Explaining Stochastic Volatility in Asset Prices*

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Abstract

We develop a theoretical model that replicates three observed phenomena in securities markets: serial correlation in trades; serial correlation in squared price changes (conditional heteroskedasticity); and more persistent serial correlation in trades than in squared price changes. In the model exogenous news is captured by signals that informed agents receive. Agents trade anonymously through a market specialist, who does not receive a signal. We show that entry and exit of informed traders following the arrival of news produces serial correlation in the number of trades and serial correlation in squared price changes. Because the bid-ask spread of the market specialist tends to shrink as individuals trade and reveal their information, the serial correlation in trades is more persistent than the serial correlation in squared price changes.

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1 Introduction

Many asset prices exhibit conditional heteroskedasticity through serial correlation in squared price changes. Statistical models that fit the observed serial correlation in squared price changes are now widely used in empirical finance.[1] Modeling serial correlation in squared price changes has important implications for option pricing and conditional return forecasting. Accurate specification of a statistical model requires knowledge of an economic model that explains why serial correlation is present. Although there is widespread speculation that the arrival of news in financial markets has an important impact on squared price changes, we know of no economic model that links the arrival of news to serial correlation in squared price changes. We provide an economic model that links the behavior of traders in a financial market following the arrival of news, to serial correlation in the squared price changes that arise from the market.

Empirical analysis of financial data reveals several additional features of the data that an economic model should explain. First, there is extensive serial correlation in the number of trades (Harris, 1987) and in total shares traded (Harris, 1987; Andersen, 1996; Brock and LeBaron, 1996). Second, serial correlation in trades is more persistent than serial correlation in squared price changes (Harris, 1987; Andersen, 1996; Steigerwald, 1997).[2] Gallant, Hsieh, and Tauchen (1991) show that if the number of trades in a calendar period is serially correlated, then squared price changes are serially correlated. Although the finding of Gallant, Hsieh, and Tauchen provides an important link between trades and squared price changes, an economic model that links the arrival of news to the empirical features of the data through the actions of traders is needed to accurately specify a statistical model.

We develop an economic model of trade in a financial market that links the arrival of news to serial correlation in trades, and hence, serial correlation in squared price changes. The exogenous arrival of private information (news) is captured by signals that informed agents receive. Agents trade anonymously with a market specialist who does not receive a signal. The market specialist faces an adverse selection problem because the specialist trades with more informed agents with positive probability.

The arrival of private signals has two important effects. First, informed traders enter the market, increasing the number of trades relative to trading periods in which there is no news (such trading periods are commonly referred to as "trading days"). Over trading periods shorter than a trading day, if informed traders have an informational advantage,

[1]For a survey, see Bollerslev, Engle, and Nelson (1993).

[2]Similarly, Tauchen, Zhang, and Liu (1996) report that a price change has more persistent effects on volume than on squared price changes.

then most likely informed traders will have an informational advantage the next period as well since information is revealed slowly over time, due to the presence of liquidity traders. Thus the entry and exit of informed traders implies trades are serially correlated. Second, the market specialist widens the bid-ask spread in response to the possible adverse selection problem. As trade occurs, the market specialist uses Bayes rule to update beliefs and hence the bid and ask. As informed traders trade and reveal their information, the bid-ask spread declines. Because the squared (calendar period) price change is determined by the number of trades in the period and the variance of the price innovation for each trade, positive serial correlation in trades leads to positive serial correlation in squared price changes. Because the bid-ask spread bounds the variance of trade-by-trade price innovations, the declining bid-ask spread reduces the serial correlation in squared price changes without affecting the serial correlation in trades. Thus, serial correlation in trades is more persistent than is serial correlation in squared price changes.

The entry and exit of informed traders after the arrival of private information is a key component of our model. The importance of private information as a determinant of stock price volatility is supported by French and Roll (1986), who conclude that revelation of private information (rather than public information or pricing errors) drives stock price changes. Our model is based on the market microstructure model of Easley and O'Hara (1992) which models the news arrival process. Market microstructure models which do not model the news arrival process generally do not exhibit serial correlation in trades. Glosten and Milgrom (1985) consider only a single news event, so trades are constant and thus serially uncorrelated. Sargent (1993) and Brock and LeBaron (1996) model traders who receive noisy signals. Because traders do not decide to leave the market, trades are serially uncorrelated, although volume generally declines through time.

Several researchers propose alternative explanations for serial correlation in squared price changes. Timmerman (1996) combines rare structural breaks in the dividend process with incomplete learning. Shorish and Spear (1996) show how moral hazard between the owner and manager of a firm generates serial correlation in squared price changes in a Lucas asset pricing model. Den Haan and Spear (1997) show how agency costs and borrowing constraints give rise to wealth effects that yield serial correlation in squared interest rate changes. Dividend based models provide an important first step by directly explaining serial correlation in squared price changes at low frequencies. Serial correlation in such models does not arise from the trading process, since the "no trade" theorems hold. In contrast our model explains how news (say about the dividend process) generates high frequency serial correlation through the trading process.

2 Market Microstructure Model

We consider a pure dealership market. By constraining attention to a pure dealership market we rule out brokerage services provided by the specialist, which in turn implies that all orders are market orders.[3] Inclusion of limit orders requires that we model the investors choice of order, which is beyond our scope. The specialist sets a bid and ask price at which he is willing to buy and sell, respectively, one share of stock. The bid and ask prices are determined so that the specialist earns zero expected profits from each trade. The zero expected profit condition is an equilibrium condition, which arises from the potential free entry of additional market specialists should the bid and ask prices lead to positive expected profits for the specialist. Thus, as in Glosten and Milgrom (1985) and Easley and O'Hara (1992), we assume a Bertrand-style market.

The information structure of the market is as follows. Informed traders have positive probability of learning the true share value before trading starts, while the specialist and uninformed traders do not learn the true share value before trading starts. We define the interval over which asymmetric information is present to be a trading day, although we recognize that the interval need not correspond to one calendar day. At the beginning of each trading day informed traders recieve the signal S_m, where m indexes trading days. Although the specialist and the uninformed do not receive the signal at the beginning of the trading day, at the end of each trading day the signal is revealed and all traders agree upon the share value.

On each trading day the random dollar value per share, V_m, takes one of two values $v_{L_m} < v_{H_m}$ with $P(V_m = v_{L_m}) = \delta$. To ensure the continuity of prices over trading days, $EV_m = v_{m-1}$ if the informed learned the true value of the stock on trading day $m-1$. If the informed did not learn the true value of the stock on trading day $m-1$, then we presume the possible share values are unchanged and $v_{L_m} = v_{L_{m-1}}$ and $v_{H_m} = v_{H_{m-1}}$.

The signals received by informed traders at the start of a trading day are independent across trading days and identically distributed. The signal S_m takes the value: s_H if the informed receive the high signal and learn $V_m = v_{H_m}$, s_L if the informed receive the low signal and learn $V_m = v_{L_m}$, and s_0 if the informed receive the uninformative signal and hence, no private information. The probability that the informed learn the true value of the stock through the signal is θ, so the probability that S_m takes the value s_L is $\delta\theta$.

The trading decisions of the informed are determined completely by the signal that they receive. Conditional on receiving the uninformative signal, informed agents do not trade

[3] Our market specialist does not keep an order book. Bollerslev and Domowitz (1991) relate the variance of prices directly to the spread existing in the order book. As such, they are able to obtain heteroskedasticity without serial correlation in the number of trades.

because of identical preferences. If informed traders receive signal s_L, then informed traders always sell as long as the specialist believes there is a positive probability that the true value is v_{H_m}. If informed traders receive signal s_H, then informed traders always buy as long as the specialist believes there is a positive probability that the true value is v_{L_m}.

While each of the informed traders receives the same signal, and so there is no heterogeneity among informed traders within a trading day, the uninformed traders are not identical. All traders and the market specialist, are risk neutral and rational. To induce uninformed rational traders to trade, there must be some disparity of preferences or endowments across traders. We let ω_i be the rate of time discount for the ith trader. As in Glosten and Milgrom each individual assigns random utility to shares of stock, s, and current consumption, c, as $\omega s V_m + c$.[4] The larger the value of ω the greater is the desire to invest and forego current consumption. We set $\omega = 1$ for the specialist and informed traders. There are three types of uninformed traders, those with $\omega = 1$, who have identical preferences and do not trade, those with $\omega = 0$, who always sell the stock, and those with $\omega = \infty$, who always buy the stock. Among the population of uninformed traders, the proportion with $\omega = 1$ is $1 - \varepsilon$, the proportion with $\omega = \infty$ is $(1 - \gamma)\varepsilon$, and the proportion with $\omega = 0$ is $\gamma\varepsilon$. The trading decisions of the uninformed are determined completely by the value of ω and do not depend on the bid and ask prices.

Traders arrive to the market one at a time, so we index traders by their order of arrival. The probability that the arriving trader is informed is α. A trader arrives, observes the bid and ask prices, and decides whether to buy, sell, or not trade. Let C_i be the random variable that corresponds to the trade decision of trader i. Then C_i takes one of three values: c_A if the ith trader buys one share at the ask, A_i; c_B if the ith trader sells one share at the bid, B_i; and c_N if the ith trader elects not to trade. The sequence of trading decisions is public information. Let Z_i be the publicly available information set after i traders have come to the market. The information set available to the specialist and the uninformed is Z_i.

Because the specialist and the uninformed have the same information set, they have the same learning process. In what follows, we simply refer to the learning process for the specialist, noting that the same process applies to the uninformed. After the action of the trader, the specialist revises beliefs about the signal received by informed traders, and thence about the true value of a share. After the ith trader has come to the market, the specialist's belief that informed traders received a high signal is,

$$P\left(S_m = s_H | Z_i\right) = y_i.$$

[4] We assume an infinite number of traders so that the probability of any player playing more than once is zero. Because V_m is realized at the end of the trading day, V_m is the random share value used to construct a trader's utility at the end of a trading day.

Correspondingly, the specialist's belief that informed traders received a low signal is,

$$P\left(S_m = s_L | Z_i\right) = x_i.$$

By construction, the specialist's belief that informed traders received an uninformative signal is,

$$P(S_m = s_0 | Z_i) = 1 - x_i - y_i.$$

The specialist's beliefs about S_m translate directly into beliefs about the value of a share. If the specialist believes $S_m = s_H$, then the accuracy of the signal implies that the specialist believes $V_m = v_{H_m}$. Similarly, if the specialist believes $S_m = s_L$, then the specialist also believes $V_m = v_{L_m}$. If the specialist believes $S_m = s_0$, then the specialist assigns the unconditional probabilities to the possible values for V_m. To summarize, after the ith trader has come to the market, the specialist's conditional probability that $V_m = v_{H_m}$ is

$$P(V_m = v_{H_m} | Z_i) = y_i + (1 - x_i - y_i)(1 - \delta),$$

while $P(V_m = v_{L_m} | Z_i) = 1 - P(V_m = v_{H_m} | Z_i)$. The action of each trader, even the decision not to trade, conveys information about the signal received by informed traders.

