NARROW-BAND ANALYSIS OF NONSTATIONARY PROCESSES*

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Abstract

The behaviour of averaged periodograms and cross-periodograms of a broad class of nonstationary processes is studied. The processes include nonstationary ones that are fractional of any order, as well as asymptotically stationary fractional ones, and the cross-periodogram can involve two nonstationary processes of possibly different orders, or a nonstationary and an asymptotically stationary one. The averaging takes place either over the whole frequency band, or on one that degenerates slowly to zero frequency as sample size increases. In some cases it is found to make no asymptotic difference, and in particular we indicate how the behaviour of the mean and variance changes across the two-dimensional space of integration orders. The results employ only local-to-zero assumptions on the spectra of the underlying weakly stationary sequences. It is shown how the results can be readily applied in case of fractional cointegration with unknown integration orders.

1 1. INTRODUCTION

In the analysis of time series that are believed prone to nonstationarity, the behaviour of bilinear and quadratic forms is of prime interest. For univariate time series, Gaussian rules of inference lead to consideration of quadratic forms, and Gaussian methods developed by Whittle (1951) and others in

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stationary short range dependent environments were extended to unit root nonstationary ones by Box and Jenkins (1971), with limit theory developed by Dickey and Fuller (1979) and many subsequent authors. In case of multivariate time series, the Gaussian approach covers not only jointly dependent modelling but also linear regression methods, and in either case bilinear and quadratic forms arise. Again, limit theory for stationary short range dependent vector processes has been extended to unit roots, activity in this direction fuelled by considerable econometric interest in the possible existence of cointegrated structures, positing the existence of a linear combination of related unit root series which has short range dependence.

The scope of time series analysis has considerably expanded with the development of methods and theory for stationary and nonstationary long range dependent or fractional processes. A fractional view of time series regards the stationary short range dependent and unit root processes as mere points (at $\beta = 0$ and $\beta = 1$, respectively) on the real line of processes indexed by integration order β. A loose definition of integration order (the paper employs a more general one) is that degree of differencing needed to corvert a stationary or nonstationary process to one with spectral density that is positive and continuous at zero frequency. Limit theory for Whittle estimates of parametric stationary long range dependent series has been developed by Fox and Taqqu (1986) and others, while recently cointegration of multiple nonstationary fractional time series has been considered by Chan and Terrin (1996), Jeganathan (1996, 1997) and others, though this topic is still in its infancy.

Narrow band frequency domain analysis has been a major focus of the long range dependence literature. A stationary long range dependent series is usually thought of as having a spectral pole at zero frequency, with spectral density behaving like λ^{-2d} nearby, where λ indicates frequency, and $0 < \beta < \frac{1}{2}$. Methods of estimating β based on a band of frequencies around zero that degenerates slowly as sample size increases were considered by Geweke and Porter-Hudak (1983), Kűnsch (1986, 1987) and Robinson (1994a,b, 1995a,b), the asymptotic theory of the latter author imposing essentially no conditions on spectral behaviour away from zero frequency and thereby demonstrating a signal advantage of such 'semiparametric' methods.

The main theoretical concern of Robinson (1994a) was the convergence of the discretely averaged periodogram of a univariate series, over a degenerating band of Fourier frequencies, but one of his applications of this theory was to cointegration of bivariate stationary long range dependent series $\{y_t, z_t, t = 0, \pm 1, ...\}$. It was envisaged that whereas y_t and z_t each has integration order $\beta \in (0, \frac{1}{2})$, there exists an unknown ν such that ζ_t in

$$y_t = \nu z_t + \zeta_t \tag{1.1}$$

has integration order $\alpha < \beta$. The ζ_t by construction thus have the character of regression errors, at least after mean correction, but there is no prior reason to suppose that they possess the classical property of orthogonality with z_t, $Cov(\zeta_t, z_t) = 0$. Were y_t, z_t nonstationary, but ζ_t stationary, or less nonstationary than y_t, z_t, such that the signal-to-noise ratio $\sum_{t=1}^{n} \zeta_t^2 / \sum_{t=1}^{n} z_t^2$ converges stochastically to zero as sample size n tends to infinity, the least squares estimate (LSE) of ν would be consistent, as demonstrated by, e.g. Stock (1987), in case y_t, z_t have a unit root but ζ_t is short range dependent ($\alpha = 0, \beta = 1$). When y_t, z_t are stationary, however, the LSE is generally inconsistent when there is correlation between z_t and ζ_t. However, Robinson (1994a) showed that the narrow-band least squares estimate (NBLSE) of ν, namely the ratio of the real part of the averaged cross-periodogram of y_t, z_t to the averaged periodogram of z_t, averaging across the m lowest Fourier frequencies where $m \to \infty$ but $m/n \to 0$ as $n \to \infty$, is consistent for ν. This is due to the spectrum of z_t dominating that of ζ_t near zero frequency, since $\alpha < \beta$, even though the respective variances are both finite and positive. Robinson (1994b) discussed optimal choice of m.

Cointegration of stationary long range dependent series has been of interest in a financial context, for example a triangle of exchange rates is likely to be cointegrated. However, financial series may also be nonstationary, as is certainly the case with macroeconomic ones, while cointegration has also been of interest in other fields, such as ecology, where nonstationarity can arise, and in general not only are integration orders likely to be unknown, but we may not even know whether or not the series is stationary. Thus, given its superiority over the LSE in stationary environments there is interest in analyzing the performance of the NBLSE in nonstationary ones.

Cointegration provides a motivation for the theoretical contribution of the present paper, an examination of the averaged cross-periodogram, and the sample covariance, of a bivariate series, one element of which is nonstationary and the other is either nonstationary or (asymptotically) stationary. We derive and compare leading terms in the asymptotic bias and variance of these statistics, leading to a qualitative classification of behaviour depending on integration orders of the time series, for example whether the integration orders sum to less than one or greater than one is important, while the case when one of them is zero and the other unity (familiar from the unit root cointegration literature) is seen to be quite special. Our modelling of the series is notably general. They are linear filters of short range dependent series.

The filters have desirable commutative properties and cover standard fractional differencing, and in general produce low frequency stochastic trends. Consequently, it is the low frequency behaviour of the short range dependent innovations that is important, as our results and conditions stress; in the spirit of Robinson (1994a,b) our conditions entail only mild restrictions at zero frequency and have little implication for higher frequencies.

The following section defines the basic averaged (cross-) periodogram statistic and its implementations of particular interest. Section 3 demonstrates an approach to modelling nonstationary and asymptotically stationary sequences, with derivation of useful properties. Sections 4 and 5 cover respectively asymptotics for the mean and variance of the averaged (cross-) periodogram under this type of model. Section 6 applies the results to LSE and NBLSE estimates of cointegrated nonstationary series. Sections 7-9 give proofs of results of Sections 3-5, respectively.

2 2. THE AVERAGED CROSS-PERIODOGRAM

For a sequence ζ_t, $t = 1, ..., n$, we define the discrete Fourier transform

$$w_\zeta(\lambda) = \frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_t \zeta_t e^{it\lambda}, \tag{2.1}$$

where $\sum_t$ will always denote $\sum_{t=1}^n$; with also a sequence ξ_t, $t = 1, ..., n$, we define the (cross-) periodogram

$$I_{\zeta\xi}(\lambda) = w_\zeta(\lambda) w_\xi(-\lambda). \tag{2.2}$$

Denoting by $\lambda_j = 2\pi j/n$, for integer j, the Fourier frequencies, and by $1(.)$ the indicator function, we define the averaged (cross-) periodogram

$$\widehat{F}_{\zeta\xi}(\ell, m) = 2\Re e \left\{ \frac{2\pi}{n} \sum_{j=\ell}^{m} I_{\zeta\xi}(\lambda_j) \right\} - \frac{2\pi}{n} I_{\zeta\xi}(\pi) 1\left(m = \frac{n}{2}\right) \tag{2.3}$$

for integers ℓ, m such that $0 \le \ell \le m \le n/2$, noting that $I_{\zeta\xi}$ has period 2π, that $\Re e\{I_{\zeta\xi}(\lambda)\}$ is symmetric about $\lambda = 0$ and $\lambda = \pi$, and that $I_{\zeta\xi}(\pi)$ is real-valued. We have for all such m

$$\widehat{F}_{\zeta\xi}(1, m) = F_{\zeta\xi}(0, m) - \overline{\zeta\xi}, \tag{2.4}$$

with the notation $\overline{a} = n^{-1} \sum_t a_t$, so that omission of zero frequency entails a sample-mean correction. We shall always consider only $\ell = 0$ or $\ell = 1$, though properties for other finite values of ℓ are the same as those for $\ell = 1$. On the other hand, the second term in (2.3) can make a non-zero contribution only when $m = n/2$, for which n must be even. Defining $\widetilde{n} = [n/2]$, where [.] denotes integer part, the orthogonality of the complex exponential implies that whether n is even or odd,

$$\widehat{F}_{\zeta\xi}(0, \widetilde{n}) = \frac{2\pi}{n} \sum_{j=1}^{n} I_{\zeta\xi}(\lambda_j) = \frac{1}{n} \sum_{t} \zeta_t \xi_t, \tag{2.5}$$

the sample second (cross-) moment, so that from (2.4), $F_{\zeta\xi}(1, \widetilde{n})$ is the corresponding statistic based on deviations from sample means.

The real part operator in (2.3) is redundant when $m = \widetilde{n}$, but not in other cases of interest. We shall sometimes generalize $m = \widetilde{n}$ to

$$m \leq \widetilde{n};\; m \to \infty, \text{ as } n \to \infty, \tag{2.6}$$

but more often contradict it by

$$m < \widetilde{n};\; \frac{1}{m} + \frac{m}{n} \to 0, \text{ as } n \to \infty, \tag{2.7}$$

so that $\widehat{F}_{\zeta\xi}$ is based on a degenerating band of frequencies.

Under (2.7), $\widehat{F}_{\zeta\xi}$ has principally been of interest in connection with estimating the (cross-) spectral density of covariance stationary processes. As a matter of notation, if ζ_t, ξ_t, $t = 0, \pm 1, ...$, are jointly covariance stationary with a (cross-) spectral density $f_{\xi\zeta}(\lambda)$, the latter satisfies

$$Cov(\zeta_0, \xi_j) = E(\zeta_0 - E\zeta_0)(\xi_j - E\xi_0) = \int_{\Pi} f_{\zeta\xi}(\lambda) e^{ij\lambda} d\lambda,\; j = 0, \pm 1, ..., \tag{2.8}$$

where $\Pi = [-\pi, \pi]$. Under regularity conditions and (2.7), $\pi n \widehat{F}_{\zeta\zeta}(1, m)/m$ consistently estimates $f_{\zeta\zeta}(0)$ (see Brillinger, 1975). When the latter is infinite (so ζ_t has long range dependence) Robinson (1994a,b) studied asymptotic properties of $\widehat{F}_{\zeta\zeta}(1, m)$, with vector generalisation given by Lobato (1997). We are concerned, however, with $\widehat{F}_{\zeta\xi}(\ell, m)$ when neither ζ_t nor ξ_t is stationary, though one of them can be asymptotically stationary, and the following section describes such processes and their properties.

Relative to the literature on quadratic forms of stationary long range dependent processes, following Fox and Taqqu (1985), $\widehat{F}_{\zeta\xi}(\ell, \widetilde{n})$ cover very specialised quadratic forms, and we can envisage how $\widehat{F}_{\zeta\xi}(\ell, m)$, for general m, can likewise be generalised. On the other hand the possible bilinear aspect, with allowance for nonstationary ζ_t, ξ_t, or a mixture of asymptotically stationary and nonstationary processes, represents in itself a considerable theoretical development, not only when $m < \widetilde{n}$ (where indeed the forms considered in the stationary literature do not even quite cover $\widehat{F}_{\zeta\xi}(0, m)$, say) but even when $m = \widetilde{n}$. As it is, our simple forms can be used to approximate ones with a factor $\sigma(\lambda_j)$ in the summand of (2.3), where $\sigma(\lambda)$ is non-zero and sufficiently well behaved at $\lambda = 0$, while allowing poles and zeros in $\sigma(\lambda)$ will affect the character of the results more interestingly, as will tapering, but require a considerably more lengthy discussion. Our possibly bivariate setting means that results for the averaged periodogram matrix are immediately covered for vector series with possibly different integration orders. Note also that while the stationary quadratic form literature focusses directly on limit distribution properties, our leading concern is with comparison of $\widehat{F}_{\zeta\xi}(\ell, m)$ satisfying (2.6) and (2.7) through their first and second moments. These comparisons vary considerably with α and β, and to the extent that $\widehat{F}_{\zeta\xi}(\ell, m)$ approximates the 'time domain' statistics $\widehat{F}_{\zeta\xi}(\ell, \widetilde{n})$ (see (2.3), (2.4)), functional limit theory for vector nonstationary fractional processes of Marinucci and Robinson (1998) can be used to characterize limit distributional theory, as mentioned in Section 6.

3 3. NONSTATIONARY SEQUENCES

We first define classes of weight sequences which will generate classes of nonstationary, including asymptotically stationary, processes.

Definition 3.1 $\Phi(\alpha)$ is the class of sequences $\{\phi_t(\alpha), t = 0, 1, ...\}$ such that

$$\phi_t(0) = 1(t = 0), \tag{3.1}$$

and for $\alpha > 0$, as $t \to \infty$

$$\phi_t(\alpha) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.2}$$

$$\left|\phi_t(\alpha) - \phi_{t+1}(\alpha)\right| = O\left(\frac{|\phi_t(\alpha)|}{t}\right), \tag{3.3}$$

where "$\sim$" means that the ratio of left- and right-hand sides tends to 1, and $\Gamma(\cdot)$ is the Gamma function.

There is no loss of generality in the scale restrictions implicit in (3.1) and (3.2). It is possible to extend the definition, and subsequent results of the paper, to cover $\alpha < 0$, but we have focused on $\alpha \geq 0$ here due to space limitations and because this covers the cases of greatest practical interest. When $0 < \alpha < 1$, (3.2), (3.3) define $\{\phi_t(\alpha)\}$ as quasi-monotonically convergent to zero and of pure bounded variation in the sense of Yong (1974, pp.2, 4). In particular, (3.2) and (3.3) are satisfied by $\phi_t(\alpha) = t^{\alpha-1}/\Gamma(\alpha)$, but only (3.2) by $\phi_t(\alpha) = t^{\alpha-1}/\Gamma(\alpha) + t^{\beta-1}1(t \text{ even})$, for $\gamma - 1 < \beta < \gamma$ (though it would be possible to show that the results of following sections hold also for the latter type of sequence).

For our purposes the class $\Phi(\alpha)$ is motivated principally by the sequence

$$\Delta_t(\alpha) = \frac{\Gamma(t+\alpha)}{\Gamma(\alpha)\Gamma(t+1)}, \qquad t \geq 0, \tag{3.4}$$

with the conventions $\Gamma(0) = \infty$, $\Gamma(0)/\Gamma(0) = 1$, given by the formal expansion

$$\Delta^{-\alpha} = \sum_{t=0}^{\infty} \Delta_t(\alpha) L^j \tag{3.5}$$

where L is the lag operator and $\Delta = 1 - L$ is the difference operator. Using Sheppard's formula, we have $\{\Delta_t(\alpha)\} \in \Phi(\alpha)$, for all $\alpha \geq 0$. For integer α, Δ^α is familiar from Box and Jenkins' (1971) "ARIMA" modelling of nonstationary series. In particular

$$\Delta_t(1) = 1, \quad t \geq 0, \tag{3.6}$$

is used to generate "unit root" series in their framework. The somewhat special nature of (3.4) relative to (3.2) and (3.3), even with α fixed at 1, is notable in view of the vast econometric literature focussing on (3.6). In fact some of our work involving $\alpha = 1$ (see Theorem 4.3) requires some strengthening of (3.3) (see (4.22) and (4.25)), but still greater generality than (3.6) is afforded. When α is nonintegral, Δ^α is the fractional difference operator arising in modelling of "FARIMA" series. A cosinusoidal modification of Definition 3.1 would enable study stationary or nonstationary cyclic or seasonal behaviour.

Practical interest in $\Phi(\alpha)$ will further be strengthened by means of the following Lemma.

Lemma 3.1 Let $\{\phi_t\} \in \Phi(\alpha)$, $\{\psi_t\} \in \Phi(\beta)$, $\alpha, \beta \geq 0$. Then

$$\chi_t \stackrel{def}{=} \sum_{j=0}^{t} \phi_j \psi_{t-j} \in \Phi(\alpha+\beta). \tag{3.7}$$

The next lemma (see also Kokoszka and Taqqu, 1986, Lemma 3.1) describes properties of the complex partial sum

$$S_{uv}(\lambda, \alpha) = \sum_{t=u}^{v} \phi_t(\alpha) e^{it\lambda}, \tag{3.8}$$

for λ real, which will be of considerable use in the sequel. Throughout the paper, C denotes a generic positive constant.

Lemma 3.2 Let $\{\phi_t(\alpha)\} \in \Phi(\alpha)$. Then for $0 \leq u < v$, $0 \leq |\lambda| \leq \pi$,

$$S_{uv}(\lambda, 0) = 1(u = 0), \tag{3.9}$$

$$|S_{uv}(\lambda, \alpha)| \leq C \min\left(v^\alpha, \frac{(u+1)^{\alpha-1}}{|\lambda|}, \frac{1}{|\lambda|^\alpha}\right), \qquad 0 < \alpha \leq 1, \tag{3.10}$$

$$|S_{uv}(\lambda, \alpha)| \leq C \min\left(v^\alpha, \frac{v^{\alpha-1}}{|\lambda|}\right), \qquad \alpha > 1. \tag{3.11}$$

Also, for $0 < \alpha < 1$, as $\lambda \to 0^+$

$$\Re e \{S_{0\infty}(\lambda, \alpha)\} \sim \cos \frac{\alpha\pi}{2} \lambda^{-\alpha} \tag{3.12}$$

$$\Im m \{S_{0\infty}(\lambda, \alpha)\} \sim \sin \frac{\alpha\pi}{2} \lambda^{-\alpha}. \tag{3.13}$$

It is useful to list the sharpest bounds yielded by (3.10) and (3.11): for $c \in (0, \pi)$ we have for $0 < \alpha \leq 1$

$$|S_{uv}(\lambda, \alpha)| \leq Cv^\alpha, \text{ if } 0 < |\lambda| \leq \frac{c}{v}, \tag{3.14}$$

$$|S_{uv}(\lambda, \alpha)| \leq \frac{C}{|\lambda|^\alpha}, \text{ if } \frac{c}{v} \leq |\lambda| \leq \frac{c}{u+1}, \tag{3.15}$$

$$|S_{uv}(\lambda, \alpha)| \leq \frac{C(u+1)^{\alpha-1}}{|\lambda|}, \text{ if } \frac{c}{u+1} \leq |\lambda| \leq \pi, \tag{3.16}$$

while for $\alpha > 1$

$$|S_{uv}(\lambda, \alpha)| \leq Cv^\alpha, \text{ if } 0 < |\lambda| \leq \frac{c}{v}, \tag{3.17}$$

$$|S_{uv}(\lambda, \alpha)| \leq \frac{Cv^{\alpha-1}}{|\lambda|}, \text{ if } \frac{c}{v} \leq |\lambda| \leq \pi. \tag{3.18}$$

Short range dependent processes are given by:

Definition 3.2 I is the class of zero-mean scalar covariance stationary sequences $\{\eta_t, t = 0, \pm 1, ...\}$ having zero mean and spectral density $f_{\eta\eta}(\lambda)$ (*cf* (2.8)) that is positive and continuous at $\lambda = 0$.

