The Occupation Density of Fractional Brownian Motion and Some of Its Applications*

Konstantin Tyurin
Department of Economics
Yale University

Peter C.B. Phillips
Cowles Foundation for Research in Economics
Yale University

October 19, 1999

Abstract

A theory of chronological local time is developed for fractionally integrated processes using conventional and new techniques. The new approach employed in the paper is based on the approximation of the chronological local time of a fractional Brownian motion by local times of continuous diffusion mixtures and can be applied to generalize the asymptotic theory of local times of Wiener processes to the case of a fractional Brownian motion. An almost sure approximation of the spatial density based on a discrete sample of observations is obtained for a wide class of long-memory processes. The methods of the paper can be used to generate empirical estimates of spatial densities of fractionally integrated time series in economics and finance and to conduct inference with nonlinear functions of such series.

1. Introduction

The identification and estimation of the stochastic processes that describe the dynamic evolution of economic and financial variables are two of the main goals of time series econometrics. Description of regularities in the data is the starting point in this process and, indeed, in much empirical work, constitutes the early stage of exploratory

*[**Preliminary and Incomplete**] Prepared for presentation at the Cowles Foundation Econometrics Conference held at Yale University, October 23-24, 1999. The paper was typed in SW2.5. The authors are fully responsible for the remaining errors. Tyurin thanks the Cowles Foundation for support under an Anderson Fellowship and Phillips thanks the NSF for research support under Grant No. SBR97-30295. Comments and suggestions are welcome and may be sent to: Konstantin Tyurin, Department of Economics, Yale University, P.O.Box 208268, New Haven, CT 06520-8268; e-mail: *tyurin@pantheon.yale.edu*.

empirical analysis when 'stylized facts' are revealed. Such analysis often becomes a reference point for subsequent research and it primarily employs descriptive statistical tools, such as sample moments, empirical densities, and distribution functions.

Identification issues become especially important if existing economic theories give little or no information about the nature and fundamental properties of the variables of interest. In such situations, empirical researchers are often encouraged to take an 'agnostic' approach to the data, in order to minimize the risk of imposing too many *a priori* restrictions on the data generating process. Among the typical restrictions that are imposed on the data, or simple temporal transforms of the data like first differences, are stationarity, together with some moment conditions and weak dependence conditions that affect the behavior of the spectral density and tails of the marginal data densities. In many situations these assumptions appear innocuous and are sufficiently mild to allow for a great variety of admissible specifications of data generating processes. At the same time they assist in building a groundwork that is sufficient for the development of a robust theory of descriptive statistics that summarize empirical behavior of the series themselves and permit some meaningful comparisons between different series.

Among the many statistical models for random processes in macroeconomics and finance, fractionally integrated time series stand out for their parsimony and flexibility. Fractional processes can capture many different empirical characteristics of economic data, from the apparent long-memory properties of interest rates, inflation rates, and asset price volatility through to the random wandering behavior of stock prices, exchange rates and technology. Beyond the theory of efficient markets, economic theory often fails to provide clear restrictions on the data generating processes that relate to the observed characteristics of many of these series. In such situations, a toolkit of simple descriptive devices that assist in characterizing the empirical behavior of such time series would seem to be particularly valuable. Since the empirical estimates of fractional integration parameters reported in applied work are often midway between zero and unity, researchers working with such data are often in the uncomfortable situation of having to rely on statistical tools that have been designed for stationary or unit root time series. One way in which the present paper seeks to contribute is by developing the groundwork for characterizing such time series data in possibly nonstationary situations without relying on unit root assumptions.

The concept of a spatial density suggested in Phillips (1998) is a new tool for describing and characterizing the spatial location of potentially nonstationary time series. In the nonstationary case, spatial densities play a role similar to that of the time-invariant marginal density of a stationary random variable. The major difference is that a spatial density is path dependent in the nonstationary case and its value at any point in the state space is a random variable. The spatial density of a continuous time stochastic process is called local time. It is a stochastic process that is indexed by two parameters—time and spatial location—and its properties depend on those of the underlying process. In spite of these differences, there is a remarkable similarity in the theory of spatial densities between stationary and nonstationary cases, since both are densities with respect to Lebesgue measure of occupation measures: in one (nonstationary) case, a spatial occupation measure which distributes variation over space; and, in the other (stationary) case, a probability occupation measure which distributes probability mass over space.[1]

The theory of spatial density estimation was developed for the case of ordinary unit root processes (with Brownian motion limits) by Phillips and Park (1998). It has been applied in Phillips (1999a) to empirical data on inflation, exchange rates, and opinion poll data, and it has been used to construct hazard functions for real interest rates, inflation and deflation (Phillips, 1998, 1999a; Matsura, 1999 [**FIND REFERENCE**]). These applications show that spatial densities can help us to characterize certain location features of time series data and measure historical risk factors, like that of price inflation above certain levels or price deflation.

No background analytical apparatus is currently available to develop a rigorous operational theory of spatial density for fractionally integrated time series. In particular, the classical Ito—Tanaka formula, which is the backbone of the theory of spatial densities for continuous semimartingales, is no longer available for processes like fractional Brownian motion that have infinite quadratic variation.[2] Thus, the development of a theory of spatial location for fractional processes must be preceded by a theory of local time for fractional processes.

The present paper shows how the notion of local time developed by Revuz and Yor (1991) and Karatzas and Shreve (1991) for continuous semimartingales can be generalized to continuous long-memory processes. We demonstrate the workings of this procedure for a relatively simple case of fractional Brownian motion with Hurst exponent H between zero and $\frac{1}{2}$. The approach is based on the approximation of the chronological local time of a fractional Brownian motion by a sequence of local times for a continuous mixture of Ornstein—Uhlenbeck diffusions. Then, we demonstrate how the asymptotic theory for local times of Wiener processes (Ikeda and Watanabe, 1981; Revuz and Yor, 1991) can be generalized to the case of a fractional Brownian motion. Some applications to nonlinear functions of fractionally integrated time series are given as illustrations. Finally, we show how the spatial density of a fractional Brownian motion can be estimated on the basis of a discrete sample of observations and how the asympotic theory for nonlinear functionals of fractional Brownian motion can be developed. Although a complete coverage of the asymptotic theory for local times and other functionals of fractional Brownian motion and its empirical applications are yet to be fully worked out, the methodology introduced in this paper can now be applied to produce empirical estimates of spatial densities of fractionally integrated time series in economics and finance and to conduct inference with nonlinear functions of such series.

[1] A summary of some applications of the local time for estimation of marginal density in the stationary case can be found in Bosq (1998, Chapter 6).

[2] The conventional development of Ito calculus relies heavily on the concept of finite quadratic variaion and we are unfamiliare with extensions of this theory to processes that have infinite quadratic variation. However, some steps towards extending the Ito calculus for semimartingales have recently been taken in work by Carmona, Coutin, and Montseny (1998) and Zähle (1997), who developed a version of Ito's lemma for functionals of fractional Brownian motion with Hurst exponent $H > \frac{1}{2}$.

2. Local Time for a Fractional Brownian Motion

A. Definition, existence, and regularity properties of local time

Let $X = (X_t)_{t\in\mathbf{R}}$ be a real measurable continuous time stochastic process on a general probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

2.1 Definition *Consider a random* ***occupation measure*** ν_T *associated with stochastic process* X *and defined as*

$$\nu_T(A) = \int_0^T \mathbf{1}_A(X_t)dt, \ \forall A \in \mathcal{B}(\mathbf{R}). \tag{2.1}$$

If measure ν_T *is absolutely continuous with respect to Lebesgue measure* da *on the real line, define the* ***local time*** *of* X *as a measurable function* $\overline{L}_T^a(\omega, X)$ *such that*

$$\overline{L}_T^a(\omega, X) = \frac{d\nu_T}{da}(\omega, X) \text{ for almost all } \omega \in \Omega.$$

Throughout the paper, we interchangeably use the terms **occupation density** and **spatial density** for the random map $a \mapsto \overline{L}_T^a(\omega, X)$ when the time variable T is fixed.

2.2 Remark Immediate implication of the above definition is the **occupation times formula**

$$\int_0^T f(X_t)dt = \int_{\mathbf{R}} f(a)\overline{L}_T^a(X)da, \tag{2.2}$$

which holds for any Borel function f and follows from the definition of Radon—Nikodym derivative and linearity combined with a monotone convergence argument.

2.3 Theorem (Geman and Horowitz, 1980) (**Criterion of existence for local time**) *The real measurable process* $X = (X_t)_{t\in[0;T]}$ *admits a square integrable local time* $\overline{L}_T^a(X)$ *if and only if*

$$\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^T \int_0^T \mathbf{P}(|X_t - X_s| \le \varepsilon)dsdt < \infty. \tag{2.3}$$

2.4 Definition ***Fractional Brownian motion*** *(fBM) with Hurst exponent* $H \in (0,1)$ *is an a.s. continuous, zero-centered Gaussian process* B_H *defined by stochastic integral*

$$B_H(r) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_0^r (r-s)^{H-\frac{1}{2}} dB(s), \tag{2.4}$$

where $B(r) \equiv \sigma W(r)$ is a scaled standard Brownian motion.[3]

It is easy to see that condition (2.3) is satisfied for fractional Brownian motion (2.4). Moreover, there exist jointly continuous and sufficiently regular versions of local time $\overline{L}_T^a(X)$ for a class of 'index γ' Gaussian processes (Geman and Gorowitz, 1980, Section 3) and for certain self-similar processes with stationary increments (Kôno and Shieh, 1993). One result in that direction is following.

2.5 Theorem (Berman, 1973; Geman and Horowitz, 1980) *Any continuous, locally nondeterministic Gaussian process X has a jointly continuous version of local time $(t,a) \mapsto \overline{L}_t^a(X)$. Moreover, if X has stationary increments and incremental variance $\sigma^2(t) \sim t^{2\gamma}$ for small $t > 0$ $(0 < \gamma \leq 1)$, then*

(i) *the map $t \mapsto \overline{L}_t^a(X)$ is Hölder continuous of order γ' for every $\gamma' < 1 - \gamma$ uniformly in a from any compact set, and*

(ii) *the map $a \mapsto \overline{L}_t^a(X)$ is Hölder continuous of order γ' for every $\gamma' < \frac{1}{2}(\frac{1}{\gamma} - 1)$ uniformly in t on every compact interval.*

2.6 Corollary *For any $0 < H < 1$, there exists a version of chronological local time of fractional Brownian motion $B_H(r)$ such that the map $(t,a) \mapsto \overline{L}_t^a(B_H)$ is a.s. continuous in both t and a. Moreover, it can be chosen so that the map $a \mapsto \overline{L}_t^a(B_H)$ is Hölder continuous of order γ_1 for every $\gamma_1 < \frac{1}{2}(\frac{1}{H} - 1)$ uniformly in t on every compact interval, while the map $t \mapsto \overline{L}_t^a(B_H)$ is Hölder continuous of order γ_2 for every $\gamma_2 < 1 - H$.*

2.7 Remark Theorem 2.5 is a manifestation of the heuristic 'Berman's principle' (Kôno and Shieh, 1993) establishing the connections between the smoothness of local times and the irregularity of sample paths. Part (ii) of Theorem 2.5 also implies that the occupation density $a \mapsto \overline{L}_t^a(X)$ of the fractional Brownian motion B_H is absolutely continuous for $H < \frac{1}{3}$, has an absolutely continuous first derivative whenever $H < \frac{1}{5}$, and so forth. It is interesting to note that the occupation densities of index $\gamma = 1$ Gaussian processes are discontinuous, whereas the occupation densities of Gaussian processes characterized by the property $\sigma^2(t) \sim (\log(1/t))^{-1}$ are real analytic functions (Berman, 1969).