2.1 Determination of Ask and Bid Prices

At the beginning of each trading day, $x_0 = \theta\delta$ and $y_0 = \theta(1 - \delta)$. Let A_1 and B_1 be the initial ask and bid prices, respectively. (Thus A_1 is the ask that the first trader faces.) The equilibrium condition that the specialist earn zero expected profit from each trade provides the equations that determine the quoted prices (B_1, A_1). In essence, the quoted prices set the specialist's expected loss from trade with an informed trader equal to the specialist's expected gain from trade with an uninformed trader. We explicilty derive A_1 (derivation of B_1 follows similar logic). If the first trader trades at the ask, then the specialist's expected loss from trade with an informed trader is

$$\alpha \cdot y_0 \left(A_1 - v_{H_m}\right),$$

where $y_0 \left(A_1 - v_{H_m}\right)$ is the expected loss if the first trader trades at the ask, given that the first trader is informed. Similarly, if the first trader trades at the ask, then the specialist's expected gain from trade with an uninformed trader is

$$(1 - \alpha)\,\varepsilon\,(1 - \gamma)\left\{\left[x_0 + \delta\left(1 - x_0 - y_0\right)\right]\left(A_1 - v_{L_m}\right) + \left[y_0 + (1 - \delta)\left(1 - x_0 - y_0\right)\right]\left(A_i - v_{H_m}\right)\right\}.$$

If expected profits equal zero, then

$$A_1 = \frac{\alpha y_0 v_{H_m} + (1-\alpha)\,\varepsilon\,(1-\gamma)\,E\,(V_m|Z_0)}{\alpha y_0 + (1-\alpha)\,\varepsilon\,(1-\gamma)},$$

where $E\,(V_m|Z_0) = x_0 v_{L_m} + y_0 v_{H_m} + (1 - x_0 - y_0)\,EV_m$. In parallel fashion

$$B_1 = \frac{\alpha x_0 v_{L_m} + (1-\alpha)\,\varepsilon\gamma E\,(V_m|Z_0)}{\alpha x_0 + (1-\alpha)\,\varepsilon\gamma}.$$

The equations for (B_i, A_i) are simply the equations for (B_1, A_1) with y_0 replaced by y_{i-1} and x_0 replaced by x_{i-1} (which implies $E\,(V_m|Z_0)$ is replaced by $E\,(V_m|Z_{i-1})$). As one would expect, both the bid and ask prices increase with y_{i-1} and decrease with x_{i-1}.

It is easy to see that $v_{L_m} \le B_i \le A_i \le v_{H_m}$, with strict inequality unless the specialist is certain the informed received a signal (no adverse selection). Mathematically, the specialist is certain the informed received a signal if $x_{i-1} = 1$ or $y_{i-1} = 1$. It is also easy to see that $B_i \le E\,(V_m|Z_{i-1}) \le A_i$, which follows directly from $v_{L_m} \le E\,(V_m|Z_{i-1}) \le v_{H_m}$.

2.2 Learning Rules

As trading occurs, information accrues to the specialist. In response, the specialist updates the probabilities (x_i, y_i).We begin by examining how the specialist learns from the action of the first trader and explicitly discuss only updating of y_1 (updating of x_1 follows similar logic).[5] The key parameters that govern the speed of learning are α and ε. Then for example, if the first trader trades at the ask

$$y_1 = y_0 \;\; \frac{\alpha + (1-\alpha)\,\varepsilon\,(1-\gamma)}{\alpha y_0 + (1-\alpha)\,\varepsilon\,(1-\gamma)}.$$

As long as $y_0 < 1$, a trade at the ask increases y_1. If $\alpha = 1$ or $\varepsilon = 0$ only informed traders trade, so learning is immediate and $y_1 = 1$. If the first trader trades at the bid

$$y_1 = y_0 \frac{(1-\alpha)\,\varepsilon\gamma}{\alpha x_0 + (1-\alpha)\,\varepsilon\gamma}.$$

As long as $\alpha x_0 > 0$, a trade at the bid decreases y_1. If $\alpha = 1$ or $\varepsilon = 0$ again learning is immediate, so $y_1 = 0$ and $x_1 = 1$. Finally, if the first trader does not trade

$$y_1 = y_0 \frac{(1-\alpha)\,(1-\varepsilon)}{\alpha\,(1 - x_0 - y_0) + (1-\alpha)\,(1-\varepsilon)}.$$

[5] The updating, or learning formulae are derived from Bayes rule in the Appendix.

As long as $\alpha(1-x_0-y_0)>0$, a decision not to trade decreases y_1. If $\alpha=1$, or if $\varepsilon=1$ in which case all uninformed traders trade, then learning is immediate with $y_1=0$ and $x_1=0$.

The learning formulae for y_i are simply the learning formulae for y_1 with y_0 replaced by y_{i-1}. The learning formulae for x_i are:

$$x_i = x_{i-1}\frac{(1-\alpha)\,\varepsilon\,(1-\gamma)}{\alpha y_{i-1}+(1-\alpha)\,\varepsilon\,(1-\gamma)},$$

if trader i trades at the ask;

$$x_i = x_{i-1}\frac{\alpha+(1-\alpha)\,\varepsilon\gamma}{\alpha x_{i-1}+(1-\alpha)\,\varepsilon\gamma},$$

if trader i trades at the bid; and

$$x_i = x_{i-1}\frac{(1-\alpha)\,(1-\varepsilon)}{\alpha\,(1-x_{i-1}-y_{i-1})+(1-\alpha)\,(1-\varepsilon)},$$

if trader i does not trade.

2.3 Effectiveness of Learning

We have posited that the signal is revealed at the end of a trading day. The trading day, which consists of a finite number of trader arrivals, is introduced because in general, an infinite number of trader arrivals are needed before the specialist's uncertainty about the value of S_m is eliminated. To ensure that the learning formulae we described above are useful, we must establish that if there were an infinite number of trader arrivals, the specialist would learn the value of S_m. We establish that the beliefs of all agents converge to the true values. As a result, the bid and ask converge to the strong form efficient value of the share, where the bid and ask prices reflect both the public and private information. Thus the transaction prices also converge to the true value of the stock.

There are three sets of beliefs in our model. The first is the specialist's belief that $S_m=s_H$, which is expressed as the sequence of conditional probabilities $\{y_i\}_{i=1}^{\infty}$. The second is the specialist's belief that $S_m=s_L$, which is expressed as the sequence of conditional probabilities $\{x_i\}_{i=1}^{\infty}$, and finally the third is the belief that $S_m=s_0$, which is expressed as the sequence $\{1-x_i-y_i\}_{i=1}^{\infty}$.

Lemma 1: *The sequence of bid and ask prices, and hence the sequence of transaction prices, converge almost surely to their strong form efficient values at an exponential rate. Formally, as i $\to\infty$:*

If $S_m = s_H$*, then* $x_i \overset{as}{\to} 0$*,* $y_i \overset{as}{\to} 1$ *and* $A_i \overset{as}{\to} v_{H_m}$*,* $B_i \overset{as}{\to} v_{H_m}$*.*
If $S_m = s_L$*, then* $x_i \overset{as}{\to} 1$*,* $y_i \overset{as}{\to} 0$ *and* $A_i \overset{as}{\to} v_{L_m}$*,* $B_i \overset{as}{\to} v_{L_m}$*.*
If $S_m = s_0$*, then* $x_i \overset{as}{\to} 0$*,* $y_i \overset{as}{\to} 0$ *and* $A_i \overset{as}{\to} EV_m$*,* $B_i \overset{as}{\to} EV_m$*.*

Proof: See Appendix.

Although the asymptotic behavior of prices is straightforward to determine, calculating the serial correlation properties requires knowledge of the distribution of share prices in each time period, a more difficult task which we turn to next.

3 Empirical Implications

With the learning rules established, we now show that the model accounts for the main empirical findings described in the introduction. We first show that one implication of the model is that the number of trades in a calendar period is serially correlated. As described in the introduction, such serial correlation leads to serial correlation in squared price changes. We then show that the serial correlation in the number of trades per calendar period is more persistent than is the serial correlation in squared price changes.

A crucial aspect of the model is how trading opportunities are aggregated. A trading day contains k calendar periods (such as an hour). A calendar period, which is indexed by t, contains η trader arrivals, which as above are indexed by i. (We can think of a trader arrival, or trading opportunity, as a unit of economic time.) The sample period consists of a large sample of trading days.

Calendar Period Trades

First we examine the covariance structure of the number of trades per calendar period. Let the number of trades in calendar period t be I_t. The distribution of trades arises from the probability that an uninformed trader does not trade and the probability that $S_m \neq s_0$. From above, $P(S_m \neq s_0) = \theta$. The distribution of trades in economic time is simply the probability of a trade at the bid or ask (the alternative is no trade), or one less the probability of a no trade. For all trader arrivals (i) on trading day m we have:

$$\begin{aligned} 1 - P(C_i = c_N | S_m \neq s_0) &= \alpha + (1-\alpha)\,\varepsilon, \\ 1 - P(C_i = c_N | S_m = s_0) &= (1-\alpha)\ \varepsilon. \end{aligned}$$

We now aggregate the distribution of trades in economic time to get the distribution of trades in calendar time. A calendar time period consists of η trading opportunities. Thus I_t takes on integer values between 0 and η. By noting that I_t is the number of "successes" (trades) in η trials, we see that for all calendar periods (t) on trading day m the distribution of I_t conditional on S_m is binomial:

$$\begin{aligned} I_t|\left(S_m \neq s_0\right) &\sim B\left(\eta, \alpha+\varepsilon\left(1-\alpha\right)\right), \\ I_t|\left(S_m = s_0\right) &\sim B\left(\eta, \varepsilon\left(1-\alpha\right)\right). \end{aligned}$$

Thus, for all calendar periods on trading day m

$$\begin{aligned} E\left[I_t|S_m \neq s_0\right] &= \mu_1 = \eta\left(\alpha+\varepsilon\left(1-\alpha\right)\right), \\ E\left[I_t|S_m = s_0\right] &= \mu_0 = \eta\varepsilon\left(1-\alpha\right), \end{aligned}$$

$$\begin{aligned} Var\left[I_t|S_m \neq s_0\right] &= \sigma_1^2 = \eta\left(\alpha+\varepsilon\left(1-\alpha\right)\right)\left(1-\alpha\right)\left(1-\varepsilon\right), \\ Var\left[I_t|S_m = s_0\right] &= \sigma_0^2 = \eta\varepsilon\left(1-\alpha\right)\left(1-\ \varepsilon\left(1-\alpha\right)\right). \end{aligned}$$

Unconditionally, we have:

$$\begin{aligned} E\left[I_t\right] &= \mu = \theta\mu_1 + (1-\theta)\mu_0 && (1) \\ Var\left[I_t\right] &= \sigma^2 = \theta\sigma_1^2 + (1-\theta)\sigma_0^2 + \theta(1-\theta)\left(\mu_1-\mu_0\right)^2 && (2) \end{aligned}$$

Given the above structure for the number of trades in a calendar period, we can derive the serial correlation properties of the number of trades.