The zero mean restriction is costless in our discussion of $\widehat{F}_{\zeta\xi}(\ell, m)$ when $\ell = 1$. Robinson and Marinucci (1999) have studied the averaged periodogram in case of additive time trends, though they obtain only upper bounds rather than our precise limits of Sections 4 and 5, and under stronger conditions on the stochastic component. We generate long range dependent processes by means of:

Definition 3.3 For $\alpha \geq 0$, $I(\alpha)$ is the class of processes $\{\zeta_t, t = 0, \pm 1, ...\}$ such that for $\{\eta_t\} \in I$ and $\{\phi_t\} \in \Phi(\alpha)$,

$$\zeta_t = \sum_{s=-\infty}^{t} \phi_{t-s} \{\eta_s 1(s \geq 1)\}. \tag{3.19}$$

Lemma 3.3 Let $\{\zeta_t\} \in I(\alpha)$ and let

$$\xi_t = \sum_{s=-\infty}^{t} \psi_{t-s} \{\zeta_s 1(s \geq 1)\}, \tag{3.20}$$

where $\{\psi_t\} \in I(\beta)$. Then $\{\xi_t\} \in I(\alpha + \beta)$.

We can thus view processes in $I(\alpha)$ as having possibly been passed through a succession of Φ-filters, whether by nature or the statistician, including the difference filter given in (3.4), (3.5).

Notice that Definition 3.3 implies $\zeta_t = 0$, $t \leq 0$, as a consequence of ζ_t being $(\eta_1, ..., \eta_t)$-measurable, which is itself motivated by the fact that, for $\{\phi_t\} \in \Phi(\alpha)$, the untruncated process

$$\rho_t = \sum_{s=-\infty}^{t} \phi_{t-s} \eta_s \tag{3.21}$$

does not have finite variance when $\alpha \geq \frac{1}{2}$. However for $\alpha < \frac{1}{2}$ ρ_t is, unlike ζ_t, covariance stationary, for example when $\alpha = 0$, $\zeta_t = \eta_t 1(t \geq 1)$. We have preferred to give a single definition for all $\alpha \geq 0$, and for $\alpha < \frac{1}{2}$ ζ_t is "asymptotically covariance stationary" in a sense indicated in the following lemma (see also Parzen, 1963, Dahlhaus, 1997), which also describes second order properties in the "purely" nonstationary case $\alpha \geq \frac{1}{2}$. Define

$$\phi(\lambda) = \sum_{s=0}^{\infty} \phi_s e^{is\lambda}, \quad \phi_t(\lambda) = \sum_{s=0}^{t-1} \phi_s e^{is\lambda}. \tag{3.22}$$

Lemma 3.4

(i) Let $0 \leq \alpha < \frac{1}{2}$. For $\{\phi_t\} \in \Phi(\alpha)$, $\{\eta_t\} \in I$, $\{\rho_t\}$ is covariance stationary with spectral density $f_{\rho\rho}(\lambda) = |\phi(\lambda)|^2 f_{\eta\eta}(\lambda)$, satisfying

$$f_{\rho\rho}(\lambda) \sim f_{\eta\eta}(0)\lambda^{-2\alpha}, \quad \text{as } \lambda \to 0^+. \tag{3.23}$$

The "time varying spectral density" of ζ_t, $f_{\zeta\zeta}^{(t)}(\lambda) = |\phi_t(\lambda)|^2 f_{\eta\eta}(\lambda)$ satisfies

$$\lim_{\lambda+(\lambda t)^{-1}\to 0+} \left\{ f_{\zeta\zeta}^{(t)}(\lambda) \big/ f_{\rho\rho}(\lambda) \right\} = 1 \tag{3.24}$$

and in addition we have, for all $j \geq 0$,

$$\lim_{t\to\infty} \left\{ Cov(\zeta_t, \zeta_{t+j}) - Cov(\rho_t, \rho_{t+j}) \right\} = 0. \tag{3.25}$$

(ii) Let $\alpha = \frac{1}{2}$. Then for all $j \geq 0$, as $t \to \infty$

$$Cov(\zeta_t, \zeta_{t+j}) \sim 2 f_{\eta\eta}(\lambda) \log t. \tag{3.26}$$

(iii) Let $\alpha > \frac{1}{2}$. Then for all $j \geq 0$, as $t \to \infty$

$$Cov(\zeta_t, \zeta_{t+j}) \sim \frac{2\pi f(0) t^{2\alpha-1}}{\Gamma(\alpha)^2 (2\alpha - 1)}. \tag{3.27}$$

Note that (3.24) holds despite $f_{\zeta\zeta}^{(t)}(\lambda)$ having no pole at $\lambda = 0$ for finite t, unlike $f_{\rho\rho}(\lambda)$ when $\alpha > 0$. By comparison (3.25) is a weak result, but a time domain version of (3.24) would require stronger conditions, in effect on $f_{\eta\eta}(\lambda)$ for all λ - an approximation for $Cov(\rho_t, \rho_{t+j})$ as $j \to \infty$ can be influenced by a pole in $f_{\eta\eta}(\lambda)$ for some $\lambda \neq 0$, for example. Lemma 3.4 presages the main results of the paper in its reliance on only mild, local-to-zero, conditions on $f_{\eta\eta}(\lambda)$.

4 4. THE MEAN OF THE AVERAGED PERIODOGRAM

We consider the statistic (2.3), where $\{\zeta_t\} \in \Phi(\alpha)$, $\{\xi_t\} \in \Phi(\beta)$ and

$$0 \leq \alpha \leq \beta, \quad \beta \geq \frac{1}{2}. \tag{4.1}$$

Thus only ζ_t can be asymptotically stationary. Strictly speaking, the case where both are asymptotically stationary in our sense has not been covered in the literature, but in view of Lemma 3.4 it is predictable that the results will be too similar to the stationary cases covered by Robinson (1994a,b), Lobato (1997) to be worth reporting. Of course when $\alpha \geq \frac{1}{2}$ also, our results include the case where $\zeta_t \equiv \xi_t$, the same nonstationary process. There is no loss of generality in the requirement $\alpha \leq \beta$.

We introduce:

Definition 4.1 I_2 is the class of jointly covariance stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, ...\}$ such that $\{\eta_t\} \in I$, $\{\theta_t\} \in I$ and $f_{\eta\theta}(\lambda)$ is continuous at $\lambda = 0$.

With ζ_t generated by (3.19) we take

$$\xi_t = \sum_{j=-\infty}^{t} \psi_{t-s} \{\theta_s 1(s \geq 1)\}, \tag{4.2}$$

where $\{\psi_t\} \in I(\beta)$.

Definition 4.2 $I(\alpha, \beta)$ is the class of bivariate processes $\{\zeta_t, \xi_t, t = 0, \pm 1, ...\}$ such that (3.19) and (4.2) hold for $\{\eta_t, \theta_t\} \in I_2$.

Depending on the values of α and β, $E\left\{\widehat{F}_{\zeta\xi}(0, m)\right\}$ may or may not differ negligibly from $E\left\{\widehat{F}_{\zeta\xi}(1, m)\right\}$, and so in view of (2.4) we first estimate $E(\overline{\zeta\xi})$ and, more generally, the covariance structure of discrete Fourier transforms $w_\zeta(\lambda_j)$, $w_\xi(\lambda_k)$ at fixed j, k, to extend results of Kũnsch (1986), Hurvitch and Beltrao (1993), Hurvitch and Ray (1995), Robinson (1995a). Denote by the superscripts R and I the real and imaginary part, respectively.

Lemma 4.1 Let $\{\zeta_t, \xi_t\} \in I(\alpha\beta)$. Then

$$\lim_{n\to\infty} n^{-\alpha-\beta} E\left\{w_\zeta^R(\lambda_j) w_\xi^R(\lambda_k)\right\} = f_{\eta\theta}(0) \int_0^1 C_j(z;\alpha) C_k(z;\beta) dz, \tag{4.3}$$

$$\lim_{n\to\infty} n^{-\alpha-\beta} E\left\{w_\zeta^R(\lambda_j) w_\xi^I(\lambda_k)\right\} = f_{\eta\theta}(0) \int_0^1 C_j(z;\alpha) S_k(z;\beta) dz, \tag{4.4}$$

$$\lim_{n\to\infty} n^{-\alpha-\beta} E\left\{w_\zeta^I(\lambda_j) w_\xi^R(\lambda_k)\right\} = f_{\eta\theta}(0) \int_0^1 S_j(z;\alpha) C_k(z;\beta) dz, \tag{4.5}$$

$$\lim_{n\to\infty} n^{-\alpha-\beta} E\left\{w_\zeta^I(\lambda_j) w_\xi^I(\lambda_k)\right\} = f_{\eta\theta}(0) \int_0^1 S_j(z;\alpha) S_k(z;\beta) dz, \tag{4.6}$$

where

$$C_j(z;\alpha) = \frac{1(\alpha > 0)}{\Gamma(\alpha)} \int_0^{1-z} y^{\alpha-1} \cos 2\pi j(y+z) dy + 1(\alpha = 0) \cos 2\pi j z \tag{4.7}$$

$$S_j(z;\alpha) \quad = \quad \frac{1(\alpha>0)}{\Gamma(\alpha)} \int_0^{1-z} y^{\alpha-1} \sin 2\pi j(y+z) dy + 1(\alpha=0) \sin 2\pi j z. \quad (4.8)$$

Thus,

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E(\overline{\zeta\xi}) = \frac{2\pi f_{\eta\theta}(0)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta+1)}. \quad (4.9)$$

For finite m, Lemma 4.1 can be applied to calculate the limit $E\left\{\widehat{F}_{\zeta\xi}(\ell,m)\right\}$. Under (2.6) or (2.7) the behaviour of $E\left\{\widehat{F}_{\zeta\xi}(\ell,m)\right\}$ varies significantly across the following five mutually exhaustive subsets of (4.1):

$$\alpha \geq 0,\ \beta \geq \frac{1}{2},\ \alpha+\beta<1, \quad (4.10)$$

$$\alpha > 0,\ \beta \geq \frac{1}{2},\ \alpha+\beta=1, \quad (4.11)$$

$$\alpha = 0,\ \beta = 1, \quad (4.12)$$

$$\alpha = 0,\ \beta > 1. \quad (4.13)$$

$$\alpha > 0,\ \beta > \frac{1}{2},\ \alpha+\beta>1. \quad (4.14)$$

In (4.10) and (4.11) ζ_t is asymptotically stationary and β is small enough that the combined memory $\alpha+\beta$ of ζ_t and ξ_t is less than one in (4.10), while in (4.11) it equals one but the familiar $I(0)/I(1)$ case (4.12) of the econometric literature is excluded. and in (4.13) and (4.14) it exceeds one. In (4.4), $\beta > \frac{1}{2}$ is implied by $\alpha+\beta>1$ in view of (4.1).

Consider first case (4.10). Define

$$\psi(\lambda) = \sum_{t=0}^{\infty} \psi_t e^{it\lambda}, \quad (4.15)$$

which (like $\phi(\lambda)$) is infinite at $\lambda=0$ but is well defined for $\lambda \neq 0$, mod(2π), from Lemma 3.2.

Theorem 4.1 Let $\{\zeta_t, \xi_t\} \in I(\alpha,\beta)$ under (4.10). Then for $\ell = 0, 1$,

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(\ell,\widetilde{n})\right\} = \int_{\Pi} \phi(\lambda)\psi(-\lambda) f_{\eta\theta}(\lambda) d\lambda, \quad (4.16)$$

where the right side is finite, and under (2.7)

$$\lim_{n\to\infty} \lambda_m^{\alpha+\beta-1} E\left\{\widehat{F}_{\zeta\xi}(\ell,m)\right\} = 2 f_{\eta\theta}(0) \frac{\cos(\alpha-\beta)\frac{\pi}{2}}{1-\alpha-\beta}. \quad (4.17)$$

Neither (4.16) nor (4.17) is affected by mean-correction. Most interestingly, the results are identical with those which may be obtained if both ζ_t and ξ_t are stationary or asymptotically stationary, so $\alpha, \beta < \frac{1}{2}$, which automatically implies $\alpha + \beta < 1$; thus it is seen that sufficiently small memory in ζ_t can compensate for the nonstationarity in ζ_t, though for given $\alpha+\beta$ (4.10) has the potential for a larger $\alpha - \beta$ and consequently smaller $\cos(\alpha-\beta)\pi/2$ factor in (4.17). The latter is positive, and so the limit (4.17) shares the sign of $f_{\eta\theta}(0)$ (which is real-valued by the continuity assumption and oddness of the quadrature spectrum).

Theorem 4.2 Let $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ under (4.11). Then for $\ell = 0, 1$,

$$\lim_{n\to\infty} \frac{1}{\log n} E\left\{\widehat{F}_{\zeta\xi}(\ell, \widetilde{n})\right\} = 2f_{\eta\theta}(0)\sin\alpha\pi = 2f_{\eta\theta}(0)\sin\beta\pi \tag{4.18}$$

and under (2.7)

$$\lim_{n\to\infty} \frac{1}{\log m} E\left\{\widehat{F}_{\zeta\xi}(\ell, m)\right\} = 2f_{\eta\theta}(0)\sin\alpha\pi = 2f_{\eta\theta}(0)\sin\beta\pi. \tag{4.19}$$

The degeneration condition (2.7) now leaves little difference between the expectations of the broad- and narrow band statistics, in fact for $m \sim n^a$, $0 < a < 1$, they have the same convergence rates. Note that just as Theorem 4.1 covered the case $\beta = \frac{1}{2}$, the border of the nonstationary region, so Theorem 4.2 covers $\alpha = \beta = \frac{1}{2}$.

Though Theorem 4.2 does not cover (4.12), putting $\alpha = 0$ or $\beta = 1$ annihilates the limits (4.18) and (4.19), suggesting a faster rate of convergence under (4.12). This is indeed the outcome, implying that the $I(0,1)$ case (4.12), which looms large in the econometric literature within an autoregressive framework, is also rather special within the fractional domain. These results do require a strengthening of the condition on $\{\zeta_t, \xi_t\}$. Define the function

$$h_{\eta\theta}(\lambda) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}(\omega_{-|j|} - \omega_{|j|+1})\cos j\lambda, \quad \lambda \in \Pi, \tag{4.20}$$

where

$$\omega_j = \sum_{\ell=-|j|}^{\infty} \gamma_{\ell sign(j)}, \quad \gamma_j = Cov(\eta_0, \theta_j), \quad j = 0, \pm 1, \dots\, , \tag{4.21}$$

with the convention that sign(0) is negative.

Theorem 4.3 Let $\{\zeta_t, \xi_t\} \in I(0,1)$.

(i) If also $h_{\eta\theta}(\lambda)$ is integrable on Π and

$$\sum_{t=0}^{\infty} \left|\psi_t - \psi_{t+1}\right| < \infty, \tag{4.22}$$

then

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(0,\widetilde{n})\right\} = \sum_{j=0}^{\infty} \psi_j \gamma_{-j}, \tag{4.23}$$

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(1,\widetilde{n})\right\} = \sum_{j=-\infty}^{\infty} \left\{(\psi_j - \frac{1}{2})\gamma_{-j} - \frac{1}{2}\gamma_{j+1}\right\}. \tag{4.24}$$

(ii) If also $h_{\eta\theta}(\lambda)$ is continuous at $\lambda = 0$, (2.7) holds and

$$\sum_{t=0}^{\infty} |\psi_t - 1| < \infty, \tag{4.25}$$

then

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(0,m)\right\} = \frac{1}{2}\sum_{j=-\infty}^{\infty} \gamma_j = \pi f_{\eta\theta}(0), \tag{4.26}$$

$$\lim_{n\to\infty} \frac{n}{m} E\left\{\widehat{F}_{\zeta\xi}(1,m)\right\} = \gamma_j + \sum_{j=1}^{\infty}(2j+1)(\gamma_{-j}-\gamma_j) + 2\sum_{t=0}^{\infty}(\psi_t - 1)\sum_{j=-\infty}^{\infty}\gamma_j. \tag{4.27}$$

It is sufficient for the conditions on $h_{\eta\theta}(\lambda)$ that $\sum \left|j\gamma_j\right| < \infty$, which is implied if $f_{\eta\theta}(\lambda)$ is differentiable with derivative satisfying a Lipschitz condition of degree greater than $\frac{1}{2}$ (see Zygmund, 1977, p.240) but a global smoothness condition is not implied, though by the Riemann-Lebesgue lemma $\omega_{-|j|} - \omega_{|j|+1} \to 0$ as $|j| \to \infty$. Note that if $\gamma_j \equiv -\gamma_j$ (as is true if $\eta_t \equiv \theta_t$, for example), we have $h_{\eta\theta}(\lambda) \equiv f_{\eta\theta}(\lambda)$, so the additional conditions are vacuous. The mean-corrected narrow-band statistic $\widehat{F}_{\zeta\xi}(1,m)$ (but not $\widehat{F}_{\zeta\xi}(0,m)$) has expectation of smaller order than that of the full band statistics. Sensitivity is found, except in (4.26), to the precise values of the sequence $\{\psi_t\}$, rather than simply their asymptotic value (in this case, 1). In the usual case $\psi_t \equiv 1$, stressed in the econometric literature, (4.23), (4.24) and (4.27) become, respectively,

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(0,\widetilde{n})\right\} = \frac{1}{2}\int_{\Pi} h_{\eta\theta}(\lambda) + \pi f_{\eta\theta}(\theta) = \sum_{j=0}^{\infty}\gamma_{-j}, \tag{4.28}$$

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(1,\widetilde{n})\right\} = \int_{\Pi} h_{\eta\theta}(\lambda) = \frac{1}{2}\sum_{j=0}^{\infty}\left(\gamma_{-j}-\gamma_{j}\right), \tag{4.29}$$

$$\lim_{n\to\infty} E\left\{\widehat{F}_{\zeta\xi}(1,m)\right\} = 2\pi h_{\eta\theta}(\theta) = \gamma_0 + \sum_{j=1}^{\infty}(2j+1)(\gamma_{-j}-\gamma_{j}). \tag{4.30}$$

The results (4.28) and (4.29) are already known though seemingly only under more global conditions, with respect to the frequency domain. Condition (4.22) is only slightly stronger than (3.3) since we have $\alpha = 1$ in Definition 3.1, while (4.25) is stronger than (4.22), by the triangle inequality.

