KANT AND LINDAHL

By

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1. Introduction

A good is public if its consumption is nonrival, and some public goods are nonexcludable. We consider three different ways by which the amounts of a public good and the distribution of its costs are determined: (i) The political system; (ii) Private supply by voluntary contributions; (iii) The supply of an excludable public good by private firms.

1.1. Public provision by the political system

For many public goods, the decisions on their provision and financing are made by the public sector. These goods are quantitatively important, their supply requiring a substantial fraction of the budget at all levels of government. Knut Wicksell (1896) and Erik Lindahl (1919) analyzed the operation of a representative parliament.[1] They envisaged a negotiation among the various parties in the parliament until all universally beneficial alternatives were exhausted. The resulting unanimous agreement yields (Pareto) efficiency. Section 2 below offers a precise model.

1.2. Private provision by voluntary contributions

Some public goods are provided outside the public sector and outside the market by individual voluntary contributions: the textbook example is public radio. These tend to be quantitatively less important than the ones provided by the government. But information is a public good *par excellence*: thanks to the internet, much information is now provided on a voluntary contribution basis (Wikipedia, consumer ratings of products…).

The voluntary provision of public goods is most naturally formalized as a game in normal formal, where each player chooses her individual strategy, namely her contribution. The early analysis (Mancur Olson, 1965, see Theodore Bergstrom, Lawrence Blume and Hal Varian, 1986,

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[*] We are deeply indebted to Andreu Mas-Colell for useful comments and suggestions, with the usual *caveat.*

[1] See Silvestre (2003) for a discussion.

and the references therein) adopted the Cournot-Nash equilibrium concept, and emphasized its typical inefficiency, the "free rider problem," which reflects the absence of cooperation.

In contrast, Roemer (2010, 2019) offers a cooperative approach to normal-form games. It is based on an alternative behavioral assumption or optimization protocol inspired by Immanuel Kant's (1758) categorical imperative (sections 3 and 5 below). The resulting equilibrium allocations are efficient.

1.3. Private provision by the market

If an individual can be prevented from accessing an existing public good, then it is possible to finance its supply by access prices or user fees. In our internet age, with encrypted passwords, it is cheap for a supplier to deny access to software, networks, news, satellite TV, Google, and all sorts of information. Hence, significant public goods are now provided by privately owned, profit maximizing firms. (Section 6 below.)

Some of the issues concerning these markets are studied in the economics of public utilities.[2] Duncan Foley (1970) (see also Françoise Fabre-Sender, 1969, and John Roberts, 1974) reformulated the insights of Lindahl (1919) eliminating the public sector and the political parties and envisioning instead a pseudo-Walrasian equilibrium where the public good is traded by price-taking buyers and sellers. The equilibrium is then blessed with efficiency by the Invisible Hand.

1.4. Our analysis

We see that three of these models (Wicksell-Lindahl, Kant, and Foley) yield efficiency. In addition, there are formal parallelisms among them. The relationship between Wicksell-Lindahl and Foley has been provided, *e. g.*, by Andreu Mas-Colell and Silvestre (1989): see Fact 6.1 below. The present paper focuses on the relationship between the Wicksell-Lindahl and Kantian models.

We emphasize the fundamental difference between these two worlds. In Wicksell and Lindahl, the public sector deliberates and decides, in a centralized way, on the supply of the public goods and on their financing: this is the world of national defense. In Roemer's Kantian approach, each person separately decides how much to contribute towards the financing of the public good: this is the world of public radio. Both models display efficiency. Moreover, the primary versions of either model are defined by the same individual optimization problem (Section 4.1 below) and,

[2] See Silvestre (2012, Ch. 5)

under differentiability, a person's cost share equals her Lindahl Ratio (*i. e.*, her marginal valuation of the public good divided by the social marginal valuation: Section 4.2 below); in other versions, a person's cost share departs from her Lindahl ratio in ways made precise in sections 5 and 6 below.

1.5. The economy

We consider a society with one private, desirable, non-produced good, and M produced public goods. A vector of public goods is denoted $y=(y^1,y^2,...,y^M)\in\Re_+^M$. The social technology is defined by a cost function $C:\Re_+^M\to\Re_+$, where $C(y)$ is the amount of the private good required to produce the vector y of public goods.

There are N persons in society. Person i is endowed with ω_i units of the private good and a utility function $u_i(x_i, y)$, strictly increasing in x_i, Person i's consumption of the private good.

A *state* of the economy is a vector $(x, y) = (x_1,\ldots, x_N; y^1,\ldots,y^M) \in\Re_+^{N+M}$.

Definition 1.1. A state (x, y) is *feasible* if $C(y)\le\sum_{i=1}^N[\omega_i-x_i]$.

Definition 1.2. A feasible state (x, y) is *efficient* if there is no other feasible state which gives every person a weakly greater utility, and some person a greater utility.

2. The political system: Wicksell and Lindahl

Wicksell (1896 [1958, pp. 89-90]) states:

" As long as the project in question permits the creation of utility beyond its cost, it would always be theoretically possible, and often feasible in practice, to find a cost-sharing scheme such that all parties consider the project beneficial, so that their unanimous approval would be possible."

The Wicksellian view is thus centralized: the parties in the parliament negotiate and find a cost-share scheme at which they unanimously agree on the vector of public goods. The idea of unanimity based on a cost-sharing scheme motivates the analyses by Mamoru Kaneko (1977, see Definition 7.3 below) and Mas-Colell and Silvestre (1989). We adapt the following definition from the latter.

Definition 2.1. A *cost share system* is a family of N functions $g_i:\Re_+^M\to\Re$, $i = 1,\ldots, N$, such that $\sum_{i=1}^N g_i(y)=C(y)$ for all y.

Definition 2.2. A *Cost Share Equilibrium* is a pair comprising a state (x^*, y^*) and a cost share system $(g_1, \ldots, g_N)$ such that, for $i = 1,\ldots, N$:

(i) $x_i^* = \omega_i - g_i(y^*)$;

(ii) $u_i(x_i^*, y^*) \geq u_i(\omega_i - g_i(y), y)$, for all $y \in \Re_+^M$.

We then say that state (x^*, y^*) is *supported* by a Cost Share Equilibrium. We shall use the terms "supported" and "supportable" for any equilibrium concept.

Fact 2.1. It is shown in Proposition 1 of Mas-Colell and Silvestre (1989) that *a Cost Share Equilibrium is efficient*. The only assumption required is the above-mentioned increasingness of utility with respect to the private good.

Lindahl's (1919) formal discussion, presented in his doctoral thesis written under Wicksell's direction, considers two representative parties (for the rich and the poor), but he notes that the analysis can be extended to any number of parties.[3] Our discussion here admits an arbitrary number of parties. It is convenient to visualize all the constituents of a given party as having exactly the same interests, so that the preferences of Party i are truly representative of the interests of its constituents.[4] For convenience of comparison within sections, here we use the notation N for the number of parties (realistically, a few) rather than the number of individuals (some millions).

Lindahl (1919) specializes Wicksell's view in a formally explicit model that appeals to proportionality or linearity. In his words, (Lindahl, 1919 [1958, p. 173]): "collective goods do not have the same order of priority for all," in which case "each party must undertake to pay a greater share than the other toward the cost of these services which each finds most useful." This motivates the following definition (Mas-Colell and Silvestre, 1989). Here and in what follows we denote by $\Delta^{N-1} \equiv \{(\alpha_1,\ldots,\alpha_N) \in \Re_+^N : \sum_{i=1}^N \alpha_i = 1\}$ the $(N - 1)$-dimensional standard simplex.

Definition 2.3. A cost share system is *linear* if it is of the form $g_i(y) = \sum_{j=1}^M a_i^j y^j + b_i C(y)$, $i = 1,\ldots, N$, satisfying

$$(b_1,\ldots,b_N) \in \Delta^{N-1}; \tag{2.1}$$

[3] Thomas Piketty (2019, pp. 226-231) describes the Swedish society of the 19th century as extremely unequal and "quaternary," with four groups represented in the *riksdag* (parliament).

[4] The cost share of an individual will then be the cost share of her party divided by the number of constituents, see Silvestre (2012, Section 4.3.1).

For $j = 1,\ldots, M$, $\sum_{i=1}^{N} a_i^j = 0$. (2.2)

Definition 2.4. A *Linear Cost Share Equilibrium* (*LCSE*) is a Cost Share Equilibrium for a linear cost share system.

It follows from (2.2) that $\sum_{j=1}^{M} y^j \sum_{i=1}^{N} a_i^j = \sum_{i=1}^{N} \sum_{j=1}^{M} a_i^j y^j = 0$, *i. e.*, $\sum_{j=1}^{M} a_i^j y^j$ represents a net transfer from (or to) Person *i*, the aggregate net transfer being zero. We can interpret a Linear Cost Share Equilibrium as distributionally impartial if every individual transfer is zero, leading to the following definition.

Definition 2.5. A Linear Cost-Share Equilibrium is *Balanced* (a *BLCSE*) if, for $i = 1,\ldots, N$, $\sum_{j=1}^{M} a_i^j y^j = 0$.

Remark 2.1. Balanced Linear Cost Share Equilibria exist under convexity, as long as the private good is indispensable and nobody considers the public good a bad (Mas-Colell and Silvestre, 1989, Proposition 7). These equilibria may or may not exist under increasing returns to scale: it may depend upon the relative curvatures of the cost and indifference curves. Theorems 4.1-2 below prove existence under increasing returns to scale in a special case.

3. Kantian decentralized cooperation and the voluntary provision of public goods

3.1. Simple and Multiplicative Kantian Equilibria in abstract games

The Cournot-Nash model has become the canonical paradigm for the theory of human interaction in modern economics. Formally, a normal-form game is defined by a list $i = 1,\ldots,N$ of players, and, for $i = 1,\ldots, N$, a strategy set I_i and a payoff function $V_i : \prod_{h=1}^{N} I_h \to \Re$. In the Cournot-Nash approach, Player *i* chooses the strategy that is best for her while taking as given the strategies played by other players. The resulting equilibrium is typically inefficient, evidencing the noncooperative character of the Cournot-Nash Equilibrium.

But humans do cooperate (Roemer, 2019, Ch. 1). There are two approaches to including cooperation within the Cournot-Nash paradigm. First, one may design complex games with an indefinite number of stages where efficient outcomes are enforced by punishments (Michihiro Kandori, 1992). A second approach is to assume that the payoff functions involve altruism or a preference for socially efficient outcomes.

Roemer's Kantian approach maintains the modelling of a normal-form game but repudiates the Cournot-Nash behavioral assumption in all its forms. Instead, it bases cooperation on the desire to do "the right thing" in a version of Kant's categorical imperative. This leads to the following notions (Roemer 2010, 2019).

Definition 3.1. Let $((1,...,N),(I_1,...,I_N),(V_1,...,V_N))$ be a normal form game satisfying $I_i = I, \forall i$. A *Simple Kantian Equilibrium* is a strategy $t^* \in I$ satisfying, for *i* = 1,… , *N*, $V_i(t^*,...,t^*) \geq V_i(t,...,t), \forall t \in I$.

Simple Kantian Equilibria (usually) exist when the game is symmetric (*i*. *e*., there is a function $\hat{V}$ such that for all *i*, $V_i(t_1,...,t_N) = \hat{V}(t_i, t_{S\backslash i})$, where $t_{S\backslash i} = \sum_{h \neq i} t_h$), as in the Prisoner's Dilemma. They may or may not exist when the set *I* is discrete. They typically fail to exist when I_i is an interval of real numbers, which we assume in what follows, and players have different payoff functions. (Simple Kantian Equilibria do exist with identical players, see Theorem 4.2 below.) But Kantian equilibria according to the following definition often exist. Other "kinds of action" are described in Section 5.2 below.

Definition 3.2. Let I_i be an interval of real numbers, *i* = 1,…, *N*. A strategy profile $(t_1,...,t_N)$ is a *Multiplicative Kantian Equilibrium* if, for *i* = 1, …, *N*, $\rho = 1$ solves $\max_{\rho \geq 0} V_i(\rho t_1,...,\rho t_N)$ subject to $\rho t_i \in I_i$.

In words, no player would like to re-scale the whole vector of strategies.