Theorem 2: *Let $r > 0$. If $r < k$, then I_{t-r} and I_t are positively serially correlated. If $k \geq r$, then I_{t-r} and I_t are uncorrelated. Further for all r, the correlation between I_{t-r} and I_t is given by:*

$$Cor(I_{t-r}, I_t) = \frac{\theta(1-\theta)\left(\alpha\eta\right)^2}{\sigma^2}\left[\frac{k-\min(r,k)}{k}\right]$$

Proof: See Appendix.

Proposition 2 gives the exact formula for the correlation. Therefore, it is straightforward to establish comparative statics, which we summarize in the following corollary. Let $\tau = k\eta$.

Corollary 3: *If $r < k$, then the correlation between I_{t-r} and I_t is decreasing in r, increasing in τ, and increasing in α.*

Proof:

Substituting in the definitions of k and σ^2 into equation (7) and assuming $r < k$, results in:

$$Cor(I_{t-r}, I_t) = \left(\frac{\tau - r\eta}{\tau}\right) \frac{\eta\theta(1-\theta)\alpha^2}{\varepsilon(1-\alpha)\left[1-\varepsilon(1-\alpha)\right] + \theta\alpha\left[(1-\alpha)(1-2\varepsilon) + \alpha\eta(1-\theta)\right]}$$

The results then follow by taking the appropriate derivatives. ∎

The comparative static calculations in Corollary 3 imply certain patterns of serial correlation in trades across markets. To accord with the results in Corollary 3, the frequency of trading opportunities must be constant. Because the serial correlation in trades is an increasing function of α, a market with many informed traders has more serial correlation in trades than does a market with fewer informed traders. The intuition is that with a larger number of informed traders, there is a wider difference between the number of trades on a trading day with news and on a trading day without news.

To understand the testable implications of Corollary 3 for τ, we need to interpret the relation between τ and η. There are two ways to interpret τ, which is the number of economic time periods between the arrival of private news and the public revelation of private news. One interpretation is that τ indicates the amount of calendar time until the revelation of private news (and the length of an economic time period as roughly constant across markets). In this case, Corollary 3 indicates that markets in which news is revealed after a long period of calendar time exhibit more serial correlation in trades than markets in which news is revealed after a relatively short period of calendar time, holding the length of a calendar time period, η, constant.

Under this interpretation, the effect of increasing the length of a calendar time period, holding τ fixed, is ambiguous. Increasing η widens the difference between the number of trades on news and no news days, thus increasing the covariance quadratically. Increasing η also increases the variance linearly as more trades generate more variance. The covariance effect is stronger, but there is a third effect. Increasing the length of the calendar time period also reduces the number of calendar time periods in a trading day, and calendar time periods in different trading days are uncorrelated, which reduces the serial correlation for an overall ambiguous effect. In simulations the third effect is usually the strongest, generating a negative relationship between serial correlation in trades and the length of a calendar period. Thus we expect to see empirically more serial correlation in 5 minute interval data than daily data, and more serial correlation in trades in daily data versus monthly data.

A second interpretation is that τ indicates the technological ability of the market maker to process trades (and the amount of calendar time until news is revealed as roughly constant across markets). Here we must increase η along with τ so that η continues to represent a fixed amount of calendar time. Under this interpretation τ and η move together, and the number of calendar time periods in a trading day is constant. The only effect of increasing τ here is an increase in η within a trading day. Increasing η increases the covariance quadratically and the variance linearly, thus increasing the serial correlation. Thus the effect of an improvement in the technological ability to process trades unambiguously increases the serial correlation in trades.

Under this interpretation, we can still increase the length of a calendar interval, η, holding the technological abilities constant. Thus we again have an ambiguous, but generally negative effect of increasing the length of a calendar period on the serial correlation in trades.

Next, consider the relationship between ε (the fraction of uninformed which do not trade) and the serial correlation in trades. To understand how ε affects the serial correlation in trades, we determine how ε affects both the covariance and the variance. Because the trading behavior of the uninformed does not vary between news days and no news days, the difference $(\mu_1 - \mu_0)$, and hence the covariance, is not a function of ε. Therefore, the effect of ε on the variance of trades determines the effect of ε on the correlation. Recall that the number of trades is a binomial random variable, so a probability of a trade equal to $1/2$ maximizes the variance of trades. As ε increases the probability of a trade increases. As the probability increases toward $1/2$, the variance grows and the serial correlation declines. As the probability increases beyond $1/2$, the variance declines and the serial correlation grows. The intuition is clear: If informed traders play a relatively small role in the market ($\alpha\theta$ is small), then increasing the likelihood that the uninformed trade can increase the variation in the number of trades. As the role that the informed play grows, increasing the likelihood of trade by uniformed actually reduces the noise in trades.

Since ε increases the probability of a success (trade) in a binomial, we expect that increasing ε has an inverted "U" effect on the variance. Moving ε towards $1/2$ increases the variance. However, on news days informed traders also trade. Thus increasing ε may increase the variance on no news days, pushing the probability of a trade towards .5, but decrease the variance on news days, pushing the probability above .5. Since the variance increases on no news days and decreases on news days, the overall effect on the variance depends on the probability of news and varies from market to market.

An interesting implication is that our model predicts correlation in a variety of markets. For example, there is serial correlation in trades in both liquid and illiquid markets. Our findings of serial correlation even in illiquid markets is also supported empirically by Lange (1998).

Of course, serial correlation in the number of trades could be artificially imposed by creating serial correlation in the *private* information arrival process. Engle *et al.* (1990) find some evidence of serial correlation in public news; although serial correlation in public news does not imply serial correlation in private news. Appealing to serial correlation in private news does not really provide an economic cause for serial correlation in squared price changes as it begs the question as to what causes serial correlation in private news.

Serial Correlation in Calendar Period Squared Price Changes

We next examine the serial correlation in squared price changes per calendar period. The price change that results from the action of i is $U_i = E(V_m|Z_i) - E(V_m|Z_{i-1})$, where $E(V_m|Z_i) = x_i v_{L_m} + y_i v_{H_m} + (1 - x_i - y_i)EV_m$.[6] To relate decisions in economic time given by our model to the calendar period measurements, we write calendar period price changes as

$$\Delta P_t = \sum_{i=(t-1)\eta+1}^{t\eta} U_i. \tag{3}$$

Price changes in economic time thus drive calendar price changes. In turn, the information content of trades (or no trades) drive price changes in economic time. The information content of a trade or no trade depends on the history of trades and the parameter values. For example, if ε is large, no trades convey relatively more information. If γ is large, a trade at the ask conveys relatively more information. Trades or no trades at early economic time periods convey more information than trades at later time intervals. In this way, serial correlation in squared price changes are serially correlated.

To provide insight, we study in detail the price change associated with the arrival of the first trader on trading day m. There are three possible values for U_1, one corresponding to each of the possible trade decisions. If $C_1 = c_A$, then $E(V_m|Z_1) = A_1$, and

$$U_1 = \frac{\alpha y_0 \left[v_{H_m} - E(V_m|Z_0)\right]}{P(C_1 = c_A|Z_0)}.$$

[6] The price is conditional on public information and is hence theoretically observable to the econometrician. In reality the set of parameters must be estimated, resulting in an estimate of the price based on the estimated parameters. However, in most empirical studies of serial correlation in squared price changes, econometricians use the bid, ask, or last trade, which may have different properties from the price.

If $C_1 = c_B$, then $E\left(V_m|Z_1\right) = B_1$ and

$$U_1 = \frac{\alpha x_0\left[v_{L_m} - E\left(V_m|Z_0\right)\right]}{P\left(C_1 = c_B|Z_0\right)}.$$

Finally, if $C_1 = c_N$, then

$$E\left(V_m|Z_1\right) = \frac{\alpha\left(1 - x_0 - y_0\right) EV_m + \left(1 - \alpha\right)\left(1 - \varepsilon\right) E\left(V_m|Z_0\right)}{\alpha\left(1 - x_0 - y_0\right) + \left(1 - \alpha\right)\left(1 - \varepsilon\right)}$$

and

$$U_1 = \frac{\alpha\left(1 - x_0 - y_0\right)\left[EV_m - E\left(V_m|Z_0\right)\right]}{P\left(C_1 = c_N|Z_0\right)}.$$

The mean price change from trader 1 is

$$E\left(U_1|Z_0\right) = \sum_{j=A,B,N} P\left(C_1 = c_j|Z_0\right) U_1(C_1 = c_j),$$

which equals

$$\alpha y_0 v_{H_m} + \alpha x_0 v_{L_m} + \alpha\left(1 - x_0 - y_0\right) EV_m - E\left(V_m|Z_0\right) = 0.$$

Because

$$P\left(C_i = c_A|s_m\right) \neq P\left(C_i = c_A|Z_i\right)$$

for any finite i, price changes are not mean zero with respect to the information set of the informed.