The case (4.13) is somewhat anomalous, the "discontinuiity" at $\alpha = 0$ in Definition 3.1 here taking effect. In the first place it can be shown under rather similar conditions on η_t, θ_t to those before that $E\left\{\widehat{F}_{\zeta\xi}(1,\widetilde{n})\right\} = o(n^{\beta-1})$ (rather than having rate $n^{\beta-1}$ as would be consistent with Theorem 4.3 and Theorem 4.4 below) while under stronger conditions a smaller order is possible, indeed for ARMA η_t, θ_t we have $E\left\{\widehat{F}_{\zeta\xi}(0,\widetilde{n})\right\} = O(1)$ for all $\beta > 1$. Moreover, $E\left\{\widehat{F}_{\zeta\xi}(1,m)\right\}$, under (2.7), turns out to be especially complicated. As the paper is already lengthy we thus omit discussion of the rather special case (4.13).

Theorem 4.4 Let $\{\zeta_t, \xi_t\} \in I(\alpha,\beta)$ under (4.14). Then under (2.6)

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E\left\{\widehat{F}_{\zeta\xi}(0,m)\right\} = \frac{2\pi f_{\eta\theta}(0)}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta)(\alpha+\beta-1)}, \tag{4.31}$$

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E\left\{\widehat{F}_{\zeta\xi}(1,m)\right\} = \frac{A(\alpha,\beta)2\pi f_{\eta\theta}(0)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{4.32}$$

where

$$A(\alpha,\beta) = \frac{\alpha\beta(\alpha+\beta-1) - \alpha(\alpha-1) - \beta(\beta-1)}{\alpha\beta(\alpha+\beta-1)(\alpha+\beta)(\alpha+\beta-1)}. \tag{4.33}$$

The distinctive feature of Theorem 4.4 is that $E\left\{\widehat{F}_{\zeta\xi}(\ell,\widetilde{n})\right\}$ is dominated by an arbitrarily slowly increasing number of low frequency components. As in some of our earlier results, the rate of convergence is improved if η_t and θ_t are fully incoherent at zero frequency, not necessarily at all frequencies. Note that only (2.6) is imposed, so that we also cover the case where m increases as fast as n.

5 5. THE VARIANCE OF THE AVERAGED PERIODOGRAM

Unlike in the case of the mean, we can give a single theorem to describe the variance of $\widehat{F}_{\zeta\xi}(\ell, m)$ for

$$0 \leq \alpha \leq \beta, \quad \beta > \frac{1}{2}, \tag{5.1}$$

though different proofs are needed over different portions of this region. Thus we cover the case $\alpha = 0$, $\beta > 1$ excluded in Section 4, but now omit the case $\alpha = 0$, $\beta = \frac{1}{2}$, which seems too special to include in view of the special treatment it requires.

We need to extend some earlier definitions.

Definition 5.1 I_3 is the class of jointly fourth order stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, ...\}$, such that $\{\eta_t, \theta_t\} \in I_2$ and the cumulant spectral density $f_{\eta\theta\eta\theta}(\lambda, \mu, \omega)$ given by

$$cum\left\{\eta_s, \theta_t, \eta_u, \theta_v\right\} = \int\limits_{\Pi}\int\limits_{\Pi}\int\limits_{\Pi} f_{\eta\theta}(\lambda, \mu, \omega) e^{i(t-s)\lambda + i(u-s)\lambda + i(v-s)\omega} d\mu d\lambda d\omega \tag{5.2}$$

is continuous at $\lambda = \mu = \omega = 0$ and satisfies

$$\sup_{\mu,\omega \in \Pi} \int\limits_{\Pi} f^2_{\eta\theta\eta\theta}(\lambda, \mu, \omega) d\lambda < \infty. \tag{5.3}$$

Definition 5.2 I_4 is the class of jointly fourth order stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, ...\}$ such that $\{\eta_t, \theta_t\} \in I_3$ and $f_{\eta\eta}(\lambda)$, $f_{\theta\theta}(\lambda)$ are square integrable.

Definition 5.3 For $j = 3, 4$, $I_j(\alpha.\beta)$ is the class of bivariate processes $\{\zeta_t, \xi_t, t = 0, 1, ...\}$ such that (3.19) and (4.2) hold for $\{\eta_t, \theta_t\} \in I_j$.

We introduce

$$P(x, y; \alpha, \beta) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_{\max(0, x-y)}^{x} z^{\alpha-1}(y - x + z)^{\beta-1} dz, \quad \alpha > 0,\ \beta > 0, \tag{5.4}$$

$$= \frac{2\pi}{\Gamma(\beta+1)} \left\{y^{\beta} - (y-x)^{\beta} 1(x < y)\right\}, \qquad \alpha = 0,\ \beta > 0, \tag{5.5}$$

$$= 2\pi \min(x, y), \qquad \alpha = \beta = 0, \tag{5.6}$$

$$Q(\alpha,\beta,\gamma,\delta) = \int_0^1\int_0^1 P(x,y;\alpha,\beta)P(y,x;\gamma,\delta)dxdy, \quad (5.7)$$

$$R(\alpha,\beta,\gamma,\delta) = \int_0^1\int_0^1\int_0^1 P(x,y;\alpha,\beta)P(z,x;\gamma,\delta)dxdydz, \quad (5.8)$$

$$S(\alpha,\beta,\gamma,\delta) = \int_0^1\int_0^1 P(x,y;\alpha,\beta)dxdy\int_0^1\int_0^1 P(x,y;\gamma,\delta)dxdy, \quad (5.9)$$

$$T(\alpha,\beta,\gamma,\delta) = Q(\alpha,\beta,\gamma,\delta) - 2R(\alpha,\beta,\gamma,\delta) + S(\alpha,\beta,\gamma,\delta). \quad (5.10)$$

Theorem 5.1 Let $\{\zeta_t,\xi_t\} \in I_3(\alpha,\beta)$ for $\alpha > \frac{1}{2}$ $\beta > \frac{1}{2}$ and $\{\zeta_t,\xi_t\} \in I_4(\alpha,\beta)$ for $0 \le \alpha \le \frac{1}{2}$, $\beta > \frac{1}{2}$. Then under (2.6)

$$\lim_{n\to 0} n^{2(1-\alpha-\beta)} Var\left\{\widehat{F}_{\zeta\xi}(0,m)\right\} = f^2_{\eta\theta}(0)Q(\alpha,\beta,\alpha,\beta) + f_{\eta\eta}(0)f_{\theta\theta}(0)Q(\alpha,\alpha,\beta,\beta), \quad (5.11)$$

$$\lim_{n\to 0} n^{2(1-\alpha-\beta)} Var\left\{\widehat{F}_{\zeta\xi}(1,m)\right\} = f^2_{\eta\theta}(0)T(\alpha,\beta,\alpha,\beta) + f_{\eta\eta}(0)f_{\theta\theta}(0)T(\alpha,\alpha,\beta,\beta), \quad (5.12)$$

$$\lim_{n\to 0} n^{2(1-\alpha-\beta)} Var\left\{\widehat{F}_{\zeta\xi}(m+1,\widetilde{n})\right\} = 0. \quad (5.13)$$

Throughout the region (5.1), $Var\left\{\widehat{F}_{\zeta\xi}(\ell,m)\right\}$ is asymptotically dominated by the contribution from an arbitrarily slowly increasing number of low frequencies. The variance is increased when $f_{\eta\theta}(0) \neq 0$, though this does not affect the rate of convergence, or divergence. The square integrability requirement on $f_{\eta\eta}$ and $f_{\theta\theta}$ (and thence on $f_{\eta\theta}$) when $\alpha \le \frac{1}{2}$ seems unavoidable and is, for example, essential for sample autocovariances of stationary sequences to be $n^{\frac{1}{2}}$-consistent (see Hannan, 1976). The fourth cumulant requirement seems rather mild by the standards of such conditions in the literature; (5.3) is milder than boundedness of $f_{\eta\theta\eta\theta}$, but stronger than square integrability. We suspect that it could be further relaxed, but the proof would further lengthen the paper and our current condition is automatically satisfied when η_t,θ_t are Gaussian. In any case the absence from the limiting variances (5.11) and (5.12) of any full cumulant contribution is fortunate, and also distinctive from the stationary situation.

6 6. COINTEGRATION APPLICATION

We define observable sequences $\{y_t, z_t, t = 0, 1, ...\}$such that

$$y_t = \zeta_t + \nu \xi_t, \qquad z_t = \xi_t \tag{6.1}$$

where ν is unknown and $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ under (4.1) with

$$\alpha < \beta, \tag{6.2}$$

subject to which we shall examine cases (4.10), (4.11), (4.12) and (4.14). From (6.1), y_t and z_t have a common, nonstationary, component ξ_t, while y_t has an additional component ζ_t that can be nonstationary or asymptotically stationary. It is readily possible to apply the results of the preceding sections to a model with additional components in y_t and z_t, with smaller memory parameters, and to a model with vector observables of arbitrary dimension, but we keep the setting as simple as possible to conserve on notation. We deduce (1.1) from (6.1) and as discussed in Section 1 consider estimating ν by

$$\widehat{\nu}_\ell = \widehat{F}_{\zeta\xi}(\ell, \widetilde{n}) / \widehat{F}_{\zeta\xi}(\ell, \widetilde{n}), \quad \ell = 0, 1, \tag{6.3}$$

and also by

$$\widetilde{\nu}_\ell = \widehat{F}_{\zeta\xi}(\ell, m) / \widehat{F}_{\zeta\xi}(\ell, m), \quad \ell = 0, 1, \tag{6.4}$$

so that $\widehat{\nu}_\ell$ the LSE with ($\ell = 1$) or without ($\ell = 0$) intercept, and under (2.7) $\widetilde{\nu}_\ell$ is the NBLSE likewise mean-corrected or not. When (2.6) holds with $m \sim cn$, $0 < c < 1$, then $\widetilde{\nu}_\ell$ is based on a nondegenerate band of frequencies, following the idea of Hannan (1963). Note that Phillips (1991) considered a spectral form of estimate in cointegration with $\alpha = 0$ or $\beta = 1$, though his proofs concern weighted periodogram estimates rather than averaged periodogram ones, and in a nonstationary environment these are not necessarily close asymptotically. For $\alpha = 0$, $\beta = 1$, Phillips (1991) and others have proposed superior estimates to the LSE and NBLSE, but while they can straightforwardly be extended to other known α, β, it is far harder to show that these properties are unaffected by substituting estimates of unknown α, β, in view of their relatively slow convergence rates, and in any case $\widehat{\nu}_\ell$ or $\widetilde{\nu}_\ell$ will still be of use at a preparatory stage.

Our main interest is in comparison of $\widehat{\nu}_\ell, \widetilde{\nu}_\ell$ across ℓ, m in terms of bias and convergence rates but we can also attempt to characterize limit distributions. It follows from Theorems 4.4 and 5.1 that $n^{1-2\beta} F_{\xi\xi}(\ell, \widetilde{n})$ and, when $\alpha + \beta > 1$, $n^{1-\alpha-\beta} F_{\zeta\xi}(\ell, \widetilde{n})$, have mean and variance which both have finite but nonzero limits. Thus we introduce:

Assumption 6.1 For $\ell = 0, 1$, there exist random variables $\Phi_\ell(\beta)$, $\Psi_\ell(\alpha, \beta)$ such that $\Phi_\ell(\beta) \neq 0$ almost surely and

$$n^{1-2\beta} \widehat{F}_{\xi\xi}(\ell, \widetilde{n}) \longrightarrow {}_d \Phi_\ell(\beta), \quad \beta > \frac{1}{2} \tag{6.5}$$

$$n^{1-\alpha-\beta} \widehat{F}_{\zeta\xi}(\ell, \widetilde{n}) \longrightarrow {}_d \Psi_\ell(\alpha, \beta), \quad \alpha + \beta > 1. \tag{6.6}$$

We can deduce (6.5) and (6.6) from the continuous mapping theorem if there exist jointly dependent processes $U(r;\alpha)$, $V(r;\beta)$, $0 \le r \le 1$, such that

$$\left\{\zeta_{[nr]}, \xi_{[nr]}\right\} \Rightarrow \{U(r;\alpha), V(r;\beta)\}, \quad \text{as } n \to \infty, \quad 0 \le r \le 1, \qquad (6.7)$$

where "⇒" denotes a suitable notion of weak convergence (see Billingsley, 1968, pp.30, 111-123). Then $\Phi_0(\beta) = \int_0^1 V(r;\beta)^2 dr$, $\Phi_1(\beta) = \Phi_0(\beta) - \left\{\int_0^1 V(r;\beta)dr\right\}^2$ and $\Psi_0(\alpha,\beta) = \int_0^1 U(r;\alpha)V(r;\beta)dr$, $\Psi_1(\alpha,\beta) - \int_0^1 U(r;\alpha)V(r;\beta)dr$. Sufficient conditions for (6.7) given by Marinucci and Robinson (1998) (which developes earlier work of Akonom and Gourieroux (1987), Silveira (1990)), are that ϕ_t, ψ_t are given by $\Delta_t(\alpha)$, $\Delta_t(\beta)$, while $(\eta_t, \theta_t)' = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{r-j}$, the A_j being 2×2 matrices such that $\sum_{j=0}^{\infty} \sum_{|k|=j+1}^{\infty} \|A_k\|^2 < \infty$ where $\|.\|$ is Euclidean norm, the ε_t being independent and identically distributed with zero mean and finite qth moment for $q > \max(2, 2/(2\alpha-1), 2/(2(\beta-1))$ while $\sum_{j=-\infty}^{\infty} A_j$ and the covariance matrix of ε_t have full rank. These conditions are implied by Gaussian "FARIMA" (ζ_t, ξ_t), such that (η_t, θ_t) is a stationary and invertible "ARMA" sequence while on the other hand implying that $(\zeta_t, \xi_t) \in I_4(\alpha, \beta)$. Then for $\alpha, \beta > \frac{1}{2}$ we have (6.7) with U, V "fractional Brownian motion"

$$(U(r;\alpha), V(r;\beta)) = \int_0^r \left\{(r-s)^{\alpha-1} dB_1(r), (r-s)^{\beta-1} dB_2(r)\right\}, \qquad (6.8)$$

where $B(r) = \{B_1(r), B_2(r)\}'$ is 2×1 Brownian motion with $EB(r) = 0$ and

$$E\left\{B(r_1)B(r_2)'\right\} = 2\pi \min(r_1, r_2) \begin{bmatrix} f_{\eta\eta}(0) & f_{\eta\theta}(0) \\ f_{\eta\theta}(0) & f_{\theta\theta}(0) \end{bmatrix}. \qquad (6.9)$$

When $\alpha \le \frac{1}{2}$, V is given as in (6.8) under a simplified version of the conditions. We cannot so characterize $\Phi_\ell(\alpha, \beta)$ when $\alpha + \beta > 1$ but $0 < \alpha \le \frac{1}{2}$ since on the one hand the continuous mapping theorem does not apply, while on the other ζ_t cannot be approximated by a semi-martingale. The latter property holds when $\alpha = 0$, $\beta = 1$ (case (4.12)) where, when $\psi_t \equiv 1$,

$$\Psi_0(0,1) = \int_0^1 B_2(1)dB_1(r) + \omega_0,\ \Psi_1(0,1) = \Psi_0(0,1) - B_1(1)B_2(1) - \pi f_{\eta\theta}(0), \qquad (6.10)$$

where ω_0 represents the limiting expectation of $\widehat{F}_{\zeta\xi}(0, \widetilde{n})$ from (4.28), and $\frac{1}{2}(\omega_0 - \omega_1) = \omega_0 - \pi f_{\eta\theta}(0)$ that of $\widehat{F}_{\zeta\xi}(1, \widetilde{n})$, from (4.29).

