} + \tilde{l}_j^{(m)}]} p_j(w_j|d)dw_j\}. \quad (14)$$

We specify an infinite sequence of strategy profiles $(s^{(m)})_{m=1}^{\infty}$ in the following way. For each $i = 1,2$,

$$s_i^{(m)}(h_i^0) = c_i,$$
$$s_i^{(m)}(h_i^t) = c_i \text{ if } s_i^{(m)}(h_i^{t-1}) = c_i \text{ and } w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}],$$
$$s_i^{(m)}(h_i^t) = c_i \text{ if } s_i^{(m)}(h_i^{t-1}) = d_i \text{ and } w_i \in (\tilde{w}_i - \tilde{l}_i^{(m)}, \tilde{w}_i + \tilde{l}_i^{(m)}],$$
$$s_i^{(m)}(h_i^t) = d_i \text{ if } s_i^{(m)}(h_i^{t-1}) = c_i \text{ and } w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}],$$

and

$$s_i^{(m)}(h_i^t) = d_i \text{ if } s_i^{(m)}(h_i^{t-1}) = d_i \text{ and } w_i \notin (\tilde{w}_i - \tilde{l}_i^{(m)}, \tilde{w}_i + \tilde{l}_i^{(m)}].$$

Note that there exist $D_i : A \to R$ such that for every $h_i^t \in H_i$ and every $h_j^{t'} \in H_j$,

$$v_i(\boldsymbol{d}^{(m)}, s_i^{(m)}|_{h_i^t}, s_j^{(m)}|_{h_j^{t'}}) = D_i(c) \text{ if } s_i^{(m)}(h_i^t) = c_i \text{ and } s_j^{(m)}(h_j^{t'}) = c_j,$$

$$v_i(\boldsymbol{d}^{(m)}, s_i^{(m)}|_{h_i^t}, s_j^{(m)}|_{h_j^{t'}}) = D_i(d) \text{ if } s_i^{(m)}(h_i^t) = d_i \text{ and } s_j^{(m)}(h_j^{t'}) = d_j,$$

$$v_i(\boldsymbol{d}^{(m)}, s_i^{(m)}|_{h_i^t}, s_j^{(m)}|_{h_j^{t'}}) = D_i(c/d_j) \text{ if } s_i^{(m)}(h_i^t) = c_i \text{ and } s_j^{(m)}(h_j^{t'}) = d_j,$$

and

$$v_i(\boldsymbol{d}^{(m)}, s_i^{(m)}|_{h_i^t}, s_j^{(m)}|_{h_j^{t'}}) = D_i(d/c_j) \text{ if } s_i^{(m)}(h_i^t) = d_i \text{ and } s_j^{(m)}(h_j^{t'}) = c_j.$$

Note

$$D_i(c) = 1 - \boldsymbol{d}^{(m)} + \boldsymbol{d}^{(m)}\{D_i(c) \int\limits_{\substack{w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \notin (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|c)dw + D_i(d) \int\limits_{\substack{w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|c)dw$$

$$+ D_i(c/d_j) \int\limits_{\substack{w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|c)dw + D_i(d/c_j) \int\limits_{\substack{w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \notin (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|c)dw\},$$

$$D_i(d) = \boldsymbol{d}^{(m)}\{D_i(c) \int\limits_{\substack{w_i \in (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \in (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|d)dw + D_i(d) \int\limits_{\substack{w_i \notin (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \notin (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|d)dw$$

$$+ D_i(c/d_j) \int\limits_{\substack{w_i \in (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \notin (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|d)dw + D_i(d/c_j) \int\limits_{\substack{w_i \notin (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \in (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|d)dw\},$$

$$D_i(c/d_j) = (1 - \boldsymbol{d}^{(m)})(-y_i) + \boldsymbol{d}^{(m)}\{D_i(c) \int\limits_{\substack{w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \in (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|c/d_j)dw$$

$$+ D_i(d) \int\limits_{\substack{w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \notin (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|c/d_j)dw + D_i(c/d_j) \int\limits_{\substack{w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \notin (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|c/d_j)dw$$

$$+ D_i(d/c_j) \int\limits_{\substack{w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}] \\ w_j \in (\tilde{w}_j - l_j^{(m)}, \tilde{w}_j + l_j^{(m)}]}} p(w|c/d_j)dw\},$$

