Gaussian Estimation of a Continuous Time Macroeconomic Model of the United Kingdom with Unobservable Stochastic Trends

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Abstract

This paper describes the formulation, analysis, and estimation of a new continuous time macroeconometric model of the United Kingdom. The model differs from earlier continuous time macroeconometric models in that it incorporates unobservable stochastic trends to represent such variables as technical progress. The estimation of its parameters is the first application of the algorithm of Bergstrom (1997, *Econometric Theory* 13, 467-505) for the Gaussian estimation of continuous time dynamic models with unobservable stochastic trends. The parameters have been estimated from quarterly data for the period 1975-94 and the post-sample forecasts tested against quarterly data for the period 1995-96. In addition to obtaining plausible estimates of the parameter values and satisfactory post-sample forecasts, we have carried out the steady state and stability analysis and shown that the model generates plausible long-run behaviour.

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1 INTRODUCTION

Unobservable trend variables are, usually, represented, in both discrete and continuous time econometric models, by simple deterministic functions of time, which are introduced into appropriate equations of the model. For example, technical progress is often allowed for by incorporating, in the production function, a trend term implying a constant rate of growth of productivity. A more flexible and realistic way of allowing for technical progress is to incorporate, in the production function, a stochastic trend in the form of an integrated random process with constant drift and volatility parameters. Similar stochastic trends can be introduced into the model to represent other unobservable phenomena that are believed to influence the behaviour of the endogenous variables.

During the last 15 years, there has been extensive work on the problems of statistical influence in cointegrated systems (see, especially, Phillips and Durlauf, 1986, Phillips, 1987, 1991a, 1991b, 1995, Engle and Granger, 1987, Johansen, 1988, 1991, Park and Phillips, 1988, 1989, and Franses 1996). Nearly all of this work is concerned with models which are formulated in discrete time and in which all the stochastic trend variables are observable. There has, however, been some work by Harvey and Stock (1988) and Bergstrom (1997) on the estimation of the parameters of continuous time dynamic models with unobservable stochastic trends.

The article of Bergstrom (1997) develops a method of obtaining exact Gaussian estimates of the parameters of a system of mixed order stochastic differential equations, with unobservable stochastic trends, from a sample of discrete stock and flow data. The method is an extension of the general method developed in a series of earlier articles (Bergstrom, 1983, 1985, 1986, 1990, Agbeyegbe, 1984, 1987, 1988, Chambers, 1991, Nowman 1991) for the estimation of the parameters of continuous time models from discrete data. The general method involves the derivation of a VARMAX model, whose coefficient matrices are explicit functions of the parameters of the continuous time model, and which is exactly satisfied by the discrete data, when either the model is closed or the exogenous variables satisfy certain conditions.

One of the major aims of this paper is to demonstrate the feasibility of using the algorithm of Bergstrom (1997) in the estimation of a new 15 equation continuous time macroeconometric model of the United Kingdom. This new model is also a development of earlier work, starting with the continuous time UK model of Bergstrom and Wymer (1976). It is most closely related to the more recent UK model of Bergstrom, Nowman, and Wymer (1992). The main changes are that the deterministic trends are replaced by stochastic trends, and the new model is formulated as a system of mixed first and second order differential equations, whereas the model of Bergstrom et. al. (1992) was formulated as a system of second order equations only.

In addition to being the first continuous time macroeconometric model, the model of Bergstrom and Wymer (1976) had several innovative features, which have been retained in most of the continuous time macroeconometric models for which it has served as a prototype, including the model of Bergstrom et. al. (1992) and models of several other leading industrial countries (see Bergstrom, 1996, for a complete list of these models.) In particular, it made intensive use of economic theory to obtain a parsimonious parametrization and cross equation restrictions, and it was designed in such a way as to permit a rigorous mathematical analysis of its steady state and asymptotic stability properties, thus providing a check on its capacity to generate plausible long-run behaviour. These features are also retained in the new model developed in this paper.

The specification of the model will be discussed in the next section, the steady state and stability analysis in Section 3 and the parameter estimates and various results derived from these estimates in Section 4. Some conclusions will be drawn in Section 5.

2 SPECIFICATION OF MODEL

The model, in its deterministic form, is set out below. It is a system of 18 mixed first and second order nonlinear differential equations with 63 structural parameters, including a vector β of 33 long-run parameters, a vector γ of 27 speed of adjustment parameters, and a vector λ of 3 trend or drift parameters. The variables and equations of the model, in its deterministic form, are as follows.

Endogenous Variables

C = real private consumption

E_n = real non-oil exports

F = real current transfers abroad

I = volume of imports

K_h = residential fixed capital

K = private non-residential fixed capital

K_a = cumulative net real investment abroad (excluding change in official reserves)

L = employment

P = real profits, interest and dividends from abroad

p = price level

Q = real net output

q = exchange rate (price of sterling in foreign currency)

r = interest rate

S = stocks

w = wage rate

<u>Exogenous Variables</u>

B = stock of bonds

d_x = dummy variable for exchange controls ($d_x = 1$ for 1974-79, $d_x = 0$ for 1980 onwards)

E_o = real oil exports

G_c = real government consumption

K_p = public non-residential fixed capital

p_f = price level in leading foreign industrial countries

p_i = price of imports (in foreign currency)

M = money supply

r_f = foreign interest rate

T_1 = total taxation policy variable ($(Q+P)/T_1$ is real private disposable income)

T_2 = indirect taxation policy variable (Q/T_2 is real output at factor cost)

Y_f = real income of leading foreign industrial countries

Unobservable Trend Variables

μ_1 = productivity trend variable ($d\mu_1/dt$ is the proportional rate of decrease in the amount of labour required to produce a given output with a given amount of capital)

μ_2 = labour supply trend variable ($d\mu_2/dt$ is the proportional rate of growth in the non-accelerating inflation level of employment)

μ_3 = trend variable allowing for the growth in the use of credit and charge cards (plastic money) ($d\mu_3/dt$ is the proportional rate of growth in the use of cards)

Structural Equations

$$D\log C = \lambda_1 + \lambda_2 + \gamma_1 \log\left[\frac{\beta_1 e^{-\{\beta_2(r - D\log p) + \beta_3 D\log p\}}(Q+P)}{T_1 C}\right] \tag{1}$$

$$D^2 \log L = \gamma_2(\lambda_2 - D\log L) + \gamma_3 \log\left[\frac{\beta_4 e^{-\mu_1}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L}\right] \tag{2}$$

$$D^2 \log K_h = \gamma_4(\lambda_1 + \lambda_2 - D\log K_h) + \gamma_5 \log\left[\frac{\beta_7 e^{-\{\beta_8(r - D\log p) + \beta_9 D\log p\}}(Q+P)}{T_1 K_h}\right] \tag{3}$$

$$D^2 \log K = \gamma_6(\lambda_1 + \lambda_2 - D\log K) + \gamma_7 \log\left[\frac{\beta_5\left(Q/K\right)^{1+\beta_6}}{r - \beta_{10} D\log p + \beta_{11}}\right] \tag{4}$$

