Modified Local Whittle Estimation of the Memory Parameter in the Nonstationary Case[*,†]

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Abstract

Semiparametric estimation of the memory parameter is studied in models of fractional integration in the nonstationary case, and some new representation theory for the discrete Fourier transform of a fractional process is used to assist in the analysis. A limit theory is developed for a new estimator of the memory parameter that covers a range of values of d commonly encountered in applied work with economic data. The new estimator is called the modified local Whittle estimator and employs a version of the Whittle likelihood based on frequencies adjacent to the origin and modified to take into account the form of the data generating mechanism in the frequency domain. The modified local Whittle estimator is shown to be consistent for $0 < d < 2$ and is asymptotically normally distributed with variance $\frac{1}{4}$ for $\frac{1}{2} < d < \frac{7}{4}$. The approach allows for likelihood-based inference about d in a context that includes nonstationary data, is agnostic about short memory components and permits the construction of valid confidence regions for d that extend into the nonstationary region.

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1 Introduction

Fractional integration and the study of the so-called $I(d)$ processes has recently attracted a good deal of attention amongst theorists and empirical researchers. In applied econometric work, $I(d)$ processes with fractional $d > 0$ have been found to provide good empirical models for certain financial time series and volatility measures, as well as certain macroeconomic time series like inflation and interest rates. Fractional processes accommodate temporal dependence in a time series that is intermediate in form between short-memory series (the so-called $I(0)$ processes) and unit root time series ($I(1)$ processes). Fractional models encompass both stationary and nonstationary processes depending on the value of the memory parameter, and include both $I(0)$ and $I(1)$ processes as limiting cases when the memory parameter takes on the values zero and unity. For these reasons, fractional integration is attractive to empirical researchers, providing some liberation from the classical dichotomy of $I(0)$ and $I(1)$ processes. Growing evidence in applied work indicates that fractionally integrated processes can describe certain long range characteristics of economic data rather well, including the volatility of financial asset returns, forward exchange market premia, interest rate differentials, and inflation rates.

The memory parameter, d, plays a central role in the definition of fractional integration and is often the focus of empirical interest. When $d \geq \frac{1}{2}$, the process is nonstationary and there are several ways of defining the observed series in terms of weakly dependent inputs. In this paper, we define fractionally integrated processes as weighted sums of short-memory input variables, which are treated nonparametrically[1]. A new representation and approximation theory for the discrete Fourier transform of a fractionally integrated time series is used here, based on recent work in Phillips (1999), which provides us with a representation that is valid in both nonstationary and (asymptotically) stationary cases. It is particularly helpful in analyzing the asymptotic behavior of the discrete Fourier transform and, hence, the periodogram of nonstationary fractionally integrated time series. So, it provides the key element in developing our theory and motivating the estimator we will use.

With this representation theory in hand, we develop a limit theory for a new estimator of the memory parameter of a fractional process allowing for nonstationary values of d. The new estimator is called the modified local Whittle estimator and employs a version of the Whittle likelihood based on frequencies adjacent to the origin and modified to take into account the form of the data generating mechanism in the frequency domain. The approach was suggested in Phillips (1999) without

[1] Some alternate definitions are discussed in Appendix B in Section 7. These include definitions based on forming partial sums of stationary long memory inputs, as well as those involving weighted sums of short memory inputs with the summation index extending into the distant, but not infinite, past .

any formal development of its properties or asymptotic behavior. The present paper takes up this study and demonstrates that the modified local Whittle estimator is consistent for $d \in (0, 2)$ and asymptotically normally distributed with variance $\frac{1}{4}$ for $d \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $d \in \left(\frac{3}{2}, \frac{7}{4}\right)$. For $d \in \left[\frac{7}{4}, 2\right)$, the limit distribution is nonnormal and the rate of convergence decreases. Thus, the approach allows for likelihood-based inference about d in a context that allows for nonstationarity, using a limit theory that is equivalent to that which applies in the stationary region for the unmodified Whittle estimator (Robinson, 1995b). In particular, it permits the construction of valid confidence regions for d that extend into the nonstationary region. In this respect, our theory complements recent work by Velasco (1999a), who extended the conventional $N\left(0, \frac{1}{4}\right)$ limit distribution theory of the unmodified Whittle estimator (given by Robinson, 1995b) to the region $d \in (-\frac{1}{2}, \frac{3}{4})$ and gave a corresponding limit theory for a version of Whittle estimator for general d that employs data tapering and where the limiting variance depends on the taper.

The remainder of the paper is organized as follows. The new representation and approximation theory that we need are developed in Section 2. Section 3 defines the modified local Whittle estimator and proves its consistency. Section 4 demonstrates asymptotic normality. Section 5 reports some simulation results and gives an empirical illustration. Proofs are collected together in Appendix A in Section 6. Some alternative nonstationary representations are discussed in Appendix B in Section 7.

2 Preliminary Representation Theory and Asymptotics

2.1 Frequency Domain Decompositions

We consider the fractional process X_t generated by the model

$$(1 - L)^d (X_t - X_0) = u_t, \quad t = 0, 1, 2, ... \tag{1}$$

where X_0 is a random variable with a certain fixed distribution. Our interest is primarily in the case where X_t is nonstationary and $\frac{1}{2} < d \leq 2$, so in (1) we work from a given initial date $t = 0$, set $u_t = 0$ for all $t \leq 0$, and assume that u_t $(t \geq 1)$ is stationary with zero mean and continuous spectrum $f_u(\lambda) > 0$. Expanding the binomial in (1) gives the form

$$\sum_{k=0}^{t} \frac{(-d)_k}{k!} (X_{t-k} - X_0) = u_t, \tag{2}$$

where

$$(d)_k = \frac{\Gamma(d + k)}{\Gamma(d)} = (d)(d + 1)...(d + k - 1),$$

is Pochammer's symbol for the forward factorial function and $\Gamma(\cdot)$ is the gamma function. When d is a positive integer, the series in (2) terminates, giving the usual formulae for the model (1) in terms of the differences and higher order differences of $X_{t.}$ An alternate form for X_t is obtained by inversion of (1), giving

$$X_t = (1-L)^{-d} u_t + X_0 = \sum_{k=0}^{t-1} \frac{(d)_k}{k!} u_{t-k} + X_0. \tag{3}$$

Throughout this paper it will be convenient to assume that the stationary component u_t in (1) is a linear process of the form

$$u_t = C(L)\,\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j\,|c_j| < \infty, \quad C(1) \neq 0, \tag{4}$$

for all t and with $\varepsilon_t = iid\left(0, \sigma^2\right)$ and $E\varepsilon_t^4 = \mu_4 < \infty$. The summability condition in (4) is satisfied by a wide class of parametric and nonparametric models for u_t and enables the use of the techniques in Phillips and Solo (1992). Under (4), the spectral density of u_t is $f_u(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} c_j e^{ij\lambda}\right|^2$.

Define the discrete Fourier transform (dft) of a time series a_t evaluated at the fundamental frequencies as

$$w_a(\lambda_s) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda_s}, \quad \lambda_s = \frac{2\pi s}{n}, s = 1, ..., n. \tag{5}$$

Our approach is to algebraically manipulate (2) so that it can be rewritten in a convenient form to accommodate dft's. The following Lemma leads to an exact expression that we can use for the model in frequency domain form.

2.2 Lemma

(a) If X_t follows (1), then

$$w_x(\lambda)\left(1 - e^{i\lambda}\right) = D_n\left(e^{i\lambda}; f\right) w_u(\lambda) - \frac{e^{in\lambda}}{\sqrt{2\pi n}} \widetilde{U}_{\lambda n}(f) - \frac{e^{i\lambda}}{\sqrt{2\pi n}}\left(e^{in\lambda} X_n - X_0\right), \tag{6}$$

where $D_n\left(e^{i\lambda}; f\right) = \sum_{k=0}^{n} \frac{(-f)_k}{k!} e^{ik\lambda}$, $f = 1 - d$, and

$$\widetilde{U}_{\lambda n}(f) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; f\right) u_n = \sum_{p=0}^{n-1} \widetilde{f}_{\lambda p} e^{-ip\lambda} u_{n-p}, \quad \widetilde{f}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-f)_k}{k!} e^{ik\lambda}. \tag{7}$$

(b) If X_t follows (1) with $d = 1$, then

$$w_x(\lambda)\left(1 - e^{i\lambda}\right) = w_u(\lambda) - \frac{e^{i\lambda}}{\sqrt{2\pi n}}\left(e^{in\lambda} X_n - X_0\right). \tag{8}$$

2.3 Component Approximations (deterministic part)

The following lemmas give approximate representations of the sinusoidal polynomials $D_n\left(e^{i\lambda_s}; d\right)$ and $\widetilde{f}_{\lambda p}$ in (6) when $\lambda_s = \frac{2\pi s}{n} \to 0$.

2.4 Lemma

For $f > -1$ *and* $\lambda_s = \frac{2\pi s}{n} \to 0$,

$$D_n\left(e^{i\lambda_s}; f\right) = \left(1 - e^{i\lambda_s}\right)^f + O\left(n^{-f} s^{-1}\right), \tag{9}$$

uniformly in s.

2.5 Lemma

For $\lambda \downarrow 0$, *uniformly in* λ,

$$\begin{aligned} \lambda^{-f}\left(1 - e^{i\lambda}\right)^f &= e^{-\frac{\pi}{2} f i} + O(\lambda), \\ \lambda^{-f}\left(1 - e^{-i\lambda}\right)^f &= e^{\frac{\pi}{2} f i} + O(\lambda). \end{aligned} \tag{10}$$

2.6 Corollary

For $f > -1$ *and* $\lambda_s = \frac{2\pi s}{n} \to 0$,

$$\begin{aligned} \lambda_s^{-f} D_n\left(e^{i\lambda_s}; f\right) &= \lambda_s^{-f}\left(1 - e^{i\lambda_s}\right)^f + \lambda_s^{-f} O\left(n^{-f} s^{-1}\right) \\ &= e^{-\frac{\pi}{2} f i} + O(\lambda_s) + O\left(s^{-1-f}\right), \end{aligned} \tag{11}$$

uniformly in s.

2.7 Lemma

Uniformly in p *and* s,

$$(a) \quad \widetilde{f}_{\lambda_s p} = \begin{cases} O\left(p^{-f}\right), & \text{for } f > 0, \\ O\left(n^{-f}\right), & \text{for } f \in (-1, 0), \end{cases} \tag{12}$$

$$(b) \quad \widetilde{f}_{\lambda_s p} = O\left(\frac{n}{p^{f+1} s}\right), \text{ f or } f > -1. \tag{13}$$

2.8 Component Approximations (stochastic part)

The following lemmas give asymptotic approximations to the term $\widetilde{U}_{\lambda n}(f)$ and X_n in (6) when $\lambda_s = \frac{2\pi s}{n} \to 0$. Note that the stochastic order of magnitude of $\widetilde{U}_{\lambda n}(f)$ changes depending on the value of f.

2.9 Lemma

For $\lambda_s = \frac{2\pi s}{n} \to 0$,

$$\widetilde{U}_{\lambda_s n}(f) = C(1)\,\widetilde{\varepsilon}_{\lambda_s n}(f) + r_{s,n}(f),$$

where

$$\widetilde{\varepsilon}_{\lambda n}(f) = \sum_{p=0}^{n-1} \widetilde{f}_{\lambda p} e^{-ip\lambda} \varepsilon_{n-p},$$

and

$$E\left|r_{s,n}(f)\right|^2 = \begin{cases} O(1), & \text{for } f > 0, \\ O\left(n^{-2f}\right) = O\left(n^{2d-2}\right), & \text{for } f \in (-1,0), \end{cases}$$

uniformly in s.

2.10 Lemma

For $f \in \left(0, \frac{1}{2}\right)$ and any number L such that $L \to \infty$ and $\frac{L}{n} \to 0$, the following holds uniformly in s:

$$\begin{aligned} (a) \quad E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= O\left(n^{1-2f}\right) = O\left(n^{2d-1}\right), \\ (b) \quad E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= O\left(L^{1-2f} + \frac{n}{s} L^{-2f}\right) = O\left(L^{2d-1} + \frac{n}{s} L^{2d-2}\right), \\ (c) \quad \left(E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2\right)^{\frac{1}{2}} &= O\left(L^{\frac{1}{2}-f} + \left(\frac{n}{s}\right)^{\frac{1}{2}} L^{-f}\right) = O\left(L^{d-\frac{1}{2}} + \left(\frac{n}{s}\right)^{\frac{1}{2}} L^{d-1}\right). \end{aligned}$$

2.11 Lemma

For $f \in \left(-\frac{1}{2}, 0\right)$ and any number L such that $L \to \infty$ and $\frac{L}{n} \to 0$, the following holds uniformly in s:

$$\begin{aligned} (a) \quad E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= O\left(n^{1-2f} s^{-1}\right) = O\left(n^{2d-1} s^{-1}\right), \\ (b) \quad E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= O\left(\frac{n^{1-f}}{s} L^{-f} + \frac{n^2}{s^2} L^{-2f-1}\right) = O\left(\frac{n^d}{s} L^{d-1} + \frac{n^2}{s^2} L^{2d-3}\right). \end{aligned}$$

2.12 Lemma

(a) For $f \in \left[\frac{1}{2}, 1\right)$, the following holds:

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 = \begin{cases} O(\log n), & \text{for } f = \frac{1}{2}, \\ O(1), & \text{for } f \in \left(\frac{1}{2}, 1\right). \end{cases}$$

(b) For $f \in \left(-1, -\frac{1}{2}\right]$, the following holds uniformly in s:

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 = \begin{cases} O\left(n^{1-2f} s^{-2}\right) = O\left(n^{2d-1} s^{-2}\right), & \text{for } f \in \left(-1, -\frac{1}{2}\right), \\ O\left(s^{-2} n^2 \log n\right), & \text{for } f = -\frac{1}{2}. \end{cases}$$

2.13 Lemma

For $d \in \left(\frac{1}{2}, 1\right)$ *and* $1 \leq t \leq n$, *we have*

(a) $X_t - X_0 = C(1) X_t^{\varepsilon} + r_t$, *where* $X_t^{\varepsilon} = \sum_{k=0}^{t-1} \frac{(d)_k}{k!} \varepsilon_{t-k}$ *and* $E|r_t|^2 = O(1)$ *uniformly in* t,

(b) $E|X_t^{\varepsilon}|^2 = O\left(n^{2d-1}\right)$,

(c) $E|X_t|^2 = O\left(n^{2d-1}\right)$.

2.14 Approximation for $w_x(\lambda_s)$

Based on the results obtained above, now we can analyze the asymptotic behavior of the "modified discrete Fourier transform"

$$v_x(\lambda_s) = w_x(\lambda_s) + \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n - X_0}{\sqrt{2\pi n}},$$

and the "modified periodogram"

$$I_v(\lambda_s) = v_x(\lambda_s) v_x(\lambda_s)^*.$$

The following lemmas give approximations of the normalized modified dft and periodogram in terms of the dft and periodogram of ε_t. These results are used in the following sections to examine asymptotic properties of the modified local Whittle estimator.

2.15 Lemma

(a) For $d \in \left(\frac{1}{2}, \frac{3}{2}\right) \setminus \{1\}$,

$$\lambda_s^d v_x(\lambda_s) = e^{\frac{\pi}{2} di} C(1) w_{\varepsilon}(\lambda_s) - \frac{\lambda_s^d C(1)}{1 - e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}} + r_{s,n}^a + r_{s,n}^b(d) + r_{s,n}^c(d),$$

where $E\left|r_{s,n}^a\right|^2 = O\left(\lambda_s^2\right)$, $E\left|r_{s,n}^b(d)\right|^2 = O\left(s^{2d-4}\right)$, *and*

$$E\left|r_{s,n}^c(d)\right|^2 = \begin{cases} O\left(s^{2d-2} n^{1-2d}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(s^{2d-2} n^{-1}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s.

(b) For $d \in \left(\frac{1}{2}, \frac{3}{2}\right) \setminus \{1\}$,

$$\lambda_s^d v_x(\lambda_s) = e^{\frac{\pi}{2} di} C(1) w_{\varepsilon}(\lambda_s) + r_{s,n}^a + r_{s,n}^b(d),$$

where $E\left|r_{s,n}^a\right|^2 = O\left(\lambda_s^2\right)$ *and*

$$E\left|r_{s,n}^b(d)\right|^2 = \begin{cases} O\left(s^{2d-2}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(s^{2d-3}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s.

(c) For $d=1$,

$$\lambda_s v_x(\lambda_s) = iC(1)\, w_\varepsilon(\lambda_s) + r^a_{s,n},$$

where $E\left|r^a_{s,n}\right|^2 = O\left(\lambda_s^2\right)$ uniformly in s.

(d) For $d \in \left(0, \frac{1}{2}\right]$,

$$\lambda_s^d v_x(\lambda_s) = e^{\frac{\pi}{2}di} C(1)\, w_\varepsilon(\lambda_s) + r^a_{s,n} + r^b_{s,n}(d),$$

where $E\left|r^a_{s,n}\right|^2 = O\left(\lambda_s^2\right)$ and $E\left|r^b_{s,n}(d)\right|^2 = O\left(s^{2d-2} n^{1-2d} \log n\right)$ uniformly in s.

(e) For $d \in \left[\frac{3}{2}, 2\right)$,

$$\lambda_s^d v_x(\lambda_s) = e^{\frac{\pi}{2}di} C(1)\, w_\varepsilon(\lambda_s) + r^a_{s,n} + r^b_{s,n}(d) + r^c_{s,n}(d),$$

where $E\left|r^a_{s,n}\right|^2 = O\left(\lambda_s^2\right)$, $E\left|r^b_{s,n}(d)\right|^2 = O\left(s^{2d-4} \log n\right)$, and $E\left|r^c_{s,n}(d)\right|^2 = O\left(s^{2d-2} n^{-1}\right)$ uniformly in s.

2.16 Corollary

(a) For $d \in \left(\frac{1}{2}, \frac{3}{2}\right) \setminus \{1\}$,

$$\lambda_s^{2d} I_v(\lambda_s) = \left| e^{\frac{\pi}{2}di} C(1)\, w_\varepsilon(\lambda_s) - \frac{\lambda_s^d C(1)}{1 - e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}} \right|^2 + R^a_{s,n} + R^b_{s,n}(d) + R^c_{s,n}(d),$$

where $E\left|R^a_{s,n}\right| = O(\lambda_s)$, $E\left|R^b_{s,n}(d)\right| = O\left(s^{d-2}\right)$, and

$$E\left|R^c_{s,n}(d)\right| = \begin{cases} O\left(s^{d-1} n^{\frac{1}{2}-d}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(s^{d-1} n^{-\frac{1}{2}}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s.

(b) For $d \in \left(\frac{1}{2}, \frac{3}{2}\right) \setminus \{1\}$,

$$\lambda_s^{2d} I_v(\lambda_s) = |C(1)|^2 I_\varepsilon(\lambda_s) + R^a_{s,n} + R^b_{s,n}(d),$$

where $I_a(\lambda_s) = w_a(\lambda_s)\, w_a(\lambda_s)^*$, $E\left|R^a_{s,n}\right| = O(\lambda_s)$, and

$$E\left|R^b_{s,n}(d)\right| = \begin{cases} O\left(s^{d-1}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(s^{d-\frac{3}{2}}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s.

(c) For $d=1$,

$$\lambda_s^2 I_v(\lambda_s) = |C(1)|^2 I_\varepsilon(\lambda_s) + R^a_{s,n},$$

where $E\left|R^a_{s,n}\right| = O(\lambda_s)$ *uniformly in* s.