The variance of the price change from trader 1 is

$$E\left(U_1^2|Z_0\right) = \sum_{j=A,B,N} P\left(C_1 = c_j|Z_0\right) U_1^2(C_1 = c_j)$$

which equals

$$\frac{\left(\alpha y_0\right)^2\left[v_{H_m} - E\left(V_m|Z_0\right)\right]^2}{P\left(C_1 = c_A|Z_0\right)} + \frac{\left(\alpha x_0\right)^2\left[v_{L_m} - E\left(V_m|Z_0\right)\right]^2}{P\left(C_1 = c_B|Z_0\right)} + \frac{\alpha^2\left(1 - x_0 - y_0\right)^2\left[EV_m - E\left(V_m|Z_0\right)\right]^2}{P\left(C_1 = c_N|Z_0\right)}.$$

To understand the impact of informed traders on the behavior of calendar period squared price changes, we must compare the variance of U_1 for $S_m = s_0$ with the variance of U_1 for

$S_m \neq s_0$. (In general, the comparison will depend on whether the low or high signal was received. If the nuisance parameters are symmetric, $\gamma = \delta = .5$, then $x_0 = y_0$ and the variance of U_1 is identical for the low and high signals. In the remainder of the section we assume $\gamma = \delta = .5$ and so we do not need to distinguish between the low and high signals.) The addition of the signal alters the variance of U_1 only through the impact on the probability with which each trade outcome is observed

$$E\left(U_1^2|S_m = s_m, Z_0\right) = \sum_{j=A,B,N} P\left(C_1 = c_j|S_m = s_m\right) U_1^2(C_1 = c_j).$$

Compare the probabilities of each trade outcome for $S_m = s_H$ with $S_m = s_0$:

$$\begin{aligned} P\left(C_1 = c_A|S_m = s_H\right) &= P\left(C_1 = c_A|S_m = s_0\right) + \alpha \\ P\left(C_1 = c_N|S_m = s_H\right) &= P\left(C_1 = c_N|S_m = s_0\right) - \alpha, \end{aligned}$$

where the probability that $C_1 = c_B$ is the same for the two values of S_m. Thus $E\left(U_1^2|S_m = s_H, Z_0\right) - E\left(U_1^2|S_m = s_0, Z_0\right)$ equals

$$\alpha\left\{\frac{\left(\alpha y_0\right)^2\left[v_{H_m} - E\left(V_m|Z_0\right)\right]^2}{\left[P\left(C_1 = c_A|Z_0\right)\right]^2} - \frac{\alpha^2\left(1 - x_0 - y_0\right)^2\left[EV_m - E\left(V_m|Z_0\right)\right]^2}{\left[P\left(C_1 = c_N|Z_0\right)\right]^2}\right\},$$

which is greater than zero because $EV_m = E\left(V_m|Z_0\right)$. The impact of trader 2 and following traders is not immediately signed because $EV_m \neq E\left(V_m|Z_i\right)$ for $i > 0$. To determine the sign of the difference we study the behavior of U_i for general i.

Behavior of Trader i Price Change

For general i there are 3^i possible values for U_i, so direct calculation of the moments of U_i is tedious. Rather, we construct analytic bounds to the moments that describe the behavior of the distribution of U_i. Let

$$\widetilde{A}_i - \widetilde{B}_i = \max\{A_i, \mathrm{E}[V_m|Z_{i-1}, C_i = c_N]\} - \min\{B_i, \mathrm{E}[V_m|Z_{i-1}, C_i = c_N]\}$$

be the "spread" or the difference between the maximum price change and the minimum price change. For most parameter values, the spread is equal to the familiar bid-ask spread. However, as noted earlier, for some parameter values a no trade may induce larger or smaller price changes than a trade at the ask or bid, respectively.

Let $\{U_i\}_{i=1}^{k\eta}$ be the sequence of trader price changes (price changes in economic time) for a trading day. With respect to the public information set, the elements of the sequence are uncorrelated but are dependent and not identically distributed. Specifically, the trader price changes are heteroskedastic and the heteroskedasticity is autoregressive.

Theorem 4: *Price changes in economic time satisfy:*

1. $E\left(U_i | Z_{i-1}\right) = 0$
2. $E\left(U_i U_j | Z_{i-1}\right) = 0$ for $i \neq j$
3. $[P(C_i = c_A)P(C_i = c_B)P(C_i = c_N)] \left(\widetilde{A}_i - \widetilde{B}_i\right)^2 \leq E\left(U_i^2 | Z_{i-1}\right) \leq \left(\widetilde{A}_i - \widetilde{B}_i\right)^2.$

Proof: See Appendix.

The spread drives the variance in U_i. Randomly arriving informed and uninformed traders trade at the bid and ask causing the sequence of prices to vary. Theorem 4 and Proposition 1 together imply that $E\left(U_i^2 | Z_{i-1}\right) \rightarrow 0$ as $i \rightarrow \infty$. As the market maker becomes certain of the true value of the share, the bid and ask converge to the true value of the share and squared price changes go to zero.

The calculations of Theorem 4 rely on the definition $E[V|Z_{i-1}] = P_{i-1}$. Of course, publicly available news about V_m may arrive after the decision of trader $i-1$ but before the decision of trader i. Because the public information set, Z_{i-1}, would include this news, in general $E[V_m|Z_{i-1}] \neq P_{i-1}$. As a consequence $E[U_i|Z_{i-1}]$, which equals $E[V_m|Z_{i-1}] - P_{i-1}$, may be nonzero. Surprisingly, the arrival of public news is less important for the variance of price changes. The conditional variance of U_i is

$$
\begin{aligned}
Var\left(U_i | Z_{i-1}\right) &= P(C_i = c_A)(A_i - E[V_m|Z_{i-1}])^2 + P(C_i = c_B)(B_i - E[V_m|Z_{i-1}])^2 \\
&\quad + P(C_i = c_N) \cdot (E[V_m|Z_{i-1}, C_N] - E[V_m|Z_{i-1}])^2 \\
&\leq (\widetilde{A}_i - \widetilde{B}_i)^2,
\end{aligned}
$$

so the bounds for the variance of U_i are not dependent on assumptions about the timing of public news announcements. Although the release of public information can result in a change in the level of U_i, the variance of U_i is still driven by the bid-ask spread which in turn is driven by the adverse selection caused by private information. Hence, the variance of price changes is bounded by the bid-ask spread independently of the path of public information. Because the results presented above indicate that private information, rather than public information, plays the key role in our model, we focus on private information in our derivation of the serial correlation properties of squared calendar price changes.

To determine how the properties of the distribution of U_i are affected by the signal received by informed traders, we compare the variance of U_i if $S_m = s_0$ with the variance of U_i if $S_m \neq s_0$. Parallel to the case for trader 1, $E\left(U_i^2 | S_m = s_H, Z_{i-1}\right) - E\left(U_i^2 | S_m = s_0, Z_{i-1}\right)$ equals

$$\alpha \left\{ \frac{\left(\alpha y_{i-1}\right)^2 \left[v_{H_m} - E\left(V_m | Z_{i-1}\right)\right]^2}{\left[P\left(C_1 = c_A | Z_{i-1}\right)\right]^2} - \frac{\alpha^2 \left(1 - x_{i-1} - y_{i-1}\right)^2 \left[EV_m - E\left(V_m | Z_{i-1}\right)\right]^2}{\left[P\left(C_1 = c_N | Z_{i-1}\right)\right]^2} \right\}. \quad (4)$$

The difference (4) depends on $E\left(V_m | Z_{i-1}\right)$, which in turn depends on (x_{i-1}, y_{i-1}).

If the proportion of informed traders is high enough, then learning takes place quickly and the entire distribution of U_i can be directly calculated. For example, if $\alpha = .9$, then the bid-ask spread is reduced very close to zero in only 10 trades. For $\alpha = .9$ and $\gamma = \delta = .5$, the columns of Table 1 contain the values of (4) corresponding to trader 1 through trader 8 and the rows of Table 1 correspond to different values of ε.

TABLE 1
VALUE OF $E\left(U_i^2 | S_m = s_H, Z_{i-1}\right) - E\left(U_i^2 | S_m = s_0, Z_{i-1}\right)$

Trader:	1	2	3	4	5	6	7	8
$\varepsilon = .9$	3.46	0.64	0.55	0.14	0.08	0.03	0.01	0.00
$\varepsilon = .8$	2.89	0.52	0.46	0.11	0.07	0.02	0.01	0.00
$\varepsilon = .7$	2.30	0.47	0.32	0.09	0.04	0.01	0.01	0.00
$\varepsilon = .6$	1.66	0.42	0.18	0.07	0.02	0.01	0.00	0.00
$\varepsilon = .5$	0.97	0.33	0.04	0.04	0.00	0.00	0.00	0.00
$\varepsilon = .4$	0.20	0.18	-.05	-.01	0.00	0.00	0.00	0.00
$\varepsilon = .3$	-.65	-.11	-.09	-.09	-.01	-.01	0.00	0.00
$\varepsilon = .2$	-1.61	-.70	-.14	-.05	-.04	-.01	-.01	0.00
$\varepsilon = .1$	-2.73	-2.12	-.32	-.15	-.09	-.02	-.01	0.00

The entries in Table 1 reveal two important features. First, as learning accumulates (moving across a row) the difference in squared price changes tends toward zero. Second, the value of ε plays a key role. For large values of ε, the behavior of transaction level price changes is such that the variance is higher, uniformly, for days in which the informed trade. As ε declines, the variance of transaction level price changes can be higher on days in which the informed do not trade. Why? With ε low, very few of the uninformed trade and so there are very few trades if $S_m = s_0$ (because the informed also do not trade). With so few trades, learning is slowed enough that the slow learning corresponding to $S_m = s_0$ actually outweighs the uncertainty associated with $S_m \neq s_0$.

For smaller values of α learning occurs more slowly, so reduction of the bid-ask spread to zero takes many more trades and calculation of the exact distribution is cumbersome. To understand the behavior of (4) with a smaller value of α, we approximate the exact distribution with simulations. In the last figure titled "mean: squared price changes" we report the simulated distribution for $\alpha = .2$ and $\varepsilon = .5$ and construct 3000 simulations. To be precise, we simulate 3000 trading days. At the outset of each trading day the probability that the informed receive a signal (θ) is .4, so slightly less than half of the simulations correspond to the line labeled "News Days".[7] A trading day is assumed to consist of 960 trader arrivals, but as the figure reveals the squared price change is effectively zero after the first 500 trader arrivals. The last attached figure contains $E\left(U_i^2 | S_m = s_H, Z_{i-1}\right) - E\left(U_i^2 | S_m = s_0, Z_{i-1}\right)$ as the difference between the lines corresponding to "News Days" and "No News Days". As the figure reveals, the difference is generally positive and shrinks to zero as the number of traders increases. (The horizontal axis measures the number of traders that have arrived, so learning is nearly complete after 500 traders have come to the market.)

As we shall see, the heteroskedasticity in U_i^2 that arises from the movements in the expected bid-ask spread play an important role in explaining the persistence puzzle. If U_i^2 is assumed to be homoskedastic, as in Gallant, Hsieh, and Tauchen (1991), then the covariance of calendar period squared price changes is driven exclusively by the covariance in calendar period trades, and the persistence in the covariance in trades should be matched by the persistence in the covariance in squared price changes. Our model breaks the persistence link because one prediction of our model is that the variance of U_i is not constant. In fact, the variance of U_i declines as trades occur because information is revealed and the bid-ask spread declines over time. If the variance of U_i declines, then the covariance in squared price changes will eventually be less than the covariance in the number of trades. We show that even during the period in which all traders are willing to trade, the variance of U_i declines so that the news arrival has a more persistent effect on the number of trades than on squared price changes.