and

$$D_i(d/c_j) = (1 - \boldsymbol{d}^{(m)})(1 + x_i) + \boldsymbol{d}^{(m)}\{D_i(c) \int\limits_{\substack{w_i \in (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \notin (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|d/c_j)dw$$

$$+ D_i(d) \int\limits_{\substack{w_i \notin (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|d/c_j)dw + D_i(c/d_j) \int\limits_{\substack{w_i \in (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \in (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|d/c_j)dw$$

$$+ D_i(d/c_j) \int\limits_{\substack{w_i \notin (\tilde{w}_i - l_i^{(m)}, \tilde{w}_i + l_i^{(m)}] \\ w_j \notin (\hat{w}_j - e_j^{(m)}, \hat{w}_j + e_j^{(m)}]}} p(w|d/c_j)dw\}.$$

From equalities (11), (12), (13) and (14), one gets that

$$w(c) = w(d/c_j) = \hat{v}_i^{(m)},$$

$$w(d) = w(c/d_j) = \tilde{v}_i^{(m)},$$

and, for every $h_i^t \in H_i$, every $h_j^{t'} \in H_j$, and for every $s_i \in S_i$ satisfying that

$s_i|_{(a_i,w_i)} = s_i^{(m)}|_{(h_i^t,(a_i,w_i))}$ for all $(a_i, w_i) \in A_i \times \Omega_i$,

$$v_i(d^{(m)}, s_i, s_j^{(m)}|_{h_j^{t'}}) = \begin{cases} \hat{v}_i^{(m)} & if\ s_j^{(m)}(h_j^{t'}) = c_j \\ \tilde{v}_i^{(m)} & if\ s_j^{(m)}(h_j^{t'}) = d_j \end{cases},$$

and, therefore,

$$v_i(d^{(m)}, s_i^{(m)}|_{h_i^t}, s_j^{(m)}|_{h_j^{t'}}) = v_i(d^{(m)}, s_i, s_j^{(m)}|_{h_j^{t'}}). \qquad (15)$$

Inequalities (15) imply that for every $h_1^t \in H_1$ and every $\tilde{h}_2^{t'} \in H_2$, $(s_1^{(m)}|_{h_1^t}, s_2^{(m)}|_{\tilde{h}_2^{t'}})$ is a Nash equilibrium in $\Gamma(d^{(m)})$. Hence, we have completed the proof of Step 1.

Step 2: We show that for every positive integer $K > 0$ and every K sustainable payoff vectors $v^{[1]}$, ..., $v^{[K]}$, $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is also sustainable.

Fix $(d^{(m)})_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} d^{(m)} = 1$. For every $k = 1,\ldots,K$, let $(s^{\{k,m\}})_{m=1}^{\infty}$ be an infinite sequence of strategy profiles satisfying that for every large enough $m = 1,2,\ldots$, $s^{\{k,m\}}$ is a Nash equilibrium in $\Gamma(d^{(m)})$, and

$$\lim_{m\to+\infty} v(d^{(m)}, s^{\{k,m\}}) = v^{[k]}.$$

We define an infinite sequence of strategy profiles $(\overset{=}{s}^{(m)})_{m=1}^{\infty}$ satisfying that

$$\overset{=}{s}_i^{(m)}(h_i^{k-1}) = s_i^{\{k,m\}}(h_i^0),$$

and for every $t \geq K+1$,

$$\overset{=}{s}_i^{(m)}(h_i^{t-1}) = s_i^{\{k,m\}}(\tilde{h}_i^{\tilde{t}}) \text{ if } t = K\tilde{t} + k \text{ and, for every } t = 1,\ldots,\tilde{t},$$
$$(\tilde{a}_i(t), \tilde{w}_i(t)) = (a_i(Kt + k), w_i(Kt + k)).$$

Note

$$v((d^{(m)})^{\frac{1}{K}}, \overset{=}{s}^{(m)}) = \frac{\sum_{k=1}^{K} (d^{(m)})^{\frac{k-1}{K}} v(d^{(m)}, s^{\{k,m\}})}{\sum_{k=1}^{K} (d^{(m)})^{\frac{k-1}{K}}},$$

which tends towards $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ as m increases. Since $s^{\{k,m\}}$ is a Nash equilibrium in $\Gamma(d^{(m)})$ for every large enough $m = 1,2,\ldots$, one gets that $\overset{=}{s}^{(m)}$ is a Nash equilibrium in $\Gamma((d^{(m)})^{\frac{1}{K}})$ for every large enough $m = 1,2,\ldots$. Hence, $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is sustainable.