Proposition 6.1 Let $(\zeta_t, \xi_t) \in I_4(\alpha, \beta)$ under (4.10) and let (6.1), (6.2) and Assumption 6.1 hold. Then as $n \to \infty$

$$n^{2\beta-1}(\widehat{\nu}_\ell - \nu) \longrightarrow_d \frac{\int_\pi \phi(\lambda)\psi(\lambda) f_{\eta\theta}(\lambda) d\lambda}{\Psi_\ell(\beta)}, \quad \ell = 0, 1, \tag{6.11}$$

and under (2.7)

$$n^{\beta-\alpha} m^{\alpha+\beta-1}(\widetilde{\nu}_\ell - \nu) \longrightarrow_d \frac{2(2\pi)^{1-\alpha-\beta} f_{\eta\theta}(0) \frac{\cos(\beta-\alpha)\frac{\pi}{2}}{1-\alpha-\beta}}{\Phi_\ell(\beta)}, \quad \ell = 0, 1. \tag{6.12}$$

Proof Write $\widehat{a}_\ell = \widehat{F}_{\zeta\xi}(\ell, \widetilde{n})$, $\widehat{b}_\ell = \widehat{F}_{\xi\xi}(\ell, \widetilde{n})$, and $\widetilde{a}_\ell = \widehat{F}_{\zeta\xi}(\ell, m)$, $\widetilde{b}_\ell = \widehat{F}_{\xi\xi}(\ell, m)$, so $\widehat{\nu}_\ell - \nu = \widehat{a}_\ell / \widehat{b}_\ell$, $\widetilde{\nu}_\ell - \nu = \widetilde{a}_\ell / b_\ell$. Now $\widetilde{b}_\ell = \widehat{b}_\ell - \left\{\widehat{b}_\ell - \widetilde{b}_\ell - E(\widehat{b}_\ell - \widetilde{b}_\ell)\right\} - E(\widehat{b}_\ell - \widetilde{b}_\ell)$. The term in braces is $o_p(n^{1-2\beta})$ from (5.12) of Theorem 5.1, while from (4.31) and (4.32) of Theorem 4.4, $E(\widehat{b}_\ell - \widetilde{b}_\ell) = o(n^{1-2\beta})$. Thus from Assumption 6.1 we have $n^{1-2\beta}\widetilde{b}_\ell$, $n^{1-2b}\beta\widehat{b}_\ell \longrightarrow_d \Psi_\ell(\beta)$. Next, from Theorem 5.1, $\widehat{a}_\ell = E\widehat{a}_\ell + O_p(n^{\alpha+\beta-1})$ and $\widetilde{a}_\ell = E\widetilde{a}_\ell + O_p(n^{\alpha+\beta-1})$, so that $\lambda_m^{\alpha+\beta-1}\widetilde{a}_\ell = \lambda_m^{\alpha+\beta-1} E\widetilde{a}_\ell + O_p(n^{\alpha+\beta-1})$. The proof is then routinely completed by means of Theorem 4.1. □

Proposition 6.2 Let $(\zeta_t, \xi_t) \in I_4(1-\beta, \beta)$, under (4.11) and let (6.2) and Assumption 6.1 hold. Then as $n \to \infty$

$$\frac{n^{2\beta-1}}{\log n}(\widehat{v}_\ell - \nu) \longrightarrow_d \frac{2 f_{\eta\theta}(0) \sin \beta\pi}{\Phi_\ell(\beta)}, \quad \ell = 0, 1, \tag{6.13}$$

and under (2.6)

$$\frac{n^{2\beta-1}}{\log m}(\widehat{v}_\ell - \nu) \longrightarrow_d \frac{2 f_{\eta\theta}(0) \sin \beta\pi}{\Phi_\ell(\beta)}, \quad \ell = 0, 1. \tag{6.14}$$

Proof From Theorem 5.1, $\widehat{a}_\ell - E\widehat{a}_\ell$, $\widetilde{a}_\ell - E\widetilde{a}_\ell$ are $O_p(1)$, so that $\widehat{a}_\ell / \log n$, $\widetilde{a}_\ell / \log m \longrightarrow_p 2 f_{\eta\theta}(0) \sin \beta\pi$ by Theorem 4.4, and the remaining proof follows from that of Proposition 6.1 □

Proposition 6.3 Let $(\zeta_t, \xi_t) \in I_4(0, 1)$ and let (6.1), Assumption 6.1, and the additional assumptions of Theorem 4.3 hold. Then as $n \to \infty$

$$n(\widehat{\nu}_\ell - \nu) \longrightarrow_d \frac{\Psi_\ell(0, 1)}{\Phi_\ell(1)}, \quad \ell = 0, 1 \tag{6.15}$$

and under (2.7)

$$n(\widetilde{\nu}_0 - \nu) \longrightarrow_d \frac{\Psi_\ell(0,1) - \sum_{j=0}^{\infty}\{(\psi_j - \frac{1}{2})\gamma_{-j} - \frac{1}{2}\gamma_{j+1}\}}{\Phi_\ell(1)}, \quad \ell = 0,1. \tag{6.16}$$

Proof We have $\widehat{a}_\ell \longrightarrow_d \Psi_\ell(0,1)$ by Assumption 6.1 and $\widetilde{a}_\ell = \widehat{a}_\ell - E\widehat{a}_\ell - \{\widehat{a}_\ell - \widetilde{a}_\ell - E(\widehat{a}_\ell - \widetilde{a}_\ell)\} + E\widetilde{a}_\ell$. For $\ell = 1$, the last two terms are $o_p(1)$ by Theorem 5.1, and $O(m/n)$ by (4.26) and (4.27) of Theorem 4.3, whereas by Assumption 6.1 and (4.23) and (4.24) of Theorem 4.3, $\widehat{a}_\ell - E\widehat{a}_\ell$ converges in distribution to the numerators on the right of (6.10), (6.12) for $\ell = 0,1$. For $\ell = 0$, the only difference is that $E\widehat{a}_0 \to \pi f(0)$, and since $E\widetilde{a}_0 \to \sum_{j=0}^{\infty}\psi_j\gamma_{-j}$ we get the same correction term in the numerator as when $\ell = 1$. The proof is again completed by that of Proposition 6.1. □

Proposition 6.4 Let $(\zeta_t, \xi_t) \in I_4(\alpha, \beta)$ and let (6.1), (6.2) and Assumption 6.1 hold. Then as $n \to \infty$, for $\ell = 0,1$

$$n^{\alpha+\beta-1}(\widehat{\nu}_\ell - \nu) \longrightarrow_d \frac{\Psi_\ell(\alpha,\beta)}{\Phi_\ell(\beta)}, \tag{6.17}$$

and under (2.7)

$$n^{\alpha+\beta-1}(\widetilde{\nu}_\ell - \widehat{\nu}_\ell) \longrightarrow_p 0, \tag{6.18}$$

and thus

$$n^{\alpha+\beta-1}(\widetilde{\nu}_\ell - \nu_\ell) \longrightarrow_d \frac{\Psi_\ell(\alpha,\beta)}{\Phi_\ell(\beta)}. \tag{6.19}$$

Proof The proof of (6.17) is routine, and (6.19) will follow from (6.17) and (6.18). We write $\widetilde{\nu}_\ell - \widehat{\nu}_\ell = (\widetilde{a}_\ell - \widehat{a}_\ell)/\widehat{b}_\ell + \widetilde{a}(\widetilde{b}_\ell^{-1} - \widehat{b}_\ell^{-1})$. Now $\widetilde{a}_\ell - \widehat{a}_\ell = o_p(n^{1-\alpha-\beta})$ and $\widetilde{a}_\ell = O_p(n^{1-\alpha-\beta})$ by Theorem 4.4, while $\widetilde{b}_\ell^{-1} - \widehat{b}_\ell^{-1} = (\widehat{b}_\ell\widetilde{b}_\ell)^{-1}(\widehat{b}_\ell - \widetilde{b}_\ell)$, which is $o_p(n^{1-\alpha-\beta})$ by the proof of Proposition 6.1 and Assumption 6.1. □

Proposition 6.4 shows that when the combined memory of the observables and cointegrating error exceeds that of the usual case $\alpha = 0$, $\beta = 1$, $\widetilde{\nu}_\ell$ has the same convergence rate of limit distribution as $\widehat{\nu}_\ell$, so that nothing is lost by neglecting high frequencies, even all those outside a band around zero that decays arbitrarily slower than n^{-1}. In Propositions 6.1-6.3, $\widetilde{\nu}_\ell$ is found to have the capacity to beat $\widehat{\nu}_\ell$ when it is less affected by the "bias" due to correlation between ζ_t and ξ_t in (1.1). In Proposition 6.3, when $\alpha = 0$, $\beta = 1$, rates of convergence are identical but $\widetilde{\nu}_1$ eliminates the "second-order bias" (see Phillips, 1991), namely the expectation of $\Psi_1(0,1)$; more particularly, the "second order bias" of $\widetilde{\nu}_1$ is only $O(m/n^2)$, which is of smaller order than $1/n$ under (2.7). Monte Carlo simulations demonstrate the consequent superiority of $\widetilde{\nu}_1$ in smallish samples. (Note that $\widetilde{\nu}_0$ does not share the desirable properties of $\widetilde{\nu}_1$.) In Proposition 6.2, $\alpha \neq 0$, $\beta \neq 1$ but again $\alpha + \beta = 1$, and here the comparison depends on m. If m increases at the same rate as n, as permitted by (2.6), so $\log m \sim \log n$, then $\widehat{\nu}_\ell$ and $\widetilde{\nu}_\ell$ have the same convergence rate and limit distribution. On the other hand if (2.7) holds there are essentially two possibilities of interest. If $m \sim cn^d$, for $c > 0$, $0 < d < 1$, then $\widetilde{\nu}_\ell$ has the same convergence rate as $\widehat{\nu}_\ell$ but it is shrunk towards ν. If $\log m = o(\log n)$, for example if $m = \log\log n$, then $\widetilde{\nu}_\ell$ converges faster than $\widehat{\nu}_\ell$. This is much more dramatically the case in Proposition 6.1, where, with $\alpha + \beta < 1$, $\widetilde{\nu}_\ell$'s bias-reducing qualities really come to the fore; the more slowly m increases the better.

7 7. PROOFS FOR SECTION 3

<u>**Proof of Lemma 3.1**</u> The proof when $\alpha = 0$ and/or $\beta = 0$ is trivial so assume $\alpha > 0$, $\beta > 0$. By integral approximation we have

$$\chi_t \sim \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \sim \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \tag{7.1}$$

to verify (3.2). For $0 < r < t$ we may write

$$\chi_t - \chi_{t+1} = \sum_{s=0}^{r} \phi_s(\psi_{t-s} - \psi_{t+1-s}) - \phi_{r+1}\psi_{t-r} + \sum_{s=0}^{t-r-1} (\phi_{t-s} - \phi_{t+1-s})\psi_s. \tag{7.2}$$

Taking $r = [t/2]$, the first and last terms are $O\left(\sum_{j=1}^{t} j^{\alpha+\beta-3}\right)$ which is $O(1)$ for $\alpha + \beta < 2$, $O(\log t)$ for $\alpha + \beta = 2$, and $O\left(t^{\alpha+\beta-2}\right)$ for $\alpha + \beta > 2$, while the second is $o\left(t^{\alpha+\beta-2}\right)$, and in each case these terms are $O(|\chi_t|/t)$, to verify (3.3). $\square$

<u>**Proof of Lemma 3.2**</u> The proof of (3.9) is trivial so we consider (3.10) and (3.11) with $\alpha > 0$. Drop the argument α from $S_{uv}(\lambda, \alpha)$. Obviously $|S_{uv}(\lambda)| \leq Cv^\gamma$. For $\alpha \in (0,1)$ we write, for $u < s < r$,

$$S_{uv}(\lambda) = \sum_{t=u}^{s-1} \varphi_t e^{it\lambda} + \sum_{t=s}^{v-1} \left(\varphi_t - \varphi_{t+1}\right) \sum_{v=s}^{t} e^{iv\lambda} + \varphi_v \sum_{t=s}^{v} e^{it\lambda} \tag{7.3}$$

by summation-by-parts. Thus because

$$\left|\sum_{v=s}^{t} e^{iv\lambda}\right| \leq \frac{C(t-s)}{1+(t-s)|\lambda|}, \qquad |\lambda| < \pi, \tag{7.4}$$

(see, e.g. Zygmund, 1977, p.51), (3.2) and (3.3) imply that $|S_{uv}(\lambda)| \leq C\left(s^{\alpha} + s^{\alpha-1}/|\lambda|\right)$. For $c \in (0,\pi)$ we may choose $s = [c/|\lambda|]$ when $c/v \leq |\lambda| \leq c/(u+1)$, because $c/|\lambda| - 1 \leq [c/|\lambda|] \leq c/|\lambda|$, which gives the bound $C/|\lambda|^{\alpha}$ for such λ. On the other hand we also have

$$S_{uv}(\lambda) = \sum_{t=u}^{v-1}(\phi_t - \phi_{t+1})\sum_{s=u}^{t} e^{is\lambda} + \phi_v \sum_{t=u}^{v} e^{it\lambda} \tag{7.5}$$

to deduce $|S_{uv}(\lambda)| \leq C(u+1)^{\alpha-1}/|\lambda|$ for $0 < \alpha < 1$ from (3.2), (3.3), (7.4). Since $v^{\alpha} \leq C/|\lambda|^{\alpha}$ for $0 < |\lambda| < c/v$ and $(u+1)^{\alpha-1}/|\lambda| \leq C/|\lambda|^{\alpha}$ for $c/(u+1) \leq |\lambda| \leq \pi$ the bound $C/|\lambda|^{\alpha}$ holds for all $\lambda \in (0,\pi]$ when $0 < \alpha < 1$, to complete the proof of (3.10). For $\alpha > 1$, (7.5) gives instead $|S_{uv}(\lambda)| \leq Cv^{\alpha-1}/|\lambda|$ to complete the proof of (3.11). Finally (3.12) and (3.13) follow directly from Theorem III-11 of Yong (1974) and reflection formula for the Gamma function. □

Proof of Lemma 3.3 We have

$$\begin{aligned}\xi_t &= \sum_{s=-\infty}^{t} \psi_{t-s}\left[1(s \geq 1)\sum_{r=-\infty}^{s} \phi_{s-r}\{\eta_r 1(r \geq 1)\}\right] \\ &= \sum_{s=1}^{t} \psi_{t-s}\sum_{r=1}^{s} \phi_{s-r}\eta_r = \sum_{s=1}^{t} \chi_{t-s}\eta_s \end{aligned} \tag{7.6}$$

for $t \geq 1$, and 0 for $t \leq 0$, where χ_t is given in (3.7). □

Proof of Lemma 3.4 (i) The first statement is standard while (3.23) follows from the stated formula for $f_{\rho\rho}$, (3.12) and (3.13) of Lemma 3.2 and $\{\eta_t\} \in I$. For $\alpha > 0$ write $\overline{\phi}_t(\lambda) = \sum_{s=t}^{\infty} \phi_s e^{is\lambda}$. From (3.15), (3.16) we have respectively $|\phi_t(\lambda)| \leq C|\lambda|^{-\alpha}$ and $\left|\overline{\phi}_t(\lambda)\right| \leq Ct^{\alpha-1}/|\lambda|$, so,

$$\begin{aligned}&|\phi(\lambda)|^2 - |\phi_t(\lambda)|^2 = \phi_t(\lambda)\overline{\phi}_t(-\lambda) + \phi_t(-\lambda)\overline{\phi}_t(\lambda) \\ &\leq C\left(t^{\alpha-1}|\lambda|^{-\alpha-1} + t^{2\alpha-2}\lambda^{-2}\right) \leq C\lambda^{-2\alpha}\left((t|\lambda|)^{\alpha-1} + (t|\lambda|)^{2(\alpha-1)}\right),\end{aligned} \tag{7.7}$$

whence (3.24) follows by reference to (3.23). To prove (3.25), note that

$$Cov(\zeta_t, \zeta_{t+j}) = \int_{\Pi} f_{\zeta\zeta}^{(t)}(\lambda)e^{ij\lambda}d\lambda, \tag{7.8}$$

so

$$\left|Cov(\zeta_t, \zeta_{t+j}) - Cov(\rho_t, \rho_{t+j})\right| \leq 2 \int\limits_{\Pi} \left\{ \left|\phi_t(\lambda)\overline{\phi}(\lambda)\right| + \left|\overline{\phi}_t(\lambda)\right|^2 \right\} f_{\eta\eta}(\lambda) d\lambda. \tag{7.9}$$

Fix $\delta > 0$. Because $\{\eta_t\} \in I$ we can choose $\varepsilon > 0$ such that

$$\sup_{|\lambda|<\varepsilon} |f_{\eta\eta}(\lambda) - f_{\eta\eta}(0)| < \delta. \tag{7.10}$$

By the Schwarz inequality the contribution to (7.9) from the integral over $(-\varepsilon, \varepsilon)$ is thus bounded by

$$4\{f(0) + \delta\} \left\{ \int\limits_{\Pi} |\phi(\lambda)|^2 d\lambda \int\limits_{\Pi} \left|\overline{\phi}_t(\lambda)\right|^2 \right\}^{\frac{1}{2}} \leq C \sum_{s=t}^{\infty} \phi_s^2 = O\left(t^{2\gamma-1}\right) \tag{7.11}$$

as $t \to \infty$, while the contribution from $[-\pi, -\varepsilon] \cup [\varepsilon, \pi]$ is bounded by

$$\frac{C}{\varepsilon^2}\left(t^{\alpha-1} + t^{2(\alpha-1)}\right) \int\limits_{\Pi} f_{\eta\eta}(\lambda) d\lambda = O\left(t^{\alpha-1}\right) \tag{7.12}$$

using (3.16).

(ii) and (iii). Write (7.9) as

$$f_{\eta\eta}(0) \int\limits_{\Pi} |\phi_t(\lambda)|^2 e^{ij\lambda} d\lambda + \int' \{f_{\eta\eta}(\lambda) - f_{\eta\eta}(0)\} |\phi_t(\lambda)|^2 e^{ij\lambda} d\lambda \tag{7.13}$$

$$+ \int'' \{f_{\eta\eta}(\lambda) - f_{\eta\eta}(0)\} |\phi_t(\lambda)|^2 e^{ij\lambda} d\lambda, \tag{7.14}$$

writing $\int'$, $\int''$ for the integrals over $(-\varepsilon, \varepsilon)$ and $[-\pi, \varepsilon] \cup [\varepsilon, \pi]$. For t sufficiently large the first term of (7.13) is

$$2\pi f(0) \sum_{s=0}^{t-1-j} \phi_s \phi_{s+j} \sim \frac{2\pi f(0)}{\Gamma(\alpha)^2} \sum_{s=1}^{t} s^{2\alpha-2} \tag{7.15}$$

as $t \to \infty$, whence the conclusions (3.26) and (3.27) are easily deduced. On the other hand from (7.15), the first term is bounded in absolute value by $\delta \int_{\Pi} |\phi_t(\lambda)|^2 d\lambda = O(\delta \log t)$ for $\alpha = \frac{1}{2}$ and $O(\delta t^{2\alpha-1})$ for $\alpha > \frac{1}{2}$, using (7.10), where δ is arbitrary. Finally using (3.18), (7.14) is bounded in absolute value by $C\varepsilon^{-2} t^{2\alpha-2} \int_{\Pi} \{f_{\eta\eta}(\lambda) + f_{\eta\eta}(0)\} d\lambda$ $= O(\log t)$ for $\alpha = \frac{1}{2}$ and $o(t^{2\alpha-1})$ for $\alpha > \frac{1}{2}$. □

8 8. PROOFS FOR SECTION 4

Proof of Lemma 4.1 Though (4.3)-(4.6) are of independent interest they are not of much direct importance to the sequel, while their proof requires introduction of notation which would not find subsequent use. We thus give the proof only of (4.9), which is equivalent to (4.3) with $j = k = 0$, the proofs of (4.3)-(4.6) being only notationally more complex. We first provide some basic derivations which will be useful also in subsequent proofs. In view of (2.1), (3.19), (3.22), (4.2), we can write

$$w_\zeta(\lambda) = \frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_t \phi_{n-t+1}(\lambda)\eta_t e^{it\lambda}, \; w_\xi(\lambda) = \frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_t \psi_{n-t+1}(\lambda)\theta_t e^{it\lambda}, \tag{8.1}$$

where $\psi_t(\lambda) = \sum_{s=0}^{t-1} \psi_s e^{is\lambda}$. Applying (3.19), (3.20),

$$EI_{\zeta\xi}(\lambda) = \frac{1}{2\pi n} \int_\Pi \chi_n(\lambda, \mu) f(\mu) d\mu, \tag{8.2}$$

where for brevity we write $f(\mu) = f_{\eta\theta}(\mu)$, and $\chi_n(\lambda,\mu) = \phi_n(\lambda, -\mu)\psi_n(-\lambda, \mu)$, where, for example,

$$\begin{aligned} \phi_n(\lambda, \mu) &= \sum_t e^{it(\lambda+\mu)} \phi_t(-\mu) \\ &= \sum_t \phi_{n-t+1}(\lambda) e^{it(\lambda+\mu)} = \sum_t \phi_{n-t} e^{i(n-t)\lambda} D_t(\lambda + \mu), \end{aligned} \tag{8.3}$$

the final equality following by summation-by-parts with

$$D_t(\lambda) = \sum_{s=1}^{t} e^{is\lambda}, \tag{8.4}$$

the Dirichlet kernel. From (7.4)

$$|\phi_n(\lambda, \mu)| \leq \frac{Cn^{\alpha+1}}{1 + n\,|\lambda + \mu|}, \;\; |\psi_n(\lambda, \mu)| \leq \frac{Cn^{\beta+1}}{1 + n\,|\lambda + \mu|}, \;\; 0 \leq |\lambda + \mu| \leq \pi. \tag{8.5}$$

