BOOTSTRAP GOODNESS-OF-FIT TESTS FOR FARIMA MODELS*

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Abstract

This paper proposes goodness-of-fit tests for the class of covariance stationary $FARIMA$ processes. They are based on functionals of weighted empirical processes, say $S_n(\cdot)$, where the weights are the relative error between the periodogram and the fitted spectral density function under the null specification of the data. Two examples of such functionals are the $T_p - Barlett$ and the Cràmer-Von Mises standardized $\omega - statistic$. We show that the tests are able to detect contiguous alternatives converging to the null at the rate $n^{-1/2}$. However, because the cumbersome covariance structure of the limiting process of $S_n(\cdot)$, tests based on its asymptotic distribution are difficult to implement in practice. To circumvent this problem, we propose a bootstrap test, showing its consistency, and studying its small sample performance by a Monte Carlo experiment.

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1. INTRODUCTION

This paper proposes goodness-of-fit tests for a covariance stationary fractional autoregressive moving average ($FARIMA$) process, whose parameters are to be estimated, against the alternative that the data is generated by a general covariance stationary linear model. The tests are based on continuous functionals of the integrated relative error between the periodogram and the spectral density function obtained under the null specification, having the following attributes. 1) They are consistent against any covariance stationary linear process. 2) They have power against contiguous alternatives converging to the null at rate $n^{-1/2}$ and 3) although the alternative model is left unspecified, as in any goodness-of-fit test, the tests do not require the choice of any bandwidth parameter.

The main motivation to look at a $FARIMA$ model, apparently originated in Adenstedt (1974), is due to their prominence in empirical studies, see e.g. Diebold and Rudebusch (1989), Porter-Hudak (1990), Sowell (1992) or Ray (1993) among others. This prominence is partly due to the work of Box and Jenkins (1976) who advocated and emphasized the use of $ARMA$ or autoregressive integrated moving average $ARIMA$ models and also to their role in the standard approach to cointegration analysis, wherein several related series may be thought to have unit roots. That is, one or more linear combinations follow an $ARMA$ process.

Goodness-of-fit tests are of long standing in the statistical literature. It goes back to the work of Kolmogorov (1933) for independent identically distributed (iid) data and extended to two sample problem, when the null is that the two populations come from the same distribution, by Smirnov (1939). In a time series framework, Grenander and Rosenblatt (1957) used Kolmogorov-Smirnov statistics to test for a specific short-range model, and latter extended by Ibragimov (1963) under the assumption of square integrable spectral density function, so allowing for the presence of long-range dependence. More recent work is Velilla (1994) and Anderson (1997) who considered tests for an autoregressive moving average $ARMA(p,q)$ and autoregressive $AR(p)$ processes respectively, (see also the review paper by Anderson (1993),) and Kokoszka and Mikosch (1997) who allowed for, possibly, infinite variance.

However, as noted by Durbin (1973), when the null model depends on a set of unknown parameters, the tests are difficult to implement in practice as they are no longer based on functionals of the Brownian motion or Brownian bridge. To overcome this difficulty, Velilla (1994) proposed a pivotal test based on a smooth estimator of the spectral density function. In contrast, Anderson (1997) used the decomposition of a Gaussian process as an infinite weighted sum of independent normal random variables, where the weights depend on the covariance structure of the process. But their approaches amount a degree of sensitivity, in the former to the choice of the bandwidth parameter to estimate the spectral density function, whereas in Anderson (1997) one needs to truncate the infinite series at some finite value, say r. So the resulting inferences and performance of the tests are subject to those choices.

We propose a bootstrap test. The critical values are estimated by the conditional quantiles, given the sample, of a bootstrap statistic. Such a statistic is the bootstrap analog of the original one, computed from a naive residual bootstrap resample. This approach has the advantage compared to the aforementioned ones that no smoothing parameter, or number r, are required to be chosen, besides that it gives a more accurate approximation to the actual finite sample distribution of the tests.

The reminder of the paper is as follows. In section 2, we describe the hypothesis testing and the tests. Section 3 discusses their asymptotic properties, showing that the tests do not have trivial power under contiguous alternatives converging to the null at the parametric rate $n^{-1/2}$. Because, under the null hypothesis, the limit distribution of the tests is difficult to tabulate, in Section 4 we propose to estimate their critical values by their bootstrap analogs based on a naive resample of the innovation sequence of the process. Section 5 presents the results of a small Monte Carlo experiment which illustrates the good level accuracy of the bootstrap tests with fairly small samples. In Section 6, we give the proofs of our results in Sections 3 and 4, which employ a Lemmata in Section 7.

2. THE TEST

Consider a covariance stationary linear process x_t which is observed at times $t = 1, ..., n$ with autocovariance function γ, and spectral density function, f, defined from

$$\gamma(j) = E(x_j x_0) \int_{-\pi}^{\pi} f(\lambda) e^{ij\lambda} d\lambda \quad j = 0, 1, 2, ..., \tag{1}$$

satisfying

$$\int_{-\pi}^{\pi} \log(f(\lambda)) d\lambda > -\infty. \tag{2}$$

It is well known that under (2), the process x_t admits a Wold decomposition

$$x_t = \sum_{j=0}^{\infty} \alpha(j) \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} \alpha^2(j) < \infty, \tag{3}$$

where the innovations ε_t are a white noise process with zero mean and variance σ_ε^2.

We wish to test the null hypothesis H_0 that x_t follows the $FARIMA(p,d,q)$ representation

$$\Phi(L,\phi_0)(1-L)^{d_0}x_t = \Xi(L,\psi_0)\varepsilon_t, \tag{4}$$

where Φ and Ξ are the autoregressive and moving average polynomials to be made more precise in Assumption A.1, against the alternative H_{1n} that x_t follows (3). When $d_0 > 0$, we say that the process x_t exhibits long-range dependence, for $d_0 = 0$, the process corresponds to weakly or short-range dependence, whereas for $d_0 < 0$, we have an example of a process x_t exhibiting the so-called negative or anti-persistent dependence.

The covariance structure of the $FARIMA$ process in (4) can be described from (1) in terms of its spectral density function

$$f(\lambda,\theta_0) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{1}{\left|1-e^{i\lambda}\right|^{2d_0}}\frac{\left|\Xi\left(e^{i\lambda},\psi_0\right)\right|^2}{\left|\Phi\left(e^{i\lambda},\phi_0\right)\right|^2}, \qquad \lambda\in[0,\pi],$$

whereas that from (3) is $f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi}\left|A(\lambda)\right|^2$ with $A(\lambda) = \sum_{j=0}^{\infty}\alpha(j)e^{ij\lambda}$. Because the models given in (3) or (4) are perfectly described by their spectral density function, the hypothesis testing can be formulated as

$$\begin{aligned} H_0 &: \forall\lambda\in[0,\pi] \text{ and for some } \theta_0\in\Theta,\ f(\lambda) = f(\lambda;\theta_0) \\ & \quad \textit{against} \\ H_1 &: \exists H(\lambda)\subset[0,\pi] \text{ such that for all } \theta\in\Theta,\ f(\lambda)\neq f(\lambda;\theta) \end{aligned} \tag{5}$$

where $H(\lambda)$, which may depend on θ, has Lebesgue measure greater than zero.

The motivation to leave the alternative model (3) unspecified comes from the observation that the $FARIMA$ model is only one example of the many possible parameterizations of a covariance stationary linear process, even, in terms of a finite number of parameters. An example is Bloomfield's (1973) (fractional integrated) exponential model, see also Robinson (1994), and which has recently been employed by Gil-Alaña and Robinson (1996) and Lobato and Robinson (1998). An earlier example is the fractional Gaussian noise model introduced by Mandelbrot and Van Ness (1968). One feature of these models is that neither possesses a finite $FARIMA$ representation.

We now describe the test. Introduce the periodogram of x_t

$$I_n(\lambda) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n}x_t e^{-it\lambda}\right|^2$$

and the weighted empirical process in $\mathbb{D}[0,1]$,

$$\widetilde{S}_n(\vartheta,\theta_0)=\frac{1}{n}\sum_{j=1}^{[n/2]}\mathcal{I}(\lambda_j\le\pi\vartheta)\left(\frac{I_{n,j}}{f_j(\theta_0)}-1\right)\text{ where }\vartheta\in[0,1], \tag{6}$$

where $\mathcal{I}(B)$ denotes the indicator function of the event B, $\lambda_j=2\pi j/n, j=1,...,[n/2]$ and where, henceforth, for a generic function $h(\lambda)$, $h_j=h(\lambda_j)$. The rational to look at statistics like (6) comes from the fact that, under H_0 and assumptions A.1 and A.2 below, by Robinson (1995*a*) and a straightforward extension to the region $[0,\pi]$, $E((2\pi)I_n(\lambda))\simeq f(\lambda,\theta_0)$, so that $E\left(\widetilde{S}_n(\vartheta,\theta_0)\right)\simeq 0$ for all $\vartheta\in[0,1]$, whereas under H_1 it would develop a mean different than zero. More specifically, one has that, denoting the periodogram of ε_t by $I_{n\varepsilon}(\lambda)=(2\pi n)\left|\sum_{t=1}^n\varepsilon_t e^{-it\lambda}\right|^2$,

$$\widetilde{S}_n(\vartheta,\theta_0)=\frac{1}{n}\sum_{j=1}^{[n\vartheta/2]}\left(\frac{2\pi I_{\varepsilon j}}{\sigma_\varepsilon^2}-1\right)+o_p\left(n^{-1/2}\right)$$

whose limiting behaviour is well known, see for instance Brillinger (1981, Theorem 7.6.1.).

As it is known, from related literature involving $\widetilde{S}_n(\vartheta,\theta_0)$, see for instance Anderson (1993) for a later reference, the limiting covariance structure of $\widetilde{S}_n(\vartheta,\theta_0)$ depends on the fourth cumulant κ_4 of the innovation process ε_t. Because of that, following Anderson's (1993), see also Klüppelberg and Mikosch (1996), to avoid this dependency on κ_4, we use the transformation

$$S_n(\vartheta,\theta_0)=\widetilde{S}_n(\vartheta,\theta_0)-\vartheta\widetilde{S}_n(1,\theta_0),\ \vartheta\in[0,1].$$

As in empirical examples, θ_0 is unlikely to be known. Thus, to make (6) feasible, we need to replace θ_0 by a suitable estimate, say the Whittle estimator $\widehat{\theta}_n$ defined in (11), obtaining

$$S_n\left(\vartheta,\widehat{\theta}_n\right)=\widetilde{S}_n\left(\vartheta,\widehat{\theta}_n\right)-\vartheta\widetilde{S}_n\left(1,\widehat{\theta}_n\right),\ \vartheta\in[0,1]. \tag{7}$$

Thus, tests for H_0 in (5) can be implemented from some functional of the statistic $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$. That is, denote by $\varphi(\cdot)$ a continuous functional, $\varphi:\mathbb{R}\to\mathbb{R}^+$, of $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$. The test is based on

$$\widehat{\eta}_n=\varphi\left(n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)\right). \tag{8}$$

Two common functionals are Barlett's T_p-test

$$B_n=\sup_{\{j:j=1,\ldots,n\}}\left|n^{1/2}S_n\left(\frac{j}{n},\widehat{\theta}_n\right)\right| \tag{9}$$

which is of the Kolmogorov-Smirnov type and the normalized Cràmer-Von Mises $\omega-$statistic

$$C_n=\frac{1}{n}\sum_{j=1}^{n}\left(n^{1/2}S_n\left(\frac{j}{n},\widehat{\theta}_n\right)\right)^2, \tag{10}$$

which are the Riemann's discrete approximation to $\sup_{\vartheta\in[0,1]}\left|n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)\right|$ and $n\int_0^1 S_n^2\left(\vartheta,\widehat{\theta}_n\right)d\vartheta$.

3. ASYMPTOTIC PROPERTIES OF THE TESTS

Introduce the following assumptions,

A.1: $\theta_0=\left(\phi_0',\psi_0',d_0,\sigma_{0\varepsilon}^2\right)'$ is an interior point of the compact parameter space $\Theta=\Theta_1\times\Theta_2\times\Theta_3\subset\mathbb{R}^{p+q}\times(-1/2,1/2)\times\mathbb{R}^+$. In addition, the polynomials Φ and Ξ are of order p and q, respectively with no common roots and lying outside the unit circle for all $\left(\phi',\psi'\right)'\in\Theta_1$.