(d) *For* $d \in \left(0, \frac{1}{2}\right]$,

$$\lambda_s^{2d} I_v(\lambda_s) = |C(1)|^2 I_\varepsilon(\lambda_s) + R^a_{s,n} + R^b_{s,n}(d) + R^c_{s,n}(d),$$

where $E\left|R^a_{s,n}\right| = O(\lambda_s)$, $E\left|R^b_{s,n}(d)\right| = O\left(s^{d-1} n^{\frac{1}{2}-d} (\log n)^{\frac{1}{2}}\right)$, *and* $E\left|R^c_{s,n}(d)\right| = O\left(s^{2d-2} n^{1-2d} \log n\right)$ *uniformly in* s.

(e) *For* $d \in \left[\frac{3}{2}, 2\right)$,

$$\lambda_s^{2d} I_v(\lambda_s) = |C(1)|^2 I_\varepsilon(\lambda_s) + R^a_{s,n} + R^b_{s,n}(d) + R^c_{s,n}(d) + R^e_{s,n}(d) + R^g_{s,n}(d),$$

where

$$\begin{array}{lll} E\left|R^a_{s,n}\right| = O(\lambda_s), & E\left|R^b_{s,n}\right| = O\left(s^{d-2}(\log n)^{\frac{1}{2}}\right), & E\left|R^c_{s,n}\right| = O\left(s^{d-1} n^{-\frac{1}{2}}\right), \\ E\left|R^e_{s,n}\right| = O\left(s^{2d-4}\log n\right), & E\left|R^g_{s,n}\right| = O\left(s^{2d-2} n^{-1}\right), & \end{array}$$

uniformly in s.

3 Local Gaussian Estimation: Consistency

The local Whittle likelihood function, suggested by Künsch (1987) and studied by Robinson (1995b), is defined in terms of the parameter d and G as follows:

$$Q^*_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(G\lambda_j^{-2d}\right) + \frac{\lambda_j^{2d}}{G} I_x(\lambda_j)\right], \tag{14}$$

where m is some integer less than n. We set up the modified local Whittle likelihood by replacing the periodogram ordinates, $I_x(\lambda_j)$, by the modified periodogram ordinates, $I_v(\lambda_j)$, as

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(G\lambda_j^{-2d}\right) + \frac{\lambda_j^{2d}}{G} I_v(\lambda_j)\right]. \tag{15}$$

We propose to estimate G and d by minimising $Q_m(G,d)$, so that

$$\left(\widehat{G}, \widehat{d}\right) = \arg\min_{0<G<\infty,\ d\in\Theta} Q_m(G,d),$$

where $\Theta = [\Delta_1, \Delta_2]$ and Δ_1 and Δ_2 are numbers such that $0 < \Delta_1 < \Delta_2 < \infty$. Δ_1 can be chosen as close as zero. It will be convenient in what follows to distinguish the true values of the parameters by the notation $G_0 = f_{uu}(0)$ and d_0.

Concentrating (15) with respect to G, we find that the estimate $\widehat{d}$ satisfies

$$\widehat{d} = \arg\min_d R(d),$$

where

$$R(d) = \log \widehat{G}(d) - 2d\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j, \quad \widehat{G}(d) = \frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d} I_v(\lambda_j).$$

The following result shows that $\widehat{d}$ is consistent in both stationary and nonstationary case. When $d_0 \in \left(\frac{1}{2}, \frac{3}{2}\right)$, no condition is required on the rate of m. When $d_0 \in \left[\frac{3}{2}, 2\right)$, an additional condition $\frac{n^{\alpha}}{m} + \frac{m^{2d_0-2}}{n} \to 0$ is necessary in order to achieve consistency. When $d_0 \in \left[\Delta_1, \frac{1}{2}\right]$, the rate condition on m becomes stringent. The condition $\frac{n^{1-2\Delta_1}\log n \log m}{m} \to 0$ implies that m has to grow fast for $\widehat{d}$ to be consistent, and it is difficult to satisfy when Δ_1 is small.

3.1 Theorem

If $\frac{1}{m} + \frac{m}{n} \to 0$ *as* $n \to \infty$, *then , for* $d_0 \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $\widehat{d} \to_p d_0$ *as* $n \to \infty$.

3.2 Theorem

If $d_0 \in \left[\frac{3}{2}, 2\right)$ *and* $\frac{1}{m} + \frac{m}{n} + \frac{n^{\alpha}}{m} + \frac{m^{2d_0-2}}{n} \to 0$ *as* $n \to \infty$ *for some* $\alpha > 0$, *then,* $\widehat{d} \to_p d_0$.

3.3 Theorem

If $d_0 \in \left[\Delta_1, \frac{1}{2}\right]$ *and* $\frac{1}{m} + \frac{m}{n} + \frac{n^{1-2\Delta_1}\log n \log m}{m} \to 0$ *as* $n \to \infty$, *then,* $\widehat{d} \to_p d_0$.

3.4 Theorem

If $\widehat{d} \to_p d_0$ *as* $n \to \infty$, *then,* $\widehat{G}\left(\widehat{d}\right) \to_p G_0$.

4 Local Gaussian Estimation: Asymptotic Normality

The following theorems establish asymptotic normality of the modified local Whittle estimator for $d_0 \in \left(\frac{1}{2}, \frac{7}{4}\right) \setminus \left\{\frac{3}{2}\right\}$ under somewhat stronger conditions on the rate of m.

4.1 Theorem

If $\frac{1}{m} + \frac{m^{\frac{3}{2}}\log m}{n} \to 0$ *as* $n \to \infty$, *then, for* $d_0 \in \left(\frac{1}{2}, 1\right]$, *we have*

$$m^{\frac{1}{2}}\left(\widehat{d} - d_0\right) \Rightarrow N\left(0, \frac{1}{4}\right).$$

If $\frac{1}{m} + \frac{m^{\frac{3}{2}}\log m}{n} + \frac{m^{2d_0-1}(\log m)^2}{n} \to 0$ *as* $n \to \infty$, *then, for* $d_0 \in \left(1, \frac{3}{2}\right)$, *we have*

$$m^{\frac{1}{2}}\left(\widehat{d} - d_0\right) \Rightarrow N\left(0, \frac{1}{4}\right).$$

4.2 Theorem

If $\frac{1}{m}+\frac{n^{\alpha}}{m}+\frac{m^{\frac{3}{2}}\log m}{n}\to 0$ *as* $n\to\infty$ *for some* $\alpha>0$, *then, for* $d_0\in\left(\frac{3}{2},\frac{7}{4}\right)$, *we have*

$$m^{\frac{1}{2}}\left(\widehat{d}-d_0\right)\Rightarrow N\left(0,\frac{1}{4}\right).$$

4.3 Remark

The variance of the limiting distribution is the same as in the nonstationary case. The rate condition $\frac{1}{m}+\frac{m^{\frac{3}{2}}\log m}{n}\to 0$ corresponds to the assumption A4$'$ of Robinson (1995b) with $\beta=1$. Indeed, since $C\left(e^{i\lambda}\right)$ is differentiable with a bounded derivative, if we define $f_x(\lambda)=\left|1-e^{i\lambda}\right|^{-2d}\left|C\left(e^{i\lambda}\right)\right|^2$, it holds that $f_x(\lambda)=|C(1)|^2\lambda^{-2d}(1+O(\lambda))$. An additional condition on the rate of m, $\frac{m^{2d_0-1}(\log m)^2}{n}\to 0$, becomes necessary when $d_0\in\left(1,\frac{3}{2}\right)$. When $d_0<\frac{5}{4}$, however, this condition is redundant because it is dominated by another condition $\frac{m^{\frac{3}{2}}\log m}{n}\to 0$. It becomes the most stringent when $d_0=\frac{3}{2}$, namely $\frac{m^2(\log m)^2}{n}\to 0$.

When $d_0\in\left[\frac{7}{4},2\right)$, $\widehat{d}$ has a nonnormal distribution and the rate of convergence decreases.

4.4 Theorem

If $\frac{1}{m}+\frac{n^{\alpha}}{m}+\frac{m^{\frac{3}{2}}\log m}{n}+\frac{m^{2d_0-2}(\log m)^{12}}{n}\to 0$ *as* $n\to\infty$ *for some* $\alpha>0$, *then*

(a) For $d_0=\frac{7}{4}$, *if* $E|\varepsilon_t|^p<\infty$ *for* $p>4$,

$$\sqrt{m}\left(\widehat{d}-d_0\right)=\xi_1+\xi_2,$$

where

$$\xi_1\Rightarrow N\left(0,\frac{1}{4}\right),\quad \xi_2\Rightarrow(2\pi)^{-\frac{1}{2}}B_{-\frac{1}{4}}(1)^2.$$

(b) For $d_0\in\left(\frac{7}{4},2\right)$,

$$m^{4-2d_0}\left(\widehat{d}-d_0\right)\Rightarrow\frac{(2-d_0)(2\pi)^{2d_0-4}}{(2d_0-3)^2}B_{d_0-2}(1)^2.$$

5 Simulations and Empirical Illustration

This section reports some simulations that were conducted to examine the finite sample performance of the modified local Whittle estimator (modified estimator hereafter) and the unmodified local Whittle estimator (unmodified estimator hereafter), though no theoretical results are available yet for the unmodified estimator. We generate $I(d)$ processes according to (3) with $X_0=0$ and $u_t\sim iidN(0,1)$. The bias, standard deviation, and mean squared error (MSE) were computed using 1,000 replications. Sample size and m were chosen to be $n = 500$ and $m = n^\alpha$ with $\alpha = 0.55$, 0.65, and 0.75, respectively.

Table 1 and 2 show the simulation results. For the values of d smaller than 0.5, the modified estimator has positive bias, and the bias decreases as m increases. This confirms the theoretical result that a large value of m is required to achieve consistency when $d < 0.5$. For all values of d, its standard deviation is larger than the theoretical one, and becomes very large when $d = 0.2$. The unmodified estimator has little bias when $d \leq 1.0$, but has a large negative bias and larger variance when $d \geq 1.2$ (see also Velasco (1999a)). For the value $0.6 \leq d \leq 1.0$, the variance of the two estimators are almost equal. In sum, the modified estimator gives a better estimate of d unless there is a strong prior belief that the value of d is smaller than 0.5.

Table 1. Modified local Whittle estimator: $n = 500$, $m = n^\alpha$

	$\alpha = 0.55$ $(m = 30)$ Theoretical s.d.= 0.0913			$\alpha = 0.65$ $(m = 56)$ Theoretical s.d.= 0.0668			$\alpha = 0.75$ $(m = 105)$ Theoretical s.d.= 0.0488		
	bias	s.d.	MSE	bias	s.d.	MSE	bias	s.d.	MSE
$d = 0.2$	0.1325	0.1608	0.0434	0.0939	0.1157	0.0222	0.0634	0.0837	0.0110
$d = 0.4$	0.0445	0.1278	0.0183	0.0269	0.0877	0.0084	0.0111	0.0621	0.0040
$d = 0.6$	0.0018	0.1163	0.0135	−0.0016	0.0784	0.0062	−0.0092	0.0530	0.0029
$d = 0.8$	−0.0146	0.1111	0.0126	−0.0124	0.0774	0.0061	−0.0186	0.0542	0.0033
$d = 1.0$	−0.0133	0.1131	0.0130	−0.0116	0.0762	0.0059	−0.0212	0.0518	0.0031
$d = 1.2$	−0.0122	0.1125	0.0128	−0.0139	0.0752	0.0058	−0.0262	0.0512	0.0033
$d = 1.4$	−0.0143	0.1201	0.0146	−0.0143	0.0788	0.0064	−0.0279	0.0555	0.0039
$d = 1.6$	0.0015	0.1200	0.0144	−0.0045	0.0806	0.0065	−0.0246	0.0551	0.0036
$d = 1.8$	0.0203	0.1219	0.0153	0.0112	0.0809	0.0067	−0.0145	0.0586	0.0036

Note: Theoretical s.d. is valid only for $d = 0.6 \sim 1.6$.

Table 2. Local Whittle estimator: $n = 500$, $m = n^\alpha$

	$\alpha = 0.55$ $(m = 30)$ Theoretical s.d.= 0.0913			$\alpha = 0.65$ $(m = 56)$ Theoretical s.d.= 0.0668			$\alpha = 0.75$ $(m = 105)$ Theoretical s.d.= 0.0488		
	bias	s.d.	MSE	bias	s.d.	MSE	bias	s.d.	MSE
$d = 0.2$	−0.0147	0.1151	0.0135	−0.0091	0.0773	0.0061	−0.0080	0.0545	0.0030
$d = 0.4$	−0.0015	0.1146	0.0131	−0.0043	0.0770	0.0059	−0.0101	0.0525	0.0029
$d = 0.6$	0.0042	0.1161	0.0135	0.0018	0.0789	0.0062	−0.0054	0.0544	0.0030
$d = 0.8$	0.0138	0.1143	0.0132	0.0127	0.0805	0.0066	0.0024	0.0588	0.0035
$d = 1.0$	−0.0103	0.1048	0.0111	−0.0098	0.0695	0.0049	−0.0204	0.0469	0.0026
$d = 1.2$	−0.1127	0.1079	0.0244	−0.1211	0.0825	0.0215	−0.1400	0.0712	0.0247
$d = 1.4$	−0.2933	0.1265	0.1020	−0.3128	0.1094	0.1098	−0.3399	0.0994	0.1254
$d = 1.6$	−0.4953	0.1482	0.2673	−0.5191	0.1330	0.2872	−0.5494	0.1176	0.3157
$d = 1.8$	−0.7124	0.1533	0.5310	−0.7370	0.1314	0.5605	−0.7666	0.1104	0.5999

Note: Theoretical s.d. is the one for the modified Whittle estimator.

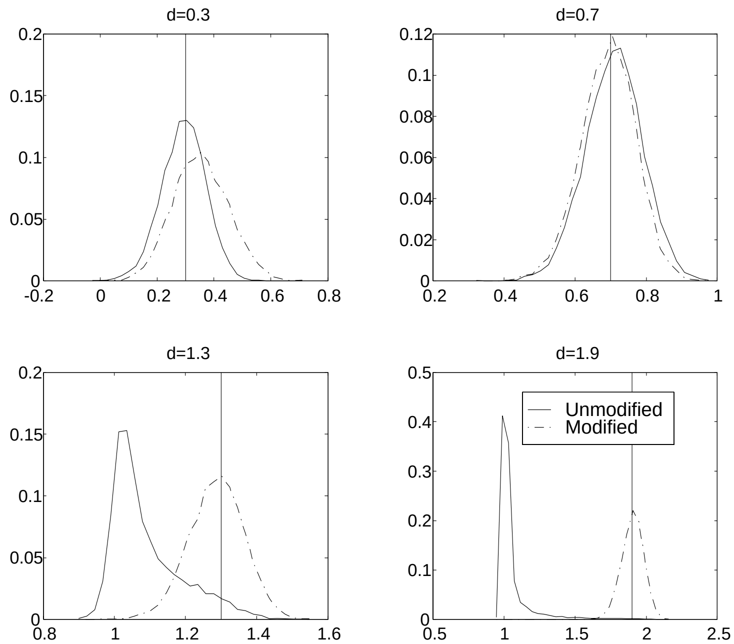

Figure 1: Modified and unmodified local Whittle estimates

Figure 1 plots the empirical probability distribution function of the modified and unmodified estimator for the values of $d = 0.3, 0.7, 1.3, 1.9$. Sample size and m were chosen to be $n = 500$, $m = n^{0.65} = 56$, and 10,000 replications are used. When $d = 0.3$, the distribution of the modified estimator is positively biased, whereas both estimator have an approximately unbiased normal pdf when $d = 0.7$. When d is larger than unity, the modified estimator still works well, whereas the unmodified estimator appears to converge to 1. The convergence to the squared fractional Brownian motion, which theoretically will occur when $d = 1.9$, does not show up with this sample size.

The modified local Whittle estimator was applied to monthly seasonally adjusted US unemployment rate series. The series constituted 621 observations over the period 1948:1-1999:9. The first panel of Figure 2 graphs the series. The second panel of Figure 2 plots $\widehat{d}$ for different values of m (Specifically, $m = n^{0.55}, ..., n^{0.75}$ were used). As m increases, $\widehat{d}$ initially increases and stays around the same level. The estimates of the memory parameter at the stable area are is in the region $(1.3, 1.5)$, indicating the series is $I(d)$ with $d > 1$.

The estimates of d higher than unity might be because of the structural change in the data. To explore this possibility, we divide the data in two periods at 1973:1 (dash-dot line in the first panel) and estimated d for each series. The third and fourth panel of Figure 2 show $\widehat{d}$ for different values of m. As the case of the whole sample, the estimate increases initially as m increases, and stays around the same level. The estimates of d are again in the region $(1.3, 1.5)$, hence the possibility of structural

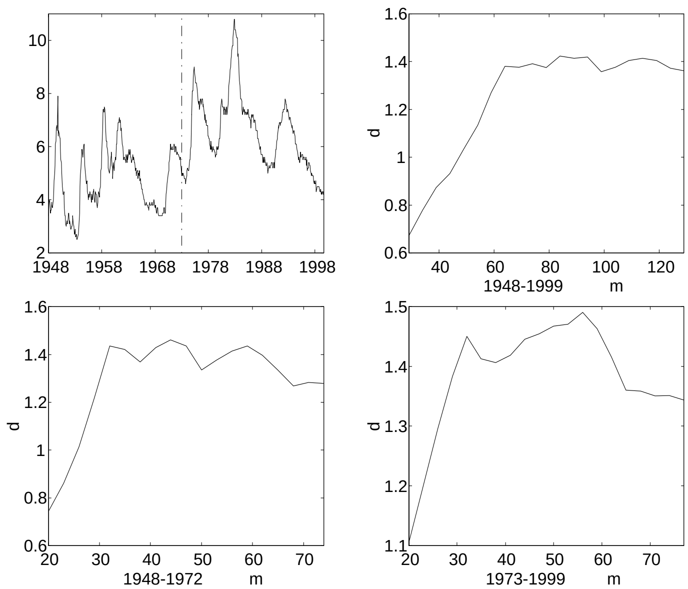

Figure 2: Unemployment rate data and estimates of d

break does not account for the high estimates of d.