Fix $\delta > 0$, then choose $\varepsilon \in (0, \pi)$ such that

$$\sup_{|\lambda|<\varepsilon} |f(\lambda) - f(0)| < \delta, \tag{8.6}$$

with the abbreviation $f = f_{\eta\theta}$. We deduce from (8.2) that

$$E(\overline{\zeta\xi}) = \frac{2\pi}{n} EI_{\zeta\xi}(0) = \frac{1}{n^2} \int_\Pi \chi_n(0, \mu) f(\mu) d\mu, \tag{8.7}$$

which can be written, for $\varepsilon \in (0, \pi)$, as

$$\frac{f(0)}{n^2} \int\limits_{\Pi} \chi_n(0,\mu) d\mu + \frac{1}{n^2} \int' \chi_n(0,\mu) \overline{f}(\mu) d\mu + \frac{1}{n^2} \int'' \chi_n(0,\mu) \overline{f}(\mu) d\mu, \quad (8.8)$$

writing $\overline{f}(\mu) = f(\mu) - f(0)$. Since

$$\int\limits_{\Pi} D_s(\lambda) D_t(-\lambda) d\lambda = 2\pi \min(s,t), \quad (8.9)$$

the first component of (8.8) is, from (8.3),

$$\frac{2\pi f(0)}{n^2} \sum_t \sum_s \phi_{n-s} \psi_{n-t} \min(s,t) = \frac{2\pi f(0)}{n^2} \sum_t \sum_{t=0}^{n-t} \phi_t \sum_{s=0}^{n-t} \psi_s. \quad (8.10)$$

For $\alpha = 0$, $\alpha > 0$ this is, as $n \to \infty$,

$$\frac{2\pi f(0)}{n^2} \sum_s s\psi_{n-s} \sim \frac{2\pi f(0) n^{\beta-1}}{\Gamma(\beta)} \int_0^1 x(1-x)^{\beta-1} dx \sim \frac{2\pi f(0) n^{\beta-1}}{\Gamma(\beta+2)} \quad (8.11)$$

$$\frac{2\pi f(0) n^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_0^{1-x} y^{\alpha-1} dy \int_0^{1-x} z^{\beta-1} dz dx (1+o(1)) \sim \frac{2\pi f(0) n^{\alpha+\beta-1}}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta+1)}, \quad (8.12)$$

respectively. The second term in (8.8) is bounded by

$$\begin{aligned} & \frac{C\delta}{n^2} \int' \left\{ \sum_s |\phi_{n-s}| \, |D_s(\mu)| \right\} \left\{ \sum_t |\psi_{n-t}| \, |D_t(\mu)| \right\} d\mu \\ \leq \; & \frac{C\delta}{n^2} \sum_s |\phi_{n-s}| \sum_t |\psi_{n-t}| \max_{1 \leq t \leq n} \int\limits_{\pi} |D_t(\mu)|^2 d\mu \leq C\delta n^{\alpha+\beta-1}, \quad (8.13) \end{aligned}$$

using (8.9). Because δ is arbitrary the second term of (8.8) can be neglected. Finally, the final term of (8.8) is bounded by (cf. (8.13))

$$\begin{aligned} \frac{C}{n^2} \sum_t |\phi_{n-t}| \sum_t |\psi_{n-t}| \int\limits_{\varepsilon}^{\pi} \mu^{-2} \left|\overline{f}(\mu)\right| d\mu \;\; & \leq \;\; \frac{C n^{\alpha+\beta-2}}{\varepsilon^2} \left[\{Var(\eta_t) Var(\theta_t)\}^{\frac{1}{2}} + |f(0)| \right] \\ & = \;\; O\left(n^{\alpha+\beta-2}\right). \quad (8.14) \end{aligned}$$

□

Proof of Theorem 4.1 Abbreviate $\widehat{F}_{\zeta\xi}$ by $\widehat{F}$. We first prove (4.17), where we can choose n such that $2\lambda_m < \varepsilon$. Take $\ell = 1$. From (8.2), $E\left\{\widehat{F}(1,m)\right\}$ is the real part of

$$\frac{2}{n^2}\sum_{j=1}^{m}\int{}' \chi_n(\lambda_j,\mu)f(\mu)d\mu + \frac{2}{n^2}\sum_{j=1}^{m}\int{}'' \chi_n(\lambda_j,\mu)f(\mu)d\mu. \qquad (8.15)$$

From (8.5), the second term is bounded in modulus by

$$Cmn^{\alpha+\beta-2}\int_{\Pi}|f(\lambda)|\,d\lambda = O(mn^{\alpha+\beta-2}) = o\left(\left(\frac{n}{m}\right)^{\alpha+\beta-1}\right). \qquad (8.16)$$

The difference between the first term of (8.15) and

$$2f(0)\sum_{j=1}^{m}\int{}' \chi_n(\lambda_j,\mu)d\mu \qquad (8.17)$$

is bounded by

$$\frac{2\delta}{n^2}\sum_{j=1}^{m}\int_{\Pi}|\chi_n(\lambda_j,\mu)|\,d\mu \le \frac{C\delta}{n^2}\sum_{j=1}^{m}\left\{\sum_t|\phi_t(\lambda_j)|^2\sum_t|\psi_t(\lambda_j)|^2\right\}^{\frac{1}{2}}, \qquad (8.18)$$

using the Schwarz inequality and, for example, from (8.3),

$$\int_{\Pi}|\phi_n(\lambda,-\mu)|^2\,d\mu = 2\pi\sum_t|\phi_t(\lambda)|^2. \qquad (8.19)$$

From (3.15), the factor in braces in (8.18) is $O\left(n^2\,|\lambda_j|^{-2(\alpha+\beta)}\right)$, so that (8.18) is bounded by

$$C\delta n^{\alpha+\beta-1}\sum_{j=1}^{m}j^{-\alpha-\beta} \le C\delta\left(\frac{m}{n}\right)^{\alpha+\beta-1}, \qquad (8.20)$$

and can thus be neglected because δ is arbitrary.

The difference between (8.17) and

$$\frac{2f(0)}{n^2}\sum_{j=1}^{m}\int_{\Pi}\chi_n(\lambda_j,\mu)d\mu \qquad (8.21)$$

is $O(n^{\alpha+\beta-2}m) = o((n/m)^{\alpha+\beta-1})$ using (8.5) again, so it remains to estimate the real part of (8.21), which is $4\pi f(0)$ times

$$\frac{1}{n^2}\sum_{j=1}^{m}\sum_{t}\chi_t(\lambda_j) = \frac{1}{n^2}\sum_{j=1}^{m}\phi(\lambda_j)\psi(-\lambda_j) \tag{8.22}$$

$$+\frac{1}{n^2}\sum_{j=1}^{m}\left\{\phi(\lambda_j)\sum_{t}\overline{\psi}_t(-\lambda_j) + \sum_{t}\overline{\phi}_t(\lambda_j)\psi(-\lambda_j)\right\} \tag{8.23}$$

$$+\frac{1}{n^2}\sum_{j=1}^{m}\sum_{t}\overline{\phi}_t(\lambda_j)\overline{\psi}_t(-\lambda_j), \tag{8.24}$$

where $\chi_t(\lambda) = \phi_t(\lambda)\psi_t(-\lambda)$ and $\overline{\psi}_t(\lambda) = \sum_{s=t}^{\infty}\psi_s e^{is\lambda}$. For $\alpha = 0$, (8.24) is zero. For $\alpha > 0$, applying (3.15) and (3.16), we bound (8.24) by

$$\frac{C}{n^2}\sum_{j=1}^{m}\left(\sum_{t=1}^{[1/2\lambda_j]}\lambda_j^{-\alpha-\beta} + \lambda_j^{-2}\sum_{t=[1/2\lambda_j]}^{n} t^{\alpha+\beta-2}\right)$$

$$\leq \frac{C}{n^2}\sum_{j=1}^{m}\lambda_j^{-1-\alpha-\beta} \leq Cn^{\alpha+\beta-1} = o\left(\left(\frac{n}{m}\right)^{\alpha+\beta-1}\right). \tag{8.25}$$

Likewise, for $\alpha > 0$, (8.23) is bounded by

$$\frac{C}{n^2}\sum_{j=1}^{m}\left(\lambda_j^{-\alpha-1}\sum_{t} t^{\beta-1} + \lambda_j^{-\beta-1}\sum_{t} t^{\alpha-1}\right) \leq Cn^{\alpha+\beta-1}\sum_{j=1}^{m}\left(j^{-\alpha-1} + j^{-\beta-1}\right)$$

$$= o\left(\left(\frac{n}{m}\right)^{\alpha+\beta-1}\right), \tag{8.26}$$

whereas for $\alpha = 0$, (8.23) is bounded by $Cn^{-2}\sum_{j=1}^{m}\lambda_j^{-1}\sum_t t^{\beta-1} \leq Cn^{\beta-1}\log m = o((n/m)^{\beta-1})$. Finally, the right side of (8.22) has, from (3.12) and (3.13), real part

$$\frac{1}{n}\left(\cos\frac{\alpha\pi}{2}\cos\frac{\beta\pi}{2} + \sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}\right)\sum_{j=1}^{m}\lambda_j^{-\alpha-\beta}(1+o(1)) \tag{8.27}$$

$$\sim \frac{\cos(\alpha-\beta)\frac{\pi}{2}}{2\pi(1-\alpha-\beta)}\cdot\lambda_m^{1-\alpha-\beta} \tag{8.28}$$

as $n \to \infty$, to complete the proof of (4.17) with $\ell = 1$. The proof for $\ell = 0$ follows from (2.4) and Lemma 4.1, due to $\alpha + \beta < 1$.

To prove (4.16) with $\ell = 0$, we can deduce from (8.2) that

$$E\left\{\widehat{F}(0,\widetilde{n})\right\} = \frac{1}{n}\sum_t \int_\Pi \chi_t(\lambda) f(\mu) d\mu, \tag{8.29}$$

which differs from (4.16) by

$$\frac{1}{n}\int_\Pi \left\{\phi(\mu)\sum_t \overline{\psi}_t(-\mu) + \sum_t \overline{\phi}_t(\mu)\psi(-\mu)\right\} f(\mu)d\mu \tag{8.30}$$

$$+\frac{1}{n}\int_\Pi \sum_t \overline{\phi}_t(\mu)\overline{\psi}_t(-\mu) f(\mu)d\mu. \tag{8.31}$$

From (3.15), (3.16) and (8.6), we can bound (8.31) by

$$C\int' |\mu|^{-\alpha-\beta} d\mu + \frac{C}{n\varepsilon^2}\sum_t t^{\alpha+\beta-2}\int_\Pi |f(\mu)|\, d\mu \le C\left(\varepsilon^{1-\alpha-\beta} + \frac{1}{n\varepsilon^2}\right) = o(1), \tag{8.32}$$

with the same bound resulting for (8.30). Finiteness of (4.16) follows similarly, by bounding it by $C\left(\varepsilon^{1-\alpha-\beta} + \varepsilon^{-2}\right)$. Thus (4.16) is proved with $\ell = 0$, and thence with $\ell = 1$ by Lemma 4.1. □

Proof of Theorem 4.2 Given (4.18) and (4.19) for $\ell = 0$, they hold also for $\ell = 1$ due to Lemma 4.1 and $\alpha + \beta = 1$, so we can ignore ℓ. The proof of (4.19) closely follows that of (4.17). In place of (8.16) we have the bound $O(m/n) = o(\log m)$, while the right side of (8.20) is $O(\delta \log m) = o(\log m)$. The argument for replacing (8.21) by (8.17) holds, as does that for neglecting (8.22) - (8.24), while (8.27) is $(\sin\alpha\pi/2\pi)\log m(1 + o(1))$. To prove (4.18), we can write (8.29) as

$$\frac{1}{n}\sum_t \left\{f(0)\int_\Pi \chi_t(\mu)d\mu + \int' \chi_t(\mu)\overline{f}(\mu)d\mu + \int'' \chi_t(\mu)\overline{f}(\mu)d\mu\right\}. \tag{8.33}$$

The contribution from the first term in braces is

$$\frac{2\pi f(0)}{n}\sum_t \sum_{s=0}^{t-1} \phi_s\psi_s \sim \frac{2\pi f(0)}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(1 + \sum_{s=1}^{n} s^{-1}\right) \sim 2\sin\alpha\pi f(0)\log n. \tag{8.34}$$

That from the remaining terms can be bounded respectively by

$$\frac{\delta}{n}\sum_t \int_{-n^{-1}}^{n^{-1}} |\chi_t(\mu)|\, d\mu + \frac{\delta}{n}\sum_t \int_{n^{-1}\le|\mu|\le\varepsilon} |\chi_t(\mu)|\, d\mu$$

$$\leq \frac{C\delta}{n^2}\sum_t t + C\delta \int_{n^{-1}}^{\varepsilon} \mu^{-1} d\mu \leq \delta C(1+\log n), \tag{8.35}$$

and by $(C/\varepsilon^2)\int_\Pi |f(\mu)|\, d\mu < C$, using (3.14) and (3.16). □

<u>Proof of Theorem 4.3</u> First note that (4.20) and (4.22) follow from Lemma 4.1 and (4.23) and (4.27), respectively, since (4.9) is $\pi f(0)$. To prove (4.19) note first that $\omega_0 = \sum_{j=-\infty}^{0} \gamma_j$, $\omega_1 = \sum_{j=1}^{\infty} \gamma_j$ are both finite, because $2\pi f(0) = \omega_0 + \omega_1$ and $\int_\Pi h(\lambda) d\lambda = \omega_0 - \omega_1$ both are, writing $h = h_{\eta\theta}$. Direct calculation gives

$$\begin{aligned} E\left\{\widehat{F}(0,\widetilde{n})\right\} &= \sum_{j=0}^{n-1}\left(1-\frac{j}{n}\right)\psi_j\gamma_{-j} \\ &= \sum_{j=0}^{\infty}\psi_j\gamma_{-j} - \sum_{j=n}^{\infty}\psi_j\gamma_{-j} - \frac{1}{n}\sum_{j=0}^{n-1} j\psi_j\gamma_{-j}. \end{aligned} \tag{8.36}$$

By summation-by-parts, the second term is bounded by

$$\sum_{j=n}^{\infty}\left|\psi_j - \psi_{j+1}\right| \left|\sum_{\ell=0}^{j}\gamma_{-\ell}\right| \leq \sum_{j=n}^{\infty}\left|\psi_j - \psi_{j+1}\right| \left(|\omega_{-n}| + |\omega_{-j-1}|\right) \to 0 \tag{8.37}$$

as $n \to \infty$, whereas the final term is bounded by

$$\left|\frac{1}{n}\sum_{j=0}^{n-2}\left\{j\psi_j - (j+1)\psi_{j+1}\right\} + \left(\frac{n-1}{n}\right)\psi_{j-1}\right| |\omega_0| \tag{8.38}$$

$$+\frac{1}{n}\sum_{j=0}^{n-1}\left(j\left|\psi_j - \psi_{j+1}\right| + \left|\psi_{j+1}\right|\right)|\omega_{-j-1}| + \left(\frac{n-1}{n}\right)\left|\psi_{n-1}\right| |\omega_{-n}| \tag{8.39}$$

which tends to 0 for the same reasons, (8.38) being identically 0. Thus (4.19) is proved.

So far as (4.27) is concerned, it is convenient to first prove the result when $\psi_t \equiv 1$, and then estimate the "error" in doing so. Write $\widetilde{\omega}_\ell = \sum_{k\leq\ell}\gamma_k$, whence

$$\begin{aligned} E\left\{I_{\zeta\xi}(\lambda_j)\right\} &= \frac{1}{2\pi n}\sum_s\sum_t(\gamma_{1-s} + \dots + \gamma_{t-s})e^{i(s-t)\lambda_j} \\ &= \frac{1}{2\pi n}\sum_s\sum_t\left(\widetilde{\omega}_{t-s} - \widetilde{\omega}_{-s}\right)e^{i(s-t)\lambda_j} \end{aligned}$$

$$= \frac{1}{2\pi} \sum_{1-n}^{n-1} \left(1 - \frac{|\ell|}{n}\right) \widetilde{\omega}_\ell e^{-i\ell\lambda_j} \tag{8.40}$$

because

$$D_n(\lambda_j) = 0, \quad 1 \leq j < n. \tag{8.41}$$

For $\ell \leq 0$, $\widetilde{\omega}_\ell = \omega_\ell$, whereas for $\ell \geq 0$, $\widetilde{\omega}_\ell = \omega_0 + \omega_1 - \omega_{\ell+1}$, so (8.40) has real part

$$\frac{1}{2\pi} \sum_{1-n}^{0} \left(1 + \frac{\ell}{n}\right) \omega_\ell \cos \ell\lambda_j + \frac{1}{2\pi} \sum_{1}^{n-1} \left(1 - \frac{\ell}{n}\right) (\omega_0 + \omega_1 - \omega_{\ell+1}) \cos \ell\lambda_j. \tag{8.42}$$

The first term can be written

$$\frac{1}{4\pi} \sum_{1-n}^{n-1} \left(1 - \frac{|\ell|}{n}\right) \omega_{-|\ell|} \cos \ell\lambda_j + \frac{\omega_0}{4\pi}. \tag{8.43}$$

To deal with the second term of (8.42) note that for $1 \leq j \leq n-1$

$$\sum_{\ell=1}^{n-1} \ell e^{i\ell\lambda_j} = \frac{e^{i\lambda_j} - 1}{(1 - e^{i\lambda_j})^2} - \frac{(n-1)}{1 - e^{i\lambda_j}} = \frac{-n}{1 - e^{i\lambda_j}}, \tag{8.44}$$

which has real part

$$-\frac{n}{2} \left(\frac{1}{1 - e^{i\lambda_j}} + \frac{1}{1 - e^{-i\lambda_j}}\right) = -\frac{n}{2} \left(\frac{2 - 2\cos\lambda_j}{2 - 2\cos\lambda_j}\right) = -\frac{n}{2}. \tag{8.45}$$

Thus, the second term in (8.42) is

$$\begin{aligned} &\frac{(\omega_0 + \omega_1)}{2\pi} \sum_{0}^{n-1} \left(1 - \frac{\ell}{n}\right) \cos \ell\lambda_j - \frac{\omega_0 + \omega_1}{2\pi} \\ &- \frac{1}{4\pi} \sum_{1-n}^{n-1} \left(1 - \frac{|\ell|}{n}\right) \omega_{|\ell|+1} \cos \ell\lambda_j + \frac{\omega_1}{4\pi} \\ &= -\frac{1}{4\pi} \sum_{1-n}^{n-1} \left(1 - \frac{|\ell|}{n}\right) \omega_{|\ell|+1} \cos \ell\lambda_j - \frac{\omega_0}{4\pi}. \end{aligned} \tag{8.46}$$

It follows that (8.40) has real part

$$\frac{1}{4\pi} \sum_{1-n}^{n-1} \left(1 - \frac{|\ell|}{n}\right) \left(\omega_{-|\ell|} - \omega_{|\ell|+1}\right) \cos \ell\lambda_j, \tag{8.47}$$

which is the Cesaro sum, to $n-1$ terms, of the Fourier series of $h(\lambda_j)/2$. Equivalently

$$E\left\{\frac{n}{m}\widehat{F}(1,m)\right\} = \frac{1}{nm}\sum_{j=1}^{m}\int_{\Pi}\left|D_n(\lambda-\lambda_j)\right|^2 h(\lambda)d\lambda. \tag{8.48}$$

Fix $\delta > 0$. There exists $\varepsilon > 0$ such that $|h(\lambda)-h(0)| < \delta$ for $0 < |\lambda| \le \varepsilon$. Let n be large enough that $2\lambda_m < \varepsilon$. Then the difference between the right hand side of (8.48) and $2\pi h(0)$ is bounded by

$$\begin{aligned}
&\frac{1}{nm}\sum_{j=1}^{m}\int_{\Pi}\left|D_n(\lambda-\lambda_j)\right|^2 |h(\lambda)-h(0)|\,d\lambda \\
\le\;& \frac{1}{n}\left\{\delta\max_{1\le j\le m}\int'\left|D_n(\lambda-\lambda_j)\right|^2 d\lambda + \sup_{\frac{\varepsilon}{2}<|\lambda|<\pi}\left|D_n(\lambda)\right|^2\left(\int_{\Pi}|h(\lambda)|\,d\lambda + 2\pi\,|h(0)|\right)\right\} \\
=\;& O\left(\delta+\frac{1}{n}\right),
\end{aligned} \tag{8.49}$$

using (8.9). Because δ is arbitrary, the proof of (4.27) when $\psi_t \equiv 1$ is complete.