A.2: In (4), the innovation sequence $\{\varepsilon_t\}$ is a stochastic process with finite eight moments, where $E\left(\varepsilon_t\mid\mathcal{F}_{t-1}\right)=0$, $E\left(\varepsilon_t^2\mid\mathcal{F}_{t-1}\right)=E\left(\varepsilon_t^2\right)=\sigma_{0\varepsilon}^2$ a.s., $E\left(\varepsilon_t^\ell\,|\mathcal{F}_{t-1}\right)=\mu_\ell<\infty$ *a.s.*, $\ell=3,...,8$, where $\mathcal{F}_t$ is the σ-algebra of events generated by $\varepsilon_s, s\leq t$, and the joint fourth cumulant of $\varepsilon_{t_1},\varepsilon_{t_2},\varepsilon_{t_3}$ and ε_{t_4} satisfies

$$cum\left(\varepsilon_{t_1},\varepsilon_{t_2},\varepsilon_{t_3},\varepsilon_{t_4}\right)=\begin{cases}\kappa_4, & t_1=t_2=t_3=t_4,\\ 0, & otherwise.\end{cases}$$

As was mentioned in the previous section, in practice θ_0 is unknown, and thus to make $n^{1/2}S_n\left(\vartheta;\theta_0\right)$ feasible, θ_0 has to be replaced by a suitable estimator, say the Whittle estimator, defined as

$$\widehat{\beta}_n=\arg\min_{\beta\in\Theta_1\times\Theta_2}\sum_{j=1}^{[n/2]}\frac{I_{n,j}}{\left|A_j\left(\beta\right)\right|^2}\text{ and }\widehat{\sigma}_\varepsilon^2=\frac{2}{n}\sum_{j=1}^{[n/2]}\frac{I_{n,j}}{\left|A_j\left(\widehat{\beta}_n\right)\right|^2},\tag{11}$$

where $\left|A\left(\lambda,\beta\right)\right|^2=2\pi f\left(\lambda,\theta\right)/\sigma_\varepsilon^2$. Observe that our definition of the Whittle estimate comes from the parameterization of the $FARIMA$ model. Indeed, under the specification (4), the parameters σ_ε^2 and $\beta=\left(\phi',\psi',d\right)'$ are functional unrelated and $\int_{-\pi}^{\pi}\log\left|A\left(\lambda,\beta\right)\right|^2d\lambda=0$ for all $\beta\in\Theta_1\times\Theta_2$. So $\sigma_{0\varepsilon}^2$ has the interpretation of being the one-step-prediction error.

The next theorem establishes the behaviour of the process of $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$.

Theorem 1 *Let $\widehat{\theta}_n$ be given by (11). Assuming A.1 and A.2, under H_0,*

$$n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)\text{ converges weakly to }S_\infty\left(\vartheta\right)\text{ in }\mathbb{D}\left[0,1\right]\text{ endowed with the Skorohod metric,}$$

where S_∞ is a Gaussian process centered at zero and covariance structure

$$\mathcal{K}\left(\vartheta_1,\vartheta_2\right)=\frac{1}{2}\left(\min\left(\vartheta_1,\vartheta_2\right)-\vartheta_1\vartheta_2\right)-\frac{1}{2\pi}\mathcal{G}\left(\vartheta_1\right)'\mathcal{A}^{-1}\mathcal{G}\left(\vartheta_2\right),\tag{12}$$

with $\mathcal{G}\left(\vartheta\right)=\int_0^\pi\left(\mathcal{I}\left(\lambda\leq\pi\vartheta\right)-\vartheta\right)\phi\left(\lambda;\theta_0\right)d\lambda$, $\phi\left(\lambda,\theta\right)=f^{-1}\left(\lambda,\theta\right)\check{f}\left(\lambda,\theta\right)$, and $\check{f}\left(\lambda,\theta\right)=\partial f\left(\lambda,\theta\right)/\partial\theta$.

From Theorem 1, the asymptotic covariance structure $\mathcal{K}(\vartheta_1, \vartheta_2)$ has two components, the second, that is, $(2\pi)^{-1} \mathcal{G}(\vartheta_1)' \mathcal{A}^{-1} \mathcal{G}(\vartheta_2)$, due to the estimation of θ_0 as in Durbin (1973) or Anderson (1997), whereas the first term of (12) is twice that of a Brownian bridge. Moreover, it is expected that tests based on $\widehat{\eta}_n$ given in (8) should be able to detect contiguous alternatives which converge to the null at the rate $n^{-1/2}$. To this end, consider the contiguous alternatives,

$$H_{1n} : f(\lambda) = f(\lambda, \theta) \left(1 + \frac{g(\lambda)}{n^{1/2}}\right) \text{ for some } \theta \in \Theta \text{ and for all } \lambda \in [-\pi, \pi],$$

where $g(\lambda)$ is some symmetric (around the origin), positive and non-constant integrable function in $[-\pi, \pi]$. This type of alternatives has also been considered, in related specification testing problems by Stute (1997) or Andrews (1997), among others.

Introduce

$$R(\vartheta) = \int_0^\pi [\mathcal{I}(\lambda \leq \pi\vartheta) - \vartheta] g(\lambda) d\lambda - \mathcal{G}(\vartheta)' \mathcal{A}^{-1} \int_0^\pi \phi(\lambda, \theta) g(\lambda) d\lambda,$$

which is a non-zero function. Notice that if $g(\lambda)$ were constant, then $R(\vartheta) = 0$ as is not surprising since in that case, H_{1n} will be included in H_0. Thus, we have the following Corollary, which shows the limiting behaviour of any continuous functional $\varphi(\cdot)$ of $n^{1/2} S_n\left(\vartheta; \widehat{\theta}_n\right)$ under H_{1n}, and by extension under H_0.

Corollary 1 *Let $\varphi(\cdot)$ be a continuous mapping in R^+ and let $\hat{\theta}_n$ be given in (11). If A1 and A2 hold, under H_{1n},*

$$\widehat{\eta}_n = \varphi\left(n^{1/2} S_n\left(\vartheta; \widehat{\theta}_n\right)\right) \overset{d}{\to} \eta'_\infty = \varphi\left(S_\infty(\vartheta) + R(\vartheta)\right),$$

where "$\overset{d}{\to}$" denotes convergence in distribution.

Observe that under H_0, $g(\lambda) = 0$, so $\widehat{\eta}_n = \varphi\left(n^{1/2} S_n\left(\vartheta; \widehat{\theta}_n\right)\right) \overset{d}{\to} \eta_\infty = \varphi\left(S_\infty(\vartheta)\right)$. Also Corollary 1 indicates that for the functionals $\varphi(\cdot)$ defined in (9) and/or (10), under H_{1n},

$$B_n \overset{d}{\to} \sup_{\vartheta \in [0,1]} |S_\infty(\vartheta) + R(\vartheta)| \text{ and } C_n \overset{d}{\to} \int_0^1 \left(S_\infty(\vartheta) + R(\vartheta)\right)^2 d\vartheta.$$

As was mentioned in the introduction, as the structure of $\mathcal{K}(\vartheta_1, \vartheta_2)$ in (12) is complicated, tests based on $\eta_\infty = \varphi(S_\infty(\vartheta))$ seems difficult to implement in practice. So, in the next section we describe and justify the bootstrap approach to $\widehat{\eta}_n$.

4. BOOTSTRAP TESTS

Since Efron's (1979) seminal paper, bootstrap methods have proved to be a very useful tool in statistics, see for instance the monographs by Hall (1992) and Shao and Tu (1995). Once a continuous functional $\varphi : \mathbb{R} \rightarrow \mathbb{R}^{+}$ has been designed to test for H_0, we propose to estimate the distribution of $\widehat{\eta}_n = \varphi\left(n^{1/2} S_n\left(\vartheta; \widehat{\theta}_n\right)\right)$ by its bootstrap analog, $\widehat{\eta}_n^*$, based on a resample $\underset{\sim}{x}^* = (x_1^*, x_2^*, ..., x_n^*)'$ of $\underset{\sim}{x} = (x_1, x_2, ..., x_n)'$. The resampling method must be such that the conditional distribution, given $\underset{\sim}{x}$, of the bootstrap statistic $\widehat{\eta}_n^*$ consistently estimates the distribution of η_∞ under H_0. That is, $\widehat{\eta}_n^* \rightarrow_{d^*} \eta_\infty$ in probability under H_0, where "$\rightarrow_{d^*}$" denotes

$$\lim_{n\rightarrow\infty} \Pr\left[\widehat{\eta}_n^* \leq z | \underset{\sim}{x}\right] \xrightarrow{p} G(z),$$

at each continuity point z of $G(z) = \Pr(\eta_\infty \leq z)$ as defined in Gine and Zinn (1990). Moreover, under contiguous alternatives H_{1n}, $\widehat{\eta}_n^*$ must also converge, in bootstrap distribution to η_∞.

We now describe the bootstrap.

STEP 1 Let $\Omega_n(\theta_0) = [\gamma(|i-j|, \theta_0)]_{i,j=1,\ldots,n}$ be the $n \times n$ autocovariance matrix of $\underset{\sim}{x}$ and $L_n(\theta_0)$ its Cholewsky's decomposition. That is, $\Omega_n(\theta_0) = L_n(\theta_0)' L_n(\theta)$ where $L_n(\theta_0)$ is a triangular matrix, so the vector of observations $\underset{\sim}{x}$ can be represented as

$$\underset{\sim}{x} = L_n(\theta_0)' \underset{\sim}{\varepsilon},$$

where $\underset{\sim}{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)'$ is the vector of innovations.

STEP 2 Let $\hat{\theta}_n$ be the Whittle given in (11) and compute the residuals as $\underset{\sim}{\widehat{\varepsilon}} = L_n\left(\widehat{\theta}_n\right)^{-1\prime} \underset{\sim}{x}$.

STEP 3 Let $\underset{\sim}{\widetilde{\varepsilon}}^* = (\widetilde{\varepsilon}_1^*, \widetilde{\varepsilon}_2^*, ..., \widetilde{\varepsilon}_n^*)'$ be a random sample with replacement from the standardized residuals

$$\widetilde{\varepsilon}_t = \frac{\widehat{\varepsilon}_t - n^{-1}\sum_{t=1}^n \widehat{\varepsilon}_t}{\widetilde{\sigma}_{\varepsilon n}},\ \widetilde{\sigma}_{\varepsilon n}^2 = \frac{1}{n}\sum_{t=1}^n \left(\widehat{\varepsilon}_t - \frac{1}{n}\sum_{t=1}^n \widehat{\varepsilon}_t\right)^2,$$

and obtain the bootstrap sample

$$\underset{\sim}{x}^* = L_n\left(\widehat{\theta}_n\right)' \underset{\sim}{\widetilde{\varepsilon}}^*.$$

STEP 4 Denote the periodogram of $\underset{\sim}{x}^*$ as

$$I_{n,j}^* = \frac{1}{2\pi n}\left|\sum_{t=1}^n x_t^* e^{-it\lambda_j}\right|^2, \tag{13}$$

and obtain the bootstrap analog of the Whittle estimate (11), that is

$$\widehat{\beta}_n^* = \arg\min_{\beta\in\Theta_1\times\Theta_2} \sum_{j=1}^{[n/2]} \frac{I_{n,j}^*}{f_j(\theta)} \text{ and } \widehat{\sigma}_{\varepsilon n}^{2*} = \frac{2}{n}\sum_{j=1}^{[n/2]} \frac{I_{n,j}^*}{f_j(\theta)}. \tag{14}$$

STEP 5 Let $\widehat{\theta}_n^* = \left(\widehat{\beta}_n^{*\prime}, \widehat{\sigma}_\varepsilon^{2*}\right)'$ be the bootstrap analogs of $\widehat{\theta}_n$ obtained with the resample $\underset{\sim}{x}^*$, and compute

$$S_n^*\left(\vartheta, \widehat{\theta}_n^*\right) = \widetilde{S}_n^*\left(\vartheta, \widehat{\theta}_n^*\right) - \vartheta \widetilde{S}_n^*\left(1, \widehat{\theta}_n^*\right), \ \vartheta \in [0,1],$$

with

$$\widetilde{S}_n^*\left(\vartheta, \widehat{\theta}_n^*\right) = \frac{1}{n} \sum_{j=1}^{[n/2]} \mathcal{I}\left(\lambda_j \leq \pi\vartheta\right) \left(f_j^{-1}\left(\widehat{\theta}_n^*\right) I_{n,j}^* - 1\right).$$

STEP 6 Finally, obtain the bootstrap test as $\widehat{\eta}_n^* = \varphi\left(n^{1/2} S_n^*\left(\vartheta; \widehat{\theta}_n^*\right)\right)$.