6 Appendix A: Proofs

6.1 Proof of Lemma 2.2

See Theorem 2.2 and 2.7 of Phillips (1999). ■

6.2 Proof of Lemma 2.4

$$\begin{aligned} D_n\left(e^{i\lambda_s}; f\right) &= \sum_{k=0}^{n} \frac{(-f)_k}{k!} e^{ik\lambda_s} \\ &= \sum_{k=0}^{\infty} \frac{(-f)_k}{k!} e^{ik\lambda_s} - \sum_{k=n+1}^{\infty} \frac{(-f)_k}{k!} e^{ik\lambda_s} \\ &= {}_2F_1\left(-f, 1; 1; e^{i\lambda_s}\right) - \frac{1}{\Gamma(-f)} \sum_{k=n+1}^{\infty} k^{-f-1} e^{ik\lambda_s} + O\left(\sum_{k=n+1}^{\infty} k^{-f-2}\right), \end{aligned} \tag{16}$$

where the third line follows from the fact that (Erdélyi, 1953, p.47)

$$\frac{(-f)_k}{k!} = \frac{\Gamma(-f+k)}{\Gamma(-f)\Gamma(k+1)} = \frac{1}{\Gamma(-f)} k^{-f-1}\left(1 + O\left(k^{-1}\right)\right). \tag{17}$$

Because $f > -1$ and $s \neq 0$, the first term in (16) converges and equals to $\left(1 - e^{i\lambda_s}\right)^f$ (Erdélyi, 1953, p.57). For the second term in (16), by Theorem 2.2 of Zygmund (1959) we have

$$\left|\sum_{k=n+1}^{\infty} k^{-f-1} e^{2\pi isk/n}\right| \leq (n+1)^{-f-1} \max_N \left|\sum_{k=n+1}^{n+N} e^{2\pi isk/n}\right|,$$

and the ordinary summation formula gives

$$\left|\sum_{k=n+1}^{n+N} e^{2\pi isk/n}\right| = \left|\sum_{k=1}^{N} e^{2\pi isk/n}\right| = O\left(\frac{n}{s}\right).$$

uniformly in N. The third term in (16) is $O\left(n^{-f}s^{-1}\right)$ because $\sum_{k=n+1}^{\infty} k^{-f-2} = O\left(n^{-f-1}\right)$ and $s/n \to 0$. ■

6.3 Proof of Lemma 2.5

Note that $\left|1 - e^{\pm i\lambda}\right| = \left|2\sin\left(\frac{\lambda}{2}\right)\right|$. An elementary geometric argument (see the attached figure) implies that, for $0 \leq \lambda < \pi$,

$$\arg\left(1 - e^{i\lambda}\right) = \frac{\lambda - \pi}{2} \quad \text{and} \quad \arg\left(1 - e^{-i\lambda}\right) = \frac{\pi - \lambda}{2}.$$

Hence we can write $\left(1 - e^{i\lambda}\right)^f$ in polar form

$$\begin{aligned}
\left(1 - e^{i\lambda}\right)^f &= \left\{\left|2\sin\left(\frac{\lambda}{2}\right)\right| e^{i\left(\frac{\lambda}{2} - \frac{\pi}{2}\right)}\right\}^f \\
&= \left|2\sin\left(\frac{\lambda}{2}\right)\right|^f e^{if\left(\frac{\lambda}{2} - \frac{\pi}{2}\right)} \\
&= \left|2\sin\left(\frac{\lambda}{2}\right)\right|^f \left[\cos\left(\frac{\lambda f}{2} - \frac{\pi f}{2}\right) + i\sin\left(\frac{\lambda f}{2} - \frac{\pi f}{2}\right)\right].
\end{aligned}$$

Taylor expansion yields

$$\begin{aligned}
2\sin\left(\frac{\lambda}{2}\right) &= 2\cos(0)\cdot\frac{\lambda}{2} - \frac{1}{3}\cos\left(\widetilde{\lambda}\right)\cdot\left(\frac{\lambda}{2}\right)^3 = \lambda + O\left(\lambda^3\right), \\
\cos\left(\frac{\lambda f}{2} - \frac{\pi f}{2}\right) &= \cos\left(-\frac{\pi f}{2}\right) - \sin\left(\widetilde{\lambda}\right)\cdot\left(\frac{\lambda f}{2}\right) = \cos\left(-\frac{\pi f}{2}\right) + O(\lambda), \\
\sin\left(\frac{\lambda f}{2} - \frac{\pi f}{2}\right) &= \sin\left(-\frac{\pi f}{2}\right) + \cos\left(\widetilde{\lambda}\right)\cdot\left(\frac{\lambda f}{2}\right) = \sin\left(-\frac{\pi f}{2}\right) + O(\lambda),
\end{aligned}$$

and all the reminder terms are uniform in λ. Therefore, uniformly in λ,

$$\lambda^{-f}\left(1 - e^{i\lambda}\right)^f = \lambda^{-f}\left(\lambda + O\left(\lambda^3\right)\right)^f \left[\cos\left(-\frac{\pi f}{2}\right) + O(\lambda) + i\sin\left(-\frac{\pi f}{2}\right) + iO(\lambda)\right]$$

$$
\begin{aligned}
&= \left(1+O\left(\lambda^2\right)\right)^f \left[e^{-\frac{\pi}{2}fi}+O(\lambda)\right] \\
&= \left(1+O\left(\lambda^2\right)\right) \left[e^{-\frac{\pi}{2}fi}+O(\lambda)\right] \\
&= e^{-\frac{\pi}{2}fi}+O(\lambda).
\end{aligned}
$$

The approximation of $\lambda^{-f}\left(1-e^{-i\lambda}\right)^f$ follows the same line of argument. ■

6.4 Proof of Lemma 2.7

The approximation (17) yields

$$
\widetilde{f}_{\lambda_s p} = \frac{1}{\Gamma(-f)} \sum_{k=p+1}^{n} k^{-f-1}\left(1+O\left(k^{-1}\right)\right) e^{2\pi isk/n} = \frac{1}{\Gamma(-f)} \sum_{k=p+1}^{n} k^{-f-1} e^{2\pi isk/n} + O\left(\sum_{k=p+1}^{n} k^{-f-2}\right).
$$

Using the results derived in the proof of Lemma 2.4, we obtain

$$
\sum_{k=p+1}^{n} k^{-f-1}\left(1+O\left(k^{-1}\right)\right) e^{2\pi isk/n} = O\left(\sum_{k=p+1}^{n} k^{-f-1}\right) = \begin{cases} O\left(p^{-f}\right), & \text{for } f>0, \\ O\left(n^{-f}\right), & \text{for } f\in(-1,0), \end{cases}
$$

giving part (a). Part (b) follows from

$$
\begin{aligned}
\left|\sum_{k=p+1}^{n} k^{-f-1} e^{2\pi isk/n}\right| &\le (p+1)^{-f-1} \max_N \left|\sum_{k=p+1}^{p+N} e^{2\pi isk/n}\right|, \\
\left|\sum_{k=p+1}^{p+N} e^{2\pi isk/n}\right| &= O\left(\frac{n}{s}\right), \\
\sum_{k=p+1}^{n} k^{-f-2} &= O\left(p^{-f-1}\right) = O\left(\frac{n}{p^{f+1}s}\right),
\end{aligned}
$$

since $s \le n$. ■

6.5 Proof of Lemma 2.9

Applying the BN decomposition

$$
u_t = C(L)\varepsilon_t = C(1)\varepsilon_t - (1-L)\widetilde{\varepsilon}_t, \quad \widetilde{\varepsilon}_t = \sum_{j=0}^{\infty} \widetilde{c}_j \varepsilon_{t-j}, \quad \widetilde{c}_j = \sum_{s=j+1}^{\infty} c_s, \tag{18}
$$

to $\widetilde{U}_{\lambda_s n}(f)$ yields

$$
\widetilde{U}_{\lambda_s n}(f) = \sum_{p=0}^{n-1} \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} u_{n-p}
$$

$$
\begin{aligned}
&= \sum_{p=0}^{n-1} \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} \left[C(1)\varepsilon_{n-p} - (1-L)\widetilde{\varepsilon}_{n-p}\right] \\
&= C(1)\widetilde{\varepsilon}_{\lambda_s n}(f) - \sum_{p=0}^{n-1} \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} (1-L)\widetilde{\varepsilon}_{n-p}.
\end{aligned}
$$

Note that the assumption $\sum_{j=0}^{\infty} j\,|c_j| < \infty$ implies that $\sum_{j=0}^{\infty} \widetilde{c}_j^2 < \infty$ hence $E\left[\widetilde{\varepsilon}_t\right]^2 < \infty$. Rewrite the second term as follows:

$$
\begin{aligned}
& \sum_{p=0}^{n-1} \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} (1-L)\widetilde{\varepsilon}_{n-p} \\
&= \widetilde{f}_{\lambda_s 0}\widetilde{\varepsilon}_n + \sum_{p=1}^{n-1} \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s}\widetilde{\varepsilon}_{n-p} - \sum_{p=1}^{n-1} \widetilde{f}_{\lambda_s (p-1)} e^{-i(p-1)\lambda_s}\widetilde{\varepsilon}_{n-p} - \widetilde{f}_{\lambda_s (n-1)} e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0 \\
&= \sum_{p=1}^{n-1} \left[\widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} - \widetilde{f}_{\lambda_s (p-1)} e^{-i(p-1)\lambda_s}\right]\widetilde{\varepsilon}_{n-p} + \widetilde{f}_{\lambda_s 0}\widetilde{\varepsilon}_n - \widetilde{f}_{\lambda_s (n-1)} e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0.
\end{aligned}
$$

In view of the results in the proof of Lemma 2.7,

$$
\begin{cases}
\widetilde{f}_{\lambda_s 0} = O(1), \quad \widetilde{f}_{\lambda_s (n-1)} = O\left(n^{-f-1}\right), & \text{for } f > 0, \\
\widetilde{f}_{\lambda_s 0} = O\left(n^{-f}\right), \quad \widetilde{f}_{\lambda_s (n-1)} = O\left(n^{-f-1}\right), & \text{for } f \in (-1, 0).
\end{cases}
$$

Hence, using the fact that $E\left[\widetilde{\varepsilon}_t\right]^2 < \infty$, we have

$$
\begin{cases}
E\left|\widetilde{f}_{\lambda_s 0}\widetilde{\varepsilon}_n\right|^2 = O(1), \quad E\left|\widetilde{f}_{\lambda_s (n-1)} e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0\right|^2 = O\left(n^{-2f-2}\right), & \text{for } f > 0, \\
E\left|\widetilde{f}_{\lambda_s 0}\widetilde{\varepsilon}_n\right|^2 = O\left(n^{-2f}\right), \quad E\left|\widetilde{f}_{\lambda_s (n-1)} e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0\right|^2 = O\left(n^{-2f-2}\right), & \text{for } f \in (-1, 0).
\end{cases}
$$

$>$From Lemma 2.7,

$$
\begin{aligned}
\widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} - \widetilde{f}_{\lambda_s (p-1)} e^{-i(p-1)\lambda_s} &= \widetilde{f}_{\lambda_s p}\left[e^{-ip\lambda_s} - e^{-i(p-1)\lambda_s}\right] + e^{-i(p-1)\lambda_s}\left[\widetilde{f}_{\lambda_s p} - \widetilde{f}_{\lambda_s (p-1)}\right] \\
&= \widetilde{f}_{\lambda_s p} e^{-ip\lambda_s}\left(1 - e^{i\lambda_s}\right) - e^{-i(p-1)\lambda_s}\frac{(-f)_p}{p!}e^{ip\lambda_s} \\
&= O\left(\frac{n}{p^{f+1}s}\right) O\left(\frac{s}{n}\right) + O\left(p^{-f-1}\right) = O\left(p^{-f-1}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
E\left|\sum_{p=1}^{n-1}\left[\widetilde{f}_{\lambda_s p} e^{-ip\lambda_s} - \widetilde{f}_{\lambda_s (p-1)} e^{-i(p-1)\lambda_s}\right]\widetilde{\varepsilon}_{n-p}\right|^2 &\leq \sum_{p=1}^{n-1}\sum_{q=1}^{n-1} O\left(p^{-f-1}\right) O\left(q^{-f-1}\right) E\left|\widetilde{\varepsilon}_{n-p}\widetilde{\varepsilon}_{n-q}\right| \\
&= \begin{cases} O(1), & \text{for } f > 0, \\ O\left(n^{-2f}\right), & \text{for } f \in (-1, 0), \end{cases}
\end{aligned}
$$

and the result follows from Loève's c_r inequality (Davidson, 1994, p.140). ■

6.6 Proof of Lemma 2.10

For part (a), it follows from Lemma 2.7 that

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 = \sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\right|^2 = O\left(\sum_{p=1}^{n-1}p^{-2f}\right) = O\left(n^{1-2f}\right).$$

For part (b), we write $\widetilde{\varepsilon}_{\lambda_s n}(f)$ as the sum of two components, the first involving $L+1$ components. Specifically,

$$\begin{aligned} E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= E\left|\sum_{p=0}^{L}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p} + \sum_{p=L+1}^{n-1}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2 \\ &\leq 2E\left|\sum_{p=0}^{L}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2 + 2E\left|\sum_{p=L+1}^{n-1}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2, \quad (19) \end{aligned}$$

where the second line follows from Loève's c_r inequality. By Lemma 2.7, each of the terms in (19) is bounded by

$$\begin{aligned} \sum_{p=0}^{L}\left(\widetilde{f}_{\lambda_s p}\right)^2 &= O\left(\sum_{p=1}^{L}p^{-2f}\right) = O\left(L^{1-2f}\right), \\ \sum_{p=L+1}^{n-1}\left(\widetilde{f}_{\lambda_s p}\right)^2 &= O\left(\sum_{p=L+1}^{n-1}\frac{1}{p^f}\frac{n}{p^{f+1}s}\right) = O\left(\frac{n}{s}\sum_{p=L+1}^{n-1}\frac{1}{p^{2f+1}}\right) = O\left(\frac{n}{s}L^{-2f}\right). \end{aligned}$$

For part (c), Minkowski's inequality yields

$$\begin{aligned} \left(E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2\right)^{\frac{1}{2}} &\leq \left(E\left|\sum_{p=0}^{L}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2\right)^{1/2} + \left(E\left|\sum_{p=L+1}^{n-1}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2\right)^{1/2} \\ &= O\left(L^{\frac{1}{2}-f}\right) + O\left(\left(\frac{n}{s}\right)^{\frac{1}{2}}L^{-f}\right), \end{aligned}$$

giving the required result. ■

6.7 Proof of Lemma 2.11

For part (a), using Lemma 2.7 we get

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 = \sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\right|^2 = O\left(\sum_{p=1}^{n-1}n^{-f}\frac{n}{p^{f+1}s}\right) = O\left(\frac{n^{1-f}}{s}\sum_{p=1}^{n-1}p^{-f-1}\right) = O\left(n^{1-2f}s^{-1}\right).$$

Part (b) is proved by the same argument as used in Lemma 2.10. Specifically, we have

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 \leq 2E\left|\sum_{p=0}^{L}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2 + 2E\left|\sum_{p=L+1}^{n-1}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\varepsilon_{n-p}\right|^2, \quad (20)$$

and

$$\begin{aligned}\sum_{p=0}^{L}\left(\widetilde{f}_{\lambda_s p}\right)^2 &= O\left(\sum_{p=1}^{L} n^{-f}\frac{n}{p^{f+1}s}\right) = O\left(\frac{n^{1-f}}{s}\sum_{p=1}^{L} p^{-f-1}\right) = O\left(\frac{n^{1-f}}{s}L^{-f}\right),\\ \sum_{p=L+1}^{n-1}\left(\widetilde{f}_{\lambda_s p}\right)^2 &= O\left(\sum_{p=L+1}^{n-1}\frac{n^2}{p^{2f+2}s^2}\right) = O\left(\frac{n^2}{s^2}\sum_{p=L+1}^{n-1} p^{-2f-2}\right) = O\left(\frac{n^2}{s^2}L^{-2f-1}\right),\end{aligned}$$

giving the required result. ■

6.8 Proof of Lemma 2.12

Using Lemma 2.7, we have, for part (a),

$$E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 = \sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\right|^2 = O\left(\sum_{p=1}^{n-1} p^{-2f}\right) = \begin{cases} O(\log n), & \text{for } f = \frac{1}{2},\\ O(1), & \text{for } f = \left(\frac{1}{2}, 1\right),\end{cases}$$

and for part (b),

$$\begin{aligned}E\left|\widetilde{\varepsilon}_{\lambda_s n}(f)\right|^2 &= \sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}\right|^2 = O\left(\sum_{p=1}^{n-1}\frac{n^2}{p^{2f+2}s^2}\right) = O\left(\frac{n^2}{s^2}\sum_{p=1}^{n-1} p^{-2f-2}\right)\\ &= \begin{cases} O\left(n^{1-2f}s^{-2}\right), & \text{for } f = \left(-1, -\frac{1}{2}\right),\\ O\left(s^{-2}n^2\log n\right), & \text{for } f = -\frac{1}{2},\end{cases}\end{aligned}$$

giving the required result. ■

6.9 Proof of Lemma 2.13

Applying the BN decomposition to u_t and substituting into (3) yields

$$X_t - X_0 = \sum_{k=0}^{t-1}\frac{(d)_k}{k!}u_{t-k} = \sum_{k=0}^{t-1}\frac{(d)_k}{k!}\left[C(1)\varepsilon_{t-k} - (1-L)\widetilde{\varepsilon}_{t-k}\right] = C(1)X_t^{\varepsilon} - \sum_{k=0}^{t-1}\frac{(d)_k}{k!}(1-L)\widetilde{\varepsilon}_{t-k}. \tag{21}$$

Let us rearrange the second term in (21) as follows:

$$\begin{aligned}\sum_{k=0}^{t-1}\frac{(d)_k}{k!}(1-L)\widetilde{\varepsilon}_{t-k} &= \sum_{k=0}^{t-1}\frac{(d)_k}{k!}\widetilde{\varepsilon}_{t-k} - \sum_{k=0}^{t-1}\frac{(d)_k}{k!}\widetilde{\varepsilon}_{t-k-1}\\ &= \frac{(d)_0}{0!}\widetilde{\varepsilon}_t + \sum_{k=1}^{t-1}\frac{(d)_k}{k!}\widetilde{\varepsilon}_{t-k} - \sum_{k=1}^{t-1}\frac{(d)_{k-1}}{(k-1)!}\widetilde{\varepsilon}_{t-k} - \frac{(d)_{t-1}}{(t-1)!}\widetilde{\varepsilon}_0\\ &= \sum_{k=1}^{t-1}\frac{(d-1)_k}{k!}\widetilde{\varepsilon}_{t-k} + \widetilde{\varepsilon}_t - \frac{(d)_{t-1}}{(t-1)!}\widetilde{\varepsilon}_0,\end{aligned}$$

where the fourth line follows from the fact that

$$\begin{aligned}
\frac{(d)_k}{k!}-\frac{(d)_{k-1}}{(k-1)!} &= \frac{1}{\Gamma(d)}\left[\frac{\Gamma(d+k)}{\Gamma(k+1)}-\frac{\Gamma(d+k-1)}{\Gamma(k)}\right] \\
&= \frac{\Gamma(d+k-1)}{\Gamma(d)\Gamma(k+1)}\left[(d+k-1)-k\right] \\
&= \frac{\Gamma(d+k-1)}{\Gamma(d-1)\Gamma(k+1)}=\frac{(d-1)_k}{k!}. \qquad (22)
\end{aligned}$$

When we take the mean squared of the first term, it is bounded by

$$E\left[\sum_{k=1}^{t-1}k^{d-2}\widetilde{\varepsilon}_{t-k}\right]^2=E\left[\sum_{k=1}^{t-1}k^{d-2}\widetilde{\varepsilon}_{t-k}\right]\left[\sum_{l=1}^{t-1}l^{d-2}\widetilde{\varepsilon}_{t-l}\right]=\sum_{k=1}^{t-1}k^{d-2}\sum_{l=1}^{t-1}l^{d-2}E\left(\widetilde{\varepsilon}_{t-k}\widetilde{\varepsilon}_{t-l}\right)=O(1), \qquad (23)$$

by the fact that $d-2<-1$, $E\widetilde{\varepsilon}_{t-k}^2<\infty$, and Cauchy-Schwartz inequality. Trivially $E\widetilde{\varepsilon}_t^2=O(1)$ and $E\left|\frac{(d)_{t-1}}{(t-1)!}\widetilde{\varepsilon}_0\right|^2=O(1)$, and part (a) follows from Loève's c_r inequality.

