A FIRST-ORDER CHARACTERIZATION OF ROBUST PARETO IMPROVEMENTS UNDER NONCONVEXITY

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Abstract

In the presence of market failures, welfare-improving policy interventions are commonly studied through the first-order conditions of a social planning program: A benevolent government maximizes social welfare, subject to feasibility constraints, by means of a limited set of instruments, such as taxes, quantity restrictions or other forms of market regulation. However, it is also well-understood that this otherwise powerful method fails due to the lack of convexity caused by market imperfections through the pecuniary externality, even though the economy's primitives satisfy canonical restrictions. We argue that, despite this failure, the first-order conditions of the social planning program still exhaustively characterize the local absence of robust welfare-improving policies. Any remaining scope for welfare enhancement must rely on an implausibly accurate knowledge of the economy's primitives. Our findings provide a rigorous foundation for the widespread use of Ramsey-type planning in macroeconomics and microeconomics, as illustrated by extended applications to macroprudential policies.

Key words: KKT conditions, robust efficiency, Ramsey planning program, macroprudential policies.

CONTENTS

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1. Introduction

In both microeconomics and macroeconomics, it is a well-established practice to characterize efficient allocations using first-order conditions involving Pareto weights and Lagrange multipliers.[1] Well-known economic textbooks (see Bewley [3, Chapter 6], Kreps [19, Chapter 8], Ljungqvist and Sargent [21, Chapter 8] and Mas-Colell et al. [23, Chapter 16]) provide thorough presentations of the conceptual and mathematical foundations of this approach. The method is crucial for assessing the effectiveness and welfare impact of economic policies aimed at repairing market failures in various second-best environments. Unfortunately, this approach is only fully consistent under convexity assumptions, which are often violated in situations where its application would be most valuable. In this paper, we provide a rigorous foundation for using first-order conditions when convexity is absent, or doubtful.

In a convex economic environment, a constrained efficient distribution can be determined by means of a Ramsey-type planning program (see Mas-Colell et al. [23, Proposition 16.E.2] or Che et al. [7] for a recent refinement). A benevolent planner is responsible for the distribution of resources among agents populating the economy subject to all restrictions arising from physical, technological, institutional and informational constraints. The planner's objective is given by the weighted sum of agents' utilities. By varying these welfare weights, the planning program attains all allocations lying on the constrained Pareto frontier, thus providing a comprehensive characterization of constrained efficiency. Relevantly, the planning program delivers suitable necessary and sufficient first-order conditions that serve both to derive empirically verifiable relations among relevant variables and as an effective computational device.

In the presence of nonconvexity, this otherwise fruitful method suffers from two major limitations. First, by varying welfare weights, the Ramsey planning program can only

[1]These conditions are commonly known as Karush-Kuhn-Tucker (KKT) conditions. The KKT conditions were originally named after Kuhn and Tucker [20]'s publication in 1951. It was later discovered that they had been previously stated by Karush [16] in his master's dissertation in 1939.

attain certain constrained efficient allocations. Thus, the weighted utilitarian objective of the planner entails an implicit judgment in terms of welfare distributions, a rather unacceptable feature from a normative perspective. Second, the first-order conditions of the planning program are necessary but might fail sufficiency. As a consequence, an allocation might be satisfying all the first-order conditions without being a social optimum. This lack of correspondence dramatically undermines the scope of the method when the planning program is inherently nonconvex, as, for instance, in certain recent studies on optimal taxation or macroprudential policies (see, among others, Aguiar et al. [1], Bianchi and Mendoza [4], Dávila and Korinek [8], Farhi and Werning [10] and Werning [28]).

We here argue that the first-order conditions of a Ramsey-type planning program are both necessary and sufficient for a *robust* form of *local* constrained efficiency, even when convexity is absent. In other terms, in the proximity of the *status quo*, an allocation cannot be robustly improved subject to feasibility if and only if the first-order conditions of the associated planning program are satisfied for a given set of (Pareto) welfare weights. Perhaps counterintuitively, such a coincidence holds true even though certain constrained efficient allocations cannot be attained as social optima given the welfare weights for which the first-order conditions of planning program are satisfied (and indeed for no other welfare weights). In fact, the *sole* role of a Ramsey-type planning program for given welfare weights is to facilitate an immediate derivation of the associated first-order conditions. At a constrained efficient allocation, these first-order conditions are fulfilled because, locally, the planner cannot improve the welfare of all agents while satisfying all feasibility requirements. In other terms, a violation of the first-order conditions would reveal an arbitrage opportunity for the planner.

A shift from global to local efficiency becomes inevitable in the presence of nonconvexity. This is intrinsic to methods that rely on first-order conditions, which are inherently designed to capture only local changes.[2] Any justification for policy intervention that involves implicit evaluations of marginal social costs and benefits (through measures such as elasticities, propensities, or wedges) inevitably adheres to a concept of local efficiency. In turn, when a locally efficient allocation fails to attain global efficiency, the remaining scope for welfare-improving interventions requires alternative assessments that cannot be based on first-order conditions alone.

We actually study the occurrence of robust welfare-improving policies in the proximity of a *status quo*. A distribution is considered robustly constrained inefficient if it admits an unambiguous welfare-improvement even under slightly inaccurate knowledge of the economy's primitives or, alternatively, under a slightly imprecise implementation of the planned redistribution. These thought experiments for robustness require a perturbation of utilities and constraints in the former case, and a sort of trembling-hand perfection criterion in the latter case. We demonstrate that, locally, the conventional KKT conditions are both

[2]For instance, several generalisations of the Second Welfare Theorem (see Guesnerie [13], Bonnisseau and Cornet [5], Kahn and Vohra [15] and Quinzii [25]) were established by investigating first-order necessary conditions and the associated *local* prices supporting a Pareto efficient allocation under nonconvexity.

necessary and sufficient for robust efficiency, provided that agents' utilities are concave when trembles are considered. Additionally, we verify that residual local welfare gains at a robustly efficient allocation (if any) are negligible, involving only second-order or higher-order terms. Importantly, in a convex environment, robust local efficiency coincides with global efficiency, leaving no scope for further refinement of our characterization through necessary and sufficient first-order conditions.

We provide a complementary approach to the local robustness of feasible Pareto improvements. Indeed, we establish that, under nondegenerate regularity, the KKT conditions hold if and only if any residual space of feasible Pareto improvements becomes almost (Lebesgue) negligible, relative to the feasible set, in the vicinity of the *status quo*. This characterization not only enhances our understanding of the geometric foundations of local efficiency but also offers a sharper criterion for the validity of first-order conditions in smooth economic environments, potentially relevant for computational methods. As a matter of fact, it corroborates the notion that any residual policy intervention would locally require an arbitrarily high degree of accuracy when the KKT conditions are satisfied, whereas a feasible Pareto improvement remains robust when these conditions are violated.

The characterization of efficient distributions is pervasive in economics. We illustrate the importance of our contribution by applying it to recent developments in the literature on macroprudential policies. In the wake of the Great Recession, a consensus emerged on the necessity to complement monetary policy with financial market interventions, such as taxes, quantity restrictions, or prudential regulations. Despite the diversity in this literature, the common objective remains the development of a comprehensive theoretical framework and a sound doctrine for policy intervention. As we briefly explain, Karush-Kuhn-Tucker (KKT) conditions are essential for achieving this goal.

A prominent theoretical justification for macroprudential policies relies on the idea of *pecuniary externality* (*e.g.*, Hart [14], Stiglitz [27] and Geanakoplos and Polemarchakis [11]). In the presence of incomplete markets, asset reallocations cause price fluctuations affecting market-spanning properties and improving insurance. As these externalities are not internalized by private agents, a competitive equilibrium is typically constrained inefficient and can be improved through policy interventions. Similar outcomes occur in economies with private information (*e.g.*, Greenwald and Stiglitz [12] and Farhi et al. [9]), price-dependent credit constraints (*e.g.*, Kiyotaki and Moore [17], Caballero and Krishnamurthy [6] and Lorenzoni [22]), and nominal rigidities (*e.g.*, Woodford [29], Schmitt-Grohé and Uribe [26] and Korinek and Simsek [18]). A common thread across these environments is that pecuniary externalities create market failures that can be corrected through optimal government interventions, such as Pigouvian taxes or quantity restrictions (see Aguiar et al. [1], Dávila and Korinek [8] and Farhi and Werning [10]).

The method shared by the recent literature requires the use of a Ramsey planning program. The planner maximizes social welfare subject to all the limitations of a competitive equilibrium arising from the underlying frictions. These constraints are inherently *nonconvex*, even though all the primitives of the economy satisfy the conventional convexity assumptions. Policy recommendations are derived from the KKT conditions associated with this Ramsey planning program. Due to the unavoidably nonconvex nature of the constraints, the KKT conditions might not be sufficient to determine a constrained social optimum.

The crucial advancement over previous literature lies in the fact that optimal monetary and macroprudential policies are characterized by *simple formulas* based on *sufficient statistics*, such as elasticities, marginal propensities and wedges. These formulas, derived from the KKT conditions, are notably invariant to the peculiar nature of the underlying frictions. Indeed, the sufficient statistics are high-level variables, rather than primitives of the economy, that determine the presence of pecuniary externalities and the nature of optimal corrective policies. The disconnection from primitives provides a major advantage and an operational guidance for the development of macroprudential policies. However, as the other side of the coin, it also amplifies the risk of failure in implementing a social optimum, because the KKT conditions are not sufficient. Formulas grounded on sufficient statistics are admittedly inadequate to separate constrained efficiency from inefficiency, as they are purposely devised only to ensure the validity of the KKT conditions. It is precisely here that our contribution offers a stronger foundation for their approach.