Remark 1 *Since* $E\left[\left.\underset{\sim}{\widetilde{\varepsilon}}^*\right| \underset{\sim}{x}\right] = 0$ *and* $E\left[\left.\underset{\sim}{\widetilde{\varepsilon}}^* \underset{\sim}{\widetilde{\varepsilon}}^*\right| \underset{\sim}{x}\right] = I_n$, $E\left[\left.\underset{\sim}{x}^*\right| \underset{\sim}{x}\right] = 0$, $E\left[\left.\underset{\sim}{x}^* \underset{\sim}{x}^{*\prime}\right| \underset{\sim}{x}\right] = \Omega_n\left(\widehat{\theta}_n\right)$ *and the spectral density function of* x_t^**, conditional on* $\underset{\sim}{x}$*, is* $f\left(\lambda, \widehat{\theta}_n\right)$.

The following theorem provides the consistency of the bootstrap test.

Theorem 2 *Let* $\varphi(\cdot)$ *be a continuous mapping in* R^+*, and let* $\widehat{\theta}_n$ *be given by* (11). *If A1 and A2 hold, under* H_0 *or* H_{1n}*,*

$$\widehat{\eta}_n^* = \varphi\left(n^{1/2} S_n^*\left(\vartheta; \widehat{\theta}_n^*\right)\right) \xrightarrow{d^*} \eta_\infty = \varphi\left(S_\infty(\vartheta)\right) \quad \textit{in probability.}$$

Theorem 2 justifies the consistency of the bootstrap test which employs the critical values computed from the conditional distribution of the bootstrap statistic, $c_{n\alpha}^*$ say, where $\Pr\left[\widehat{\eta}_n^* \geq c_{n\alpha}^* \middle| \underset{\sim}{x}\right] = \alpha$. As the bootstrap critical values are computationally difficult to obtain, they are approximated, as accurately as desired, by Monte Carlo simulation. That is, let $\left\{\underset{\sim}{x}^{*(1)}, \underset{\sim}{x}^{*(2)}, ..., \underset{\sim}{x}^{*(B)}\right\}$ be B resamples as generated in STEP 3 and be $\left\{\widehat{\eta}_n^{*(1)}, \widehat{\eta}_n^{*(2)}, ..., \widehat{\eta}_n^{*(B)}\right\}$ their corresponding bootstrap statistics as given in STEPS 5 and 6. Then, $c_{n\alpha}^*$ is approximated by $c_{n\alpha}^{*B}$, defined from

$$\frac{1}{B} \sum_{j=1}^{B} 1\left(\widehat{\eta}_n^{*(j)} \geq c_{n\alpha}^{*B}\right) = \alpha.$$

5. MONTE CARLO EXPERIMENTS

In all the experiments, we have generated 5000 Monte Carlo samples and we have used 2000 bootstrap resamples. We have considered sample sizes of $n = 25, 50, 100$ and 150.

To compare the performance of the bootstrap test with respect to the asymptotic one, when this is feasible, we have performed the test when the null hypothesis is that x_t follows a white noise process. To this end, the observations x_t were generated as $iid\ N(0,1)$ and $Uniform(-0.5, 0.5)$. The empirical level of the Monte Carlo experiments is reported in Table 1. The results of Table 1 illustrates that the bootstrap tests exhibit an excellent accuracy level for both distributions considered for x_t, even for sample sizes as small as $n = 25$. In contrast, the performance of the tests based on their limit distribution is worst than that of the bootstrap the smaller the sample size is. In addition, we observe that the Cràmer-von Mises, C_n, works better than the Kolmogorov-Smirnov test, B_n, a well known fact (see e.g. D'Agostino and Stephens, 1986). This illustrates that even in situations where the limit distribution of the statistic can be approximated, as in Anderson (1997), its performance would not be better than that of the bootstrap test by means of a better level accuracy.

Table 2 studies the performance of the level of the bootstrap test when the null hypothesis is an $AR(1)$ process with parameter 0.5 and the innovations ε_t were $iid\ N(0,1)$ or $Uniform(-0.5, 0.5)$. In both situations, the bootstrap tests perform very well, even for sample sizes of $n = 25$.

In Table 3, we examined the performance of the test under H_0 when the model follows a $FARIMA(0,d,0)$ process with $d = 0.2, 0.3,$ and $0.4,$ and where the innovations ε_t are $iid\ N(0,1)$. As could be expected, larger sample sizes, at least of $n = 100$, are needed to obtain a reasonable level accuracy than when testing for a short-range specification, across the spectrum of values of d.

Tables 4 and 5 illustrates the power of the tests. In Table 4, we describe the empirical power when testing that the model is an $AR(1)$ process, but the true model is a $FARIMA(0,d,0)$ process with parameter $d = 0.2, 0.3,$ or $0.4,$ whereas in Table 5, we report the power of the tests when testing that the data follows a $FARIMA(0,d,0)$ process but the true model is an $AR(1)$ with parameter 0.5. In both cases, the innovations ε_t were generated as $iid\ N(0,1)$. Not surprisingly, the power increases with the sample size. Of course, this is what one can expect, as, in finite samples, the power depends very much on how far away the true model is from the hypothetical one. This fact is illustrated when testing an $AR(1)$ model against the $FARIMA$ process. There, the greater the parameter d is, and thus the far away the model is from the $AR(1)$ structure, the smaller the sample sizes are required to achieve a reasonable power behaviour.

6. PROOFS

In this and next sections, for notational simplicity, we write $[n/2]$ as $n/2$.

Proof of Theorem 1

Write $\mathcal{J}(\vartheta, j) = \mathcal{I}(j \leq [n\vartheta/2]) - \vartheta$. By definition

$$S_n\left(\vartheta, \widehat{\theta}_n\right) = S_n(\vartheta, \theta_0) + \widehat{P}_n(\vartheta),$$

where

$$\widehat{P}_n(\vartheta) = \frac{1}{n}\sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j)\, I_{nj}\left[f_j^{-1}\left(\hat{\theta}_n\right) - f_j^{-1}(\theta_0)\right]. \tag{15}$$

Since see, for instance Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990), Hosoya (1997) or Velasco and Robinson (1999) who allow also $d < 0$, under A.1 and A.2,

$$\widehat{\theta}_n = \theta_0 + (2\pi)\,\mathcal{A}^{-1} b_n + o\left(n^{-1/2}\right) \;\; a.s., \tag{16}$$

where

$$b_n = \frac{1}{n}\sum_{j=1}^{[n/2]} \phi_j(\theta_0)\left[f_j(\theta_0)^{-1} I_{n,j} - 1\right] \quad \text{and} \quad \mathcal{A} = \int_0^{\pi} \phi(\lambda, \theta_0)\, \phi(\lambda, \theta_0)'\, d\lambda,$$

uniformly in $\vartheta \in [0,1]$. By standard linearization of $f_j^{-1}\left(\widehat{\theta}_n\right) - f_j^{-1}(\theta_0)$

$$\widehat{P}_n(\vartheta) = -\frac{1}{n}\sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j)\, \phi_j(\theta_0)'\, \frac{I_{nj}}{f_j(\theta_0)}\left(\widehat{\theta}_n - \theta_0\right)\left(1 + O_p\left(n^{-1/2}\right)\right).$$

Because by A.1 and A.2, $\phi(\lambda, \theta_0)$ is continuously differentiable outside any neighborhood containing the origin, for any $\delta > 0$ and uniformly in $\vartheta \in [0,1]$,

$$\frac{1}{n}\sum_{j=\delta n/2}^{n/2} \mathcal{J}(\vartheta, j)\, \phi_j(\theta_0) - \frac{1}{(2\pi)}\int_{\delta\pi}^{\pi} \left(\mathcal{I}(\lambda \leq \pi\vartheta) - \vartheta\right) \phi(\lambda; \theta_0)\, d\lambda = O\left(n^{-1}\right)$$

and

$$\frac{1}{n}\sum_{j=1}^{\delta n/2} \mathcal{J}(\vartheta, j)\, \phi_j(\theta_0) - \frac{1}{(2\pi)}\int_{0}^{\delta\pi} \left(\mathcal{I}(\lambda \leq \pi\vartheta) - \vartheta\right) \phi(\lambda; \theta_0)\, d\lambda = O\left(n^{-1}\log n\right),$$

by Brillinger (1981, p.15) and Lemma 2 of Robinson (1995*b*), respectively.

Thus uniformly in $\vartheta \in [0,1]$,

$$n^{1/2} S_n\left(\vartheta, \widehat{\theta}_n\right) = R_n(\vartheta) + H_n(\vartheta) + o_p(1),$$

where

$$R_n(\vartheta) = \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j(\vartheta) \left(\frac{I_{nj}}{f_j(\theta_0)} - 1 \right),$$

with $\psi_j(\vartheta) = \mathcal{J}(\vartheta, j) - \mathcal{G}(\vartheta)\mathcal{A}^{-1}\phi_j(\theta_0)$, and

$$H_n(\vartheta) = -\frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j)\, \phi_j(\theta_0)' \left(\frac{I_{nj}}{f_j(\theta_0)} - 1 \right) \left(\widehat{\theta}_n - \theta_0 \right).$$

Now the proof follows by Propositions 1 to 3 below. $R_n(\vartheta)$ is

$$\frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j(\vartheta) \left(\frac{I_{nj}}{f_j(\theta_0)} - \frac{(2\pi) I_{\varepsilon j}}{\sigma_{0\varepsilon}^2} \right) + \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j(\vartheta) \left(\frac{(2\pi) I_{\varepsilon j}}{\sigma_{0\varepsilon}^2} - 1 \right) = R_n^1(\vartheta) + R_n^2(\vartheta),$$

where $\sup_{\vartheta \in (0,1)} \left| R_n^1(\vartheta) \right| = o_p(1)$ by Proposition 1 and $R_n^2(\vartheta) \overset{weakly}{\Rightarrow} S_\infty(\vartheta)$ in $\mathbb{D}[0,1]$ by Propositions 2 and 3. Finally, $\sup_{\vartheta \in (0,1)} |H_n(\vartheta)| = o_p(1)$ from Propositions 1 to 3 and $\widehat{\theta}_n - \theta_0 = o_p(1)$. $\square$

Henceforth, to simplify the notation, we assume, without loss of generality, that $\sigma_{0\varepsilon}^2 = 1$.

Proposition 1 *Assuming A1 and A2,* $\sup_{\vartheta \in (0,1)} \left| R_n^1(\vartheta) \right| = o_p(1)$.

Proof. By definition of $R_n^1(\vartheta)$ and the triangle inequality,

$$\begin{aligned} \sup_{\vartheta \in (0,1)} \left| R_n^1(\vartheta) \right| \leq & \sup_{\vartheta \in (0,1)} \left| \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{I}(j \leq [n\vartheta/2]) \left(\frac{I_{nj}}{f_j(\theta_0)} - (2\pi) I_{\varepsilon j} \right) \right| \\ & + \sup_{\vartheta \in (0,1)} \left| \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \left(\vartheta + \mathcal{G}(\vartheta)\mathcal{A}^{-1}\phi_j(\theta_0) \right) \left(\frac{I_{nj}}{f_j(\theta_0)} - (2\pi) I_{\varepsilon j} \right) \right|. \end{aligned} \tag{17}$$

We only prove that the first term on the right vanishes asymptotically. The proof for the second term is easier, as $\sup_{\vartheta \in (0,1)} |G(\vartheta)| < C$, where henceforth C is a generic finite positive constant.

Let $u_j = A_j^{-1} w_j$, $v_j = w_{\varepsilon,j}$ and $A_j = A(\lambda_j; \beta_0)$ where

$$w_j = (2\pi n)^{-1/2} \sum_{t=1}^{n} x_t e^{-it\lambda_j} \text{ and } w_{\varepsilon,j} = (2\pi n)^{-1/2} \sum_{t=1}^{n} \varepsilon_t e^{-it\lambda_j}.$$

The first term of (17) is thus bounded by

$$\sup_{\vartheta \in (0,1)} \frac{1}{n^{1/2}} \sum_{j=1}^{[n\vartheta/2]} |u_j - v_j|^2 + 2 \sup_{\vartheta \in (0,1)} \left| \frac{1}{n^{1/2}} \sum_{j=1}^{[n\vartheta/2]} v_j \left(\overline{u}_j - \overline{v}_j \right) \right|, \tag{18}$$

where $\overline{c}$ denote the conjugate of the complex number c.