Covariance of Calendar Period Squared Price Changes

From the results above, we form a structure for the expectation of the calendar period squared price change. The analytic and simulation results lead to the following structure. If on trading day m the informed do not trade, $S_m = s_0$ and $E\left[(\Delta P_t)^2 | S_m = s_0\right]$ is constant

[7] To ensure that our results are stable, we find little variation in results when the number of simulations is doubled.

for all periods in the trading day. If on trading day m the informed do trade, $S_m \neq s_0$ and $E\left[(\Delta P_t)^2 | S_m \neq s_0\right]$ declines over the course of the trading day. Let j index the calendar periods in a trading day. We represent the structure as

$$
\begin{aligned}
E\left[(\Delta P_t)^2 | S_m = s_0\right] &= \sigma_0 \\
E\left[(\Delta P_t)^2 | S_m \neq s_0\right] &= \sum_{j=1}^{k} \sigma_j 1\,(t = j)\,,
\end{aligned}
$$

where $\sigma_1 > \sigma_2 > \cdots > \sigma_k > \sigma_0$. If calendar period t is the first period of trading day m on which $S_m \neq s_0$, the expected squared price change is σ_1. The inequality $\sigma_k > \sigma_0$ arises from the observation that the informed have an information advantage and continue to trade until the trading day ends. The unconditional expectation of calendar period squared price changes is

$$
E\left[(\Delta P_t)^2\right] = \theta\bar{\sigma}_k + (1-\theta)\,\sigma_0,
$$

where $\bar{\sigma}_k = \frac{1}{k}\sum_{j=1}^{k}\sigma_j$.

For general k the covariance of calendar period squared price changes, $Cov\left[(\Delta P_{t-r})^2, (\Delta P_t)^2\right]$, equals

$$
\left[\frac{k - \min(r,k)}{k}\right]\{\theta\,(1-\theta)\sum_{j=1}^{k-r}(\sigma_j - \sigma_0)\,(\sigma_{j+r} - \sigma_0) + \theta^2\sum_{j=1}^{k}(\bar{\sigma}_k - \sigma_j)\,(\bar{\sigma}_k - \sigma_{j+r})\},
$$

where the addition is wrapped at k. That is, if $j + r > k$, then replace $j + r$ with $j + r - k$.

As for the covariance of calendar period trades, the covariance of calendar period squared price changes is zero if $r \geq k$. To determine the sign of the covariance if $r < k$, we must examine each term in detail. The first term in brackets is the sum of the conditional covariances, which is positive. The second term in brackets is the sum of the covariances of the conditional means, which is generally negative. Determination of the sign of the covariance depends on the relative magnitudes of the two terms.

We begin our analytic derivations with a trading day in which there are two periods, so $k = 2$. For example, if a trading day corresponded to a calendar day, then empirical study of mornings versus afternoons would yield $k = 2$. For this case an important condition emerges that is needed to ensure the covariance is positive.

Condition 1 is said to hold for period j^*, with $1 < j^* \leq k$, if j^* is the largest value of j for which

$$
\sigma_j > \theta\bar{\sigma}_k + (1-\theta)\,\sigma_0.
$$

If $k = 2$, the only possible value of j^* for which Condition 1 can hold is 2, so we shorten the phrase "Condition 1 holds for period 2" to "Condition 1 holds" when discussing a trading day with $k = 2$.

Condition 1 is intuitive. Recall that positive covariance between two random variables, with the same unconditional mean, implies that if one random variable is below the unconditional mean, then the other random variable tends to be below the unconditional mean. Correspondingly, if one random variable is above the unconditional mean, then the other random variable tends to be above the unconditional mean. From the structure for the expectation of calendar period squared price changes it follows that σ_1 lies above the unconditional mean and σ_0 lies below the unconditional mean. Let $(\Delta P_{t-1})^2$ be the first period of trading day m. If $S_m = s_0$, then in expectation $(\Delta P_{t-1})^2$ equals σ_0 and so tends to be below the unconditional mean. Yet if $S_m = s_0$, then in expectation $(\Delta P_t)^2$ also equals σ_0 and so also tends to be below the unconditional mean. If $S_m \neq s_0$, then in expectation $(\Delta P_{t-1})^2$ equals σ_1 and so tends to be above the unconditional mean. With $S_m \neq s_0$, then in expectation $(\Delta P_t)^2$ equals σ_2 and the behavior of the covariance depends on the relative magnitude of σ_2 and the conditional mean. If Condition 1 holds, then σ_2 is larger than the unconditional mean, so if $(\Delta P_{t-1})^2$ tends to be above the unconditional mean, then $(\Delta P_t)^2$ also tends to be above the unconditional mean.

Lemma 5: *Let Condition 1 hold (for period 2) with $k = 2$. The covariance of calendar period squared price changes is*

$$Cov\left[(\Delta P_{t-r})^2, (\Delta P_t)^2\right] = \left[\frac{k - \min(r, k)}{k}\right]\left\{\theta(1-\theta)(\sigma_1 - \sigma_0)(\sigma_2 - \sigma_0) + \frac{\theta^2}{2}(\sigma_1 - \sigma_2)^2\right\} \geq 0.$$

Proof: See Appendix.

In more detail, if $r \geq 2$ the covariance is zero while if $r = 1$ the covariance is positive.

To shed further light on the behavior of the covariance of calendar period squared price changes, we derive analytic results for a trading day with 3 periods. If $k = 3$, then the second term in the formula for $Cov\left[(\Delta P_{t-r})^2, (\Delta P_t)^2\right]$, which corresponds to the covariance of the conditional means, is identical for $r = 1$ and $r = 2$. As the conditional covariance for $r = 1$ exceeds the conditional covariance for $r = 2$,

$$Cov\left[(\Delta P_{t-1})^2, (\Delta P_t)^2\right] > Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right].$$

We begin by establishing under what condition $Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right]$ is positive.

Lemma 6: *Let Condition 1 hold for period 3 with $k = 3$. For $r < k$ the covariance of calendar period squared price changes is positive.*

Proof: See Appendix.

While the definition of positive covariance leads naturally to Condition 1 holding for period 2, there is no such natural intuition for extending Condition 1 to hold for period 3. If there are 3 periods in a trading day, then in the last period of a trading day enough of the information of the traders may have been revealed that the expected squared price change for that period need not exceed the unconditional expected squared price change for a period. If Condition 1 holds only for period 2, then the decline of the covariance in calendar period squared price changes can be dramatically rapid.

More Rapid Decay of Covariance for Calendar Period Squared Price Changes

Close study of the case in which $k = 3$ reveals much about the relative decay of the correlation in calendar period trades and squared price changes. One of the empirical features brought forward in the introduction is that the correlation of calendar period trades decays more slowly than does the correlation of calendar period squared price changes. Close study of the case in which $k = 3$ reveals why this is so. As the variance of either quantity is constant as the lag of the correlation changes, the decay of the correlation is driven by the decay of the covariance.

Suppose Condition 1 holds for period 3 so that $Cov\left[(\Delta P_{t-r})^2, (\Delta P_t)^2\right]$ is positive for $r = 1, 2$. Define the following quantities: $c_1 = (\alpha\eta)^2$, $c_2 = (\sigma_2 - \sigma_0)(\sigma_3 - \sigma_0)$, $c_3 = (\sigma_2 - \sigma_0)(\sigma_1 - \sigma_3)$, $c_4 = (\sigma_3 - \sigma_0)(\sigma_1 - \sigma_2)$, and $c_5 = \frac{1}{3}\theta^2 \sum_{j=1}^{3} (\bar{\sigma}_3 - \sigma_i)(\bar{\sigma}_3 - \sigma_{i+1})$.

Lemma 7: *Let Condition 1 hold for period 3 with $k = 3$. The covariance, and hence the correlation, of calendar period squared price changes decays more rapidly than the covariance of calendar period trades.*

Proof: The decay rates are revealed by direct calculation from the covariances. For calendar period trades

$$
\begin{aligned}
Cov(I_{t-1}, I_t) &= \frac{2}{3}\theta(1-\theta)c_1 \\
Cov(I_{t-2}, I_t) &= \frac{1}{3}\theta(1-\theta)c_1.
\end{aligned}
$$

For calendar period squared price changes

$$\begin{aligned} Cov\left[(\Delta P_{t-1})^2, (\Delta P_t)^2\right] &= \frac{2}{3}\theta(1-\theta)c_2 + \frac{1}{3}\theta(1-\theta)c_3 + c_5 \\ Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right] &= \frac{1}{3}\theta(1-\theta)c_2 + \frac{1}{3}\theta(1-\theta)c_4 + c_5. \end{aligned}$$

The first term in the expression for the covariance for calendar period squared price changes captures the same proportional decay as in the covariance for calendar period trades. But there is a further decay, from the second term that arises because the definition of $\{\sigma_j\}_{j=1}^k$ implies $c_3 > c_4$. It is the additional decay from the second term that leads to the greater proportional decay in calendar period squared price changes. ■

If Condition 1 holds for period 2, it is possible that the covariance of calendar period squared price changes at lag 2 is negative. As the covariance of calendar period trades is always nonnegative, such a finding further enforces the more rapid decay of the covariance of calendar period squared price changes.

Proposition 8: *Let Condition 1 hold for period 2 with $k = 3$. There is an open subset of parameter values for which $Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right]$ is negative.*

Proof: We need only establish that there exist parameter values for which the covariance is negative. Consider the set $\{\sigma_j\}_{j=1}^3 = \{20, 7, 3\}$ so $\bar{\sigma}_3 = 10$. Let $\sigma_0 = 1$. From the definition of the covariance of calendar period squared price changes, $Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right]$ is negative if $\theta > .33$. Let $\theta = .4$, so Condition 1 holds for period 2. As Condition 1 continues to hold for period 2 for an open set of values of θ above .4, Proposition 8 is established. ■

Note that for the set of parameter values that establish Proposition 8, Condition 1 does not hold for period 3, as must be the case from Lemma 4.

What information set should be used to form conditional expectations of $(\Delta P_t)^2$? The above calculations seem to indicate that prediction of the variance of price changes hinges only on prediction of $\frac{T_t'}{T_{t-1}'}$. Such a conclusion is too strong for two reasons. First, and most importantly, our model captures some of the features that lead to trading, but certainly not all. Second, the specific form of the variance derived above depends on averaging over sample paths of quotes. For a given sample path, prices may contain useful information. At the least though, our model gives reason as to why T_{t-1}' should be included in the information set used to predict squared price changes. In addition, further lags of T could augment lagged values of ΔP to the degree to which they predict current transactions.