3.2. Simple and Multiplicative Kantian Equilibria for the voluntary contribution game

Many of our points are most clearly expressed in a model with a single public good. Accordingly, we focus on the case *M* = 1 in sections 3-6 here and below. We suppress the superindices in the notation of sections 1.5 and 2 above. We take the cost function $C: \Re_+ \to \Re_+$ to be strictly increasing, and, hence, invertible.

Definition 3.3. The *Voluntary Contribution Game* is the game in normal form where, for $i = 1,...,N$, Player *i*'s strategy is $t_i \in [0, \omega_i] \equiv I_i$ and her payoff function is

$$V_i(t_1,...,t_N) = u_i(\omega_i - t_i, (C)^{-1}(\sum_{h=1}^{N} t_h)).$$

The Cournot-Nash equilibria of this game are typically inefficient (*free rider problem*).

Definitions 3.1 and 3.2 can be applied to the Voluntary Contribution Game as follows.

Definition 3.4. The vector $((x^*,y^*),(t_1^*,...,t_N^*)\in \Re_+^{N+1+N}$ is a *Simple Kantian Equilibrium* (*SKE*) *for M* = 1 if, for *i* = 1,…, *N,*

(i). $t_i^* = t^* \in [0,\omega_i]$, $x_i^* = \omega_i - t^*$;

(ii). (t^*,y^*) solves max $u_i(\omega_i - t_i, y)$ subject to $y = C^{-1}(Nt_i)$, $t_i \in [0,\omega_i]$. (3.1)

Definition 3.5. A *Multiplicative Kantian Equilibrium* (*MKE*) *for M* = 1 is a vector $((x^*,y^*),(t_1^*,...,t_N^*))\in \Re_+^{N+1+N}$ such that, for *i* = 1,…, N, $\rho = 1$ solves

$$\max_{\rho\in\Re_+} u_i(\omega_i - \rho t_i^*, (C^{-1})(\rho\sum_{h=1}^N t_h^*)) \text{ subject to } \omega_i - \rho t_i^* \in [0,\omega_i],$$

with $y^* = (C^{-1})(\sum_{h=1}^N t_h^*)$ and $x_i^* = \omega_i - t_i^*$.

In words, no person would like to transform the vector $(t_1^*,...,t_N^*)$ into the vector $(\rho t_1^*,...,\rho t_N^*)$ with the corresponding change in the level of the public good.[5]

4. Balanced Cost Share Equilibria and Simple or Multiplicative Kantian Equilibria

4.1. Equivalence results

Our next definition specializes definitions 2.1-5 above.

Definition 4.1. A *Balanced Linear Cost Share Equilibrium* (*BLCSE*) *for M* = 1 is a state $(x^*,y^*)\in \Re_+^{N+1}$ and a vector $(b_1,...,b_N)\in \Delta^{N-1}$ such that, writing $t_i^* = \omega_i - x_i^*$,

(t_i^*,y^*) solves max $u_i(\omega_i - t_i, y)$ subject to $t_i = b_i C(y)$, i = 1,…, *N*. (4.1)

Because $\sum_{h=1}^N b_h = 1$, $\sum_{h=1}^N t_h^* = C(y^*)$, *i. e.*, $t_i^* = b_i\sum_{h=1}^N t_h^*$. Hence, as long as $y^* > 0$, (4.1) can be written:

$$\max\ u_i(\omega_i - t_i, y) \text{ subject to } t_i = \frac{t_i^*}{\sum_{h=1}^N t_h^*} C(y). \qquad (4.2)$$

Theorem 4.1. Equivalence result for Simple Kantian Equilibria. *Let M* = 1. $((x^*,y^*),(t_1^*,...,t_N^*))\in \Re_+^{N+1+N}$ *is a Simple Kantian Equilibrium if and only if* (x^*,y^*) *is a Balanced Linear Cost Share Equilibrium with* $b_i = \frac{1}{N}, i = 1,...,N$.

[5] A related idea appears in Silvestre (1984) .

Proof. In a BLCSE with $b_i = 1/N$, we have [by (4.1)] that (t_i^*, y^*) maximizes $u_i(\omega_i - t_i, y)$ subject to $t_i = \frac{1}{N}C(y)$ .It follows that $t_i^* = \frac{1}{N}C(y^*) \equiv t^*$, and hence

$$(t^*, y^*) \text{ solves } \max\ u_i(\omega_i - t_i, y) \text{ subject to } t_i = \frac{1}{N}C(y). \qquad (4.3).$$

But this defines, by (3.3), a SKE: the objective function is the same in (3.1) and (4.3), and so is the constraint, because $y = C^{-1}(Nt) \Leftrightarrow \frac{C(y)}{N} = t$. ■

As noted in Section 3, Simple Kantian Equilibria typically fail to exist. We next show that they do in the special case where all N persons are identical.

Assumption 4.1. Identical persons. Let $M = 1$, $\omega_i \equiv \omega, \forall i$, and let the utility function be the same for $\forall i$, *i. e.*, $u_i \equiv u : \Re_+^2 \to \Re, i = 1,...,N$.

Theorem 4.2. *Assume $M = 1$ and identical persons. If u and C are continuous, and C is unbounded, then a Simple Kantian Equilibrium exists.*

Proof. Because C is unbounded, continuous and increasing, C and C^{-1} are defined and continuous on $\Re_+$. Thus, the function $\psi(t) \equiv u(\omega - t, (C^{-1})(Nt))$ is continuous on the compact interval $[0, \omega]$. ■

Consider Figure 4.1. The equilibrium, identical for all i, is found by maximizing $u(\omega - t_i, y)$ on the (thick) constraint curve. When reading the result with y as an independent variable, it is Cost-Share. When read with t_i as an independent variable it is Kant. In words, everybody is treated equally in terms of contributions: in one case (Cost-Share) by the imposition of equal cost shares; in the other by an internalized moral hypothesis (Kant). But then the result of the optimization must be the same.

We can further simplify the illustration letting $C(y) = y$ (constant returns to scale). Then the thick curve in Figure 4.1 becomes the straight line "$y = Nt_i$" or "$t_i = \frac{y}{N}$." Cost Share goes from y to t_i by dividing, while Kant goes from t_i to y by multiplying.

We now exit the identical person case: Simple Kantian Equilibria typically fail to exist when people have different endowments or preferences. Consider a Multiplicative Kantian

Equilibrium of Definition 3.5 above satisfying $(t_1^*,...,t_N^*) \in \Re_{++}^N$. We can then write $\rho = \frac{t_i}{t_i^*}$ and reword Definition 3.5 as follows.

Definition 4.2. A *Multiplicative Kantian Equilibrium for* $M = 1$ *with strictly positive contributions* is a vector $((x^*, y^*), (t_1^*,...,t_N^*)) \in \Re_+^{N+1} \times \Re_{++}^N$ such that, for $i = 1,\ldots, N$, (t_i^*, y^*) solves,

$$\max\ u_i(\omega_i - \frac{t_i}{t_i^*} t_i^*, y) \text{ subject to } y = C^{-1}\left(\frac{t_i}{t_i^*}\sum_{h=1}^N t_h^*\right),\ \ t_i \in [0, \omega_i], \tag{4.4}$$

with $y^* = (C)^{-1}\left(\sum_{h=1}^N t_h^*\right)$ and $x_i^* = \omega_i - t_i^*$.

Theorem 4.3. Equivalence result for Multiplicative Kantian Equilibria. *Let* $M = 1$; $((x^*, y^*), (t_1^*,...,t_N^*)) \in \Re_+^{N+1} \times \Re_{++}^N$ *is a Multiplicative Kantian Equilibrium if and only if* (x^*, y^*) *is a Balanced Linear Cost Share Equilibrium with* $b_i = \frac{\omega_i - x_i^*}{\sum_{h=1}^N [\omega_h - x_h^*]} = \frac{t_i^*}{\sum_{h=1}^N t_h^*}, i = 1,..., N.$

Proof. The proof parallels that of Theorem 4.2 by appealing to the optimization problems given by (4.2) for the BLCSE and (4.4) for the MKE. Again, their objective functions are the same, and so are their constraints, since

$$y = C^{-1}\left(\frac{t_i}{t_i^*}\sum_{h=1}^N t_h^*\right) \Leftrightarrow C(y) = \frac{t_i}{t_i^*}\sum_{h=1}^N t_h^* \Leftrightarrow t_i = \frac{t_i^*}{\sum_{h=1}^N t_h^*} C(y). \quad \blacksquare$$

Figure 4.2 illustrates. Again, when reading the result with *y* as an independent variable it is Cost-Share. When read with t_i as an independent variable it is Kant.

Remark 4.1. Theorem 4.3 generalizes Theorem 4 in Roemer (2010).

Remark 4.2. Note that Theorem 4.2 does not require convexity (or differentiability). Thus, Kantian Equilibria and Cost Share Equilibria are not incompatible with increasing returns: see Remark 2.1. By Fact 2.1, they are all efficient: this contrasts with the Cournot-Nash equilibria, typically inefficient even when all persons are identical (see, *e. g.*, Silvestre, 2012, Section 4.4.4).

4.2. The Lindahl Ratio

Assumption 4.2. Differentiability. The functions *C* and u_i, $i = 1, \ldots, N$, are differentiable. The derivative *C*', called the *marginal cost* and denoted $m(y)$ or simply *m*, is assumed positive.

We denote by $r_i \equiv \frac{\partial u_i}{\partial y} \Big/ \frac{\partial u_i}{\partial x_i}$ *i's marginal rate of valuation* of the public good. The following fact is well known.

Fact 4.1. (Samuelson Condition). *Assume differentiability and let the state* $(x, y) \in \Re^{N+1}_{++}$ *be efficient. Then* $\sum_{h=1}^{N} r_h(x_h, y) = m(y)$.

Definition 4.3. Person *i*'s *Lindahl Ratio at* $(x_i, y) \in \Re^2_{++}$ is $\frac{r_i(x_i, y)}{m(y)} \equiv L_i(x_i, y)$, the ratio of her marginal valuation of the public good to the social marginal valuation.

Theorem 4.4. *Let* $M = 1$. *Assume differentiability. Let the state* $(x^*, y^*) \in \Re^{N+1}_{++}$, *with* $t_i^* \equiv \omega_i - x_i^* > 0, \forall i$, *be supported by a Balanced Linear Cost Share Equilibrium or equivalently* (thanks to Theorem 4.3) *by a Multiplicative Kantian Equilibrium. Then for i* = 1,…, *N*, *i*'s *relative contribution towards the cost of the public good equals her Lindahl ratio*, *i. e.*,

$$\frac{\omega_i - x_i^*}{C(y^*)} \equiv \frac{t_i^*}{C(y^*)} = L_i(x_i^*, y^*) \quad . \tag{4.5}$$

Proof. The first order conditions of the (4.4) Lagrangean (we can equivalently use (4.2))

$$u_i(\omega_i - \frac{t_i}{t_i^*} t_i^*, y) \ - \ \lambda \left[y - C^{-1}\left(\frac{t_i}{t_i^*} \sum_{h=1}^{N} t_h^* \right) \right],$$

yield (4.5) , noting that $(C^{-1})' = \frac{1}{m}$ and that

$$C(y^*) = \sum_{h=1}^{N} t_h^* . \qquad \blacksquare \tag{4.6}$$

By the equivalence between Multiplicative Kantian Equilibrium and Balanced Linear Cost Share Equilibrium, either concept offers a theory of cost sharing. Here Kant meets Lindahl in a strong way: your relative payment agrees with your relative marginal benefit, a version of the "marginal benefit principle" which Lindahl defended normatively: see Section 8.1 below.

Remark 4.3. By the Samuelson Condition, $\sum_{h=1}^{N} L_h = 1$, and hence the *average* Lindahl Ratio is 1/*N*. Therefore, we will typically have $\frac{t_i}{C} = L_i > \frac{1}{N}$ for some *i* , whom we can call a "*High* (Marginal) *Valuator*," whereas $\frac{t_h}{C} = L_h < \frac{1}{N}$ for some *h*, a "*Low* (Marginal) *Valuator*."