The first term of (18) is $o_p(1)$ since its expectation is

$$\frac{1}{n^{1/2}}\sum_{j=1}^{[n/2]}\left\{\left(E\left|u_j\right|^2-1\right)-\left(E\left(u_j\overline{v}_j\right)-1\right)-\left(E\left(\overline{u}_jv_j\right)-1\right)+\left(E\left|v_j\right|^2-1\right)\right\}=O\left(\frac{1}{n^{1/2}}\sum_{j=1}^{[n/2]}\frac{\log j}{j}\right)$$

because $E\left|v_j\right|^2=1$ and by Theorems 1 and 2 of Robinson (1995*a*).

Next, to show that the second term of (18) is $o_p(1)$, it suffices to show that the finite dimensional distributions of the term inside the absolute value converge to zero and tightness. First,

$$E\left|n^{-1/2}\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]}v_j\left(\overline{u}_j-\overline{v}_j\right)\right|^2=a_1+a_2+b_1+b_2,$$

where

$$\begin{aligned}
a_1 &= \frac{1}{n}\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]}\Big\{E\left|u_j\right|^2E\left|v_j\right|^2+\left|E\left(v_j\overline{u}_j\right)\right|^2+\left|E\left(v_ju_j\right)\right|^2+2\left(E\left|v_j\right|^2\right)^2+\left(Ev_j^2\right)^2\\
&\quad -2E\left|v_j\right|^2E\left(v_j\overline{u}_j\right)-\left|Ev_j^2\right|E\left(\overline{u}_j\overline{v}_j\right)-2E\left|v_j\right|^2E\left(u_j\overline{v}_j\right)-E\left(u_jv_j\right)E\left(\overline{v}_j^2\right)\Big\},
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{1}{n}\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]}\left\{cum\left(v_j,\overline{v}_j,u_j,\overline{u}_j\right)+cum\left(v_j,\overline{v}_j,v_j,\overline{v}_j\right)-cum\left(v_j,\overline{v}_j,v_j,\overline{u}_j\right)\right.\\
&\quad \left.-\,cum\left(v_j,\overline{v}_j,\overline{v}_j,\overline{u}_j\right)\right\},
\end{aligned}$$

$$\begin{aligned}
b_1 &= \frac{1}{n}\sum_{[n\vartheta_1/2]<j<k\leq[n\vartheta_2/2]}\left\{E\left(v_j\overline{v}_k\right)E\left(u_j\overline{u}_k\right)+\left[E\left(v_j\overline{u}_j\right)-1\right]\left[E\left(v_k\overline{u}_k\right)-1\right]\right.\\
&\quad +\left[E\left(v_j\overline{u}_j\right)-1\right]+\left[E\left(v_k\overline{u}_k\right)-1\right]+E\left(v_ju_k\right)E\left(\overline{u}_j\overline{v}_k\right)+E\left(v_j\overline{v}_k\right)E\left(\overline{v}_jv_k\right)\\
&\quad +\left(E\left|v_k\right|^2-1\right)+E\left(v_jv_k\right)E\left(\overline{v}_j\overline{v}_k\right)-E\left(v_j\overline{v}_k\right)E\left(\overline{u}_jv_k\right)-\left[E\left(v_j\overline{u}_j\right)-1\right]\\
&\quad \left.-E\left(v_jv_k\right)E\left(\overline{u}_j\overline{v}_k\right)-E\left(v_j\overline{u}_k\right)E\left(\overline{v}_ju_k\right)-\left[E\left(\overline{v}_ku_k\right)-1\right]-E\left(v_ju_k\right)E\left(\overline{v}_k\overline{v}_j\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
b_2 &= \frac{1}{n}\sum_{[n\vartheta_1/2]<j<k\leq[n\vartheta_2/2]}\left\{cum\left(v_j,\overline{v}_k,u_j,\overline{u}_k\right)+cum\left(v_j,\overline{v}_j,v_k,\overline{v}_k\right)-cum\left(v_j,\overline{v}_k,\overline{u}_j,v_k\right)\right.\\
&\quad \left.-\,cum\left(v_j,\overline{v}_k,\overline{v}_j,u_k\right)\right\}.
\end{aligned}$$

By routine extension of the proof of the term (4.8) in Robinson (1995*b*) to $[0,\pi]$, it follows that

$$a_1+a_2=\frac{1}{n}\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]}\frac{\log j}{j}\leq Cn^{-1}\left(\log^2\left(n\vartheta_2\right)-\log^2\left(n\vartheta_1\right)\right)=o(1),$$

and

$$b_1 + b_2 \leq C\left(n^{-1}\log^2 n\vartheta_2 + n^{-1/2}\left(\vartheta_2^{1/2} - \vartheta_1^{1/2}\right)\log n\vartheta_2 + n^{-1/2}\left(\vartheta_2^{1/2} - \vartheta_1^{1/2}\right)^2\right).$$

So, the finite dimensional distributions of the second term of (18) converge to zero in probability.

To complete the proof we need to show tightness. By Billingsley's (1968) Theorem 15.6 it suffices to show that

$$E\left|n^{-1/2}\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]} v_j\left(\overline{u}_j - \overline{v}_j\right)\right|^4 \leq C\left(F\left(\vartheta_2\right) - F\left(\vartheta_1\right)\right)^{1+\delta} \tag{19}$$

where $\delta > 0$ and $F(\vartheta)$ is a nondecreasing function. The left side of (19) is bounded by

$$\frac{1}{n^2}\left(|M_4| + 3M_2^2\right)$$

where M_r denotes the rth cumulant of $\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]} v_j\left(\overline{u}_j - \overline{v}_j\right)$. From the proof of the first term of (18), $n^{-2}M_2^2 \leq C\left(F\left(\vartheta_2\right) - F\left(\vartheta_1\right)\right)^{1+\delta}$, so it remains to examine the behaviour of $n^{-2}\left|M_4\right|$ which is

$$\frac{1}{n^2}\sum_{j_1,j_2,j_3,j_4=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]} cum\left(v_{j_1}w_{j_1}, v_{j_2}w_{j_2}, v_{j_3}w_{j_3}, v_{j_4}w_{j_4}\right) \tag{20}$$

where $w_j = \left(\overline{u}_j - \overline{v}_j\right)$. By Theorem 2.3.2 of Brillinger (1981) and denoting $X_{j1} = v_j$ and $X_{j2} = w_j$,

$$(20) = \frac{1}{n^2}\sum_{\upsilon} cum\left(X_{j\ell}; j\ell \in \upsilon_1\right) ... cum\left(X_{j\ell}; j\ell \in \upsilon_p\right)$$

where the summation is over all indecomposable partitions $\upsilon = \upsilon_1 \cup ... \cup \upsilon_p$. A typical component in $cum\left(X_{j\ell}; j\ell \in \upsilon_1\right)$ has k_1 elements v_j and k_2 elements w_j, so applying formulae of Brillinger [(1981), (2.6.3), page 26 and (2.10.3), page 39], we deduce after straightforward calculations that $cum\left(X_{j\ell}; j\ell \in \upsilon_1\right)$ is

$$\frac{\mu_{k_1+k_2}}{n^{(k_1+k_2)/2}}\int_{[-\pi,\pi]^{k_1+k_2-1}}\left(\frac{\alpha\left(\lambda^1 + ... + \lambda^{(k_1-1)} + \nu^1 + ... + \nu^{k_2}\right)\alpha\left(-\lambda^1\right)...\alpha\left(-\lambda^{k_1-1}\right)}{\alpha_{j_1}...\alpha_{j_{k_1}}}\right.$$
$$\left.\times\widetilde{\alpha}\left(-\nu^1\right)...\widetilde{\alpha}\left(-\nu^{k_2}\right)E_{j_1...j_{k_1}\ell_1...\ell_{k_2}}\left(\lambda^1, ..., \lambda^{(k_1-1)}, \nu^1, ..., \nu^{k_2}\right)d\lambda^1...d\lambda^{(k_1-1)}d\nu^1...d\nu^{k_2}\right.,$$

where

$$\begin{aligned} E_{j_1...j_{k_1}\ell_1...\ell_{k_2}}\left(\lambda^1, ..., \lambda^{(k_1-1)}, \nu^1, ..., \nu^{k_2}\right) &= D\left(\lambda_{j_1} - \left[\lambda^1 + ... + \lambda^{(k_1-1)} + \nu^1 + ... + \nu^{k_2}\right]\right)D\left(\lambda_{j_2} + \lambda^1\right) \\ &\times ... D\left(\lambda_{j_{k_1}} + \lambda^{(k_1-1)}\right)D\left(\nu^1 - \lambda_{\ell_1}\right)D\left(\nu^{k_2} - \lambda_{\ell_{k_2}}\right), \end{aligned}$$

$D(\lambda)=\sum_{t=1}^{n}e^{it\lambda}$ is the Dirichlet kernel and, say, $\widetilde{\alpha}\left(-\nu^{1}\right)=\alpha_{\ell_1}^{-1}\alpha\left(-\nu^{1}\right)-1$. But by a routine extension of Lemma 3 of Robinson (1995*b*) and observing that in each partitioned υ, each subindex $j_i, i=1,...,4$, appears only once

$$(20)\leq Cn^{-2}\left(\sum_{j=[n\vartheta_1/2]+1}^{[n\vartheta_2/2]}\frac{1}{j^{1/2}}\right)^{4}\leq C\left(\vartheta_2^{1/2}-\vartheta_1^{1/2}\right)^{4}.$$

Then conclude since $F(\vartheta)=\vartheta^{1/2}$ is a nondecreasing continuous function on $\vartheta\in[0,1]$. □

Define $g(\vartheta_1,\vartheta_2)=\int_0^{1/2}\psi(2\pi u,\vartheta_1)\psi(2\pi u,\vartheta_2)\,du-2\Phi(\vartheta_1)\Phi(\vartheta_2)$, where $\Phi(\vartheta)=\int_0^{1/2}\psi(2\pi u,\vartheta)\,du$ and write

$$c_s(\vartheta)=2n^{-3/2}\sum_{j=1}^{n/2}\psi_j(\vartheta)\cos(s\lambda_j). \tag{21}$$

Proposition 2 *Assuming A.1 and A.2, the finite dimensional distributions of $R_n^2(\vartheta)$ converge to those of a Gaussian process with covariance structure $g(\vartheta_1,\vartheta_2)+(\kappa_4+2)\,\Phi(\vartheta_1)\,\Phi(\vartheta_2)$.*

Proof. Fix $\vartheta_1,...,\vartheta_q$ and constants $a_1,...,a_q$. Observing that

$$R_n^2(\vartheta)=\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta)\right)\left(\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\varepsilon_{t-1}^2-1\right)\right)+\sum_{t=2}^{n}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s}(\vartheta), \tag{22}$$

by Cràmer-Wold device, it suffices to investigate the limiting distribution of

$$\sum_{p=1}^{q}a_pR_n^2(\vartheta_p)=\frac{1}{n}\sum_{j=1}^{n/2}\left(\sum_{p=1}^{q}a_p\psi_j(\vartheta_p)\right)\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\varepsilon_t^2-1\right)+\sum_{t=2}^{n}z_t, \tag{23}$$

where

$$z_t=\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s\left(\sum_{p=1}^{q}a_pc_{t-s}(\vartheta_p)\right),$$

suppressing any reference to n in z_t and $c_{t-s}(\vartheta_p)$, $p=1,...,q$.