The paper is organized as follows. In section 3, we present our abstract framework. In section 4, we provide the first-order characterizations of robust local efficiency. In section 5, we illustrate how the theory extends to environments where fundamentals are subject to further economic restrictions. In section 6, we show how the analysis relates to well-established approaches. In section 7, we uncover the implications of our characterization in applications to macroprudential policies. We then conclude with a few comments. All proofs, irrespectively of their relevance, are collected in Appendix A. To further enhance readability, the most technical material is available in Appendix B.

2. A MOTIVATING EXAMPLE

We demonstrate the significance of our approach through an application to the recent literature on macroprudential policies, drawing on works such as Farhi and Werning [10] and, more specifically, Dávila and Korinek [8]. We consider a simple economy with a financial constraint in which a pecuniary externality arises, justifying policy intervention. We argue that, even under canonical assumptions on the underlying primitives, the environment is inherently nonconvex. The planning program yields first-order conditions that may not correspond to a social optimum. Therefore, policy prescriptions based solely on first-order conditions may be misleading if one adheres to a conventional notion of Pareto efficiency. However, as we will argue in this paper, these same first-order conditions are reliable in excluding the presence of robust welfare-improving policies.

Consider an economy populated by a finite set I of agents extending over a three-period horizon $T = \{0, 1, 2\}$. We assume that the utility of agent i in I is given by

$$U_i(c_{i0}, c_{i1}, c_{i2}) = u_{i0}(c_{i0}) + u_{i1}(c_{i1}) + u_{i2}(c_{i2}),$$

where $u_{it} : \mathbb{R}_{++} \to \mathbb{R}$ is a smooth, increasing, concave utility function, and c_{it} in $\mathbb{R}_+$ is consumption in period t in T. In the initial period, the budget constraint of agent i in I is given by

$$k_i + c_{i0} \leq e_{i0},$$

where k_i in $\mathbb{R}_+$ is the capital investment, and e_{it} in $\mathbb{R}_+$ is the endowment. In the following periods, the budget constraints are given by

$$qb_i + c_{i1} \leq e_{i1} + (1 + \rho)\, k_i \text{ and } c_{i2} \leq e_{i2} + b_i,$$

where b_i in $\mathbb{R}$ represents the savings, or the borrowing whenever negative, and ρ in $\mathbb{R}_+$ is the net return on capital. At a competitive equilibrium, the plan of each agent is individually-optimal and the price of the bond q in $\mathbb{R}_+$ clears the market,

$$\sum_{i \in I} b_i = 0.$$

Notice that the only market imperfection in this economy is a borrowing limit in the initial period.

Beginning from the second period, the value function of agent i in I can be expressed as $v_i(k_i, q(k))$, where $q(k)$ in $\mathbb{R}_+$ is the price of the bond at a competitive equilibrium, conditional on the distribution of capital across agents k in $\mathbb{R}_+^I$. A hypothetical benevolent planner is responsible for initial capital accumulation plans, anticipating the determination of a competitive equilibrium in the continuation economy, so as to maximize social welfare. In other terms, the social planner maximizes

$$\sum_{i \in I} \lambda_i \left(u_{i0}(c_{i0}) + v_i(k_i, q(k))\right)$$

subject to

$$\sum_{i \in I} k_i + \sum_{i \in I} c_{i0} \leq \sum_{i \in I} e_{i0},$$

where each k_i lies in $\mathbb{R}_+$ and λ_i in $\mathbb{R}_+$ is the welfare weight for agent i in I. Unlike private agents, the planner internalizes the externality arising from the effect on the interest rate induced by the amount of accumulated capital.

To derive the KKT conditions, let μ in $\mathbb{R}_+$ be the Lagrange multiplier for the feasibility constraint. For every agent i in I with $\lambda_i > 0$, first-order conditions require

$$\lambda_i u_{i0}'(c_{i0}) = \mu$$

and

$$\mu \geq \left(\lambda_i \frac{\partial v_i}{\partial k_i} + \sum_{j \in I} \lambda_j \frac{\partial v_j}{\partial q} \frac{\partial q}{\partial k_i}\right),$$

with the equality if $k_i > 0$. Furthermore, by the Envelope Theorem, we obtain

$$\frac{\partial v_i}{\partial k_i} = (1+\rho)\, u'_{i1}(c_{i1}) \text{ and } \frac{\partial v_i}{\partial q} = -u'_{i1}(c_{i1})\, b_i.$$

These yield that, for every agent i in I,

$$u'_{i0}(c_{i0}) \geq (1+\rho)\, u'_{i1}(c_{i1}) - \left(\frac{\mu}{\lambda_i}\right) \sum_{j \in I} \left(\frac{u'_{j1}(c_{j1})}{u'_{j0}(c_{j0})}\right) b_j \frac{\partial q}{\partial k_i},$$

with the equality if $k_i > 0$.

A competitive equilibrium is generically locally inefficient when at least one agent is constrained in the initial period and at least one agent is accumulating capital in the first period. Indeed, for this latter agent, the previous KKT conditions would imply

$$\sum_{j \in I} \left(\frac{u'_{j1}(c_{j1})}{u'_{j0}(c_{j0})}\right) b_j = 0.$$

As the marginal rates of substitution are different across agents, due to the no borrowing constraint, and the holdings are generically nonzero, this condition cannot be satisfied. A policy intervention may reduce the welfare loss due to market failure and, as argued in Farhi and Werning [10] and Dávila and Korinek [8], policy prescriptions must be based on the first-order conditions of the Ramsey-type planning program. Through a numerically calibrated example, we demonstrate that, even in such a simplified environment, a certain degree of nonconvexity arises, so compromising the reliability of first-order conditions for social optimality.

Assume that there are only two agents, $I = \{1, 2\}$, that there is no endowment in the intermediate period, $e_{i1} = 0$, that agent 1 has no endowment in the last period, $e_{12} = 0$, and that the net return on capital is unitary, $\rho = 1$. Furthermore, agent 1's utilities are $u_{10}(c) = \alpha \log(c)$ and $u_{11}(c) = u_{12}(c) = \log(c)$. These restrictions allow us to immediately determine the optimal individual plan for agent 1 conditional on accumulated capital k in $\mathbb{R}_+$. Assuming the planner only cares about the welfare of agent 1, $\lambda_2 = 0$, the planning objective is given by

$$U_1(k) = \alpha \log(e_0 - k) + \log(k) + \log\left(\frac{k}{q(k)}\right).$$

To determine the equilibrium price of the bond, we exploit the first-order condition of agent 2, that is,

$$q(k)\, u'_{21}(k) = u'_{22}\left(e_2 - \frac{k}{q(k)}\right).$$

To simplify the presentation, we pose $u_{22}(c) = \log(c)$ and $u'_{21}(c)\, g(c) = 1$. The price obtains explicitly as

$$q(k) = \frac{k + g(k)}{e_2}, \tag{2.1}$$

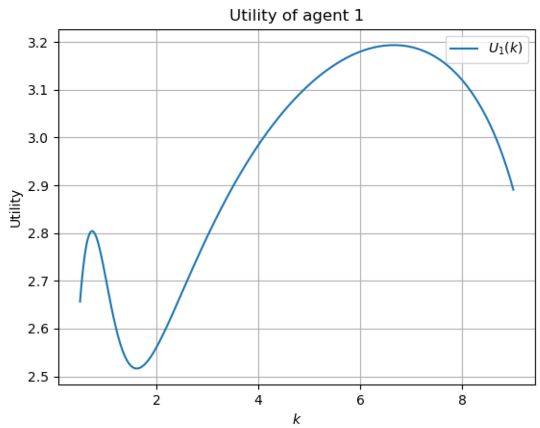

FIGURE 1. Local minimum $\left(\alpha = 0.5, e_0 = e_2 = 10, \frac{1}{g(c)} = \left(\frac{1}{4}\frac{1}{c} + \frac{3}{4}\frac{1}{c^5}\right)\right)$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a smoothly increasing map with $g(0) = 0$.[3] Consequently, agent 1's utility is given by

$$U_1(k) = \alpha \log(e_0 - k) + 2\log(k) - \log(k + g(k)) + \log(e_2)\,. \tag{2.2}$$

We evaluate (2.2) numerically under the maintained assumptions.

Our computations indicate that an allocation can satisfy the first-order conditions of the social planning program even when it is not locally Pareto efficient. In particular, the local minimum of the curve in Figure 1 represents a situation in which a slight increase in capital investment would generate welfare gains for both agents. In fact, agent 1's utility would increase due to the local minimum, while agent 2's utility would rise as well. This is because an increase in capital would lead to a reduction in the interest rate (see condition (2.1)), creating more favorable market conditions for a net borrower, such as agent 2.

To illustrate the fragility of this Pareto improvement, we enrich the environment with two identical copies of each type of agent, $I = \{1, 2\} \times \{a, b\}$, and asymmetrically perturb the utility parameters (α_a, α_b) in $\mathbb{R}_+ \times \mathbb{R}_+$.[4] In particular, the perturbed initial-period utility is

$$u_{1j0}(c) = \alpha_j \log(c) - \alpha_j \log\left(\frac{2\alpha_j}{\alpha_a + \alpha_b}\right) + \Phi_j,$$

where Φ_j in $\mathbb{R}$ is an innocuous constant used for a more convenient graphical illustration. Assuming equal Pareto weights for agents of type 1, time-additive $\log$-utilities ensure that, at any social optimum, initial consumption and capital investment of each copy of agent 1

[3] Notice that allocating some of the capital stock to agent 2 is socially inefficient when $\lambda_2 = 0$, as it would depress the welfare of agent 1 directly (by reducing her capital income) and indirectly (by reducing the interest rate on her savings).