Step 3: Note that $(0,0)$ is sustainable, because the repetition of the choices of d is the Nash equilibrium in $\Gamma(d)$ for all $d \in [0,1)$. Step 1 implies that $\overline{v}$, $\underline{v}$, $(\overline{v}_1, \underline{v}_2)$, and $(\underline{v}_1, \overline{v}_2)$ are all sustainable. Since the set of sustainable payoff vectors is compact, one gets from Step 2 that the set of sustainable payoff vectors is convex. Hence, every payoff vector in the convex hull of the set $\{(0,0), \overline{v}, (\overline{v}_1, \underline{v}_2), (\underline{v}_1, \overline{v}_2)\}$., i.e., in V^*, is sustainable.

From these observations, we have completed the proof of Theorem 1.

5. The Folk Theorem

We define

$$b^{[i]} \equiv \frac{\bar{v}_i - \underline{v}_i}{y_i + \bar{v}_i - \underline{v}_i} \text{ for each } i = 1,2 ,$$

$$z^{[1]} = (z_1^{[1]}, z_2^{[1]}) \equiv b^{[1]}(-y_1, 1 + x_2) + (1 - b^{[1]})\bar{v} ,$$

$$z^{[2]} = (z_1^{[2]}, z_2^{[2]}) \equiv b^{[2]}(1 + x_1, -y_2) + (1 - b^{[2]})\bar{v} ,$$

and we define a subset $V^{**} \subset V$ by the convex hull of the set $\{(0,0), \bar{v}, z^{[1]}, z^{[2]}\}$.

Theorem 3: *Suppose that for each* $i = 1,2$ *and for* $j \neq i$,

$$p_i(\cdot|d) \neq p_i(\cdot|d / c_j) , \tag{16}$$

and for every $a \in A / \{c\}$*, the private signals are independent, i.e.,*

$$p(w|a) = p_1(w_1|a) p_2(w_2|a) \text{ for all } w \in \Omega . \tag{17}$$

Then, every $v \in V^{**}$ *is sustainable.*

We provide the proof of Theorem 3 in the next section.

A feasible payoff vector $v \in V$ is said to be *individually rational* if it is more than or equal to the minimax payoff vector, i.e., $v \geq (0,0)$. The following theorem implies that the full folk theorem holds when players' private signals are independent, $(1,1)$ is efficient, and the minimum likelihood ratios $\underline{L}_i(c, c / d_j)$ and $\underline{L}_i(d, d / c_i)$ are both equal to zero.

Theorem 4 (Folk Theorem): *Suppose that* $x_1 + x_2 < y_1 + y_2$*, the conditions in Theorem 3 hold, and for each* $i = 1,2$ *and for* $j \neq i$,

$$\underline{L}_i(c, c / d_j) = \underline{L}_i(d, d / c_j) = 0 . \tag{18}$$

Then, every feasible and individually rational payoff vector is sustainable.

Proof: Equalities (18) imply that $\bar{v} = (1,1)$, $\underline{v} = (0,0)$, $b^{[i]} = \dfrac{1}{1 + y_i}$, $z^{[1]} = (0, \dfrac{1 + y_1 + x_2}{1 + y_1})$, and $z^{[2]} = (\dfrac{1 + y_2 + x_1}{1 + y_2}, 0)$. Inequality $x_1 + x_2 < y_1 + y_2$ implies that V^{**} is equivalent to the set of feasible and individually rational payoff vectors.

Q.E.D.

6. Proof of Theorem 3

The proof of Theorem 3 is divided into four steps.

Step 1: Step 1 in the proof of Theorem 1 has proved that for every $v^+ \in V^*$ and every $v^- \in V^*$, if $\overline{v} \geq v^+ > v^- \geq \underline{v}$, then, v^+, v^-, (v_1^+, v_2^-), and (v_1^-, v_2^+) are all sustainable.

Step 2: We show that $z^{[1]}$ and $z^{[2]}$ are sustainable. Consider $z^{[1]}$ only. We can prove that $z^{[2]}$ is sustainable in the same way as $z^{[1]}$.