For part (b) and (c), $E\left[X_t^{\varepsilon}\right]^2$ is bounded by $\sigma^2\sum_{k=1}^{t-1}k^{2(d-1)}=O\left(t^{2d-1}\right)=O\left(n^{2d-1}\right)$, giving the required result. ■

6.10 Proof of Lemma 2.15

Multiplying both sides of (6) by $\lambda_s^d\left(1-e^{i\lambda_s}\right)^{-1}$ yields

$$\lambda_s^d w_x(\lambda_s)+\frac{\lambda_s^d}{1-e^{i\lambda_s}}\frac{e^{i\lambda_s}(X_n-X_0)}{\sqrt{2\pi n}}=\frac{\lambda_s^d D_n\left(e^{i\lambda_s};f\right)}{1-e^{i\lambda_s}}w_u(\lambda_s)-\frac{\lambda_s^d}{1-e^{i\lambda_s}}\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}}. \qquad (24)$$

Using Lemma 2.5 and Corollary 2.6, we have

$$\begin{aligned}
\frac{\lambda_s^d D_n\left(e^{i\lambda_s};f\right)}{1-e^{i\lambda_s}} &= \frac{\lambda_s^{-f}D_n\left(e^{i\lambda_s};f\right)}{\lambda_s^{-1}\left(1-e^{i\lambda_s}\right)}=\frac{e^{-\frac{\pi}{2}fi}+O(\lambda_s)+O\left(s^{-1-f}\right)}{e^{-\frac{\pi}{2}i}+O(\lambda_s)} \\
&= e^{\frac{\pi}{2}di}+O(\lambda_s)+O\left(s^{-1-f}\right), \qquad (25) \\
\frac{\lambda_s^d}{1-e^{i\lambda_s}} &= O\left(\lambda_s^{d-1}\right). \qquad (26)
\end{aligned}$$

Since $E\varepsilon^4<\infty$ and $\sum_{j=0}^{\infty}j\left|c_j\right|<\infty$, $w_u(\lambda_s)$ can be approximated as follows (Hannan, 1970, p.248):

$$w_u(\lambda_s)=C\left(e^{i\lambda_s}\right)w_\varepsilon(\lambda_s)+r_n(\lambda_s),$$

where

$$E\left|w_\varepsilon(\lambda_s)\right|^2=\frac{\sigma^2}{2\pi},\quad E\left|r_n(\lambda_s)\right|^2=O\left(n^{-2}\right),$$

uniformly in s. $C\left(e^{i\lambda}\right)$ is differentiable with a bounded derivative because $\sum_{j=0}^{\infty}j\left|c_j\right|<\infty$. Therefore, Taylor expansion gives $C\left(e^{i\lambda_s}\right)=C(1)+O(\lambda_s)$ uniformly in s. It follows that

$$w_u(\lambda_s) = C(1)\, w_\varepsilon(\lambda_s) + O(\lambda_s)\, w_\varepsilon(\lambda_s) + r_n(\lambda_s) = C(1)\, w_\varepsilon(\lambda_s) + r_n^1(\lambda_s), \quad (27)$$

where $E\left|r_n^1(\lambda_s)\right|^2 = O\left(\lambda_s^2\right)$. Combining (25) and (27), we obtain the approximation of the first term in (24), viz.

$$\begin{aligned}
& \frac{\lambda_s^d D_n\left(e^{i\lambda_s}; f\right)}{1-e^{i\lambda_s}} w_u(\lambda_s) \\
= \; & e^{\frac{\pi}{2}di} C(1)\, w_\varepsilon(\lambda_s) + e^{\frac{\pi}{2}di} r_n^1(\lambda_s) + \left[O(\lambda_s) + O\left(s^{-1-f}\right)\right]\left[C(1)\, w_\varepsilon(\lambda_s) + r_n^1(\lambda_s)\right] \\
= \; & e^{\frac{\pi}{2}di} C(1)\, w_\varepsilon(\lambda_s) + r_n^a(\lambda_s) + r_n^2(\lambda_s),
\end{aligned}$$

where $E\left|r_n^a(\lambda_s)\right|^2 = O\left(\lambda_s^2\right)$ and $E\left|r_n^2(\lambda_s)\right|^2 = O\left(s^{-2-2f}\right) = O\left(s^{2d-4}\right)$.

Now we derive the bound of the second term in (24). It follows from Lemma 2.9 and (26) that

$$\frac{\lambda_s^d}{1-e^{i\lambda_s}} \frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}} = \frac{\lambda_s^d}{1-e^{i\lambda_s}} \frac{C(1)\, \widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}} + \frac{\lambda_s^d}{1-e^{i\lambda_s}} \frac{r_{s,n}(f)}{\sqrt{2\pi n}},$$

where

$$E\left|\frac{\lambda_s^d}{1-e^{i\lambda_s}} \frac{r_{s,n}(f)}{\sqrt{2\pi n}}\right|^2 = \begin{cases} O\left(\lambda_s^{2d-2} n^{-1}\right) = O\left(s^{2d-2} n^{1-2d}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(\lambda_s^{2d-2} n^{2d-3}\right) = O\left(s^{2d-2} n^{-1}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s, giving part (a).

For part (b), using Lemma 2.10 (a) and 2.11 (a), we get

$$E\left|\frac{\lambda_s^d C(1)}{1-e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2 = \begin{cases} O\left(\lambda_s^{2d-2} n^{2d-2}\right) = O\left(s^{2d-2}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(\lambda_s^{2d-2} n^{2d-2} s^{-1}\right) = O\left(s^{2d-3}\right), & \text{for } d \in \left(1, \frac{3}{2}\right). \end{cases}$$

It follows from Minkowski's inequality that

$$E\left|r_n^2(\lambda_s) + \frac{\lambda_s^d}{1-e^{i\lambda_s}} \frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2 = \begin{cases} O\left(s^{2d-2}\right), & \text{for } d \in \left(\frac{1}{2}, 1\right), \\ O\left(s^{2d-3}\right), & \text{for } d \in \left(1, \frac{3}{2}\right), \end{cases}$$

uniformly in s, giving the required result.

For part (c), a straightforward application of Lemma 2.2 (b) yields

$$\begin{aligned}
\lambda_s w_x(\lambda_s) + \frac{\lambda_s}{1-e^{i\lambda_s}} \frac{e^{i\lambda_s}(X_n - X_0)}{\sqrt{2\pi n}} &= \frac{\lambda_s}{1-e^{i\lambda_s}} w_u(\lambda_s) \qquad (28) \\
&= \left(e^{\frac{\pi}{2}i} + O(\lambda_s)\right)\left[C(1)\, w_\varepsilon(\lambda_s) + r_n^1(\lambda_s)\right] \\
&= iC(1)\, w_\varepsilon(\lambda_s) + O(\lambda_s)\left[C(1)\, w_\varepsilon(\lambda_s) + r_n^1(\lambda_s)\right] + e^{\frac{\pi}{2}i} r_n^1(\lambda_s) \\
&= iC(1)\, w_\varepsilon(\lambda_s) + r_{s,n}^c.
\end{aligned}$$

For part (d), using Lemmas 2.9 and 2.12, we have

$$E\left|\frac{\lambda_s^d}{1-e^{i\lambda_s}}\frac{r_{s,n}(f)}{\sqrt{2\pi n}}\right|^2=O\left(\lambda_s^{2d-2}n^{-1}\right)=O\left(s^{2d-2}n^{1-2d}\right),$$

$$E\left|\frac{\lambda_s^d C(1)}{1-e^{i\lambda_s}}\frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2=\begin{cases} O\left(\lambda_s^{2d-2}n^{-1}\right)=O\left(s^{2d-2}n^{1-2d}\right), & \text{for } d\in\left(0,\frac{1}{2}\right), \\ O\left(\lambda_s^{2d-2}n^{-1}\log n\right)=O\left(s^{2d-2}n^{1-2d}\log n\right), & \text{for } d=\frac{1}{2}. \end{cases}$$

It follows that

$$E\left|r_n^2(\lambda_s)+\frac{\lambda_s^d}{1-e^{i\lambda_s}}\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2=O\left(s^{2d-2}n^{1-2d}\log n\right),$$

giving the required result.

For part (e), a similar calculation yields

$$E\left|\frac{\lambda_s^d}{1-e^{i\lambda_s}}\frac{r_{s,n}(f)}{\sqrt{2\pi n}}\right|^2=O\left(\lambda_s^{2d-2}n^{2d-3}\right)=O\left(s^{2d-2}n^{-1}\right),$$

$$\begin{aligned} & E\left|\frac{\lambda_s^d C(1)}{1-e^{i\lambda_s}}\frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2 \\ = & \begin{cases} O\left(\lambda_s^{2d-2}n^{2d-2}s^{-2}\right)=O\left(s^{2d-4}\right), & \text{for } d\in\left(\frac{3}{2},2\right), \\ O\left(\lambda_s^{2d-2}s^{-2}n\log n\right)=O\left(s^{2d-4}n^{3-2d}\log n\right)=O\left(s^{2d-4}\log n\right), & \text{for } d=\frac{3}{2}. \end{cases} \end{aligned}$$

Thus

$$E\left|r_n^2(\lambda_s)+\frac{\lambda_s^d C(1)}{1-e^{i\lambda_s}}\frac{\widetilde{\varepsilon}_{\lambda_s n}(f)}{\sqrt{2\pi n}}\right|^2=O\left(s^{2d-4}\log n\right),$$

and the stated result follows. ■

6.11 Proof of Theorem 3.1

Define $G(d)=G_0\frac{1}{m}\sum_{j=1}^m\lambda_j^{2(d-d_0)}$ and $S(d)=R(d)-R(d_0)$. Rewrite $S(d)$ as follows:

$$\begin{aligned} S(d) &= R(d)-R(d_0) \\ &= \log\widehat{G}(d)-\log\widehat{G}(d_0)-(2d-2d_0)\frac{1}{m}\sum_{j=1}^m\log\lambda_j \\ &= \log\frac{\widehat{G}(d)}{G_0\frac{1}{m}\sum_{j=1}^m\lambda_j^{2(d-d_0)}}-\log\frac{\widehat{G}(d_0)}{G_0}+\log\left(\frac{1}{m}\sum_{j=1}^m\lambda_j^{2(d-d_0)}\right) \\ &\quad -(2d-2d_0)\frac{1}{m}\sum_{j=1}^m\log\lambda_j \end{aligned}$$

$$
\begin{aligned}
&= \log \frac{\widehat{G}(d)}{G_0 \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2(d-d_0)}} - \log \frac{\widehat{G}(d_0)}{G_0} + \log \left(\left(\frac{2\pi}{n} \right)^{2d-2d_0} \frac{1}{m} \sum_{j=1}^{m} j^{2d-2d_0} \right) \\
&\quad -(2d-2d_0) \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j \\
&= \log \frac{\widehat{G}(d)}{G(d)} - \log \frac{\widehat{G}(d_0)}{G_0} + \log \left(\frac{1}{m} \sum_{j=1}^{m} j^{2d-2d_0} / \frac{m^{2(d-d_0)}}{2(d-d_0)+1} \right) \\
&\quad -(2d-2d_0) \frac{1}{m} \sum_{j=1}^{m} \log j + \log \left(\frac{m^{2(d-d_0)}}{2(d-d_0)+1} \right) \\
&= \log \frac{\widehat{G}(d)}{G(d)} - \log \frac{\widehat{G}(d_0)}{G_0} + \log \left(\frac{1}{m} \sum_{j=1}^{m} j^{2d-2d_0} / \frac{m^{2(d-d_0)}}{2(d-d_0)+1} \right) \\
&\quad -(2d-2d_0) \left[\frac{1}{m} \sum_{j=1}^{m} \log j - (\log m - 1) \right] \\
&\quad +(2d-2d_0) - \log\left(2(d-d_0)+1\right).
\end{aligned}
$$

For arbitrary small $\Delta > 0$, define $\Theta_1 = \left\{ d : d_0 - \frac{1}{2} + \Delta < d \leq \Delta_2 \right\}$ and $\Theta_2 = \left\{ d : \Delta_1 \leq d \leq d_0 - \frac{1}{2} + \Delta \right\}$. Without loss of generality, we assume $\Delta < \frac{1}{4}$ hereafter. In view of the arguments in Robinson (1995b), $\widehat{d} \to_p d_0$ if

$$
\sup_{\Theta_1} |T(d)| \to_p 0,
$$

and

$$
\Pr\left(\inf_{\Theta_2} S(d) \leq 0 \right) \to 0,
$$

as $n \to \infty$, where

$$
\begin{aligned}
T(d) &= \log \frac{\widehat{G}(d_0)}{G_0} - \log \frac{\widehat{G}(d)}{G(d)} - \log \left(\frac{1}{m} \sum_{j=1}^{m} j^{2d-2d_0} / \frac{m^{2(d-d_0)}}{2(d-d_0)+1} \right) \\
&\quad +(2d-2d_0) \left[\frac{1}{m} \sum_{j=1}^{m} \log j - (\log m - 1) \right].
\end{aligned}
$$

>From Lemma 1 and Lemma 2 of Robinson (1995b), for $d \in \Theta_1$, we have

$$
\begin{aligned}
\frac{1}{m} \sum_{j=1}^{m} \log j - (\log m - 1) &= O\left(\frac{\log m}{m} \right), \\
\frac{2(d-d_0)+1}{m} \sum_{j=1}^{m} \left(\frac{j}{m} \right)^{2d-2d_0} - 1 &= O\left(\frac{1}{m^{2\Delta_1}} \right).
\end{aligned} \tag{29}
$$

Note that

$$
\begin{aligned}
& \frac{\widehat{G}\left(d\right)-G(d)}{G\left(d\right)} \\
= & \frac{\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d}I_v\left(\lambda_j\right)-G_0\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2(d-d_0)}}{G_0\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2(d-d_0)}} \\
= & \frac{\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d_0}\left(\frac{j}{m}\right)^{2(d-d_0)}I_v\left(\lambda_j\right)-G_0\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2(d-d_0)}}{G_0\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2(d-d_0)}} \\
= & \frac{\left[2\left(d-d_0\right)+1\right]\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0}\left[\lambda_j^{2d_0}I_v\left(\lambda_j\right)-G_0\right]}{\left[2\left(d-d_0\right)+1\right]G_0\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2(d-d_0)}} \\
= & \frac{A\left(d\right)}{B\left(d\right)}. \qquad (30)
\end{aligned}
$$

Therefore, by the fact that $\Pr\left(\left|\log Y\right|\geq\varepsilon\right)\leq\Pr\left(\left|Y-1\right|\geq\varepsilon/2\right)$ for any nonnegative random variable Y and $\varepsilon\leq 1$, $\sup_{\Theta_1}\left|T\left(d\right)\right|\rightarrow_p 0$ if

$$
\sup_{\Theta_1}\left|\frac{A\left(d\right)}{B\left(d\right)}\right|\rightarrow_p 0.
$$

>From Corollary 2.16 (b) and (c), we have

$$
\lambda_j^{2d_0}I_v\left(\lambda_j\right)=\left|C\left(1\right)\right|^2I_\varepsilon\left(\lambda_j\right)+R_{j,n}^a+R_{j,n}^b\left(d_0\right),
$$

where $E\left|R_{j,n}^a\right|=O\left(\lambda_j\right)$ and

$$
E\left|R_{j,n}^b\left(d_0\right)\right|=\begin{cases} O\left(j^{d_0-1}\right), & \text{for } d_0\in\left(\frac{1}{2},1\right), \\ 0, & \text{for } d_0=1, \\ O\left(j^{d_0-\frac{3}{2}}\right), & \text{for } d_0\in\left(1,\frac{3}{2}\right), \end{cases}
$$

uniformly in j. Thus, in view of the fact that $G_0=f_u\left(0\right)=\frac{\sigma^2}{2\pi}\left|C\left(1\right)\right|^2$, $A\left(d\right)$ can be written as

$$
\begin{aligned}
A\left(d\right) &= \left[2\left(d-d_0\right)+1\right]\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0}\left[\lambda_j^{2d_0}I_v\left(\lambda_j\right)-G_0\right] \\
&= A_1\left(d\right)+A_2\left(d\right)+A_3\left(d\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1\left(d\right) &= g\frac{\left|C\left(1\right)\right|^2}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0}\left[I_\varepsilon\left(\lambda_j\right)-\frac{\sigma^2}{2\pi}\right], \\
A_2\left(d\right) &= g\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0}R_{j,n}^a, \quad A_3\left(d\right)=g\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0}R_{j,n}^b\left(d_0\right),
\end{aligned}
$$

and $g = 2(d - d_0) + 1$. We proceed to consider the successive terms $A_i(d)$ $i = 1, ..., 3$. For the first term $A_1(d)$, since $E\varepsilon^4 < \infty$, we have, uniformly in j and k, (Priestley, 1981, p.405)

$$\begin{aligned} EI_\varepsilon(\lambda_j) &= \frac{\sigma^2}{2\pi}, & (31)\\ Var(I_\varepsilon(\lambda_j)) &= O(1), & (32)\\ Cov(I_\varepsilon(\lambda_j), I_\varepsilon(\lambda_k)) &= O\left(n^{-1}\right), \quad j \neq k. & (33)\end{aligned}$$

>From (31), (32), and (33) and the fact that $\frac{1}{m}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0} = O(1)$ for $d \in \Theta_1$ (see (29)), it follows that

$$\begin{aligned} & E\left[\frac{1}{m}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0}\left[I_\varepsilon(\lambda_j) - \frac{\sigma^2}{2\pi}\right]\right]^2 \\ = & \frac{1}{m^2}\sum_{j=1}^m \left(\frac{j}{m}\right)^{4d-4d_0} Var\left[I_\varepsilon(\lambda_j)\right] \\ & + \frac{1}{m^2}\sum_{j\neq k}^m \left(\frac{j}{m}\right)^{2d-2d_0}\left(\frac{k}{m}\right)^{2d-2d_0} Cov\left[I_\varepsilon(\lambda_j), I_\varepsilon(\lambda_k)\right] \\ = & O\left(\frac{1}{m^2}\sum_{j=1}^m \left(\frac{j}{m}\right)^{4d-4d_0}\right) + O\left(\frac{1}{m^2}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0}\sum_{k=1}^m \left(\frac{k}{m}\right)^{2d-2d_0} n^{-1}\right) \\ = & \left\{\begin{array}{ll} O(\log m/m) & \text{for } 4d - 4d_0 \geq -1 \\ O\left(m^{-2-4d+4d_0}\right) & \text{for } -2 + 4\Delta < 4d - 4d_0 < -1 \end{array}\right\} + O\left(n^{-1}\right) \\ = & O\left(m^{-4\Delta} + m^{-1}\log m + n^{-1}\right) = O\left(m^{-4\Delta} + n^{-1}\right). \qquad (34)\end{aligned}$$

Therefore, for all $d \in \Theta_1$ we have

$$A_1(d) = O_p\left(m^{-2\Delta} + n^{-\frac{1}{2}}\right). \qquad (35)$$

Next consider $A_2(d)$ and $A_3(d)$. $E\left|A_2(d)\right|$ is bounded by

$$\frac{1}{m}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0}\frac{j}{n} = O\left(\frac{1}{m}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0}\frac{m}{n}\right) = O\left(\frac{m}{n}\right).$$

$E\left|A_3(d)\right| = 0$ for $d_0 = 1$, and for $d_0 \in \left(\frac{1}{2}, 1\right)$, we get

$$\begin{aligned} E\left|A_3(d)\right| &= O\left(\frac{1}{m}\sum_{j=1}^m \left(\frac{j}{m}\right)^{2d-2d_0} j^{d_0-1}\right) \\ &= O\left(m^{2d_0-2d-1}\sum_{j=1}^m j^{2d-d_0-1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} O\left(m^{d_0-1}\right) & \text{for } 2d-d_0>0 \\ O\left(m^{2d_0-2d-1}\log m\right) & \text{for } 2d-d_0\leq 0 \end{cases} \\
&= O\left(m^{d_0-1}+m^{-2\Delta}\log m\right).
\end{aligned}
$$

For $d_0 \in \left(1, \frac{3}{2}\right)$, we obtain

$$
\begin{aligned}
E\left|A_3(d)\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} j^{d_0-\frac{3}{2}}\right) \\
&= O\left(m^{2d_0-2d-1}\sum_{j=1}^{m} j^{2d-d_0-\frac{3}{2}}\right) \\
&= \begin{cases} O\left(m^{d_0-\frac{3}{2}}\right) & \text{for } 2d-d_0-\frac{3}{2}>-1 \\ O\left(m^{2d_0-2d-1}\log m\right) & \text{for } 2d-d_0-\frac{3}{2}\leq -1 \end{cases} \\
&= O\left(m^{d_0-\frac{3}{2}}+m^{-2\Delta}\log m\right).
\end{aligned}
$$

Thus, $A_2(d) = O_p\left(n^{-1}m\right)$ and

$$
A_3(d) = \begin{cases} O_p\left(m^{d_0-1}+m^{-2\Delta}\log m\right), & \text{for } d_0\in\left(\frac{1}{2},1\right), \\ 0, & \text{for } d_0=1, \\ O_p\left(m^{d_0-\frac{3}{2}}+m^{-2\Delta}\log m\right), & \text{for } d_0\in\left(1,\frac{3}{2}\right), \end{cases}
$$

for all $d \in \Theta_1$.