$$\begin{aligned} D\log Q = {} & \lambda_1 + \lambda_2 \\ & + \gamma_8 \log\left[\frac{\{1 - \beta_{12}(qp/p_i)^{\beta_{13}}\}\{1 + \beta_{14}(\lambda_1 + \lambda_2)\}(C + G_c + DK + DK_h + DK_p + E_n + E_o)}{Q}\right] \\ & + \gamma_9 \log\left[\frac{\beta_{14}(C + G_c + DK + DK_h + DK_p + E_n + E_o)}{S}\right] \end{aligned} \tag{5}$$

$$D^2 \log p = \gamma_{10}\left(D\log(w/p) - \lambda_1\right) + \gamma_{11}\log\left[\frac{\beta_{15}\beta_4 T_2 w e^{-\mu_1}\left\{1-\beta_5 (Q/K)^{\beta_6}\right\}^{-(1+\beta_6)/\beta_6}}{p}\right] \tag{6}$$

$$D^2 \log w = \gamma_{12}\left(\lambda_1 - D\log(w/p)\right) + \gamma_{13} D\log(p_i/qp) + \gamma_{14}\log\left[\frac{\beta_4 e^{-\mu_1}\left\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\right\}^{-1/\beta_6}}{\beta_{16} e^{\mu_2}}\right] \tag{7}$$

$$D^2 r = -\gamma_{15} Dr + \gamma_{16}\left[\beta_{17} + \beta_{18} r_f - \beta_{19} D\log q + \beta_{20}\left\{p(Q+P)/Me^{\mu_3}\right\} + \beta_{21}(B/M) - r\right] \tag{8}$$

$$\begin{aligned} D\log I &= \lambda_1 + \lambda_2 \\ &+ \gamma_{17}\log\left[\frac{\beta_{12}(qp/p_i)^{\beta_{13}}\{1+\beta_{14}(\lambda_1+\lambda_2)\}(C + G_c + DK + DK_h + DK_p + E_n + E_o)}{(p_i/qp)I}\right] \\ &+ \gamma_{18}\log\left[\frac{\beta_{14}(C + G_c + DK + DK_h + DK_p + E_n + E_o)}{S}\right] \end{aligned} \tag{9}$$

$$D\log E_n = \lambda_1 + \lambda_2 + \gamma_{19}\log\left[\frac{\beta_{22} Y_f^{\beta_{23}} (p_f/qp)^{\beta_{24}}}{E_n}\right] \tag{10}$$

$$DF = \gamma_{20}\left\{\beta_{25}(Q+P) - F\right\} \tag{11}$$

$$DP = \gamma_{21}\left[\left\{\beta_{26} + \beta_{27}\left(r_f - D\log p_f\right)\right\}K_a - P\right] \tag{12}$$

$$D^2 K_a = -\gamma_{22} DK_a + \gamma_{23}\left[\left\{\beta_{28} + \beta_{29}(r_f - r) - \beta_{30} D\log q - \beta_{31} d_x\right\}(Q+P) - K_a\right] \tag{13}$$

$$\begin{aligned} D^2 \log q &= \gamma_{24} D\log\left(\frac{p_f}{qp}\right) + \gamma_{25}\log\left[\frac{\beta_{32} p_f}{qp}\right] + \gamma_{26}(r - \beta_{33}) \\ &+ \gamma_{27}\log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I}\right] \end{aligned} \tag{14}$$

$$DS = Q + (p_i/qp)I - C - DK - DK_h - DK_p - E_n - E_o - G_c \tag{15}$$

$$D\mu_1 = \lambda_1 \quad (16)$$

$$D\mu_2 = \lambda_2 \quad (17)$$

$$D\mu_3 = \lambda_3 \quad (18)$$

The above model can be simplified by solving (16)-(18) subject to the normalization conditions $\mu_1(0)=0, \mu_2(0)=0, \mu_3(0)=0,$ to obtain $\mu_1 = \lambda_1 t,\ \mu_2 = \lambda_2 t,\ \mu_3 = \lambda_3 t$ and substituting these expressions into (2), (6), (7), and (8). But, this simplification will not be possible in the stochastic form of the model, where (16), (17), and (18) will be replaced by stochastic differential equations and white noise innovations will be added to each of the other equations.

A general feature of the model is that nearly every equation has the form of a partial adjustment (error correction) equation, in which the dependent variable is adjusting, continuously, in response to the deviation of its current level from its partial equilibrium level, which is a function of the other variables in the model. This was also a feature of the model of Bergstrom and Wymer (1976) and all of the continuous time models for which that model has served as a prototype. Another general feature of the model is that it incorporates the assumption of long-run rational expectations by assuming that the various agents in the economy know the long-run growth rate $\lambda_1 + \lambda_2$ (the rate of technical progress plus the rate of growth of the labour supply) and take this into account in adjusting the variables that they control. In particular, if a variable is, currently at its partial equilibrium level, the agent controlling that variable will continue to increase it at the expected long-run growth rate of its partial equilibrium level, which, for most variables, is $\lambda_1 + \lambda_2$.

We turn now to a discussion of the individual equations of the model. Equation (1) is the consumption adjustment equation. It assumes that the excess of the rate of increase in consumption over the expected long-run growth rate $\lambda_1 + \lambda_2$ of real income depends on the ratio of the partial equilibrium level of consumption to the current level of consumption. The partial equilibrium level of consumption is assumed to depend on real disposable income $(Q+P)/T_1,$ the real interest rate $r - D\log p,$ and the inflation rate $D\log p$. The reason for assuming that inflation has an independent effect on consumption (in addition to its effect through the real interest rate) is that inflation reduces the real disposable income of holders of government bonds (like a hidden tax), and hence the variable $(Q+P)/T_1$ overstates the true level of real disposable income. The partial equilibrium level of consumption is really assumed, therefore, to be a function of only the real interest rate and real disposable income correctly measured. Consumption is assumed to adjust gradually, rather than instantaneously, to its partial equilibrium level because of adjustment costs, which include both physical and psychological costs.

It is convenient, at this stage, to discuss, equation (3), which is the residential (housing) investment equation. Although it is a capital adjustment equation, it is, essentially, a demand equation for residential services, since the consumption of residential services is, implicitly, assumed to be proportional to the stock of residential capital. The equation is similar in form, therefore, to equation (1), except that it is a second order differential equation. It assumes that the acceleration in the logarithm of the stock of residential capital depends on the excess of the expected long-run growth rate $\lambda_1+\lambda_2$ of real income over the current rate of growth of the stock of residential capital and the ratio of the partial equilibrium stock of residential capital to the current stock. The formulation as a second order differential equation allows for the fact that adjustment costs will have a more important influence on the adjustment of residential capital and the consumption of the services of this capital than on the general level of consumption. Indeed, a second order equation can be, formally, derived by assuming that adjustment costs depend on both the rate of change and the acceleration of the capital stock.

Equations (2), (4), (5), and (6) relate to the production sector of the economy. It will be useful, before discussing them in detail, to say something about the underlying assumptions about firm behaviour and market structure. It is assumed that the economy is made up of a large number of monopolistic competitors with identical production functions and uniformly differentiated products. In a position of static equilibrium, each firm would be employing labour and capital in the proportions that make the marginal technical rate of substitution between these factors equal to the ratio of the real wage to the real interest rate, setting the price of its product equal to marginal cost plus a margin depending on the own-price elasticity of the partial demand function for its product (on the assumption that all other firms hold their prices constant), and producing the quantity demanded at that price. Since the prices charged by the different firms in the economy are equal, it can be treated as a single product economy, and it tends to a perfectly competitive economy as the degree of differentiation between the products tends to zero, that is as the price elasticity of the partial demand function for each firm's product tends to minus infinity.