B. Alternative definition of local time based on a diffusion representation of fractional Brownian motion

[3] We use the initialization $B(s) = 0$ for all $s \leq 0$, which implies $B_H(r) = 0$ for $r \leq 0$. The initialization issue is of secondary importance to the contents of the theory developed in the paper. All our results can be replicated with minor modifications for the alternate definition of fractional Brownian motion (Mandelbrot and Van Ness, 1968),

$$B_H(r) = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^{r} ((r-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}) dB(s)$$

with $B(0) = 0$.

In the remainder of this section we analyze the case of fractional Brownian motion 2.4 with Hurst exponent $0 < H < \frac{1}{2}$. Using the integral representation

$$\frac{1}{z^{\gamma}} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} e^{-z\tau} \tau^{\gamma-1} d\tau, \quad \mathrm{Re}(\gamma) > 0 \tag{2.5}$$

we can write the kernel of stochastic integral (2.4) as

$$(r-s)^{H-\frac{1}{2}} = \frac{1}{(r-s)^{\frac{1}{2}-H}} = \frac{1}{\Gamma(\frac{1}{2}-H)} \int_0^{\infty} e^{-(r-s)\tau} \tau^{-(\frac{1}{2}+H)} d\tau.$$

Then, the stochastic integral (2.4) can be written as a mixture of Ornstein—Uhlenbeck diffusions as follows:

$$\begin{aligned} B_H(r) &= \frac{1}{\Gamma(\frac{1}{2}+H)} \int_0^{r} \frac{1}{(r-s)^{\frac{1}{2}-H}} dB(s) \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^{r} \int_0^{\infty} e^{-(r-s)\tau} \tau^{-(\frac{1}{2}+H)} d\tau dB(s) \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^{\infty} \left[\int_0^{r} e^{-(r-s)\tau} dB(s) \right] \tau^{-(\frac{1}{2}+H)} d\tau \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^{\infty} J_\tau(r) \tau^{-(\frac{1}{2}+H)} d\tau. \end{aligned} \tag{2.6}$$

In the above representation, all the Ornstein—Uhlenbeck diffusions

$$J_\tau(r) = \int_0^{r} e^{-(r-s)\tau} dB(s), \; 0 < \tau < \infty, \tag{2.7}$$

are generated by a common Brownian motion $B(r)$. The diffusions $J_\tau(r)$ satisfy Langevin's equation

$$dJ_\tau(r) = -\tau J_\tau(r) dr + dB(r)$$

with common stochastic disturbances $dB(r)$. Note that the mixture of diffusions (2.6) is 'improper' in the sense that the 'weights' $\tau^{-(\frac{1}{2}+H)}$ of the mixing distribution do not integrate to a finite number.[4] Nevertheless, it is still possible to justify the above representation by the following heuristic convergence argument.

[4] A similar representation of fBM with Hurst exponent between zero and $\frac{1}{2}$ has recently been established by Carmona, Coutin, and Montseny (1998):

$$B_H(r) = \int_{-\infty}^{+\infty} X(\xi, r) d\xi,$$

Consider a β-parametric family of processes

$$B_H^\beta(r) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_0^r (\beta + r - s)^{H-\frac{1}{2}} dB(s), \ \beta \to 0, \tag{2.8}$$

converging[5] to fractional Brownian motion (2.4) as $\beta \to 0$. Using the integral formula (2.5), we can easily prove that for any $\beta > 0$, processes (2.8) can be written as a 'proper' mixture of Ornstein—Uhlenbeck diffusions originating from the same Brownian motion $B(r)$. In particular,

$$\begin{aligned} B_H^\beta(r) &= \frac{1}{\Gamma(\frac{1}{2}+H)} \int_0^r \frac{1}{(\beta + r - s)^{\frac{1}{2}-H}} dB(s) \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^r \int_0^\infty e^{-(\beta+r-s)\tau} \tau^{-(H+\frac{1}{2})} d\tau dB(s) \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^\infty J_\tau(r) e^{-\beta\tau} \tau^{-(H+\frac{1}{2})} d\tau, \end{aligned} \tag{2.9}$$

and the mixing distribution is gamma with parameters $\frac{1}{2} - H$ and $\frac{1}{\beta}$ scaled by factor

$$\frac{1}{\Gamma(\frac{1}{2}+H)\beta^{\frac{1}{2}-H}} = \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)} \int_0^\infty e^{-\beta\tau} \tau^{-(H+\frac{1}{2})} d\tau.$$

Moreover, quadratic variation of the mixture process (2.8) is *a.s.* finite and is given by formula

$$[B_H^\beta](r) = \frac{r}{\Gamma(\frac{1}{2}+H)^2 \beta^{1-2H}}.$$

Since the Ornstein—Uhlenbeck diffusions (2.7) are continuous semimartingales, their intrinsic local times $L_t^a[J_\tau]$ can be defined as in Chapter VI of Revuz and Yor (1991) (see also Ikeda and Watanabe, 1981):

$$L_t^a(J_\tau) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}[|J_\tau(s) - a| < \varepsilon] d[J_\tau](s) = \lim_{\varepsilon \to 0} \frac{\sigma^2}{2\varepsilon} \int_0^t \mathbf{1}[|J_\tau(s) - a| < \varepsilon] ds. \tag{2.10}$$

---

where

$$X(\xi, r) = \frac{(2\pi)^{1-2H}}{\Gamma(\frac{1}{2}-H)\Gamma(\frac{1}{2}+H)} |\xi|^{-2H} \int_0^r e^{-(r-s)(2\pi\xi)^2} dB(s),$$

which can derived from (2.3) by a simple change of the integration variable.

[5] The mode of convergence is yet to be established. This important step will be performed in the near future.

Similarly the chronological local times $\overline{L}_t^a(J_\tau)$ (c.f. Phillips and Park, 1998) are defined as

$$\overline{L}_t^a(J_\tau) \equiv \frac{1}{\sigma^2} L_t^a(J_\tau).$$

For diffusions (2.7) generated by the common Brownian motion $B(r)$, the intrinsic and chronological local times of the 'proper' mixtures $B_H^\beta(r)$ are also well defined as follows:

$$\begin{aligned} L_t^a(B_H^\beta) &= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}[|B_H^\beta(s) - a| < \varepsilon] d[B_H^\beta](s) \\ &= \lim_{\varepsilon\to 0} \frac{\sigma^2}{2\varepsilon} \int_0^t \mathbf{1}[|B_H^\beta(s) - a| < \varepsilon] \frac{ds}{\Gamma(\frac{1}{2}+H)^2 \beta^{1-2H}} \end{aligned} \tag{2.11}$$

and

$$\overline{L}_t^a(B_H^\beta) = \frac{\Gamma(\frac{1}{2}+H)^2 \beta^{1-2H}}{\sigma^2} L_t^a(B_H^\beta).$$

We can justify formula (2.11) by the following heuristic argument. Consider the vector random process $J_\tau(r) \equiv (J_{\tau_1}(r), J_{\tau_2}(r), ..., J_{\tau_n}(r))'$ in $\mathbf{R}^n$, where the components of the random vector $\tau \equiv (\tau_1, \tau_2, ..., \tau_n)'$ are mutually independent draws from a gamma distribution with parameters $\frac{1}{2} - H$ and $\frac{1}{\beta}$, scaled by the factor $\frac{1}{\Gamma(\frac{1}{2}+H)\beta^{\frac{1}{2}-H}}$ and independent of $B(r)$. Given the vector τ, the local time spent by process $\overline{J}_\tau \equiv \frac{1}{n}\iota' J_\tau(r)$ at the point a is

$$\begin{aligned} L_t^a(\overline{J}_\tau) &= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}[|\overline{J}_\tau(s) - a| < \varepsilon] d[\overline{J}_\tau](s) \\ &= \sigma^2 \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}[|\overline{J}_\tau(s) - a| < \varepsilon] ds. \end{aligned}$$

As n grows large, the random variable $\overline{J}_\tau$ converges in distribution to the scaled gamma mixture of the diffusions J_{τ_i}, i.e. to B_H^β, and the quadratic variation process $\sigma^2 s$, associated with the discrete diffusion mixture $\overline{J}_\tau(s)$, converges to $\sigma^2 \cdot s$, scaled by the factor $\frac{1}{\Gamma(\frac{1}{2}+H)^2\beta^{1-2H}}$.

Finally, we arrive to a natural definition of chronological local time $\overline{L}_t^a(B_H)$ for fractional Brownian motion in terms of the following limit

$$B_H(r) = \lim_{\beta\to 0} B_H^\beta(r).$$

2.8 Proposition *The chronological local time of the fractional Brownian motion (2.4) can be alternatively represented by a limit of chronological local times of continuous mixtures (2.9) as $\beta \to 0$:*

$$\begin{aligned}\overline{L}_t^a(B_H) &\equiv \lim_{\beta\to 0} \overline{L}_t^a(B_H^\beta) = \lim_{\beta\to 0} \frac{\Gamma(\frac{1}{2}+H)^2\beta^{1-2H}}{\sigma^2} L_t^a(B_H^\beta) \\ &= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\left[\lim_{\beta\to 0}\int_0^t \mathbf{1}[|B_H^\beta(s)-a|<\varepsilon]ds\right]. \qquad (2.12)\end{aligned}$$

Since, by the Ito—Tanaka formula,

$$L_t^a(B_H^\beta) = |B_H^\beta(t)-a| - |B_H^\beta(0)-a| - \int_0^t \operatorname{sgn}[B_H^\beta(s)-a]dB_H^\beta(s),$$

the chronological local time of fractional Brownian motion $B_H(r)$ can be equivalently defined as

$$\overline{L}_t^a(B_H) = \lim_{\beta\to 0}\frac{\Gamma(\frac{1}{2}+H)^2\beta^{1-2H}}{\sigma^2}\left[|B_H^\beta(t)-a| - |B_H^\beta(0)-a| - \int_0^t \operatorname{sgn}[B_H^\beta(s)-a]dB_H^\beta(s)\right], \qquad (2.13)$$

where

$$\begin{aligned}dB_H^\beta(s) &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)}\int_0^\infty dJ_\tau(s)e^{-\beta\tau}\tau^{-(H+\frac{1}{2})}d\tau \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)}\left[-\int_0^\infty J_\tau(s)e^{-\beta\tau}\tau^{\frac{1}{2}-H}d\tau ds + \int_0^\infty e^{-\beta\tau}\tau^{-(H+\frac{1}{2})}d\tau dB(s)\right] \\ &= \frac{1}{\Gamma(\frac{1}{2}+H)}\left[-(\frac{1}{2}-H)B_{H-1}^\beta(s)ds + \frac{1}{\beta^{\frac{1}{2}-H}}dB(s)\right].\end{aligned}$$

Note that the limit in the right-hand side of (2.13) exists and is *a.s.* finite by construction (it is *a.s.* bounded by t).[6]

In the next section we review asymptotic properties of the local time functional (2.12), and sketch the proofs of limit theorems for other continuous functionals of fractional Brownian motion. Extensive coverage of the limit theory and kernel integrals of local times, as well as non-parametric density estimation for non-stationary fractional Brownian motions and potential applications of our results in economics and finance, are beyond the scope of the present paper. A comprehensive treatment of these issues is ongoing and will be covered in future papers.

[6]The appropriate mode of convergence is likely to be the same as in footnote 4.