Remark 4.4. Under differentiability and interiority, Multiplicative Kantian Equilibria are characterized by the system of $2N + 1$ equations (4.5) ($2N$ equations) and (4.6) in the $2N + 1$ unknowns $\{x_i\}$, $\{t_i\}$ and y. Hence, its solutions typically are locally unique and the set of states supportable by a Multiplicative Kantian Equilibrium, or, equivalently, by a Balanced Linear Cost Share Equilbrium, generically is a 0-dimensional manifold.

5. Additive and K^{β} - Equilibria and their relation to Linear Cost Share Equilibria

5.1. The case of Constant Returns to Scale

Theorem 4.2 (resp., 4.1) above shows the equivalence between Balanced Linear Cost Share Equilibria and Multiplicative Kantian equilibria (resp., Simple Kantian Equilibria, when they exist). Despite their different rationales, these primary formulations of Kant's and Lindahl's ideas are mathematically identical, and, as just noted in Remark 4.4, determinate.

We can consider broader versions of these equilibrium concepts which may in principle support larger sets of states (all efficient). On Lindahl's side, we can dispense with balancedness and consider Linear Cost Share Equilibria (Definition 2.4 above). And on Kant's side we can contemplate a more general notions of "same kind of action" formalized as β-Kantian or K^{β} equilibria (Roemer, 2019, and Section 5.2 below), and look at the set of states supported by such equilibria as β ranges over $[0,\infty)$.

It turns out that the connection between these broader Lindahlian and Kantian notions loses the simplicity of the primary notions in Section 4 above. We start with the special case of constant returns to scale. We apply definitions 2.4-5 to the one-public-good case, and adopt the notation of the first paragraph of Section 3.2.

Theorem 5.1. *Let $M = 1$. Assume constant returns to scale, i. e., $C(y) = \bar{m}y$ for some $\bar{m} \in \Re_{++}$, and that u_i is nondecreasing in y, $\forall i$. The state $(x^*, y^*) \in \Re_{+}^{N+1}$ is supported by a Linear Cost Share Equilibrium if and only if it is supported by a Balanced Linear Cost Share Equilibrium.*

Proof. Trivially, a BLCSE is a LCSE. So let (x^*, y^*) be supported by a LCSE with parameters $\{a_i\}$, $\{b_i\}$, *i. e.*, $\forall i, (x_i^*, y^*)$ solves max $u_i(x_i, y)$ s. to $x_i = \omega_i - a_i y - b_i C(y) =$ $\omega_i - [a_i + b_i \bar{m}]y$. Define, for $i = 1,\dots, N$, $\bar{a}_i = 0, \bar{b}_i = \frac{a_i}{\bar{m}} + b_i$. Then the constraint $x_i = \omega_i - \bar{b}_i \bar{m} y$ is the same as in the original LCSE, and hence the solutions are the same. We are left with checking that $\bar{b}_i \geq 0$, satisfied since otherwise the original LCSE optimization would not have a solution (because u_i is nondecreasing in y), and that $\sum_{h=1}^{N} \bar{b}_h = 1$, satisfied because, by (2.1) and (2.2), $\sum_{h=1}^{N} b_h = 1$ and $\sum_{h=1}^{N} a_h = 0$. Thus $\sum_{h=1}^{N} \bar{b}_h = \frac{1}{\bar{m}} \sum_{h=1}^{N} a_h + \sum_{h=1}^{N} b_h = 1$. ■

Remark 5.1. By Theorem 5.1, Linear Cost Share Equilibria do not expand, under constant returns, the 0-dimensional set of states supported by Balanced Linear Cost Share Equilibria (see Remark 4.4 above). Remarks 5.7 and 6.2 below offer some interpretation.

It follows from theorems 4.3 and 5.1 that, under constant returns to scale, a state is supported by a Linear Cost Share Equilibrium if and only if it is supported by a Multiplicative Kantian Equilibrium.

5.2. Additive and β - Kantian Equilibria in abstract games

We follow Roemer (2019) and return to the abstract normal-form games of Section 3.1 above. We now postulate that the strategy set I_i is an interval of real numbers, $i = 1,\ldots, N$.

Definition 5.1. A strategy profile $(t_1,...,t_N)$ is an *Additive Kantian Equilibrium* if, for $i = 1, \ldots, N$, $\tau = 0$ solves $\max_{\tau\in\Re} V_i(t_1+\tau,...,t_N+\tau)$ subject to $t_i + \tau \in I_i$.

Multiplicative and Additive Kantian optimization contemplate the set of counterfactuals that every player considers to a given vector of strategies $(t_1,...,t_N)$ to be either all *re-scalings* of that vector, or all *translations* of that vector, respectively. In both cases, all players contemplate choosing a preferred counterfactual which lies in a *common set* of counterfactuals. This is in contrast to the player who optimizes in the Nash manner: she contemplates choosing an altered profile of strategies in which *only her* strategy changes, while the strategies of all other players remain fixed. It is the consideration of a common set of counterfactuals which is the mathematical expression of cooperation.

Finally, we can consider an optimization protocol where the set of counterfactuals – again common to all – is generated by applying an affine transformation to the existing vector of strategies. This gives the following equilibrium concept.

Definition 5.2. Given $\beta \geq 0$, define $\varphi(t,\rho) \equiv \rho t + \beta[\rho - 1]$. A strategy profile $(t_1,...,t_N)$ is a $\beta-$*Kantian*, or K^β-, *Equilibrium* if, for $i = 1, \ldots, N$, $\rho = 1$ solves

$\max_{\rho\in\Re_+} V_i(\varphi(t_1,\rho),...,\varphi(t_N,\rho))$ subject to $\varphi(t_i,\rho)\in I_i$.

Remark 5.2. It immediately follows from definitions 3.1, 5.1 and 5.2 that a Simple Kantian Equilibrium, when it exists, is an Additive Kantian Equilibrium and a K^β-Equilibrium for any $\beta \geq 0$. This in particular applies to the Identical Persons case of Theorem 4.2 above.

We now show that K^β optimization comprises a continuum of possible optimization protocols with additive and multiplicative Kantian optimization as its two endpoints:

Theorem 5.2. *Suppose that the game in normal form* $\{V_i\}$ *is concave and continuously differentiable, where the strategy spaces* I_i *are closed intervals. Suppose, for sufficiently large* $\beta\in\Re_+$ *that* $t^\beta=(t_1^\beta,...,t_N^\beta)$ *is an interior* K^β*- equilibrium of* $\{V_i\}$ *and that* $t^*=(t_1^*,...,t_N^*)$ *is a limit point of* $\{t^\beta\}$ *as* $\beta\to\infty$ *. Then* t^* *is an Additive Kantian Equilibrium of* $\{V_i\}$ *.*

Proof. By interiority and concavity, the first-order condition for t^β to be a K^β - equilibrium is:

$$\sum_{h=1}^{N}\frac{\partial V_i}{\partial t_h}(t^\beta)[t_h+\beta]=0=\nabla V_i(t^\beta)\cdot t^\beta+\beta\nabla V_i(t^\beta)\cdot\mathbf{1},$$

where $\nabla V_i(t^\beta)$ is the gradient of V_i evaluated at t^β and **1** is the vector of 1's in $\Re^N$. Dividing this equation by β implies, as we let $\beta\to\infty$, that $\nabla V_i(t^*)\cdot\mathbf{1}=0$. But this is the necessary and sufficient condition for t^* to be an Additive Kantian Equilibrium of $\{V_i\}$. Note that t^* may itself not be interior. The stated first-order condition for t^* to be an additive Kantian equilibrium is correct because t^* is the limit of interior K^β equilibria. ■

Hence Multiplicative Kantian Equilbrium (the case $\beta=0$) and Additive Kantian Equilibrium are the two poles of the continuum of K^β equilibria.

Cooperation, defined as Kantian optimization, differs from altruism. Altruism is represented by a consumption externality in preferences: an altruistic player values increasing the utility or consumption of other players. Behavioral economists typically place 'exotic' arguments in players' preferences (*e. g*., the consumption of others), and then explain the apparently altruistic or cooperative outcome of the game with altered preferences as Cournot-Nash equilibria of that game. In contrast, the Kantian approach does not alter preferences from classical ones, but changes the optimization protocol. The cooperative morality, if you will, is not displayed in preferences, but is represented in how people optimize.

The appellation 'Kantian' is derived from the 'simple' case: here, t is the strategy that each player would like all players to play. In Immanuel Kant's language, each player is taking the action he 'would will be universalized.' The more general formulation is that all players agree on a strategy profile to be chosen from a common set of profiles.

A game is *strictly monotone increasing* (*decreasing*) if each payoff function V_i is a strictly increasing (decreasing) function of the strategies of the players other than i. Strictly increasing games are games with positive externalities, and strictly decreasing games are games with negative externalities. The Voluntary Contribution Game of Definition 3.3 is monotone increasing (as long as nobody dislikes the public good).

Fact 5.1. (Roemer, 2019). *In any strictly monotone game, Simple Kantian Equilibria, Additive Kantian equilibria, K^β - Equilibria with $\beta > 0$, and Multiplicative Kantian Equilibria with $t_i \in \text{Int} I_i, \forall i$, are efficient.*

Fact 5.1 justifies calling Kantian optimization a protocol of 'cooperation,' for it resolves efficiently the free rider problem (in monotone increasing games) and the "tragedy of the commons" (in monotone decreasing games, such as externality-causing activities) that characterize Cournot-Nash optimization.

5.3. Additive and β- Kantian Equilibria for the voluntary contributions game

Definitions 5.1-5.2 are adapted as follows.

Definition 5.3. An *Additive Kantian Equilibrium for M* = 1 is a vector $((x,y),(t_1,...,t_N)) \in \Re_+^{N+1+N}$ such that, for i = 1,…, N, $\tau = 0$ solves

$$\max_{\tau\in\Re} u_i(\omega_i - [t_i + \tau], (C^{-1})(N\tau + \sum_{h=1}^N t_h)) \text{ subject to } t_i + \tau \in [0, \omega_i] ,$$

with $y = (C^{-1})(\sum_{h=1}^N t_h)$ and $x_i = \omega_i - t_i, i = 1,...,N$.

Definition 5.4. Let $\beta \in \Re_+$ be given. A β- *Kantian*, or K^β-*Equilibrium for M* = 1 is a vector $((x,y),(t_1,...,t_N)) \in \Re_+^{N+1+N}$ such that, for i = 1,…, N, $\rho = 1$ solves

$$\max_{\rho\in\Re_+} u_i(\omega_i - [\rho t_i + \beta[\rho - 1]], (C^{-1})(N\beta[\rho - 1] + \sum_{h=1}^N \rho t_h)) \text{ subject to } \rho t_i + \beta[\rho - 1] \in [0, \omega_i],$$

with $$y = (C^{-1})(\sum_{h=1}^{N} t_h),\ x_i = \omega_i - t_i, i = 1,...,N. \quad (5.1)$$

By Theorem 4.3, and recalling Remark 2.1, Multiplicative Kantian Equilibria exist under standard assumptions. The following theorem covers existence for $\beta > 0$.

<u>Assumption 5.1. Convexity</u>. The function $C(y)$ is convex; the functions $u_i(x_i, y)$, $i = 1,\ldots, N$, are concave.