The first and second terms on the right of (23) are uncorrelated since, by A.2, for all $t<s$, $E\left(\varepsilon_t\varepsilon_s\left(\varepsilon_r^2-1\right)\right)=0$. Next, by standard CLT for martingale differences, the first term on the right of (23) converges in distribution to a normal random variable with variance $(\kappa_4+2)\left(\sum_{p=1}^{q}a_p\Phi(\vartheta_p)\right)^2$. So, it remains to examine the behaviour of the second term on the right of (23). Because z_t forms a triangular array of a martingale difference sequence it suffices, see for instance Hall and Heyde (1980), to check

$$(a)\quad \sum_{t=2}^{n}E\left(z_t^2\,|\mathcal{F}_{t-1}\right)-\sum_{p_1=1}^{q}\sum_{p_2=1}^{q}a_{p_1}g(\vartheta_{p_1},\vartheta_{p_2})\,a_{p_2}\overset{P}{\to}0$$

$$(b)\quad \sum_{t=2}^{n}E\left(z_t^2\mathcal{I}(|z_t|>\delta)\right)\overset{P}{\to}0\quad\text{for all }\delta>0.$$

We begin with (a), whose left side is

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}\varepsilon_s^2\left(\sum_{p=1}^{q}a_p c_{t-s}(\vartheta_p)\right)^2-\sum_{p_1=1}^{q}\sum_{p_2=1}^{q}a_{p_1}g(\vartheta_{p_1},\vartheta_{p_2})a_{p_2} \tag{24}$$

$$+\sum_{t=2}^{n}\sum_{1=s_1\neq s_2}^{t-1}\varepsilon_{s_1}\varepsilon_{s_2}\left\{\left(\sum_{p=1}^{q}a_p c_{t-s_1}(\vartheta_p)\right)\left(\sum_{p=1}^{q}a_p c_{t-s_2}(\vartheta_p)\right)\right\}. \tag{25}$$

First we examine (24), which is

$$\sum_{t=1}^{n-1}\left(\varepsilon_t^2-1\right)\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2+\left(\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2-\sum_{p_1=1}^{q}\sum_{p_2=1}^{q}a_{p_1}g(\vartheta_{p_1},\vartheta_{p_2})a_{p_2}\right).$$

By Lemma 1 the second term converges to zero whereas the first term has zero mean and variance

$$(2+\kappa_4)\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2\right)^2$$

by A.2. Next, for $\vartheta\in[0,1]$,

$$|c_s(\vartheta)|\le n^{-1/2} \tag{26}$$

and by summation by parts, it is also $O(n/s)$, because by Zygmund (1977),

$$\left|\sum_{\ell=1}^{j}\psi_\ell(\vartheta)\cos(s\lambda_\ell)\right|=O\left(\frac{n}{s}\right) \tag{27}$$

if $1\le s\le n/2$, whereas for $[n/2]\le s\le n-1$, it follows because $\cos(s\lambda_\ell)=(-1)^\ell\cos((s-[n/2])\lambda_\ell)$ and $\psi(2\pi u,\vartheta)$ is an integrable function in u for all ϑ. So, we can restrict ourselves to the sum on $s\le[n/2]$. But,

$$\begin{aligned}\sum_{s=1}^{n/2}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2 &= \sum_{s=1}^{n/m}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2+\sum_{s=n/m+1}^{n/2}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2 \\ &= O\left(\frac{1}{n}\frac{n}{m}+\frac{1}{n}\sum_{s=n/m}s^{-2}\right)=O\left(\frac{1}{m}+\frac{m}{n^2}\right)\end{aligned} \tag{28}$$

because $\sum_{p=1}^{q}c_s(\vartheta_p)=\sum_{p=1}^{q}c_{n-s}(\vartheta_p)$, and where for the first and second terms on the right of (28) we have used (26) and (27) respectively and the definition of $c_s(\vartheta_p)$. So,

$$\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2\right)^2=O\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n/2}\left(\sum_{p=1}^{q}a_p c_s(\vartheta_p)\right)^2\right)^2\right)=O\left(\frac{n}{m^2}+\frac{m^2}{n^3}\right).$$

Then, by Markov's inequality we conclude that (24) $= o_p(1)$, choosing $m = n^{\xi}$ with $1/2 > \xi$.

To complete the proof of part (a), we need to examine (25), whose expectation is zero and its second moment has as typical element

$$\begin{aligned}
&\sum_{t,u=2}^{n} \sum_{s_1,s_2=1}^{\min(t-1,u-1)} c_{t-s_1}(\vartheta_{p_1})\, c_{u-s_1}(\vartheta_{p_2})\, c_{t-s_2}(\vartheta_{p_3})\, c_{u-s_2}(\vartheta_{p_4}) \\
= &\sum_{t=2}^{n} \sum_{s_1 \neq s_2 = 1}^{t-1} c_{t-s_1}(\vartheta_{p_1})\, c_{t-s_1}(\vartheta_{p_2})\, c_{t-s_2}(\vartheta_{p_3})\, c_{t-s_2}(\vartheta_{p_4}) \\
&+ \sum_{t=3}^{n} \sum_{u=2}^{t-1} \sum_{s_1 \neq s_2 = 1}^{u-1} c_{t-s_1}(\vartheta_{p_1})\, c_{u-s_1}(\vartheta_{p_2})\, c_{t-s_2}(\vartheta_{p_3})\, c_{u-s_2}(\vartheta_{p_4}).
\end{aligned}$$

The first term on the right is $o(1)$ proceeding as in the proof of (28), and by Schwarz inequality, the second term is bounded by

$$\begin{aligned}
&\sum_{t=3}^{n} \sum_{u=2}^{t-1} \left(\sum_{s=1}^{u-1} c_{t-s}(\vartheta_{p_1})\, c_{t-s}(\vartheta_{p_3}) \sum_{s=1}^{u-1} c_{u-s}(\vartheta_{p_2})\, c_{u-s}(\vartheta_{p_4}) \right) \\
\leq &\left(\sum_{t=1}^{n} c_t(\vartheta_{p_1})\, c_t(\vartheta_{p_3}) \right) \left(\sum_{t=3}^{n} \sum_{u=2}^{t-1} \sum_{s=t-u+1}^{t-1} c_s(\vartheta_{p_2})\, c_s(\vartheta_{p_4}) \right). \qquad (29)
\end{aligned}$$

Proceeding as with (28), the second bracketed term on the right of (29) is

$$\sum_{s=1}^{n-2} s(n-s-1)\, c_s(\vartheta_{p_2})\, c_s(\vartheta_{p_4}) \leq 2n \sum_{s=1}^{n/2} s c_s(\vartheta_{p_2})\, c_s(\vartheta_{p_4}) = O\left(\frac{nn^2}{nm^2} + \log\left(\frac{n}{m}\right) \right),$$

using (26) and (27) for the first and second terms on the right of the above inequality, respectively. Also, by (26) and (27), the first bracketed term on the right of (29) is $O\left(m^{-1} + mn^{-2}\right)$, so

$$(29) = O\left(\frac{n^2}{m^3} + \frac{\log n}{m} \right) = o(1),$$

by choosing $m = n^{\xi}$ with $\xi > 2/3$. (Observe that this choice of m is also valid for (28).) Then by Markov's inequality, (25) $= o_p(1)$, which concludes the proof of part (a).

To finish the proof, we need to prove part (b). To that end, it suffices to show the sufficient condition $\sum_{t=2}^{n} E\left(z_t^4\right) \to 0$, whose proof is similar to that in Robinson (1995b), so is omitted. □

Proposition 3 *Assuming A.1 and A.2, the process $R_n^2(\vartheta)$ equipped with the Skorohod's metric in $\mathbb{D}[0,1]$ is tight.*

Proof. The first term on the right of (22) is tight since $\left| n^{-1} \sum_{j=1}^{n/2} \left(\psi_j(\vartheta_1) - \psi_j(\vartheta_2) \right) \right| \leq |\vartheta_1 - \vartheta_2|^{\zeta}$ for $\zeta > 1/2$ by the definition of $\psi(\lambda, \vartheta)$ and $E\left(n^{-1/2} \sum_{t=2}^{n} \left(\varepsilon_t^2 - 1 \right) \right)^2 < C$ by A.2. So, it suffices to examine the tightness condition for the second term of (22). To that end, write

$$\mathcal{E}_n(\vartheta) = \sum_{t=2}^{n} \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s c_{t-s}(\vartheta).$$

First, by definition of $\psi_j(\vartheta)$, $\mathcal{E}_n(\vartheta)$ is a process which belongs to $\mathbb{D}[0,1]$, so by Billingsley's (1968) Theorem 15.6, it suffices to show the moment condition

$$E\left|\mathcal{E}_n(\vartheta_2) - \mathcal{E}_n(\vartheta_1)\right|^4 \le C\left(F(\vartheta_2) - F(\vartheta_1)\right)^{1+\delta}$$

for some $\delta > 0$ and where $F(\vartheta)$ is a nondecreasing function on $\vartheta \in [0,1]$.

Writing $\widetilde{c}_t = c_t(\vartheta_2) - c_t(\vartheta_1)$, the left side of the last display inequality is

$$E\left[\sum_{2=t_1\le t_2\le t_3\le t_4}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\varepsilon_{t_3}\varepsilon_{t_4}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=1}^{t_4-1}\varepsilon_{s_4}\widetilde{c}_{t_4-s_4}\right)\right].$$

By A.2, the above expectation is zero if $t_3 < t_4$, so it is

$$E\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\varepsilon_{t_3}^2\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=1}^{t_3-1}\varepsilon_{s_4}\widetilde{c}_{t_3-s_4}\right)\right]$$

$$= E\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=1}^{t_2-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)^2\right] \tag{30}$$

$$+2E\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_2-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=t_2+1}^{t_3-1}\varepsilon_{s_4}\widetilde{c}_{t_3-s_4}\right)\right] \tag{31}$$

$$+E\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=t_2+1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)^2\right]. \tag{32}$$

Since s_4 is greater than t_2 and so is than s_1, s_2, s_3 and t_1, by A.2, (31) $= 0$.

Because $s_3 > t_2$ and by A.2, (32) is

$$E\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=t_2+1}^{t_3-1}\widetilde{c}_{t_3-s_3}^2\right)\right],$$

which is zero unless $t_1 = t_2$, in which case, it becomes

$$\sum_{2=t_1\le t_3}^{n}\left(\sum_{s_1=1}^{t_1-1}\widetilde{c}_{t_1-s_1}^2\right)\left(\sum_{s_3=t_1+1}^{t_3-1}\widetilde{c}_{t_3-s_3}^2\right) \le \left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right)\right)^2 \tag{33}$$

because the quantities $\widetilde{c}_{t-s}^2$ are nonnegative and by Schwarz's inequality.

Next (30), which is zero unless $t_1 = t_2$, and thus it is

$$E\left(\varepsilon_t^4\right)\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}^2\widetilde{c}_{t_3-s}^2\right)+\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}^2\right)\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_3-s}^2\right)+2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}\widetilde{c}_{t_3-s}\right)^2. \tag{34}$$

The first term of (34) is

$$E\left(\varepsilon_t^4\right)\sum_{t_1=2}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}^2\right)\left(\sum_{t_3=t_1+1}^{n}\widetilde{c}_{t_3-s}^2\right)\le\frac{E\left(\varepsilon_t^4\right)}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right)$$

because $\widetilde{c}_{t-s}^2 \le n^{-1}\left|t-s\right|^{-2}$ by (27). The second term of (34) is bounded by $\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right)\right)^2$ since $\widetilde{c}_{t-s}^2 \ge 0$. Finally, the third term of (34), by the Schwarz inequality, is bounded by

$$2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}^2\right)\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_3-s}^2\right)\le 2\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right)\right)^2.$$

Thus (30) + (32) is bounded by

$$4\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right)\right)^2+\frac{E\left(\varepsilon_t^4\right)}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}_{t-s}^2\right),$$

which, proceeding as in Lemma 1, is bounded by

$$\begin{aligned}
&4\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)^2\\
&+\frac{E\left(\varepsilon_t^4\right)}{n}\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)\\
\le\ & D\left(\vartheta_2-\vartheta_1\right)^{2\zeta}
\end{aligned}$$

from the definition of $\psi\left(\lambda,\vartheta\right)$ and squared integrability. This concludes the proof of the proposition.□

Proof of Corollary 1

Proceeding as in the proof of Theorem 1, under H_{1n},

$$\begin{aligned}
S_n\left(\vartheta,\widehat{\theta}_n\right) &= \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}\left(\vartheta,j\right)\left(\frac{I_{nj}}{f_j}-1\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}\left(\vartheta,j\right)g_j\\
&\quad -\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}\left(\vartheta,j\right)\phi_j\left(\theta_0\right)\frac{I_{nj}}{f_j\left(\theta_0\right)}\left(\widehat{\theta}_n-\theta_0\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}\left(\vartheta,j\right)g_j\left(\frac{I_{nj}}{f_j}-1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f_j}-1\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\,g_j \\
&\quad -\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\,\phi_j(\theta_0)\left(1+\frac{1}{n^{1/2}}g_j\right)\left(\widehat{\theta}_n-\theta_0\right)+o_p\left(n^{-1/2}\right).
\end{aligned}
$$

Thus from the proof of Theorem 1, the right side is

$$
\begin{aligned}
n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right) &= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f_j}-1\right)+\int_0^{\pi}\left(\mathcal{I}(\lambda\leq\pi\vartheta)-\vartheta\right)g(\lambda)\,d\lambda \\
&\quad -\mathcal{G}(\vartheta)'\,n^{1/2}\left(\widehat{\theta}_n-\theta_0\right)+o_p(1).
\end{aligned}
$$