Choose $b > 0$ larger than but close to $b^{[1]}$, and let

$$v^* \equiv (1-b)(-y_1, 1+x_2) + b\overline{v}.$$

Note $v_1^* > z_1^{[1]}$. Fix $(d^{(m)})_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} d^{(m)} = 1$. Let $(s^{(m)})_{m=1}^{\infty}$ be the infinite sequence of strategy profiles specified in Step 1 in the proof of Theorem 1, where we assume

$$v^+ = \overline{v} \text{ and } v^- = \underline{v}.$$

Note

$$\lim_{m\to\infty} v(d^{(m)}, s^{(m)}) = \overline{v}, \tag{19}$$

$$\lim_{m\to\infty} v(d^{(m)}, s_1^{(m)}, s_2^{(m)}|_{(c_1, \hat{w}_2^{(m)})}) = (\underline{v}_1, \overline{v}_2), \tag{20}$$

and

$$v_2(d^{(m)}, s^{(m)}) = v_2(d^{(m)}, s_1^{(m)}, s_2^{(m)}|_{(c_1, \hat{w}_2^{(m)})}). \tag{21}$$

Let $(T^{(m)})_{m=1}^{\infty}$ be an infinite sequence of positive integers satisfying

$$\lim_{m\to\infty} (d^{(m)})^{T^{(m)}} = b.$$

Inequalities (16) imply that there exists an interval $(\underline{w}_2, \overline{w}_2) \subset \Omega_2$ such that

$$\int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 < \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d) dw_2 .$$

Fix $m > 0$ close to 0. We denote by $T(h_2^{T^{(m)}})$ the number of the periods t between period 1 and period $T^{(m)}$ which satisfies $w_2(t) \in (\underline{w}_2, \overline{w}_2)$.

We specify an infinite sequence of strategy profiles $(s^{[1,m]})_{m=1}^{\infty}$ in the following way.

$$s_2^{[1,m]}(h_2^t) = d_2 \text{ if } t < T^{(m)},$$

$$s_2^{[1,m]}|_{h_2^{T(m)}} = s_2^{(m)} \text{ if } \frac{T(h_2^{T^{(m)}})}{T^{(m)}} < \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 + m,$$

$$\overline{s}_2^{[1,m]}|_{h_2^{T(m)}} = s_2^{(m)}|_{(c_2, \hat{w}_2^{(m)})} \text{ if } \frac{T(h_2^{T^{(m)}})}{T^{(m)}} \geq \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 + m,$$

$$s_1^{[1,m]}|_{h_1^{T(m)}} = s_1^{(m)} \text{ for all } h_1^{T^{(m)}},$$

and $s_1^{[1,m]}$ is the best response to $s_2^{[1,m]}$, i.e.,

$$v_1(d^{(m)}, s^{[1,m]}) \geq v_1(d^{(m)}, s_1, s_2^{[1,m]}) \text{ for all } s_1 \in S_1 .$$

Since the private signals are independent for all $a \neq c$, i.e., equalities (17) hold, and since player 2 plays a pure strategy, we can choose $s_1^{[1,m]}$ so as to be history-independent between period 1 and period $T^{(m)}$, i.e., so as to satisfy that for every $m = 1,2,\ldots$, every $t \in \{1,\ldots,T^{(m)}\}$, every $h_1^t \in H_1$, and every $\tilde{h}_1^t \in H_1$,

$$s_1^{[1,m]}(h_1^t) = s_1^{[1,m]}(\tilde{h}_1^t) .$$

For every $t = 1,\ldots,T^{(m)}$, let $a_1^{(m)}(t)$ be the action for player 1 which satisfies that for every $h_1^t \in H_1$,

$$a_1^{(m)}(t) = s_1^{[1,m]}(h_1^t) .$$

Since d is the dominant action profile and both $s^{(m)}$ and $(s_1^{(m)}, s_2^{(m)}|_{(c_2,\hat{w}_2^{(m)})})$ are Nash equilibria in $\Gamma(d^{(m)})$ for every large enough m, one gets that $s^{[1,m]}$ is also a Nash equilibrium in $\Gamma(d^{(m)})$ for every large enough m.