<u>Theorem 5.3</u>. *Let $M = 1$. Assume convexity and continuity, and let $u_i(x_i, y)$, $i = 1, \ldots, N$, be nondecreasing in y (nobody dislikes the public good).*

(i). *If $\beta > 0$, then a K^{β}-Equilibrium exists.*

(ii). *An Additive Kantian (i. e., $K^{\infty}-$) Equilibrium exists.*

<u>Proof</u>. **(i).** Let $\beta > 0$ be given. Write $\varphi : \Re_+^2 \to \Re : \varphi(t,\rho) = \rho t + \beta[\rho - 1]$, and $\Omega = \prod_{i=1}^{N}[0,\omega_i]$. Given $(t_1,...,t_N) \in \Omega$ and $i \in \{1,...,N\}$, define the function

$$\xi_i[t_1,...,t_N] : \Re_+ \to \Re_+ : \ \xi_i[t_1,...,t_N](\rho) = u_i(\omega_i - \varphi(t_i,\rho), C^{-1}(\sum_{h=1}^{N}\varphi(t_h,\rho))).$$

Because u_i is concave and nondecreasing in (x_i, y), and φ is linear in ρ, and C^{-1} is concave, we have that $\xi_i[t_1,...,t_N]$ is a concave, continuous function of ρ. Therefore, since $\beta > 0$, the set

$$\rho_i^*(t_1,...,t_N) \equiv \underset{\frac{\beta}{t_i+\beta} \le \rho \le \frac{\omega_i+\beta}{t_i+\beta},}{\arg\max}\ u_i(\omega_i - \varphi(t_i,\rho), C^{-1}(\sum_{h=1}^{N}\varphi(t_h,\rho)))$$

is nonempty, compact and convex. Note that if $\rho_i \in \rho_i^*(t_1,...,t_N)$, then $\varphi(t_i,\rho_i) \in [0,\omega_i]$, and $\frac{\beta}{\omega_i + \beta} \le \frac{\beta}{t_i + \beta} \le \rho \le \frac{\omega_i + \beta}{t_i + \beta} \le \frac{\omega_i + \beta}{\beta}$. By Berge's Theorem, the correspondence ρ_i^* that assigns the set $\rho_i^*(t_1,...,t_N)$ to a vector $(t_1,...,t_N) \in \Omega$ is bounded, upper hemicontinuous and convex valued. Define the correspondence

$$B_i : \Omega \to\to [0,\omega_i] : B_i(t_1,...,t_N) = \{\tilde{t}_i \in [0,\omega_i] : \tilde{t}_i = \varphi(t_i,\rho_i) \text{ for some } \rho_i \in \rho_i^*(t_1,...,t_N)\}.$$

In words, the best reply correspondence of Player *i* at a proposed vector of strategies $(t_1,...,t_N)$ is to transform her strategy by the 'factor' ρ that is ideal for *i,* restricting ρ to values that generate feasible strategies *for Player i.*

Again because φ is linear in ρ, the set $B_i(t_1,\ldots, t_N)$ inherits compactness and convexity from the set $\rho_i^*(t_1,...,t_N)$, and the correspondence B_i inherits upper hemicontinuity from the correspondence ρ_i^*.

Define the correspondence $\Phi:\Omega \to\to \Omega$ by $\Phi(t_1,...,t_N) = (B_1(t_1,...,t_N),...,B_N(t_1,...,t_N)).$ By Kakutani's Theorem Φ has a fixed point $(t_1^*,...,t_N^*) \in \Omega$, *i. e.*, for $i = 1,\ldots, N$, $t_i^* = \rho_i[t_i^* + \beta] - \beta$, *i.e.*, $t_i^*[1-\rho_i] + \beta[1-\rho_i] = 0$, for some $\rho_i \in \rho_i^*(t_1^*,...,t_N^*)$. Since $\beta > 0$ and $t_i^* \geq 0$, it follows that, for all *i*, such $\rho_i = 1$. So a fixed point is a K^β - Equilibrium.

(ii). Adapt the argument in **(i)** by replacing $\rho_i^*(t_1,...,t_N)$ by the set

$$\tau_i^*(t_1,...,t_N) = \underset{-t_i \leq \tau \leq \omega_i - t_i}{\arg\max}\, u_i(\omega_i - (t_i + \tau), C^{-1}(t_S + N\tau)).$$

At a fixed point $(t_1^*,...,t_N^*)$, $t_i^* = t_i^* + \tau_i$, *i. e.*, $\tau_i = 0, i = 1,...,N$. Thus, a fixed point is an Additive Kantian Equilibrium. ■

Remark 5.3. Equilibria may involve corner solutions $x_i \in \{0,\omega_i\}$, for some *i*.

Remark 5.4. In Section 3.1, we contrasted Kantian equilibria in games to the typical approach in behavioral economics to explaining non-classical Cournot-Nash equilibria in real situations, which is defining preferences over non-standard arguments (*e. g.*, other people's consumption or utility). Typically this is done in an experimental setting, where the game being played is quite a simple one. Often, indeed, these experimental games are symmetric. Theorem 5.2, in contrast, tells us that, in the voluntary contribution game, Kantian equilibria exist for very complex and heterogeneous preferences (assuming only they are continuous and convex). How would one design exotic preferences whose *Nash* equilibrium would, in such a complex, altered game, be Pareto efficient? Roemer (2019, Chapter 6) argues there is no acceptable way of accomplishing this: the move of including exotic arguments in preferences is *ad hoc* and insufficiently general.

Lemma 5.1. *Let M =* 1 *and assume differentiablility and convexity, so that the first order conditions are necessary and sufficient. The first order condition for an interior solution, i. e.,* $t_i \in (0,\omega_i)$, *to i's maximization problem, i =* 1*,…, N, are as follows.*

For a K^β*-Equilibrium,* $\beta \geq 0$ (Definition 5.4):

$$L_i \equiv \frac{r_i}{m} = \frac{t_i + \beta}{\sum_{h=1}^{N} t_h + N\beta}; \tag{5.2}$$

For an Additive Kantian-Equilibrium (Definition 5.3) :

$$L_i \equiv \frac{r_i}{m} = \frac{1}{N}. \tag{5.3}$$

Proof. Differentiate the relevant objective function and note that $(C^{-1})' = \frac{1}{C'} = \frac{1}{m}$. ■

We can rewrite (5.2) as $\omega_i - x_i = t_i = L_i[C + N\beta] - \beta$, which if divided by C yields:

$$\frac{\omega_i - x_i}{C} = \frac{t_i}{C} = L_i + \frac{\beta N}{C}\left[L_i - \frac{1}{N}\right] \quad . \tag{5.4}$$

Whenever $\beta > 0$, from (5.4) we obtain that:

* If i is a High Valuator (*i. e.*, $L_i > \frac{1}{N}$), then: $\frac{\omega_i - x_i}{C} > L_i$,

* If h is a Low Valuator (*i. e.*, $L_h < \frac{1}{N}$), then: $\frac{\omega_h - x_h}{C} < L_h$.

In words, when $\beta > 0$ a High (resp. Low) Valuator's share in the cost is higher (resp. less) than her Lindahl Ratio -- and also than the average share. Compare with Theorem 4.4 above, where (4.5) is (5.4) for $\beta = 0$, and a person's cost share is her Lindahl Ratio.

Remark 5.5. An interior K^{β}- (resp. Additive Kantian) Equilibrium is determinate (generically locally unique) in the sense that the equations defining it, namely the $2N+1$ equations (5.1) & (5.2) (resp., (5.1) & (5.3)) in the $2N+1$ unknowns $\{t_i\},\{x_i\}$ and y admit zero degrees of freedom. To the extent that changes in β change the equilibrium state, local uniqueness carries over, and thus the set of states supportable by K^{β}-equilibria as β ranges over $[0,\infty)$ is 1-dimensional. See the discussion at the end of Section 5.4.

What can we say about the K^{β}- equilibria as β becomes larger? Because, $\forall i, t_i$ is bounded, as long as all the solutions are interior, (5.2) yields [6]

[6] Equalities (5.3) and (5.5) illustrate the identification of a K^{∞}-Equilibrium with an Additive Kantian Equilibrium, see Theorem 5.2. The identification of a K^{0}-Equilibrium with a Multiplicative Kantian Equilibrium is displayed by (4.5) and (5.2).

$$\lim_{\beta\to\infty} L_i = \frac{1}{N}. \tag{5.5}$$

Remark 5.6. Intuitively, high valuators pay more, hence they consume a smaller amount of the private good, which typically increases its marginal utility, and hence decreases the marginal rate of valuation $r_i(x_i, y) \equiv \frac{\partial u_i}{\partial y} \Big/ \frac{\partial u_i}{\partial x_i}$. Conversely for low valuators. Of course, if the marginal utility of the private good is not decreasing, as in the quasilinear case, then this process does not work. Indeed, it may well happen that solutions are not interior, and (5.5) may fail. (See also Remark 6.3 below.)

Even if (5.4) holds for all β, we do not know how the RHS of (5.4) behaves as $\beta \to \infty$, since, by (5.5), its limit is indeterminate of the type $\infty \bullet 0$. We have computed examples where (5.4) holds for all β but, as $\beta \to \infty$, $\frac{t_i}{C}$ converges to a limit other than $\frac{1}{N}$.

5.4. Linear Cost Share Equilibria and β-Kantian Equilibria

The constant-returns-to-scale case has been analyzed in Section 5.1. We now assume differentiability, convexity and interiority in order to analyze the relationship between Linear Cost Share Equilibria and K^{β}- (β-Kantian **)** Equilibria when $my > C(y)$ (decreasing returns to scale). Applying Definition 2.4 to the $M = 1$ case and appealing to the first-order conditions, an interior Linear Cost Share Equilibrium is characterized by the following equalities:

$$a_i + b_i m = r_i, i = 1,..., N; \tag{5.6}$$

$$x_i = \omega_i - a_i y - b_i C(y), i = 1,..., N. \tag{5.7}$$

Remark 5.7. The set of states supportable as a Linear Cost Share Equilibrium under constant returns to scale typically is 0-dimensional (Remark 5.1). Indeed, we can then collapse equations (5.6) & (5.7) into the N equations: $x_i = \omega_i - r_i(x_i, y)y, i = 1,..., N$, which together with $\sum_{h=1}^{N}[\omega_i - x_i] = \bar{m}y$ constitute a determinate system of $N + 1$ equations in the $N + 1$ unknowns (x, y). But for nonconstant returns to scale with $C(y) \neq my$, then, as argued in Mas-Colell and Silvestre (1989), the $2N + 2$ equations (5.6), (5.7), (2.1) and (2.2) in the $3N + 1$ unknowns $\{x_i\}$, $\{a_i\}$, $\{b_i\}$ and y display N - 1 degrees of freedom, indicating that the set of states supportable as a Linear Cost Share Equilibrium typically is (N - 1)-dimensional. See Remark 6.2 below.

Consider a state supported by a K^{β}- Equilibrium or by an Additive Kantian Equilibrium. When does it also belong to the set of states supported by Linear Cost Share Equilibria?

<u>Theorem 5.4</u>. *Let M* = 1. *Assume differentiability and convexity. Let* $((x,y),(t_1,...,t_N)) \in \Re_{++}^{N+1+N}$ *satisfy* $t_i \in (0,\omega_i)$ *and* $t_i = \omega_i - x_i, \forall i$, $\sum_{h=1}^{N} t_h = C(y)$ *and* $my > C(y)$.

(i) *Let* $((x,y),(t_1,...,t_N))$ *be a* $K^{\beta}-$*Equilibrium. Then the state* (*x*, *y*) *can be supported by a Linear Cost Share Equilibrium if and only if*

$$my - C(y) \geq \beta\left[N - \frac{1}{L_i}\right], \forall i . \qquad (5.8)$$

(ii) *Let* $((x,y),(t_1,...,t_N))$ *be an Additive KantianEquilibrium. Then the state* (*x*, *y*) *can be supported by a Linear Cost Share Equilibrium if and only if*

$$\frac{my}{N} \geq t_i, \forall i . \qquad (5.9)$$

<u>Proof</u>. Note that, under (5.6), (5.7) holds if and only if $t_i = [r_i - b_i m]y + b_i C$, *i. e.*,

$$b_i = \frac{r_i y - t_i}{my - C(y)} . \qquad (5.10)$$

At a Kantian Equilibria (efficient by Fact 5.1), the Samuelson Condition (Fact 4.1) then yields $\sum_{i=1}^{N} b_i = 1$ and thus $\sum_{i=1}^{N} a_i = 0$. Hence (*x*, *y*) is supportable by a LCSE if an only if b_i as defined by (5.10), is nonnegative for all *i*, so that $(b_1,...,b_N) \in \Delta^{N-1}$.

(i) K^{β}<u>*E*</u>. Using (5.4), compute $r_i y - t_i = mL_i y - L_i C - N\beta\left[L_i - \frac{1}{N}\right]$, *i. e.*,

$\frac{r_i y - t_i}{L_i} = my - C - \beta\left[N - \frac{1}{L_i}\right]$, nonnegative if and only if (5.8) holds.