But, under H_{1n}, $n^{1/2}\left(\widehat{\theta}_n-\theta_0\right)\xrightarrow{d}N\left(\mathcal{A}^{-1}\int_0^{\pi}\phi(\lambda)\,g(\lambda)\,d\lambda;(4\pi)\,\mathcal{A}^{-1}\right)$. From here, the conclusion of the Corollary is standard. □

Proof of Theorem 2

The technique of proof uses arguments in Stute et al. (1998) and those of Theorem 1, but instead of applying Propositions 1 to 3, we apply Propositions 4 to 7 below. First, by Lemma 5, and the continuity of $\phi(\lambda,\theta)$ in θ,

$$
\begin{aligned}
\widehat{\theta}_n^{*} &= \widehat{\theta}_n-\left(\frac{1}{n}\sum_{j=1}^{n/2}\left(\phi_j\left(\widehat{\theta}_n\right)\phi_j\left(\widehat{\theta}_n\right)'\right)\right)^{-1}\frac{1}{n}\sum_{j=1}^{n/2}\left(f_j^{-1}\left(\widehat{\theta}_n\right)I_{nj}^{*}-1\right)(1+o_{p^*}(1)) \\
&= \widehat{\theta}_n-(2\pi)\,\mathcal{A}^{-1}\frac{1}{n}\sum_{j=1}^{n/2}\left(f_j^{-1}\left(\widehat{\theta}_n\right)I_{nj}^{*}-1\right)(1+o_{p^*}(1)) \qquad (35)
\end{aligned}
$$

where for the second equality, we have used the consistency of $\widehat{\theta}_n$ and the definition of $\mathcal{A}$. Then, proceeding as in the proof of Theorem 1, we have that

$$
n^{1/2}S_n^{*}\left(\vartheta,\widehat{\theta}_n^{*}\right)=n^{1/2}S_n^{*}\left(\vartheta,\widehat{\theta}_n\right)-\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\,\phi_j\left(\widehat{\theta}_n\right)'\frac{I_{nj}^{*}}{f_j\left(\widehat{\theta}_n\right)}\left(\widehat{\theta}_n^{*}-\widehat{\theta}_n\right)(1+o_{p^*}(1)).
$$

Proceeding as in the proof of Theorem 1 and using that $n^{-1}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\,\phi_j\left(\widehat{\theta}_n\right)\xrightarrow{P}(2\pi)^{-1}\mathcal{G}(\vartheta)$, we obtain

$$
n^{1/2}S_n^{*}\left(\vartheta,\widehat{\theta}_n^{*}\right)=R_n^{*}(\vartheta)+H_n^{*}(\vartheta)+o_{p^*}(1)
$$

where

$$
R_n^{*}(\vartheta)=\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta,\widehat{\theta}_n\right)\left(f_j^{-1}\left(\widehat{\theta}_n\right)I_{nj}^{*}-1\right),
$$

with $\psi_j\left(\vartheta,\widehat{\theta}_n\right)=\mathcal{J}\left(\vartheta,j\right)-\mathcal{G}\left(\vartheta\right)'\mathcal{A}^{-1}\phi_j\left(\widehat{\theta}_n\right)$ and

$$H_n^*\left(\vartheta\right)=\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}\left(\vartheta,j\right)\phi_j\left(\widehat{\theta}_n\right)\left(\frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n^*\right)}-1\right)\left(\widehat{\theta}_n^*-\widehat{\theta}_n\right).$$

Now the proof follows by Propositions 4 to 7. First, by Propositions 4 to 7 and Lemma 5, $\sup_{\vartheta\in(0,1)}\left|H_n^*\left(\vartheta\right)\right|=o_{p^*}\left(1\right)$. Next,

$$\begin{aligned}R_n^*\left(\vartheta\right) &= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta,\widehat{\theta}_n\right)\left(\frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n\right)}-(2\pi)I_{\varepsilon^*j}^*\right)+\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta,\widehat{\theta}_n\right)\left(\left(2\pi\right)I_{\varepsilon^*j}^*-1\right)\\ &= R_n^{1*}\left(\vartheta\right)+R_n^{2*}\left(\vartheta\right),\end{aligned}$$

where $I_{\varepsilon^*,j}^*=(2\pi n)^{-1}\left|\sum_{t=1}^n\varepsilon_t^*e^{-it\lambda_j}\right|^2$. Proposition 4 shows that $\sup_{\vartheta\in(0,1)}\left|R_n^{1*}\left(\vartheta\right)\right|=o_{p^*}\left(1\right)$. Proposition 5 shows that R_n^{2*} has a covariance structure, conditional on $\underset{\sim}{x}$, that converges in probability to $\mathcal{K}\left(\vartheta_1,\vartheta_2\right)$. Proposition 6 shows that the finite dimensional limiting distribution of R_n^{2*} is Gaussian centered at zero. Finally, Proposition 7 shows tightness of R_n^{*2}. Thus, combining Propositions 5, 6 and 7, $R_n^{2*}\overset{weakly}{\Longrightarrow}S_\infty$ in $\mathbb{D}\left[0,1\right]$ in probability, as defined by Gine and Zinn (1990). (Observe that the arguments used are valid under H_0 and H_{1n}.) Then, apply the continuous mapping theorem to conclude. □

Proposition 4 *Under the same conditions of Proposition 1,* $\sup_{\vartheta\in[0,1]}\left|R_n^{1*}\left(\vartheta\right)\right|=o_{p^*}\left(1\right)$.

Proof. The proof is omitted since it follows by identical steps as those in Proposition 1, but instead of Lemma 3 of Robinson (1995*b*) and Theorem 2 of Robinson (1995*a*), we use Lemmas 2 and 3 respectively where necessary. □

Proposition 5 *Under the same conditions of Proposition 4,*

$$E^*\left(R_n^{2*}\left(\vartheta\right)\;R_n^{2*}\left(\vartheta\right)\right)\overset{P}{\rightarrow}g\left(\vartheta_1,\vartheta_2\right)+\left(\kappa_4+2\right)\Phi\left(\vartheta_1\right)\Phi\left(\vartheta_2\right). \tag{36}$$

Proof. The left side of (36) is

$$\begin{aligned}&4E^*\left(\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^*\cos\left(\left(t-s\right)\lambda_j\right)\right. &(37)\\ &\left.\times\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^*\cos\left(\left(t-s\right)\lambda_j\right)\right)\\ &+\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\right)E^*\left(\frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)\right).\end{aligned}$$

By Lemma 4 and the properties of ε_t^*, the second term of (37) converges in probability to $(\kappa_4 + 2)\,\Phi(\vartheta_1)\,\Phi(\vartheta_2)$, because the empirical distribution function of ε_t^* converges uniformly to the distribution function of ε_t, implying that $cum^*(\varepsilon_t^*, \varepsilon_t^*, \varepsilon_t^*, \varepsilon_t^*) - \kappa_4 = o_p(1)$. Thus, we are left with the behaviour of the first term of (37). To this end, write

$$c_s\left(\vartheta;\widehat{\theta}_n\right) = 2n^{-3/2}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\cos(s\lambda_j)$$

and the triangular array of martingale difference sequence, conditional on $\underset{\sim}{x}$,

$$z_t^*(\vartheta) = \varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^* c_{t-s}\left(\vartheta;\widehat{\theta}_n\right),$$

where, for notational convenience, reference to n and $\widehat{\theta}$ in $z_t^*(\vartheta)$ and to n in $c_{t-s}\left(\vartheta;\widehat{\theta}_n\right)$ have been suppressed. Let $\mathcal{F}_t^*$ be the smallest sigma algebra generated by $\{\varepsilon_s^*, s \le t\}$conditional on $\underset{\sim}{x}$. Since $E^*(\varepsilon_t^*) = 0$ and $E^*\left(\varepsilon_t^{*2}\right) = 1$,

$$\begin{aligned} E^*\left(\sum_{t=2}^{n} z_t^*(\vartheta_1)\sum_{t=2}^{n} z_t^*(\vartheta_2)\middle|\mathcal{F}_{t-1}^*\right) &= \sum_{t=2}^{n}\sum_{s=1}^{t-1}\varepsilon_s^{*2}\left\{c_{t-s}\left(\vartheta_1;\widehat{\theta}_n\right)c_{t-s}\left(\vartheta_2;\widehat{\theta}_n\right)\right\} \\ &\quad + \sum_{t=2}^{n}\sum_{1=s_1\neq s_2}^{t-1}\left(\varepsilon_{s_1}^*\varepsilon_{s_2}^* c_{t-s_1}\left(\vartheta_1;\widehat{\theta}_n\right)c_{t-s_2}\left(\vartheta_2;\widehat{\theta}_n\right)\right). \end{aligned} \tag{38}$$

The first term on the right of (38) is

$$\sum_{t=1}^{n-1}\left(\varepsilon_t^{*2} - 1\right)\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right) + \sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right).$$

By Lemma 4, the second term converges in probability to $g(\vartheta_1, \vartheta_2)$, whereas the first term, conditional on $\underset{\sim}{x}$, has mean zero and variance which converges in probability to

$$(2 + \kappa_4)\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2,$$

because $cum^*(\varepsilon_t^*, \varepsilon_t^*, \varepsilon_t^*, \varepsilon_t^*) \xrightarrow{P} \kappa_4$.

Next, by the continuously differentiability of $\psi_j\left(\vartheta;\widehat{\theta}_n\right)$ for all $\vartheta \in [0,1]$ and $\left(\widehat{\theta}_n - \theta_0\right) = o_p(1)$,

$$\left|c_s\left(\vartheta;\widehat{\theta}_n\right)\right| = n^{-1/2}\left(1 + o_p(1)\right), \tag{39}$$

and by summation by parts, it is also $o_p(n/s)$, because by $\left(\widehat{\theta}_n - \theta_0\right) = o_p(1)$ and Zygmund (1977),

$$\left|\sum_{\ell=1}^{j}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\cos(s\lambda_\ell)\right| = O_p\left(\frac{n}{s}\right), \tag{40}$$

if $1 \le s \le n/2$, whereas for $[n/2] \le s \le n-1$ because $\cos(s\lambda_\ell) = (-1)^\ell \cos((s-[n/2])\lambda_\ell)$ and $\psi(2\pi u, \vartheta) = \psi(2\pi u, \vartheta; \theta_0)$ is an integrable function for all $\vartheta \in [0,1]$. So, we can restrict ourselves to the sum on $s \le [n/2]$. But,

$$\begin{aligned}\sum_{s=1}^{n/2} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right) &= \sum_{s=1}^{n/m} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right) + \sum_{s=n/m+1}^{n/2} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right) \\ &= O_p\left(\frac{1}{n}\frac{n}{m} + \frac{1}{n}\sum_{s=n/m+1} s^{-2}\right) = O_p\left(\frac{1}{m} + \frac{m}{n^2}\right), \qquad (41)\end{aligned}$$

because $c_s\left(\vartheta;\widehat{\theta}_n\right) = c_{n-s}\left(\vartheta;\widehat{\theta}_n\right)$ and by (39) and (40). Thus,

$$\begin{aligned}\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2 &= O\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n/2} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2\right) \\ &= O_p\left(\frac{n}{m^2} + \frac{m^2}{n^3}\right) = o_p(1),\end{aligned}$$

choosing $m = n^\zeta$ with $1/2 > \zeta$. That finishes the proof that the first term on the right of (38) converges in probability to $g(\vartheta_1, \vartheta_2)$.