In order to limit the number of parameters to be estimated, we have formulated the firm's dynamic adjustment equations in such a way as to take account of our _a prior_ knowledge of the relative speeds at which different factors of production can be adjusted (or the relative costs of adjusting them at a given speed). We know that it is easier to adjust output, by varying the intensity with which the employed labour force is used (varying the average number of hours per week worked by each employee) than to vary the number of persons employed, and it is easier to vary the number of persons employed than to vary the stock of fixed capital. We assume, therefore, that, at each point of time, output is adjusting in response to sales, the number of persons employed is adjusting in response to output, and capital is adjusting in response to the marginal product of capital and the real interest rate. The assumed forms of the adjustment equations imply that, in the steady state, the equilibrium conditions mentioned in the previous paragraph are satisfied.

The employment adjustment equation (2) assumes that the acceleration of the logarithm of employment depends on the excess of the growth rate λ_2 of the labour supply (the non-accelerating inflation level of employment) over the current growth rate of employment and on the ratio of the partial equilibrium level of employment to the current level of employment. The partial equilibrium level of employment depends on output and capital, through a constant elasticity of substitution production function with Harrod neutral technical progress. The parameter β_5 is a measure of the importance of capital in the production process, $1/(1+\beta_6)$ is the elasticity of substitution between labour and capital, and λ_1 (the drift parameter in the trend variable μ_1) is the rate of decrease, through technical progress, in the amount of labour required to produce a given output with a given amount of capital. The equation is consistent with the assumption of long-run rational expectations, since it implies that, if employment in the representative firm is, currently, at its partial equilibrium level, then the firm will increase its employment at a rate equal to the expected long-run growth rate $\lambda_1 + \lambda_2$ in the demand for its product minus the rate of decrease λ_1 in the amount of labour required to produce a given output.

The private non-residential fixed capital adjustment equation (4) assumes that the acceleration of the logarithm of the capital stock depends on the excess of the expected long-run growth rate $\lambda_1 + \lambda_2$ of output over the current growth rate of the capital stock and on the ratio of the marginal product of capital to the real interest rate plus a risk premium. If $\beta_{10} = 1,$ then the risk premium is a constant β_{11}. But, we do not impose this restriction, since the risk premium is likely to be related to the inflation rate.

The output adjustment equation (5) assumes that the excess of the rate of increase in output over the expected long-run growth rate $\lambda_1 + \lambda_2$ of aggregate sales depends on the ratio of the partial equilibrium level of output to the current level of output and the ratio of the partial equilibrium level of stocks (inventories) to the current level of stocks. The partial equilibrium level of output is equal to aggregate sales (for private and public consumption, capital formation and exports) multiplied by the terms $1 + \beta_{14}(\lambda_1 + \lambda_2)$ and $1 - \beta_{12}(qp/p_i)^{\beta_{13}}$. The term $1 + \beta_{14}(\lambda_1 + \lambda_2)$, which also occurs in the import adjustment equation (9), ensures that, in the state of partial equilibrium, total supply, from output and imports, is sufficient to meet current sales and keep stocks growing at the rate $\lambda_1 + \lambda_2$, maintaining a constant ratio β_{14} of stocks to sales. The term $1 - \beta_{12}(qp/p_i)^{\beta_{13}}$ is the proportion of total supply met from domestic output, while the term, $\beta_{12}(qp/p_i)^{\beta_{13}}$, which also occurs in the import adjustment equation (9), is the proportion met from imports. As will be evident from equation (9), the price elasticity of demand for imports is $-(1 + \beta_{13})$.

The next group of equations are the adjustment equations for the price level, the wage rate, and the interest rate. The price adjustment equation (6) assumes that the acceleration of the logarithm of the price level depends on the excess of the current rate of increase in the real wage over the rate of technical progress λ_1 and on the ratio of the partial equilibrium price level to the current price level. The partial equilibrium price level equals marginal cost $w(\partial L/\partial Q)$, derived from the production function in equation (2), multiplied by the indirect tax variable T_2 and the mark-up parameter $\beta_{15} = e/(1+e)$ where e is the own price elasticity of the partial demand function for each firm's product. The limiting case $\beta_{15} = 1$ corresponds to perfect competition. It should be noted, incidentally, that p is defined as the implicit price deflator of the gross domestic product and is not, directly, affected by the import price level p_i. The speed of adjustment parameters depend, not only on adjustment costs (which, for price variables, are relatively minor), but also on the time lags in the competitive adjustment process, in which each firm is adjusting its price in small steps, taking account of the current prices of its competitor's products (see Chamberlin, 1946, Chapter 5, Section 3).

The wage adjustment equation (7) assumes that the acceleration of the logarithm of the nominal wage rate depends on the excess of the rate of technical progress over the current rate of increase in the real wage, on the current rate of increase in the ratio of import prices to domestic prices (both measured in sterling), and on the ratio of the partial equilibrium level of employment to the labour supply $\beta_{16}e^{\mu_2}$. The labour supply can be interpreted as the non-accelerating inflation level of employment. It is not directly measurable, since it depends, not only on demographic factors, but also on conditions in the labour market. It is assumed, therefore, to be an exponential function of the unobservable trend variable μ_2, with constant drift parameter λ_2 and, in the stochastic version of the model, constant volatility parameter. The term $D\log(p_i/qp)$ allows for the pressure for higher wages to compensate for the loss of welfare caused by the fall in the real exchange rate (since, although w/p is the measure of the real wage that is relevant for firms employing labour, it is not a measure of welfare). The complete equation implies there is no long-run trade off between unemployment and inflation.

The interest rate adjustment equation (8) represents the dynamic behaviour of the market for long dated bonds. It is, essentially, a portfolio balance equation, which allows for substitution between money, domestic bonds, and foreign bonds. The acceleration of the interest rate (the yield on long dated bonds) is assumed to depend on the current rate of change of the interest rate and the excess of the partial equilibrium interest rate over the current rate. The partial equilibrium interest rate depends on the foreign interest rate (yield on foreign bonds), the rate of change of the exchange rate, real liquidity, and the ratio of the total stock of bonds to the money supply. The measure of real liquidity $p(Q+P)/Me^{\mu_3}$ takes account of the growth of the use of plastic money through the trend term e^{μ_3}.

The final group of behavioural equations are the adjustment equations for the balance of payments variables and the exchange rate. The import adjustment equation (9) assumes that the excess of the rate of increase in the volume of imports over the expected long-run growth rate $\lambda_1+\lambda_2$ of aggregate sales depends on the ratio of the partial equilibrium real value of imports to the current real value of imports and the ratio of the partial equilibrium level of stocks to the current level of stocks. This equation, together with equation (5), ensures that, in the steady state, the real value of imports plus domestic output is sufficient to meet total sales and keep stocks growing at the rate $\lambda_1+\lambda_2$. It should be noticed that we have distinguished between the volume of imports I and the real value of imports $p_i I/qp$. In order to avoid introducing too many price variables (and to be consistent with our assumption of, essentially, a single product economy) all other real variables in the model (for example, C) have been defined as values at current prices deflated by p, the implicit prices deflator of the gross domestic product.