We denote by $P^{(m)}$ the probability that the realized private history for player 2 up to period $T^{(m)}$, $h_2^{T^{(m)}}$, satisfies

$$\frac{T(h_2^{T^{(m)}})}{T^{(m)}} < \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 + m .$$

We denote by $g^{(m)}$ the proportion of the periods between period 1 and period $T^{(m)}$ in which player 1 chooses action c_1, i.e., $a_1^{(m)}(t) = c_1$. We define

$$p^{(\infty)} \equiv \lim_{m \to \infty} P^{(m)} \text{ and } g^{(\infty)} \equiv \lim_{m \to \infty} g^{(m)} .$$

Note that

$$v_1(d^{(m)}, s^{[1,m]}) = (1 - d^{(m)}) \sum_{t=1}^{T^{(m)}} (d^{(m)})^{t-1} u_1(a_1^{(m)}(t), d_2)$$
$$+ (d^{(m)})^{T^{(m)}} \{P^{(m)} \hat{v}_1^{(m)} + (1 - P^{(m)}) \tilde{v}_1^{(m)}\},$$
$$v_2(d^{(m)}, s^{[1,m]}) = (1 - d^{(m)}) \sum_{t=1}^{T^{(m)}} (d^{(m)})^t u_2(a_1(t), d_2) + (d^{(m)})^{T^{(m)}} \hat{v}_2^{(m)},$$

and

$$\lim_{m \to \infty} v(d^{(m)}, s^{[1,m]}) = (1 - b)(-y_1, 1 + x_2) g^{(\infty)}$$
$$+ b\{P^{(\infty)} \overline{v} + (1 - P^{(\infty)})(\underline{v}_1, \overline{v}_2)\} .$$

If

$$g^{(\infty)} \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 + (1 - g^{(\infty)}) \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d) dw_2$$
$$> \int_{w_2 \in (\underline{w}_2, \overline{w}_2)} p_2(w_2 | d / c_1) dw_2 + m,$$

then, the Law of Large Numbers implies $p^{(\infty)} = 0$, and therefore,

$$\lim_{m\to\infty} v_1(d^{(m)}, s^{[1,m]}) = (1-b)(-y_1)g^{(\infty)} + b\underline{v}_1 \le b\underline{v}_1 .$$

On the other hand, if

$$g^{(\infty)} \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d/c_1)dw_2 + (1-g^{(\infty)}) \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d)dw_2$$
$$< \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d/c_1)dw_2 + m,$$

then, the Law of Large Numbers implies $p^{(\infty)} = 1$, and therefore,

$$\lim_{m\to\infty} v_1(d^{(m)}, s^{[1,m]}) = (1-b)(-y_1)g^{(\infty)} + b\overline{v}_1 \ge (1-b)(-y_1) + b\overline{v}_1 ,$$

which is larger than $b\underline{v}_1$ because of $b > b^{[1]}$ and the definition of $b^{[1]}$. Hence, one gets

$$g^{(\infty)} \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d/c_1)dw_2 + (1-g^{(\infty)}) \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d)dw_2$$
$$\le \int_{w_2\in(\underline{w}_2,\overline{w}_2)} p_2(w_2|d/c_1)dw_2 + m .$$

Since we can choose m as close to 0 as possible, one gets that $g^{(\infty)}$ is approximately equal to 1, and therefore, for every large enough m, $v(d^{(m)}, s^{[1,m]})$ is approximately equal to

$$v^* = (1-b)(-y_1, 1+x_2) + b\overline{v} .$$

Since we can choose v^* as close to $z^{[1]}$ as possible, we have proved that $z^{[1]}$ is sustainable.

Step 3: Step 2 in the proof of Theorem 1 has proved that for every positive integer $K > 0$ and every K sustainable payoff vectors $v^{[1]}$, ..., $v^{[K]}$, $\dfrac{\sum_{k=1}^{K} v^{[k]}}{K}$ is also sustainable.

Step 4: Note that $(0,0)$ is sustainable. Step 1 implies that $\overline{v}$ is sustainable. Step 2 implies that $z^{[1]}$ and $z^{[2]}$ are sustainable. Since the set of sustainable payoff vectors is compact, one gets from Step 3 that the set of sustainable payoff vectors is convex. Hence, every payoff vector in the convex hull of the set $\{(0,0), \overline{v}, z^{[1]}, z^{[2]}\}$, i.e., in V^{**}, is sustainable.

From these observations, we have completed the proof of Theorem 3.