To complete the proof of the proposition, we are left to prove that the second term on the right of (38) $= o_{p^*}(1)$. But, conditional on $\underset{\sim}{x}$, its first moment is 0, and its second moment is

$$\begin{aligned}&\sum_{t,u=2}^{n}\sum_{s_1,s_2=1}^{\min(t-1,u-1)} c_{t-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{t-s_2}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_2}\left(\vartheta;\widehat{\theta}_n\right) \\ =\ &\sum_{t=2}^{n}\sum_{s_1\neq s_2=1}^{t-1} c_{t-s_1}^2\left(\vartheta;\widehat{\theta}_n\right) c_{t-s_2}^2\left(\vartheta;\widehat{\theta}_n\right) \\ &+\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s_1\neq s_2=1}^{u-1} c_{t-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{t-s_2}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_2}\left(\vartheta;\widehat{\theta}_n\right).\end{aligned}$$

The first term on the right of the above equation is $o_p(1)$, by (41), while the second term on the right, by Schwarz inequality, is bounded by

$$\sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{s=1}^{u-1} c_{t-s}^2\left(\vartheta;\widehat{\theta}_n\right)\sum_{s=1}^{u-1} c_{u-s}^2\left(\vartheta;\widehat{\theta}_n\right)\right) \le \left(\sum_{t=1}^{n} c_t^2\left(\vartheta;\widehat{\theta}_n\right)\right)\left(\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=t-u+1}^{t-1} c_s^2\left(\vartheta;\widehat{\theta}_n\right)\right). \qquad (42)$$

The second bracketed term on the right of (42) is

$$\sum_{s=1}^{n-2} s(n-s-1)\, c_s^2\left(\vartheta;\widehat{\theta}_n\right) \quad\le\quad 2n\sum_{s=1}^{n/2} s c_s^2\left(\vartheta;\widehat{\theta}_n\right)$$

$$\leq 2n \sum_{s=2}^{n/m} sc_s^2 \left(\vartheta; \widehat{\theta}_n\right) + 2n \sum_{s=n/m+1} sc_s^2 \left(\vartheta; \widehat{\theta}_n\right)$$
$$= O_p \left(\frac{nn^2}{nm^2} + \log\left(\frac{n}{m}\right) \right),$$

because $c_s^2 \left(\vartheta; \widehat{\theta}_n\right) = n^{-1} s^{-2} (1 + o_p(1))$. Thus, together with (41), it implies that

$$(42) = O_p \left(\frac{n^2}{m^3} + \frac{\log n}{m} \right) = o_p(1),$$

by choosing $m = n^{\zeta}$ with $\zeta > 2/3$. That concludes the proof of Proposition 5. □

Introduce the following

Definition 1 *We say that* $Z_n^* = o_{p^*}(1)$*, if for all* $\delta > 0$, $\Pr\left\{ |Z_n^*| > \delta | \underset{\sim}{x} \right\} \overset{P}{\to} 0$.

Proposition 6 *Under the same conditions of Proposition 4, the finite dimensional distributions of* R_n^{2*} *converge in bootstrap law to those of a centered Gaussian process.*

Proof. Fix $\vartheta_1, ..., \vartheta_q$ and constants $a_1, ..., a_q$. By Cràmer-Wold device, it suffices to examine the limit distribution of

$$\sum_{p=1}^{q} a_p \left(\frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j \left(\vartheta_p; \widehat{\theta}_n\right) \left((2\pi) I_{\varepsilon^*,j}^* - 1 \right) \right)$$
$$= \frac{1}{n} \sum_{j=1}^{n/2} \left(\sum_{p=1}^{q} a_p \psi_j \left(\vartheta_p; \widehat{\theta}_n\right) \right) \frac{1}{n^{1/2}} \sum_{t=1}^{n} \left(\varepsilon_t^{*2} - 1\right) + \sum_{t=2}^{n} z_t^* \left(\underset{\sim}{\vartheta}\right), \tag{43}$$

where

$$z_t^* \left(\underset{\sim}{\vartheta}\right) = \varepsilon_t^* \frac{2}{n^{3/2}} \sum_{j=1}^{n/2} \left(\sum_{p=1}^{q} \left(a_p \psi_j \left(\vartheta_p; \widehat{\theta}_n\right) \right) \left(\sum_{s=1}^{t-1} \varepsilon_s^* \cos\left((t-s)\lambda_j\right) \right) \right)$$

with $\underset{\sim}{\vartheta} = (\vartheta_1, ..., \vartheta_p)'$. Proceeding as in the proof of Proposition 2, the terms on the right of (43) are uncorrelated. Moreover, it is straightforward to show that the second moment of the first term of (43) converges in probability to $(\kappa_4 + 2) \left(\sum_{p=1}^{q} a_p \Phi(\vartheta_p) \right)^2$.

So, we are left to examine the second term of (43). Proceeding as in the proof of Proposition 5,

$$\sum_{j=1}^{n/2} E^* \left(z_t^{*2} \left(\underset{\sim}{\vartheta}\right) | \mathcal{F}_{t-1}^* \right) \overset{P}{\to} \sum_{p_1=1}^{q} \sum_{p_2=1}^{q} a_{p_1} g\left(\vartheta_{p_1}, \vartheta_{p_2}\right) a_{p_2},$$

and so, it remains to verify the Lindeberg's condition, that is $\forall \delta > 0$,

$$\sum_{t=2}^{n} E^* \left[z_t^{*2} \left(\underset{\sim}{\vartheta}\right) \mathcal{I} \left(\left| z_t^* \left(\underset{\sim}{\vartheta}\right) \right| > \delta \right) \right] \overset{P}{\to} 0,$$

or the sufficient condition $\sum_{t=2}^{n} E^{*}\left[z_t^{*4}\left(\underset{\sim}{\vartheta}\right)\right] \overset{P}{\to} 0$, whose proof is essentially that of Robinson (1995*b*) proceeding as in Proposition 2, and thus is omitted. □

Proposition 7 *Under the same conditions of Proposition 4, conditional on* $\underset{\sim}{x}$, $R_n^{2*}(\vartheta)$ *is tight.*

Proof. Let $\mathcal{E}_n^{*}\left(\vartheta;\widehat{\theta}_n\right) = n^{-1/2}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right) n^{-1}\sum_{t=2}^{n}\varepsilon_t^{*}\sum_{s=1}^{t-1}\varepsilon_s^{*}\cos\left((t-s)\lambda_j\right)$. Then

$$R_n^{2*}(\vartheta) = \left(\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\right) + \mathcal{E}_n^{*}\left(\vartheta;\widehat{\theta}_n\right). \tag{44}$$

The first term on the right of (44) is tight since $n^{-1/2}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right) = O_p(1)$ and

$$\left|\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)-\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\right| \overset{P}{\to} v\left(\vartheta_1,\vartheta_2\right) \le \left|\vartheta_1-\vartheta_2\right|^{\zeta},$$

with $\zeta > 1/2$ as we now prove. The left side is bounded by

$$\left|\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\theta_0\right)-\psi_j\left(\vartheta_2;\theta_0\right)\right| + \left|\frac{1}{n}\sum_{j=1}^{n/2}\left(\mathcal{G}\left(\vartheta_1\right)-\mathcal{G}\left(\vartheta_2\right)\right)'\mathcal{A}^{-1}\left(\phi_j\left(\widehat{\theta}_n\right)-\phi_j\left(\theta_0\right)\right)\right|,$$

where the first term is as in Proposition 3, bounded by $C\left|\vartheta_1-\vartheta_2\right|^{\zeta}$ and the second as $\widehat{\theta}_n-\theta_0 = O_p\left(n^{-1/2}\right)$ and differentiability of $\mathcal{G}(\vartheta)$ also bounded by $C\left|\vartheta_1-\vartheta_2\right|^{\zeta}$. To show tightness for the second term on the right of (44) it suffices to check the moment condition

$$\begin{aligned} E^{*}\left|\mathcal{E}_n^{*}\left(\vartheta_2;\widehat{\theta}_n\right)-\mathcal{E}_n^{*}\left(\vartheta_1;\widehat{\theta}_n\right)\right|^4 &\le C\left[F_n\left(\vartheta_2;\widehat{\theta}_n\right)-F_n\left(\vartheta_1;\widehat{\theta}_n\right)\right]^{1+\delta} \\ &\overset{P}{\to} C\left(F\left(\vartheta_2;\theta_0\right)-F\left(\vartheta_1;\theta_0\right)\right)^{1+\delta}. \end{aligned}$$

Denoting $\widetilde{c}_t\left(\widehat{\theta}_n\right) = c_t\left(\vartheta_2;\widehat{\theta}_n\right)-c_t\left(\vartheta_1;\widehat{\theta}_n\right)$, the left side of the above inequality is bounded by

$$\begin{aligned} CE^{*}\Bigg[\sum_{2=t_1\le t_2\le t_3\le t_4}^{n} &\varepsilon_{t_1}^{*}\varepsilon_{t_2}^{*}\varepsilon_{t_3}^{*}\varepsilon_{t_4}^{*}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}^{*}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}^{*}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right) \\ &\times\left(\sum_{s_3=1}^{t_3-1}\varepsilon_{s_3}^{*}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_4=1}^{t_4-1}\varepsilon_{s_4}^{*}\widetilde{c}_{t_4-s_4}\left(\widehat{\theta}_n\right)\right)\Bigg]. \end{aligned}$$

From here, the reminder of the proof is identical to that of Proposition 3 but instead of Lemma 1, we use by Lemma 3, that $c_t\left(\vartheta;\widehat{\theta}_n\right)-c_t\left(\vartheta,\theta_0\right)\overset{a.s}{\to} 0$ and ε_t^{*}, conditional on $\underset{\sim}{x}$, is iid with mean 0 and variance 1. □

7. TECHNICAL LEMMAS

Lemma 1 *Let $c_s(\vartheta)$ be as given in (21). Then, for all ϑ_1 and $\vartheta_2 \in [0,1]$,*

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s(\vartheta_1)\, c_s(\vartheta_2) = g(\vartheta_1,\vartheta_2)(1+o(1)). \tag{45}$$

Proof. The left side of (45) is

$$\begin{aligned}
& 4n^{-3}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\psi_j(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) \\
& +4n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{j_2=1,\neq j_2}^{n/2}\psi_{j_2}(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s\lambda_{j_1})\cos(s\lambda_{j_2}) \\
= \; & 4n^{-3}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\psi_j(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) \qquad (46)\\
& +2n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{j_2=1,\neq j_2}^{n/2}\psi_{j_2}(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s(\lambda_{j_1}+\lambda_{j_2}))+\cos(s(\lambda_{j_1}-\lambda_{j_2})).
\end{aligned}$$

Because, see for instance Robinson (1995*b*),

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) = \frac{(n-1)^2}{4} \tag{47}$$

and for $\lambda_{j_1} \neq \lambda_{j_2}$

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s(\lambda_{j_1}+\lambda_{j_2}))+\cos(s(\lambda_{j_1}-\lambda_{j_2})) = -n, \tag{48}$$

the right side of (46) is

$$\left(\frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\psi_j(\vartheta_2)\right) - 2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{j_2=1,\neq j_2}^{n/2}\psi_{j_2}(\vartheta_2)\right) = g(\vartheta_1,\vartheta_2)(1+o(1)),$$

from the definition of $\psi_j(\vartheta)$ and the continuity of $\mathcal{G}(\vartheta)$. □

Lemma 2 *Let $K(\lambda)$ be the Fèjer kernel. Assuming A1 and A2, as $n\to\infty$,*

$$\int_{-\pi}^{\pi}\left|\frac{\alpha\left(\lambda;\widehat{\theta}_n\right)}{\alpha\left(\lambda_j;\widehat{\theta}_n\right)}-1\right|^2 K(\lambda-\lambda_j)\,d\lambda = O_p\left(\frac{1}{j}\right).$$

Proof. The proof is essentially the same as that of Lemma 3 of Robinson (1995*b*). For $\delta\in(2\lambda_j,\pi)$, we split the integral up as follows:

$$\int_{-\pi}^{-\delta}+\int_{-\delta}^{-\lambda_j/2}+\int_{-\lambda_j/2}^{\lambda_j/2}+\int_{\lambda_j/2}^{2\lambda_j}+\int_{2\lambda_j}^{\delta}+\int_{\delta}^{\pi}. \tag{49}$$

As in Robinson's (1995b) Lemma 3, conditional on the sample, the first integral is bounded by

$$\begin{aligned} & \frac{1}{\pi f\left(\lambda_j;\widehat{\theta}_n\right)}\left\{\frac{1}{n\delta^2}\int_{-\pi}^{\pi} f\left(\lambda;\widehat{\theta}_n\right)d\lambda + f\left(\lambda_j;\widehat{\theta}_n\right)\frac{2\pi}{n\delta^2}\right\} \\ = & \frac{1}{\pi f\left(\lambda_j;\widehat{\theta}_n\right)}\left\{\frac{\widehat{\sigma}_\varepsilon^2}{n\delta^2} + f\left(\lambda_j;\widehat{\theta}_n\right)\frac{2\pi}{n\delta^2}\right\}\end{aligned}$$

since $\int_{-\pi}^{\pi} f\left(\lambda;\widehat{\theta}_n\right)d\lambda = \widehat{\sigma}_\varepsilon^2$. Because, by A.1, $f(\lambda;\theta)$ is differentiable in θ and $\left(\widehat{\theta}_n-\theta_0\right) = O_p\left(n^{-1/2}\right)$, the right side of the above equation is $O_p\left(j^{-1}\right)$, also $\left|\int_\delta^\pi\right|$ has the same bound by the same arguments. Proceeding as in Robinson (1995b), conditional on $\underset{\sim}{x}$, the contribution due to second integral in (49) is bounded by

$$\frac{1}{2\pi n f\left(\lambda_j;\widehat{\theta}_n\right)}\left\{\int_{\lambda_j/2}^{\pi}\lambda^{2\widehat{d}_n}d\lambda + f\left(\lambda_j;\widehat{\theta}_n\right)\int_{\lambda_j/2}^{\pi}\lambda^{-2}d\lambda\right\}.$$