In sum, $A(d)$ is bounded uniformly for all $d \in \Theta_1$ as follows:

$$
A(d) = O_p\left(m^{-2\Delta}+n^{-\frac{1}{2}}+n^{-1}m\right) + \begin{cases} O_p\left(m^{d_0-1}\right), & \text{for } d_0\in\left(\frac{1}{2},1\right), \\ 0, & \text{for } d_0=1, \\ O_p\left(m^{d_0-\frac{3}{2}}\right), & \text{for } d_0\in\left(1,\frac{3}{2}\right). \end{cases} \tag{36}
$$

Finally, observe that

$$
B(d) = \left[2\left(d-d_0\right)+1\right]G_0\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2(d-d_0)} = G_0 + O\left(m^{-2\Delta}\right), \tag{37}
$$

uniformly for all $d \in \Theta_1$, hence $\Pr\left(\inf_{\Theta_1} B(d) \leq G_0/2\right) \to 0$.

>From (36)-(37) we deduce that, uniformly over $d \in \Theta_1$,

$$
\sup_{\Theta_1}\left|\frac{A(d)}{B(d)}\right| = o_p(1). \tag{38}
$$

Also we have established

$$
\frac{\widehat{G}(d)}{G(d)} = 1 + \frac{\widehat{G}(d)-G(d)}{G(d)} \to_p 1.
$$

Now we consider $\Theta_2 = \left\{ d : \Delta_1 \leq d \leq d_0 - \frac{1}{2} + \Delta \right\}$. We proceed to prove that, using the same notation and technique as those in Robinson (1995b),

$$\Pr\left(\inf_{\Theta_2} S(d) \leq 0 \right) \to 0.$$

Let $p = \exp\left(m^{-1} \sum_1^m \log j\right)$ and $S(d) = \log\left\{ \widehat{D}(d) / \widehat{D}(d_0) \right\}$, where

$$\widehat{D}(d) = \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{p} \right)^{2(d-d_0)} j^{2d_0} I_v(\lambda_j).$$

It follows that

$$\inf_{\Theta_2} \widehat{D}(d) \geq \frac{1}{m} \sum_{j=1}^{m} a_j j^{2d_0} I_v(\lambda_j),$$

where

$$a_j = \begin{cases} \left(\frac{j}{p}\right)^{2\Delta - 1}, & \text{for } 1 \leq j \leq p, \\ \left(\frac{j}{p}\right)^{-2d_0 - 1}, & \text{for } p < j \leq m. \end{cases}$$

Then,

$$\Pr\left(\inf_{\Theta_2} S(d) \leq 0 \right) \leq \Pr\left(\frac{1}{m} \sum_{j=1}^{m} (a_j - 1) j^{2d_0} I_v(\lambda_j) \leq 0 \right) = \Pr\left(\frac{1}{m} \sum_{j=1}^{m} (a_j - 1) \frac{I_v(\lambda_j)}{\lambda_j^{-2d_0}} \leq 0 \right). \tag{39}$$

>From Corollary 2.16 (b) and (c), (39) is equal to

$$\Pr\left(B_1 + B_2 + B_3 \leq 0\right),$$

where

$$\begin{aligned} B_1 &= \frac{|C(1)|^2}{m} \sum_{j=1}^{m} (a_j - 1) I_\varepsilon(\lambda_j), \\ B_2 &= \frac{1}{m} \sum_{j=1}^{m} (a_j - 1) R_{j,n}^a, \quad B_3 = \frac{1}{m} \sum_{j=1}^{m} (a_j - 1) R_{j,n}^b(d_0). \end{aligned}$$

We proceed to consider the successive terms as above. For B_1,

$$B_1 = \frac{|C(1)|^2}{m} \sum_{j=1}^{m} (a_j - 1) \left[I_\varepsilon(\lambda_j) - \frac{\sigma^2}{2\pi} \right] + G_0 \frac{1}{m} \sum_{j=1}^{m} (a_j - 1).$$

As $m \to \infty$, $p \sim m/e$ and $\sum_{1 \leq j \leq p} a_j \sim \frac{m}{2\Delta e}$. In view of the magnitudes of the moments of $I_\varepsilon(\lambda_j)$ discussed above and the fact that (note that $\Delta < 1/4$)

$$\begin{aligned} \sum_{j=1}^{m} a_j &= \sum_{1 \leq j \leq p} a_j + \sum_{p+1 \leq j \leq m} a_j = O(m) + O\left(p^{2d_0+1} \int_p^m x^{-2d_0-1} dx \right) = O(m), \\ \sum_{j=1}^{m} {a_j}^2 &= p^{2-4\Delta} \sum_{j=1}^{p} j^{4\Delta - 2} + p^{4d_0+2} \sum_{j=p+1}^{m} j^{-4d_0-2} = O\left(m^{2-4\Delta} + m\right), \end{aligned}$$

we have

$$\begin{aligned}
& E\left[\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)\left[I_{\varepsilon}\left(\lambda_j\right)-\frac{\sigma^2}{2\pi}\right]\right]^2 \\
= & O\left(\frac{1}{m^2}\sum_{j=1}^{m}\left(a_j-1\right)^2\right)+O\left(\frac{1}{m^2}\sum_{j=1}^{m}\left(a_j-1\right)\sum_{k=1}^{m}\left(a_k-1\right)\frac{1}{n}\right) \\
= & O\left(m^{-4\Delta}+m^{-1}\right)+O\left(n^{-1}\right). \qquad (40)
\end{aligned}$$

Thus

$$B_1 \to_p G_0\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right).$$

$E\left|B_2\right|$ is bounded by

$$\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)\frac{j}{n}=O\left(\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)\frac{m}{n}\right)=O\left(\frac{m}{n}\right).$$

For B_3, we have

$$E\left|B_3\right|=\begin{cases} O\left(\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)j^{d_0-1}\right), & \text{for } d_0\in\left(\frac{1}{2},1\right), \\ 0, & \text{for } d_0=1, \\ O\left(\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)j^{d_0-\frac{3}{2}}\right), & \text{for } d_0\in\left(1,\frac{3}{2}\right). \end{cases}$$

This is $o\left(1\right)$ because

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}a_j j^{d_0-1} &= \frac{p^{1-2\Delta}}{m}\sum_{j=1}^{p}j^{2\Delta+d_0-2}+\frac{p^{1+2d_0}}{m}\sum_{j=p+1}^{m}j^{-2-d_0}=O\left(m^{-2\Delta}\log m+m^{d_0-1}\right), \\
\frac{1}{m}\sum_{j=1}^{m}a_j j^{d_0-\frac{3}{2}} &= \frac{p^{1-2\Delta}}{m}\sum_{j=1}^{p}j^{2\Delta+d_0-\frac{5}{2}}+\frac{p^{1+2d_0}}{m}\sum_{j=p+1}^{m}j^{-\frac{5}{2}-d_0}=O\left(m^{-2\Delta}\log m+m^{d_0-\frac{3}{2}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}j^{d_0-1} &= O\left(m^{d_0-1}\right), \\
\frac{1}{m}\sum_{j=1}^{m}j^{d_0-\frac{3}{2}} &= O\left(m^{d_0-\frac{3}{2}}\right).
\end{aligned}$$

Choose $\Delta<1/\left(2e\right)<1/4$ with no loss of generality, then for sufficiently large m,

$$\frac{1}{m}\sum_{j=1}^{m}\left(a_j-1\right)\geq\frac{1}{m}\sum_{1\leq j\leq p}a_j-1\sim\frac{1}{2\Delta e}-1>\delta>0.$$

Hence,

$$B_1 + B_2 + B_3 \to_p G_0 \frac{1}{m} \sum_{j=1}^{m} (a_j - 1) \geq G_0 \delta > 0.$$

It follows that

$$\Pr(B_1 + B_2 + B_3 \leq 0) \to 0 \quad \text{as } m \to \infty. \tag{41}$$

Therefore, $\widehat{d} \to_p d_0$, giving the stated result. ■

6.12 Proof of Theorem 3.2

Because the proof has the same structure as that of Theorem 3.1 , we deal only with the relevant parts. First, it follows from Corollary 2.16 (e) that

$$\begin{aligned} A(d) &= [2(d-d_0)+1] \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} \left[\lambda_j^{2d_0} I_v(\lambda_j) - G_0\right] \\ &= A_1(d) + A_2(d) + A_3(d) + A_4(d) + A_5(d) + A_6(d), \end{aligned}$$

where

$$\begin{aligned} A_1(d) &= g \frac{|C(1)|^2}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} \left[I_\varepsilon(\lambda_j) - \frac{\sigma^2}{2\pi}\right], \\ A_2(d) &= g \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^a, \quad A_3(d) = g \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^b(d_0), \\ A_4(d) &= g \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^c(d_0), \quad A_5(d) = g \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^e(d_0), \\ A_6(d) &= g \frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^g(d_0). \end{aligned}$$

It has already shown that $A_1(d) \to_p 0$ and $A_2(d) \to_p 0$. For $A_i(d_0)$ $i = 3, ..., 6$, we obtain

$$\begin{aligned} E|A_3(d)| &= O\left(\frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} j^{d_0-2} (\log n)^{\frac{1}{2}}\right) = O\left((\log n)^{\frac{1}{2}} m^{2d_0-2d-1} \sum_{j=1}^{m} j^{2d-d_0-2}\right) \\ &= O\left((\log n)^{\frac{1}{2}} m^{d_0-2} + (\log n)^{\frac{1}{2}} m^{-2\Delta} \log m\right), \\ E|A_4(d)| &= O\left(\frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} j^{d_0-1} n^{-\frac{1}{2}}\right) = O\left(n^{-\frac{1}{2}} m^{2d_0-2d-1} \sum_{j=1}^{m} j^{2d-d_0-1}\right) \\ &= O\left(n^{-\frac{1}{2}} m^{d_0-1} + n^{-\frac{1}{2}} m^{-2\Delta} \log m\right), \\ E|A_5(d)| &= O\left(\frac{1}{m} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} j^{2d_0-4} \log n\right) = O\left((\log n) m^{2d_0-2d-1} \sum_{j=1}^{m} j^{2d-4}\right) \end{aligned}$$

$$
\begin{aligned}
&= O\left((\log n)\, m^{2d_0-4} + (\log n)\, m^{-2\Delta}\log m\right), \\
E\left|A_6(d)\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} j^{2d_0-2}n^{-1}\right) = O\left(n^{-1}m^{2d_0-2d-1}\sum_{j=1}^{m} j^{2d-2}\right) \\
&= O\left(n^{-1}m^{2d_0-2} + n^{-1}m^{-2\Delta}\log m\right).
\end{aligned}
$$

It follows that $\sum_{i=1}^{6} A_i(d) \to_p 0$ if $\frac{n^{\alpha}}{m} + \frac{m^{2d_0-2}}{n} \to 0$ as $n \to \infty$.

Similarly, we have

$$
\begin{aligned}
B_1 &= \frac{|C(1)|^2}{m}\sum_{j=1}^{m}(a_j-1)\, I_{\varepsilon}(\lambda_j) \to_p G_0 \frac{1}{m}\sum_{j=1}^{m}(a_j-1), \\
B_2 &= \frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, R_{j,n}^{a} \to_p 0, \\
B_3 &= \frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, R_{j,n}^{b}(d_0), \quad B_4 = \frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, R_{j,n}^{c}(d_0), \\
B_5 &= \frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, R_{j,n}^{e}(d_0), \quad B_6 = \frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, R_{j,n}^{g}(d_0).
\end{aligned}
$$

$B_3, ..., B_6$ converge to zero in probability, because

$$
\begin{aligned}
E\left|B_3\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{d_0-2}(\log n)^{\frac{1}{2}}\right) = O\left((\log n)^{\frac{1}{2}} m^{-2\Delta}\log m + (\log n)^{\frac{1}{2}} m^{d_0-2}\right), \\
E\left|B_4\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{d_0-1}n^{-\frac{1}{2}}\right) = O\left(n^{-\frac{1}{2}}m^{-2\Delta}\log m + n^{-\frac{1}{2}}m^{d_0-1}\right), \\
E\left|B_5\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{2d_0-4}\log n\right) = O\left((\log n)\, m^{-2\Delta}\log m + (\log n)\, m^{2d_0-4}\right), \\
E\left|B_6\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{2d_0-2}n^{-1}\right) = O\left(n^{-1}m^{-2\Delta}\log m + n^{-1}m^{2d_0-2}\right).
\end{aligned}
$$

Hence $\sum_{i=1}^{6} B_i \to_p G_0 \frac{1}{m}\sum_{j=1}^{m}(a_j-1) > 0$ if $\frac{n^{\alpha}}{m} + \frac{m^{2d_0-2}}{n} \to 0$ as $n \to \infty$ and $\widehat{d} \to_p d_0$ follows. ■

6.13 Proof of Theorem 3.3

As above, we deal only with the relevant parts. It follows from Corollary 2.16 (d) that

$$
A(d) = A_1(d) + A_2(d) + A_3(d) + A_4(d),
$$

where $A_1(d) \to_p 0$, $A_2(d) \to_p 0$, and

$$
A_3(d) = g\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^{b}(d_0), \quad A_4(d) = g\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} R_{j,n}^{c}(d_0).
$$

For $A_3(d)$ and $A_4(d)$, we obtain

$$\begin{aligned}
E|A_3(d)| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} j^{d_0-1} n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}}\right) = O\left(n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}} m^{2d_0-2d-1}\sum_{j=1}^{m} j^{2d-d_0-1}\right) \\
&= \begin{cases} O\left(n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}} m^{d_0-1}\right), & \text{for } 2d-d_0>0, \\ O\left(n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}} m^{2d_0-2d-1}\log m\right), & \text{for } 2d-d_0\le 0, \end{cases} \\
E|A_4(d)| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2d-2d_0} j^{2d_0-2} n^{1-2d_0}\log n\right) = O\left(n^{1-2d_0}(\log n)\, m^{2d_0-2d-1}\sum_{j=1}^{m} j^{2d-2}\right) \\
&= \begin{cases} O\left(n^{1-2d_0}(\log n)\, m^{2d_0-2}\right), & \text{for } d>1/2, \\ O\left(n^{1-2d_0}(\log n)\, m^{2d_0-2d-1}\log m\right), & \text{for } d\le 1/2. \end{cases}
\end{aligned}$$

Note that, for $d_0 \le \frac{1}{2}$ we have

$$\begin{aligned}
n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}} m^{2d_0-2d-1}\log m &= n^{d_0-\frac{1}{2}}(\log n)^{-\frac{1}{2}}\left(n^{1-2d_0}(\log n)\, m^{2d_0-2d-1}\log m\right) \\
&= O\left(n^{1-2d_0}(\log n)\, m^{2d_0-2d-1}\log m\right).
\end{aligned}$$

Hence, $E|A_3(d)| \to 0$ and $E|A_4(d)| \to 0$ if (note that $d \ge \Delta_1$)

$$\frac{n^{1-2d_0}\log n}{m^{2-2d_0}} \to 0 \quad \text{and} \quad \frac{n^{1-2d_0}\log n\log m}{m^{1-2d_0+2\Delta_1}} \to 0.$$

Since $\Delta_1 \le d_0 \le \frac{1}{2}$, this is satisfied if

$$\frac{n^{1-2\Delta_1}\log n\log m}{m} \to 0.$$

For the parameter space Θ_2, we change the definition of a_j as follows:

$$a_j = \begin{cases} \left(\frac{j}{p}\right)^{2(d-d_0)}, & \text{for } 1\le j\le p, \\ \left(\frac{j}{p}\right)^{-2d_0-1}, & \text{for } p<j\le m. \end{cases}$$

It still holds that

$$\inf_{\Theta_2}\widehat{D}(d) \ge \frac{1}{m}\sum_{j=1}^{m} a_j j^{2d_0} I_v(\lambda_j),$$

and hence

$$\Pr\left(\inf_{\Theta_2} S(d)\le 0\right) \le \Pr\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\frac{I_v(\lambda_j)}{\lambda_j^{-2d_0}} \le 0\right).$$

Since $d \ge \Delta_1 > 0$ and $d_0 \le \frac{1}{2}$, we have $2d-2d_0 > -1$. Thus

$$\begin{aligned}
\sum_{1\le j\le p} a_j &= p^{2(d_0-d)}\sum_{1\le j\le p} j^{2(d-d_0)} \sim \frac{p}{2(d-d_0)+1} \sim \frac{m}{e}\frac{1}{2(d-d_0)+1} \ge \frac{m}{2\Delta e}, \\
\sum_{1\le j\le p} {a_j}^2 &= p^{4(d_0-d)}\sum_{1\le j\le p} j^{4(d-d_0)} = \begin{cases} O(m) & \text{for } 4(d-d_0)>-1 \\ O\left(m^{4(d_0-d)}\log m\right) & \text{for } 4(d-d_0)\le -1 \end{cases} \\
&= O\left(m^2\left(m^{-1}+m^{4(d_0-d)-2}\log m\right)\right) = o\left(m^2\right),
\end{aligned}$$

and it follows that $\sum_{j=1}^{m} a_j = O(m)$ and $\sum_{j=1}^{m} a_j^2 = o(m^2)$. Therefore, using the same argument as above, we have $B_1 \to_p G_0 \frac{1}{m} \sum_{j=1}^{m} (a_j - 1)$, $B_2 \to_p 0$, and

$$\begin{aligned}
E|B_3(d)| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{d_0-1} n^{\frac{1}{2}-d_0} (\log n)^{\frac{1}{2}}\right) \\
&= O\left(\frac{1}{m}\sum_{j=1}^{p}\left(\frac{j}{p}\right)^{2(d-d_0)} j^{d_0-1} n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}} + \frac{1}{m}\sum_{j=p+1}^{m}\left(\frac{j}{p}\right)^{-2d_0-1} j^{d_0-1} n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}}\right) \\
&\quad + O\left(\frac{1}{m}\sum_{j=1}^{m} j^{d_0-1} n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}}\right) \\
&= O\left(n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}}\left(m^{2(d_0-d)-1}\sum_{j=1}^{p} j^{2d-d_0-1} + m^{2d_0}\sum_{j=p+1}^{m} j^{-d_0-2} + \frac{1}{m}\sum_{j=1}^{m} j^{d_0-1}\right)\right) \\
&= O\left(n^{\frac{1}{2}-d_0}(\log n)^{\frac{1}{2}}\left(m^{d_0-1} + m^{2d_0-2d-1}\log m\right)\right) \\
&= O\left(n^{1-2d_0}\log n\left(m^{d_0-1} + m^{2d_0-2d-1}\log m\right)\right) = o(1), \\
E|B_4(d)| &= O\left(\frac{1}{m}\sum_{j=1}^{m}(a_j-1)\, j^{2d_0-2} n^{1-2d_0}\log n\right) \\
&= O\left(\frac{1}{m}\sum_{j=1}^{p}\left(\frac{j}{p}\right)^{2(d-d_0)} j^{2d_0-2} n^{1-2d_0}\log n + \frac{1}{m}\sum_{j=p+1}^{m}\left(\frac{j}{p}\right)^{-2d_0-1} j^{2d_0-2} n^{1-2d_0}\log n\right) \\
&\quad + O\left(\frac{1}{m}\sum_{j=1}^{m} j^{2d_0-2} n^{1-2d_0}\log n\right) \\
&= O\left(n^{1-2d_0}\log n\left(m^{2d_0-2d-1}\sum_{j=1}^{p} j^{2d-2} + m^{2d_0}\sum_{j=p+1}^{m} j^{-3} + \frac{1}{m}\sum_{j=1}^{m} j^{2d_0-2}\right)\right) \\
&= O\left(n^{1-2d_0}\log n\left(m^{2d_0-2} + m^{2d_0-2d-1}\log m + m^{-1}\right)\right) \\
&= O\left(n^{1-2d_0}\log n\left(m^{-1} + m^{2d_0-2d-1}\log m\right)\right) = o(1),
\end{aligned}$$

because $\frac{n^{1-2\Delta_1}\log m \log n}{m} \to 0$. Therefore, $\sum_{i=1}^{4} B_i \to_p G_0 \frac{1}{m}\sum_{j=1}^{m}(a_j-1) > 0$ and $\widehat{d} \to_p d_0$ follows. ■