7. Machine Games and Uniform Equilibrium

This section regards $\Gamma(d)$ as a machine game. For each $i = 1,2$, fix the sets of states of machine for player i, Q_i, arbitrarily, where $|Q_i| \geq 2$. Let $Q \equiv Q_1 \times Q_2$. A *behavioral rule*, or simply a *rule*, for player i is defined by $s_i \equiv (f_i, t_i)$, where $f_i : Q_i \to A_i$ is a behavior function, $t_i : Q_i \times \Omega_i \to Q_i$ is a transition function, and t_i is measurable w. r. t. Ω_i. The set of rules for player i is denoted by Σ_i. Let $\Sigma \equiv \Sigma_1 \times \Sigma_2$. A *machine for player* i is defined as a combination of a rule and an initial state of machine $\boldsymbol{q}_i = (s_i, q_i) \in \Sigma_i \times Q_i$. The set of all machines for player i is denoted by Θ_i. Let $\Theta \equiv \Theta_1 \times \Theta_2$. In every period t, player i chooses the action $a_i(t) = f_i(q_i(t))$ when $q_i(t)$ is the state for player i. The state for player i transits from $q_i(t)$ to $q_i(t+1) = t_i(q_i(t), w_I(T))$ when she observes her own private signal $w_i(t) \in \Omega_i$. Player $i's$ normalized long-run payoff induced by a machine profile $\boldsymbol{q} \in \Theta$ is defined by $v_i(d, \boldsymbol{q}) \equiv (1-d)E[\sum_{t=1}^{\infty} d^{t-1} u_i(a(t)) | \boldsymbol{q}]$. A machine profile $\boldsymbol{q} \in \Theta$ is said to be a *Nash equilibrium* in $\Gamma(d)$ if for each $i = 1,2$ and every $\boldsymbol{q}_i' \in \Theta_i$, $v_i(d_i, \boldsymbol{q}) \geq v_i(d_i, \boldsymbol{q} / \boldsymbol{q}_i')$. A machine profile $\boldsymbol{q} \in \Theta$ is sometimes denoted by $(s, q) \in \Sigma \times Q$.

For every $i = 1,2$ and every machine for player i, $\boldsymbol{q}_i = (s_i, q_i) \in \Theta_i$, we define a strategy for player i, $s_i(\boldsymbol{q}_i) \in S_i$, and a function $q_i(\boldsymbol{q}_i) : H_i \to Q_i$, by

$$q_i(\boldsymbol{q}_i)(h_i^0) = q_i \in Q_i,$$

$$s_i(\boldsymbol{q}_i)(h_i^0) = f_i(q_i) \in A_i,$$

and for every $t \geq 1$ and every $h_i^t \in H_i$,

$$q_i(\boldsymbol{q}_i)(h_i^t) = t_i(q_i(\boldsymbol{q}_i)(h_i^{t-1}), w_i(t)) \in Q_i$$

and

$$s_i(\boldsymbol{q}_i)(h_i^t) = f_i(q_i(\boldsymbol{q}_i)(h_i^t)) \in A_i .$$

Let $s(\boldsymbol{q}) \equiv (s_1(\boldsymbol{q}_1), s_2(\boldsymbol{q}_2)) \in S$. Note that $\boldsymbol{q}$ is a Nash equilibrium in $\Gamma(d)$ if $s(\boldsymbol{q})$ is a Nash equilibrium in $\Gamma(d)$.

The following definition implies that players always play Nash equilibria irrespective of their initial states of machine.

Definition 2: A rule profile $s \in \Sigma$ is a *uniform equilibrium* in $\Gamma(d)$ if (s, q) is a Nash equilibrium in $\Gamma(d)$ for all $q \in Q$.

Note that s is a uniform equilibrium in $\Gamma(d)$ if $s(s, q)$ is a Nash equilibrium in $\Gamma(d)$ for all $q \in Q$. Note also that

$$v(d, (s, q)) = v(d, s(s, q)) \text{ for all } q \in Q .$$

The following theorem provides an upper-bound of the uniform equilibrium payoff vectors.

Theorem 5: *If* $s \in \Sigma$ *is a uniform equilibrium in* $\Gamma(d)$*, then for each* $i = 1,2$,

$$\max[0, \bar{v}_i] \geq v_i(d, s, q) \text{ for all } q \in Q.$$

Proof: Suppose that there exists $i = 1,2$ such that $f_i(q_i) = d_i$ for all $q_i \in Q_i$. Since d is the dominant action profile, it must hold that for $j \neq i$,

$$f_j(q_j) = d_j \text{ for all } q_j \in Q_j.$$

Hence, players repeatedly choose d, and therefore,

$$v(d, s, q) = 0.$$

Suppose that for each $i = 1,2$, there exists $\tilde{q}_i \in Q_i$ such that $f_i(\tilde{q}_i) = c_i$. Fix $i = 1,2$ arbitrarily, and let

$$D_i(q_j) \equiv \max_{q_i' \in \Theta_i} v_i(d_i, q_i', (s_j, q_j)).$$

The uniform equilibrium property of s implies that for every $q \in Q$,

$$v_i(d, s, q) = D_i(q_j),$$

and therefore,

$$D_i(q_j) = (1-d)u_i(c_i, f_j(q_j))$$
$$+ d \int_{w_j \in \Omega_j} p_j(w_j | c_i, f_j(q_j)) D_i(t_j(q_j, w_j)) dw_j.$$