[4] This replication is only a straightforward method of having two agents with strictly positive Pareto weights, while maintaining the simplicity of our calculations.

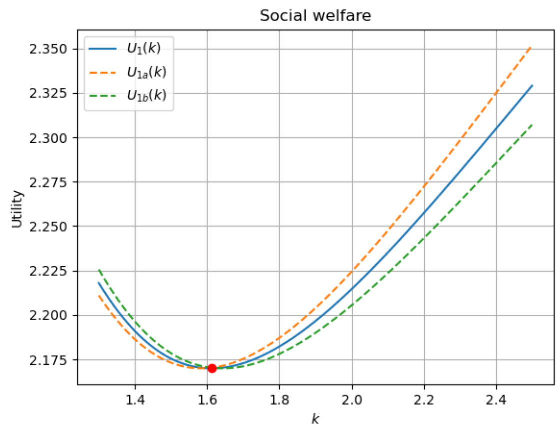

FIGURE 2. Fragility of the Pareto improvement ($\alpha_a = 0, 3$, $\alpha_b = 0.7$)

are given by

$$c_{1j0} = \left(\frac{2\alpha_j}{\alpha_a + \alpha_b}\right)(e_0 - k) \text{ and } k_j = k,$$

where e_0 in $\mathbb{R}_+$ and k in $\mathbb{R}_+$ are the *per-capita* initial-period endowment and the *per-capita* capital investment of agents of type 1. As a consequence,

$$U_{1j}(k) = \alpha_j \log(e_0 - k) + \log(k) + \log\left(\frac{k}{q(k)}\right) + \Phi_j.$$

The social welfare is exactly as described by Figure 1, provided that

$$\frac{\alpha_a + \alpha_b}{2} = \alpha \text{ and } \frac{\Phi_a + \Phi_b}{2} = 0.$$

The local minimum persists, though the Pareto improvement disappears for any slightly perturbed parameters (α_a, α_b) in $\mathbb{R}_+ \times \mathbb{R}_+$. Indeed, increasing capital would be welfare-improving for one copy of agent 1, while turning welfare-depressing for the other copy, as illustrated by Figure 2.

To conclude this introductory digression, when the KKT conditions are satisfied, residual welfare gains (if any) are only of second or higher order. Any policy intervention aiming at restoring efficiency locally needs an incredibly accurate knowledge of the marginal impact on each agent in the economy, so as to calibrate the distribution of welfare gains. Absent the necessary accuracy, a Pareto improvement cannot be ensured, and no intervention can be unambiguously justified based on the Pareto criterion.

3. FUNDAMENTALS

3.1. **Preliminaries.** We consider canonical Euclidean spaces of the form $\mathbb{R}^N$, where N is a finite set, endowed with the supremum norm and the natural ordering. In particular, the norm is given by

$$\|x\| = \sup_{j \in N} |x_j| .$$

Furthermore, adhering to a common notational practice, given x and y in $\mathbb{R}^N$, $x \geq y$ and $x \gg y$ mean, respectively, $x_j \geq y_j$ and $x_j > y_j$ for all j in N, while $x > y$ is short-notation for $x_j \geq y_j$ for all j in N and $x_j > y_j$ for some j in N.

Let N and M be finite sets, and let X be an open set in $\mathbb{R}^N$. Given a differentiable map $f : X \to \mathbb{R}^M$, the linear map $Df(x) : \mathbb{R}^N \to \mathbb{R}^M$ is its (Fréchet) derivative at x in X, that is,

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Df(x)(h)\|}{\|h\|} = 0.$$

Thus, $Df(x)(h)$ in $\mathbb{R}^M$ measures the first-order change along the direction h in $\mathbb{R}^N$. As usual, the operator norm is given by

$$\|Df(x)\| = \sup_{h \in \mathbb{R}^N} \{\|Df(x)(h)\| : \|h\| \leq 1\} .$$

Let $\mathcal{E}$ be the space of all smooth maps $f : X \to \mathbb{R}^M$, where $X \subset \mathbb{R}^N$ is an open set. This space $\mathcal{E}$ can be endowed with the weak topology induced by the seminorms

$$p_K(f) = \max \left\{ \sup_{x \in K} \|f(x)\| , \sup_{x \in K} \|Df(x)\| \right\},$$

where $K \subset X$ is understood to be a compact set. Any neighborhood $\mathcal{N}$ of f in $\mathcal{E}$ is such that, for some $\epsilon > 0$ in $\mathbb{R}$ and some compact set $K \subset X$,

$$p_K\left(\hat{f} - f\right) < \epsilon \text{ implies } \hat{f} \in \mathcal{N}.$$

In other terms, proximity in this topology is captured by the uniform convergence of maps and their derivatives over compact sets $K \subset X$.

A robustness analysis unavoidably requires a topology in order to express the idea of proximity of economies. We measure closeness in terms of the distance of maps and their first derivatives. To our ends, any finer topology would be justly satisfactory. However, evaluating the distance only up to first-order terms seems the more natural choice if our aim is to assess the degree of discipline imposed of first-order conditions. Due to the enforcement of certain first-order conditions at equilibrium, a policy authority can often rely on some indirect and perhaps imprecise knowledge of first-order terms. Second- or higher-order terms, instead, are not revealed by first-order conditions and it seems legitimate to assume that little or no information is available about them.

3.2. **Planning program.** We use $\mathbb{R}^N$ for the linear space of allocations, $\mathbb{R}^J$ for the linear space associated with constraints, and $\mathbb{R}^I$ for the linear space of distributions of utility values, where N, J and I are finite sets. In particular, I is interpreted as the set of agents populating the economy, whereas J represents the set of constraints imposed by feasibility restrictions in the underlying economic environment. The nature of allocations will depend on the specific application of the theory.

We provide a parsimonious description of the economic environment. The primitive elements are given by smooth maps $f : X \to \mathbb{R}^I$ and $g : X \to \mathbb{R}^J$, where $X \subset \mathbb{R}^N$ is an open set. We refer to (f, g) as an economy, and let $\mathcal{E}$ be the space of all such economies. An allocation x in X is *feasible* if it lies in the set $\mathcal{F}$, where

$$\mathcal{F} = \{x \in X : g(x) \leq 0\}.$$

The distribution of utility values at an allocation x in $\mathcal{F}$ is given by $f(x)$ in $\mathbb{R}^I$, that is, $f_i(x)$ in $\mathbb{R}$ is the utility value of agent i in I at allocation x in $\mathcal{F}$. A large variety of conventional frameworks is encompassed in this abstract setting. Relevantly, we impose no convexity restrictions because this property fatally fails in many relevant applications of our theory. In a supplemental appendix (Appendix B), we will provide extensions of our theory by encompassing additional equality constraints, as they naturally arise in the welfare evaluations of certain economic policies at a competitive equilibrium.

3.3. **Efficiency.** An allocation x^* in $\mathcal{F}$ is *inefficient* if there exists an alternative allocation x in $\mathcal{F}$ such that $f(x) > f(x^*)$. It is *locally inefficient* if there exists a sequence of allocations $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that $f(x_n) > f(x^*)$ for every n in $\mathbb{N}$. These notions are canonical. We study a robust form of local inefficiency: a welfare-improving reallocation must exist locally even under a slightly inaccurate knowledge of fundamentals.

To ensure robustness, we require the existence of local reallocations that are Pareto improving for any economy in the proximity of the benchmark economy (f, g) in $\mathcal{E}$. To this end, we consider a space of perturbations $\mathcal{N}(x^*)$, containing all economies $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{E}$ lying in a neighborhood of the benchmark economy (f, g) in $\mathcal{E}$ and satisfying the feasibility restriction

$$\hat{g}(x^*) \leq 0.$$

This is the space of economies consistent with the observation of the given allocation x^* in $\mathcal{F}$, conditional on a slightly imprecise knowledge of fundamentals. We will refer to $\mathcal{N}(x^*)$ simply as a *neighborhood* of economies.

An allocation x^* in $\mathcal{F}$ is *robustly locally inefficient* if there exist a neighborhood of economies $\mathcal{N}(x^*)$ and a sequence of allocations $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that, for every n in $\mathbb{N}$, $\hat{g}(x_n) \leq 0$ and $\hat{f}(x_n) > \hat{f}(x^*)$ are satisfied for all economies $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}(x^*)$. In other terms, arbitrarily close to the *status quo* allocation, there exists an alternative allocation that is feasible and Pareto-improving in every economy in the neighborhood.

Our efficiency concepts are clearly nested. The notion of efficiency reflects the canonical Pareto criterion. Local efficiency is its natural restriction to reallocations remaining in the proximity of the *status quo*. Yet, local efficiency is perhaps more relevant than global efficiency for the design and the assessment of economic policies. Indeed, any nonlocal failure of efficiency requires policy interventions involving abrupt changes in the economic conditions, whereas restoring local efficiency only entails a more conventional comparison of marginal social benefits and marginal social costs. The reinforcement of robust local efficiency mirrors the idea that measurements or estimates of fundamental characteristics are imprecise. Indeed, when local inefficiency is not robust, welfare gains are ensured only conditional on an extremely accurate knowledge of the economic conditions in the proximity of the *status quo*. Therefore, implementing a supposedly welfare-improving policy entails the risk of reducing welfare when measurements or estimates of fundamental characteristics are imprecise.