6.14 Proof of Theorem 3.4

Since $\widehat{d} \to_p d_0$ and $\widehat{G}(d)$ is a continuous function of d, we may analyse $\widehat{G}(d_0)$. We have

$$\frac{\widehat{G}(d_0) - G(d_0)}{G(d_0)} = \frac{\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d_0} I_v(\lambda_j) - G_0}{G_0} = \frac{A(d_0)}{B(d_0)} \to_p 0.$$

So

$$\widehat{G}(d_0) \to_p G_0,$$

which gives the required result. ■

6.15 Proof of Theorem 4.1

We work from the first order conditions for $\widehat{d}$, viz.

$$0 = R'\left(\widehat{d}\right) = R'(d_0) + R''(d^*)\left(\widehat{d} - d_0\right), \tag{42}$$

where $|d^* - d_0| \leq \left|\widehat{d} - d_0\right|$. As in the proof of Theorem 2 of Robinson (1995b) we get the following expression for $R''(d)$

$$R''(d) = \frac{4\left[\widehat{F}_2(d)\widehat{F}_0(d) - \widehat{F}_1(d)^2\right]}{\widehat{F}_0(d)^2},$$

where

$$\widehat{F}_k(d) = \frac{1}{m}\sum_{j=1}^{m}(\log j)^k \lambda_j^{2d} I_v(\lambda_j).$$

>From (36) and (37), we have

$$\sup_{\Theta_1}\left|\frac{A(d)}{B(d)}\right| = o_p\left((\log m)^{-6}\right),$$

thus by the argument in pages 1642-43 of Robinson (1995b), $R''(d^*) = R''(d_0) + o_p(1)$. Now, from Corollary 2.16 (b), we find

$$\widehat{F}_k(d_0) = \frac{1}{m}\sum_{j=1}^{m}(\log j)^k \lambda_j^{2d_0} I_v(\lambda_j) = C_1 + C_2 + C_3,$$

where

$$\begin{aligned} C_1 &= \frac{|C(1)|^2}{m}\sum_{j=1}^{m}(\log j)^k I_\varepsilon(\lambda_j), \\ C_2 &= \frac{1}{m}\sum_{j=1}^{m}(\log j)^k R_{j,n}^a, \quad C_3 = \frac{1}{m}\sum_{j=1}^{m}(\log j)^k R_{j,n}^b(d_0). \end{aligned}$$

We proceed to consider the successive terms as above for $k = 0, 1, 2$. For C_1,

$$C_1 = \frac{|C(1)|^2}{m}\sum_{j=1}^{m}(\log j)^k\left[I_\varepsilon(\lambda_j) - \frac{\sigma^2}{2\pi}\right] + G_0\frac{1}{m}\sum_{j=1}^{m}(\log j)^k.$$

Similar argument as above and the fact that

$$\frac{1}{m}\sum_{j=1}^{m}\log j \sim \log m, \quad \frac{1}{m}\sum_{j=1}^{m}(\log j)^2 \sim (\log m)^2,$$

yield

$$\begin{aligned}
& E\left[\frac{|C(1)|^2}{m}\sum_{j=1}^{m}(\log j)^k\left[I_\varepsilon(\lambda_j)-\frac{\sigma^2}{2\pi}\right]\right]^2 \\
&= O\left(\frac{1}{m^2}\sum_{j=1}^{m}(\log j)^{2k}\right)+O\left(\frac{1}{m^2}\sum_{j=1}^{m}(\log j)^k\sum_{l=1}^{m}(\log l)^k\frac{1}{n}\right) \\
&= O\left(\frac{(\log m)^{2k}}{m}\right)+O\left(\frac{(\log m)^{2k}}{n}\right)=O\left(\frac{(\log m)^{2k}}{m}\right),
\end{aligned}$$

giving $C_1=\frac{1}{m}\sum_{j=1}^{m}(\log j)^k(G_0+o_p(1))$.

For C_2, $E|C_2|$ is bounded by

$$\frac{1}{m}\sum_{j=1}^{m}(\log j)^k\frac{j}{n}=O\left(\frac{1}{m}\sum_{j=1}^{m}(\log j)^k\frac{m}{n}\right)=o\left((\log m)^k\right).$$

$E|C_3|=0$ for $d_0=1$, and for $d_0\neq 1$, we obtain

$$E|C_3|=\begin{cases} O\left(\frac{1}{m}\sum_{j=1}^{m}(\log j)^k j^{d_0-1}\right)=O\left(\frac{(\log m)^k}{m}\sum_{j=1}^{m}j^{d_0-1}\right), & \text{for } d_0\in\left(\frac{1}{2},1\right), \\ O\left(\frac{1}{m}\sum_{j=1}^{m}(\log j)^k j^{d_0-\frac{3}{2}}\right)=O\left(\frac{(\log m)^k}{m}\sum_{j=1}^{m}j^{d_0-\frac{3}{2}}\right), & \text{for } d_0\in\left(1,\frac{3}{2}\right). \end{cases}$$

Hence

$$\widehat{F}_k(d_0)=G_0\left[\frac{1}{m}\sum_{j=1}^{m}(\log j)^k\right][1+o_p(1)].$$

Then

$$\begin{aligned}
R''(d_0) &= \frac{4G_0^2\left[\frac{1}{m}\sum_{j=1}^{m}(\log j)^2-\left(\frac{1}{m}\sum_{j=1}^{m}(\log j)\right)^2\right]}{G_0^2}[1+o_p(1)] \\
&= \left\{4\left[\frac{1}{m}\left\{\left(m+\frac{1}{2}\right)(\log m)^2-2m\log m+2m+O(1)\right\}\right]\right. \\
&\quad \left.-4\left[\frac{1}{m^2}\left\{\left(m+\frac{1}{2}\right)(\log m)-m+O(1)\right\}^2\right]\right\}[1+o_p(1)] \\
&= 4+o_p(1). \qquad (43)
\end{aligned}$$

Next we consider the first term on the right side of (42). We have

$$R'(d_0)=2\frac{\widehat{G}_1(d_0)}{\widehat{G}(d_0)}-\frac{2}{m}\sum_{j=1}^{m}\log\lambda_j,$$

where

$$\widehat{G}_1(d_0)=\frac{1}{m}\sum_{j=1}^{m}(\log\lambda_j)\lambda_j^{2d_0}I_v(\lambda_j),\quad \widehat{G}(d_0)=\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d_0}I_v(\lambda_j).$$

Then

$$
\begin{aligned}
m^{\frac{1}{2}} R'(d_0) &= m^{\frac{1}{2}} \left[2 \frac{\widehat{G}_1(d_0)}{\widehat{G}(d_0)} - \frac{2}{m} \sum_{j=1}^{m} \log \lambda_j \right] \\
&= \frac{2}{\sqrt{m}} \frac{\sum_{j=1}^{m} (\log \lambda_j) \lambda_j^{2d_0} I_v(\lambda_j) - \left(\sum_{j=1}^{m} \log \lambda_j \right) \widehat{G}(d_0)}{\widehat{G}(d_0)} \\
&= \frac{2}{\sqrt{m}} \frac{\sum_{j=1}^{m} (\log \lambda_j) \lambda_j^{2d_0} I_v(\lambda_j) - \left(\frac{1}{m} \sum_{j=1}^{m} \log \lambda_j \right) \sum_{j=1}^{m} \lambda_j^{2d_0} I_v(\lambda_j)}{\widehat{G}(d_0)} \\
&= \frac{2}{\sqrt{m}} \frac{\sum_{j=1}^{m} \left(\log \lambda_j - \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j \right) \lambda_j^{2d_0} I_v(\lambda_j)}{\widehat{G}(d_0)} \\
&= \frac{\frac{2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j \left[\lambda_j^{2d_0} I_v(\lambda_j) - G_0 \right]}{\widehat{G}(d_0)},
\end{aligned}
$$

where

$$
\nu_j = \log \lambda_j - \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j = \log j - \frac{1}{m} \sum_{j=1}^{m} \log j,
$$

and $\sum_{j=1}^{m} \nu_j = 0$. For the denominator, from Theorem 3.4 we have

$$
\widehat{G}(d_0) \to_p G_0. \tag{44}
$$

By Corollary 2.16 (a) and (c), the numerator can be decomposed as follows:

$$
\frac{2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j \left[\lambda_j^{2d_0} I_v(\lambda_j) - G_0 \right] = \begin{cases} D_1 + D_2 + D_3 + D_4 + D_5 + D_6, & \text{for } d_0 \in \left(\frac{1}{2}, \frac{3}{2} \right) \setminus \{1\}, \\ D_1 + D_4, & \text{for } d_0 = 1, \end{cases}
$$

where

$$
\begin{aligned}
D_1 &= \frac{2 |C(1)|^2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j \left[I_\varepsilon(\lambda_j) - \frac{\sigma^2}{2\pi} \right], \\
D_2 &= \frac{2 |C(1)|^2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j \frac{\lambda_j^{2d_0}}{|1 - e^{i\lambda_j}|^2} \frac{\left| \widetilde{\varepsilon}_{\lambda_j n}(f_0) \right|^2}{2\pi n}, \\
D_3 &= -\frac{2 |C(1)|^2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j \left[e^{\frac{\pi}{2} d_0 i} w_\varepsilon(\lambda_j) \frac{\lambda_j^{d_0}}{1 - e^{-i\lambda_j}} \frac{\widetilde{\varepsilon}_{\lambda_j n}(f_0)^*}{\sqrt{2\pi n}} + \frac{\lambda_j^{d_0}}{1 - e^{i\lambda_j}} \frac{\widetilde{\varepsilon}_{\lambda_j n}(f_0)}{\sqrt{2\pi n}} e^{-\frac{\pi}{2} d_0 i} w_\varepsilon(\lambda_j)^* \right], \\
D_4 &= \frac{2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j R_{j,n}^a, \quad D_5 = \frac{2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j R_{j,n}^b(d_0), \quad D_6 = \frac{2}{\sqrt{m}} \sum_{j=1}^{m} \nu_j R_{j,n}^c(d_0).
\end{aligned}
$$

For D_1, $\{\varepsilon_t\}$ satisfies the assumptions

$$
E(\varepsilon_t | F_{t-1}) = 0, \quad E\left(\varepsilon_t^2 | F_{t-1}\right) = \sigma^2, \quad E\left(\varepsilon_t^3 | F_{t-1}\right) = \mu_3, \quad a.s., \quad E\left(\varepsilon_t^4\right) = \mu_4,
$$

for $t=1,2,...$, thus we can apply the result in Robinson (1995b) pp.1644-1647 to conclude

$$D_1=\frac{2G_0}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left[\frac{2\pi}{\sigma^2}I_\varepsilon(\lambda_j)-1\right]\to_d N\left(0,4G_0^2\right). \quad (45)$$

>From Lemma 2.10 (b), for $d_0\in\left(\frac{1}{2},1\right)$, $E\left|D_2\right|$ is bounded by

$$\begin{aligned}
&\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\left|\nu_j\right|\lambda_j^{2d_0-2}\frac{E\left|\widetilde{\varepsilon}_{\lambda_j n}(f_0)\right|^2}{2\pi n}\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2d_0-2}\frac{1}{n}\left(L^{2d_0-1}+\frac{n}{j}L^{2d_0-2}\right)\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\left(m^{2d_0-1}n^{1-2d_0}L^{2d_0-1}+n^{2-2d_0}L^{2d_0-2}\right)\right)\\
&= O\left(\log m\left(m^{-\frac{1}{2}}\left(\frac{mL}{n}\right)^{2d_0-1}+m^{\frac{3}{2}-2d_0}\left(\frac{mL}{n}\right)^{2d_0-2}\right)\right).
\end{aligned}$$

When $d_0>\frac{3}{4}$, this is $o(1)$ by choosing $L=\frac{n}{m}$. When $d_0\leq\frac{3}{4}$, choose $L=n(\log m)^{\frac{-2}{2d_0-1}}$, then we have

$$\begin{aligned}
(\log m)\,m^{-\frac{1}{2}}\left(\frac{mL}{n}\right)^{2d_0-1} &= m^{2d_0-\frac{3}{2}}(\log m)^{-1}\to 0,\\
(\log m)\,m^{\frac{3}{2}-2d_0}\left(\frac{mL}{n}\right)^{2d_0-2} &= m^{-\frac{1}{2}}(\log m)^{\frac{4-4d_0}{2d_0-1}+1}\to 0.
\end{aligned}$$

Therefore, $D_2=o_p(1)$ for $d_0\in\left(\frac{1}{2},1\right)$. For $d_0\in\left(1,\frac{5}{4}\right)$, from Lemma 2.11 (a), $E\left|D_2\right|$ is bounded by

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\left|\nu_j\right|\lambda_j^{2d_0-2}\frac{E\left|\widetilde{\varepsilon}_{\lambda_j n}(f_0)\right|^2}{2\pi n}=O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}j^{2d_0-3}\right)=O\left(\log m\left(m^{2d_0-\frac{5}{2}}\right)\right)=o(1),$$

and for $d_0\in\left[\frac{5}{4},\frac{3}{2}\right)$, from Lemma 2.11 (b), we have

$$\begin{aligned}
&\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\left|\nu_j\right|\frac{\lambda_j^{2d_0}}{\left|1-e^{i\lambda_j}\right|^2}\frac{E\left|\widetilde{\varepsilon}_{\lambda_j n}(f_0)\right|^2}{2\pi n}\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\left(\frac{j}{n}\right)^{2d_0-2}\frac{L^{d_0-1}n^{d_0-1}}{j}+\left(\frac{j}{n}\right)^{2d_0-2}\frac{L^{2d_0-3}n}{j^2}\right)\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(j^{2d_0-3}L^{d_0-1}n^{1-d_0}+j^{2d_0-4}L^{2d_0-3}n^{3-2d_0}\right)\right)\\
&= O\left(\log m\left(\left(\frac{L}{n}\right)^{d_0-1}m^{2d_0-\frac{5}{2}}+\left(\frac{L}{n}\right)^{2d_0-3}m^{-\frac{1}{2}}\right)\right)
\end{aligned}$$

$$= O\left(\log m\left(\left(\frac{Lm}{n}\right)^{d_0-1} m^{d_0-\frac{3}{2}}+\left(\frac{Lm}{n}\right)^{2d_0-3} m^{\frac{5}{2}-2d_0}\right)\right)=o\left(1\right),$$

by letting $L=\frac{n}{m}\left(\log m\right)^{\frac{-2}{2d_0-3}}$, giving $D_2=o_p\left(1\right).$

For D_3, in view of the fact that

$$w_{\varepsilon}\left(\lambda_j\right)^*=\frac{1}{\sqrt{2\pi n}}\sum_{p=1}^{n}e^{-ip\lambda_j}\varepsilon_p=\frac{1}{\sqrt{2\pi n}}\sum_{n-p=0}^{n-1}e^{i(n-p)\lambda_j}\varepsilon_{n-(n-p)}=\frac{1}{\sqrt{2\pi n}}\sum_{q=0}^{n-1}e^{iq\lambda_j}\varepsilon_{n-q},$$

we can obtain a decomposition as follows:

$$\begin{aligned}&\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{d_0}}{1-e^{i\lambda_j}}\frac{\widetilde{\varepsilon}_{\lambda_j n}(f_0)}{\sqrt{2\pi n}}w_{\varepsilon}\left(\lambda_j\right)^*\\ =\;&\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{d_0}}{1-e^{i\lambda_j}}\frac{1}{2\pi n}\left(\sum_{p=0}^{n-1}\widetilde{f}_{\lambda_j p}e^{-ip\lambda_j}\varepsilon_{n-p}\right)\left(\sum_{q=0}^{n-1}e^{iq\lambda_j}\varepsilon_{n-q}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&E\left|\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{d_0}}{1-e^{i\lambda_j}}\frac{\widetilde{\varepsilon}_{\lambda_j n}(f_0)}{\sqrt{2\pi n}}w_{\varepsilon}\left(\lambda_j\right)^*\right|^2\\ =\;&\frac{1}{\pi^2 mn^2}E\left[\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{d_0}}{1-e^{i\lambda_j}}\left(\sum_{p=0}^{n-1}\widetilde{f}_{\lambda_j p}e^{-ip\lambda_j}\varepsilon_{n-p}\right)\left(\sum_{q=0}^{n-1}e^{iq\lambda_j}\varepsilon_{n-q}\right)\right]\\ &\times\left[\sum_{k=1}^{m}\nu_k\frac{\lambda_k^{d_0}}{1-e^{-i\lambda_k}}\left(\sum_{r=0}^{n-1}\widetilde{f}_{-\lambda_k r}e^{ir\lambda_k}\varepsilon_{n-r}\right)\left(\sum_{s=0}^{n-1}e^{-is\lambda_k}\varepsilon_{n-s}\right)\right].\end{aligned}\tag{46}$$

Because $\{\varepsilon_t\}$ are independent,

$$E\left(\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s\right)=\begin{cases}\mu_4, & \text{if } p=q=r=s,\\ \sigma^4, & \text{if } p=q\neq r=s,\ p=s\neq q=r,\ p=r\neq q=s,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, (46) is bounded by

$$\frac{\mu_4}{mn^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\left|\nu_j\right|\left|\nu_k\right|\lambda_j^{d_0-1}\lambda_k^{d_0-1}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\left|\widetilde{f}_{-\lambda_k p}\right|\right)\tag{47}$$

$$+\frac{2\sigma^4}{mn^2}\sum_{j=1}^{m}\left|\nu_j\right|\lambda_j^{d_0-1}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\right)\sum_{k=1}^{m}\left|\nu_k\right|\lambda_k^{d_0-1}\left(\sum_{q=0}^{n-1}\left|\widetilde{f}_{-\lambda_k q}\right|\right)\tag{48}$$

$$+\frac{\sigma^4}{mn^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\left|\nu_j\right|\left|\nu_k\right|\lambda_j^{d_0-1}\lambda_k^{d_0-1}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\left|\widetilde{f}_{-\lambda_k p}\right|\right)\left(\sum_{q=0}^{n-1}e^{iq(\lambda_j-\lambda_k)}\right).\tag{49}$$