3. Limit Theorems for Functionals of Fractional Brownian Motion

3.1 Lemma (A version of occupation time formula (2.2)) *Let $B_H^\beta(r)$ be a continuous mixture (*2.9*) of Ornstein—Uhlenbeck diffusions approximating fractional Brownian motion $B_H(r)$, $0 < H < \frac{1}{2}$. Then*

$$\int_0^t f(B_H^\beta(s), s)ds = \frac{\Gamma(\frac{1}{2}+H)^2\beta^{1-2H}}{\sigma^2}\int_{-\infty}^{+\infty} da \int_0^t f(a,s)d_s L_s^a(B_H^\beta) \tag{3.1}$$

for every Borel function f. Taking the limit of (3.1), we obtain

$$\int_0^t f(B_H(s), s)ds = \int_{-\infty}^{+\infty} da \int_0^t f(a,s)d_s \overline{L}_s^a(B_H). \tag{3.2}$$

If f has only a single argument a, formula (3.2) simplifies to:

$$\int_0^t f(B_H(s))ds = \int_{-\infty}^{+\infty} [f(a) \lim_{\beta\to 0} \overline{L}_t^a(B_H^\beta)]da = \int_{-\infty}^{+\infty} f(a)\overline{L}_t^a(B_H)da. \tag{3.3}$$

3.2 Theorem *If f is a piecewise continuous and integrable function of spatial location a with*

$$\overline{f} \equiv \int_{-\infty}^{+\infty} f(a)da \neq 0,$$

then

$$\frac{1}{\lambda^{1-H}}\int_0^{\lambda t} f(B_H(s))ds \xrightarrow{d} \overline{f}\ \overline{L}_t^0(B_H) \tag{3.4}$$

as $\lambda \to \infty$.

Development of the asymptotic theory for integral functionals like

$$\int_0^{\lambda t} f(B_H(s))ds \tag{3.5}$$

in the zero charge case $\overline{f} = 0$ is substantially more difficult. The crucial obstacle here is the absense of a simple analogue of Ito's lemma for functionals of a fractional Brownian motion with integration parameter $H < \frac{1}{2}$. In Conjecture 3.3 below, we conjecture that the behavior of the zero charge integral functional (3.5) of fractional Brownian motion is analogous to the case of ordinary Brownian motion, which is covered extensively in the existing literature (e.g., Ikeda and Watanabe, 1981) and used in Phillips and Park (1998).

3.3 Conjecture *If f is a piecewise continuous and integrable zero-charge function and*

$$\int_{-\infty}^{+\infty} a^2 f(a)da < \infty,$$

then

$$\lambda^{-\frac{1}{2}(1-H)} \int_0^{\lambda t} f(B_H(s))ds \xrightarrow{d} \frac{\sqrt{\{f,f\}}}{\sigma} U(\overline{L}_t^0(B_H))$$

as $\lambda \to \infty$, where $\{f,f\}$ is a constant and U is an ordinary Brownian motion independent of B.

3.4 Remark In the case of ordinary Brownian motion, the value of constant $\{f,f\}$ is the double 'energy' functional

$$2\langle f,f\rangle \equiv -2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |a-b| f(a)f(b)dadb$$

applied to the 'charge' distribution f.

3.5 Remark Theorem 3.2 and Conjecture 3.3 formulated above can be expected to play important role in the development of an asymptotic theory for nonlinear functionals of normalized integrated processes ζ_t of the form

$$\frac{1}{n}\sum_{t=1}^{n} f(\zeta_t).$$

Such quantities arise in many statistical applications, including nonlinear regression, and they can be approximated by integrals of the form (3.5) as $n \to \infty$. A comprehensive analysis of such asymptotic approximations in the case where $\zeta_t = \frac{1}{\sqrt{n}}B(t)$ has been conducted by Park and Phillips (1999). Theorem 3.2 and expression (3.4) are important in determining the asymptotic distribution of the functional and the rate of convergence, the latter approaching the stationary rate when H is small. We illustrate the analysis in the following discussion.

3.6 Sample Mean Asymptotics We start with the consideration of sample means of nonlinear functions of a fractionally integrated time series. Such functions arise naturally in the study of nonlinear regressions with fractional processes. Let x_t be a fractionally integrated time series generated by

$$(1-L)^d x_t = \varepsilon_t, \text{ with } \varepsilon_t \equiv iid\,(0,1) \text{ and } 1 \geq d > \frac{1}{2},$$

and define the process $X_n(r) = n^{-(d-1/2)}x_{[nr]}$ for $0 \leq r \leq 1$ with $[\cdot]$ denoting the greatest integer part. Then, under certain conditions (see Akonom and Gourieroux, 1987, and Theorem **[ADD THEOREM]** below) $X_n \to_d B_H$, where B_H is a fractional Brownian motion with Hurst exponent $H = d - \frac{1}{2}$. If f is an integrable function (with integral $\overline{f} \equiv \int_{-\infty}^{\infty} f(s)ds \neq 0$) and $d = 1$, then Park and Phillips (1999) show that the sample mean function $\sum_{t=1}^{n} f(x_t)$ has the following asymptotic behavior as $n \to \infty$

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} f(x_t) \to_d \int_{-\infty}^{\infty} f(a) da \overline{L}_t^0(B), \tag{3.6}$$

where $\overline{L}_t^0(B)$ is the chronological local time of standard Brownian motion B at the origin over the time interval $(0,1)$. When $d = 0$ and x_t is strictly stationary and ergodic with time invariant density $pdf\,(x)$, by the ergodic theorem we have

$$\frac{1}{n} \sum_{t=1}^{n} f(x_t) \to_{a.s.} E\left(f\left(x_t\right)\right) = \int_{-\infty}^{\infty} f(a) pdf(a) da. \tag{3.7}$$

The two results are quite different, and even involve different rates of convergence. Nevertheless, the forms of (3.6) and (3.7) are closely related, both involving a similar representation in terms of (differently defined) spatial distributions. So, what is the behavior of the sample mean function $\sum_{t=1}^{n} f(x_t)$ when $\frac{1}{2} < d < 1$, or, equivalently, when $0 < H < \frac{1}{2}$? This can be answered by the following heuristic development.

Write

$$\begin{aligned}
& \frac{1}{n^{1-H}} \sum_{t=1}^{n} f(x_t) \\
\sim\ & \frac{1}{n^{1-H}} \sum_{a=\min_{t \leq n}(x_t)}^{\max_{t \leq n}(x_t)} f\left(a\right) 2\delta \times \frac{1}{2\delta} \#\{t : x_t \in (a-\delta; a+\delta];\ t = 1, ..., n\} \\
=\ & \frac{1}{n^{1-H}} \sum_{a=\min_{t \leq n}(x_t)}^{\max_{t \leq n}(x_t)} f\left(a\right) 2\delta \times \frac{1}{n^H} \frac{1}{2\delta/n^H} \#\left\{t : \frac{x_t}{n^H} \in \left(\frac{a}{n^H} - \frac{\delta}{n^H}; \frac{a}{n^H} + \frac{\delta}{n^H}\right];\ t = 1, ..., n\right\} \\
=\ & \sum_{a=\min_{t \leq n}(x_t)}^{\max_{t \leq n}(x_t)} f\left(a\right) 2\delta \times \frac{1}{n} \frac{1}{2\varepsilon} \#\left\{t : \frac{x_t}{n^H} \in \left(\frac{a}{n^H} - \varepsilon; \frac{a}{n^H} + \varepsilon\right];\ t = 1, ..., n\right\} \\
=\ & \sum_{a=\min_{t \leq n}(x_t)}^{\max_{t \leq n}(x_t)} f\left(a\right) 2\delta \times \frac{1}{n} \frac{1}{2\varepsilon} \sum_{t=1}^{n} \mathbf{1}\left\{\frac{x_t}{n^H} \in \left(\frac{a}{n^H} - \varepsilon; \frac{a}{n^H} + \varepsilon\right]\right\},
\end{aligned} \tag{3.8}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. As explained in Park and Phillips (1999), the essential simplification that is involved in the transition to the form (3.8) is that it converts a nonlinear function of x_t into a formulation which involves x_t linearly, inside the indicator function, so that it can be standardized by an appropriate power of n, here n^{1-H}.

Now, as $n \to \infty$, we note that $\max_{t \le n}(x_t) \to \infty$, $\min_{t \le n}(x_t) \to -\infty$, so that for large n and small δ we have the spatial approximation

$$\sum_{a=\min_{t \le n}(x_t)}^{\max_{t \le n}(x_t)} f(a)\, 2\delta \sim \int_{-\infty}^{\infty} f(a)\, da. \tag{3.9}$$

Also for all finite a we have

$$\begin{aligned} & \frac{1}{n}\frac{1}{2\varepsilon}\sum_{t=1}^{n} \mathbf{1}\left\{\frac{x_t}{n^H} \in \left(\frac{a}{n^H} - \varepsilon; \frac{a}{n^H} + \varepsilon\right]\right\} \\ \sim \quad & \frac{1}{n}\frac{1}{2\varepsilon}\sum_{t=1}^{n} \mathbf{1}\left\{\frac{x_t}{n^H} \in (-\varepsilon; \varepsilon]\right\} \\ \sim \quad & \frac{1}{2\varepsilon}\int_0^1 \mathbf{1}\left\{|X_n(r)| \le \varepsilon\right\} dr \\ \sim \quad & \frac{1}{2\varepsilon}\int_0^1 \mathbf{1}\left\{|B_H(r)| \le \varepsilon\right\} dr. \end{aligned} \tag{3.10}$$

>From these heuristics we get the approximate expression

$$\frac{1}{n^{1-H}}\sum_{t=1}^{n} f(x_t) \sim \left(\int_{-\infty}^{\infty} f(a)\, da\right)\left(\frac{1}{2\varepsilon}\int_0^1 1(|B_H(r)| \le \varepsilon) dr\right), \tag{3.11}$$

which is given in terms of the product of a spatial integral and a functional of the limiting fractional Brownian motion process. Note that the resulting formula is free of the sample size, so that the order of the magnitude of the sample function $\sum_{t=1}^{n} f(x_t)$ is now properly determined as being of $O_p(n^{1-H})$.

The final step in these heuristics is to simplify (3.11). Noting that ε was arbitrary, we can let $\varepsilon \to 0$ in (3.11). The natural limit as $\varepsilon \to 0$ measures the spatial density of the fractional Brownian motion limit process B_H over the time interval $[0, 1]$. Specifically, the limit is

$$\overline{L}_t^0(B_H) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon}\int_0^1 1(|B_H(r)| \le \varepsilon) dr, \tag{3.12}$$

or the chronological local time of fractional Brownian motion at the origin (c.f. equation (2.10) above). This analysis suggests the result

$$\frac{1}{n^{1-H}}\sum_{t=1}^{n} f(x_t) \to_d \left(\int_{-\infty}^{\infty} f(a)\, da\right)\overline{L}_t^0(B_H). \tag{3.13}$$

Formula (3.13) provides an interface in terms of both the rate of convergence and the limiting result between the stationary long memory ($d < \frac{1}{2}$) case and the nonstationary unit root case($d = 1$). However, a remaining issue is the transition in the integral (3.13), which involves spatial characteristics of the limit process only at the origin (via the local time $\overline{L}_t^0(B_H)$), to the integral (3.7), which involves the stationary density over the full support of x_t. This will involve a careful analysis of the simultaneous use of the approximations in (3.9) and (3.10).