4. First-order conditions

We now identify first-order conditions for robust local inefficiency. As in the traditional approach, we need to impose a regularity condition in order to exploit variational methods. It is well known that this regularity condition essentially eliminates the possibility of the constraint boundary forming a cusp at a point.

Smooth regularity condition. *An allocation x^* in $\mathcal{F}$ is* regular *if there does not exist $\mu > 0$ in $\mathbb{R}^{J(x^*)}$ such that*

$$\sum_{j \in J(x^*)} \mu_j Dg_j(x^*) = 0,$$

where $J(x^*) \subset J$ is the set of binding constraints, i.e.*, of all j in J such that $g_j(x^*) = 0$.*

Lemma 1 (Stiemke's Lemma)**.** *An allocation x^* in $\mathcal{F}$ is regular if and only if there exists a direction h in $\mathbb{R}^N$ such that, for every j in J,*

$$g_j(x^*) + Dg_j(x^*)(h) < 0. \tag{4.1}$$

We say that an allocation x^* in $\mathcal{F}$ satisfies the *KKT conditions* if there exist an element $\lambda > 0$ of $\mathbb{R}^I$ and an element $\mu \geq 0$ of $\mathbb{R}^J$ such that

$$\sum_{i \in I} \lambda_i Df_i(x^*) = \sum_{j \in J} \mu_j Dg_j(x^*) \tag{4.2}$$

and

$$\sum_{j \in J} \mu_j g_j(x^*) = 0. \tag{4.3}$$

We argue that KKT conditions are in fact equivalent to the absence of robust local inefficiency.

The KKT conditions must be satisfied whenever robust Pareto improvements are not locally feasible. Despite the complexity introduced by robustness, this is not entirely surprising, as first-order conditions are required even in the presence of nonconvexity. The sufficiency of the KKT conditions in ruling out robust local Pareto improvements depends on the risk of misrepresentation: Any supposedly welfare-improving policy would induce welfare losses if the underlying primitives are slightly perturbed. As is typical in perturbation analyses, our abstract framework enhances the clarity of the argument, but at the cost of limiting its broader applicability. In practical settings, the underlying primitives may be subject to additional constraints that reflect the particular features of the economic environment. In such cases, the simple perturbations considered in our analysis may not always be feasible. To address this, we complement our characterization by specifying certain conditions that must be locally satisfied whenever an allocation is robustly inefficient under the KKT conditions. As further detailed in Sections 5.1–5.2, the failure to meet these conditions can often be operationally verified in many applications, thereby ruling out robust local inefficiency.

Proposition 4.1 (KKT characterization)**.** *A regular allocation x^* in $\mathcal{F}$ satisfies the KKT conditions if and only if it is* not *robustly locally inefficient. In particular, it is robustly locally inefficient only if there exists a (nonzero) direction h in $\mathbb{R}^N$ such that*

$$\lambda_i Df_i\left(x^*\right)(h) = 0 \text{ and } \mu_j Dg_j\left(x^*\right)(h) = 0; \tag{4.4}$$

and, for any nearby economy $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}\left(x^*\right)$,

$$\lambda_i D\hat{f}_i\left(x^*\right)(h) \geq 0 \text{ and } \mu_j D\hat{g}_j\left(x^*\right)(h) \leq 0, \tag{4.5}$$

provided that $\hat{g}_j\left(x^*\right) = 0$ for every j in $J\left(x^*\right)$.

The necessity of the KKT conditions for robust local efficiency arises from the fact that they prevent the existence of a direction h in $\mathbb{R}^N$ such that

$$0 \ll Df\left(x^*\right)(h) \text{ and } g\left(x^*\right) + Dg\left(x^*\right)(h) \ll 0. \tag{4.6}$$

As a matter of fact, the first-order effects along the direction h in $\mathbb{R}^N$ would lead to a strict increase in all agents' utilities while moving towards the interior of the feasible set. Furthermore, condition (4.6) is notably robust to any slight smooth perturbation of the primitives. Thus, if an allocation is not robustly locally inefficient, condition (4.6) cannot hold, so implying that the KKT conditions must necessarily be satisfied.

The argument for the sufficiency of the KKT conditions is more subtle and warrants a brief discussion. Assuming the KKT conditions hold, any local welfare improvement would eventually approach a direction h in $\mathbb{R}^N$ such that, for at least one agent i in I,

$$Df_i\left(x^*\right)(h) = 0. \tag{4.7}$$

This occurs because the KKT conditions preclude any first-order increase in agents' utilities, implying that any welfare improvement must be of second or higher order. Condition (4.7) is apparently fragile: A slight perturbation of the primitives would yield

$$D\hat{f}_i\left(x^*\right)(h) < 0,$$

thereby transforming the supposed welfare increase into an actual decrease. Therefore, locally, any welfare improving reallocation necessitates an extremely accurate knowledge of the fundamentals.

In certain circumstances, the KKT conditions are derived from equilibrium constraints (or, more fundamentally, from no-arbitrage restrictions), and their validity is ensured even when the exact characteristics of the economy are not precisely known. Consequently, when evaluating the robust local inefficiency of an equilibrium allocation, the space of allowable perturbations to the economy must be restricted to those that preserve equilibrium, and thereby, the KKT conditions. An appropriate counterfactual for robust inefficiency would require a restricted neighborhood $\mathcal{N}_{\text{KKT}}(x^*) \subset \mathcal{N}(x^*)$ consisting of all economies $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}(x^*)$ satisfying KKT conditions for some $\hat{\lambda} > 0$ in $\mathbb{R}^I$ and $\hat{\mu} \geq 0$ in $\mathbb{R}^J$.

Consistently, an allocation x^* in $\mathcal{F}$, satisfying the KKT conditions, is *KKT-robustly locally inefficient* if there exist a neighborhood of economies $\mathcal{N}(x^*)$ and a sequence of allocations $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that, for every n in $\mathbb{N}$, $\hat{g}(x_n) \leq 0$ and $\hat{f}(x_n) > \hat{f}(x^*)$ are satisfied for all economies $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}_{\text{KKT}}(x^*)$. In other terms, even subject to the validity of KKT conditions, any residual welfare-improvement requires an extremely accurate knowledge of the primitives.

Proposition 4.2 (KKT characterization strengthened)**.** *A regular allocation x^* in $\mathcal{F}$ satisfies the KKT conditions only if it is* not *KKT-robustly locally inefficient, provided there exist at least two agents i in I with $\lambda_i > 0$. In particular, it is robustly locally inefficient only if there exists a (nonzero) direction h in $\mathbb{R}^N$ such that, for any economy $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}_{\text{KKT}}(x^*)$,*

$$\hat{\lambda}_i D\hat{f}_i(x^*)(h) = 0 \text{ and } \hat{\mu}_j D\hat{g}_j(x^*)(h) = 0. \tag{4.8}$$

Our previous argument for the sufficiency of the KKT conditions requires modification. The validity of these conditions, under the maintained assumptions, is guaranteed whenever

$$\sum_{i\in I} \lambda_i \left(D\hat{f}_i(x^*) - Df_i(x^*)\right) = 0. \tag{4.9}$$

In fact, condition (4.9) ensures that the KKT conditions are satisfied, in line with the objectives of our robustness experiment. According to the Pareto criterion, to demonstrate a lack of robustness, it is sufficient to identify a perturbation that reduces the welfare of one agent, $D\hat{f}_{i_1}(x^*)(h) < 0$, even if the same perturbation increases the welfare of another agent, $D\hat{f}_{i_2}(x^*)(h) > 0$. Condition (4.9) is suitably permissive to allow for such a sort of perturbation.

We complement our characterization by arguing that robust local inefficiency is in fact equivalent to the presence of first-order welfare gains. In other terms, whenever an allocation is not robustly locally inefficient, residual welfare gains can only be negligible, that is, entailing only second- or higher-order terms. These terms might be difficult to estimate, even relying on restrictions on fundamentals implied by first-order conditions of optimizing agents in the economy.

Proposition 4.3 (First-order welfare gains)**.** *A regular allocation x^* in $\mathcal{F}$ is robustly locally inefficient if and only if there exists a sequence of allocations $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that*

$$\lim_{n\to\infty} \frac{f(x_n) - f(x^*)}{\|x_n - x^*\|} \gg 0.$$

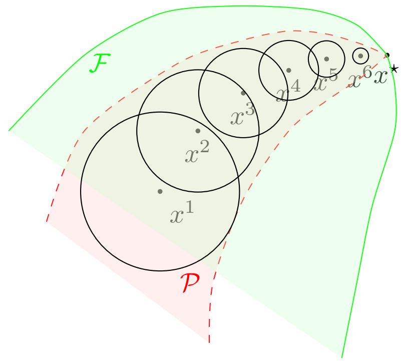

FIGURE 3. Trembling-hand local inefficiency

We have established that, under KKT conditions, welfare gains are only negligible locally and the devise of a welfare-improving policy (if any) requires an extremely accurate local knowledge of the primitives. We now perform another robustness experiment: Even though the knowledge of the primitives is extremely accurate and reliable, the implementation of a planned reallocation is subject to small errors or, alternatively, the reallocation process entails a small welfare-reducing real cost. It turns out that, when utilities are strictly concave, this sort of robustness to trembles coincides with our previous idea of robust local inefficiency and, hence, it is exhaustively characterized by first-order conditions.