The export adjustment equation (10) assumes that the excess of the rate of increase in exports over the expected long-run growth rate $\lambda_1+\lambda_2$ of demand for exports depends on the ratio of the partial equilibrium level of exports to the current level of exports. The partial equilibrium level of exports depends on foreign income and the ratio of the foreign price level to the domestic price level (both measured in sterling). The parameters β_{23} and β_{24} are the foreign income and price elasticities of demand for UK products.

Equations (11)-(13) are the adjustment equations for real current transfers abroad F, real profits, interest, and dividends from abroad P, and cumulative net real investment abroad K_a. These equations are formulated in natural values rather than logarithms, because each of the variables F, P, and K_a can be either positive or negative. The rate of change in each of the variables F and P is assumed to depend on the excess of its partial equilibrium level over its current level, while the acceleration of K_a is assumed to depend on its current rate of change and the excess of its partial equilibrium level over its current level. The partial equilibrium level of real transfers abroad is proportional to real income. The partial equilibrium level of real profits, interest, and dividends from abroad is equal to cumulative real net investment abroad multiplied by a linear function of the real foreign interest rate; and the partial equilibrium level of cumulative real net investment abroad is equal to real income multiplied by a linear function of the difference between the foreign and domestic interest rates, the rate of change in the exchange rate, and a dummy variable to allow for exchange controls (which were abolished in 1979).

The exchange rate adjustment equation (14) represents the dynamic behaviour of the foreign exchange market. The acceleration of the logarithm of the exchange rate is assumed to depend on the rate of decrease in the real exchange rate $D\log(p_f/qp)$, on the ratio of the expected steady state real exchange rate β_{32} to the current real exchange rate, on the excess of the current interest rate over the expected steady state interest rate β_{33}, and on the ratio of the balance of payments surplus (including capital items other than then change in official reserves of gold and foreign exchange) to the real value of imports. When solving the complete model for the steady state in Section 3, we shall use the assumption of rational long-run expectations by assuming that β_{32} and β_{33} are equal to the actual steady state values of the real exchange rate and the interest rate. But this assumption will not be imposed when estimating the parameters of the model. The last four equations of the model include (15), which is an identity, and (16)-(18) which generate the unobservable trend variables.

3 STEADY STATE AND STABILITY ANALYSIS

In the steady state and stability analysis, we will use the deterministic form of the model, as specified in (1)-(18). A realistic macroeconomic model should, at least, generate plausible long-run behaviour when all the white noise innovations (including those in the stochastic trends) have their expected value zero. The steady state will be derived under the following assumptions:

$$B = B^* e^{\lambda_4 t}, \tag{19}$$

$$d_x = 0, \tag{20}$$

$$E_o = E_o^* e^{(\lambda_1+\lambda_2)t}, \tag{21}$$

$$G_c = g^*(Q+P), \tag{22}$$

$$K_p = K_p^* e^{(\lambda_1+\lambda_2)t}, \tag{23}$$

$$M = M^* e^{\lambda_4 t}, \tag{24}$$

$$p_f = p_f^* e^{\lambda_5 t}, \tag{25}$$

$$p_i = p_i^* e^{\lambda_5 t} = p_f^* e^{\lambda_5 t}, \tag{26}$$

$$r_f = r_f^*, \tag{27}$$

$$T_1 = T_1^*, \tag{28}$$

$$T_2 = T_2^*, \tag{29}$$

$$Y_f = Y_f^* e^{\left(\frac{\lambda_1+\lambda_2}{\beta_{23}}\right)t}, \tag{30}$$

$$\beta_{32} = q^* p^* / p_f^*, \tag{31}$$

$$\beta_{33} = r^*, \tag{32}$$

$$\mu_1(0) = 0, \tag{33}$$

$$\mu_2(0) = 0, \tag{34}$$

$$\mu_3(0) = 0, \tag{35}$$

where $B^*, E_o^*, g^*, K_p^*, M^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, \lambda_4,$ and λ_5 are given constants and $q^*, p^*,$ and r^* are functions of the parameters of the model given by the steady state solution.

Equations (19) and (24) assume that the stock of bonds and the money supply grow at the same constant rate λ_4, while (20) assumes that there are no exchange controls. Equations (21) and (23) assume that oil exports and the stock of public non-residential fixed capital grow at the rate $\lambda_1 + \lambda_2$ which, as will be shown, is the steady state rate of growth of output and other real variables in the economy. Equation (22) assumes that real government consumption is a constant proportion of real income, so that, for the purpose of the steady state analysis, it becomes an endogenous variable. Equation (25) and (26) assume that the foreign price indices p_f and p_i (which are normalized to be equal in the base period of the sample) follow identical paths with a constant growth rate λ_5. Equations (27)-(29) assume that the foreign interest rate and the two tax rate variables are all constant. Equation (30) assumes that foreign income grows at a constant rate $(\lambda_1+\lambda_2)/\beta_{23}$. This implies that, when the real exchange rate is constant, the rate of growth of demand for exports is equal to the steady state growth rate of the economy $\lambda_1+\lambda_2$. Although the last assumption is rather special, it is not unrealistic, and we should, at least, ensure that the model generates plausible long-run behaviour under this assumption. It facilitates the analysis by ensuring the existence of a steady state solution in which the real exchange rate is constant.

Under the above assumptions, the system (1)-(18) has a particular solution in which all variables grow at constant proportional rates (possible zero). This solution, which we call the steady state solution, is given by equations (36)-(53).

$$C = C^* e^{(\lambda_1+\lambda_2)t}, \quad (36)$$

$$E_n = E_n^* e^{(\lambda_1+\lambda_2)t}, \quad (37)$$

$$F = F^* e^{(\lambda_1+\lambda_2)t}, \quad (38)$$

$$I = I^* e^{(\lambda_1+\lambda_2)t}, \quad (39)$$

$$K = K^* e^{(\lambda_1+\lambda_2)t}, \quad (40)$$

$$K_a = K_a^* e^{(\lambda_1+\lambda_2)t}, \quad (41)$$

$$K_h = K_h^* e^{(\lambda_1+\lambda_2)t}, \quad (42)$$

$$L = L^* e^{\lambda_2 t}, \quad (43)$$

$$p = p^* e^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2)t}, \quad (44)$$

$$P = P^* e^{(\lambda_1+\lambda_2)t}, \quad (45)$$

$$q = q^* e^{(\lambda_1+\lambda_2+\lambda_5-\lambda_3-\lambda_4)t}, \quad (46)$$

$$Q = Q^* e^{(\lambda_1+\lambda_2)t}, \tag{47}$$

$$r = r^*, \tag{48}$$

$$S = S^* e^{(\lambda_1+\lambda_2)t}, \tag{49}$$

$$w = w^* e^{(\lambda_3+\lambda_4-\lambda_2)t}, \tag{50}$$

$$\mu_1 = \lambda_1 t, \tag{51}$$

$$\mu_2 = \lambda_2 t, \tag{52}$$

$$\mu_3 = \lambda_3 t, \tag{53}$$

where C^*, E_n^*, F^*, I^*, K^*, K_a^*, K_h^*, L^*, p^*, P^*, q^*, Q^*, r^*, S^*, and w^* are functions of the vector (β, γ, λ) of 63 parameters in (1)-(18) and the additional parameters B^*, E_o^*, g^*, K_p^*, M^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, λ_4, and λ_5 in (19)-(35).