4. Strong Approximation of Local Time

A. Notation and basic assumptions

Let ξ_k be a sequence of independent, identically distributed random variables with

$$\mathbf{E}\xi_k = 0,\ \mathbf{Var}(\xi_k) = 1 \tag{4.1}$$

and c_k a sequence of real numbers, $k = 0, \pm 1, \pm 2, ...,$

$$\sum_{k=-\infty}^{+\infty} c_k^2 = 1,$$

$$X_u = \sum_{k=-\infty}^{+\infty} c_{u-k}\xi_k,\ S_n^m = \sum_{u=1}^{n} X_{m+u},\ S_n \equiv S_n^0. \tag{4.2}$$

Construct the sequence of stochastic processes

$$\zeta_n^m(t) = \frac{1}{\sqrt{\mathbf{Var}(S_n^m)}} \sum_{u=1}^{[nt]} X_{m+u} = \sum_{k=-\infty}^{+\infty} a_{k,[nt]}^{m,n}\xi_k,\ \zeta_n(t) \equiv \zeta_n^0(t), \tag{4.3}$$

$$a_{k,u}^{m,n} = (c_{m+1-k} + c_{m+2-k} + ... + c_{m+u-k})/\sqrt{\mathbf{Var}(S_n^m)}, \tag{4.4}$$

which is a sequence of random step functions on the interval $[0;1]$ with jumps at points

$$(\frac{u}{n}, \frac{S_u^m}{\sqrt{\mathbf{Var}(S_n^m)}}),\ u = 1, 2, ..., [nt]. \tag{4.5}$$

The following additional assumptions are useful throughout the remainder of the paper.

4.1 Assumption A (A1) *For some* $q > 2p > 2\max(\frac{1}{H}, 2)$, $\mathbf{E}|\xi_k|^q < \infty$.

(A2) $\mathbf{Var}(S_n) = n^{2H}L(n)$, *where* $L(n)$ *is a slowly varying function.*

(A3) *The distribution of* ξ_k *is absolutely continuous with respect to the Lebesque measure and has characteristic function* $\phi(t) = \mathbf{E}\left[e^{it\xi_k}\right]$ *for which*

$$\lim_{t\to\infty} t^\beta \phi(t) = 0$$

for some $\beta > 0$.

Denote by $W_H(t)$, $H \in (0;1)$, a Gaussian process on $\mathbf{R}_+$ with covariance kernel

$$R_H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right), \tag{4.6}$$

initialized by $W_H(t) \equiv 0$ for $t \leq 0$. By definition, the stochastic process $W_H(t)$ with this covariance kernel is a continuous-time fractional Brownian motion with Hurst exponent H.

As shown by Gorodetskii (1977), under assumptions (A1), (A2), and (A3), $\zeta_n^0(t)$ weakly converges in distibution to a fractional Brownian motion $W_H(t)$ as $n \to \infty$. Similarly, the vector process

$$(\zeta_{n_1}^0(t), \zeta_{n_2}^{N_1}(t), ..., \zeta_{n_q}^{N_{q-1}}(t))$$

converges in distribution to

$$(b_1 W_H(n_1 t), b_2 W_H(N_1 + n_2 t), ..., b_q W_H(N_{q-1} + n_q t)), \tag{4.7}$$

where factors $(b_1, b_2, ..., b_q)$ are chosen to satisfy the condition

$$\mathbf{Var}(b_i(W_H(N_i) - W_H(N_{i-1}))) = 1$$

for $i = 1, 2, ..., q$, that is,

$$b_i = (N_i^{2H} - N_{i-1}^{2H})^{-1/2},\ i = 1, 2, ..., q.$$

B. Local limit theorem for densities of dependent processes

The purpose of this section is to develop a version of a local limit theorem. This asserts that under certain conditions the multivariate density of a partial sum process together with all partial derivatives of the density up to a fixed (and arbitrarily large) order converge uniformly to the multivariate density of the weak limit of these partial sums and, correspondingly, to the partial derivatives of the density. In the next section, the theorem is used to obtain asymptotic bounds for moments of empirical occupation measures $\mathbf{Z}_n^k$ defined by equation (4.14) below.

Denote by f the density of the scalar random variable ξ_k and its characteristic function by $\phi(t) = \mathbf{E}\left[e^{it\xi_k}\right]$. For any multi-index $\mathbf{n} = (n_1, n_2, ..., n_r)$, let $\mathbf{N} = (N_1, N_2, ..., N_r)$, where

$$N_j = \sum_{i=1}^{j} n_i \text{ for } j = 1, 2, ..., r, \text{ and } N_0 \equiv 0.$$

Let $g_{\mathbf{n}}$ be the multivariate density of the vector of partial sums

$$(\zeta_{n_1}^0(1), \zeta_{n_2}^{N_1}(1), \zeta_{n_3}^{N_2}(1), ..., \zeta_{n_r}^{N_{r-1}}(1)), \tag{4.8}$$

and $\varphi_{\mathbf{n}}$ be the density of a zero-mean Gaussian random vector with covariance matrix $\Sigma_r(\mathbf{n})$ defined by its elements

$$\sigma_r(\mathbf{n})_{ij} = \begin{cases} 1, & i = j, \\ \frac{1}{2}\{\mathbf{Var}(S_{N_j - N_{i-1}}^{N_{i-1}}) + \mathbf{Var}(S_{N_{j-1} - N_i}^{N_i}) - \mathbf{Var}(S_{N_j - N_i}^{N_i}) & \\ -\mathbf{Var}(S_{N_{j-1} - N_{i-1}}^{N_{i-1}})\}/(\mathbf{Var}(S_{n_i}^{N_{i-1}}))^{1/2}(\mathbf{Var}(S_{n_j}^{N_{j-1}}))^{1/2}, & i < j, \\ \sigma_r(\mathbf{n})_{ji}, & i > j. \end{cases} \tag{4.9}$$

Note that $\Sigma_r(\mathbf{n})$ is the covariance matrix of vector (4.8).

4.2 Theorem *Let $(\xi_k)_{k=-\infty}^{+\infty}$ be a sequence of i.i.d. random variables (4.1). Let $(S_t)_{t\in\mathbb{N}}$ defined by (4.2) satisfy assumptions (A1), (A2), and (A3). Then for all $\varepsilon > 0$, any integer $r \geq 1$, and an arbitrary multiindex $\mathbf{k} = (k_1, k_2, ..., k_r)$ there exists an integer $m = m(\mathbf{k}, \phi)$ such that for all $\mathbf{n} = (n_1, n_2, ..., n_r) > \mathbf{m}(\mathbf{k}, \phi) \equiv (m, m, ..., m)$ the density $g_{\mathbf{n}}(y_1, y_2, ..., y_r)$ belongs to the class $C^{\mathbf{k}}(\mathbf{R}^r)$, and for all $\mathbf{k}' \leq \mathbf{k}$,*

$$\sup_{y\in\mathbf{R}^r} \left| D^{\mathbf{k}'} g_{\mathbf{n}}(y_1, y_2, ..., y_r) - D^{\mathbf{k}'} \varphi_{\mathbf{n}}(y_1, y_2, ..., y_r) \right| < \varepsilon. \tag{4.10}$$

4.3 Remark The local limit theorem for densities in the form given above generalizes the result developed originally by Gnedenko (1954) for sequences of sums S_n generated by i.i.d. random variables $(\zeta_t)_{t\in\mathbb{N}}$. In our setup, Gnedenko's theorem appears as a special case when $c_0 = 1$ and $c_k = 0$ for all $k \neq 0$. In the case of scalar dependent variables ($r = 1$) the convergence of $g_n(y)$ to the normal density was investigated by many authors (Davydov and Shukri, 1975; Gorodetskii, 1977, and others). The relationship between the weak convergence of distributions of normalized sums to an absolutely continuous distribution and the validity of local limit theorems for densities and domains was studied by Mukhin (1996). It can be verified that the necessary conditions given by Mukhin are satisfied in our case.

4.4 Corollary *Under assumptions of Theorem 2.1, denote by $f_{\mathbf{n}}$ the density of random vector $(S_{n_1}^0, S_{n_2}^{N_1}, S_{n_3}^{N_2}, ..., S_{n_r}^{N_{r-1}})$. Then for all integer $r \geq 1$ and arbitrary $\mathbf{k} = (k_1, k_2, ..., k_r)$,*

$$\sup_{y\in\mathbf{R}^r} \left| D^{\mathbf{k}} f_{\mathbf{n}}(y_1, y_2, ..., y_r) \right| = O\left(\prod_{i=1}^{r} n_i^{-H(1+k_i)} \right) \tag{4.11}$$

as $\mathbf{n} \to \infty$.

C. Comparison of occupation measures for finite intervals

This section, whose development follows Akonom (1993), establishes an auxiliary property for even moments of the variable $\mathbf{Z}_n$ defined below by (4.14). This property allows us to derive the main result of the next section, giving uniform upper bounds for the difference between the observed discrete occupation measure of empirical process X_t at the origin and the local time of its strong limit in an appropriate probability space.

Start by defining the integer random variable

$$\mathbf{N}_n(a, \delta) = \sum_{t=1}^{n} \mathbf{1}_{[a;a+\delta)}(S_t), \tag{4.12}$$

representing the number of visits of the accumulated sum process S_t to the interval $[a; a + \delta)$ up to time n. Also define the random function

$$\mathbf{M}_n(z; a, \delta) = \sum_{t=1}^{n} \exp(iz \cdot S_t) \mathbf{1}_{[a;a+\delta)}(S_t) \tag{4.13}$$

Denote by $\mathbf{I}_k(x)$ the indicator function of interval $[ka;(k+1)a)$, $k = 0,1,2,....$ Let $\mathbf{U}_k(x) = \mathbf{I}_0(x) - \mathbf{I}_k(x)$, so that

$$\mathbf{Z}_n^k = \mathbf{N}_n(a,\delta) - \mathbf{N}_n(a+k\delta,\delta) = \sum_{t=1}^{n} \mathbf{U}_k(S_t), \tag{4.14}$$

and

$$\mathbf{Y}_n^k(z) = \mathbf{M}_n(z;a,\delta) - \mathbf{M}_n(z;a+k\delta,\delta) = \exp(iza)\cdot\sum_{t=1}^{n}\exp(iz\cdot S_t)\mathbf{U}_k(S_t). \tag{4.15}$$

Our first technical result is the following.

4.5 Lemma (a) *Let assumptions (*A1*), (*A2*), and (*A3*) hold, $a \in \mathbf{R}$, and $0 < H < \frac{1}{2}$. Then for all $r \in \mathbb{N}$, there exists a constant $c(r,\phi)$, such that for all $k \in \mathbb{N}$ and $\delta > 0$ satisfying condition $\delta n^{1-H} \geq 1$,*

$$\mathbf{E}(\mathbf{Z}_n^k)^{2r} \leq c(r,\phi)\cdot(\delta n^{1-H})^r(1+k\delta^2 n^{1-2H})^r. \tag{4.16}$$

(b) *Under assumptions of part (a), for all $r \in \mathbb{N}$ there exists a constant $c(r,\phi)$, such that for all $k \in \mathbb{N}$ and $\delta > 0$ satisfying condition $\delta n^{1-H} \geq 1$,*

$$\mathbf{E}(\mathbf{Y}_n^k)^{2r} \leq c(r,\phi)\cdot(\delta n^{1-H})^r(1+k\delta^2 n^{1-2H})^r \tag{4.17}$$

uniformly in $z \in \mathbf{R}$.

4.6 Proposition *Let $a \in \mathbf{R}$, $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying inequality $\delta_n n^{1-H} \geq 1$, and $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers $k_n \leq n$. Let assumptions (A1), (A2), and (A3) hold. Then $\forall\varepsilon > 0$, the number of visits to the intervals $[a;a+\delta_n)$ and $[a+\delta_n;a+(k_n+1)\delta_n)$ by accumulated sums S_t $(t = 1,2,...,n)$ satisfies*

$$\mathbf{N}_n(a,\delta_n) - \frac{1}{k_n}\mathbf{N}_n(a+\delta_n,k_n\delta_n) = o((\delta_n n^{1-H}(1+k_n\delta_n^2 n^{1-2H}))^{1/2}n^{\varepsilon}) \text{ a.s.} \tag{4.18}$$

as $n \to \infty$. The similar relationship

$$\mathbf{M}_n(z;a,\delta_n) - \frac{1}{k_n}\mathbf{M}_n(z;a+\delta_n,k_n\delta_n) = o((\delta_n n^{1-H}(1+k_n\delta_n^2 n^{1-2H}))^{1/2}n^{\varepsilon}) \text{ a.s.} \tag{4.19}$$

holds uniformly in $z \in \mathbf{R}$ for random function $\mathbf{M}_n(z;\cdot)$.