Formally, we say that an allocation x^* in $\mathcal{F}$ is *trembling-hand locally inefficient* if there exist ϵ in $(0, 1) \subset \mathbb{R}$ and a sequence of allocations $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that, for every n in $\mathbb{N}$, $g(y_n) \leq 0$ and $f(y_n) > f(x^*)$ are satisfied for all y_n in X such that

$$\|y_n - x_n\| \leq \epsilon \|x_n - x^*\| .$$

In the spirit of trembling-hand robustness, arbitrarily close to the *status quo*, a planner is able to implement a welfare-improving feasible reallocation even when the reallocation is subject to small errors proportional to the size of moved resources. In this sense, any planned reallocation along the sequence is robust to errors due to some degree of inaccuracy in its implementation. We refer to Figure 3 for a graphical illustration, where $\mathcal{P} = \{x \in \mathcal{F} : f(x) > f(x^*)\}$.

Proposition 4.4 (Trembling-hand robustness)**.** *A regular allocation x^* in $\mathcal{F}$ is robustly locally inefficient if and only if it is trembling-hand locally inefficient, provided that each $f_i : X \to \mathbb{R}$ is smoothly strictly concave (*i.e.*, its Hessian matrix is negative definite).*

To complete our characterization, we finally argue that any residual feasible welfare-improvement has an arbitrarily negligible (Lebesgue) measure in the proximity of the *status quo* whenever the KKT conditions are satisfied. The intuition is as follows. By regularity (see Lemma 1), in every small neighborhood around x^* in $\mathcal{F}$, the set of feasible allocations $\mathcal{F}$ occupies a uniformly positive fraction of the space. On the other hand, if

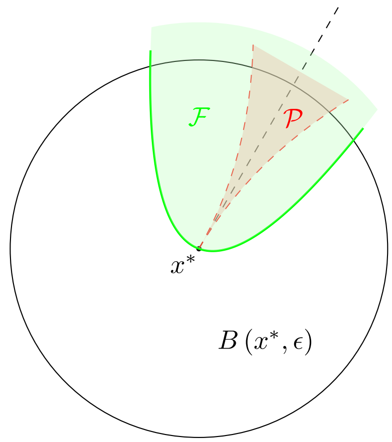

FIGURE 4. Neglible Pareto improvements

the utility gradients satisfy a mild nondegeneracy condition, then the KKT conditions force the set of attainable Pareto improvement to locally lie almost in a lower-dimensional space. Since any lower-dimensional set in $\mathbb{R}^N$ has zero Lebesgue measure, the space of feasible Pareto improvements is negligible compared to the full space of nearby feasible allocations. Figure 4 provides a heuristic graphical illustration, while Proposition 4.5 presents the formal statement. As far as notation is concerned, $\mathcal{P} = \{x \in \mathcal{F} : f(x) > f(x^*)\}$ is the space of feasible Pareto improvements and $B(x^*, \epsilon) = \{x \in X : \|x - x^*\| < \epsilon\}$ is an open ball around x^* in $\mathcal{F}$ of radius ϵ in $\mathbb{R}_{++}$.

Proposition 4.5 (KKT characterization revised)**.** *Provided that $Df_i(x^*) \neq 0$ for every agent i in I, a regular allocation x^* in $\mathcal{F}$ satisfies the KKT conditions if and only if*

$$\lim_{\epsilon \to 0} \frac{\lambda^*(\mathcal{P} \cap B(x^*, \epsilon))}{\lambda^*(\mathcal{F} \cap B(x^*, \epsilon))} = 0, \tag{4.10}$$

where λ^* is the Lebesgue measure on $\mathbb{R}^N$.

5. FURTHER COMPLICATIONS

5.1. **Allowed pertubations.** The simplicity of our environment enhances the clarity of our overall analysis, particularly in relation to the sufficiency of the KKT conditions for ensuring robust local efficiency. However, this parsimony comes at the expense of limiting the broader applicability of our propositions. In many applications, the underlying economic environment imposes further discipline on the fundamentals. As a consequence, the space of permissible perturbations must be more narrowly defined to account for this increased complexity. While our arguments may not directly extend to such cases in their current form, they could be adapted to these more specific contexts. In fact, our characterization in Propositions 4.1-4.2 provides conditions that can be operationally verified through perturbations permitted by the economic environment.

It is worth to further discuss the nature of the appropriate perturbations that can be used to establish the sufficiency of the KKT conditions for robust local efficiency. To prove Proposition 4.2, we consider a set of linear maps $\phi_i : \mathbb{R}^N \to \mathbb{R}$ such that the condition $\sum_{i \in I} \lambda_i \phi_i = 0$ holds, where $\phi_i(h) < 0$ for at least one agent i in I with $\lambda_i > 0$, and $\|\phi_i\|$ is sufficiently small for all agents in I. The local perturbation $\hat{f}_i(x) = f_i(x) + \phi_i(x - x^*)$ is then appropriate to support our argument. Unfortunately, in certain contexts, such a perturbation may not be feasible because the utilities of agents depend solely on their private consumptions. As an alternative, we can assume that the collection of linear maps $\phi_i : \mathbb{R}^N \to \mathbb{R}$ satisfies the more permissive condition

$$\sum_{i \in I} \lambda_i \phi_i = \sum_{j \in J} (\hat{\mu}_j - \mu_j) \nabla g_j(x^*),$$

where $\hat{\mu} \geq 0$ in $\mathbb{R}^J$ are slightly adjusted Lagrange multipliers. This formulation introduces additional degrees of freedom, enabling the argument to proceed, as illustrated by a simple example.

Example 5.1 (Allocation of goods with nonconvex utilites)**.** Consider the allocation of certain goods among agents with nonconvex preferences. To integrate this scenario into our general framework, let J denote the finite set of goods, and define $N = I \times J$, $f_i(x) = u_i(x_i)$, and

$$g(x) = \sum_{i \in I} x_i,$$

where $u_i : \mathbb{R}^J \to \mathbb{R}$ is a smoothly increasing function. In this setting, the KKT conditions are expressed, for each agent i in I, as

$$\lambda_i \nabla u_i(x_i^*) = \nu,$$

where $\nu \gg 0$ in $\mathbb{R}^J$ represents the Lagrange multipliers associated with the resource constraints. Within our general framework, the gradients and the Lagrange multipliers are given by

$$\nabla f_i(x^*) = (0, \ldots, 0, \nabla u_i(x_i^*), 0, \ldots, 0) \in \mathbb{R}^N,$$
$$\nabla g_j(x^*) = (\mathbf{1}_j, \ldots, \mathbf{1}_j, \ldots, \mathbf{1}_j) \in \mathbb{R}^N,$$

and

$$\mu = (\nu, \ldots, \nu, \ldots, \nu) \in \mathbb{R}^N,$$

where $\mathbf{1}_j$ in $\mathbb{R}^J$ is the indicator vector corresponding to commodity j in J. Additionally, the direction h in $\mathbb{R}^N$ of a potential local Pareto improvement decomposes as $h = (\ldots, h_i, \ldots)$, with each h_i in $\mathbb{R}^J$. We now demonstrate the existence of a suitable perturbation that violates condition (4.8).

The utility gradients are defined over disjoint consumption spaces, preventing us from constructing perturbations that satisfy $\sum_{i \in I} \lambda_i \phi_i = 0$, as required in our benchmark proof of Proposition 4.2. However, for any agent i in I, we can define a small linear map $\phi_i : \mathbb{R}^N \to \mathbb{R}$ as

$$\phi_i(x) = \frac{1}{\lambda_i} \sum_{j \in J} \xi_j x_{ij},$$

where $\xi > 0$ in $\mathbb{R}^J$ is chosen such that $\sum_{j \in J} \xi_j h_{ij} < 0$ for at least one agent i in I. We then consider a local perturbation of all utilities given by

$$\hat{u}_i(x_i) = u_i(x_i) + \phi_i(x_i - x_i^*).$$

The KKT conditions remain valid under this perturbation with updated Lagrange multipliers $\hat{\nu} = \nu + \xi$. The existence of such a perturbation precludes any unambiguous local Pareto improvement.

Conventional arguments for genericity can be used to demonstrate the sufficiency of the KKT conditions under appropriate regularity assumptions. The KKT conditions (4.2) entail N equations in $(I + J) \times (N + 1) - 1$ unknowns. These unknowns include the Pareto weights and the Lagrange multipliers, assumed to be strictly positive for simplicity, as well as the gradients of the utility functions and constraints. Specifically, there are I Pareto weights and J Lagrange multipliers, both determined only up to a normalization, resulting in $(I + J - 1)$ unknowns. Moreover, the utility functions contribute I gradients, each of dimension N, while the constraints involve J gradients, also of dimension N. The gradients can be treated as unknowns because the primitives are susceptible to slight perturbations.

Under the assumption of robust local inefficiency, the additional $(I + J)$ equations provided by (4.8) are necessarily satisfied. However, this configuration often becomes infeasible due to the presence of $(I + J - 1) \times N - 1$ residual degrees of freedom. Under regularity assumptions, the Implicit Function Theorem implies that these additional unknowns would uniquely determine the other variables. This prevents the validity of the equations (4.8) for all perturbed economies. While additional assumptions may reduce the dimensionality of the gradients, some degrees of freedom are likely to persist. Therefore, if the system is sufficiently rich, the KKT conditions rule out the occurrence of robust local inefficiency.