The steady state growth rates implied by (36)-(53) are summarized in Table 1. The growth rates of consumption, output, stocks, exports, imports, and other components of the balance of payments are all equal to the rate of technical progress plus the rate of growth of the labour supply. The rate of growth of employment equals the rate of growth of the labour supply. The rate of increase in the price level equals the rate of growth of the money supply plus the rate of growth in the use of plastic money minus the rate of growth of output. The rate of increase in the wage rate equals the rate of increase in the price level plus the rate of technical progress. The rate of interest is constant, and the rate of change in the nominal exchange rate is such that the real exchange rate is constant.

It can be verified, by substituting the expressions given by (19)-(53) into (1)-(15), that the steady state level parameters C^*, E_n^*, F^*, I^*, K^*, K_a^*, K_h^*, L^*, p^*, P^*, q^*, Q^*, r^*, S^*, and w^* must satisfy equations (54)-(68).

$$C^* = \beta_1 e^{\{(\beta_2-\beta_3)(\lambda_3+\lambda_4-\lambda_1-\lambda_2)-\beta_2 r^*\}} \left\{ \frac{Q^* + P^*}{T_1^*} \right\}, \tag{54}$$

$$L^* = \beta_4 \left\{ (Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6} \right\}^{-1/\beta_6}, \tag{55}$$

$$K_h^* = \beta_7 e^{\{(\beta_8-\beta_9)(\lambda_3+\lambda_4-\lambda_1-\lambda_2)-\beta_8 r^*\}} \left\{\frac{Q^*+P^*}{T_1^*}\right\}, \tag{56}$$

$$r^* = \beta_{10}(\lambda_3+\lambda_4-\lambda_1-\lambda_2) - \beta_{11} + \beta_5 \left(\frac{Q^*}{K^*}\right)^{1+\beta_6}, \tag{57}$$

$$\gamma_8 \log\left[\frac{\left\{1-\beta_{12}\left(q^* p^*/p_i^*\right)^{\beta_{13}}\right\}\left\{1+\beta_{14}(\lambda_1+\lambda_2)\right\}\left(C^*+G_c^*+(\lambda_1+\lambda_2)\left(K^*+K_h^*+K_p^*\right)+E_n^*+E_o^*\right)}{Q^*}\right]$$
$$+\gamma_9 \log\left[\frac{\beta_{14}\left(C^*+G_c^*+(\lambda_1+\lambda_2)\left(K^*+K_h^*+K_p^*\right)+E_n^*+E_o^*\right)}{S^*}\right] = 0, \tag{58}$$

$$p^* = \beta_{15}\beta_4 T_2^* w^* \left\{1-\beta_5\left(Q^*/K^*\right)^{\beta_6}\right\}^{-(1+\beta_6)/\beta_6}, \tag{59}$$

$$\left(Q^*\right)^{-\beta_6} - \beta_5\left(K^*\right)^{-\beta_6} = \left(\frac{\beta_{16}}{\beta_4}\right)^{-\beta_6}, \tag{60}$$

$$r^* = \beta_{18} r_f^* + \beta_{20}\left\{p^*\left(Q^*+P^*\right)/M^*\right\} + \beta_{21}\left(B^*/M^*\right)$$
$$+ \beta_{17} + \beta_{19}(\lambda_3+\lambda_4-\lambda_1-\lambda_2-\lambda_5), \tag{61}$$

$$\gamma_{17} \log\left[\frac{\left\{\beta_{12}\left(q^* p^*/p_i^*\right)^{\beta_{13}}\right\}\left\{1+\beta_{14}(\lambda_1+\lambda_2)\right\}\left(C^*+G_c^*+(\lambda_1+\lambda_2)\left(K^*+K_h^*+K_p^*\right)+E_n^*+E_o^*\right)}{\left(p_i^*/q^* p^*\right) I^*}\right]$$
$$+\gamma_{18} \log\left[\frac{\beta_{14}\left(C^*+G_c^*+(\lambda_1+\lambda_2)\left(K^*+K_h^*+K_p^*\right)+E_n^*+E_o^*\right)}{S^*}\right] = 0, \tag{62}$$

$$E_n^* = \beta_{22}\left(Y_f^*\right)^{\beta_{23}} \left(\frac{p_f^*}{q^* p^*}\right)^{\beta_{24}}, \tag{63}$$

$$\beta_{25}\left(\frac{Q^*+P^*}{F^*}\right) = \frac{\gamma_{20}+\lambda_1+\lambda_2}{\gamma_{20}}, \tag{64}$$

$$\left\{\beta_{26}+\beta_{27}\left(r_f^*-\lambda_5\right)\right\}\frac{K_a^*}{P^*} = \frac{\gamma_{21}+\lambda_1+\lambda_2}{\gamma_{21}}, \tag{65}$$

$$\left\{\beta_{28} + \beta_{29}\left(r_f^* - r^*\right) + \beta_{30}\left(\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2 - \lambda_5\right)\right\}\left\{\frac{Q^* + P^*}{K_a^*}\right\}$$
$$= \frac{\gamma_{23} + \left(\lambda_1 + \lambda_2\right)^2 + \gamma_{22}\left(\lambda_1 + \lambda_2\right)}{\gamma_{23}}, \tag{66}$$

$$\frac{p_i^* I^*}{q^* p^*} = E_n^* + E_o^* + P^* - F^* - \left(\lambda_1 + \lambda_2\right)K_a^*, \tag{67}$$

$$\left(\lambda_1 + \lambda_2\right)S^* = Q^* + \frac{p_i^* I^*}{q^* p^*} - \left\{C^* + G_c^* + \left(\lambda_1 + \lambda_2\right)\left(K^* + K_h^* + K_p^*\right) + E_n^* + E_o^*\right\}, \tag{68}$$

We have been able to obtain, from (54)-(68), a pair of nonlinear simultaneous equations in r^* and x^* where x^* is the steady state real exchange rate defined by

$$x^* = \frac{q^* p^*}{p_f^*} = \frac{q^* p^*}{p_i^*}. \tag{69}$$

These can be solved, numerically, for r^* and x^*. We have also derived explicit formulae expressing C^*, E_n^*, F^*, I^*, K^*, K_a^*, K_h^*, L^*, p^*, P^*, q^*, Q^*, r^*, S^*, and w^* in terms of r^*, x^*, the structural parameters of the model (1)-(18), and the parameters of (19)-(35), which generate the assumed paths of the exogenous variables. These formulae have been used in the computation of the steady state paths of the endogenous variables from the estimated parameters of the model. The parameter estimates and the steady state paths derived from these estimates will be presented and discussed in Section 4.