(47) is bounded by, for $d_0 \in \left(\frac{1}{2}, 1\right)$,

$$\begin{aligned}
& \frac{(\log m)^2}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \left(\frac{j}{n}\right)^{d_0-1} \left(\frac{k}{n}\right)^{d_0-1} \frac{1}{n^2} \sum_{p=1}^{n-1} p^{-2f_0} \\
= & \; O\left(\frac{(\log m)^2}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} j^{d_0-1} k^{d_0-1} n^{-2d_0} n^{1-2f_0}\right) \\
= & \; O\left(\frac{(\log m)^2}{m} m^{2d_0} n^{-1}\right) \\
= & \; O\left(\frac{m^{2d_0-1} (\log m)^2}{n}\right) \\
= & \; O\left(\left(\frac{m}{n}\right)^{2d_0-1} \frac{(\log m)^2}{n^{2-2d_0}}\right) = o(1),
\end{aligned}$$

and for $d_0 \in \left(1, \frac{3}{2}\right)$,

$$\begin{aligned}
& \frac{(\log m)^2}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \left(\frac{j}{n}\right)^{d_0-1} \left(\frac{k}{n}\right)^{d_0-1} \frac{1}{n^2} \sum_{p=1}^{n-1} n^{-f_0} \frac{n}{p^{f_0+1} j} \\
= & \; O\left(\frac{(\log m)^2}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} j^{d_0-2} k^{d_0-1} n^{-2d_0} n^{1-2f_0}\right) \\
= & \; O\left(\frac{(\log m)^2}{m} m^{d_0-1} m^{d_0} n^{-1}\right) \\
= & \; O\left(\frac{m^{2d_0-2} (\log m)^2}{n}\right) \\
= & \; O\left(\left(\frac{m}{n}\right)^{2d_0-2} \frac{(\log m)^2}{n^{3-2d_0}}\right) = o(1).
\end{aligned}$$

For $d_0 \in \left(\frac{1}{2}, 1\right)$, (48) is bounded by

$$\begin{aligned}
& \left[\frac{\log m}{\sqrt{m}} \sum_{j=1}^{m} \left(\frac{j}{n}\right)^{d_0-1} \frac{1}{n} \left(\sum_{p=0}^{n-1} \left|\widetilde{f}_{\lambda_j p}\right|\right)\right]^2 \\
= & \left[\frac{\log m}{\sqrt{m}} \sum_{j=1}^{m} \left[\left(\frac{j}{n}\right)^{d_0-1} \frac{1}{n} \sum_{p=0}^{L} \left|\widetilde{f}_{\lambda_j p}\right| + \left(\frac{j}{n}\right)^{d_0-1} \frac{1}{n} \sum_{p=L+1}^{n-1} \left|\widetilde{f}_{\lambda_j p}\right|\right]\right]^2.
\end{aligned}$$

>From Lemma 2.7, the first term in the bracket is

$$O\left(\left(\frac{j}{n}\right)^{d_0-1} \frac{1}{n} \sum_{p=0}^{L} \frac{1}{p^{f_0}}\right) = O\left(\left(\frac{j}{n}\right)^{d_0-1} \frac{1}{n} L^{1-f_0}\right) = O\left(j^{d_0-1} n^{-d_0} L^{d_0}\right),$$

and the second term is

$$O\left(\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{n}\sum_{p=L+1}^{n-1}\frac{n}{p^{f_0+1}j}\right)=O\left(\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{j}L^{-f_0}\right)=O\left(j^{d_0-2}n^{1-d_0}L^{d_0-1}\right).$$

It follows that

$$\begin{aligned}
&\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{n}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(j^{d_0-1}n^{-d_0}L^{d_0}+j^{d_0-2}n^{1-d_0}L^{d_0-1}\right)\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}m^{d_0}\left(\frac{L}{n}\right)^{d_0}+\frac{\log m}{\sqrt{m}}\left(\frac{n}{L}\right)^{1-d_0}\right).
\end{aligned}$$

Choose $L=\frac{n}{\sqrt{m}}$, and we obtain

$$\begin{aligned}
m^{d_0-\frac{1}{2}}\left(\frac{L}{n}\right)^{d_0} &= m^{d_0-\frac{1}{2}}m^{-\frac{d_0}{2}}=m^{\frac{2d_0-1-d_0}{2}}=m^{\frac{d_0-1}{2}}=o\left(\frac{1}{\log m}\right),\\
m^{-\frac{1}{2}}\left(\frac{n}{L}\right)^{1-d_0} &= m^{-\frac{1}{2}}m^{\frac{1-d_0}{2}}=m^{\frac{-d_0}{2}}=o\left(\frac{1}{\log m}\right),
\end{aligned}$$

thus (48) is $o(1)$. For $d_0\in\left(1,\frac{3}{2}\right)$, (48) is bounded by

$$\left[\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{n}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\right)\right]^2.$$

Using Lemma 2.7, we have

$$\begin{aligned}
&\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{n}\left(\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{d_0-1}\frac{1}{n}\sum_{p=1}^{n-1}\frac{n}{p^{f_0+1}j}\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{d_0-1}\frac{n^{-f_0}}{j}\right)\\
&= O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}j^{d_0-2}\right)=O\left(m^{d_0-\frac{3}{2}}\log m\right),
\end{aligned}$$

hence (48) is $o(1)$.

In view of the fact that $\sum_{q=0}^{n-1} e^{iq(\lambda_j-\lambda_k)} = n\mathbf{1}(j=k)$, (49) is bounded by, for $d_0 \in \left(\frac{1}{2}, 1\right)$,

$$\begin{aligned}
\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\lambda_j^{2d_0-2}\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|^2 &= O\left(\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2d_0-2}\sum_{p=1}^{n}p^{-2f_0}\right) \\
&= O\left(\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2d_0-2}n^{1-2f_0}\right) \\
&= O\left((\log m)^2 m^{2d_0-2}\right),
\end{aligned}$$

and for $d_0 \in \left(1, \frac{3}{2}\right)$,

$$\begin{aligned}
\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\lambda_j^{2d_0-2}\sum_{p=0}^{n-1}\left|\widetilde{f}_{\lambda_j p}\right|^2 &= O\left(\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2d_0-2}\left(\sum_{p=1}^{n-1}n^{-f_0}\frac{n}{p^{f_0+1}j}\right)\right) \\
&= O\left(\frac{(\log m)^2}{mn}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2d_0-2}n^{2d_0-1}j^{-1}\right) \\
&= O\left(\frac{(\log m)^2}{m}\sum_{j=1}^{m}j^{2d_0-3}\right) \\
&= O\left((\log m)^2 m^{2d_0-3}\right).
\end{aligned}$$

Therefore, (46) converges to zero, and thus $D_3 = o_p(1)$.

D_4, D_5 and D_6 are all $o_p(1)$ because

$$\begin{aligned}
E|D_4| &= O\left(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}|\nu_j|\lambda_j\right) = O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}\frac{j}{n}\right) = O\left(\frac{m^{\frac{3}{2}}\log m}{n}\right), \\
E|D_5| &= O\left(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}|\nu_j|j^{d_0-2}\right) = \begin{cases} O\left(\frac{\log m}{\sqrt{m}}\right), & \text{for } d_0 \in \left(\frac{1}{2}, 1\right), \\ O\left(m^{d_0-\frac{3}{2}}\log m\right), & \text{for } d_0 \in \left(1, \frac{3}{2}\right), \end{cases} \\
E|D_6| &= \begin{cases} O\left(\frac{2}{\sqrt{m}}\sum_{j=1}^{m}|\nu_j|j^{d_0-1}n^{\frac{1}{2}-d_0}\right) = O\left(n^{\frac{1}{2}-d_0}m^{d_0-\frac{1}{2}}\log m\right), & \text{for } d_0 \in \left(\frac{1}{2}, 1\right), \\ O\left(\frac{2}{\sqrt{m}}\sum_{j=1}^{m}|\nu_j|j^{d_0-1}n^{-\frac{1}{2}}\right) = O\left(n^{-\frac{1}{2}}m^{d_0-\frac{1}{2}}\log m\right), & \text{for } d_0 \in \left(1, \frac{3}{2}\right). \end{cases}
\end{aligned}$$

Furthermore,

$$\frac{m^{d_0-\frac{1}{2}}\log m}{n^{d_0-\frac{1}{2}}} = \left(\frac{m^{\frac{3}{2}}}{n}\right)^{d_0-\frac{1}{2}} m^{-\frac{1}{2}\left(d_0-\frac{1}{2}\right)}\log m \to 0,$$

if $\frac{m^{\frac{3}{2}}\log m}{n} \to 0$.

Therefore, we obtain

$$m^{\frac{1}{2}}R'(d_0) \Rightarrow \frac{1}{G_0}N\left(0, 4G_0^2\right). \tag{50}$$

It follows from (42), (43) and (50) that

$$m^{\frac{1}{2}}\left(\widehat{d}-d_0\right)=-\frac{m^{\frac{1}{2}}R'\left(d_0\right)}{R''\left(d^*\right)}\Rightarrow\frac{1}{4G_0}N\left(0,4G_0^2\right)\equiv N\left(0,\frac{1}{4}\right),$$

giving the required result. ■

6.16 Proof of Theorem 4.2

The proof follows the same line of argument as Theorem 4.1. The condition $\frac{m^{\frac{3}{2}}\log m}{n}\to 0$ implies that

$$\frac{m^{2d_0-2}\left(\log m\right)^{12}}{n}=\frac{m^{\frac{3}{2}}\log m}{n}\frac{\left(\log m\right)^{11}}{m^{\frac{7}{2}-2d_0}}\to 0.$$

Thus, $\widehat{d}$ is consistent and also we have $\sup_{\Theta_1}\left|\frac{A(d)}{B(d)}\right|=o_p\left(\left(\log m\right)^{-6}\right)$, which gives $R''\left(d^*\right)=R''\left(d_0\right)+o_p\left(1\right)$. It follows from Corollary 2.16 (e) that

$$\widehat{F}_k\left(d_0\right)=\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k\lambda_j^{2d_0}I_v\left(\lambda_j\right)=\sum_{j=1}^{6}C_j$$

where $C_1=\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k\left(G_0+o_p\left(1\right)\right)$, $C_2=o_p\left(\left(\log m\right)^k\right)$, and

$$\begin{aligned}
E\left|C_3\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k j^{d_0-2}\left(\log n\right)^{\frac{1}{2}}\right)=O\left(\left(\log m\right)^k m^{d_0-2}\left(\log n\right)^{\frac{1}{2}}\right),\\
E\left|C_4\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k j^{d_0-1}n^{-\frac{1}{2}}\right)=O\left(\left(\log m\right)^k m^{d_0-1}n^{-\frac{1}{2}}\right),\\
E\left|C_5\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k j^{2d_0-4}\log n\right)=O\left(\left(\log m\right)^k m^{2d_0-4}\log n\right),\\
E\left|C_4\right| &= O\left(\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k j^{2d_0-2}n^{-1}\right)=O\left(\left(\log m\right)^k m^{2d_0-2}n^{-1}\right),
\end{aligned}$$

giving $\widehat{F}_k\left(d_0\right)=G_0\left[\frac{1}{m}\sum_{j=1}^{m}\left(\log j\right)^k\right]\left[1+o_p\left(1\right)\right]$.

Before we evaluate $\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left[\lambda_j^{2d_0}I_v\left(\lambda_j\right)-G_0\right]$, we derive the approximation of $\lambda_j^{2d_0}I_v\left(\lambda_j\right)$ for $d_0\in\left(\frac{3}{2},2\right)$. First, note that (see (22))

$$\begin{aligned}
\Delta X_t &= X_t-X_{t-1}\\
&= \sum_{k=0}^{t-1}\frac{(d)_k}{k!}u_{t-k}-\sum_{k=0}^{(t-1)-1}\frac{(d)_k}{k!}u_{t-k-1}\\
&= \sum_{k=0}^{t-1}\frac{(d)_k}{k!}u_{t-k}-\sum_{k+1=1}^{t-1}\frac{(d)_{k+1-1}}{(k+1-1)!}u_{t-(k+1)}
\end{aligned}$$

$$
\begin{aligned}
&= u_t + \sum_{k=1}^{t-1} \frac{(d)_k}{k!} u_{t-k} - \sum_{k=1}^{t-1} \frac{(d)_{k-1}}{(k-1)!} u_{t-k} \\
&= u_t + \sum_{k=1}^{t-1} \frac{(d-1)_k}{k!} u_{t-k} = \sum_{k=0}^{t-1} \frac{(d-1)_k}{k!} u_{t-k},
\end{aligned}
$$

and $\Delta X_0 = 0$. Let $\overline{d}_0 = d_0 - 1 \in \left(\frac{1}{2}, 1\right)$ and $\overline{f}_0 = 1 - \overline{d}_0$. From Lemma 2.15 (a) we have

$$
\lambda_s^{\overline{d}_0} w_{\Delta x}(\lambda_s) = e^{\frac{\pi}{2}\overline{d}_0 i} C(1) w_\varepsilon(\lambda_s) - \frac{\lambda_s^{\overline{d}_0} C(1)}{1 - e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}\left(\overline{f}_0\right)}{\sqrt{2\pi n}} - \frac{\lambda_s^{\overline{d}_0} e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{\Delta X_n}{\sqrt{2\pi n}} + r_{s,n}^a + r_{s,n}^b\left(\overline{d}_0\right) + r_{s,n}^c\left(\overline{d}_0\right),
$$

where $E\left|r_{s,n}^a\right|^2 = O\left(\lambda_s^2\right)$, $E\left|r_{s,n}^b\left(\overline{d}_0\right)\right|^2 = O\left(s^{2\overline{d}_0 - 4}\right)$, and $E\left|r_{s,n}^c\left(\overline{d}_0\right)\right|^2 = O\left(s^{2\overline{d}_0 - 2} n^{1 - 2\overline{d}_0}\right)$ uniformly in s. Lemma 2.13 (c) yields

$$
E\left|\frac{\lambda_s^{\overline{d}_0} e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{\Delta X_n}{\sqrt{2\pi n}}\right|^2 = O\left(\lambda_s^{2\overline{d}_0 - 2} n^{2\overline{d}_0 - 2}\right) = O\left(s^{2\overline{d}_0 - 2}\right) = O(1),
$$

hence we obtain

$$
\begin{aligned}
\lambda_s^{2\overline{d}_0} I_{\Delta x}(\lambda_s) &= \left|e^{\frac{\pi}{2}\overline{d}_0 i} C(1) w_\varepsilon(\lambda_s) - \frac{\lambda_s^{\overline{d}_0} C(1)}{1 - e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}\left(\overline{f}_0\right)}{\sqrt{2\pi n}} - \frac{\lambda_s^{\overline{d}_0} e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{\Delta X_n}{\sqrt{2\pi n}}\right|^2 \\
&\quad + R_{s,n}^a + R_{s,n}^b\left(\overline{d}_0\right) + R_{s,n}^c\left(\overline{d}_0\right),
\end{aligned}
$$

where $E\left|R_{s,n}^a\right| = O(\lambda_s)$, $E\left|R_{s,n}^b\left(\overline{d}_0\right)\right| = O\left(s^{\overline{d}_0 - 2}\right)$, and $E\left|R_{s,n}^c\left(\overline{d}_0\right)\right| = O\left(s^{\overline{d}_0 - 1} n^{\frac{1}{2} - \overline{d}_0}\right)$ uniformly in s.

>From Lemma 2.2 (b), we have

$$
v_x(\lambda_s) = \left(1 - e^{i\lambda_s}\right)^{-1} w_{\Delta x}(\lambda_s).
$$

Thus

$$
I_v(\lambda_s) = \left|1 - e^{i\lambda_s}\right|^{-2} I_{\Delta x}(\lambda_s),
$$

and in view of the fact that $E\left|\lambda_s^{2\overline{d}_0} I_{\Delta x}(\lambda_s)\right| = O(1)$, it follows that

$$
\begin{aligned}
\lambda_s^{2d_0} I_v(\lambda_s) &= \lambda_s^{2d_0 - 2}\left(1 + O(\lambda_s)\right) I_{\Delta x}(\lambda_s) \\
&= \left|e^{\frac{\pi}{2}\overline{d}_0 i} C(1) w_\varepsilon(\lambda_s) - \frac{\lambda_s^{\overline{d}_0} C(1)}{1 - e^{i\lambda_s}} \frac{\widetilde{\varepsilon}_{\lambda_s n}\left(\overline{f}_0\right)}{\sqrt{2\pi n}} - \frac{\lambda_s^{\overline{d}_0} e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{\Delta X_n}{\sqrt{2\pi n}}\right|^2 \\
&\quad + R_{s,n}^a + R_{s,n}^b\left(\overline{d}_0\right) + R_{s,n}^c\left(\overline{d}_0\right),
\end{aligned}
$$

where the order of magnitude of the reminder terms is the same as above.

Finally we obtain an expression

$$\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left[\lambda_j^{2d_0}I_v\left(\lambda_j\right)-G_0\right]=\sum_{k=1}^{9}D_k,$$

where

$$\begin{aligned}
D_1 &= \frac{2\left|C\left(1\right)\right|^2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left[I_\varepsilon\left(\lambda_j\right)-\frac{\sigma^2}{2\pi}\right],\\
D_2 &= \frac{2\left|C\left(1\right)\right|^2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{2\overline{d}_0}}{\left|1-e^{i\lambda_j}\right|^2}\frac{\left|\widetilde{\varepsilon}_{\lambda_j n}(\overline{f}_0)\right|^2}{2\pi n},\quad D_3=\frac{2}{\sqrt{m}}\frac{\Delta X_n^2}{2\pi n}\sum_{j=1}^{m}\nu_j\frac{\lambda_j^{2\overline{d}_0}}{\left|1-e^{i\lambda_j}\right|^2}\\
D_4 &= -\frac{2\left|C\left(1\right)\right|^2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left[e^{\frac{\pi}{2}\overline{d}_0 i}w_\varepsilon\left(\lambda_j\right)\frac{\lambda_j^{\overline{d}_0}}{1-e^{-i\lambda_j}}\frac{\widetilde{\varepsilon}_{\lambda_j n}\left(\overline{f}_0\right)^*}{\sqrt{2\pi n}}+\frac{\lambda_j^{\overline{d}_0}}{1-e^{i\lambda_j}}\frac{\widetilde{\varepsilon}_{\lambda_j n}\left(\overline{f}_0\right)}{\sqrt{2\pi n}}e^{-\frac{\pi}{2}\overline{d}_0 i}w_\varepsilon\left(\lambda_j\right)^*\right],\\
D_5 &= -\frac{2C\left(1\right)}{\sqrt{m}}\frac{\Delta X_n}{\sqrt{2\pi n}}\sum_{j=1}^{m}\nu_j\left[e^{\frac{\pi}{2}\overline{d}_0 i}w_\varepsilon\left(\lambda_j\right)\frac{\lambda_j^{\overline{d}_0}e^{-i\lambda_j}}{1-e^{-i\lambda_j}}+\frac{\lambda_j^{\overline{d}_0}e^{i\lambda_j}}{1-e^{i\lambda_j}}e^{-\frac{\pi}{2}\overline{d}_0 i}w_\varepsilon\left(\lambda_j\right)^*\right],\\
D_6 &= \frac{2C\left(1\right)}{\sqrt{m}}\frac{\Delta X_n}{2\pi n}\sum_{j=1}^{m}\nu_j\left[\frac{\lambda_j^{2\overline{d}_0}e^{-i\lambda_j}}{\left|1-e^{i\lambda_j}\right|^2}\widetilde{\varepsilon}_{\lambda_j n}\left(\overline{f}_0\right)+\frac{\lambda_j^{2\overline{d}_0}e^{i\lambda_j}}{\left|1-e^{i\lambda_j}\right|^2}\widetilde{\varepsilon}_{\lambda_j n}\left(\overline{f}_0\right)^*\right],\\
D_7 &= \frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j R_{j,n}^a,\quad D_8=\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j R_{j,n}^b\left(\overline{d}_0\right),\quad D_9=\frac{2}{\sqrt{m}}\sum_{j=1}^{m}\nu_j R_{j,n}^c\left(\overline{d}_0\right).
\end{aligned}$$

It has already shown that $D_1\to_d N\left(0,4G_0^2\right)$ and $D_2+D_4+D_7+D_8+D_9=o_p\left(1\right).$ For D_3, from Lemma 2.13 (c) we have

$$E\left|D_3\right|=E\left(\frac{\Delta X_n^2}{n^{2\overline{d}_0-1}}\right)O\left(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j j^{2\overline{d}_0-2}\right)=O\left(\frac{\log m}{\sqrt{m}}\sum_{j=1}^{m}j^{2\overline{d}_0-2}\right)=O\left(m^{2\overline{d}_0-\frac{3}{2}}\log m\right),$$

which is $o\left(1\right)$ if $\overline{d}_0<\frac{3}{4}\Leftrightarrow d_0<\frac{7}{4}$.