But, again by A1 and $\left(\widehat{\theta}_n-\theta_0\right) = O_p\left(n^{-1/2}\right)$, the above expression is $O_p\left(j^{-1}\right)$, and also $\left|\int_{2\lambda_j}^{\delta}\right|$ has the same bound. Finally, the contribution due to the third integral in (49) is, as in Lemma 3 of Robinson (1995b) and the previous arguments, $O_p\left(j^{-1}\right)$. That completes the proof. □

Lemma 3 *Denote* $v_j^* = I_{n,j}^*/\left|\alpha_j\left(\widehat{\theta}_n\right)\right|$. *Then, under A.1 and A.2 and* $k<j$,

$$\begin{aligned}(a)\ E^*\left(v_j^*\overline{v}_j^*\right)-1 &= O_p\left(\frac{\log j}{j}\right),\quad (b)\ E^*\left(v_j^*v_j^*\right) = O_p\left(\frac{\log j}{j}\right),\\ (c)\ E^*\left(v_j^*\overline{v}_k^*\right) &= O_p\left(\frac{\log j}{k}\right)\ \text{and}\ \ (d)\ E^*\left(v_j^*v_k^*\right) = O_p\left(\frac{\log j}{k}\right).\end{aligned}$$

Proof. The proofs follows by using the same steps as in Theorem 2 of Robinson and Lemma 3, and so is omitted. □

Lemma 4 *Let* $c_s\left(\vartheta;\widehat{\theta}_n\right) = 2n^{-3/2}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\cos\left(s\lambda_j\right)$, *where* $\psi_j\left(\vartheta;\theta\right) = \psi\left(j/n,\vartheta;\theta\right)$. *Then, for all* ϑ_1 *and* $\vartheta_2\in[0,1]$,

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right) - g\left(\vartheta_1,\vartheta_2\right)\xrightarrow{P} 0.$$

Proof. As in Lemma 1, the first term on the left side of the last display expression is

$$\begin{aligned}& 4n^{-3}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2\left(s\lambda_j\right) \qquad (50)\\ & +2n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{j_2=1,\neq j_2}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos\left(s\left(\lambda_{j_1}+\lambda_{j_2}\right)\right)+\cos\left(s\left(\lambda_{j_1}-\lambda_{j_2}\right)\right).\end{aligned}$$

By Lemma 1, c.f. (47) and (48), the left side of (50) is

$$\left(\frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\right)-2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{j_2=1,\neq j_2}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\right). \quad (51)$$

Because $\psi_j(\vartheta;\theta)$ is continuously differentiable in θ, by the Mean Value Theorem, continuity of the derivative and that, by Giraitis and Surgailis (1990), $\left(\widehat{\theta}_n-\theta_0\right)\xrightarrow{P}0$,

$$(51)-\frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta_1;\theta_0)\,\psi_j(\vartheta_2;\theta_0)\right)+2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1;\theta_0)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}(\vartheta_2;\theta_0)\xrightarrow{P}0.$$

But, by Lemma 1, the last two terms on the left of the above equation converges to $-g(\vartheta_1,\vartheta_2)$, which concludes the proof of the lemma. □

Lemma 5 *Let $\widehat{\theta}_n$ be such that it converges almost surely to $\theta_1\in\Theta$. Then*

$$\theta_n^*-\widehat{\theta}_n=o_{p^*}(1).$$

Proof. Conditional on the sample $\underset{\sim}{x}$, x_t^* is, by construction, a linear covariance stationary process with spectral density $f\left(\lambda;\widehat{\theta}_n\right)$, where the innovations ε_t^* are iid with mean 0 and variance 1. Moreover, because $f\left(\lambda;\widehat{\theta}_n\right)$ satisfies $\int_{-\pi}^{\pi}\log\left(f\left(\lambda;\widehat{\theta}_n\right)\right)d\lambda>-\infty$, it implies that the sequence x_t^* is ergodic, because it possesses a spectral distribution function which does not have atom at frequency 0. Then, proceeding as in the proof of Lemma 1 of Hannan (1973), uniformly in $\theta\in\Theta$,

$$\frac{1}{n}\sum_{j=1-n/2}^{n/2}\frac{I_{nj}^*}{f_j(\theta)}-\int_{-\pi}^{\pi}\frac{f\left(\lambda;\widehat{\theta}_n\right)}{f(\lambda;\theta)}d\lambda\xrightarrow{P}0.$$

Now proceed as in the proof of Theorem 1 of Hannan (1973) to conclude. □

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TABLE 1

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is white noise. Observations generated according to a $N(0,1)$ and a $Uniform(-0.5, 0.5)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		Asymptotic				Bootstrap			
		$Normal$		$Uniform$		$Normal$		$Uniform$	
		C_n	B_n	C_n	B_n	C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0064	0.0026	0.0094	0.0034	0.0102	0.0108	0.0126	0.0124
	$\alpha = 0.05$	0.0368	0.0148	0.0452	0.0172	0.0492	0.0478	0.0534	0.0510
	$\alpha = 0.10$	0.0852	0.0330	0.0894	0.0408	0.0960	0.0976	0.0982	0.0978
$n = 50$	$\alpha = 0.01$	0.0172	0.0104	0.0184	0.0092	0.0120	0.0120	0.0116	0.0110
	$\alpha = 0.05$	0.0594	0.0296	0.0662	0.0340	0.0476	0.0486	0.0514	0.0538
	$\alpha = 0.10$	0.1140	0.0624	0.1210	0.0734	0.0976	0.0930	0.1010	0.0988
$n = 100$	$\alpha = 0.01$	0.0118	0.0064	0.0122	0.0070	0.0096	0.0100	0.0096	0.0098
	$\alpha = 0.05$	0.0592	0.0346	0.0584	0.0366	0.0536	0.0540	0.0512	0.0542
	$\alpha = 0.10$	0.1100	0.0732	0.1152	0.0772	0.1010	0.1010	0.1042	0.1042
$n = 150$	$\alpha = 0.01$	0.0120	0.0064	0.0112	0.0070	0.0108	0.0108	0.0098	0.0102
	$\alpha = 0.05$	0.0596	0.0370	0.0610	0.0380	0.0580	0.0518	0.0568	0.0516
	$\alpha = 0.10$	0.1136	0.0796	0.1148	0.0818	0.1076	0.1064	0.1066	0.1102

TABLE 2

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is an $AR(1)$. Observations generated as $x_t = 0.5x_{t-1} + \varepsilon_t$, $\varepsilon_t \sim iid\ N(0,1)$ and $\varepsilon_t \sim iidUniform(-0.5, 0.5)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$Normal\ \varepsilon_t$		$Uniform\ \varepsilon_t$	
		C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0066	0.0070	0.0056	0.0064
	$\alpha = 0.05$	0.0438	0.0396	0.0432	0.0408
	$\alpha = 0.10$	0.0840	0.0832	0.0856	0.0844
$n = 50$	$\alpha = 0.01$	0.0092	0.0104	0.0098	0.0122
	$\alpha = 0.05$	0.0518	0.0496	0.0498	0.0470
	$\alpha = 0.10$	0.0952	0.0964	0.0944	0.0964
$n = 100$	$\alpha = 0.01$	0.0088	0.0080	0.0086	0.0082
	$\alpha = 0.05$	0.0458	0.0466	0.0490	0.0484
	$\alpha = 0.10$	0.0944	0.0954	0.0946	0.1000
$n = 150$	$\alpha = 0.01$	0.0116	0.0130	0.0108	0.0116
	$\alpha = 0.05$	0.0482	0.0524	0.0516	0.0564
	$\alpha = 0.10$	0.0962	0.0996	0.1030	0.1016

TABLE 3

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is a $FARIMA\,(0,d,0)$ process with $d=0.2, 0.3, 0.4$ and the innovations ε_t are $N\,(0,1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$d=0.2$		$d=0.3$		$d=0.4$	
		C_n	B_n	C_n	B_n	C_n	B_n
$n=25$	$\alpha=0.01$	0.0028	0.0044	0.0034	0.0048	0.0064	0.0070
	$\alpha=0.05$	0.0290	0.0334	0.0332	0.0362	0.0460	0.0500
	$\alpha=0.10$	0.0680	0.0758	0.0766	0.0810	0.0952	0.0968
$n=50$	$\alpha=0.01$	0.0046	0.0064	0.0056	0.0070	0.0068	0.0074
	$\alpha=0.05$	0.0340	0.0376	0.0366	0.0408	0.0448	0.0464
	$\alpha=0.10$	0.0766	0.0808	0.0854	0.0864	0.0942	0.0958
$n=100$	$\alpha=0.01$	0.0080	0.0094	0.0100	0.0108	0.0088	0.0102
	$\alpha=0.05$	0.0408	0.0452	0.0448	0.0464	0.0438	0.0442
	$\alpha=0.10$	0.0882	0.0892	0.0926	0.0938	0.0862	0.0912
$n=150$	$\alpha=0.01$	0.0072	0.0074	0.0082	0.0080	0.0054	0.0058
	$\alpha=0.05$	0.0480	0.0466	0.0498	0.0476	0.0414	0.0430
	$\alpha=0.10$	0.0952	0.0972	0.0968	0.1004	0.0890	0.0914

TABLE 4

Proportion of rejections, in 5000 Monte Carlo experiments under H_1, when testing that the process is an $AR(1)$ and the observations are generated according to a $FARIMA(0, d, 0)$ process with $d = 0.2, 0.3, 0.4$, and the innovations ε_t are $N(0, 1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$d = 0.2$		$d = 0.3$		$d = 0.4$	
		C_n	B_n	C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0334	0.0280	0.0538	0.0442	0.0514	0.0398
	$\alpha = 0.05$	0.1112	0.0990	0.1634	0.1444	0.1684	0.1404
	$\alpha = 0.10$	0.1786	0.1714	0.2454	0.2264	0.2614	0.2292
$n = 50$	$\alpha = 0.01$	0.0602	0.0412	0.1106	0.0856	0.1412	0.0994
	$\alpha = 0.05$	0.1476	0.1374	0.2536	0.2212	0.3402	0.2816
	$\alpha = 0.10$	0.2218	0.2048	0.3552	0.3180	0.4594	0.4050
$n = 100$	$\alpha = 0.01$	0.0982	0.0680	0.2536	0.1978	0.4278	0.3426
	$\alpha = 0.05$	0.2224	0.1958	0.4344	0.3766	0.6410	0.5792
	$\alpha = 0.10$	0.3202	0.2898	0.5344	0.4914	0.7290	0.6840
$n = 150$	$\alpha = 0.01$	0.1328	0.0968	0.3440	0.2786	0.5998	0.5188
	$\alpha = 0.05$	0.2816	0.2344	0.5356	0.4668	0.7662	0.7042
	$\alpha = 0.10$	0.3786	0.3368	0.6360	0.5740	0.8350	0.7878

TABLE 5

Proportion of rejections, in 5000 Monte Carlo experiments, under H_1 when testing that the process is a $FARIMA\,(0,d,0)$ and the observations are generated according to an $AR\,(1)$ with parameter 0.5 and the innovations ε_t are $N\,(0,1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		C_n	B_n
$n=25$	$\alpha=0.01$	0.0260	0.0318
	$\alpha=0.05$	0.1344	0.1358
	$\alpha=0.10$	0.2448	0.2316
$n=50$	$\alpha=0.01$	0.0560	0.0538
	$\alpha=0.05$	0.2082	0.1892
	$\alpha=0.10$	0.3402	0.3116
$n=100$	$\alpha=0.01$	0.1350	0.1156
	$\alpha=0.05$	0.3890	0.3412
	$\alpha=0.10$	0.5436	0.4862
$n=150$	$\alpha=0.01$	0.2540	0.2122
	$\alpha=0.05$	0.5518	0.4792
	$\alpha=0.10$	0.6982	0.6340