We turn now to the stability analysis. For this purpose, we use the variables $y_1(t), y_2(t), \ldots, y_{15}(t)$ defined by

$$y_1(t) = \log\left\{C(t)/C^* e^{(\lambda_1+\lambda_2)t}\right\},$$

$$y_2(t) = \log\left\{L(t)/L^* e^{\lambda_2 t}\right\},$$

$$y_3(t) = \log\left\{K_h(t)/K_h^* e^{(\lambda_1+\lambda_2)t}\right\},$$

$$y_4(t) = \log\left\{K(t)/K^* e^{(\lambda_1+\lambda_2)t}\right\},$$

$$y_5(t) = \log\left\{Q(t)/Q^* e^{(\lambda_1+\lambda_2)t}\right\},$$

$$y_6(t) = \log\{p(t)/p^* e^{(\lambda_3+\lambda_4-\lambda_1-\lambda_2)t}\},$$

$$y_7(t) = \log\{w(t)/w^* e^{(\lambda_3+\lambda_4-\lambda_2)t}\},$$

$$y_8(t) = r(t) - r^*,$$

$$y_9(t) = \log\{I(t)/I^* e^{(\lambda_1+\lambda_2)t}\},$$

$$y_{10}(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1+\lambda_2)t}\},$$

$$y_{11}(t) = \log\{F(t)/F^* e^{(\lambda_1+\lambda_2)t}\},$$

$$y_{12}(t) = \log\{P(t)/P^* e^{(\lambda_1+\lambda_2)t}\},$$

$$y_{13}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1+\lambda_2)t}\},$$

$$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1+\lambda_2+\lambda_5-\lambda_3-\lambda_4)t}\},$$

$$y_{15}(t) = \log\{S(t)/S^* e^{(\lambda_1+\lambda_2)t}\},$$

We have obtained from (1)-(15) a system of mixed first and second order nonlinear differential equations in the variables $y_1(t), y_2(t),..., y_{15}(t),$ and the stability analysis is based on a linear approximation to this system about the origin. (We note that each of the variables $y_1(t), y_2(t),..., y_{15}(t)$ has the value zero in the steady state.) The system of mixed order differential equations in $y_1(t), y_2(t),..., y_{15}(t)$ can be represented as a system of first order differential equations in the variables $w_1(t),..., w_{23}(t)$ defined by

$$w_1(t) = y_1(t),$$

$$w_2(t) = y_5(t),$$

$$w_3(t) = y_9(t),$$

$$w_4(t) = y_{10}(t),$$

$$w_5(t) = y_{11}(t),$$

$$w_6(t) = y_{12}(t),$$

$$w_7(t) = y_{15}(t),$$

$$w_8(t) = y_2(t),$$

$$
\begin{aligned}
w_9(t) &= y_3(t), \\
w_{10}(t) &= y_4(t), \\
w_{11}(t) &= y_6(t), \\
w_{12}(t) &= y_7(t), \\
w_{13}(t) &= y_8(t), \\
w_{14}(t) &= y_{13}(t), \\
w_{15}(t) &= y_{14}(t), \\
w_{16}(t) &= Dy_2(t), \\
w_{17}(t) &= Dy_3(t), \\
w_{18}(t) &= Dy_4(t), \\
w_{19}(t) &= Dy_6(t), \\
w_{20}(t) &= Dy_7(t), \\
w_{21}(t) &= Dy_8(t), \\
w_{22}(t) &= Dy_{13}(t), \\
w_{23}(t) &= Dy_{14}(t).
\end{aligned}
$$

A linear approximation to this system, obtained by omitting the higher order terms in a Taylor series expansion about the origin, can be written in the form

$$Dw(t) = Cw(t), \tag{70}$$

where $w(t) = [w_1(t), w_2(t), ..., w_{23}(t)]'$ and C is a 23×23 matrix whose elements are functions of the parameters of the system (1)-(18) and the steady state level parameters $C^*, ..., w^*$.

The omitted higher order terms do not involve t and are of a lower order of smallness than the elements of $w(t)$ as these tend to zero. The solution $y = 0$ to the nonlinear system in

$y(t) = [y_1(t), y_2(t),..., y_{15}(t)]$ is asymptotically stable, therefore, provided that the eigenvalues of C all have negative real parts (see Coddington and Levinson, 1955, p.314). In other words, the steady state solution of the model (1)-(18) is asymptotically stable provided that the eigenvalues of C all have negative real parts. These eigenvalues, derived from the estimated values of the parameters of the model are presented in Section 4.

4 EMPIRICAL RESULTS

Estimates of the parameters of the model were obtained from a sample of UK quarterly observations for the years 1975-94 inclusive, and its post-sample predictive performance was tested against quarterly data for the two-year period 1995-96. The exact Gaussian estimation algorithm of Bergstrom (1997) is applicable to a model which is linear in the variables, although nonlinear in the parameters. In order to use this algorithm, therefore, we obtained a linear approximation to the model (1)-(18) about the sample means. Alternatively, we could have applied the algorithm to the linear approximation about the steady state derived in the preceding section. But, if the variables are not very close to their steady state paths during the sample period, better estimates are likely to be obtained by using the linear approximation about the sample means. It should be emphasized, however, that nothing can be, rigorously, deduced about the stability properties of the model from the linear approximation about the sample means. The stability analysis must be based on the linear approximation about the steady state.

In order to put the model into the form used in Bergstrom (1997), we have defined a 7×1 vector $x_1(t)$ of the endogenous variables that adjust through first order differential equations, an 8×1 vector $x_2(t)$ of the variables that adjust through second order differential equations, a 15×1 vector $z(t)$ of exogenous variables, and a 3×1 vector $\mu(t)$ of stochastic trends. These vectors are precisely defined by

$$x_1(t) = [\log C(t), \log E_n(t), F(t), \log I(t), P(t), \log Q(t), \log S(t)],$$

$$x_2(t) = [\log K(t), K_a(t), \log K_h(t), \log L(t), \log p(t), \log q(t), r(t), \log w(t)],$$

$$z(t) = [\log B(t), \log K_p(t), \log M(t), D\log K_p(t), d_x, \log E_o(t), \log G_c(t), \log p_f(t), \\ \log p_i(t), r_f(t), \log T_1(t), \log T_2(t), \log Y_f(t), D\log p_f(t), D\log p_i(t)],$$

$$\mu(t) = [\mu_1(t), \mu_2(t), \mu_3(t)].$$

By adding white noise innovations to the linear approximation about the sample means of the variables of the model (1)-(18), we obtain the mixed order linear stochastic differential equation system

$$dx_1(t) = [A_1 x_1(t) + A_2 x_2(t) + A_3 Dx_2(t) + B_1 z(t) + C_1 \mu(t) + b_1]dt + \zeta_1(dt) \quad (t \geq 0), \tag{71}$$

$$d[Dx_2(t)] = [A_4 x_1(t) + A_5 x_2(t) + A_6 Dx_2(t) + B_2 z(t) + C_2 \mu(t) + b_2]dt + \zeta_2(dt) \quad (t \geq 0), \tag{72}$$

$$d\mu(t) = \lambda dt + \zeta_3(dt), \tag{73}$$

where the elements of the matrices $A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, C_1,$ and C_2 are functions of the sample means of the variables and the parameter vector $\theta = [\beta, \gamma],$ and the elements of the vectors b_1 and b_2 are functions of the sample means of the variables and the parameter vector $[\theta, \lambda]$.

The system (71)-(73) has the same form as the model of Bergstrom (1997, equations (97), (98), and (3)), and the algorithm developed in that article is, therefore, directly applicable in the estimation of the parameters of (71)-(73), which are also the parameters of the nonlinear model (1)-(18). The estimates are derived by maximizing the Gaussian likelihood function, which is derived from a VARMAX model satisfied by the discrete observations generated by (71)-(73).