4 Simulations

To provide an idea of the pattern of serial correlation that is implied by our model, we simulate sequences of trades and the associated price changes over a period of many trading days.

Let the unit of economic time be one second and assume that information is revealed at the end of each day, so that there are $\tau = 2880$ trading opportunities in an 8 hour trading day. Note that the model could also be interpreted as for example, with a unit of economic time being 2 seconds with news revealed at the end of the second day.[8]

Suppose $\eta = 30$, so that calendar time periods are 30 seconds in length. Our simulated sample consists of 100 trading days, each of which has probability of news $\theta = .4$. Given news, the probability of good news is $\delta = .55$. To ensure that asymmetries in the model are not driving our results we set $\gamma = \varepsilon = \frac{1}{2}$, so that the uninformed are equally likely to buy or sell. A key parameter that remains is the proportion of traders with private information. We initially set $\alpha = .2$, but vary this parameter, along with η in various simulations.

Figure 1 contains the results of single sequence of trade data. The upper left graph depicts the path of squared price changes, where the prices are sampled at five minute intervals. The vertical axis measures the actual squared price change, in logarithmic scale, and the horizontal axis measures the passage of calendar time (in 30 second intervals). Following the introduction of news the squared price change is high, but the magnitude of squared price changes declines exponentially over time. The lower left graph depicts the pattern of trading. Initially the number of trades is higher, on average 16 trades per five minute interval, but after a group of partially informed traders leaves, the average number of trades falls to 13. The pattern of actual trades reflects the fact that the partially informed leave during the 45th five minute interval, or about half way through the first trading day. Finally, the upper right graph depicts the sequence of bid and ask quotes and shows that convergence is reached in the 57th five minute interval.

Figures 2-4 depict the results over 100 trading days. Figure 2 shows the autocorrelation function for squared price changes. Figure 3 depicts the autocorrelation function for the number of trades. The magnitude of the autocorrelation for trades dominates the serial correlation in squared price changes, as can be seen in Figure 4. The ratio of the autocorrelation function for squared price changes to the autocorrelation function for trades declines by two thirds over the first 30 lags.

An interesting example is to use the estimation results in [8]. In a similar model, [8]

[8]Easley and O'Hara (1993) in a model similar to ours, assume that a trading opportunity is five minutes and a trading day is one day for Ashland Oil, based on the number of trades observed daily (a maximum of 73).

obtain estimates of α = .172, ε=.33, θ = .75, τ = 96, and δ = .502. We also set η = 4, equivalent to 5 minute calendar intervals. Results are detailed in Figures 5-8. The serial correlation in trades is small but quite persistent. The serial correlation in squared price changes is small and barely persists for two lags.

5 Conclusions

In this paper we provide an economic model that generates serial correlation in trades and serial correlation in squared price changes. Further, serial correlation in trades is more persistent than serial correlation in squared price changes. We propose that serial correlation in trades arises simply from the entry and exit of informed traders, who receive a private signal. Given that informed traders are trading in the current period, informed trader will most likely trade in the following period, which generates serial correlation in trades. The serial correlation in trades is quite strong and persistent. In the simulations after 30 lags the correlation was still above .8.

In our model serial correlation in trades generates serial correlation in squared price changes. Given that the informed traders are trading, there is more variance in squared price changes simply because there are more trades in a calendar period. More trades implies that the price change is the sum of more random trades, which in turn implies that the price change has a larger variance. Because there is serial correlation in trades, there is serial correlation in squared price changes. However, there is an additional effect on the serial correlation in squared price changes, the decline in the bid-ask spread. All trades are at the bid-ask spread, hence price expected price changes are bounded by the bid-ask spread. The bid-ask spread declines as learning proceeds, which reduces the variance and the persistence of the serial correlation in squared price changes. Given there are more trades in a calendar period, there are most likely more trades in the next calendar period, which implies higher variance in both periods. However, the trades in the second calendar period are from a random variable with a smaller variance, due to the smaller bid-ask spread. Hence the serial correlation is smaller and less persistent. Our simulations indicate that the correlation coefficient at one lag is .35, and declines to .1 after 30 lags. Hence our model replicates the observed empirical features of the data and explains serial correlation through the entry and exit of informed traders and the associated revelation of information in the prices.

Future research includes expanding our model to the possibility of multiple trading periods with randomly arriving private news events. Multiple trading periods allows for the fully informed not to trade (if no news is released) hence the serial correlation in trades will likely be even stronger.

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6 Appendix

Derivation of Learning Formulae

We explicilty derive the learning formula for y_i given that trader i trades at the ask. All other learning formulae follow the same logic. From Bayes rule

$$P\left(S_m = s_H | Z_i\right) = \frac{P(S_m = s_H | Z_{i-1}) P(C_i = c_A | S_m = s_H)}{\sum\limits_{j=s_L, s_H, s_0} P(S_m = j | Z_{i-1}) P(C_i = c_A | S_m = j)}.$$

We must calculate $P(C_i = c_A | S_m = j)$ for $j = s_L, s_H, s_0$. If the informed receive the signal s_H, the the informed will trade at the ask. Further, the fraction $\varepsilon\left(1-\gamma\right)$ of the uninformed will also trade at the ask. Hence

$$P\left(C_i = c_A | S_m = s_H\right) = \alpha + \left(1-\alpha\right)\varepsilon\left(1-\gamma\right).$$

If the informed receive the signal s_L or do not receive a signal, then the informed will not trade at the ask. Because only the uninformed trade at the ask if S_m equals s_L or s_0, both $P(C_i = c_A | S_m = s_L)$ and $P(C_i = c_A | S_m = s_0)$ equal $\left(1-\alpha\right)\varepsilon\left(1-\gamma\right)$.

6.1 Proof of Lemma 1

The learning formulae for x_i and y_i are nonlinear in (x_{i-1}, y_{i-1}) and are not recursive, which make it difficult to determine the asymptotic behavior of x_i and y_i. Because the denominator of the learning formula, conditional on the decision of trader i, is identical for x_i, y_i and $1 - x_i - y_i$, the learning formulae for ratios of x_i and y_i are linear in ratios of (x_{i-1}, y_{i-1}) and recursive. We work with ratios x_i and y_i and begin with the case $S_m = s_H$, for which the relevant ratios are $\frac{x_i}{y_i}$ and $\frac{1-x_i-y_i}{y_i}$. Consider $\frac{x_i}{y_i}$. If trader i trades at the ask

$$\frac{x_i}{y_i} = \frac{x_{i-1}}{y_{i-1}} \cdot \frac{P(C_i = c_A | S_m = s_L)}{P(C_i = c_A | S_m = s_H)}.$$

If trader i trades at the bid, then the expression for $\frac{x_i}{y_i}$ is as above with $C_i = c_A$ replaced by $C_i = c_B$. If trader i does not trade

$$\frac{x_i}{y_i} = \frac{x_{i-1}}{y_{i-1}},$$

because $P(C_i = c_N|S_m = s_L)$ equals $P(C_i = c_N|S_m = s_H)$. We have

$$\ln\left(\frac{x_i}{y_i}\right) = \ln\left(\frac{x_0}{y_0}\right) + n_A \ln\left[\frac{P(C_i = c_A|S_m = s_L)}{P(C_i = c_A|S_m = s_H)}\right] + n_B \ln\left[\frac{P(C_i = c_B|S_m = s_L)}{P(C_i = c_B|S_m = s_H)}\right],$$

where n_A is the number of the first i trading opportunities for which there was a trade at the ask, n_B is the number of the first i trading opportunities for which there was a trade at the bid, and n_N is the number of the first i trading opportunities for which there was no trade.

Because the trader arrival process is i.i.d.,

$$\frac{1}{i}\ln\left(\frac{x_i}{y_i}\right) \overset{as}{\rightarrow} \sum_{j=c_A,c_B} P(C_i = j|S_m = s_H) \ln\left[\frac{P(C_i = j|S_m = s_L)}{P(C_i = j|S_m = s_H)}\right] \tag{5}$$

as $i \rightarrow \infty$. The right-hand side of (5), multiplied by minus one, is a measure of distance between the probability measure $P(\cdot|S_m = s_H)$ and the probability measure $P(\cdot|S_m = s_L)$, which is termed the entropy of $P(\cdot|S_m = s_H)$ relative to $P(\cdot|S_m = s_L)$ and is denoted $\mathcal{J}(s_H, s_L)$. By construction the entropy is nonnegative and equals zero only if the probability measures differ solely on a set with measure zero. Hence

$$\frac{1}{i}\ln\left(\frac{x_i}{y_i}\right) \overset{as}{\rightarrow} -\mathcal{J}(s_H, s_L) < 0$$

as $i \rightarrow \infty$, so that $\frac{x_i}{y_i}$ behaves as $e^{-i\mathcal{J}(s_H,s_L)}$. Thus $\frac{x_i}{y_i}$ converges almost surely to zero at the exponential rate $i\mathcal{J}(s_H, s_L)$. A similar argument shows that $\frac{1-x_i-y_i}{y_i}$ converges almost surely to zero at the exponential rate $i\mathcal{J}(s_H, s_N)$.

If $S_m = s_H$, then $\frac{x_i}{y_i} \overset{as}{\rightarrow} 0$ and $\frac{1-x_i-y_i}{y_i} \overset{as}{\rightarrow} 0$ as $i \rightarrow \infty$.

If $S_m = s_L$, then the relevant ratios are $\frac{y_i}{x_i}$ and $\frac{1-x_i-y_i}{x_i}$. If $S_m = s_0$, then the relevant ratios are $\frac{x_i}{1-x_i-y_i}$ and $\frac{y_i}{1-x_i-y_i}$. Again the fact that the trader arrival process is i.i.d. is sufficient to establish that

if $S_m = s_L$, then $\frac{y_i}{x_i} \overset{as}{\rightarrow} 0$ and $\frac{1-x_i-y_i}{x_i} \overset{as}{\rightarrow} 0$,

if $S_m = s_0$, then $\frac{x_i}{1-x_i-y_i} \overset{as}{\rightarrow} 0$ and $\frac{y_i}{1-x_i-y_i} \overset{as}{\rightarrow} 0$,

as $i \rightarrow \infty$.