Example 5.2 (Allocation of goods with nonconvex utilites (continued))**.** Consider again the economy in Example 5.1. The KKT conditions are given by

$$\lambda_i \nabla u_i (x_i^*) = \nu,$$

which can be interpreted as a system of $I \times J$ equations involving $I \times J$ unknowns corresponding to the gradients of the agents' utilities, and J unknowns associated with the Lagrange multipliers. Furthermore, the equations in (4.8) provide additional I equations of the form

$$\nabla u_i\left(x_i^*\right)\left(h_i\right)=0.$$

Altogether the system reduces to I equations of the form

$$\sum_{j\in J}\nu_j h_{ij}=0,$$

where ν in $\mathbb{R}^J$ are the Lagrange multipliers with $\nu\gg 0$. Note that $\sum_{i\in I}h_i=0$ by the material balance and, therefore, at most $\min\{I,J\}-1$ equations are independent. Since $h_i=0$ cannot hold for all i in I, this system of equations cannot be satisfied for all perturbations of the Lagrange multipliers.

5.2. **Interdependent constraints.** In certain relevant applications, the constraints are not independent of the utilities. For instance, some of the constraints might be enforcing incentive-compatibility, and are so intimately related to utilities. Alternatively, some of the enforceability requirements might involve Euler equations or some other equality of weighted marginal utilities in order to account for a parametric optimal behavior of certain agents. In all such instances, any perturbation of $f:X\to\mathbb{R}^I$ necessarily translates into a corresponding perturbation of $g:X\to\mathbb{R}^J$.[5] Consequently, our arguments for the sufficiency of KKT conditions require some amendments. In several cases, these adjustments arise naturally, as illustrated by examples.

Example 5.3 (Dynamic Ramsey taxation)**.** Consider the classical Ramsey-taxation framework over a finite horizon. A typical implementability constraint is of the form

$$g\left(x\right)=\left(\begin{array}{c}Df\left(x\right)\left(x\right)-v\\ \Gamma\left(x\right)\end{array}\right)\leq 0,$$

where $\Gamma:X\to\mathbb{R}^K$ accounts for the material balance given consumption and labor supply over time, while v in $\mathbb{R}$ is the initial government debt (denominated in terms of initial marginal utility of consumption). In this case, one could consider any perturbation of the form

$$\hat{g}\left(x\right)=\left(\begin{array}{c}D\hat{f}\left(x\right)\left(x\right)-\hat{v}\\ \Gamma\left(x\right)\end{array}\right),$$

where

$$\hat{v}=v+Df\left(x^*\right)\left(x^*\right)-D\hat{f}\left(x^*\right)\left(x^*\right).$$

As these derivatives remain close, $|v-\hat{v}|$ in $\mathbb{R}$ can be taken arbitrarily small, so that it is in fact a slight perturbation of the benchmark economy. This ensures that $g\left(x^*\right)\leq 0$ if and only if $\hat{g}\left(x^*\right)\leq 0$, as required to prove the sufficiency of KKT conditions.

[5] We incidentally notice that Farhi and Werning [10, Proposition 3]'s genericity analysis suffers of an analogous limitation. Indeed, they argue that, if an efficient allocation can be implemented without financial taxes, a necessary first-order condition would be violated by an appropriate utility perturbation, so that the new optimum would require financial taxes. In many of their applications, however, this utility perturbation would also affect the constraints, thus inducing additional indirect effects on first-order conditions that are neglected by their arguments.

Example 5.4 (Incentive-compatibility constraints)**.** In certain environments, the implementability of an allocation requires the enforcement of incentive-compatibility constraints. We parsimoniously describe a typical framework in which such constraints are needed. Letting Θ be the finite space of types, an allocation is of the form $X = \mathbb{R}^{I\times\Theta}$, with $c_i(\theta)$ in $\mathbb{R}$ being the consumption of agent i in I conditional on θ in Θ. The planning objective is of the form

$$f_i(x) = \sum_{\theta\in\Theta} \pi(\theta)\, u_i(c_i(\theta)),$$

where $\pi(\theta) > 0$ in $\mathbb{R}$ is the probability of state θ in Θ, so that $\sum_{\theta\in\Theta}\pi(\theta) = 1$, and $u_i : \mathbb{R} \to \mathbb{R}$ is a strictly concave map. Untruthful revelations might be captured by finitely many maps of the form $\sigma : \Theta \to \Theta$, with implied incentive-compatibility constraints

$$u_i(c_i(\theta)) \geq u_i(c_i(\sigma(\theta))).$$

We can then consider open neighborhoods $U_i(\theta)$ of $c_i(\theta)$ in $\mathbb{R}$ and assume that $c_i(\sigma(\theta))$ does not lie in $U_i(\theta)$, except when $c_i(\theta) = c_i(\sigma(\theta))$. A local perturbation of utilities is given by

$$\hat{u}_i(\hat{c}_i(\theta)) = u_i(c_i(\theta)) + \phi_i(\theta)(\hat{c}_i(\theta) - c_i(\theta)).$$

As neighborhoods are disjoint, this enables us to perturb the economy as needed for the proof of sufficiency of KKT conditions. In particular, feasibility is preserved by any of such perturbations.

5.3. **Indirect utilities.** In certain applications, the objectives of the Ramsey planner are determined as indirect utilities of the agents (*e.g.*, Farhi and Werning [10]). To adapt our approach to such a circumstance, suppose that Z_i is an open set in $\mathbb{R}^{N_i}$, and let

$$f_i(x) = \max u_i(z_i)$$

subject to

$$b_i(z_i, x) \leq 0,$$

where $b_i : Z_i \times X \to \mathbb{R}^{J_i}$ is a smooth map. We postulate that $u_i : Z_i \to \mathbb{R}$ is smoothly concave and $b_i : Z_i \times X \to \mathbb{R}^{J_i}$ is smoothly quasiconvex given x in X. Furthermore, in addition to our previous assumptions, we also impose, for every x in X, the existence of some z_i in Z_i such that $f_i(x) = u_i(z_i)$ and $b_i(z_i, x) \leq 0$. Hence, an optimal individual plan always exists.

We can redefine all the elements of this environment so that they immediately fit our general framework. To this end, consider the expanded allocation space $\tilde{X} = X \times \prod_{i\in I} Z_i \subset \mathbb{R}^{\tilde{N}}$, with implied objectives defined as $\tilde{f}_i(\tilde{x}) = u_i(z_i)$. In turn, the expanded constraints are given by

$$\tilde{g}(\tilde{x}) = \left(g(x), (b_i(z_i, x))_{i\in I}\right) \in \mathbb{R}^{\tilde{J}}.$$

In fact, the expanded sets for allocations and constraints are given by $\tilde{N} = N \times \prod_{i\in I} N_i$ and $\tilde{J} = J \times \prod_{i\in I} J_i$. We now argue that, if allocation $\tilde{x}$ in $\tilde{\mathcal{F}}$ satisfies the KKT conditions of the expanded environment with all welfare weights $\lambda_i > 0$, then $\tilde{f}_i(\tilde{x}) = f_i(x)$ for every agent i in I. This implies that, subject to regularity of the individual programs, Pareto efficiency in the original economy corresponds to Pareto efficiency in the expanded economy, thus permitting an application of our characterization.

By direct identification, the validity of KKT conditions in the expanded economy requires, for each agent i in I and each constraint j in J_i,

$$\begin{aligned} \lambda_i \nabla u_i(z_i) - \sum_{j \in J_i} \mu_{ij} \nabla_{z_i} b_{ij}(z_i, x) &= 0, \\ \sum_{j \in J_i} \mu_{ij} b_{ij}(z_i, x) &= 0, \\ b_{ij}(z_i, x) &\leq 0. \end{aligned}$$

In turn, as the individual program for each agent i in I is convex, by the canonical KKT Theorem, these conditions are sufficient to established that the individual plan is optimal in the associated individual program, thus confirming our claim.

6. COMMENTS

6.1. **A Lagrangian-type approach.** It is worth noticing that the KKT conditions are essentially the first-order conditions associated with the traditional Lagrangian $\mathcal{L} : X \times \mathbb{R}_+^I \times \mathbb{R}_+^J \to \mathbb{R}$ given by

$$\mathcal{L}(x, \lambda, \mu) = \sum_{i \in I} \lambda_i f_i(x) - \sum_{j \in J} \mu_j g_j(x).$$

In turn, this Lagrangian naturally arises from a Negishi-type planning program of the form

$$\max_{x \in X} \sum_{i \in I} \lambda_i f_i(x)$$

subjet to, for every j in J,

$$g_j(x) \leq 0.$$

Social welfare is measured by the weighted sum of agents' utilities, and the planner maximizes this social welfare subject to all feasibility restrictions. Notwithstanding the strong analogy, in general, a locally efficient allocation is neither a local maximum nor a local minimum of this social planning program.

6.2. **Failure of local efficiency.** As illustrated by a simple example (Example 6.1), robust local efficiency occurs even when an allocation is locally Pareto dominated by an alternative allocation. In this case, however, local welfare gains are negligible, in the sense that they involve quadratic or higher-order terms.