(ii) <u>Additive KE</u>. Same argument, using (5.3) and (5.9) instead of (5.4) and (5.8). ■

Informally, (5.8) suggests that a large *my* – *C* gap, modest differences in individual valuations (or Lindahl ratios) and a small β helps put a $K^{\beta}-$Equilibrium in the Linear Cost Share Equilibrium set. For the Additive Kantian Equilibrium, (5.9) indicates that what matters is that individual contributions be not too different.

As noted in Section 5.1 above, K^{β}- Equilibria (resp., Linear Cost Share Equilibria) broaden the notion of Multiplicative Kantian Equilibria (resp., Balanced Linear Cost Share Equilibria). But we submit that they expand the supportable set along different directions.

Consider first the extreme case of cons*tant returns to scale*. Then relaxing balancedness does not expand the 0-dimensional set of states supportable by Linear Cost Share Equilibria (Remark 5.1). But it can be evidenced by simple examples that varying β does change the equilibrium state (see Remark 5.5).

Next, consider another extreme case, namely *identical persons*. Simple examples show that, as long as $my > C(y)$, a $(N - 1)$-dimensional set of states can then be supported as Linear Cost Share Equilibria (with different $\{a_i\}$, $\{b_i\}$ parameters: see remarks 5.7 above and 6.2 below). But K^{β}- Equilibria behave quite differently. By Remark 5.2, a Simple Kantian Equilibrium exists and is a K^{β}-Equilibrium for any $\beta \geq 0$. No other K^{β}- equilibria exist for the examples that we have worked out, so that varying β changes nothing there and the set of supportable states is 0-dimensional.[7]

Last, the LHS of (5.8) depends on the degree of convexity in the cost function, whereas its RHS, positive for high valuators, is likely higher the more valuations differ among people.

These comments suggest that the broadening of the set of states supportable as Linear Cost Share Equilibria is based on the technology, *i. e.*, on the gap $my – C(y)$, while that of K^{β}- Equilibria relies on the diversity in people's preferences.

6. Private vs. public ownership of the technology

6.1. Procuring the public good from a privately-owned firm

We can visualize the operation of the Wicksell-Lindahl parliament of Section 2 above (for $M = 1$) as follows: after an agreement is reached, an agency transfers the budget $T = C(y)$ to the production sector. Similarly, in the Voluntary Contribution Game of Definition 3.3 somebody collects the sum $T \equiv \sum_{h=1}^{N} t_h$ and sends it to the production sector, which then makes

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[7] In general, we cannot rule out in the identical persons case the possibility of K^{β}-equilibria which are not Simple Kantian equilibria. But Remark 5.5 indicates that the set of states supported as β ranges over $[0,\infty)$ would be at most unidimensional.

the amount $C^{-1}(T)$ available. These descriptions fit well with the notion that the technology $C(y)$ is publicly owned.

Consider now a *privately* owned technology, with an exogenously given vector $\theta \equiv (\theta_1,...,\theta_N) \in \Delta^{N-1}$, of profit share parameters reflecting private property rights. Section 6.2 below contemplates an *excludable* public good, directly sold to users by a privately-owned firm (see Section 1.3 above). But there is no market when the public good is nonexcludable. We may then extend as follows the concepts in Sections 3 and 4 above.

The Wicksell-Lindahl government budget T or the Kantian sum $T = \sum_{h=1}^{N} t_h$ of voluntary contributions is now devoted to *buy* or *procure*, the public good from the privately owned firm. Either a government agency or an agent for the Kantian contributors negotiates a procurement contract with the firm. The contract specifies two magnitudes: the quantity y to be supplied and the total payment T. (Dividing T by y defines the average price.)

Postulate a *procurement rule* $\tilde{T}: \Re_+ \to \Re_+ : y \mapsto \tilde{T}(y)$, strictly increasing, with inverse $\tilde{T}^{-1}(T) \equiv \tilde{y}(T)$. The firm's profits under a procurement contract $(y, \tilde{T}(y))$ are $\tilde{T}(y) - C(y) =$ $T - C(\tilde{y}(T))$. Some simple examples.

<u>Example 6.1. Cost-plus-fixed-fee contract</u>. $\tilde{T}(y) = F + C(y), \forall y > 0$.

<u>Example 6.2. Affine payment function</u>. $\tilde{T}(y) = F + qy, \forall y > 0$.

<u>Example 6.3. Markup contract</u> (also called Cost-plus-percent-of-cost contract). $\tilde{T}(y) = \gamma C(y), \gamma > 1, \forall y \geq 0$.

We modify our previous equilibrium notions to take into account *both* the individual share in the budget and the distribution of profits.

<u>Definition 6.1</u>. Let $M = 1$. A *Balanced Cost Share Equilibrium for a privately owned technology with the vector* θ *of profit-share parameters and with procurement rule* $\tilde{T}$ *(or* $\tilde{y} \equiv \tilde{T}^{-1}$*)* is a state $(x^*, y^*) \in \Re_+^{N+1}$ and a vector of parameters $(b_1,...,b_N) \in \Delta^{N-1}$ such that, defining

$$\Lambda_i(y \mid b_i) \equiv u_i(\omega_i - b_i\tilde{T}(y) + \theta_i\left[\tilde{T}(y) - C(y)\right], y),\ i = 1,\dots N, \quad (6.1)$$

we have that
$$\Lambda_i(y^* \mid b_i) \geq \Lambda_i(y \mid b_i), \forall y \geq 0, \forall i, \quad (6.2)$$

with $x_i^* = \omega_i - b_i\tilde{T}(y^*) + \theta_i[\tilde{T}(y^*) - C(y^*)] \geq 0, \forall i$.

Remark 6.1. Writing $g_i(y) = b_i\tilde{T}(y) - \theta_i\left[\tilde{T}(y) - C(y)\right]$, the equilibrium of Definition 6.1 satisfies Definition 2.2 and hence, by Fact 2.1, is efficient.

Our next definition parallels Definition 4.2 above.

Definition 6.2. Let $M = 1$. *A Multiplicative Kantian Equilibrium with strictly positive contributions for a privately owned technology with the vector* θ *of profit-share parameters and procurement rule* $\tilde{y}$ *(or* $\tilde{T} \equiv \tilde{y}^{-1}$) is a state $(x^*, y^*) \in \Re_+^{N+1}$ and a vector $(t_1^*, ..., t_N^*) \in \Re_{++}^N$ such that, writing $T^* = \sum_{h=1}^N t_h^*$ and defining

$$K_i(t_i \mid t_1^*, ..., t_N^*) \equiv u_i\left(\omega_i - \frac{t_i}{t_i^*}t_i^* + \theta_i\left[\frac{t_i}{t_i^*}T^* - C\left(\tilde{y}\left(\frac{t_i}{t_i^*}T^*\right)\right)\right], \tilde{y}\left(\frac{t_i}{t_i^*}T^*\right)\right), i = 1, ..., N, \quad (6.3)$$

we have that

$$K_i(t_i^* \mid t_1^*, ..., t_N^*) \geq K_i(t_i \mid t_1^*, ..., t_N^*), \forall t_i \geq 0, \forall i, \quad (6.4)$$

with $y^* = \tilde{y}\left(T^*\right)$ and $x_i^* = \omega_i - t_i^* + \theta_i[T^* - C(y^*)], \forall i$.

Theorem 6.1. Equivalence Result for definitions 6.2 and 6.1. *Let* $M = 1$. *Let the vector* θ *of profit-share parameters and the procurement rule* $\tilde{T}$ *(or* $\tilde{y} \equiv \tilde{T}^{-1}$*) be given. The state* $(x^*, y^*) \in \Re_+^{N+1}$ *can be supported by a Multiplicative Kantian Equilibrium of Definition* 6.2 *if and only if it can be supported by a Balanced Linear Cost Share Equilibrium of Definition* 6.1 *with* $b_i > 0, \forall i$.

Proof. **(i)** Definition 6.2 (MKE) $\Rightarrow$ Definition 6.1 (BLCSE). Define $b_i^* \equiv t_i^* / T^*, \forall i$. Refer to (6.3) and define $\hat{K}_i\left(y \middle| \frac{t_i^*}{T^*}\right) \equiv K_i\left(\frac{t_i^*}{T^*}\tilde{T}(y) \middle| t_1^*, ..., t_N^*\right), i = 1, ..., N$. Compute

$$\hat{K}_i\left(y \middle| \frac{t_i^*}{T^*}\right) = u_i\left(\omega_i - \frac{t_i^*}{T^*}\tilde{T}(y) + \theta_i\left[\frac{t_i^*}{T^*}\tilde{T}(y)\frac{T^*}{t_i^*} - C\left(\tilde{y}\left(\frac{t_i^*}{T^*}\tilde{T}(y)\frac{T^*}{t_i^*}\right)\right)\right], \tilde{y}\left(\frac{t_i^*}{T^*}\tilde{T}(y)\frac{T^*}{t_i^*}\right)\right) = \Lambda_i\left(y \middle| b_i^*\right),$$

by the fact that $\tilde{y}(\tilde{T}(y)) \equiv \tilde{T}^{-1}(\tilde{T}(y)) = y$ and the definition b_i^*. Thus, (6.4) implies (6.2).

(ii) Definition 6.1 (BLCSE) $\Rightarrow$ Definition 6.2 (MKE). Define $t_i^* = b_i\tilde{T}(y^*), \forall i$, *i. e.*, $\sum_{h=1}^N t_h^* = \tilde{T}(y^*) \equiv T^*$. Refer to (6.1) and define $\hat{\Lambda}_i\left(t_i \middle| t_1^*, ..., t_N^*\right) \equiv \Lambda_i\left(\tilde{y}\left(t_i\frac{T^*}{t_i^*}\right) \middle| \frac{t_i^*}{T^*}\right)$. A similar argument shows that $\hat{\Lambda}_i\left(t_i \middle| t_1^*, ..., t_N^*\right) = K_i\left(t_i \middle| t_1^*, ..., t_N^*\right)$, and hence (6.2) implies (6.4). ■

The optimization problems of definitions 6.1 and 6.2 are often nonconvex, and we have not pursued a general existence theorem. But equilibria can be shown to exist in some simple cases: see Appendix below. In any event, Remark 6.1 and theorems 6.1 and 6.2 do not require convexity (nor differentiability or even continuity), and when such an equilibrium does exist it is efficient and can equivalently be characterized by definitions 6.1 or 6.2.

Assume differentiability. Let an interior state be supported, equivalently, by an equilibrium of Definition 6.1 or 6.2. Setting to zero the derivative of (6.3), we obtain $t_i = \theta_i[1 - m\tilde{y}']T + L_i m\tilde{y}'T$,

i. e., $\dfrac{\omega_i - x_i}{C} = \dfrac{t_i - \theta_i[T-C]}{C} = \dfrac{-\theta_i m\tilde{y}'T + L_i m\tilde{y}'T}{C} + \theta_i + L_i - L_i$,

or
$$\frac{\omega_i - x_i}{C} = L_i + \left[L_i - \theta_i\right]\left[\frac{m\tilde{y}'T - C}{C}\right], i = 1,\ldots,N, \tag{6.5}$$

i. e., if θ_i happens to coincide with L_i, then i's share in the social cost equals her Lindahl Ratio.

6.2. Quasi-Walrasian markets

In Foley's version of Lindahl's ideas (see Section 1.3), the seller of the excludable public good is a privately owned, price taking, profit maximizing firm, which distributes its profits according to the exogenously given vector θ, and the buyers are price-taking consumers. The firm charges the personalized price p_i per unit of the public good accessed by Person *i*.

Definition 6.3. Let $M = 1$. A *Lindahl-Foley Equilibrium for the vector* θ *of profit-share parameters* is a state $(x^*, y^*) \in \Re_+^{N+1}$, a vector $(p_1,\ldots,p_N) \in \Re_+^N$ of personalized prices and a scalar production price p such that:

* $p = \sum_{h=1}^{N} p_h$; (6.6)
* y^* solves $\max_{y\in\Re_+} py - C(y)$; write $\Pi^* \equiv py^* - C(y^*)$;
* Defining $F_i\left(y \middle| p_i, \Pi^*\right) \equiv u_i\left(\omega_i - p_i y + \theta_i \Pi^*, y\right)$, $i = 1,\ldots, N$, (6.7)

  we have that $F_i\left(y^* \middle| p_i, \Pi^*\right) \geq F_i\left(y \middle| p_i, \Pi^*\right), \forall y \geq 0, i = 1,\ldots,N$; (6.8)
* For $i = 1,\ldots,N, x_i^* = \omega_i - p_i y^* + \theta_i \Pi^*$.