From the convergence properties of the ratios, we can easily deduce the convergence properties of x_i and y_i. We continue with the case $S_m = s_H$ and note that similar arguments hold for $S_m = s_L$ and $S_m = s_0$. The statement $\frac{1-x_i-y_i}{y_i} \overset{as}{\rightarrow} 0$ is equivalently written as

$$\frac{1}{y_i} - \frac{x_i}{y_i} - 1 \overset{as}{\rightarrow} 0. \tag{6}$$

Because $\frac{x_i}{y_i} \overset{as}{\to} 0$, the statement (6) is equivalent to

$$\frac{1}{y_i} - 1 \overset{as}{\to} 0,$$

which directly implies $y_i \overset{as}{\to} 1$. If $y_i \overset{as}{\to} 1$, then the statement $\frac{x_i}{y_i} \overset{as}{\to} 0$ implies $x_i \overset{as}{\to} 0$. From the definition of A_i and B_i, if $x_i \overset{as}{\to} 0$ and $y_i \overset{as}{\to} 1$, then $A_i \overset{as}{\to} v_{H_m}$ and $B_i \overset{as}{\to} v_{H_m}$. ■

6.2 Proof of Theorem 2

The proof is a straightforward, but tedious calculation of the correlation. By definition, the covariance is

$$Cov(I_{t-r}, I_t) = E\left(I_{t-r} I_t\right) - EI_{t-r} \cdot EI_t.$$

If $r \geq k$, then the independence of the signal process implies that I_{t-r} is independent of I_t, so $E\left(I_{t-r} I_t\right) = EI_{t-r} \cdot EI_t$ and the covariance is zero.

If $r < k$, then there are three possible conditional expectations $(I_{t-r} I_t)$. First, if I_{t-r} and I_t are measured on the same trading day the conditional expectation of $(I_{t-r} I_t)$ is

$$\theta \mu_1^2 + (1-\theta)\, \mu_0^2,$$

which occurs with probability $\frac{k-r}{k}$. Second, if I_{t-r} and I_t are measured on consecutive trading days and $S_{m+1} \neq s_0$, the conditional expectation of $(I_{t-r} I_t)$ is

$$\theta \mu_1^2 + (1-\theta)\, \mu_0 \mu_1,$$

which occurs with probability $\frac{r}{k}\theta$. Third, if I_{t-r} and I_t are measured on consecutive trading days and $S_{m+1} = s_0$, the conditional expectation of $(I_{t-r} I_t)$ is

$$\theta \mu_0 \mu_1 + (1-\theta)\, \mu_0^2,$$

which occurs with probability $\frac{r}{k}(1-\theta)$. We combine the three conditional expectations to yield

$$E\left(I_{t-r} I_t\right) = \frac{k-r}{k}\left[\theta \mu_1^2 + (1-\theta)\, \mu_0^2\right] + \frac{r}{k}\left[\theta \mu_1 + (1-\theta)\, \mu_0\right]^2.$$

Because the process for calendar period trades is stationary, EI_{t-r} equals EI_t. As noted in the text

$$EI_t = \theta \mu_1 + (1-\theta)\, \mu_0,$$

so

$$\begin{aligned} Cov\left(I_{t-r}, I_t\right) &= \frac{k-r}{k}\theta\left(1-\theta\right)\left(\mu_1-\mu_0\right)^2 \\ &= \frac{k-r}{k}\theta\left(1-\theta\right)\left(\alpha\eta\right)^2. \end{aligned}$$

Combining the two possible cases for r relative to k yields

$$Cov(I_{t-r}, I_t) = \begin{cases} \theta(1-\theta)\left(\alpha\eta\right)^2\left[\frac{k-r}{k}\right] & r < k \\ 0 & r \geq k \end{cases} \tag{7}$$

Combining the covariance and variance of I_t given by (1) gives the desired correlation. Because all terms are postive for $r < k$, the correlation is positive. ∎

6.3 Proof of Theorem 4

The expected value of U_i is

$$\begin{aligned} \mathrm{E}\, U_i &= \mathrm{E}\,\mathrm{E}\left(U_i|Z_{i-1}\right) \\ &= \mathrm{E}[P(C_i = c_A)(A_i - P_{i-1}) + P(C_i = c_B)(B_i - P_{i-1}) + P(C_i = c_N)\cdot(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1}] \\ &= \mathrm{E}\left[P(C_i = c_A)A_i + P(C_i = c_B)B_i + P(C_i = c_N)\,\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1}\right] \\ &= \mathrm{E}\left[\mathrm{E}\left(V_m|Z_{i-1}\right) - P_{i-1}\right] = 0. \end{aligned}$$

In similar fashion we find that U_i is a serially uncorrelated random variable. Let i and j be distinct values with $i < j$,

$$\mathrm{E}\left(U_iU_j\right) = \mathop{\mathrm{E}}_{Z_{j-1}}\left\{U_i\left[\mathrm{E}\left(V_m|Z_{j-1}\right) - P_{j-1}\right]\right\} = 0.$$

The variance of U_i is

$$\begin{aligned} \mathrm{E}\left(U_i^2|Z_{i-1}\right) &= P(C_i = c_A)(A_i - P_{i-1})^2 + P(C_i = c_B)(B_i - P_{i-1})^2 + P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N] - \\ &\leq P(C_i = c_A)(\max(A_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1})^2 \\ &\quad + P(C_i = c_B)(\min(B_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1})^2 + P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N]) - P_i \\ &\leq \left[P(C_i = c_A) + P(C_i = c_N)\right]\left(\max(A_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1}\right)^2 + \\ &\quad \left[P(C_i = c_B) + P(C_i = c_N)\right]\left(\min(B_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1}\right)^2 \\ &\leq (\max(A_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1})^2 + (\min(B_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1})^2 \\ &\leq [(\max(A_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1}) - (\min(B_i, \mathrm{E}[V_m|Z_{i-1}, C_N]) - P_{i-1})]^2 \\ &= \left[\widetilde{A}_i - \widetilde{B}_i\right]^2. \end{aligned}$$

Here the first and second inequalities follow since $B_i \leq \mathrm{E}[V_m|Z_{i-1}] \leq A_i$. So the unconditional variance is bounded by the expected bid-ask spread and the conditional variance is bounded by the bid-ask spread. Alternatively, from Jensen's inequality:

$$\mathrm{E}\, U_i^2 \leq \mathrm{E}\left(\widetilde{A}_i - \widetilde{B}_i\right)^2 \leq \left(\mathrm{E}\, \widetilde{A}_i - \mathrm{E}\, \widetilde{B}_i\right)^2.$$

The lower bound for the conditional variance requires three cases. First, let $B_i \leq A_i \leq \mathrm{E}[V_m|Z_{i-1}, C_N]$ (the other two cases follow analagously). Then:

$$P(C_i = c_A)P(C_i = c_B)P(C_i = c_N)\left(\widetilde{A}_i - \widetilde{B}_i\right)^2 \leq P(C_i = c_N)P(C_i = c_B)\left[(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1}) - (\right.$$

$$\begin{aligned}
&= (P(C_i = c_N) + P(C_i = c_B))P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1})^2 + (P(C_i = c_N) + P(C_i = c_B))\\
&\quad [P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1}) + P(C_i = c_B)(B_i - P_{i-1})]^2
\end{aligned}$$

$$\begin{aligned}
&\leq (P(C_i = c_N) + P(C_i = c_B))\left[P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1})^2 + P(C_i = c_B)(B_i - P_{i-1})^2\right]\\
&\leq (P(C_i = c_N) + P(C_i = c_B))\left[\begin{array}{c} P(C_i = c_A)(A_i - P_{i-1})^2 + P(C_i = c_B)(B_i - P_{i-1})^2 + \\ P(C_i = c_N)(\mathrm{E}[V_m|Z_{i-1}, C_N] - P_{i-1})^2 \end{array}\right]\\
&= (1 - P(C_i = c_A))\,\mathrm{E}\left(U_i^2|Z_{i-1}\right) \leq \mathrm{E}\left(U_i^2|Z_{i-1}\right).
\end{aligned}$$

The unconditional variance thus satisfies:

$$\mathrm{E}\left[P(C_i = c_A)P(C_i = c_B)P(C_i = c_N)\left(\widetilde{A}_i - \widetilde{B}_i\right)^2\right] \leq \mathrm{E}\left(U_i^2\right)$$

>From the definition of the bid and ask:

$$\mathrm{E}\left[P(C_i = c_A)P(C_i = c_B)P(C_i = c_N)\right]\left(\mathrm{E}\, \widetilde{A}_i - \mathrm{E}\, \widetilde{B}_i\right)^2 \leq \mathrm{E}\left(U_i^2\right)$$

6.4 Proof of Lemma 5.

We first carefully derive $Cov\left[(\Delta P_{t-1})^2, (\Delta P_t)^2\right]$ for $k=2$. Let $N=1$ if $t-1$ is the first period of a trading day and let $N=2$ if $t-1$ is the second period. Then

$$\begin{aligned}
&E\left[(\Delta P_{t-1})^2 | N=1\right] = \theta\sigma_1 + (1-\theta)\sigma_0 = E\left[(\Delta P_t)^2 | N=2\right],\\
&E\left[(\Delta P_{t-1})^2 | N=2\right] = \theta\sigma_2 + (1-\theta)\sigma_0 = E\left[(\Delta P_t)^2 | N=1\right],\\
&E\left[(\Delta P_t)^2\right] = \theta\left(\tfrac{\sigma_1}{2} + \tfrac{\sigma_2}{2}\right) + (1-\theta)\sigma_0.
\end{aligned}$$

Because N is equally likely to take the values 1 or 2, the conditional covariance is

$$\begin{aligned}
&\tfrac{1}{2}\{E\left[(\Delta P_{t-1})^2 (\Delta P_t)^2 | N=1\right] - E\left[(\Delta P_{t-1})^2 | N=1\right] E\left[(\Delta P_t)^2 | N=1\right]\}\\
&+\tfrac{1}{2}\{E\left[(\Delta P_{t-1})^2 (\Delta P_t)^2 | N=2\right] - E\left[(\Delta P_{t-1})^2 | N=2\right] E\left[(\Delta P_t)^2 | N=2\right]\}.
\end{aligned} \quad (A4.1)$$

From the formulae for the expected calendar period squared price change given the value of N, $(A4.1)$ equals

$$\begin{aligned}
&\tfrac{1}{2}\{[\theta\sigma_1\sigma_2 + (1-\theta)\sigma_0^2] - (\theta\sigma_1 + (1-\theta)\sigma_0)(\theta\sigma_2 + (1-\theta)\sigma_0) + \theta[\theta\sigma_2\sigma_1 + (1-\theta)\sigma_2\sigma_0]\\
&\quad + (1-\theta)[\theta\sigma_0\sigma_1 + (1-\theta)\sigma_0^2] - (\theta\sigma_2 + (1-\theta)\sigma_0)(\theta\sigma_1 + (1-\theta)\sigma_0)\},
\end{aligned}$$

which is simplified as

$$\tfrac{1}{2}\theta(1-\theta)(\sigma_1-\sigma_0)(\sigma_2-\sigma_0). \quad (A4.2)$$