Example 6.1 (Trivial example)**.** Suppose that $X = \mathbb{R}$, $g(x) = x^2 - 1$ and $f(x) = x^2$. Clearly, the two efficient allocations require $|x^*| = 1$, and they obviously satisfy the KKT conditions for $\lambda = \mu$. The allocation $x^* = 0$ is locally inefficient. However, it is not first-order inefficient because $Df(0) = 0$. Indeed, considering any sufficiently small alternative allocation x in $\mathbb{R}$,

$$\lim_{x \to x^*} \frac{f(x) - f(x^*)}{|x - x^*|} = \lim_{x \to 0} \frac{x^2}{|x|} = 0,$$

thus confirming robust local efficiency (by Proposition 4.3).

6.3. **Pareto frontier.** Any allocation on the canonical Pareto frontier is obviously not robustly locally inefficient. As a consequence, it satisfies the KKT conditions for some welfare weights. However, as already noticed, this property does not imply that allocations on the Pareto frontier maximize social welfare for the given welfare weights. These basic facts are illustrated by simple examples.

Example 6.2 (Pareto frontier)**.** Suppose that $I = \{1, 2\}$, $X = \mathbb{R}_{++} \times \mathbb{R}_{++}$, $f_i(x_1, x_2) = 2^{-1}(x_i)^2$ and $g(x_1, x_2) = x_1 + x_2 - 1$. The KKT conditions take the form

$$\lambda_i x_i = \mu$$

and

$$\mu(x_1 + x_2 - 1) = 0.$$

These conditions cannot be satisfied if the only constraint is nonbinding, because $\mu = 0$ would imply $\lambda_i x_i = 0$, contradicting the fact that $\max\{\lambda_1, \lambda_2\} > 0$ and $\min\{x_1, x_2\} > 0$. Assuming that $x_1 + x_2 = 1$, given any $\mu > 0$ in $\mathbb{R}$, the KKT conditions are satisfied by welfare weights

$$\lambda_i = \frac{\mu}{x_i}.$$

Hence, the space of robustly locally efficient allocations coincides with the space of efficient allocations, and the KKT conditions fully characterize such allocations.

Example 6.3 (Planning program)**.** Continuing with the previous example, we now clarify that satisfying the KKT conditions does not imply that the allocation maximizes the social welfare given by the weighted sum of agents' utilities. To this purpose, consider the symmetric allocation with the associated symmetric welfare weights. A small transfer ϵ in $\mathbb{R}$ from one agent to the other agent affects social welfare according to

$$W(\epsilon) = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2} + \epsilon\right)^2 + \frac{1}{2}\left(\frac{1}{2} - \epsilon\right)^2\right]$$

By direct computation,

$$W'(\epsilon) = \frac{1}{2}\left[\left(\frac{1}{2} + \epsilon\right) - \left(\frac{1}{2} - \epsilon\right)\right] = \epsilon$$

and

$$W''(\epsilon) = 1.$$

The robustly locally efficient symmetric allocation is actually a local minimum of the social welfare function for the welfare weights satisfying the KKT conditions.

6.4. **Convex environments.** The environment is convex if $f : X \to \mathbb{R}^I$ is *concave* and $g : X \to \mathbb{R}^J$ is *quasiconvex*. Under convexity, as we argue in Proposition 6.1, robust local efficiency collapses to weak Pareto efficiency, that is, the absence of an alternative allocation x in $\mathcal{F}$ such that $f(x) \gg f(x^*)$. In particular, this implies that our Proposition 4.1 also provides an exhaustive characterization of weakly Pareto efficiency in a convex environment. More importantly, as Proposition 4.1 reduces to canonical statements under convexity, it reveals that our characterization for nonconvex environments is tight.

Proposition 6.1 (Concave objective)**.** *In a convex environment with a concave objective, an allocation x^* in $\mathcal{F}$ is robustly locally inefficient if and only if it is weakly inefficient.*

We finally consider an environment in which $f : X \to \mathbb{R}^I$ is *strictly quasiconcave*, that is, given x^* in $\mathcal{F}$, for every x in $\mathcal{F}$,

$$f_i(x) \geq f_i(x^*) \text{ only if } Df_i(x^*)(x - x^*) \geq 0$$

and

$$f_i(x) > f_i(x^*) \text{ only if } Df_i(x^*)(x - x^*) > 0.$$[6]

Strict quasiconcavity of the objective, along with convexity of the feasible set, is sufficient to enforce the canonical characterization.

Proposition 6.2 (Strict quasiconcave objective)**.** *In a convex environment with a strictly quasiconcave objective, an allocation x^* in $\mathcal{F}$ is robustly locally inefficient if and only if it is weakly inefficient.*

7. Macroprundential policies

7.1. **Nominal rigidities.** Farhi and Werning [10] introduced an abstract framework for studying a variety of macroprudential policies. Their approach is based on the first-order conditions of an intrinsically nonconvex planning program and is well-suited to effectively illustrate the nature and significance of our methodological contribution. We enhance their analysis by proving that their use of first-order conditions rests upon a sound theoretical foundation.

We briefly present a streamlined version of their framework and derive the planning program. Assume S and L are the finite sets, respectively, of states of nature and physical commodities. Each agent i in I is endowed with a state-separable utility function of the from

$$U_i(x_i) = \sum_{s \in S} u_{i,s}(x_{i,s}),$$

where $x_{i.s}$ in $\mathbb{R}^L$ is the consumption plan in state s in S. Agent i in I is subject to an *ex ante* budget constraint,

$$\sum_{s \in S} q_{i,s} z_{i,s} \leq -\tau_{i,0},$$

[6] We are implicitly assuming that $Df_i(x^*) \neq 0$ (see Mas-Colell et al. [23, Theorem M.C.3].

and an *ex post* budget constraint for every state s in S,

$$\sum_{\ell \in L} p_{s,\ell} x_{i,s,\ell} \leq w_{i,s},$$

with

$$w_{i,s} = z_{i,s} - \tau_{i,s} + \pi_{i,s},$$

where $\tau_{i,0}$ and $\tau_{i,s}$ in $\mathbb{R}$ are contingent taxes (or transfers) and $\pi_{i,s}$ in $\mathbb{R}$ is the share into profits. A complete set of contingent claims is traded *ex ante*, whereas spot markets for commodities are available *ex post* conditional on the resolution of uncertainty. Farhi and Werning [10] consider agent-specific tax (or subsidy) rates in the asset market, so that each agent effectively faces individualized asset prices q_i in $\mathbb{R}^S$. In state s in S, the prices of commodities are p_s in $\mathbb{R}^L$. These are to be interpreted as consumption prices net of taxation.

Commodities are produced by means of a conventional technology. In particular, production plan y in $\mathbb{R}^{S\times L}$ is restricted by

$$\Phi(y) \leq 0,$$

where $\Phi : \mathbb{R}^{S\times L} \to \mathbb{R}$ is a smooth convex map. Profit maximization is unessential because the government levies distorsive taxes on commodities and, as a consequence, production prices are distinct from consumption prices and can be determined so as to ensure profit maximization. Importantly, allocations are subject to nominal rigidities and borrowing frictions, as imposed by

$$\Gamma(w, p) \leq 0,$$

where $\Gamma : \mathbb{R}^{S\times I} \times \mathbb{R}^{S\times L} \to \mathbb{R}^K$ is a smooth map. Equilibrium requires market clearing for all commodities, that is,

$$\sum_{i \in I} x_{i,s,\ell} = y_{s,\ell}.$$

Labor is represented as one of the commodities that is supplied by the agents and demanded by the production sector.

Consider the Marshallian demand $x_{i,s}(w_{i,s}, p_s)$ in $\mathbb{R}^L$ and the associated indirect utility function $v_{i,s}(w_{i,s}, p_s)$ in $\mathbb{R}$, both determined by the utility-maximization subject to the budget constraint

$$\sum_{\ell \in L} p_{s,\ell} x_{i,s.\ell} \leq w_{i,s}.$$

To characterize constrained efficient allocation, Farhi and Werning [10, §2.3] study the Ramsey planning program given by

$$\max \sum_{i \in I} \lambda_i \sum_{s \in S} v_{i,s}(w_{i,s}, p_s)$$

subject to the resource constraint,

$$\Phi\left(\sum_{i \in I} \left(x_{i,s}(w_{i,s}, p_s)\right)_{s \in S}\right) \leq 0,$$

and constraints due to price rigidities,

$$\Gamma\left(\left((w_{i,s})_{i\in I}, p_s\right)_{s\in S}\right) \leq 0.$$

The Ramsey planner chooses prices of commodities p in $\mathbb{R}^{L\times S}$ and wealth distributions w in $\mathbb{R}^{I\times S}$ so as to maximize the weighted sum of utilities given welfare weights. The implied competitive equilibrium with financial taxes is constrained efficient (Proposition 2).

This planning program fits our abstract framework and is unavoidably nonconvex. Indeed, in general, constraints arising from frictions and rigidities involve nonconvex restrictions. Furthermore, the indirect utility can only fortuitously be concave in wealth and prices, whereas the dependance of consumptions on wealth and prices introduces further elements of nonlinearity, because well-behaved utility-maximization imposes little discipline on demand curves. These sources of nonconvexity inevitably bear two relevant consequences: (a) only certain constrained efficient allocations obtain by means of the Ramsey planning program; (b) the first-order conditions of the Ramsey planning program do not imply constrained efficiency of the allocation. Both features critically weaken the welfare implications of Farhi and Werning [10]'s analysis, and here is the clarifying role of our characterization.