The difference between $E\left\{\widehat{F}(1,m)\right\}$, and the same thing with $\psi_t \equiv 1$, is, from (8.3), the real part of

$$\frac{2}{n^2}\sum_{j=1}^{m}\int_{\Pi}R_n(\lambda_j,\mu)f(\mu)d\mu, \tag{8.50}$$

where

$$R_n(\lambda,\mu) = D_n(\lambda-\mu)\sum_t(\psi_{n-\ell}-1)e^{it\lambda}D_t(\mu-\lambda). \tag{8.51}$$

Using (8.9), we may write (8.50) as

$$\frac{4\pi f(0)}{n^2}\sum_t t(\psi_{n-t}-1)\sum_{j=1}^{m}e^{it\lambda_j} \tag{8.52}$$

$$+\frac{2}{n^2}\sum_{j=1}^{m}\int' R_n(\lambda_j,\mu)\overline{f}(\mu)d\mu + \frac{2}{n^2}\sum_{j=1}^{m}\int'' R_n(\lambda_j,\mu)\overline{f}(\mu)d\mu. \tag{8.53}$$

To consider (8.51), we have

$$\frac{1}{n}\left|\sum_{t=1}^{n-r}t(\psi_{n-t}-1)\sum_{j=1}^{m}e^{it\lambda_j}\right| \le \frac{m}{n}\sum_{t=r}^{\infty}|\psi_t-1| = o\left(\frac{m}{n}\right) \tag{8.54}$$

as $r \to \infty$. On the other hand,

$$\frac{1}{n^2}\sum_{t=n-t+1}^{n} t(\psi_{n-t}-1)\sum_{j=1}^{m} e^{it\lambda_j} = \frac{1}{n}\sum_{t=0}^{r-1}(\psi_t-1)\sum_{j=1}^{m} e^{-it\lambda_j} \quad (8.55)$$

$$-\frac{1}{n^2}\sum_{t=0}^{r-1} t(\psi_t-1)\sum_{j=1}^{m} e^{-it\lambda_j}. \quad (8.56)$$

We can bound (8.56) by $Crm/n^2 = o(m/n)$, taking $r = o(n)$, whereas, using the inequality $|\cos x - 1| \le x^2$, the real part of (8.55) differs from

$$\frac{m}{n}\sum_{t=0}^{r-1}(\psi_t-1) = \frac{m}{n}\sum_{t=0}^{\infty}(\psi_t-1)(1+o(1)) \quad (8.57)$$

by something bounded by

$$\frac{C}{n}\sum_{t=0}^{r-1}|\psi_t-1|\sum_{j=1}^{m}(t\lambda_j)^2 \le \frac{Cm^3r^2}{n^3}\sum_{t=0}^{\infty}|\psi_t-1| = o\left(\frac{m}{n}\right), \quad (8.58)$$

since we can at the same time choose $r = o(n/m)$. Since (8.57) delivers the correction term $2\sum_{t=0}^{\infty}(\psi_t-1)\sum_{j=-\infty}^{\infty}\gamma_j$ in (4.23), it remains to show that the contribution from (8.53) is $o(m/n)$. Using (8.9) and the Schwarz inequality, the first term of (8.53) is bounded by

$$\frac{\delta}{n}\sum_{j=1}^{m}\left\{\int_{\Pi}|D_n(\lambda_j-\mu)|^2\,d\mu\int_{\Pi}\left|\sum_t(\psi_{n-t}-1)e^{it\lambda_j}D_t(\mu-\lambda_j)\right|^2 d\mu\right\}^{\frac{1}{2}}$$

$$\le \frac{C\delta}{n^{3/2}}\sum_{j=1}^{m}\left\{\sum_s\sum_t(\psi_{n-s}-1)(\psi_{n-t}-1)e^{i(s-t)\lambda_j}\min(s,t)\right\}^{\frac{1}{2}}$$

$$\le \frac{C\delta m}{n}\left\{\sum_{t=0}^{\infty}|\psi_t-1|\right\}^2 = O\left(\frac{\delta m}{n}\right), \quad (8.59)$$

whereas, with $\varepsilon > 2\lambda_m$, the second term is bounded by

$$\frac{Cm}{n^2}\sum_{t=0}^{\infty}|\psi_t-1|\left\{\int_{\Pi}|f(\mu)|\,d\mu + |f(0)|\right\} = o\left(\frac{m}{n}\right) \quad (8.60)$$

to complete the proof of (4.27). □

Proof of Theorem 4.4 To prove (4.31) with $m = \widetilde{n}$ we use (8.29) and (8.33). The left side of (8.34) is

$$\frac{2\pi f(0)}{n\Gamma(\alpha)\Gamma(\beta)}\sum_t\sum_{s=0}^{t-1}s^{\alpha+\beta-2}\left(1+o(1)\right)\sim\frac{2\pi f(0)n^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta)(\alpha+\beta-1)}. \tag{8.61}$$

By the Schwarz inequality, the contribution from the second term in braces in (8.37) is bounded by

$$C\delta n^{\beta-3/2}\sum_t\left\{\int_\Pi\left|\phi_t(\lambda)\right|^2d\lambda\right\}^{\frac{1}{2}}, \tag{8.62}$$

since $\int_\Pi\left|\psi_t(\lambda)\right|^2d\lambda = 2\pi\sum_{s=1}^t\psi_s^2 \le Cn^{2\beta-1}$, because $\beta > \frac{1}{2}$. For $\alpha > \frac{1}{2}$, (8.62) is thus clearly $O\left(\delta n^{\alpha+\beta-1}\right)$, while the same bound holds for $\alpha < \frac{1}{2}$ because the integral in (8.62) is bounded by

$$Ct^{2\alpha}\int_0^{n^{-1}}d\lambda + C\int_{n^{-1}}^{\pi}\lambda^{-2\alpha}d\lambda \le Cn^{2\alpha-1}, \tag{8.63}$$

using (3.14), (3.15). Finally, the contribution from the final term in braces in (8.33) is bounded by

$$\frac{C}{n^2}\int_\varepsilon^\pi\sum_t\left|\phi_t(\mu)\psi_t(\mu)\right|\left\{\left|f(\mu)\right|+\left|f(0)\right|\right\}d\mu, \tag{8.64}$$

which is $O\left(n^{\alpha+\beta-1}\right)$ for $\alpha > 1$ and $O\left(n^{\beta-1}\right)$ for $\alpha \le 1$, on applying (3.18) and (3.19). Thus (4.31) is proved for $m = \widetilde{n}$, whence (4.32) easily follows by incorporating Lemma 4.1. Now taking $m < \widetilde{n}$, note that

$$\widehat{F}(\ell,m) = \widehat{F}(\ell,\widetilde{n}) - \widehat{F}(m+1,\widetilde{n}) \tag{8.65}$$

and we show that the contribution from the second term on the right is negligible. We can bound $E\left\{\widehat{F}(m+1,\widetilde{n})\right\}$ by

$$\frac{C}{n^2}\sum_j{}''\int_{-\varepsilon}^{\varepsilon}\left|\chi_n(\lambda_j,\mu)\right|\left|f(\mu)\right|d\mu + \frac{C}{n^2}\sum_j{}''\int''\left|\chi_n(\lambda_j,\mu)\right|\left|f(\mu)\right|d\mu, \tag{8.66}$$

where $\sum_j''$ denotes $\sum_{j=m+1}^{\widetilde{n}}$. Applying the Schwarz inequality and (8.6), (8.19), the first term is bounded by

$$\frac{C}{n^2}\sum_j'' \left\{ \sum_t |\phi_t(\lambda_j)|^2 \sum_t |\psi_t(\lambda_j)|^2 \right\}^{\frac{1}{2}}. \tag{8.67}$$

On applying (3.17) to $|\psi_t(\lambda_j)|$, and (3.19) to $|\phi_t(\lambda_j)|$ when $\alpha > \frac{1}{2}$ and (3.15) when $\alpha \le \frac{1}{2}$ we find that (8.11) is $O\left(n^{\alpha+\beta-1}\sum_{j=m}^{\infty} j^{-2}\right) = O\left(n^{\alpha+\beta-1}m^{-1}\right)$ in the former case and $O\left(n^{\alpha+\beta-1}\sum_{j=m}^{\infty} j^{-1-\alpha}\right) = o\left(n^{\alpha+\beta-1}m^{-\alpha}\right)$ in the latter, so both are $o\left(n^{\alpha+\beta-1}\right)$ from (2.6). Finally the second term of (8.66) is bounded by

$$\frac{C}{n^2}\int'' \left\{ \sum_{j=1}^{n} |\phi_n(\lambda_j,\mu)|^2 \sum_{j=1}^{n} |\psi_n(\lambda_j,-\mu)|^2 \right\}^{\frac{1}{2}} |f(\mu)|\, d\mu. \tag{8.68}$$

From (8.3) and

$$\sum_{j=1}^{n} e^{it\lambda_j} = n, \qquad t = 0, \ \mathrm{mod}(n), \tag{8.69}$$

$$= 0, \qquad \text{otherwise}, \tag{8.70}$$

the term in braces is $n^2 \sum_t |\phi_t(\mu)|^2 \sum_t |\psi_t(\mu)|^2$, so (8.68) is bounded in the same way as (8.67). □

9 9. Proofs for Section 5

Proof of Theorem 5.1 We first consider $\widehat{F}(0,\widetilde{n})$, which has variance

$$\frac{1}{n^2}\sum_s\sum_t\sum_1^s \phi_{s-q}\sum_1^s \psi_{s-r}\sum_1^t \phi_{t-u}\sum_1^t \psi_{t-v}\left\{\gamma_{r-u}\gamma_{v-q} + \gamma_{u-q}^{(\eta)}\gamma_{v-r}^{(\theta)} + \kappa_{qruv}\right\}, \tag{9.1}$$

where $\gamma_j^{(\eta)} = Cov\left(\eta_0, \eta_j\right)$, $\gamma_j^{(\theta)} = Cov\left(\theta_0, \theta_j\right)$, $\kappa_{qruv} = cum\left\{\eta_q, \theta_r, \eta_u, \theta_v\right\}$. The contribution of the first term in braces to (9.1) can be written

$$\frac{1}{n^2}\sum_s\sum_t a_{st}a_{ts} = \frac{1}{n^2}\sum_s\sum_t \left(b_{st}+c_{st}+d_{st}\right)\left(b_{ts}+c_{ts}+d_{ts}\right), \tag{9.2}$$

where

$$a_{st} = \int_{\Pi} \chi_{st}(\mu) f(\mu) d\mu, \;\; \chi_{st}(\mu) = \phi_s(\mu)\psi_t(-\mu) e^{i(t-s)\mu}, \tag{9.3}$$

$$b_{st} = f(0) \int_{\Pi} \chi_{st}(\mu) d\mu, \;\; c_{st} = \int_{-\varepsilon}^{\varepsilon} \overline{f}(\mu)\chi_{st}(\mu) d\mu, \;\; d_{st} = \int' \overline{f}(\mu)\chi_{st}(\mu) d\mu. \tag{9.4}$$

We shall show that

$$\frac{1}{n^2}\sum_s \sum_t a_{st}a_{ts} \sim \frac{1}{n^2}\sum_s \sum_t b_{st}b_{ts} \sim Q(\alpha,\beta,\alpha,\beta) n^{2(\alpha+\beta-1)}. \tag{9.5}$$

The last relation follows from

$$\int_{\Pi} \chi_{st}(\mu) d\mu = 2\pi \sum_{j(s,t)}{}' \phi_j \psi_{j+t-s}, \tag{9.6}$$

where $\sum'_{j(s,t)} = \sum_{j=\max(0,s-t)}^{s-1}$, and by integral approximation, in view of Definition 3.1. To prove the first relation in (9.5), we first consider the case $\alpha+\beta > 1$. By elementary inequalities the difference between the first two parts of (9.5) is bounded by n^{-2} times

$$\left\{\sum_s \sum_t |b_{st}|^2\right\}^{\frac{1}{2}} \left\{2\sum_s \sum_t \left(|c_{st}|^2 + |d_{st}|^2\right)\right\}^{\frac{1}{2}} + 2\sum_s \sum_t \left(|c_{st}|^2 + |d_{st}|^2\right), \tag{9.7}$$

whence it suffices to show that

$$\sum_s \sum_t |b_{st}|^2 = O\left(n^{2(\alpha+\beta)}\right), \;\; \sum_s \sum_t |c_{st}|^2 = o\left(n^{2(\alpha+\beta)}\right), \;\; \sum_s \sum_t |d_{st}|^2 = o\left(n^{2(\alpha+\beta)}\right). \tag{9.8}$$

Now

$$\begin{aligned}\sum_s \sum_t |b_{st}|^2 &\le C\sum_t \int |\phi_t(\mu)|^2 d\mu \sum_t |\psi_t(\mu)|^2 d\mu \\ &\le Cn^2 \sum_0^n \phi_j^2 \sum_o^n \psi_j^2 \le Cn^{2(\alpha+\beta)},\end{aligned} \tag{9.9}$$

and clearly $\sum_s \sum_t |c_{st}|^2$ has the same bound, times δ^2, where δ is arbitrary, to prove the first two components of (9.8). Finally, the last component of (9.8) follows from the bound (due to (3.15), (3.16), (3.18))

$$|\chi_{st}(\mu)| \le Cn^{\max(\alpha-1,0)+\max(\beta-1,0)} |\mu|^{-\min(\alpha,1)-\min(\beta,1)}, \tag{9.10}$$

for $0 < |\mu| \leq \pi$, since then $\sum_s \sum_t |d_{st}|^2$ is $O(n^2)$ for $\alpha, \beta \leq 1$, $O(n^{2\beta})$ for $\alpha \leq 1$, $\beta > 1$, and $O(n^{2(\alpha+\beta-1)})$ for $\alpha, \beta > 1$, to complete the proof of (9.8) in case $\alpha + \beta > 1$.