The covariance of the conditional means is

$$E\{\left(E(\Delta P_{t-1})^2 - E\left[(\Delta P_{t-1})^2 | N\right]\right)\left(E(\Delta P_t)^2 - E\left[(\Delta P_t)^2 | N\right]\right)\},$$

which equals

$$\begin{aligned}
&P(N=1)\{\left(E(\Delta P_{t-1})^2 - E\left[(\Delta P_{t-1})^2 | N=1\right]\right)\left(E(\Delta P_t)^2 - E\left[(\Delta P_t)^2 | N=1\right]\right)\}\\
&+P(N=2)\{\left(E(\Delta P_{t-1})^2 - E\left[(\Delta P_{t-1})^2 | N=2\right]\right)\left(E(\Delta P_t)^2 - E\left[(\Delta P_t)^2 | N=2\right]\right)\}.
\end{aligned}$$

Note

$$\begin{aligned}
&\left(E(\Delta P_{t-1})^2 - E\left[(\Delta P_{t-1})^2 | N=1\right]\right) = \theta(\bar{\sigma}_2 - \sigma_1),\\
&\left(E(\Delta P_t)^2 - E\left[(\Delta P_t)^2 | N=1\right]\right) = \theta(\bar{\sigma}_2 - \sigma_2),\\
&\left(E(\Delta P_{t-1})^2 - E\left[(\Delta P_{t-1})^2 | N=2\right]\right) = \theta(\bar{\sigma}_2 - \sigma_2),\\
&\left(E(\Delta P_t)^2 - E\left[(\Delta P_t)^2 | N=2\right]\right) = \theta(\bar{\sigma}_2 - \sigma_1).
\end{aligned}$$

Thus, the covariance of the conditional means is

$$\left[\theta\left(\frac{\sigma_1}{2}+\frac{\sigma_2}{2}-\sigma_1\right)\right]\left[\theta\left(\frac{\sigma_1}{2}+\frac{\sigma_2}{2}-\sigma_2\right)\right]=\theta^2\left(\frac{\sigma_1}{2}-\frac{\sigma_2}{2}\right)\left(\frac{\sigma_2}{2}-\frac{\sigma_1}{2}\right). \qquad (A4.3)$$

From $(A4.2)$ and $(A4.3)$ the $Cov\left[(\Delta P_{t-1})^2,(\Delta P_t)^2\right]$ equals

$$\tfrac{1}{2}\theta(1-\theta)(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)+\theta^2\left(\frac{\sigma_1}{2}-\frac{\sigma_2}{2}\right)\left(\frac{\sigma_2}{2}-\frac{\sigma_1}{2}\right).$$

Because $\sigma_1>\sigma_2$, the second term of the covariance is negative (while the first term is positive) and the covariance is positive if

$$(1-\theta)(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)>\frac{\theta}{2}(\sigma_1-\sigma_2)^2. \qquad (A4.4)$$

First, by inspection

$$(\sigma_1-\sigma_0)>(\sigma_1-\sigma_2).$$

Thus to verify $(A4.4)$, we need only show

$$(1-\theta)(\sigma_2-\sigma_0)>\frac{\theta}{2}(\sigma_1-\sigma_2).$$

Because $\frac{\theta}{2}(\sigma_1-\sigma_2)=\theta(\bar{\sigma}_2-\sigma_2)$, to verify the preceding inequality, we must show

$$(1-\theta)(\sigma_2-\sigma_0)-\theta(\bar{\sigma}_2-\sigma_2)>0.$$

Condition 1 implies

$$(1-\theta)(\sigma_2-\sigma_0)>\theta(1-\theta)(\bar{\sigma}_2-\sigma_0).$$

Hence

$$(1-\theta)(\sigma_2-\sigma_0)-\theta(\bar{\sigma}_2-\sigma_2)>\theta(1-\theta)(\bar{\sigma}_2-\sigma_0)-\theta(\bar{\sigma}_2-\sigma_2).$$

The right-hand side of the preceding inequality equals

$$\theta\left[(\sigma_2-\sigma_0)-\theta(\bar{\sigma}_2-\sigma_0)\right],$$

and Condition 1 implies

$$(\sigma_2-\sigma_0)-\theta(\bar{\sigma}_2-\sigma_0)>0. \qquad \blacksquare$$

6.5 Proof of Lemma 6.

If $k=3$, then the covariance of calendar period squared price changes is larger for $r=1$ than for $r=2$, so Lemma 4 is established if $Cov\left[(\Delta P_{t-2})^2,(\Delta P_t)^2\right]$ is positive. We have $Cov\left[(\Delta P_{t-2})^2,(\Delta P_t)^2\right]$ equals

$$\tfrac{1}{3}\theta\{(1-\theta)(\sigma_1-\sigma_0)(\sigma_3-\sigma_0)+\theta(\bar{\sigma}_3-\sigma_1)(\bar{\sigma}_3-\sigma_3)+\theta(\bar{\sigma}_3-\sigma_2)(\bar{\sigma}_3-\sigma_1)+\theta(\bar{\sigma}_3-\sigma_3)(\bar{\sigma}_3-\sigma_2)\}$$

The first term is positive, the second negative, and the remaing two terms are opposite in sign and depend on the sign of $(\bar{\sigma}_3-\sigma_2)$. We condsider each of the three cases: $(\bar{\sigma}_3-\sigma_2)>0$, $(\bar{\sigma}_3-\sigma_2)<0$, and $(\bar{\sigma}_3-\sigma_2)=0$ in turn.

Case 1: $(\bar{\sigma}_3-\sigma_2)>0$

If $(\bar{\sigma}_3-\sigma_2)>0$, then $\sigma_1>\bar{\sigma}_3>\sigma_2>\sigma_3>\sigma_0$. Define $d_1=\sigma_1-\bar{\sigma}_3$, $d_2=\bar{\sigma}_3-\sigma_2$, $d_3=\sigma_2-\sigma_3$, and $d_4=\sigma_3-\sigma_0$. The $Cov\left[(\Delta P_{t-2})^2,(\Delta P_t)^2\right]$ is positive if

$$(1-\theta)(\sigma_1-\sigma_0)(\sigma_3-\sigma_0)+\theta(\bar{\sigma}_3-\sigma_3)(\bar{\sigma}_3-\sigma_2)>\left|\theta(\bar{\sigma}_3-\sigma_1)(\bar{\sigma}_3-\sigma_3)+\theta(\bar{\sigma}_3-\sigma_2)(\bar{\sigma}_3-\sigma_1)\right|,$$

which is equivalently expressed as

$$(1-\theta)\left(\textstyle\sum_{j=1}^{4}d_j\right)d_4+\theta(d_2+d_3)d_2>\theta d_1(2d_2+d_3)\,. \qquad (A5.1)$$

Rewrite $(A5.1)$ as

$$d_1(1-\theta)d_4+(1-\theta)\left(\textstyle\sum_{j=2}^{4}d_j\right)d_4+\theta(d_2+d_3)d_2>d_1\theta(d_2+d_3)+\theta d_1 d_2.$$

If Condition 1 holds for period 3, then

$$(1-\theta)d_4>\theta(d_2+d_3)\,,$$

and $(A5.1)$ is satisified if

$$d_2(1-\theta)d_4+\theta d_2(d_2+d_3)+(1-\theta)\left(\textstyle\sum_{j=3}^{4}d_j\right)d_4>\theta d_1 d_2. \qquad (A5.2)$$

If Condition 1 holds for period 3, then

$$(1-\theta)d_4>\theta d_2,$$

and $(A5.2)$ is satisfied if

$$\theta d_2(2d_2+d_3)-\theta d_1 d_2=\theta d_2(2d_2+d_3-d_1)=0$$

From the definition of $\bar{\sigma}_3$, $\sum_{j=1}^{3}(\sigma_j-\bar{\sigma}_3)=d_1-2d_2-d_3$, so

$$(2d_2+d_3-d_1)=0.$$

Case 2: $(\bar{\sigma}_3-\sigma_2)<0$

If $(\bar{\sigma}_3 - \sigma_2) < 0$, then $\sigma_1 > \sigma_2 > \bar{\sigma}_3 > \sigma_1 > \sigma_0$. Define $d_1 = \sigma_1 - \sigma_2$, $d_2 = \sigma_2 - \bar{\sigma}_3$, $d_3 = \bar{\sigma}_3 - \sigma_3$, and $d_4 = \sigma_3 - \sigma_0$. The $Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right]$ is positive if

$$(1-\theta)(\sigma_1 - \sigma_0)(\sigma_3 - \sigma_0) + \theta(\bar{\sigma}_3 - \sigma_1)(\bar{\sigma}_3 - \sigma_2) > \left|\theta(\bar{\sigma}_3 - \sigma_1)(\bar{\sigma}_3 - \sigma_3) + \theta(\bar{\sigma}_3 - \sigma_2)(\bar{\sigma}_3 - \sigma_3)\right|,$$

which is equivalently expressed as

$$(1-\theta)\left(\textstyle\sum_{j=1}^{4} d_j\right) d_4 + \theta(d_1 + d_2) d_2 > \theta d_3 (2d_2 + d_1). \tag{A5.3}$$

From the definition of $\bar{\sigma}_3$,

$$2d_2 + d_1 = d_3,$$

so $(A5.3)$ is satisfied if

$$(1-\theta) d_3 d_4 - \theta d_3^2 > 0.$$

Note $(1-\theta) d_3 d_4 - \theta d_3^2 = d_3 (d_4 - \theta(d_3 + d_4))$. If Condition 1 holds for period 3

$$\sigma_3 - \sigma_0 > \theta(\bar{\sigma}_3 - \sigma_0),$$

which is equivalently expressed as

$$d_4 > \theta(d_3 + d_4).$$

Case 3: $(\bar{\sigma}_3 - \sigma_2) = 0$

The $Cov\left[(\Delta P_{t-2})^2, (\Delta P_t)^2\right]$ is positive if

$$(1-\theta)(\sigma_1 - \sigma_0)(\sigma_3 - \sigma_0) > \left|\theta(\bar{\sigma}_3 - \sigma_1)(\bar{\sigma}_3 - \sigma_3)\right|.$$

First, by inspection

$$(\sigma_1 - \sigma_0) > (\sigma_1 - \bar{\sigma}_3).$$

Second, if Conditon 1 holds for period 3

$$(1-\theta)(\sigma_3 - \sigma_0) > \theta(\bar{\sigma}_3 - \sigma_3). \qquad \blacksquare$$

7 Figures