4.7 Corollary *If, in addition to the previous assumptions, the sequences $(k_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ satisfy inequality $k_n\delta_n^2 \leq n^{-(1-2H)}$, then $\forall\varepsilon > 0$,*

$$\mathbf{N}_n(a,\delta_n) - \frac{1}{k_n}\mathbf{N}_n(a+\delta_n,k_n\delta_n) = o(\left(\delta_n n^{1-H}\right)^{1/2} n^{\varepsilon}) \text{ a.s.} \tag{4.20}$$

as $n \to \infty$.

D. Strong approximation for the local time of a fractional Brownian motion

This section compares the occupation measure of the partial sum process S_n corresponding to the interval $[0;\delta)$ to the local time of the scaled limit of the partial sums, a continuous-time fractional Brownian motion $W_H(t)$ with Hurst exponent H. Our main result is the following.

4.8 Theorem *Let $(\xi_k)_{k=-\infty}^{+\infty}$ be a sequence of i.i.d. random variables (4.1). Let $(S_t)_{t\in\mathbb{N}}$ be defined by (4.2) and satisfy assumptions (A1), (A2), and (A3). Then one can construct an appropriate probability space with a standard Brownian motion $(B(t),\ t\geq 0)$ and a partial sum process $(S'_t)_{t\in\mathbb{N}}$ defined on that space, for which $(S'_t)_{t\in\mathbb{N}} \overset{\mathcal{L}}{\sim} (S_t)_{t\in\mathbb{N}}$, and for any sequence $(\delta_n)_{n\in\mathbb{N}}$ of real positive numbers satisfying inequalities $(1/n^{1-H}) \leq \delta_n \leq (1/n^{1-H})\cdot n^{\frac{2}{3}(H-\frac{1}{p})}$, it is true that $\forall\varepsilon>0$,*

$$\mathbf{N}_n[0;\delta_n) - \int_0^n \mathbf{1}_{[0;\delta_n)}(W_H(s))ds = o((\delta_n n^{1-H})^{1/2} n^{\varepsilon}) \text{ a.s.} \tag{4.21}$$

Moreover, for $\delta_n \geq (1/n^{1-H})\cdot n^{\frac{2}{3}(H-\frac{1}{p})}$, it is also true that $\forall\varepsilon>0$,

$$\mathbf{N}_n[0;\delta_n) - \int_0^n \mathbf{1}_{[0;\delta_n)}(W_H(s))ds = o(\delta_n n^{(1-H)-\frac{1}{3}(H-\frac{1}{p})+\varepsilon}) \text{ a.s.,} \tag{4.22}$$

where

$$W_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}} dB(s)$$

is a fractional Brownian motion denerated for $0\leq t\leq 1$ on the same probability space by the Brownian motion $B(t)$.

In order to prove the theorem, we need to establish an almost sure invariance principle for fractional Brownian motion.

4.9 Proposition *For a sequence of partial sums $(S_t)_{t=1}^n$ defined by formula (4.2) and satisfying assumptions (A1), (A2), and (A3), one can construct an extended probability space and define a standard Brownian motion $(B(t),\ t\geq 0)$ on that space and a partial sum process $(S'_t)_{t\in\mathbb{N}} \overset{\mathcal{L}}{\sim} (S_t)_{t\in\mathbb{N}}$ on the same space, such that $\forall\varepsilon>0$,*

$$\sup_{0\leq t\leq n} |S'_t - W_H(t)| = o(n^{\frac{1}{p}+\varepsilon}) \text{ a.s.} \tag{4.23}$$

where

$$W_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}} dB(s)$$

is a fractional Brownian motion generated on the same probability space by generic Brownian motion $B(t)$ and initialized by $W_H(t)=0$ for $t\leq 0$.

4.10 Remark The above strong approximation is based on a standard embedding argument and employs the approach by Akonom and Gourieroux (1987), used for the proof of a functional central limit theorem to a fractional Brownian motion.

E. Reformulation of results for rescaled processes

Consider the sequence of normalized partial sums $\zeta_n(t)$ defined by (4.3) for all $t \in [0;1]$. The following invariance principle is a rescaled version of Proposition 4.9.

4.11 Corollary *For a sequence of stochastic processes $\zeta_n(t)$ taking values in $D[0;1]$, the set of cadlag functions on the interval $[0;1]$, one can construct an extended probability space with standard Brownian motion $(B(t),\ 0 \le t \le 1)$ and a partial sum process $\zeta'_n(t) \overset{\mathcal{L}}{=} \zeta_n(t)$ on the same space, such that $\forall \varepsilon > 0$,*

$$\sup_{0\le t\le 1} |\zeta'_n(t) - W_H(t)| = o(n^{-(H-\frac{1}{p})+\varepsilon}) \text{ a.s.} \tag{4.24}$$

where

$$W_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} dB(s)$$

is a fractional Brownian motion generated for $0 \le t \le 1$ on the same probability space by Brownian motion $B(t)$.

For a sequence of positive real numbers $(\theta_n)_{n\in\mathbb{N}}$, the scaled occupation measure of process $\zeta'_n(t)$ is defined as

$$\nu_n(\theta_n; a, \delta) = \int_0^1 \mathbf{1}_{[a;a+\delta)}(\theta_n \zeta'_n(t))dt, \tag{4.25}$$

and, similarly, the scaled occupation measure of $W_H(t)$ is defined as

$$\nu(\theta_n; a, \delta) = \int_0^1 \mathbf{1}_{[a;a+\delta)}(\theta_n W_H(t))dt. \tag{4.26}$$

Since

$$\nu_n(\theta_n; a, \delta) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{1}_{[a;a+\delta)}\left(\frac{\theta_n}{n^H} S'_t\right) = \frac{1}{n} \cdot \mathbf{N}_n\left(\frac{an^H}{\theta_n}, \frac{\delta n^H}{\theta_n}\right)$$

and

$$\begin{aligned}\nu(\theta_n; a, \delta) &= \frac{1}{n}\int_0^n \mathbf{1}_{[a;a+\delta)}\left(\theta_n W_H\left(\frac{s}{n}\right)\right) ds \\ &= \frac{1}{n}\int_0^n \mathbf{1}_{[a;a+\delta)}\left(\frac{\theta_n}{n^H} W'_H(s)\right) ds\end{aligned}$$

$$= \frac{1}{n}\int_0^n \mathbf{1}_{[an^H/\theta_n;(a+\delta)n^H/\theta_n)}(W'_H(s))ds,$$

where $W'_H(s) \equiv n^H W_H\left(\frac{s}{n}\right)$ by definition, one can reformulate Lemma 4.5, Proposition 4.6, and Theorem 4.8 in terms of the quantities $\nu_n(\theta_n; a, \delta)$ and $\nu(\theta_n; a, \delta)$ as follows. The approach is the same as that of Lemma 2.5 in Park and Phillips (1999).

4.12 Lemma *For all $r \in \mathbb{N}$, there exists a constant $c(r, \phi)$, such that for all $k_n \in \mathbb{N}$ and $\delta_n > 0$ satisfying condition $\delta_n/\theta_n \geq 1/n$,*

$$\mathbf{E}(\nu_n(\theta_n; 0, \delta_n) - \nu_n(\theta_n; k_n\delta_n, \delta_n))^{2r} \leq c(r, \phi) \cdot \left(\frac{\delta_n}{n\theta_n}\right)^r \left(1 + \frac{k_n n \delta_n^2}{\theta_n^2}\right)^r. \quad (4.27)$$

4.13 Proposition *Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying condition $\delta_n/\theta_n \geq 1/n$, and $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers $k_n \leq n$. Then $\forall \varepsilon > 0$,*

$$\nu_n(\theta_n; 0, \delta_n) - \frac{1}{k_n}\nu_n(\theta_n; \delta_n, k_n\delta_n) = o((\delta_n/(n\theta_n))^{1/2}(1 + k_n n(\delta_n/\theta_n)^2))^{1/2} n^\varepsilon) \text{ a.s.} \quad (4.28)$$

as $n \to \infty$.

4.14 Theorem *Under the assumptions of Proposition 4.13, if $1/n \leq \delta_n/\theta_n \leq (1/n) \cdot n^{\frac{2}{3}(H-\frac{1}{p})}$*, then $\forall \varepsilon > 0$,

$$\nu_n(\theta_n; 0, \delta_n) = \nu(\theta_n; 0, \delta_n) + o((\delta_n/n\theta_n)^{1/2} n^\varepsilon) \text{ a.s.}; \quad (4.29)$$

if $\delta_n/\theta_n \geq (1/n) \cdot n^{\frac{2}{3}(H-\frac{1}{p})}$, then $\forall \varepsilon > 0$,

$$\nu_n(\theta_n; 0, \delta_n) = \nu(\theta_n; 0, \delta_n) + o((\delta_n/\theta_n) n^{-\frac{1}{3}(H-\frac{1}{p})+\varepsilon}) \text{ a.s.} \quad (4.30)$$

4.15 Remark >From the definition of local time,

$$(\theta_n/\delta_n)\nu(\theta_n; 0, \delta_n) = (\theta_n/\delta_n)\nu(1; 0, \delta_n/\theta_n) \to L_1^0(W_H) \text{ a.s.}$$

as $\delta_n/\theta_n \to 0$. Theorem 4.14 implies that $\forall \varepsilon > 0$,

$$(\theta_n/\delta_n)\nu_n(\theta_n; 0, \delta_n) = L_1^0(W_H) + o(n^{-\frac{1}{3}(H-\frac{1}{p})+\varepsilon}) \text{ a.s.}$$

as $n \to \infty$, as long as $n^{\frac{2}{3}(H-\frac{1}{p})} \leq n\delta_n/\theta_n \leq n^{1-\varepsilon'}$ with some $\varepsilon' > 0$. While $n\nu_n(\theta_n; a, \delta_n)$ is the number of visits of process $\theta_n\zeta'_n(t)$ to the interval $[a; a+\delta_n)$, quantity ν_n approximates the local time of an appropriately defined fractional Brownian motion, when n is large.

5. Conclusion

This paper is an attempt to generalize to the case of fractional Brownian motion the theory of local times developed by Phillips (1998), Park and Phillips (1999), and Phillips and Park (1998) for nonlinear functions of Wiener processes. The contribution of the paper is twofold. First, it suggests an operational extension of semimartingale local times to the case when an intrinsic time scale characterized by the quadratic variation of stochastic process does not exist. Our conceptual approach is based on a new representation of fractional Brownian motion in terms of a continuous mixture of diffusions. The chronological local time of a fractional Brownian motion is then defined by the limit of the sequence of local times of this mixture of diffusions. Second, the paper gives the rate of convergence for a spatial density estimator based on a discrete sample of observations for a wide class of long-memory processes.

Although the focus of our research is on the case of fractional Brownian motion with Hurst exponent H between zero and $\frac{1}{2}$, the methods developed in the paper can be applied to a wider class of nonstationary continuous time processes. A potentially important application, which is beyond the scope of the present paper, involves an investigation of the asymptotic properties of a spatial density estimator based on a large sample of high-frequency observations of a time series characterized by two structural parameters—one describing the high-frequency dynamics (e.g. the local Hölder exponent of trajectories) and the other characterizing the long-range behavior (the asymptotics of the spectral density around the origin).