Now consider the case $\alpha + \beta \leq 1$, which implies $\alpha < \frac{1}{2}$ and $\beta \leq 1$, and is considerably more delicate, as the bound (9.7) is now too crude and we have to consider the remaining components of (9.2) individually. First assume $\alpha > 0$. Writing

$$G_n(\lambda, \mu, \omega) = \sum_t \phi_t(\lambda)\psi_t(\mu)e^{it\omega}, \tag{9.11}$$

we have first

$$\left|\sum_s \sum_t c_{st}c_{ts}\right| = \left|\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} G_n(\lambda, -\mu, \lambda+\mu)G_n(\lambda, -\mu, \lambda-\mu)\overline{f}(\lambda)\overline{f}(\mu)d\mu d\lambda\right|$$

$$\leq \delta^2 \left\{\int_\Pi\int_\Pi |G_n(\lambda, -\mu, \lambda+\mu)|^2 d\mu d\lambda \int_\Pi\int_\Pi |G_n(\lambda, -\mu, \lambda-\mu)|^2 d\mu d\lambda\right\}^{\frac{1}{2}}. \tag{9.12}$$

Both factors in braces are

$$4\pi^2 \sum_s \sum_t \sum_{j(s,t)}{}' \phi_j\phi_{j+t-s} \sum_{j)s,t)}{}' \psi_j\psi_{j+t-s}. \tag{9.13}$$

The contribution to (9.13) from terms $s = t$ is $O\left(n \sum_j^\infty \phi_j^2 \sum_j^n \psi_j^2\right) = O(n^{2\beta})$. The contribution from terms $s \neq t$ is

$$O\left(\sum_{\substack{s=t \\ s\neq t}}\sum |s-t|^{\alpha-1} \sum_0^n |\phi_j| \sum_0^n \psi_j^2\right) = O\left(n^{2(\alpha+\beta)}\right), \tag{9.14}$$

so that (9.12) is $O\left(\delta^2 n^{2(\alpha+\beta)}\right)$, where δ is arbitrary. Next,

$$\sum_s \sum_t b_{st}c_{ts} = f(0)\int_\Pi\int_{-\varepsilon}^{\varepsilon} G_n(\lambda, -\mu, \lambda+\mu)G_n(\lambda, -\mu, \lambda-\mu)\overline{f}(\mu)d\mu d\lambda, \tag{9.15}$$

which is clearly bounded by (9.12) with δ^2 replaced by $\delta\,|f(0)|$, in other words by $O\left(\delta n^{2(\alpha+\beta)}\right)$. Now write d_{st} as

$$d_{st} = \int'' \left\{\phi(\lambda) - \overline{\phi}_s(\lambda)\right\}\left\{\psi(-\lambda) - \overline{\psi}_t(-\lambda)\right\} e^{i(t-s)\lambda} f(\lambda)d\lambda. \tag{9.16}$$

Then $\sum_s \sum_t d_{st} d_{ts} \leq \sum_s \sum_t |d_{st}|^2$ which is bounded by

$$C \int'' \int'' \left\{ |D_n(\lambda - \mu)| + \left| \sum_t \overline{\phi}_s(\lambda) \overline{\phi}_s(-\mu) e^{is(\mu - \lambda)} \right| \right\}$$
$$\times \left\{ |D_n(\lambda - \mu)| + \left| \sum_t \overline{\psi}_t(-\lambda) \psi_t(\mu) e^{it(\lambda - \mu)} \right| \right\} \left| \overline{f}(\lambda) \overline{f}(\mu) \right| d\lambda d\mu. \quad (9.17)$$

The four terms are bounded respectively by

$$C \int_\Pi \int_\Pi |D_n(\mu - \lambda)|^2 \left| \overline{f}(\mu) \right|^2 d\lambda d\mu \leq Cn, \quad (9.18)$$

$$C \sum_t t^{2(\beta - 1)} \left\{ \int_\Pi \int_\Pi |D_n(\lambda - \mu)|^2 \left| \overline{f}(\mu) \right|^2 d\lambda d\mu \right\}^{\frac{1}{2}} \leq C n^{2\beta - \frac{1}{2}}, \quad (9.19)$$

and likewise $O\left(n^{\frac{1}{2}} \sum_s s^{2(\alpha-1)}\right) = O(n^{\frac{1}{2}})$ and $O\left(\sum s^{2(\alpha-1)} \sum t^{2(\beta-1)}\right) = O\left(n^{2\beta-1}\right)$. It follows that $\sum_s \sum_t d_{st} d_{ts} = o\left(n^{2(\alpha+\beta+1)}\right)$. For $\sum_s \sum_t c_{st} d_{ts}$ (and likewise $\sum_s \sum_s b_{st} d_{ts}$) we also use the representation (9.16), but we will give only the proof for the term corresponding to $\phi(\lambda)\phi(-\lambda)$, which is dominant in $\sum_s \sum_t d_{st} d_{ts}$. The term is bounded by

$$C\delta \int' \int'' |\phi(\mu)\psi(-\mu)| \left| \sum_s \sum_t \chi_{st}(\lambda) e^{i(s-t)\mu} \right| \left| \overline{f}(\mu) \right| d\mu d\lambda$$
$$\leq C\delta \int_\Pi \int'' \left| \sum_s \sum_t \chi_{st}(\lambda) e^{i(s-t)\mu} \right| \left| \overline{f}(\mu) \right| d\mu d\lambda$$
$$\leq C\delta \left\{ \int'' \int_\Pi |\phi_n(\lambda, -\mu)|^2 \left| \overline{f}(\mu) \right|^2 d\lambda d\mu \int_\Pi \int_\Pi |\psi_n(-\lambda, \mu)|^2 d\lambda d\mu \right\}^{\frac{1}{2}} \quad (9.20)$$

The first factor in braces is $2\pi \int'' \sum_t |\phi_t(\mu)|^2 \left| \overline{f}(\mu) \right|^2 d\mu \leq Cn$ whereas the second is $O\left(n \sum_0^n \psi_j^2\right) = O(n^{2\beta})$ so that (9.20) is $O(n^{\beta + \frac{1}{2}}) = o\left(n^{2(\alpha+\beta)}\right)$.

Now take $\alpha = 0$. In the first place, the contribution from terms $s \neq t$ in (9.13) is zero, so we clearly have $\sum_s \sum_t c_{st} c_{ts} = O\left(\delta^2 n^{2\beta}\right)$, $\sum_s \sum_t b_{st} c_{ts} = O\left(\delta n^{2\beta}\right)$ as before. After substituting (9.16), we have

$$\sum_s \sum_t d_{st} d_{ts} = \int'' \int'' \left\{ \sum_t \sum_s \left\{ \psi(-\lambda) - \overline{\psi}_t(-\lambda) \right\} \left\{ \psi(\mu) - \overline{\psi}_s(\mu) \right\} \right\}$$
$$\times e^{i(t-s)(\lambda-\mu)} \overline{f}(\lambda) \overline{f}(\mu) d\lambda d\mu. \quad (9.21)$$

The contribution from the term in $\psi(-\lambda)\psi(\mu)$ is clearly $O(n)$, as before, while that from the term in $\overline{\psi}_t(-\lambda)\overline{\psi}_s(\mu)$ is bounded by

$$\int'' \int_{\Pi} \left| \sum_t \overline{\psi}_t(-\lambda) e^{it(\lambda-\mu)} \right|^2 \left| \overline{f}(\lambda) \right|^2 d\lambda d\mu$$

$$\leq C \int'' \sum_t \left| \overline{\psi}_t(\lambda) \right|^2 \left| \overline{f}(\lambda) \right|^2 d\mu d\lambda$$

$$\leq \varepsilon^{-2} \int_{\Pi} \left(\sum_t t^{2(\beta-1)} \right) \left| \overline{f}(\mu) \right|^2 d\mu \leq C n^{2\beta-1} = o\left(n^{2\beta}\right). \qquad (9.22)$$

In a similar way, the remaining terms in (9.21) are seen to be $O\left(n^{\beta+\frac{1}{2}}\right) = o\left(n^{2\beta}\right)$. As for $\sum_s \sum_t c_{st} d_{ts}$, the bound $O\left(n^{\beta+\frac{1}{2}}\right) = o\left(n^{2\beta}\right)$ for the first contribution to (9.18) follows as before, while the additional term is bounded by

$$\delta \int' \int'' \left| \psi_n(-\lambda, \mu) \right| \left| \sum_s \overline{\psi}_s(\lambda) e^{is(\lambda-\mu)} \right| \left| \overline{f}(\lambda) \right| d\lambda d\mu$$

$$\leq \delta \left\{ \int_{\Pi} \int_{\Pi} \left| \psi_n(-\lambda, \mu) \right|^2 d\lambda d\mu \int'' \int_{\Pi} \left| \sum_s \overline{\psi}_s(\lambda) e^{is(\lambda-\mu)} \right|^2 \left| \overline{f}(\lambda) \right|^2 d\mu d\lambda \right\}$$

$$\leq \delta \left\{ \int_{\Pi} \sum_t \left| \psi_t(\mu) \right|^2 s\mu \int'' \sum_s \left| \overline{\psi}_s(\lambda) \right|^2 \left| \overline{f}(\lambda) \right|^2 d\lambda \right\}^{\frac{1}{2}}$$

$$\leq \frac{C\delta}{\varepsilon} \left(n^{2\beta} \sum_s s^{2\beta-2} \right)^{\frac{1}{2}} \leq C\delta n^{2\beta-\frac{1}{2}}, \qquad (9.23)$$

so that $\sum\sum c_{st} d_{ts} = o(n^{2\beta})$. It is then easily seen that $\sum\sum c_{st} d_{ts} = o(n^{2\beta})$, the bound (9.23) applying to the second component without the δ factor.

The proof that the contribution to (9.1) from the second term in braces is very similar to the preceding proof, and in thus omitted. We write the contribution to (9.1) from the final, fourth-cumulant, term as

$$\frac{1}{n^2} \int_{\Pi} \int_{\Pi} \int_{\Pi} H_n(\lambda, \mu, \omega) f(\lambda, \mu, \omega) d\lambda d\mu d\omega, \qquad (9.24)$$

where

$$H_n(\mu, \lambda, \omega) = G_n(\lambda + \mu + \omega, -\lambda, -\mu, -\omega) G_n(-\mu, -\omega, \mu + \omega). \tag{9.25}$$

Now, to extend the approach used previously, we can write (9.24) as the sum of terms

$$\frac{f(0,0,0)}{n^2} \int\limits_{\Pi} \int\limits_{\Pi} \int\limits_{\Pi} H_n(\mu, \lambda, \omega) d\lambda d\mu d\omega \tag{9.26}$$

$$+ \int\limits_{\Pi} \int\limits_{\Pi} \int\limits_{\Pi} H_n(\lambda, \mu, \omega) \{f(\lambda, \mu, \omega) - f(0,0,0)\} d\lambda d\mu d\omega. \tag{9.27}$$

It is readily verified that (9.27) is $\delta\pi^2 f(0,0,0)/n^2$ times something bounded by

$$\sum_s \sum_t \sum_{j(s,t)}' \phi_j \psi_j \phi_{j+t-s} \psi_{j+t-s} \le \sum_{j=0}^n |\phi_j \psi_j| \sum_s \sum_t |\phi_{j+t-s} \psi_{j+t-s}|. \tag{9.28}$$

For $\alpha + \beta < 1$, the sum over s, t is $O(n)$, uniformly in j, so that (9.28) is $O(n)$ also. For $\alpha + \beta = 1$, the sum over s, t is $O(n \log n)$ and (9.28) is $O(n(\log n)^2)$. For $\alpha + \beta > 1$, (9.28) is clearly $O(n^{2(\alpha+\beta)-1})$. It follows that (9.26) is $o(n^{2(\alpha+\beta-1)})$.

We now consider (9.27). For any $\delta > 0$, we can choose ε such that

$$\sup_{|\lambda|<\varepsilon, |\mu|<\varepsilon, |\lambda| \in \varepsilon} |f(\lambda, \mu, \omega) - f(0,0,0)| < \delta. \tag{9.29}$$

Then

$$\int' \int' \int' H_n(\lambda, \mu, \omega) \{f(\lambda, \mu, \omega) - f(0,0,0)\} d\lambda d\mu d\omega \tag{9.30}$$

is bounded by

$$\frac{\delta}{n^2} \left\{ \int\limits_{\Pi} \int\limits_{\Pi} \int\limits_{\Pi} |G_n(\lambda + \mu + \omega, -\lambda, -\mu, -\omega)|^2 d\lambda d\mu d\omega \int\limits_{\Pi} \int\limits_{\Pi} \int\limits_{\Pi} |G_n(-\mu, -\omega, \mu + \omega)|^2 d\lambda d\mu d\omega \right\}. \tag{9.31}$$

Both triple integrals are easily shown to be 2π times (9.13), which is $O(n^{2(\alpha+\beta)})$ (see (9.14)). By arbitrariness of δ it follows that (9.31) is $o(n^{2(\alpha+\beta-1)})$, so that (9.30) can be neglected. The difference between (9.27) and (9.29) is bounded by

$$\sum_{j=1}^{3} \int \int\limits_{U_j} \int |H_n(\lambda, \mu, \omega)| \, |f(\lambda, \mu, \omega) - f(0,0,0)| \, d\lambda d\mu d\omega, \tag{9.32}$$

where

$$\begin{aligned} U_1 &= \{\lambda : \varepsilon \le |\lambda| \le \pi\} \times V_1, \quad V_1 = \{\mu : \mu \in \Pi\} \times \{\omega : \omega \in \Pi\} && (9.33) \\ U_2 &= \{\lambda : \lambda \in \Pi\} \times V_2, \quad V_2 = \{\mu : \varepsilon \le |\lambda| \in \pi\} \times \{\omega : \omega \in \Pi\} && (9.34) \\ U_3 &= \{\lambda : \lambda \in \Pi\} \times V_3, \quad V_3 = \{\mu : \mu \in \Pi\} \times \{\omega : \varepsilon \le |\omega| \le \pi\} && (9.35) \end{aligned}$$

Then (9.32) is bounded by

$$\left\{ \sup_{\mu,\varepsilon \in \Pi} \int_{\Pi} f^2(\lambda,\mu,\omega) d\lambda + 2\pi f^2(0,0,0) \right\} \tag{9.36}$$

$$\times \frac{1}{n^2} \sum_{j=1}^{3} \left\{ \int\int_{U_j}\int |G_n(\lambda+\mu+\omega, -\lambda, -\mu, -\omega)|^2 \, d\omega d\mu d\lambda \int\int_{V_j} |G_n(-\mu,-\omega,\mu+\omega)|^2 \, d\omega d\mu \right\}^{\frac{1}{2}}. \tag{9.37}$$

Since (9.36) is finite it suffices to show that each of the summands in (9.37) is $o(n^{2(\alpha+\beta)})$. In each case, we proceed by using the fact, already established, that one of the factors in (9.37) is $O(n^{2(\alpha+\beta)})$, and show that the other is $o(n^{2(\alpha+\beta)})$.

For $j = 1$, the first factor of (9.37) is bounded by

$$2\pi \sum_s \sum_t \left(\sum_{j(s,t)}' \phi_j \phi_{j+t-s} \right) \int'' \psi_s(-\lambda) \psi_t(\lambda) e^{i(t-s)\lambda} d\lambda \tag{9.38}$$

which is $O\left(n^{2\alpha+1}\right)$ for $\beta < 1$ and $O\left(n^{2(\alpha+\beta)-1}\right)$ for $\beta \ge 1$. Thus the summand of (9.37) for $j = 1$ is $O(n^{2(\alpha+\beta)-\frac{1}{2}}) = o(n^{2(\alpha+\beta)})$, is desired. It is easily seen that the same outcome holds for $j = 3$ (with bounds for first and second factors reversed).

Now consider the summand of (9.37) for $j = 2$. Its second factor is bounded by

$$2\pi \sum_s \sum_t \left(\sum_{j(s,t)}' \psi_j \psi_{j+t-s} \right) \int'' \phi_s(-\mu) \phi_t(\mu) e^{i(t-s)\mu} d\mu. \tag{9.39}$$

For $\frac{1}{2} < \alpha < 1$ this is $O(n^{2\beta+1}) = o(n^{2(\alpha+\beta)})$ while for $\alpha \ge 1$ it is $O(n^{2(\alpha+\beta)-1}) = o(n^{2(\alpha+\beta)})$. Thus the proof is completed for $\alpha > \frac{1}{2}$. We now deal separately with the cases $0 < \alpha \le \frac{1}{2}$ and $\alpha = 0$. For $0 < \alpha \le \frac{1}{2}$ the second factor of (9.37) is

$$\int'' \int_{\Pi} \left| \sum_t \left\{ \phi(-\mu) - \overline{\phi}_t(-\mu) \psi_t(-\omega) e^{it(\mu+\omega)} \right\} \right|^2 d\omega d\mu$$

$$= \int'' \int_{\Pi} |\phi(\mu)|^2 \left| \sum_t \phi_t(-\omega) e^{it(\mu+\omega)} \right|^2 d\omega d\mu \tag{9.40}$$

$$- \int'' \int_{\Pi} \phi(-\mu) \sum_s \psi_s(-\omega) e^{it(\mu+\omega)} \sum_t \overline{\phi}_t(\mu) \psi_t(\omega) e^{-it(\mu+\omega)} d\omega d\mu \tag{9.41}$$

$$- \int'' \int_{\Pi} \phi(\mu) \sum_s \psi_s(\omega) e^{-is(\mu+\omega)} \sum_t \overline{\phi}_t(-\mu) \psi_t(-\omega) e^{it(\mu+\omega)} d\omega d\mu \tag{9.42}$$

$$+ \int'' \int_{\Pi} \left| \sum_t \overline{\phi}_t(-\mu) \psi_t(-\omega) e^{it(\mu+\omega)} \right|^2 d\omega d\mu. \tag{9.43}$$

From previous arguments (9.40) $= O(n^{2\beta}) = o(n^{2(\alpha+\beta)})$, whereas (9.43) is bounded by

$$2\pi \sum_s \sum_t \left(\sum_{j(s,t)}{}' \psi_j \psi_{t+t-s} \right) \int' \overline{\phi}_s(-\mu) \overline{\phi}_t(\mu) e^{i(s-t)\mu} d\mu$$

$$\leq C n^{2(\alpha+\beta)-1} = o\left(n^{2(\alpha+\beta)}\right). \tag{9.44}$$

It then follows by the Schwarz inequality that (9.41) and (9.42) are $o(n^{2\beta}.n^{\alpha+\beta-\frac{1}{2}}) = o(n^{2(\alpha+\beta)})$, and so the summand for $j = 2$ is $o(n^{2(\alpha+\beta)})$ when $0 < \alpha \leq \frac{1}{2}$. In case $\alpha = 0$, we subdivide U_2, V_2 into

$$U_2' = \{\lambda : \lambda \in \Pi\} \times V_2',\ V_2' = \{\mu : \varepsilon \leq |\mu| \leq \pi\} \times \{\omega : |\omega| < \varepsilon/2\} \tag{9.45}$$

$$U_2'' = \{\lambda : \lambda \in \Pi\} \times V_2'',\ V_2'' = \{\mu : \varepsilon \leq |\mu| \leq \pi\} \times \{\omega : \varepsilon/2 \leq |\omega| \leq \pi\} \tag{9.46}$$

Noting that by summation by parts

$$\sum_t \psi_t(\lambda) e^{it(\mu+\omega)} = \psi_n(\lambda) D_n(\mu+\omega) - \sum_{t=1}^{n-1} \psi_t e^{it\lambda} D_t(\mu+\omega) \tag{9.47}$$

so integrating over (9.45), the first factor of (9.37) is bounded by

$$\int \int_{U_2'} \int \Bigg\{ |\psi_n(\lambda)|^2 |D_n(\mu+\omega)|^2 - \psi_n(\lambda) D_n(\mu+\omega) \sum_{t=1}^{n-1} \psi_t e^{-t\lambda} D_t(-\mu-\omega)$$

$$-\psi_n(-\lambda) D_t(-\mu-\omega) \sum_{t=1}^{n-1} \psi_t e^{it\lambda} D_t(\mu+\omega)$$

$$+\sum_{s=1}^{n-1}\sum_{t=1}^{n-1}\psi_s\psi_t e^{i(s-t)\lambda}D_s(\mu+\omega)D_t(-\mu-\omega)\Bigg\} d\omega d\mu d\lambda$$

$$= 2\pi\int\int_{V_2'}\Bigg\{|D_n(\mu+\omega)|^2\sum_{t=0}^{n-1}\psi_t^2 - D_n(\mu+\omega)\sum_{t=1}^{n-1}\psi_t^2 D_t(-\mu-\omega)$$

$$-D_n(-\mu-\omega)\sum_{t=1}^{n-1}\psi_t^2 D_t(\mu+\omega)+\sum_{t=1}^{n-1}\psi_t^2\,|D_t(\mu+\omega)|^2\Bigg\} d\omega d\mu.(9.48)$$

On (9.48), $|D_t(\mu+\omega)| \leq C|\mu+\omega|^{-1} \leq C|\mu|^{-1}$, (9.49) $\leq C\sum_{t=0}^{n-1}\psi_t^2 = O(n^{2\beta-1})$, and the contribution to (9.37) is $o(n^{2\beta})$. For (9.46), on the other hand, the second factor of (9.37) is bounded by

$$C\int\int_{V_2'}\left|\sum_t \psi_t(-\omega)e^{it(\mu+\omega)}\right|^2 d\omega d\mu \leq C\sum_t\int_\Pi |\psi_t(\omega)|^2\, d\omega, \qquad (9.49)$$

which is $O(n)$ for $\beta < 1$, and $O(\sum_t t^{2(\beta-1)}) = O(n^{2\beta-1})$ for $\beta \geq 1$, where the contribution to (9.37) is again $o(n^{2\beta})$. We have thereby completed the proof of (5.11) for $m = \widetilde{n}$.