Choose $q_j^* \in Q_j$ which maximizes $D_i(q_j)$, and suppose

$$D_i(q_j^*) > 0.$$

Note

$$D_i(q_j^*) = u_i(c_i, f_j(q_j^*))$$
$$+ \frac{d}{1-d} \int_{w_j \in \Omega_j} p_j(w_j | c_i, f_j(q_j^*)) \{D_i(t_j(q_j^*, w_j)) - D_i(q_i^*)\} dw_j.$$

Since $D_i(t_j(q_j^*, w_j)) - D_i(q_i^*) \leq 0$ for all $w_j \in \Omega_j$, one gets that $D_i(q_i^*)$ is less than or equal to the value induced by the following conditional maximization.

$$\max_{e: \Omega_j \to R_+ \cup \{0\}, a_j \in A_j} [u_i(c_i, a_j) - \int_{w_j \in \Omega_j} p_j(w_j | c_i, a_j) e(w_j) dw_j]$$

subject to

$$u_i(c_i, a_j) - \int_{w_j \in \Omega_j} p_j(w_j | c_i, a_j) e(w_j) dw_j$$
$$\geq u_i(d_i, a_j) - \int_{w_j \in \Omega_j} p_j(w_j | d_i, a_j) e(w_j) dw_j.$$

Since $D_i(q_j^*) > 0$ and $u_i(c / d_j) = -y_i < 0$, one gets that $a_j = c_j$ must hold. Hence, the value induced by the above conditional maximization is equivalent to

$$\max_{e:\Omega_j \to R_+ \cup \{0\}} [1 - \int_{w_j \in \Omega_j} p_j(w_j|c)e(w_j)dw_j]$$

subject to

$$\int_{w_j \in \Omega_j} \{p_j(w_j|d_i, c_j) - p_j(w_j|c)\}e(w_j)dw_j \geq x_i .$$

The value induced by this conditional maximization is equal to

$$1 - \frac{\underline{L}_i(c, c/d_j)x_j}{1 - \underline{L}_i(c, c/d_j)},$$

which is equal to $\overline{v}_i$.

Q.E.D.

8. Uniform Sustainability

The following definition implies that players always play Nash equilibria and always obtain virtually the same payoff vector irrespective of their initial states of machine.

Definition 3: A payoff vector $(v_1, v_2) \in R^2$ is *uniformly sustainable* if for every infinite sequence of discount factors $(d^{(m)})_{m=1}^{\infty}$ satisfying $\lim_{m\to+\infty} d^{(m)} = 1$, and for every $e > 0$, there exists an infinite sequence of rule profiles $(s^{(m)})_{m=1}^{\infty}$ such that for every large enough m, $s^{(m)}$ is a uniform equilibrium in $\Gamma(d)$, and for every $q \in Q$,

$$v - (e, e) < \lim_{m\to+\infty} v(d^{(m)}, s^{(m)}, q) < v + (e, e).$$

By definition, the set of uniformly sustainable payoff vectors is compact.

Theorem 6: *If* $v = (v_1, v_2) \in R^2$ *satisfies*

$$\bar{v} \geq v \geq \underline{v},$$

then it is uniformly sustainable.

Proof: Fix $v^+ \in V^*$ and $v^- \in V^*$ arbitrarily, which satisfies $\bar{v} \geq v^+ > v^- \geq \underline{v}$. Fix $(d^{(m)})_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} d^{(m)} = 1$. Let $(s^{(m)})_{m=1}^{\infty}$ be the infinite sequence of strategy profiles specified in Step 1 in the proof of Theorem 1. Let $Q_i = \{q_{i,1}, q_{i,2}, ..., q_{i,b_i}\}$, where $b_i \equiv |Q_i|$. We define an infinite sequence of rule profiles $(s^{(m)})_{m=1}^{\infty}$ in the following way. For each $i = 1,2$,