As in Bergstrom et al. (1992), we have placed prior bounds on all 63 of the structural parameters and maximized the likelihood subject to the constraint that the parameter vector belongs to a compact rectangular set defined by these bounds. In addition to facilitating the computation of the estimates, this constraint (particularly, the bounds on the speed of adjustment parameters) provides a way of avoiding the lack of identification that can result from the aliasing phenomenon: that is the inability to distinguish between cycles whose frequencies (per unit observation period) differ by integers. (See Phillips, 1973, Hansen and Sargent, 1983, and Bergstrom et. al. 1992.)

The Gaussian estimates of the parameters are presented in Table 2. They all have plausible values, and only one of them is on a bound. The mark-up parameter β_{15} is on its lower bound, which implies perfect competition. The column headed "Standard error" contains the square roots of the diagonal elements of minus the inverse of the Hessian of the logarithm of the Gaussian likelihood function, evaluated at the estimated values of the parameters. They cannot, however, be interpreted in the same way as under classical assumptions, since, because of the stochastic trends, the parameter estimates are not asymptotically normally distributed. They can, however be given a simple Bayesian interpretaton. Assuming that the innovations in the continuous time model are Brownian motion and a uniform prior distribution of the parameters, the likelihood function is the posterior probability density function of the parameters, and minus the inverse of the Hessian of the logarithm of the likelihood function is the covariance matrix of a normal approximation to this distribution in the neighbourhood of its mode.

We turn now to the analysis of the steady state solution of the model for the estimated values of the parameters. For this purpose, we have adjusted 3 of the estimated structural parameters, the remaining 60 parameters being set at their estimated values. The parameters adjusted are β_{13}, which determines the price elasticity of demand for imports, λ_2 the rate of growth of the labour supply, and λ_3 the rate of growth in the use for plastic money. We have set $\beta_{13} = 0.2$, which implies a price elasticity of demand for imports of -1.2. In view of the greater liberalization of trade, which is now taking place, this seems more realistic for the future than the price elasticity of -0.67 implied by the estimated value of β_{13}. The parameter λ_2 has been set at the value 0.002. This implies a rate of growth of the labour supply of 0.8 per cent per annum, which seems more realistic for the future than the rate of growth of 2.7 per cent per annum implied by the estimated value of λ_2. The parameter λ_3 has been set at the value 0.0, for simplicity, and the assumed rate of growth of the money supply adjusted accordingly.

In order to complete the steady state solution, we must also assume values of the parameters in (19)-(35), which determine the assumed paths of the exogenous variables. The parameter λ_4, representing the rate of growth of the money supply, has been set at the value 0.0048, which, in conjunction with the estimated value 0.0028 of λ_1 and the assumed value 0.0020 of λ_2, implies a steady state rate of increase in UK prices of zero. The parameter λ_5, representing the rate of increase in foreign prices, has been set equal to 0.0, which, in conjunction with the zero steady state rate of increase in UK prices, implies a constant steady state exchange rate. The level parameters $B^*,\ E_o^*,\ K_p^*,\ M^*,\ p_f^*,\ p_i^*,\ Y_f^*,$ have been set at the values that ensure that the levels of the steady state paths of the corresponding variables are equal to the actual levels of the variables in the base period. The parameters $T_1^*, T_2^*,$ and g^* have been set at the average values of $T_1, T_2,$ and $G_c/(Q+P)$ over the sample period and r_f^* at the average value of the foreign real interest rate over the sample period. The numerical values of the parameters of the assumed steady state paths of the exogenous variables are shown in Table 3.

The steady state growth rates of the endogenous variables, under the above assumptions, are shown in Table 4 and the steady state level parameters in Table 5. The actual and steady state levels of the endogenous variables in the last quarter of 1996 are also shown in Table 5. It can be seen that the greatest difference between the actual and steady state levels of the real variables, in that quarter, was for the capital stock K. This is not surprising in view of the parameter changes. In particular, the reduction in the value of the parameter λ_2 from the estimated value 0.0068 to the value 0.002 assumed in the computation of the steady state paths will cause an increase in the ratio of capital to output. This is a well known neoclassical relationship, which is a result of the fact that the proportion of output that must be invested in order to maintain a given ratio of capital to output is smaller the lower is the growth rate. Another change which will tend to increase the ratio of capital to output is an apparent increase in the savings propensity in recent years. Some evidence for this is provided by the estimate of the consumption equation parameter $\beta_1,$ which is 0.915, as compared with the estimate 0.95 of the corresponding parameter obtained by Nowman (1996) using less recent data and the model of Bergstrom et. al. (1992).

The eigenvalues of the linear approximation about the steady state are shown in Table 6. The short-run dynamic behaviour of the system, in the neighbourhood of the steady state growth path, is dominated by the pair of complex roots $0.162 \pm 0.155i$. These imply that it generates an explosive cycle, about the steady state growth path, with a period of approximately 10 years. The cycle can be prevented from exploding by taking account of constraints, such as a "full employment" ceiling, as proposed by Hicks (1950). Economists who have studies the historical data are generally agreed that the average period of the UK trade cycle, during the last 200 years, has been in the range 7 to 10 years. We conclude, therfore, that the model generates plausible long-run behaviour.

We turn, finally, to the post-sample forecasts. The root mean squares errors (RMSE) of the multiperiod forecasts for the 8 post-sample quarters, 1995 Quarter 1 to 1996 Quarter 4, are shown in Table 7. Also shown in Table 7 are the RMSE of the multiperiod forecasts obtained from a VARX model (vector autoregressive model with exogenous variables) with two lags in the endogenous variables and current values of the exogenous variables. The VARX model contains only the 12 variables measured in logarithms.

The results in Table 7 show that, although the forecasting performance of the continuous time model is satisfactory, it is not as good as that of the VARX model. The continuous time model provides the more accurate forecasts of exports, capital, and the interest rate, while the VARX model provides the more accurate forecasts of the other 9 variables. This could be just a random property of the stochastic innovations in the particular 8 quarters that we have chosen to forecast, or it could indicate some misspecification of the continuous time model.

5 CONCLUSION

We have described the formulation, analysis, and estimation of a new continuous time macroeconometric model of the United Kingdom. The main innovative feature of the model is that it incorporates unobservable stochastic trends to represent such trend variables as technical progress, which are not directly observable, but are known to have an influence on the observable variables in the model. The parameters have been estimated using the algorithm of Bergstrom (1997) for obtaining exact Gaussian estimates of the parameters of mixed order continuous time dynamic models with unobservable stochastic trends, and the paper provides the first demonstration of the feasibility of using that algorithm in the estimation of a macroeconometric model.

Apart from the incorporation of the unobservable stochastic trends, the model is a development of the earlier models of Bergstrom and Wymer (1976) and Bergstrom, Nowman, and Wymer (1992). Like those models, it uses economic theory to obtain a parsimonious parametrization and cross equation restrictions and is designed in such a way as to permit a rigorous mathematical analysis of its steady state and asymptotic stability properties.

The parameters of the model have been estimated from quarterly data for the period 1975-94, and the post-sample forecasts have been tested against quarterly data for the period 1995-96. In addition to obtaining plausible estimates of the parameter values and satisfactory post-sample forecasts, we have carried out the steady state and stability analysis and shown that the model generates plausible long-run behaviour.