By elementary inequalities and (8.69), (5.11) for $m < \widetilde{n}$ will follow from the above proof and (5.13), so we prove the latter. $Var\left\{\widehat{F}(m+1,\widetilde{n})\right\}$ is bounded by the real part of

$$\frac{1}{4\pi n^4}\sum_j''\sum_k''\sum_q\sum_r\sum_s\sum_t \phi_{n-q+1}(\lambda_j)\psi_{n-r+1}(-\lambda_j)\phi_{n-s+1}(-\lambda_k)\psi_{n-t+1}(\lambda_k)$$
$$\times e^{i(q-r)\lambda_j - i(s-t)\lambda_k}\left\{\gamma_{t-q}\gamma_{r-s}+\gamma_{s-q}^{(\eta)}\gamma_{t-r}^{(\theta)}+\kappa_{qrst}\right\}. \qquad (9.50)$$

The contribution from the first term in braces may be written as $(4\pi^2 n^4)^{-1}$ times

$$\int_\Pi\int_\Pi V_n(\lambda,\mu)f(\lambda)f(\mu)d\lambda d\mu, \qquad (9.51)$$

where

$$V_n(\lambda,\mu) = \sum_j''\sum_k''\phi_n(\lambda_j,-\lambda)\psi_n(-\lambda_j,\mu)\phi_n(-\lambda_k,-\mu)\psi_n(\lambda_k,\lambda). \qquad (9.52)$$

We subdivide the integral (9.51) into components $\int'\int'$, $\int''\int''$, $\int'\int''$ and

$\int'' \int'$. First

$$\left| \int' \int' V_n(\lambda, \mu) f(\lambda) f(\mu) d\lambda d\mu \right| \leq C \int_\Pi \int_\Pi \left| \sum_j'' \phi_n(\lambda_j, -\lambda) \psi_n(-\lambda_j, \mu) \right|^2 d\lambda d\mu. \tag{9.53}$$

The double integral is evaluated as

$$4\pi^2 \sum_s \sum_t \left| \sum_j'' \phi_s(\lambda_j) \psi_t(-\lambda_j) e^{i(t-s)\lambda_j} \right|^2. \tag{9.54}$$

For $\alpha + \beta > 1$ and $\alpha > 0$ this is bounded by

$$\begin{aligned} & C \sum_s s^{2\max(\alpha-1,0)} \sum_t t^{2\max(\beta-1,0)} n^{2\min(\alpha,1)+2\min(\beta,1)} \left(\sum_t'' j^{-\min(\alpha,1)-\min(\beta,1)} \right)^2 \\ \leq \; & C n^{2(\alpha+\beta+1)} m^{-\min(\alpha+\beta-1,\alpha,1)} = o\left(n^{2(\alpha+\beta+1)}\right) \end{aligned} \tag{9.55}$$

as desired. For $\alpha + \beta > 1$ but $\alpha = 0$, (9.54) is, from (8.68), (8.69),

$$4\pi^2 n \sum_j'' \sum_t |\psi_t(\lambda_j)|^2 \leq C n^{2(\beta+1)} \sum_j'' j^{-2} = o\left(n^{2(\alpha+\beta+1)}\right). \tag{9.56}$$

For $\alpha + \beta \leq 1$ we write (9.54) as

$$\sum_s \sum_t \left| \sum_j'' \left\{ \phi(\lambda_j) - \overline{\phi}_s(\lambda_j) \right\} \psi_t(-\lambda_j) e^{i(t-s)\lambda_j} \right|^2 \tag{9.57}$$

$$= \sum_j'' \sum_k'' \phi(\lambda_j) \phi(-\lambda_j) \sum_t \psi_t(-\lambda_k) \psi_t(\lambda_k) e^{it(\lambda_j - \lambda_k)} D_n(\lambda_k - \lambda_j) \tag{9.58}$$

$$- \sum_j'' \sum_k'' \phi(\lambda_j) \sum_t \psi_t(-\lambda_j) \psi_t(\lambda_k) e^{it(\lambda_j - \lambda_k)} \sum_s \overline{\phi}_s(-\lambda_k) e^{is(\lambda_k - \lambda_j)} \tag{9.59}$$

$$- \sum_j'' \sum_k'' \phi(-\lambda_k) \sum_t \psi_t(-\lambda_j) \psi_t(\lambda_k) e^{it(\lambda_j - \lambda_k)} \sum_s \overline{\phi}_s(\lambda_k) e^{is(\lambda_k - \lambda_j)} \tag{9.60}$$

$$+ \sum_j'' \sum_k'' \sum_s \overline{\phi}_s(\lambda_j) \overline{\phi}_s(-\lambda_k) e^{it(\lambda_j - \lambda_k)} \sum_t \psi_t(-\lambda_j) \psi_t(\lambda_k) e^{it(\lambda_j - \lambda_k)} \tag{9.61}$$

Because $D_n(\lambda_k - \lambda_j) = n$ for $j \geq k$ and zero otherwise, (9.58) is bounded by $n^{2(1+\alpha+\beta)}$ times

$$C \sum_j j^{-2(\alpha+\beta)} \leq C m^{1-2(\alpha+\beta)} = o(1). \tag{9.62}$$

On the other hand (9.59) and (9.60) are bounded by $n^{2(1+\alpha+\beta)}$ times

$$C\sum_j'' j^{-\alpha-2\beta-1} + C\sum_j'' j^{-\alpha-\beta}\sum_{k>j} k^{-1-\beta}$$

$$\leq Cm^{-\alpha-2\beta} + C\sum_j'' j^{-\alpha-2\beta} \leq Cm^{1-\alpha-2\beta} = o(1). \tag{9.63}$$

Finally, bounding $\overline{\phi}_s(\lambda_j)$ by $Cs^{\beta-1}/\lambda_j$ and $\overline{\phi}_s(-\lambda_k)$ by $Cs^{\beta-1}/\lambda_k$ we get the same bound for (9.66). Thus the component $\int'\int'$ of (9.51) in $o(n^{2(\alpha+\beta+1)})$. Next, the component $\int''\int''$ is bounded by

$$\int''\int''\left|\sum_j''\phi_n(\lambda_j,-\lambda)\psi_n(-\lambda_j,\mu)\right|^2|f(\lambda)f(\mu)|\,d\lambda d\mu \tag{9.64}$$

$$\begin{aligned}
&\leq \int''\sum_j''|\phi_n(\lambda_j-\lambda)|^2|f(\lambda)|\,d\lambda\int''\sum_k''|\psi_n(-\lambda_k,\mu)|^2|f(\mu)|\,d\mu\\
&\leq \int''\sum_{j=1}^{n}|\phi_n(\lambda_j-\lambda)|^2|f(\lambda)|\,d\lambda\int''\sum_{k=1}^{n}|\psi_n(-\lambda_k,\mu)|^2|f(\mu)|\,d\mu\\
&\leq Cn^2\int''\sum_s|\phi_s(\lambda)|^2|f(\lambda)|\,d\lambda\int''\sum_t|\psi_t(\mu)|^2|f(\mu)|\,d\mu
\end{aligned} \tag{9.65}$$

applying (8.68), (8.69). But (cf. (9.9)) (9.50) is already known to be $o(n^{(\alpha+\beta+1)})$ for $\alpha > \frac{1}{2}$. For $\alpha \leq \frac{1}{2}$, whereas when $m = \widetilde{n}$ we gave the corresponding proof with $\beta < 1$, here we will give it instead for $\beta \geq 1$, the treatment of $\beta < 1$ straightforwardly combining ideas from the two proofs. We write (9.64) as

$$\int''\int''\left|\sum_j''\left[\sum_s e^{is(\lambda_j-\lambda)}\left\{\phi(\lambda)-\overline{\phi}_s(\lambda)\right\}\right]\psi_n(-\lambda_j,\mu)\right|^2|f(\lambda)f(\mu)|\,d\lambda d\mu. \tag{9.66}$$

The contribution from $\phi(\lambda)$ is bounded by

$$C\int''\int''\left|\sum_j''D_n(\lambda_j-\lambda)\psi_n(-\lambda_j,\mu)\right|^2|f(\lambda)f(\mu)|\,d\lambda d\mu$$

$$\leq C\left\{\int''\int_{\Pi}\left|\sum_j'' D_n(\lambda_j-\lambda)\psi_n(-\lambda_j,\mu)\right|^2 d\lambda d\mu\right\}^{\frac{1}{2}}$$

$$\times\left\{\int''\sum_{j=1}^n|D_n(\lambda_j-\lambda)|^2|f(\lambda)|^2 d\lambda\int''\sum_{j=1}^n|\psi_n(-\lambda_j,\mu)|^2|f(\mu)|^2 d\mu\right\}^{\frac{1}{2}}. \quad (9.67)$$

Now

$$\int_{\Pi} D_n(\lambda_j-\lambda)D_n(\lambda-\lambda_k)=2\pi D_n(\lambda_j-\lambda_k), \quad (9.68)$$

which is $2\pi n$ for $j=k$ and zero otherwise, while

$$\sum_j''|\psi_n(-\lambda_j,\mu)|^2\leq\sum_{j=1}^n|\psi_n(-\lambda_j,\mu)|^2=n\sum_t|\psi_t(\mu)|^2, \quad (9.69)$$

which for $|\mu|>\varepsilon$ is $O(n^{2\beta})$, and, for all λ,

$$\sum_{j=1}^n|D_n(\lambda_j-\lambda)|^2=n^2. \quad (9.70)$$

It easily follows that (9.67) is $O\left(n^{2\beta+3/2}\right)=o\left(n^{2(\alpha+\beta+1)}\right)$. The contribution to (9.66) from $\overline{\phi}_s(\lambda)$ is bounded by

$$\int''\int''\sum_{j=1}^n\left|\sum_s e^{is(\lambda_j-\lambda)}\overline{\phi}_s(\lambda)\right|^2\sum_{k=1}^n|\psi_n(-\lambda_k,\mu)|^2|f(\lambda)f(\mu)|\,d\lambda d\mu$$

$$\begin{aligned}&\leq Cn^2\int''\int''\sum_s\left|\overline{\phi}_s(\lambda)\right|^2\sum_t|\psi_t(\mu)|^2|f(\lambda)f(\mu)|\,d\lambda d\mu\\&\leq Cn^{2(\alpha+\beta)+1}=o\left(n^{2(\alpha+\beta+1)}\right).\end{aligned} \quad (9.71)$$

Thus we can neglect the component $\int''\int''$ to (9.51), as we can also $\int'\int''$ and $\int''\int'$ in view of the proofs given so far.

Next the contribution of κ_{qrst} to (9.50) is

$$\frac{1}{4\pi^2n^4}\int_{\Pi}\int_{\Pi}\int_{\Pi}\left\{\sum_j''\phi_n(\lambda_j,-\lambda-\mu-\omega)\psi_n(-\lambda_j,\lambda)\right\}\left\{\sum_k''\phi_n(-\lambda_k,\mu)\psi_n(\lambda_k,\omega)\right\}$$
$$\times f(\lambda,\mu,\omega)d\lambda d\mu d\omega. \quad (9.72)$$

The contribution of $\int' \int' \int'$ is bounded by

$$\frac{C}{n^4} \left\{ \int_\Pi \int_\Pi \int_\Pi \left| \sum_j'' \phi_n(\lambda_j, -\lambda - \mu - \omega) \psi_n(-\lambda_j, \lambda) \right|^2 d\lambda d\mu d\omega \right.$$

$$\left. \times 2\pi \int_\Pi \int_\Pi \left| \sum_k'' \phi_n(-\lambda_k, \mu) \psi_n(\lambda_k, \mu) \right|^2 d\mu d\omega \right\}^{\frac{1}{2}} . \qquad (9.73)$$

But both factors in braces are bounded by (9.53), noting that in the first factor we may substitute for $\lambda + \mu + \omega$ and use periodicity of period 2π. Thus (9.73) $= o\left(n^{2(\alpha+\beta-1)}\right)$. We omit the proof for the remainder of (9.77) as it is so similar to earlier proofs. This completes the proof for $\widehat{F}(0, m)$.

For $\widehat{F}(1, m)$ we note that

$$Var\left\{\widehat{F}(1, m)\right\} = Var\left\{\widehat{F}(0, m)\right\} - 2Cov\left\{\widehat{F}(0, m), \overline{\zeta\xi}\right\} + Var(\overline{\zeta\xi}). \quad (9.74)$$

Then the proof proceeds by showing that

$$\lim_{n\to\infty} n^{2(1-\alpha-\beta)} Cov\left\{\widehat{F}(0, m), \overline{\zeta\xi}\right\} = f_{\eta\theta}^2(0) R(\alpha, \beta, \alpha, \beta) + f_{\eta\eta}(0) f_{\theta\theta}(0) R(\alpha, \alpha, \beta, \beta) \quad (9.75)$$

$$\lim_{n\to\infty} n^{2(1-\alpha-\beta)} Var(\overline{\zeta\xi}) = f_{\eta\theta}^2 R(\alpha, \beta, \alpha, \beta) + f_{\eta\eta}(0) f_{\theta\theta}(0) S(\alpha, \alpha, \beta, \beta). \quad (9.76)$$

These proofs follow very closely the previous pattern, where we established them first for $m = \widetilde{n}$ and then that the effect of taking $m < \widetilde{n}$ makes no difference, the details being so similar as not to be worth reporting. □

10 References

Akonom, J. and Gourieroux, C. (1987). A functional central limit theorem for fractional processes, preprint.

Billingsley, P. (1968). *Convergence of Probability Measures*, Wiley, New York.

Box, G.E.P. and Jenkins, G.M. (1971). *Time Series Analysis. Forecasting and Control*. Heilden-Day, San Francisco.

Brillinger, D.R. (1975). *Time Series, Data Analysis and Theory*, Holt, Rinehart and Winston, New York.

Chan, N.H. and Terrin, N. (1995). Inference for unstable long-memory processes with applications to fractional unit root autoregressions, *Annals of Statistics*, 23, 1662-1683.

Dahlhaus, R. (1997). Fitting time series models to nonstationary processes. *Annals of Statistics*, 25, 1-37.

Dickey, D.A. and Fuller, W.A. (1979). Distribution of the estimators for autoregressive time series with a unit root, *Journal of the American Statistical Association*, 74, 427-431.

Fox, R. and Taqqu, M.S. (1985). Non-central theorems for quadratic forms in random variables having long-range dependence, *Annals of Probability*, 13, 428-476.

Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *Annals of Statistics*, 14, 517-532.

Geweke, J. and Porter-Hudak, S. (1983). The estimation and application of long memory time series models, *Journal of Time Series Analysis*, 4, 221-238.

Hannan, E.J. (1963). Regression for time series with errors of measurement, *Biometrika*, 50, 293-302.

Hannan, E.J. (1976). The asymptotic distributions of serial covariances, *Annals of Statistics*, 4, 396-399.

Hurvich, C.M. and Bellrao, K. (1993). Asymptotics for the low-frequency ordinates of the periodogram of a long-memory time series. *Journal of Time Series Analysis*, 14, 455-472.

Hurvich, C.M. and B.K. Ray (1995). Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes, *Journal of Time Series Analysis*, 16, 17-41.

Jeganathan, P. (1996). On asymptotic inference in cointegrated time series with fractionally integrated errors, preprint.

Jeganathan, P. (1997). On asymptotic inference in cointegrated time series in linear cointegrated time series systems, *Econometric Theory*, 13, 692-745.

Kokoszka, P.S. and Taqqu, M.S. (1996). Parameter estimation for infinite variance fractional ARIMA, *Annals of Statistics*, 24, 1880-1913.

Künsch, H.R. (1986). Discrimination between monotonic trends and long-range dependence, *Journal of Applied Probability*, 23, 1025-1030.

Künsch, H.R. (1987). Statistical aspects of self-similar processes, *Proceedings of the First World Congress of the Bernoulli Society*, VNU Science Press, 1, 67-74.

Lobato, I.G. (1997). Consistency of the averaged cross-periodogram in long memory series, *Journal of Time Series Analysis*, 18, 137-156.

Marinucci, D. and Robinson, P.M. (1998). Weak convergence to fractional Brownian motion, preprint.

Parzen, E. (1963). On spectral analysis with missing observations and amplitude modulation, *Sankhya A*, 25, 383-392.

Phillips, P.C.B. (1991). Spectral regression for cointegrated time series, *Nonparametric and Semiparametric Methods in Econometrics and Statistics* (W.A. Barnett, J. Powell and G. Tauchen, eds.), Cambridge Univesity Press, Cambridge, 413-435.

Robinson, P.M. (1994a). Semiparametric analysis of long-memory time series, *Annals of Statistics*, 22, 515-539.

Robinson, P.M. (1994b). Rates of convergence and optimal spectral bandwidth for long range dependence, *Probability Theory and Related Fields*, 99, 443-473.

Robinson, P.M. (1995a). Log-periodogram regression of time series with long range dependence, *Annals of Statistics, 23*, 1048-1072.

Robinson, P.M. (1995b). Gaussian semiparametric estimation of long-range dependence, *Annals of Statistics*, 23, 1630-1661.

Robinson, P.M. and Marinucci, D. (1997). Semiparametric frequency-domain analysis of fractional cointegration, preprint.

Robinson, P.M. and Marinucci, D. (1999). The averaged periodogram for nonstationary vector stochastic processes, preprint.

Silveira, G. (1991). *Contributions to Strong Approximations in Time Series with Applications in Nonparametric Statistics and Functional Central Limit Theorems*, Ph.D. Thesis, University of London.

Stock, J.H. (1987). Asymptotic properties of least squares estimators of cointegrating vectors, *Econometrica*, 55, 1035-1056.

Whittle, P. (1951). *Hypothesis Testing in Time Series Analysis*, Thesis, Uppsala University, Uppsala.

Yong, C.H. (1974). *Asymptotic Behaviour of Trigonometric Series*, Chinese University, Hong Kong.

Zygmund, A. (1977). *Trigonometric Series*, Cambridge University Press, Cambridge.