6. Appendix

6.1 Proof of Theorem 3.2 The proof almost identically replicates the justification of an analogous result of Ikeda and Watanabe (1981) for integral functionals of ordinary Brownian motion B. By the scaling property of FBM, $B_H(\lambda r) \overset{d}{\sim} \lambda^H B_H(r)$ for each $\lambda > 0$. Therefore

$$\begin{aligned}
\frac{1}{\lambda^{1-H}} \overline{L}_{\lambda t}^{\lambda^H a}(B_H) &= \frac{1}{\lambda^{1-H}} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left[\lim_{\beta\to 0} \int_0^{\lambda t} \mathbf{1}[|B_H^\beta(s) - \lambda^H a| < \varepsilon] ds \right] \\
&= \frac{1}{\lambda^{1-H}} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left[\lim_{\beta\to 0} \int_0^{t} \mathbf{1}[|B_H^\beta(\lambda s) - \lambda^{H-1} a| < \varepsilon] \lambda ds \right] \\
&\overset{d}{=} \frac{1}{\lambda^{-H}} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left[\lim_{\beta\to 0} \int_0^{t} \mathbf{1}[|B_H^\beta(s) - a| < \frac{\varepsilon}{\lambda^H}] ds \right] \\
&\overset{d}{=} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left[\lim_{\beta\to 0} \int_0^{t} \mathbf{1}[|B_H^\beta(s) - a| < \varepsilon] ds \right] \\
&= \overline{L}_t^a(B_H)
\end{aligned}$$

for each $\lambda > 0$. Then, by the occupation times formula,

$$\frac{1}{\lambda^{1-H}} \int_0^{\lambda t} f(B_H(s))ds \stackrel{d}{=} \frac{1}{\lambda^{1-H}} \int_{-\infty}^{+\infty} f(a)\overline{L}^a_{\lambda t}(B_H)da$$

$$= \int_{-\infty}^{+\infty} f(a)\overline{L}_t^{a\lambda^{-H}}(B_H)da \stackrel{d}{\to} \overline{f}\ \overline{L}_t^0(B_H).$$

Thus the assertion of theorem 3.2 is proved.■

6.2 Proof of Corollary 4.4 Follows immediately from equality

$$f_{\mathbf{n}}(y_1, ..., y_r) = \left[\prod_{i=1}^{r} \left(\mathbf{Var}(S_{n_i}^{N_{i-1}})\right)^{-1/2}\right] g_{\mathbf{n}}\left((\mathbf{Var}(S_{n_1}))^{-1/2}y_1, ..., (\mathbf{Var}(S_{n_r}^{N_{r-1}}))^{-1/2}y_r\right), \tag{6.1}$$

properties of multivariate Hermite polynomials (Holmquist (1996)), and the result of theorem 4.2.■

6.3 Proof of Lemma 4.5(a) The proof follows closely the approach developed by Akonom (1993) for ordinary Brownian motion and weakly dependent processes.

First, introduce additional notation. For any fixed positive integer vector $\mathbf{r} = (r_1, r_2, ..., r_q)$ characterized by $\dim(\mathbf{r}) \equiv q$ and norm

$$||\mathbf{r}||_1 = \sum_{j=1}^{q} r_j = 2r,$$

and for an arbitrary q-vector $\mathbf{t} = (t_1, t_2, ..., t_q)$ with $1 \le t_1 < t_2 < ... < t_q \le n$, define

$$\mathbf{T}(\mathbf{r}; \mathbf{t}) = \mathbf{T}(r_1, ..., r_q; t_1, ..., t_q) = \mathbf{E}\prod_{j=1}^{q}(\mathbf{U}_k(X_{t_j}))^{r_j}$$

and

$$\mathbf{T}(\mathbf{r}) = \sum_{1 \le t_1 < t_2 < ... < t_q \le n} \mathbf{T}(\mathbf{r}; \mathbf{t}).$$

Note that for any $r \in \mathbb{N}$, the number of distinct positive integer vectors $\mathbf{r} = (r_1, r_2, ..., r_q)$ with fixed norm $||\mathbf{r}||_1 = 2r$ does not exceed 2^{2r-1}. Therefore, in order to prove part (a) of the lemma, it is sufficient to give an upper bound for $\max_{||\mathbf{r}||_1 = 2r} \mathbf{T}(\mathbf{r})$.

It follows immediately from the definition of $(\mathbf{U}_k(x))^{r_j}$ that

$$(\mathbf{U}_k(x))^{r_j} = \begin{cases} \mathbf{I}_0(x) + \mathbf{I}_k(x) \text{ if } r_j \text{ is even,} \\ \mathbf{U}_k(x) \text{ if } r_j \text{ is odd.} \end{cases}$$

Therefore it is sufficient to analyze the case of integer vectors $\mathbf{r} = (r_1, r_2, ..., r_q)$ with $r_j = 1$ or 2 and $||\mathbf{r}||_1 = \sum_{j=1}^{q} r_j \le 2r$, which we assume for the remainder of the proof. For a given vector $\mathbf{r}$, denote by $J_1(\mathbf{r})$ the subset of indices $j \in \{1, 2, ..., q(\mathbf{r})\}$ with $r_j = 1$, and by $J_2(\mathbf{r})$ the subset of indices with $r_j = 2$.

Denote $n_1 = t_1$ and $n_j = t_j - t_{j-1}$ for $2 \le j \le q$. Let $f_{n_1 n_2 ... n_q}(y_1, y_2, ..., y_q)$ be the density function of random vector $(S_{n_1}, S_{t_1+n_2} - S_{t_1}, S_{t_2+n_3} - S_{t_2}, ..., S_{t_{q-1}+n_q} - S_{t_{q-1}})$. The following representations of $\mathbf{T}(\mathbf{r}; \mathbf{t})$ will be useful in the future:

$$\begin{aligned}
\mathbf{T}(\mathbf{r}; \mathbf{t}) &= \int_{\mathbf{R}^q} f_{n_1 n_2 ... n_q}(x_1, x_2 - x_1, ..., x_q - x_{q-1}) \prod_{j=1}^{q} (\mathbf{U}_k(x_j))^{r_j} dx_1 dx_2 ... dx_q \\
&= \int_{\mathbf{R}^q} f_{n_1 n_2 ... n_q}(y_1, y_2, ..., y_q) \prod_{j=1}^{q} (\mathbf{U}_k(y_1 + y_2 + ... + y_j))^{r_j} dy_1 dy_2 ... dy_q \\
&= \int_{\mathbf{R}^q} \prod_{j=1}^{q} f_{n_j | n_1 n_2 ... n_{j-1}}(y_j | y_1, ..., y_{j-1}) (\mathbf{U}_k(y_1 + y_2 + ... + y_j))^{r_j} dy_1 dy_2 ... dy_q.
\end{aligned}$$

Corollary 4.4 to the local limit theorem for densities of dependent processes implies that for any integer vector $\mathbf{r} > 0$ characterized by norm $||\mathbf{r}||_1 = 2r$ and dimension $\dim(\mathbf{r}) = q$, there exist a constant $c(r, \phi)$ and an index $n_0(r, \phi)$ such that for all increasing integer q-sequences $1 \le t_1 < t_2 < ... < t_q \le n$ satisfying inequality

$$(t_1, t_2 - t_1, ..., t_q - t_{q-1}) = (n_1, n_2, ..., n_q) > (n_0, n_0, ..., n_0),$$

the value of $\mathbf{T}(\mathbf{r}; \mathbf{t}) = \mathbf{E} \prod_{j=1}^{q} (\mathbf{U}_k(X_{t_j}))^{r_j}$ does not exceed

$$c(r, \phi) \cdot \prod_{j \in J_1(\mathbf{r})} \min(\delta n_j^{-H}, k\delta^2 n_j^{-2H}) \cdot \prod_{j \in J_2(\mathbf{r})} (\delta n_j^{-H}). \tag{6.2}$$

Moreover, if we denote by $J_0(\mathbf{r})$ the subset of indices $j \in \{1, 2, ..., q(\mathbf{r})\}$ for which $n_j \le n_0$, a similar inequality will be valid for an arbitrary integer q-sequence $1 \le t_1 < t_2 < ... < t_q \le n$:

$$\mathbf{T}(\mathbf{r}; \mathbf{t}) \le c(r, \phi) \cdot \prod_{j \in J_1(\mathbf{r}) \setminus J_0(\mathbf{r})} \min(\delta n_j^{-H}, k\delta^2 n_j^{-2H}) \cdot \prod_{j \in J_2(\mathbf{r}) \setminus J_0(\mathbf{r})} (\delta n_j^{-H}). \tag{6.3}$$

Once we establish (6.2) and (6.3) and take into account condition $\delta n^{1-H} \ge 1$, the desired upper bound for $\mathbf{T}(\mathbf{r})$ is obtained immediately:

$$\begin{aligned}
\mathbf{T}(\mathbf{r}) &= \sum_{1 \le t_1 < t_2 < ... < t_q \le n} \mathbf{T}(\mathbf{r}; \mathbf{t}) \\
&\le c(r, \phi) \prod_{j \in J_1(\mathbf{r})} (n_0 + \sum_{n_j = n_0 + 1}^{n} \min(\delta n_j^{-H}, k\delta^2 n_j^{-2H})) \prod_{j \in J_2(\mathbf{r})} (n_0 + \sum_{n_j = n_0+1}^{n} \delta n_j^{-H}) \\
&\le c(r, \phi) \cdot (\delta n^{1-H})^r (1 + k\delta^2 n^{1-2H})^r.
\end{aligned}$$

The second inequality in the last formula is obtained if we assign δn^{1-H} to one half of the indices $j \in J_1(\mathbf{r})$ and assign $k\delta^2 n^{1-2H}$ to the other half. Note that the set $J_1(\mathbf{r})$ always contains an even number of elements.

To prove inequalities (6.2) and (6.3), we first analyze the case $r = 1$. Then we briefly review of the general case.

Case $r = 1$.

Two subcases are possible, $\mathbf{r} = (2)$ and $\mathbf{r} = (1, 1)$. In the first subcase, consider $\mathbf{t} = (n_1)$, $1 \le n_1 \le n$, and

$$\mathbf{T}((2);(n_1)) = \int f_{n_1}(x_1)(\mathbf{U}_k(x_1))^2 dx_1 = \int f_{n_1}(x_1)(\mathbf{I}_0(x_1) + \mathbf{I}_k(x_1))dx_1.$$

By the corollary of the local limit theorem for dependent variables, there exist integer n_0 and a real constant C, such that for all $n_1 > n_0$,

$$\sup_{x_1 \in \mathbf{R}} |f_{n_1}(x_1)| \le C \cdot n_1^{-H}$$

and

$$\int f_{n_1}(x_1)(\mathbf{I}_0(x_1) + \mathbf{I}_k(x_1))dx_1 \le C \cdot n_1^{-H} \int (\mathbf{I}_0(x_1) + \mathbf{I}_k(x_1))dx_1 = 2C\delta \cdot n_1^{-H}.$$

For $n_1 \le n_0$, using the fact that $|\mathbf{U}_k(x_1)| \le 1$, we obtain

$$\int f_{n_1}(x_1)(\mathbf{U}_k(x_1))^2 dx_1 \le \int f_{n_1}(x_1)dx_1 = 1.$$

Therefore,

$$\mathbf{T}((2);(n_1)) \le \begin{cases} C\delta \cdot n_1^{-H}, & \text{for } n_1 > n_0; \\ 1, & \text{for } n_1 \le n_0. \end{cases}$$