$$f_i^{(m)}(q_{i,1}) = c_i,$$

$$f_i^{(m)}(q_i) = d_i \text{ for all } q_i \neq q_{i,1},$$

$$t_i^{(m)}(q_{i,1}, w_i) = q_{i,1} \text{ if } w_i \notin (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}],$$

$$t_i^{(m)}(q_{i,1}, w_i) = q_{i,2} \text{ if } w_i \in (\hat{w}_i - e_i^{(m)}, \hat{w}_i + e_i^{(m)}],$$

and for every $q_i \neq q_{i,1}$,

$$t_i^{(m)}(q_i, w_i) = q_{i,1} \text{ if } w_i \in (\tilde{w}_i - \tilde{l}_i^{(m)}, \tilde{w}_i + \tilde{l}_i^{(m)}],$$

and

$$t_i^{(m)}(q_i, w_i) = q_{i,2} \text{ if } w_i \notin (\tilde{w}_i - \tilde{l}_i^{(m)}, \tilde{w}_i + \tilde{l}_i^{(m)}],$$

where $\hat{w}_i$, $\tilde{w}_i$, $e_i^{(m)}$, and $l_i^{(m)}$ were specified in Step 1 in the proof of Theorem 1. Note that

$$s_i(s_i^{(m)}, q_{i,1}) = s_i^{(m)}|_{h_i^t} \text{ if } s_i^{(m)}(h_i^t) = c_i,$$

and for every $q_i \neq q_{i,1}$,

$$s_i(s_i^{(m)}, q_i) = s_i^{(m)}|_{h_i^t} \text{ if } s_i^{(m)}(h_i^t) = d_i .$$

Since $(s_1^{(m)}|_{h_1^t}, s_2^{(m)}|_{h_2^{t'}})$ is a Nash equilibrium in $\Gamma(d^{(m)})$ for every $h_1^t \in H_1$ and every $h_2^{t'} \in H_2$, one gets that $s(s^{(m)}, q)$ is a Nash equilibrium in $\Gamma(d^{(m)})$ for all $q \in Q$, and therefore, $s^{(m)}$ is a uniform equilibrium in $\Gamma(d^{(m)})$. Since for each $i = 1,2$,

$$v_i(d, (s^{(m)}, q)) = v_i(d, s(s^{(m)}, q)) = \hat{v}_i^{(m)} \text{ if } q_i = q_{i,1},$$
$$v_i(d, (s^{(m)}, q)) = v_i(d, s(s^{(m)}, q)) = \tilde{v}_i^{(m)} \text{ if } q_i \neq q_{i,1},$$

and we can choose v^- as close to v^+ as possible, we have proved that v^+ is uniformly sustainable. Since the set of uniformly sustainable payoff vectors is compact, we have proved that every v satisfying $\overline{v} \geq v \geq \underline{v}$ is uniformly sustainable.

Q.E.D.

Theorems 5 and 6 imply that $\overline{v}$ is the unique uniformly sustainable payoff vector which Pareto-dominates all other uniformly sustainable payoff vectors. This uniqueness of Pareto-undominance is in contrast with the folk theorem in the study of Nash, or perfect Nash, equilibrium, because the folk theorem implies that there exist multiple Pareto-undominated equilibrium payoff vectors.

Theorem 7: *Suppose that for each $i = 1,2$, inequality (3) holds, i.e.,*

$$\underline{L}_i(d, d / c_j) < \frac{1}{y_i + 1}.$$

Then, $(1,1)$ is uniformly sustainable if and only if for each $i = 1,2$, equality (2) holds, i.e.,

$$\underline{L}_i(c, c / d_j) = 0.$$

Proof: We show the "if" part. Theorem 6, the definition of $\overline{v}$, and equalities (2) imply that if $(1,1) \geq v > \underline{v}$, then v is uniformly sustainable. Inequalities (3) and the definition of $\underline{v}$ imply

$$(1,1) > \underline{v} .$$

Hence, $(1,1)$ is uniformly sustainable.

We show the "only if" part. Theorem 5 implies that for each $i = 1,2$,

$$\max[0, \overline{v}_i] \geq 1.$$

Hence, $\overline{v} = (1,1)$ must hold, which implies equalities (2).

Q.E.D.

When inequality $x_1 + x_2 > y_1 + y_2$ holds, Theorem 7 implies that the sufficient condition for efficiency in sustainability addressed by Theorem 2, i.e., the zero likelihood ratio condition, is also necessary and sufficient for efficiency in uniform sustainability.

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