Definition 6.3 requires price-taking profit-maximizing, incompatible with increasing returns to scale. Profits are positive (resp., zero) under decreasing (resp., constant) returns.[8]

In both the Multiplicative Kantian Equilibrium of Definition 6.2 and the Lindahl-Foley Equilibrium, the public good is produced by a privately owned firm, with exogenously given profit distribution shares. But these two concepts diverge in two aspects.

First, Lindahl-Foley buyers are (personalized) price takers; they also take their profit income parametrically. Contributors in the Kantian Equilibrium of Definition 6.2, on the contrary, take into account all the effects of their voluntary contributions through the procurement rule. Second, the Lindahl-Foley firm is a price taker, and thus its profits are $my - C$. Profits are defined instead by the procurement rule in the Kantian Equilibrium of Definition 6.2.

Within the space of privately-owned firms, the Equilibrium of Definition 6.2 fully relies on the Kantian optimization protocol, whereas the Lindahl-Foley one is (pseudo)-Walrasian. Our next notion has both Walrasian and Kantian features: the firm is a price-taking profit maximizer, persons treat parametrically their profit income, and demand must equal supply. But whereas public-good supply is fully Walrasian, demand is obtained by a Kantian optimization protocol rather than by Walrasian constrained-utility maximization. The approach is similar to the equilibrium concept in Roemer (In Press, Definition 6) to model cooperation in the presence of global negative externalities from carbon emissions.

The following definition parallels definitions 4.2 and 6.2 above.

<u>Definition 6.4</u>. Let $M = 1$. A *Walras-Kant Equilibrium with strictly positive contributions for the vector* θ *of profit-share parameters* is a state $(x^*, y^*) \in \Re_+^{N+1}$, a price scalar $p > 0$ and a vector $(t_1^*,...,t_N^*) \in \Re_{++}^N$ of voluntary contributions such that

* $py^* = \sum_{h=1}^N t_h^*$; (6.9)

* y^* solves $\max_{y \in \Re_+} py - C(y)$; write $\Pi^* \equiv py^* - C(y^*)$;

* Defining $W_i\,(t_i | t_1^*,...,t_N^*, p, \Pi^*) \equiv u_i\left(\omega_i - \frac{t_i}{t_i^*} t_i^* + \theta_i \Pi^*, \frac{t_i}{t_i^*} \frac{\sum_{h=1}^N t_h^*}{p}\right), i = 1,...,N$, (6.10)

  we have that $W_i\,(t_i^* | t_1^*,...,t_N^*, p, \Pi^*) \geq W_i\,(t_i | t_1^*,...,t_N^*, p, \Pi^*), \forall t_i \geq 0, i = 1,...,N$; (6.11)

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[8] Foley (1971) assumed constant returns to scale, whereas Roberts (1974) admitted decreasing returns.

* $x_i^* = \omega_i - t_i^* + \theta_i \Pi^*$, $i = 1, \ldots, N$.

Theorem 6.2. Equivalence between Walras-Kant and Lindahl-Foley Equilibria. *Let M* = 1, *and let the vector* θ *of profit-share parameters be given. The state* $(x^*, y^*) \in \Re_+^{N+1}$ *can be supported by a Walras-Kant Equilibrium* (*Definition* 6.4) *if an only if it can be supported by a Lindahl-Foley Equilibrium* (*Definition* 6.3) *with* $p_i > 0, \forall i$.

Proof. The profit maximization condition is the same in definitions 6.3 and 6.4.

(i). Walras-Kant (Definition 6.4) $\Rightarrow$ Lindahl-Foley (Definition 6.3). Define $p_i \equiv t_i^* / y^*, \forall i$. Refer to (6.10) and define $\hat{W}_i\left(y \middle| \frac{t_i^*}{y^*}, \Pi^*\right) \equiv W_i\left(\frac{t_i^*}{y^*} y \middle| t_1^*, \ldots, t_N^*, p, \Pi^*\right), \forall i$. Compute

$$W_i\left(\frac{t_i^*}{y^*} y \middle| t_1^*, \ldots, t_N^*, p, \Pi^*\right) = u_i\left(\omega_i - \frac{t_i^*}{y^*} y + \theta_i \Pi^*, \frac{t_i^*}{y^*} y \frac{\sum_{h=1}^N t_h^*}{t_i^* p}\right) = u_i(\omega_i - p_i y + \theta_i \Pi^*, y) \text{ [using (6.9)]},$$

i. e., $\hat{W}_i\left(y \middle| t_i^* / y^*, \Pi^*\right) = F_i(y | p_i, \Pi^*)$, because $p_i \equiv t_i^* / y^*$. Thus, (6.11) implies (6.8).

(ii). Lindahl-Foley (Definition 6.3) $\Rightarrow$ Walras-Kant (Definition 6.4). Define

$$t_i^* \equiv p_i y^*, \forall i, \tag{6.12}$$

which by (6.6) implies that

$$\sum_{h=1}^N t_h^* = p y^*. \tag{6.13}$$

Refer to (6.7) and define $\hat{F}_i\left(t_i \middle| p_1 y^*, \ldots, p_N y^*, p, \Pi^*\right) \equiv F_i\left(\frac{t_i}{p_i} \middle| p_i, \Pi^*\right)$. Compute

$$F_i\left(\frac{t_i}{p_i} \middle| p_i, \Pi^*\right) = u_i(\omega_i - p_i \frac{t_i}{p_i} + \theta_i \Pi^*, \frac{t_i}{p_i}) = u_i\left(\omega_i - t_i + \theta_i \Pi^*, \frac{t_i}{t_i^*} y^*\right) \qquad \text{[by (6.12)]}$$

$$= u_i\left(\omega_i - t_i + \theta_i \Pi^*, \frac{t_i}{t_i^*} \frac{\sum_{h=1}^N t_h^*}{p}\right). \qquad \text{[by (6.13)]}$$

Hence, $\hat{F}_i\left(t_i \middle| p_1 y^*, \ldots, p_N y^*, p, \Pi^*\right) = W_i\left(t_i \middle| t_1^*, \ldots, t_N^*, p, \Pi^*\right)$ [by (6.12)], and (6.8) implies (6.11). ■

Equality (6.5) has a parallel here. The argument leading to (6.5), but using instead the first order conditions in the optimizations of Definition 6.3 yields:

$\frac{\omega_i - x_i}{C} = L_i + \left[L_i - \theta_i\right]\left[\frac{my - C}{C}\right], i = 1, \ldots, N$. As long as $my > C(y)$, *i*'s share in the social cost equals her Lindahl Ratio if and only if the exogenous θ_i happens to coincide with her Lindahl Ratio. If $L_i > \theta_i$ (resp., $L_i < \theta_i$), then i's cost share exceeds (resp., falls short of) L_i.

The following table summarizes the differences among these equilibrium concepts.

		BLCSE for $(\theta, \tilde{T})$ (Def. 6.1)	Mult. Kantian for $(\theta, \tilde{T})$ (Def. 6.2)	Walras-Kant for θ (Def. 6.4)	Lindahl-Foley for θ (Def. 6.3)
Walrasian Features	Firm is a price taking profit maximizer	NO	NO	YES	YES
	Person *i* treats profit income parametrically	NO	NO	YES	YES
	Person *i* is a price taker	NO	NO	NO	YES
Kantian Feature	Person *i* adopts Kantian optimization protocol	NO	YES	YES	NO

Table 6.1. Walrasian and Kantian features of equilibrium concepts for privately owned technologies

Lindahl-Foley equilibria display a straightforward relation with the Linear Cost Share equilibria of Section 2.4.

Fact 6.1. (Mas-Colell and Silvestre, 1989).[9] *Let* $M = 1$ *and assume differentiability and convexity. Let* (x, y) *be interior with* $my > C(y)$. $((x, y), \{a_i\}, \{b_i\})$ *is a Linear Cost Share Equilibrium* (Definition 2.4) *if and only if* $((x, y), (p_1, ..., p_N), p)$ *is a Lindahl-Foley equilibrium for the privately owned technology with the vector* θ *of profit distribution parameters*, *where for* $i = 1, ..., N$, $\theta_i = b_i$ *and* $p_i = a_i + b_i m$.

It follows that the Person i's cost share at a Lindahl-Foley Equilibrium for θ equals her Lindahl Ratio if and only if the corresponding Linear Cost Share Equilibrium is balanced.

Remark 6.2. Since the exogenous vector θ can in principle be any point in Δ^{N-1}, Fact 6.1 suggests that the set of states supportable as Linear Cost Share Equilibria is $(N-1)$-

[9] The result there covers the general case $M \geq 1$. Because the existence of Lindahl-Foley equilibrium for any given θ is well established under standard conditions (Roberts, 1974), this result implies the existence of Linear Cost Share equilibria where the vector $(b_1, \ldots, b_N)$ can be chosen to be any point in Δ^{N-1}.

dimensional, see Mas-Colell and Silvestre (1989), and Remark 5.7 above. This also illustrates the role of returns to scale on the set of states supportable by a Linear Cost Share Equilibrium: the higher the potential profits, the more relevant are the profit-share parameters θ. Zero profits, as in the constant returns case of Theorem 5.1 above, means that nothing can be changed by altering the parameters θ.

Remark 6.3. Under the conditions of Theorem 5.4 above, a K^{β} - Equilibrium is a LCSE, and hence, by Fact 6.1, it provides a theory of profit sharing in a quasi-Walrasian world. Indeed, from Fact 6.1, (5.10) and (5.4), we compute $\theta_i = b_i = \frac{r_i - t_i}{my - C} = L_i + \frac{\beta N}{my - C}\left[-L_i + \frac{1}{N}\right]$. As β increases, the profit share of low (resp., high) valuators tends to increase (resp., decrease). Compare with Remark 5.6 above.

7. Many public goods

The Kantian equilibrium notions of Section 3 above are unidimensional in nature. Indeed, it is not easy to generalize them for the $M > 1$ economy of Section 2, except under the following assumption.

Assumption 7.1. Cost Separability. The cost function is *additively separable*, *i. e.*, there exist M increasing functions $C^j : \Re_+ \to \Re_+, j = 1,...,M$, such that $C(y^1,...,y^M) = \sum_{j=1}^{M} C^j(y^j)$.

Under differentiability, we denote $m^j \equiv (C^j)'$, assumed positive, and $r_i^j \equiv \frac{\partial u_i}{\partial y^j} \Big/ \frac{\partial u_i}{\partial x_i}$.

Definition 7.1. A *Multiplicative Kantian Equilibrium for an economy with M public goods and cost separability* is a state $(x,(y^1,...,y^M)) \in \Re_+^{N+M}$ and a MN vector $(t_1^1,...,t_1^M;...;t_N^1,...,t_N^M) \in \Re_+^{NM}$ of contributions such that:

For $j = 1,\ldots, M$, $\sum_{h=1}^{N} t_h^j = C^j(y^j)$ *i. e.*, $y^j = (C^j)^{-1}(\sum_{h=1}^{N} t_h^j)$; (7.1)

For $i = 1,\ldots, N$, $x_i = \omega_i - \sum_{j=1}^{M} t_i^j$; (7.2)

For $i = 1,\ldots, N$ and $j = 1,\ldots, M$, $\rho = 1$ solves

$$\max_{\rho\in\Re_+} u_i(\omega_i - \rho t_i^j - \sum_{k\neq j} t_i^k, (C^1)^{-1}(\sum_{h=1}^{N} t_h^1),...,(C^j)^{-1}(\rho\sum_{h=1}^{N} t_h^j),...,(C^M)^{-1}(\sum_{h=1}^{N} t_h^M)) \qquad (7.3)$$

subject to $\rho t_i^j \geq 0$.