Next analyze the second subcase, $\mathbf{r} = (1, 1)$, $\mathbf{t} = (t_1, t_2) = (n_1, n_1 + n_2)$, and $1 \le t_1 < t_2 \le n$. Then

$$\mathbf{T}((1,1);\mathbf{t}) = \iint f_{n_1 n_2}(x_1, x_2 - x_1)\mathbf{U}_k(x_1)\mathbf{U}_k(x_2)dx_1 dx_2$$

$$= \iint f_{n_1 n_2}(x_1, x_2 - x_1)(\mathbf{I}_0(x_1) - \mathbf{I}_k(x_1))(\mathbf{I}_0(x_2) - \mathbf{I}_k(x_2))dx_1 dx_2$$

$$= \iint (f_{n_1 n_2}(x_1, x_2 - x_1) - f_{n_1 n_2}(x_1, x_2 - x_1 + k\delta))\mathbf{I}_0(x_1)\mathbf{I}_0(x_2)dx_1 dx_2$$

$$+ \iint (f_{n_1 n_2}(x_1 + k\delta, x_2 - x_1) - f_{n_1 n_2}(x_1 + k\delta, x_2 - x_1 - k\delta))\mathbf{I}_0(x_1)\mathbf{I}_0(x_2)dx_1 dx_2. \quad (6.4)$$

By Corollary 4.4, there exist an integer n_0 and a real constant C, such that for all $n_1 > n_0$ and $n_2 > n_0$, function $f_{n_1 n_2}$ is differentiable and

$$\sup_{(y_1, y_2) \in \mathbf{R}^2} |D_{y_2} f_{n_1 n_2}(y_1, y_2)| \le C \cdot n_1^{-H} n_2^{-2H}.$$

By the mean value theorem, for all pairs $(n_1, n_2) > (n_0, n_0)$,

$$\begin{aligned}
&\iint |f_{n_1 n_2}(x_1, x_2 - x_1) - f_{n_1 n_2}(x_1, x_2 - x_1 + k\delta)|\mathbf{I}_0(x_1)\mathbf{I}_0(x_2)dx_1 dx_2 \\
\leq\ & Cn_1^{-H} n_2^{-2H} \cdot k\delta \iint \mathbf{I}_0(x_1)\mathbf{I}_0(x_2)dx_1 dx_2 = Ck\delta^3 \cdot n_1^{-H} n_2^{-2H}. \qquad (6.5)
\end{aligned}$$

For $n_1 \leq n_0$ and $n_2 > n_0$, apply the corollary to the local limit theorem to the first derivative of $f_{n_2}(y_2)$, which satisfies inequality

$$\sup_{y_2 \in \mathbf{R}} |D_{y_2} f_{n_2}(y_2)| \leq C \cdot n_2^{-2H}.$$

After the change of variables $y_1 = x_1$ and $y_2 = x_2 - x_1$, we obtain

$$\begin{aligned}
&\iint |f_{n_1 n_2}(y_1, y_2) - f_{n_1 n_2}(y_1, y_2 + k\delta)|\mathbf{I}_0(y_1)\mathbf{I}_0(y_1 + y_2)dy_1 dy_2 \\
=\ & \iint |f_{n_1|n_2}(y_1|y_2) f_{n_2}(y_2) - f_{n_1|n_2}(y_1|y_2 + k\delta) f_{n_2}(y_2 + k\delta)|\mathbf{I}_0(y_1)\mathbf{I}_0(y_1 + y_2)dy_1 dy_2 \\
\leq\ & \sup_{y_2 \in \mathbf{R}} \int f_{n_1|n_2}(y|y_2)\mathbf{I}_0(y)dy \cdot \sup_{y_1 \in \mathbf{R}} \int |f_{n_2}(y) - f_{n_2}(y + k\delta)|\mathbf{I}_0(y_1 + y)dy \\
\leq\ & Cn_2^{-2H} k\delta \cdot \max_{y_1 \in \mathbf{R}} \int \mathbf{I}_0(y_1 + y)dy \leq Ck\delta^2 n_2^{-2H}. \qquad (6.6)
\end{aligned}$$

Whenever $n_2 \leq n_0$, the corollary to the local limit theorem yields, in a similar manner,

$$\begin{aligned}
&\iint |f_{n_1 n_2}(y_1, y_2) - f_{n_1 n_2}(y_1, y_2 + k\delta)|\mathbf{I}_0(y_1)\mathbf{I}_0(y_1 + y_2)dy_1 dy_2 \\
\leq\ & \iint (f_{n_2|n_1}(y_2|y_1) + f_{n_2|n_1}(y_2 + k\delta|y_1)) f_{n_1}(y_1)\mathbf{I}_0(y_1)\mathbf{I}_0(y_1 + y_2)dy_1 dy_2 \\
\leq\ & 2 \cdot \sup_{z \in \{0; k\delta\}} \sup_{y_1 \in \mathbf{R}} \int f_{n_2|n_1}(y + z|y_1)\mathbf{I}_0(x_1 + y)dy \cdot \int f_{n_1}(y)\mathbf{I}_0(y)dy \\
\leq\ & \begin{cases} 2 \cdot \int f_{n_1}(y)\mathbf{I}_0(y)dy \leq C\delta n_1^{-H} & \text{for } n_1 > n_0; \\ 2 & \text{for } n_1 \leq n_0. \end{cases} \qquad (6.7)
\end{aligned}$$

Integrals

$$\iint (f_{n_1 n_2}(x_1 + k\delta, x_2 - x_1) - f_{n_1 n_2}(x_1 + k\delta, x_2 - x_1 - k\delta))\mathbf{I}_0(x_1)\mathbf{I}_0(x_2)dx_1 dx_2$$

in (**??**) are evaluated by a virtually identical argument.

To summarize the above analysis, substitute upper bounds (6.5), (6.6), and (6.7) in the expression for $\mathbf{T}((1,1);\mathbf{t})$ to obtain

$$\mathbf{T}((1,1);(n_1, n_1 + n_2)) = \begin{cases} Ck\delta^3 \cdot n_1^{-H} n_2^{-2H} & \text{for } \min(n_1, n_2) > n_0; \\ Ck\delta^2 \cdot n_2^{-2H} & \text{for } n_1 \leq n_0 \text{ and } n_2 > n_0; \\ C\delta \cdot n_1^{-H} & \text{for } n_1 > n_0 \text{ and } n_2 \leq n_0; \\ C & \text{for } \max(n_1, n_2) \leq n_0. \end{cases}$$

General case $r > 1$.

Use the representation of density $f_{n_1 n_2 ... n_q}(y_1, y_2, ..., y_q)$ by the product of conditional densities which are all well defined for sufficiently large $(n_1, n_2, ..., n_q) > (n_0, n_0, ..., n_0)$:

$$f_{n_1 n_2 ... n_q}(y_1, y_2, ..., y_q) = \prod_{j=1}^{q} f_{n_j | n_1 n_2 ... n_{j-1}}(y_j | y_1, ..., y_{j-1}).$$

To evaluate

$$\mathbf{T}(\mathbf{r}; \mathbf{t}) = \int_{\mathbf{R}^q} \prod_{j=1}^{q} f_{n_j | n_1 n_2 ... n_{j-1}}(y_j | y_1, ..., y_{j-1})(\mathbf{U}_k(y_1 + y_2 + ... + y_j))^{r_j} dy_1 dy_2 ... dy_q, \tag{6.8}$$

analyze sequentially each of the q terms

$$\int f_{n_j | n_1 n_2 ... n_{j-1}}(y_j | y_1, ..., y_{j-1})(\mathbf{U}_k(y_1 + y_2 + ... + y_j))^{r_j} dy_j$$

on the right-hand side of (6.8).

For any fixed $j \in J(\mathbf{r}) \equiv J_1(\mathbf{r}) \cup J_2(\mathbf{r}) \setminus J_0(\mathbf{r})$, denote by $J(j)$ the subset of indices $\{j' \in J(\mathbf{r}) : j' < j\}$. Let $f(y_j | \mathbf{y}_{J(j)})$ be the density of component y_j conditional on the values of all $y_{j'}$ with j' from $J(j)$. Then for any multiindex $\mathbf{n} = (n_1, n_2, ..., n_q)$, the density $f_{n_1 n_2 ... n_q}(y_1, y_2, ..., y_q)$ can be represented by the product of two terms,

$$f(\mathbf{y}_{J_0(\mathbf{r})} | \mathbf{y}_{J(\mathbf{r})}) \tag{6.9}$$

and

$$f(\mathbf{y}_{J(\mathbf{r})}) = \prod_{j \in J(\mathbf{r})} f(y_j | \mathbf{y}_{J(j)}), \tag{6.10}$$

both of which are well defined.

First, analyze the second term (6.10). If $j \in J_2(\mathbf{r}) \setminus J_0(\mathbf{r})$, then

$$\int f(y_j | \mathbf{y}_{J(j)})(\mathbf{U}_k(y_j + \sum_{j' \in J(j)} y_{j'}))^{r_j} dy_j \tag{6.11}$$

is majorized by $c(r, \phi) \cdot \delta n_j^{-H}$ uniformly in $\mathbf{y}_{J(j)}$, just like it has been done for the case $\mathbf{r} = (2)$. If $j \in J_1(\mathbf{r}) \setminus J_0(\mathbf{r})$, the integral (6.11) can be majorized by $c(r, \phi) \cdot \delta n_j^{-H}$ and by $c(r, \phi) \cdot k\delta^2 n_j^{-2H}$ uniformly in $\mathbf{y}_{J(j)}$, similarly to the case $\mathbf{r} = (1, 1)$. Therefore,

$$\int \prod_{j \in J(\mathbf{r})} f(y_j | \mathbf{y}_{J(j)})(\mathbf{U}_k(y_j + \sum_{j' \in J(j)} y_{j'}))^{r_j} d\mathbf{y}_{J(\mathbf{r})}$$

does not exceed

$$c(r, \phi) \cdot \prod_{j \in J_1(\mathbf{r}) \setminus J_0(\mathbf{r})} \min(\delta n_j^{-H}, k\delta^2 n_j^{-2H}) \cdot \prod_{j \in J_2(\mathbf{r}) \setminus J_0(\mathbf{r})} (\delta n_j^{-H})$$

for some constant $c(r,\phi)$ independent of multiindex $\mathbf{n} = (n_1, n_2, ..., n_q)$.