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TABLE 1: Steady State Growth Rates

Variable	Steady State Growth Rate
$C, E_n, F, I, K, K_a, K_h, P, Q, S$	$\lambda_1 + \lambda_2$
L	λ_2
p	$\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2$
w	$\lambda_3 + \lambda_4 - \lambda_2$
r	0
q	$\lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4$

TABLE 2: Gaussian Estimates of United Kingdom Model

Parameter	Prior Lower Bound	Prior Upper Bound	Estimate	Standard Error
β_1	0.8000	1.0000	0.9151	0.0083
β_2	0.0000	4.0000	0.2014	0.0047
β_3	0.0000	4.0000	1.9971	0.0618
β_4	0.1000	10.000	0.1494	0.0166
β_5	0.0100	1.0000	0.2664	0.0018
β_6	0.0100	1.0000	0.2965	0.0016
β_7	1.0000	10.000	3.5000	0.3166
β_8	0.0000	4.0000	0.1815	0.0022
β_9	0.0000	4.0000	3.3971	0.1048
β_{10}	0.0000	1.0000	0.1535	0.0075
β_{11}	0.0000	0.0500	0.0070	0.0005
β_{12}	0.1000	0.3000	0.1972	0.0112
β_{13}	-0.5000	0.5000	-0.3333	0.0207
β_{14}	0.0000	0.5000	0.4511	0.0191
β_{15}	1.0000	1.1000	1.0000	
β_{16}	20000	40000	28580.7	100.000
β_{17}	-1.0000	-0.0100	-0.0832	0.0313
β_{18}	0.5000	1.0000	0.9312	1.8023
β_{19}	0.0000	1.0000	0.1593	0.0070
β_{20}	0.0050	0.0500	0.0073	0.0016
β_{21}	0.0050	0.0500	0.0060	0.0007
β_{22}	10000	35000	29500	2282.71
β_{23}	0.1000	1.0000	0.5000	0.0595
β_{24}	0.5000	1.5000	1.0000	0.0692
β_{25}	0.0000	0.0100	0.0057	0.0004
β_{26}	0.0000	0.0200	0.0100	0.0016
β_{27}	0.0000	1.0000	0.2520	0.0175
β_{28}	-2.0000	2.0000	-0.1185	0.0045
β_{29}	0.0000	100.00	57.0548	1.3396
β_{30}	0.0000	10.000	0.1577	0.0051
β_{31}	0.0000	0.1000	0.0007	0.0001
β_{32}	0.5000	1.5000	0.8959	0.1045
β_{33}	0.0000	10.000	0.0089	0.0016
γ_1	0.0010	4.0000	0.5882	0.0695
γ_2	0.1000	4.0000	0.1001	0.0177

γ_3	0.0010	4.0000	0.0100	0.0017
γ_4	0.1000	4.0000	3.9999	0.0751
γ_5	0.0010	4.0000	0.2760	0.0010
γ_6	0.1000	4.0000	0.0976	0.0053
γ_7	0.0001	4.0000	0.0009	0.00004
γ_8	0.0010	4.0000	0.1029	0.0021
γ_9	0.0010	4.0000	0.0944	0.0035
γ_{10}	0.1000	4.0000	0.2384	0.0216
γ_{11}	0.0010	4.0000	0.0021	0.00002
γ_{12}	0.1000	4.0000	3.9590	0.0253
γ_{13}	0.0010	4.0000	0.6494	0.0102
γ_{14}	0.0010	4.0000	0.0245	0.0003
γ_{15}	0.1000	4.0000	0.4932	0.0061
γ_{16}	0.0010	4.0000	0.0292	0.0002
γ_{17}	0.0010	4.0000	0.1039	0.0025
γ_{18}	0.0010	4.0000	0.1019	0.0005
γ_{19}	0.0010	4.0000	0.1008	0.0005
γ_{20}	0.0010	4.0000	3.9184	0.0765
γ_{21}	0.0010	4.0000	1.4944	0.0126
γ_{22}	0.1000	4.0000	0.2033	0.0016
γ_{23}	0.0010	4.0000	0.0075	0.0001
γ_{24}	0.1000	4.0000	3.9345	0.0088
γ_{25}	0.0010	4.0000	0.1644	0.0020
γ_{26}	0.0010	4.0000	0.1244	0.0015
γ_{27}	0.0000	4.0000	0.0009	0.00002
λ_1	0.0000	0.0100	0.0028	0.00002
λ_2	-0.0100	0.0100	0.0068	0.00005
λ_3	0.0000	0.0500	0.0098	0.0001

TABLE 3: Values of Exogenous Variable Time Path Parameters Assumed in Steady State Analysis

Parameter	Assumed Value
B^*	178786
E_o^*	1580
g^*	0.2400
K_p^*	172434
M^*	17143
p_f^*	1.0000
p_i^*	1.0000
r_f^*	0.00984
T_1^*	1.3103
T_2^*	1.1526
Y_f^*	0.5514
λ_4	0.0048
λ_5	0.0000

TABLE 4: Derived Steady State Growth Rates

Variable	Instantaneous Growth Rate (% pa)
$C, E_n, F, I, K, K_a, K_h, P, Q, S$	1.92
L	0.80
p	0.00
w	1.12
r	0.00
q	0.00

TABLE 5: Derived Steady State Levels

Variable	Steady State Level Parameter	Steady State Level in Last Quarter of 1996	Actual Level in Last Quarter of 1996
C	79441	121196	100619
E_n	26456	40361	42999
F	648	989	1096
I	22152	33797	47103
K_h	303900	463636	579636
K	953253	1454301	713621
K_a	-12837	-19584	-18154
L	28580	34080	26244
P	-159	-243	1570
p	0.4422	0.4422	1.2449
Q	114138	174131	142808
q	1.8724	1.8724	0.9139
r	0.0100	0.0100	0.0184
S	63918	97514	116833
w	1.3146	1.6819	4.5106

TABLE 6: Eigenvalues for Linear Approximation about Steady State

Real Part	Imaginary Part
-3.9952	0.0000
-3.9232	0.0000
-1.4992	0.0000
-0.5834	0.0000
-0.4928	0.0000
-0.1604	0.0000
-0.1036	0.0000
-0.1003	0.0000
-0.0597	0.0000
0.0058	0.0000
-0.0069	0.0000
-0.0428	0.0000
-0.0021	0.0000
-4.0449	± 0.3646
0.1622	± 0.1546
-0.0500	± 0.0866
0.0047	± 0.0302
-0.0524	± 0.0196

TABLE 7: Root Mean Square Errors of Dynamic Forecasts 1995Q1-1996Q4

Variable	Continuous Time	VARX
$\log C$	0.0177	0.0136
$\log E_n$	0.0875	0.1119
$\log I$	0.1508	0.1308
$\log K_h$	0.0082	0.0021
$\log K$	0.0013	0.0078
$\log L$	0.0177	0.0054
$\log p$	0.0410	0.0257
$\log Q$	0.0371	0.0185
$\log q$	0.0809	0.0439
r	0.0036	0.0080
$\log S$	0.3223	0.0100
$\log w$	0.0543	0.0256