Lemma 7.1. *Let* $M \geq 1$ *and assume differentiability, convexity and cost separability. The state* $(x,(y^1,...,y^M)) \in \Re^{N+M}_{++}$ *can be supported by a Multiplicative Kantian Equilibrium of Definition* 7.1 *with* $t_i^j > 0, \forall i, \forall j$ *if an only if* (7.1) *and* (7.2) *hold, and*

$$\frac{t_i^j}{C^j(y^j)} = \frac{r_i^j}{m^j},\ i = 1,\ldots,N, j = 1,\ldots,M. \qquad (7.4)$$

Proof. Immediate by differentiating the objective function in (7.3), using (7.1), and noticing that, because $\rho t_i^j > 0$, the Lagrange multiplier is zero. ■

Definition 7.2. (Kaneko, 1977). A *Ratio Equilibrium for an economy with M public goods and cost separability* is a state $(x^*,(y^{1*},...,y^{M*})) \in \Re^{N+M}_{+}$ and an *MN* vector $(b_1^1,...,b_1^M;...;b_N^1,...,b_N^M) \in \Re^{NM}_{+}$ of cost shares such that:

For $j = 1, \ldots, M$, $(b_1^j,...,b_N^j) \in \Delta^{N-1}$; (7.5)

For $i = 1,\ldots, N$, $(y^{1*},...,y^{M*})$ solves $\max_{y^1,...,y^M} u_i(\omega_i - \sum_{j=1}^M b_i^j C^j(y^j), y^1,...,y^M)$; (7.6)

For $i = 1,\ldots, N$, $x_i^* = \omega_i - \sum_{j=1}^M b_i^j C^j(y^{j*})$. (7.7)

Remark 7.1. Kaneko (1977, Theorem 1) proves the existence of Ratio Equilibrium under standard assumptions. His discussion of the core can be used to show efficiency. In fact, efficiency also follows from Fact 2.1 above, since a Ratio Equilibrium satisfies Definition 2.2.

Remark 7.2. For $M = 1$, Ratio Equilibrium and Balanced Cost Share Equilibrium coincide. But they are different notions and typically support different states when $M > 1$.

Lemma 7.2. *Let* $M \geq 1$ *and assume differentiability, convexity and cost separability. The state* $(x,(y^1,...,y^M)) \in \Re^{N+M}_{++}$ *can be supported by a Ratio Equilibrium if and only if* (7.7) *holds, and*

for $i = 1,\ldots, N, j = 1,\ldots, M,\ b_i^j m^j = r_i^j$. (7.8)

Proof. Differentiate the objective function in (7.6) and appeal to the *M*-dimensional version of the Samuelson Condition. ■

Theorem 7.1. *Let* $M \geq 1$. *Assume differentiability, convexity and cost separability. A state* $(x,(y^1,...,y^M)) \in \Re^{N+M}_{++}$ *can be supported by a Ratio Equilibrium with* $b_i^j > 0, \forall i, \forall j$, *if and only if it can be supported by Multiplicative Kantian Equilibrium of Definition* 7 *with* $t_i^j > 0, \forall i, \forall j$.

Proof. Sufficiency. For $i = 1,\ldots, N, j = 1,\ldots,M$, define $b_i^j = t_i^j / C^j(y^j)$. By (7.4), it is (7.8). Necessity. For $i = 1,\ldots, N, j = 1,\ldots,M$, define $t_i^j = b_i^j C^j(y^j)$. By (7.8), it is (7.4). ■

Remark 7.3. At an equilibrium of Definitions 7.1 or 7.2 one can separately specify Person *i*'s contribution to the cost of public good *j*, $j = 1,\ldots, M$, which coincides with *i*'s good-specific Lindahl Ratio $L_i^j = r_i^j / m^j$.

8. Further comments

8.1. Welfare economics

Wicksell and Lindahl do not distinguish between positive and normative economics with today's sharpness (Silvestre, 2003). They were concerned with the "justness" of the distribution of initial wealth (being quite aware of Sweden's extreme inequality) and with the "fairness" of taxation, based in turn on the "benefit principle," *i. e.*, a person's taxes should be in line with the benefit that she derives from the public goods. Lindahl's formulation, more precise than Wicksell's, yields a marginal version of the benefit principle, a version which, thanks to our equivalence results, also obtains at a Multiplicative Kantian Equilibrium.[10]

Of course, as is well known since Jules Dupuit (1844), two persons with the same marginal valuation of a public good may well have widely different total valuations.

Alternative versions of the benefit principle appealing to a person's total, rather than marginal, benefit could no doubt be defined within the general notion of a cost share equilibrium (Definition 2.2). Instead of a linear system, as in Definition 2.3, we could have an affine system, along the lines of two-part tariffs, that puts Person *i*'s share in the social cost in line with her relative total benefit. To illustrate, assume $M = 1$ and quasilinerity, *i. e.*, $u_i(x_i, y) = x_i + v_i(y), i = 1,\ldots, N$. Then an affine cost share system $\{g_i\}$ could be defined such that, at the (efficient, by Fact 2.1) equilibrium, $\frac{\omega_i - x_i}{C(y)} \equiv \frac{g_i(y)}{C(y)} = \frac{v_i(y)}{\sum_{h=1}^{N} v_h(y)}$.

We do not agree that marginal-cost-pricing is associated with distributive justice, except in so far as efficiency may be one desideratum of justice. The natural view would be that contributions should be proportional to total, not marginal, benefits. But, as Liam Murphy and

[10] The Austrian and Italian originators of the benefit principle in the late 1880's seem to have had marginal, not total, benefit in mind, see Richard Musgrave (1985).

Thomas Nagel (2002) argue, it is hardly possible to contemplate just taxation apart from justice in the entire distribution of endowments, contributions, and income. Equalizing utilities, for instance, will often require wealth transfers beyond what can be achieved by a two-part tariff that obeys participation constraints.

8.2. The core

The notion of the core strengthens the idea of efficiency by adding a condition of social stability: no group of persons can "improve upon" the state. The argument in the proof of Proposition 1 in Mas-Colell and Silvestre (1989) can be easily adapted to show that, if the cost shares $g_i(y)$ are always nonnegative, then a cost-share equilibrium is in the core (in the sense of Foley, 1970, *i. e.*, all coalitions have access to the technology $C(y)$; see also Kaneko, 1977). It follows that (under some conditions) states supportable as Kantian Equilibria belong to the core.

8.3. Externalities

Externalities display common elements with public goods, although it is not easy to draw a sharp distinction. (One discriminating characteristic relies on *intentionality*.) Cost Share Equilibria can be applied to externalities (Mas-Colell and Silvestre, 1989, IV.3). Kantian optimization is indeed relevant for externalities (Roemer, 2010, III; 2019, Ch. 11, In Press: see Section 5.2 above). Our results can help establish correspondences between these approaches.

8.4. Decentralization

Kantian optimization applies to individual decisions in a game of voluntary contributions, whereas Wicksell-Lindahl covers negotiation within the national parliament. Hence, in a basic way, the former is *decentralized*, and the latter *centralized*. Because they apply to different worlds, there isn't much room for debating which system is generally superior: to make decisions on the nation's space program, a centralized system is required, but a decentralized scheme may be preferable for financing a classical music radio station.

The issue of centralization vs. decentralization appeared in the 1940's debate on the virtues of the market vs. central planning. (Leading figures were Ludwig Von Mises, Friedrich Hayek, Oskar Lange and Abba Lerner.) Leonid Hurwicz (1960, 1972) pioneered an extensive literature on mechanism design: individual decision makers are given the incentives to act in a way, including the revelation of private information, that results in a satisfactory overall outcome. Of course, the design of the mechanism, and its enforcement, is itself a centralized activity.

Some attention was devoted to the case of public goods supplied by a central authority that lacks the relevant information on technology or preferences privatey held by agents who have incentives not to reveal it (Hurwicz, 1979). We submit that any informational difficulties in the Wicksell-Lindahl and Kant approaches are less pronounced than what is assumed in the mechanism design literature. True, the cost function must be known by each participant.[11] But the Wicksell-Lindahl parties that negotiate in the parliament are faithful representatives of the interests of their constituents, and the interests of each social class may well be public knowledge rather than private information. And there is no need for Kantian players to know anything about other players' preferences.[12]

The mechanism design literature appealed to implementation through noncooperative equilibrium concepts (dominant strategy, Bayes, Nash, subgame perfection). Kantian optimization yields efficient outcomes directly from the behavior of participants, without the need for designing complex mechanisms.

8.5. Coordination

The ideal market produces both efficiency and coordination, but outside it there is often a need for coordination even in decentralized contexts. For example, a Kantian recycles her trash because she wants everybody to recycle. But should polystyrene be recycled? Recycling works thanks to both a central agency and Kantian behavior. Moreover, ouside the world of Simple Kantian equilibria people would have to coordinate on which β - Kantian variation to adopt.

Some local public goods are in fact supplied by voluntary contributions: think of an irrigation system locally created and organized. But the size of the public good and its financing must be negotiated among the interested parties, which is the heart of the Wicksell-Lindahl approach, whether at the national or local level.

[11] Except when public goods are supplied by Walrasian firms, as in definitions 6.3-4 above, where only the profit maximizing firm has to know the cost function.

[12] In the Kantian equilibria defined above they must know the contributions of the other players, as is the case in the Cournot-Nash equilibrium. The exception is the Simple Kantian Equilibrium where, in a manner parallel to the noncooperative Dominant Strategy Equilibrium, no player needs to know the strategies of the other players.

9. Conclusion

Wicksell (1896) and Lindahl (1919) contemplated a society comprising a few homogeneous classes, each represented by a party in a democratic parliament. These representative parties negotiated among themselves the amount of public goods to be supplied and the way of allocating their costs among the classes. Wicksell and Lindahl assumed that these negotiations exhausted all possibilities of mutually beneficial agreements, so that the outcome was efficient. Lindahl's model is more determinate and matches the cost share of each class with the marginal valuation of the public goods by the party's constituency. The world of Wicksell and Lindahl was that of a national government, centrally deciding on the public goods that governments *publicly* provide.

A literature emerged in the mid 1970's concerning the *private* provision of public goods by voluntary contributions. They focused on the Cournot-Nash equilibria of a normal-form game named the *Voluntary Contribution Game*. This captures a decentralized operation, where each player independently decides on her action: these equilibria are typically inefficient (the "free rider" problem). Roemer (2010, 2019) emphasized the cooperative nature of the human species and proposed an alternative equilibrium for the voluntary contribution game based on Kant's categorical imperative: act the way you want everybody to act. Kantian equilibria are efficient, a fact which follows from the behavior, or optimization protocol, of the contributors.

Thus, we have two different worlds: the public, centralized provision of Wicksell and Lindahl, and the private, decentralized provision by Kantian players. But the primary versions of the models are mathematically equivalent. Both equilibria are efficient and identically distribute social costs in accordance to the *Lindahl Ratio*. Generalized versions of the Wicksell-Lindahl and Kant-Roemer models depart from the Lindahl Ratio in related manners, which can often be interpreted as the role played by the private ownership of the technology.

In summary, two worlds: provision is public in one, and private in the other one. Yet the primary versions of their respective models are mathematical equivalent.

Appendix

Existence of equilibria with procurement rules under quasilinearity and differentiability

Let Person i's take $b_i, \tilde{T}(y)$ and θ_i as given, and choose y in order to maximize

$$\omega_i - \tilde{b}_i \tilde{T}(y) + \theta_i \left[\tilde{T}(y) - C(y) \right] + v_i(y) \ , \tag{A.1}$$

where v_i is assumed strictly increasing and strictly concave.