For D_5, rewrite

$$\frac{1}{\sqrt{m}}\frac{\Delta X_n}{\sqrt{n}}\sum_{j=1}^{m}\nu_j w_\varepsilon\left(\lambda_j\right)\frac{\lambda_j^{\overline{d}_0}e^{-i\lambda_j}}{1-e^{-i\lambda_j}}$$

as

$$\frac{\Delta X_n}{n^{\overline{d}_0-\frac{1}{2}}}\times\frac{n^{\overline{d}_0-1}}{\sqrt{m}}\sum_{j=1}^{m}\nu_j w_\varepsilon\left(\lambda_j\right)\frac{\lambda_j^{\overline{d}_0}e^{-i\lambda_j}}{1-e^{-i\lambda_j}}=D_{51}\times D_{52}.$$

In view of the fact that

$$E\left[w_\varepsilon\left(\lambda_j\right)^*w_\varepsilon\left(\lambda_k\right)\right]=\frac{1}{2\pi n}\sum_{p=1}^{n}\sum_{q=1}^{n}E\left[\varepsilon_p e^{-ip\lambda_j}\right]\left[\varepsilon_q e^{iq\lambda_k}\right]=\frac{\sigma^2}{2\pi n}\sum_{p=1}^{n}e^{i\left(\lambda_k-\lambda_j\right)p}=\frac{\sigma^2}{2\pi}\mathbf{1}\left(j=k\right),$$

we have

$$E\left|D_{52}\right|^{2}=O\left(\frac{(\log m)^{2}}{m}\sum_{j=1}^{m}j^{2\overline{d}_{0}-2}\right)=O\left((\log m)^{2}\,m^{2\overline{d}_{0}-2}\right)=o\left(1\right),$$

hence $D_{52}=o_{p}\left(1\right)$. $D_{51}=O_{p}\left(1\right)$ by Corollary 2.13 (c), and $D_{5}=o_{p}\left(1\right)$ follows.

For D_{6}, rewrite

$$\frac{1}{\sqrt{m}}\frac{\Delta X_{n}}{n}\sum_{j=1}^{m}\nu_{j}\frac{\lambda_{j}^{2\overline{d}_{0}}e^{-i\lambda_{j}}}{\left|1-e^{i\lambda_{j}}\right|^{2}}\widetilde{\varepsilon}_{\lambda_{j}n}\left(\overline{f}_{0}\right)$$

as

$$\frac{\Delta X_{n}}{n^{\overline{d}_{0}-\frac{1}{2}}}\times\frac{n^{\overline{d}_{0}-\frac{3}{2}}}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\frac{\lambda_{j}^{2\overline{d}_{0}}e^{-i\lambda_{j}}}{\left|1-e^{i\lambda_{j}}\right|^{2}}\widetilde{\varepsilon}_{\lambda_{j}n}\left(\overline{f}_{0}\right)=D_{61}\times D_{62}.$$

We have $E\left|D_{61}\right|^{2}=O_{p}\left(1\right)$, and from Lemma 2.10 (c), $\left(E\left|D_{62}\right|^{2}\right)^{\frac{1}{2}}$ is bounded by

$$\begin{aligned}
&\frac{n^{\overline{d}_{0}-\frac{3}{2}}\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2\overline{d}_{0}-2}L^{\overline{d}_{0}-\frac{1}{2}}+\frac{n^{\overline{d}_{0}-\frac{3}{2}}\log m}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2\overline{d}_{0}-2}\left(\frac{n}{j}\right)^{1/2}L^{\overline{d}_{0}-1}\\
=\;&O\left(n^{\frac{1}{2}-\overline{d}_{0}}\left(\log m\right)m^{2\overline{d}_{0}-\frac{3}{2}}L^{\overline{d}_{0}-\frac{1}{2}}+n^{1-\overline{d}_{0}}\left(\log m\right)^{2}m^{-\frac{1}{2}}m^{\max\left\{2\overline{d}_{0}-\frac{3}{2},0\right\}}L^{\overline{d}_{0}-1}\right)\\
=\;&O\left(\log m\left(\frac{mL}{n}\right)^{\overline{d}_{0}-\frac{1}{2}}m^{\overline{d}_{0}-1}+\left(\log m\right)^{2}\left(\frac{mL}{n}\right)^{\overline{d}_{0}-1}m^{\max\left\{\overline{d}_{0}-1,\frac{1}{2}-\overline{d}_{0}\right\}}\right)=o\left(1\right),
\end{aligned}$$

by setting $L=\frac{n}{m}$. $D_{6}=o_{p}\left(1\right)$ follows from Cauchy-Schwartz inequality. Therefore, $m^{\frac{1}{2}}R'\left(d_{0}\right)\Rightarrow\frac{1}{G_{0}}N\left(0,4G_{0}^{2}\right)$, giving the required result. ■

6.17 Proof of Theorem 4.4

Recall

$$\begin{aligned}
m^{4-2d_{0}}\left(\widehat{d}-d_{0}\right) &= -\frac{m^{4-2d_{0}}R'\left(d_{0}\right)}{R''\left(d^{*}\right)}\\
&= -\frac{m^{\frac{7}{2}-2d_{0}}\left(D_{1}+D_{2}+D_{4}+D_{5}+D_{6}+D_{7}+D_{8}+D_{9}\right)}{4G_{0}+o_{p}\left(1\right)} \qquad (51)\\
&\quad -\frac{m^{\frac{7}{2}-2d_{0}}D_{3}}{4G_{0}+o_{p}\left(1\right)}, \qquad (52)
\end{aligned}$$

where

$$\begin{aligned}
D_{3} &= \frac{2}{\sqrt{m}}\frac{\Delta X_{n}^{2}}{2\pi n}\sum_{j=1}^{m}\nu_{j}\frac{\lambda_{j}^{2\overline{d}_{0}}}{\left|1-e^{i\lambda_{j}}\right|^{2}}\\
&= \frac{2\left(2\pi\right)^{2\overline{d}_{0}-2}}{\sqrt{m}}\frac{\Delta X_{n}^{2}}{2\pi n^{2\overline{d}_{0}-1}}\sum_{j=1}^{m}\nu_{j}j^{2\overline{d}_{0}-2}\left(1+O\left(\lambda_{j}\right)\right)\\
&= \frac{2\left(2\pi\right)^{2\overline{d}_{0}-2}}{\sqrt{m}}\frac{\Delta X_{n}^{2}}{2\pi n^{2\overline{d}_{0}-1}}\sum_{j=1}^{m}\nu_{j}j^{2\overline{d}_{0}-2}+\frac{\Delta X_{n}^{2}}{n^{2\overline{d}_{0}-1}}O\left(\frac{\log m}{n\sqrt{m}}\sum_{j=1}^{m}j^{2\overline{d}_{0}-1}\right).
\end{aligned}$$

The second term is

$$O_p(1)\, O\left(\frac{m^{2\overline{d}_0-\frac{1}{2}}\log m}{n}\right) = o_p(1).$$

Now evaluate the summation:

$$\sum_{j=1}^{m} \nu_j j^{2\overline{d}_0-2} = \sum_{j=1}^{m}\left(\log j - \frac{1}{m}\sum_{j=1}^{m}\log j\right) j^{2\overline{d}_0-2} = \sum_{j=1}^{m} j^{2\overline{d}_0-2}\log j - \sum_{j=1}^{m} j^{2\overline{d}_0-2}\left(\frac{1}{m}\sum_{j=1}^{m}\log j\right).$$

For the first term, we have

$$\begin{aligned}
\sum_{j=1}^{m} j^{2\overline{d}_0-2}\log j &= \int_1^m x^{2\overline{d}_0-2}\log x dx + \frac{1}{2}\left(m^{2\overline{d}_0-2}\log m\right) + O\left(\log m \int_1^m x^{2\overline{d}_0-3}dx\right) \\
&= \left[\frac{x^{2\overline{d}_0-1}\log x}{2\overline{d}_0-1}\right]_1^m - \int_1^m \frac{x^{2\overline{d}_0-2}}{2\overline{d}_0-1}dx + O(\log m) \\
&= \frac{m^{2\overline{d}_0-1}\log m}{2\overline{d}_0-1} - \frac{m^{2\overline{d}_0-1}}{\left(2\overline{d}_0-1\right)^2} + O(\log m),
\end{aligned}$$

and for the second term,

$$\begin{aligned}
\sum_{j=1}^{m} j^{2\overline{d}_0-2}\left(\frac{1}{m}\sum_{j=1}^{m}\log j\right) &= \left(\int_1^m x^{2\overline{d}_0-2}dx + \frac{1}{2}\left(m^{2\overline{d}_0-2}+1\right) + O\left(\int_1^m x^{2\overline{d}_0-3}dx\right)\right) \\
&\quad \times \left(\log m - 1 + O\left(\frac{\log m}{m}\right)\right) \\
&= \left(\frac{m^{2\overline{d}_0-1}}{2\overline{d}_0-1} + O(1)\right)\left(\log m - 1 + O\left(\frac{\log m}{m}\right)\right) \\
&= \frac{m^{2\overline{d}_0-1}\log m}{2\overline{d}_0-1} - \frac{m^{2\overline{d}_0-1}}{2\overline{d}_0-1} + O(\log m).
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{m} \nu_j j^{2\overline{d}_0-2} = -\frac{m^{2\overline{d}_0-1}}{\left(2\overline{d}_0-1\right)^2} + \frac{m^{2\overline{d}_0-1}}{2\overline{d}_0-1} + O(\log m) = \frac{\left(2\overline{d}_0-2\right)m^{2\overline{d}_0-1}}{\left(2\overline{d}_0-1\right)^2} + O(\log m).$$

>From Akonom and Gourieroux (1987), if $E|\varepsilon_t|^p < \infty$ for $p > \max\left\{\frac{1}{\overline{d}_0-\frac{1}{2}}, 2\right\}$, we have

$$\frac{\Delta X_n^{\varepsilon}}{n^{\overline{d}_0-\frac{1}{2}}} \to_d \sigma B_{\overline{d}_0-1}(1) = \frac{\sigma}{\Gamma\left(\overline{d}_0\right)}\int_0^1 (1-s)^{\overline{d}_0-1} dB(s).$$

When $d_0 = \frac{7}{4}$, $\frac{1}{\overline{d}_0-\frac{1}{2}} = 4$ and we need an additional moment condition $E|\varepsilon_t|^p < \infty$ for $p > 4$ for convergence. When $d_0 > \frac{7}{4}$, $\frac{1}{\overline{d}_0-\frac{1}{2}} < 4$ and the condition $E|\varepsilon_t|^4 < \infty$

suffice. It follows that, for $d_0 \in \left[\frac{7}{4}, 2\right)$,

$$
\begin{aligned}
m^{\frac{7}{2}-2d_0} D_3 &= 2\left(2\pi\right)^{2\overline{d}_0-2} \frac{\Delta X_n^2}{2\pi n^{2\overline{d}_0-1}} m^{1-2\overline{d}_0} \sum_{j=1}^{m} \nu_j j^{2\overline{d}_0-2} \\
&= 2\left(2\pi\right)^{2\overline{d}_0-2} \left[\frac{C\left(1\right)^2 \left(\Delta X_n^{\varepsilon}\right)^2}{2\pi n^{2\overline{d}_0-1}} + o_p\left(1\right)\right] m^{1-2\overline{d}_0} \sum_{j=1}^{m} \nu_j j^{2\overline{d}_0-2} \\
&\to {}_d 2\left(2\pi\right)^{2\overline{d}_0-2} \frac{C\left(1\right)^2 \sigma^2}{2\pi} B_{\overline{d}_0-1}\left(1\right)^2 \frac{2\overline{d}_0-2}{\left(2\overline{d}_0-1\right)^2}.
\end{aligned}
$$

For $d_0 = \frac{7}{4,}$ (51) converges to $N\left(0, \frac{1}{4}\right)$ and

$$
-\frac{m^{\frac{7}{2}-2d_0} D_3}{4G_0 + o_p\left(1\right)} \to_d \frac{\left(1-\overline{d}_0\right)\left(2\pi\right)^{2\overline{d}_0-2}}{\left(2\overline{d}_0-1\right)^2} B_{\overline{d}_0-1}\left(1\right)^2 \equiv \left(2\pi\right)^{-\frac{1}{2}} B_{-\frac{1}{4}}\left(1\right)^2.
$$

For $d_0 \in \left(\frac{3}{4}, 1\right)$, (51) is $o_p\left(1\right)$ and

$$
-\frac{m^{\frac{7}{2}-2d_0} D_3}{4G_0 + o_p\left(1\right)} \to_d \frac{\left(1-\overline{d}_0\right)\left(2\pi\right)^{2\overline{d}_0-2}}{\left(2\overline{d}_0-1\right)^2} B_{\overline{d}_0-1}\left(1\right)^2 = \frac{\left(2-d_0\right)\left(2\pi\right)^{2d_0-4}}{\left(2d_0-3\right)^2} B_{d_0-2}\left(1\right)^2,
$$

giving the required result. ■

7 Appendix B: Different Characterizations of Nonstationary $I\left(d\right)$ Processes

Two main approaches to defining a nonstationary $I\left(d\right)$ process have been used in the literature to date. They are by no means exhaustive. The first, which is used in Hurvich and Ray (1995) and Velasco (1999a, 1999b), is to define the observed process X_t as the partial sum of a stationary fractionally integrated process, viz.

$$
X_t = X_0 + \sum_{j=1}^{t} z_j, \quad t \geq 1, \tag{53}
$$

where z_j is a stationary $I\left(d-1\right)$ process and satisfies

$$
z_t = \left(1-L\right)^{1-d} \varepsilon_t = \sum_{j=0}^{\infty} \frac{\left(d-1\right)_j}{j!} \varepsilon_{t-j}, \tag{54}
$$

where ε_t is a short-memory stationary process. Combining (53) and (54), we obtain

$$
\left(1-L\right)\left(X_t - X_0\right) = \left(1-L\right)^{1-d} \varepsilon_t,
$$

leading to a definition of the operator equation

$$(1-L)^d (X_t - X_0) = \varepsilon_t, \qquad t \geq 1, \tag{55}$$

in terms of (53) and (54). X_t is said to be integrated of order d.

A second definition (Phillips, 1999) defines the nonstationary fractionally integrated process X_t directly in terms of the short memory inputs by using a finite order expansion of the operator $(1-L)^{-d}$, viz.

$$X_t = X_0 + \sum_{k=0}^{t-1} \frac{(d)_k}{k!} \varepsilon_{t-k}, \tag{56}$$

where ε_t is a short-memory stationary process. This leads to the operator expression

$$(1-L)^d (X_t - X_0) = \varepsilon_t, \qquad t \geq 1, \tag{57}$$

and again X_t is integrated of order d. The two definitions (55) and (57) are different, however, because the stationary input formulation (54) implies that, by the first definition, $X_t \left(= X_0 + \sum_{j=1}^{t} (1-L)^{1-d} \varepsilon_j\right)$ involves inputs ε_s with $s \leq 0$. In fact, for each t we have that

$$(1-L)^{1-d} \varepsilon_t = \frac{(d-1)_0}{0!} \varepsilon_t + \frac{(d-1)_1}{1!} \varepsilon_{t-1} + \ldots + \frac{(d-1)_{t+k}}{(t+k)!} \varepsilon_{-k} + \ldots$$

so that the infinite past history of the short memory stationary inputs ε_s figures in X_t.

Some further comparisons involving the impulse responses may be helpful. When $d \in \left(\frac{1}{2}, 1\right)$, according to the first definition, X_t is integrated of order $d < 1$ and the increments z_t constitute an $I(f)$ process with negative $f = 1 - d$. In other words, the increments have negative correlation and are often described as antipersistent. On the other hand, according to the second definition, X_t is integrated of order $d < 1$ because the coefficients of ε_{t-k} are not unity but decay slowly, too slowly for the process to be stationary and have finite variance. Thus, the second definition gives the anticipated slow decay of the impulse responses directly, and as such is more apparently intermediate in form between a unit root process and a stationary long memory process or a short-memory process (but see (58) below).

In some cases, the empirical context may be helpful in motivating the formative process. Suppose that $d \in \left(1, \frac{3}{2}\right)$. Then, according to the first definition, X_t is integrated of order $d > 1$ because it is the accumulation of stationary increments z_t that have long memory with $f = 1 - d > 0$. According to the second definition, X_t is integrated of order $d > 1$ because the coefficients of ε_{t-k} increase as k increases. When it is known that the process of interest is the result of an accumulation of past long-memory shocks (perhaps, like the diameter of a tree), the first definition would seem to be appropriate. However, when it is expected that the shocks each period have short memory but may have increasing impulse responses over time on the observed variable, then the second definition seems more appropriate. For instance, in seeking to characterize a time series like GDP as a nonstationary $I(d)$ process with $d > 1$, the first definition posits GDP as the sum of past shocks which have long memory, whereas the second definition posits that the shocks to GDP each period have short memory but the cumulative effect of these shocks is allowed to increase over time, perhaps by way of some internal feedback mechanism.

Whether the first or the second definition is used, it will often be useful to extract the impulse responses from the short memory components to the observed series. In the second definition these appear directly as the coefficients $\frac{(d)_k}{k!}$ in (56). By rearrangement of the series in the first definition, one finds that the impulse responses are the same in this case as well. In particular, it can be shown that an $I(d)$ process by the first definition can be written as

$$X_t = X_0 + \xi_0(d) + \sum_{k=0}^{t-1} \frac{(d)_k}{k!} \varepsilon_{t-k}, \tag{58}$$

where the term $\xi_0(d)$ has an order of magnitude that is dominated by that of the third term asymptotically. Thus, the essential difference between the definitions can be interpreted as one relating to initialization.

As with the definition of unit root processes, there are alternative ways of dealing with initial conditions for nonstationary fractional processes and these may or may not affect large sample behavior. If X_0 is taken to be any $O_p(1)$ random variable then its value has no affect on large sample behavior. Similar considerations apply to the term $\xi_0(d)$ in (58). However, when X_0 has the same stochastic order as X_t for $t = O(n)$ then initializations do matter, as indeed has been found to be the case for unit root time series (e.g., Phillips and Lee, 1996, and Canjels and Watson, 1997). In the present case, the generalization might involve a distant past initialization of the form

$$X_0 = X_0^\kappa = \sum_{k=0}^{[n\kappa]} \frac{(d)_k}{k!} \varepsilon_{-k},$$

or one might extend (56) directly by writing

$$X_t = \sum_{k=0}^{t+[n\kappa]} \frac{(d)_k}{k!} \varepsilon_{-k}.$$

In both these cases, the effective initialization is pushed into the distant past and is parameterized by κ, which measures the extent of the pre-sample history on the current data X_t. While κ is not estimable, it will generally figure in the asymptotic theory, just as it does in the case of unit root asymptotics (Phillip and Lee, 1996). The present paper does not deal with this additional level of difficulty, but works from the definition (56) with $X_0 = O_p(1)$.

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