Finally, evaluate the first term (6.9). Since $f(\mathbf{y}_{J_0(\mathbf{r})}|\mathbf{y}_{J(\mathbf{r})})$ is a density and $|\mathbf{U}_k(y_1 + ... + y_j)| \le 1$ for all j, it is obvious that

$$\int f(\mathbf{y}_{J_0(\mathbf{r})}|\mathbf{y}_{J(\mathbf{r})}) \prod_{j\in J_0(\mathbf{r})} (\mathbf{U}_k(\sum_{j'\le j} y_{j'}))^{r_j} d\mathbf{y}_{J_0(\mathbf{r})} \le 1.$$

Thus, inequality (6.3) and therefore part (a) of Lemma 4.5 are proved.■

6.4 Proof of Lemma 4.5(b) Follows immediately from the result of part (a), since

$$\begin{aligned} |\mathbf{Y}_n^k|^2 &= \sum_{t=1}^{n}\sum_{u=1}^{n} \exp(iz(S_u - S_t)) \cdot \mathbf{U}_k(S_t)\overline{\mathbf{U}}_k(S_u) \\ &\le \sum_{t=1}^{n}\sum_{u=1}^{n} \mathbf{U}_k(S_t)\overline{\mathbf{U}}_k(S_u) = (\mathbf{Z}_n^k)^2. \blacksquare \end{aligned}$$

6.5 Proof of Proposition 4.6 For an arbitrary $\beta > 0$, denote by $A_{n,k}(\beta)$ and $A_n(\beta)$ events

$$A_{n,k}(\beta) = \{|\mathbf{N}_n(a,\delta_n) - \mathbf{N}_n(a + k\delta_n, \delta_n)| > \beta(\delta_n n^{1-H})^{1/2}(1 + k\delta_n^2 n^{1-2H})^{1/2} n^{\varepsilon}\}$$

and

$$A_n(\beta) = \{|\mathbf{N}_n(a,\delta_n) - (1/k_n)\mathbf{N}_n(a+\delta_n, k_n\delta_n)| > \beta(\delta_n n^{1-H})^{1/2}(1 + k_n\delta_n^2 n^{1-2H})^{1/2} n^{\varepsilon}\}.$$

Obviously,

$$A_n(\beta) \subset \bigcup_{k\le k_n} A_{n,k}(\beta).$$

After the application of Markov inequality of order $2r$ and the previous lemma, we obtain

$$\mathbf{P}(A_n(\beta)) \le k_n \frac{c(r,\phi)}{\beta^{2r} n^{2r\varepsilon}}$$

for an arbitrary $\varepsilon > 0$. Finally, relationship (4.18) follows immediately from Borel−Cantelli lemma applied for sufficiently large r to a sequence $\beta_n \to 0$ satisfying

$$\sum_{n=1}^{\infty} \beta_n^{-2r} n^{1-2r\varepsilon} < \infty.$$

The proof of (4.19) replicates identically the above argument, with $\mathbf{N}_n(\cdot)$ substituted everywhere by $\mathbf{M}_n(z;\cdot)$.■

6.6 Proof of Theorem 4.8 Our proof closely follows the line of argument developed in Akonom (1993) and Park and Phillips (1999) for the ordinary Brownian motion. The idea is to majorize the difference

$$\left| \mathbf{N}_n(0,\delta_n) - \int_0^n \mathbf{1}_{[0;\delta_n)}(W_H(s))ds \right| \le A_1(n) + \frac{1}{2k_n} A_2(n) + A_3(n),$$

sequentially evaluating the terms

$$A_1(n) = \left| \mathbf{N}_n(0,\delta_n) - \frac{1}{2k_n}(\mathbf{N}_n(-k_n\delta_n, 2k_n\delta_n)) \right|,$$

$$A_2(n) = \left| \mathbf{N}_n(-k_n\delta_n, 2k_n\delta_n) - \int_0^n \mathbf{1}_{(-k_n\delta_n;k_n\delta_n)}(W_H(s))ds \right|,$$

$$A_3(n) = \left| \frac{1}{2k_n} \int_0^n \mathbf{1}_{(-k_n\delta_n;k_n\delta_n)}(W_H(s))ds - \int_0^n \mathbf{1}_{[0;\delta_n)}(W_H(s))ds \right|$$

using Proposition 4.9 (invariance principle), Proposition 4.6 (comparison of occupation measures), and the law of iterated logarithm for the local time of fractional Brownian motion (Nasyrov, 1984) formulated below. Analyze each term separately in what follows.

By Proposition 4.6,

$$A_1(n) = o\left((\delta_n n^{1-H})^{1/2} (1 + k_n \delta_n^2 n^{1-2H})^{1/2} n^{\varepsilon} \right) \text{ a.s.}$$

To find the order of magnitude of $A_2(n)$, set up intervals $J'_n \subset J_n \subset J''_n$:

$$J_n = (-k_n\delta_n; k_n\delta_n),$$

$$J'_n = \left(-k_n\delta_n + n^{\frac{1}{p}+\varepsilon}; k_n\delta_n - n^{\frac{1}{p}+\varepsilon} \right),$$

$$J''_n = \left(-k_n\delta_n - n^{\frac{1}{p}+\varepsilon}; k_n\delta_n + n^{\frac{1}{p}+\varepsilon} \right),$$

and define the event

$$G_n = \left\{ \int_0^n \mathbf{1}_{J'_n}(W_H(s))ds \le \mathbf{N}_n(-k_n\delta_n, 2k_n\delta_n) \le \int_0^n \mathbf{1}_{J''_n}(W_H(s))ds \right\}$$
$$= \left\{ -\int_0^n \mathbf{1}_{J_n \setminus J'_n}(W_H(s))ds \le \mathbf{N}_n(-k_n\delta_n, 2k_n\delta_n) - \int_0^n \mathbf{1}_{J_n}(W_H(s))ds \le \int_0^n \mathbf{1}_{J_n \setminus J''_n}(W_H(s))ds \right\}.$$

In view of the strong approximation (4.23), $(S_t)_{t=1}^n$ is almost surely of distance less than $n^{\frac{1}{p}+\varepsilon}$ from $W_H(s)$ uniformly over each of the intervals $[t-1;t)$, $t = 1, 2, ..., n$. Then for the event G_n,

$$\mathbf{P}\left(\liminf_{n\to\infty} G_n \right) = 1$$

and

$$\mathbf{P}\left(\limsup_{n\to\infty}\left\{A_2(n)\geq\int\limits_0^n \mathbf{1}_{J_n''\setminus J_n'}(W_H(s))ds\right\}\right)=0.$$

By the occupation times formula for fractional Brownian motion,

$$\int\limits_0^n \mathbf{1}_{J_n''\setminus J_n'}(W_H(s))ds=\int\limits_{J_n''\setminus J_n'} L_n^a(W_H)da.$$

The law of iterated logarithm applied to the local time process $L_n^a(W_H)$ yields

$$\limsup_{n\to\infty}\frac{L_n^a(W_H)}{n^{1-H}(\log\log n)}<c \text{ a.s.}$$

for some constant $c>0$ uniformly in a from any compact subset of $\mathbf{R}$. Therefore, we deduce

$$L_n^a(W_H)=o(n^{(1-H)+\varepsilon}) \text{ a.s.}$$

and

$$A_2(n)=o(n^{\frac{1}{p}+(1-H)+\varepsilon}) \text{ a.s.}$$

The order of magnitude of $A_3(n)$ is found from the application of alternate versions of Lemma 4.5 and Proposition 4.6 with the partial sum process S_t replaced by the fractional Brownian motion $W_H(t)$. The upper bound for $A_3(n)$ is identical to that of $A_1(n)$.

>From the majoration argument presented above, we conclude that

$$\begin{aligned}\eta_n &= \left|\mathbf{N}_n(0,\delta_n)-\int\limits_0^n \mathbf{1}_{[0;\delta_n)}(W_H(s))ds\right| \\ &= o\left((\delta_n n^{1-H})^{1/2}(1+k_n\delta_n^2 n^{1-2H})^{1/2}n^{\varepsilon}+\frac{1}{k_n}n^{\frac{1}{p}+(1-H)+\varepsilon}\right) \text{ a.s.}\end{aligned}$$

If $1/n^{1-H}\leq\delta_n\leq n^{\frac{2}{3}(H-\frac{1}{p})}/n^{1-H}$, then we choose sequence $k_n=n^{\frac{1}{2}(1-H)+\frac{1}{p}}\delta_n^{-\frac{1}{2}}$ to satisfy

$$k_n\delta_n^2 n^{1-2H}=(k_n/n)(\delta_n n^{1-H})^2=(\delta_n n^{1-H})^{\frac{3}{2}}n^{-(H-\frac{1}{p})}\leq 1,$$

and obtain

$$\eta_n=o((\delta_n n^{1-H})^{1/2}n^{\varepsilon}) \text{ a.s.}$$

If $\delta_n\geq n^{\frac{2}{3}(H-\frac{1}{p})}/n^{1-H}$, then we choose sequence $k_n=n^{\frac{2}{3p}+\frac{1}{3}H}\delta_n^{-1}$ to obtain

$$k_n\delta_n^2 n^{1-2H}=(\delta_n n^{1-H})n^{-\frac{2}{3}(H-\frac{1}{p})}\geq 1$$

and

$$\eta_n=o(\delta_n n^{1-H}n^{-\frac{1}{3}(H-\frac{1}{p})+\varepsilon}) \text{ a.s.}$$

Therefore $\forall \varepsilon > 0$,

$$\left| \mathbf{N}_n[0;\delta_n) - \int\limits_0^{\delta_n} L_t^a(W_H)da \right| = o(\delta_n n^{1-\frac{4}{3}H+\frac{1}{3p}+\varepsilon}) \text{ a.s.}$$

The proof of Theorem 4.8 is completed.■

7. References

- Akonom, J. (1993) Comportement asymptotique du temps d'occupation du processus des sommes partielles, *Annales de l'Institut Henri Poincare,* **29**(1), 57–81.
- Akonom, J. and C.Gourieroux (1987) A functional limit theorem for fractional processes. Technical Report #8801, CEPREMAP, Paris.
- Berman, S.M. (1969) Local times and sample function properties of stationary Gaussian processes, *Transactions of American Mathematical Society*, **137**, 277–299.
- Berman, S.M. (1973) Local nondeterminism and local times of Gaussian processes, *Indiana University Mathematical Journal*, **23**, 69–94.
- Bosq, D. (1998) Nonparametric Statistics for Stochastic Processes. New York, Springer-Verlag.
- Carmona, Ph., L.Coutin, and G.Montseny (1998) Applications of a representation of long-memory Gaussian processes. Working Paper, Universite Paul Sabatier, Toulouse.
- Davydov, Yu.A. and E.M.Shukri (1975) Local limit theorems for weighted sums of independent random variables, *Vestnik Leningradskogo Gosudarstvennogo Universiteta,* **13**, 140–142.
- Geman, D. and J.Horowitz (1980) Occupation densities, *Annals of Probability*, **8**(1), 1–67.
- Gnedenko, B.V. (1954) Local limit theorem for densities, *Doklady Akademii Nauk SSSR,* **95**(1), 5–7.
- Gorodetskii, V.V. (1977) On convergence to semi-stable Gaussian processes, *Theory of Probability and Its Applications,* **22**(3), 498–508.
- Gorodetskii, V.V. (1977) Estimates in limit theorems for linearly generated random variables, *Vestnik Leningradskogo Universiteta,* **1**, 23–27.
- Holmquist, B. (1996) The d-variate vector Hermite polynomial of order k, *Linear Algebra and Its Applications,* **237/238**, 155–190.

- Ikeda, N. and S.Watanabe (1981) Stochastic Differential Equations and Diffusion Processes. Amsterdam, North Holland.
- Karatzas, I. and S.Shreve (1991) Brownian Motion and Stochastic Calculus. New York, Springer-Verlag.
- Kôno, N. and N.-R. Shieh (1993) Local times and related sample path properties of certain self-similar properties, *Journal of Mathematics Kyoto University*, **33**(1), 51–64.
- Mandelbrot, B. and J.Van Ness (1968) Fractional Brownian motion, fractional noises and applications, *SIAM Review* **10**(4), 422–437.
- Mukhin, A.B. (1996) Relationship between local and integral limit theorems, *Theory of Probability and Its Applications,* **40**(1), 92–103.
- Nasyrov, F.S. (1984) Iterated logarithm law for local times for a class of Gaussian processes, *Mathematical Notes,* **35**(5/6), 475–477.
- Park, J.Y. and P.C.B.Phillips (1999) Asymptotics for nonlinear transformations of integrated time series, *Econometric Theory,* **15**(2), 269–298.
- Phillips, P.C.B. (1998) Econometric analysis of Fisher's equation. Working Paper, Yale University.
- Phillips, P.C.B. (1999a) Descriptive econometrics for nonstationary time series with empirical illustrations. Working Paper, Yale University.
- Phillips, P.C.B. (1999b) Unit root log periodogram regression. Working Paper, Yale University.
- Phillips, P.C.B. and J.Y.Park (1998) Nonstationary density estimation and kernel autoregression. Working Paper, Yale University.
- Revuz, D. and M.Yor (1991) Continuous Martingales and Brownian Motion. New York, Springer-Verlag.
- Zähle, M. (1997) Integration with respect to fractal functions and stochastic calculus. Working Paper, Universität Jena.