Let $y^* > 0$ maximize $\sum_{h=1}^{N} v_h(y) - C(y)$. Then (Samuelson Condition, Fact 4.1 above):

$$\sum_{h=1}^{N} v_h'(y^*) = C'(y^*) \, . \tag{A.2}$$

If $y^* > 0$ maximizes (A.1), then it must be the case that

$$-\tilde{b}_i \tilde{T}'(y^*) + \theta_i \left[\tilde{T}'(y^*) - C'(y^*) \right] + v_i'(y^*) = 0 \, ,$$

i. e.,

$$\tilde{b}_i = \frac{\theta_i \left[\tilde{T}'(y^*) - C'(y^*) \right] + v_i'(y^*)}{\tilde{T}'(y^*)} \equiv b_i^* \, . \tag{A.3}$$

Substituting (A.3) into (A.1), define the function

$$\tilde{u}_i(y) \equiv \omega_i - b_i^* \tilde{T}(y) + \theta_i \left[\tilde{T}(y) - C(y) \right] + v_i(y) \, ,$$

with

$$\tilde{u}_i'(y) = -b_i^* \tilde{T}'(y) + \theta_i \left[\tilde{T}'(y) - C'(y) \right] + v_i'(y) \, , \tag{A.4}$$

and

$$\tilde{u}_i''(y) = -b_i^* \tilde{T}''(y) + \theta_i \left[\tilde{T}''(y) - C''(y) \right] + v_i''(y) \, . \tag{A.5}$$

For Definition 6.1 to be satisfied, it is necessary and sufficient that, first, $(b_1^*, ..., b_N^*) \in \Delta^{N-1}$ and, second, that y^* maximize $\tilde{u}_i(y), \forall i$. By (A.2), $\sum_{h=1}^{N} b_h^* = 1$. Hence, $(b_1^*, ..., b_N^*) \in \Delta^{N-1}$ if and only if

$$b_i^* = \frac{\theta_i \left[\tilde{T}'(y^*) - C'(y^*) \right] + v_i'(y^*)}{\tilde{T}'(y^*)} \geq 0 \, , \forall i \, . \tag{A.6}$$

It is clear from (A.3) and (A.4) that $\tilde{u}_i'(y^*) = 0$. Thus, a sufficient condition for y^* to maximize $\tilde{u}_i(y), \forall i$, is that $\tilde{u}_i(y)$ be a concave function, *i. e.*,

$$\tilde{u}_i''(y) \leq 0, \forall y, \forall i \, . \tag{A.7}$$

Thus, (A.6) and (A.7) are sufficient for the existence of equilibrium with $y^* > 0$. We now consider the examples of Section 6.1 above.

Example 1. Cost-plus-fixed-fee contract. Let $\tilde{T}(y) = F + C(y), \forall y > 0$. We can check that (A.6) is satisfied, since (A.3) yields $b_i^* = \dfrac{v_i'(y^*)}{C'(y^*)} = L_i(y^*) > 0$. And so is (A.7), because $\tilde{u}_i''(y) = -b_i^* C''(y) + v_i''(y) < 0, \forall i$, as long as C is convex.

Note that for $F = 0$ we are back to the the case covered by Sections 3 and 4 above, where we know that equilibria exist under convexity (without the need for quasilinearity or differentiability).

Example 2. Affine payment function. Let $\tilde{T}(y) = F + qy, \forall y > 0$. Then $\tilde{T}''(y) = 0$ and (A.7) is satisfied as long as C is convex. A sufficient condition for (A.6) is that $q \geq C'(y^*)$, *i. e.*, that the marginal payment be not lower than the marginal cost at the efficient output level, a not unnatural assumption.

Example 3. Markup contract. Let $\tilde{T}(y) = \gamma C(y), \forall y \geq 0, \gamma > 1$, *i. e.*, $\tilde{T}'(y) = \gamma C'(y)$ and $\tilde{T}''(y) = \gamma C''(y)$. Now (A.3) becomes $b_i^* = \dfrac{\theta_i[\gamma - 1]C'(y^*) + v_i'(y^*)}{\gamma C'(y^*)} > 0$, so that (A.6) is satisfied. Substituting this expression into (A.5) we obtain

$$\tilde{u}_i''(y) \equiv -\left[\frac{\theta_i[\gamma-1]}{\gamma} + \frac{v_i'(y^*)}{\gamma C'(y^*)}\right]\gamma C''(y) + \theta_i\left[\gamma - 1\right]C''(y) + v_i''(y)$$

$$= -\theta_i[\gamma-1]C''(y) - \left[\frac{v_i'(y^*)}{C'(y^*)}\right]C''(y) + \theta_i\left[\gamma-1\right]C''(y) + v_i''(y) = -\left[\frac{v_i'(y^*)}{C'(y^*)}\right]C''(y) + v_i''(y),$$

guaranteed negative as long as C is convex, so that (A.7) is also satisfied.

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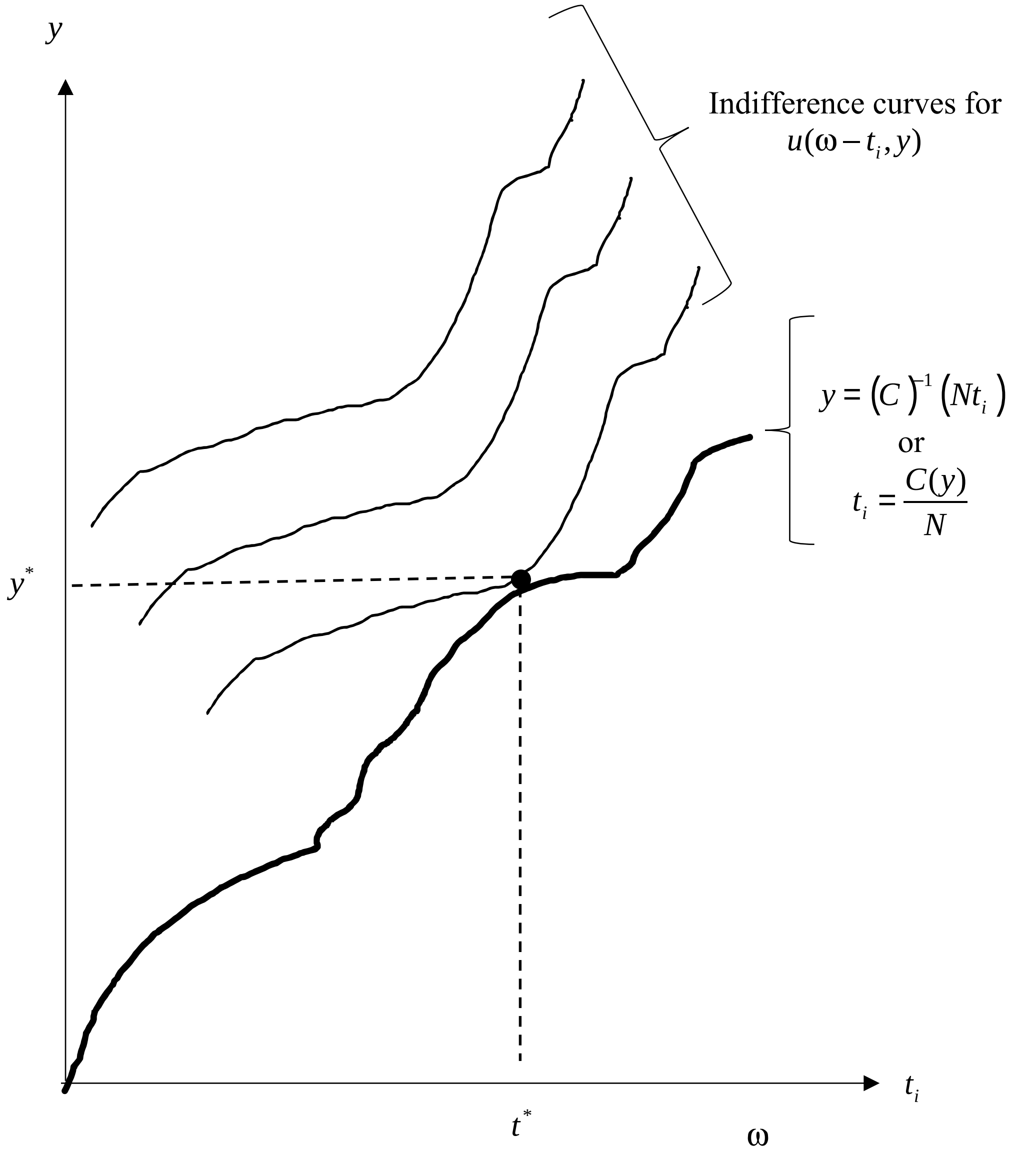

Figure 4.1

$M = 1$, Identical Person Case

The equivalence between Simple Kantian Equilibrium (Definition 3.4)

and Balanced Linear Cost Share Equilibrium (Definition 4.1)

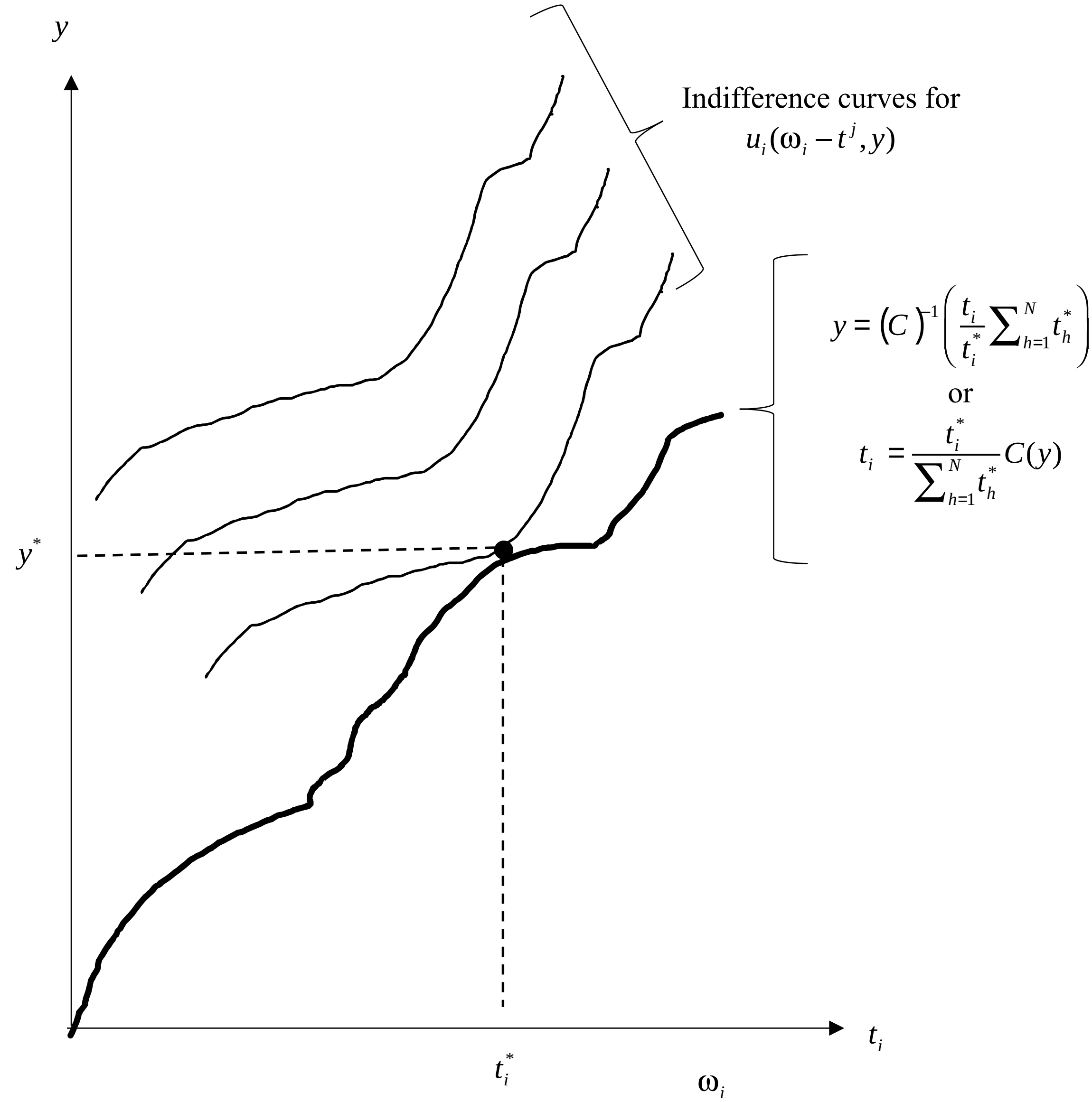

Figure 4.2

$M = 1$. The equivalence between Multiplicative Kantian Equilibrium (Definition 4.2) and Balanced Linear Cost Share Equilibrium (Definition 4.1)