Due to the nonconvex nature of the environment, by varying welfare weights, only certain constrained efficient allocations are attained as Ramsey optimal plans. This is a normatively disputable feature, because it implies that the proposed welfare criterion might entail an implicit judgement in terms of distribution, favoring certain social groups while harming others. Adopting a more primitive Pareto criterion and relying directly on constrained efficiency would avoid this drawback. However, passing through the Ramsey planning program is an essential element of Farhi and Werning [10]'s analysis, because it yields the relevant first-order conditions (*i.e.*, their equations (6)-(7)) and, hence, 'a simple formula for the required financial interventions that depends on a small number of measurable sufficient statistics'. In this regard, we prove that *all* constrained efficient allocations necessarily satisfy the first-order conditions of a Ramsey-type planning program for some welfare weights, even though they might not correspond to the actual optima arising from the underlying social planning. Hence, we ensure that no normative judgement is implied and first-order conditions can be legitimately used to study macroprudential policies restoring constrained efficiency.

Besides providing a new foundation to macroprudential policies based on aggregate demand externalities, Farhi and Werning [10]'s analysis offers a characterization of macroprudential policies by means of explicit formulas combining minimal sufficient statistics: elasticities of substitution, marginal propensities to spend, and good-specific wedges. In fact, these formulas are derived from the first-order conditions of the Ramsey planning program. Arguably, they can accurately accomplish their operational role and serve as an empirical guidance for policy only if some sort of *sufficiency* of first-order conditions for constrained efficiency is established. Otherwise, the knowledge of these statistics remains inconclusive in terms of correction of the underlying externalities. In this regard, we indeed prove that these first-order conditions are necessary and sufficient for a robust instance of local constrained efficiency. Hence, whenever sufficient statistics enforce the derived formulas, no substantial welfare improvement is feasible in the proximity of the *status quo*. Any further welfare-enhancing policy intervention cannot be justified in terms of elasticities, marginal propensities and wedges, and requires an alternative foundation.

7.2. **Financial frictions.** Dávila and Korinek [8] provide another relevant application of our theory. They characterize the efficiency properties of competitive economies with financial constraints. These competitive economies lead to several types of pecuniary externalities related to the distributive effects of incomplete insurance markets and to price-dependent collateral constraints. By means of a first-order conditions, they also identify sufficient statistics determining optimal taxes to implement constrained efficient allocations. We argue that our theory offers a more solid foundation for their marcoprundential policies.

To illustrate the social planning approach in Dávila and Korinek [8, §3], we adapt our previous notation. The constrained planning program is given by

$$\max \sum_{i \in I} \lambda_i \left(u_i \left(x_{i,0} \right) + \beta \sum_{s \in S} \mu \left(s \right) v_{i,s} \left(z_s, k_s \right) \right)$$

subject to a resource constraint,

$$\sum_{i \in I} x_{i,0} - \sum_{i \in I} h_i \left(k_i \right) - \sum_{i \in I} e_{i,0} \leq 0,$$

a market clearing constraint for contingent claims,

$$\sum_{i \in I} z_{i,s} = 0,$$

and a financial constraint,

$$\Phi_i \left(\left(z_{i,s} \right)_{s \in S}, k_i \right) \leq 0.$$

For each agent i in I, the planner chooses initial consumption $x_{i,0}$ in $\mathbb{R}_+$, capital investment k_i in $\mathbb{R}_+$, subject to a convex cost $h_i \left(k_i \right)$ in $\mathbb{R}_+$, and the holdings of contingent securities z_i in $\mathbb{R}^S$ anticipating the determination of a competitive equilibrium in the continuation. Conditional on the resolution of uncertainty s in S, and depending on accumulated capital stocks k_s in $\mathbb{R}_+^I$ and the distribution of net claims z_s in $\mathbb{R}^I$, each agent's indirect utility is given by $v_{i,s} \left(z_s, k_s \right)$ in $\mathbb{R}$. The planner maximizes the social welfare given by the weighted sum of agents' overall utilities.

Although the constraints in this social planning program are convex, the program itself is inherently nonconvex. While agents' indirect utilities are concave with respect to capital stock and net wealth when competitive prices are treated as exogenous, calculating these utility values involves determining a unique competitive equilibrium as the economy evolves. This introduces complex highly nonlinear relationships among aggregate economic variables, undermining the assumption of concavity. As a result, the objective function in this planning program is neither concave nor quasiconcave, and first-order conditions are generally insufficient to guarantee a social optimum.

Dávila and Korinek [8, Proposition 1] provide a characterization of constrained efficiency using the KKT conditions. However, due to the nonconvex nature of the social planning program, their assertion that KKT conditions are sufficient is not entirely accurate. Their approach to implementing constrained efficiency through contingent and individualized tax rates is also on uncertain footing, as the validity of KKT conditions is consistent with local constrained inefficiency, leading to potential implementation failures. Our theory addresses these issues as far as it is feasible: the KKT conditions are both necessary and sufficient for ensuring robust local efficiency.

7.3. **Idiosyncratic risk.** We also illustrate the application of our theory to Aguiar et al. [1]. They study *robust* Pareto improvements in an economy with uninsurable idiosyncratic risks and capital accumulation. A policy intervention relies on labor and capital taxation instruments, and the implied Pareto improvement is robust in the sense that it induces an expansion of the budget space of all competitive agents, thus ensuring uniform welfare gains. We encompass a (finite-time) streamlined version of their economy in our framework and derive the KKT conditions for robust local efficiency according to our idea.

Time is discrete and indexed by t in a finite set T, while uncertainty is governed by a Markov process over a finite state space S, with initial state s_0 in S. At a competitive equilibrium, each agent i in I is subject to a sequence of budget constraints of the form

$$k_{i,t+1} + c_{i,t} \leq (1 + r_t)\, k_{i,t} + w_t n_{i,t},$$

where $k_{i,t}$ in $\mathbb{R}_+$ is the investment in capital, $n_{i,t}$ in $(0,1) \subset \mathbb{R}$ is the stochastic share of labor endowment, so that $\sum_{i \in I} n_{i,t} = 1$, w_t in $\mathbb{R}_+$ is the wage rate and r_t in $\mathbb{R}$ is the interest rate. Wages and interest rates are interpreted as net of taxation (or subsidies), and are thus distinct from production prices. Market clearing requires, at every t in T,

$$k_t = \sum_{i \in I} k_{i,t} \text{ and } k_{t+1} + \sum_{i \in I} c_{i,t} = f(k_t)$$

Moving from a given competitive equilibrium, Aguiar et al. [1]'s policy experiment consists in an increase in interest rates, leaving wage rates unvaried. We thus set, for some small ϵ in $\mathbb{R}$,

$$r_t = r_t^0 + \epsilon \left(r_t^1 - r_t^0 \right),$$

where $r_t^1 > r_t^0$ for every t in T. Wage rates are instead given as $w_t = w_t^0$ for every t in T. To simplify our arguments, we assume that, for each agent i in I, $k_{i,t}^0 > 0$ for some t in T, thus ensuring that a slight increase in interest rates yields a strict utility gain to all agents.

A permanent variation of interest rates induces optimal consumption $c_{i,t}(\epsilon)$ in $\mathbb{R}_+$ of agent i in I at time t in T. To verify the presence of a welfare-improving policy, we study the planning program given by

$$\max \sum_{i \in I} \lambda_i \mathbb{E}_0 \sum_{t \in T} \beta^t u_i \left(c_{i,t}\left(\epsilon\right)\right)$$

subject to

$$k_{t+1} + \sum_{i \in I} c_{i,t}\left(\epsilon\right) \leq f\left(k_t\right),$$

where initial capital k_0^0 in $\mathbb{R}_+$ is given exogenously. Letting μ_t in $\mathbb{R}_+$ be the Lagrange multiplier for the resource constraint at time t in T, the KKT conditions are given by, for any non-terminal t in T,

$$\mu_t = f'\left(k_{t+1}^0\right) \mathbb{E}_t \mu_{t+1}$$

and

$$\sum_{i \in I} \lambda_i \mathbb{E}_0 \sum_{t \in T} \beta^t u_i'\left(c_{i,t}^0\right) \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon} = \mathbb{E}_0 \sum_{t \in T} \mu_t \sum_{i \in I} \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon}.$$

We now argue that a competitive equilibrium is *not* robustly locally inefficient if and only if there exists a stochastic process for Lagrange multipliers μ_t in $\mathbb{R}_+$ such that, for any non-terminal t in T,

$$\mu_t = f'\left(k_{t+1}^0\right) \mathbb{E}_t \mu_{t+1} \tag{7.1}$$

and

$$\mathbb{E}_0 \sum_{t \in T} \mu_t \sum_{i \in I} \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon} > 0. \tag{7.2}$$

As an increase in ϵ in $\mathbb{R}$ raises interest rates, and no borrowing is allowed, it expands the budget set of each agent. As a consequence, the impact on welfare is strictly positive for all agents, so that

$$\sum_{i \in I} \lambda_i \mathbb{E}_0 \sum_{t \in T} \beta^t u_i'\left(c_{i,t}^0\right) \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon} > 0.$$

If conditions (7.1)-(7.2) are fulfilled, an appropriate choice of Pareto weights will permit to satisfy the KKT conditions. *Vice versa*, whenever the KKT conditions are satisfied, conditions (7.1)-(7.2) are also certainly true. This simple argument establishes the claimed equivalence.

We are now able to compare our characterization with Aguiar et al. [1, Proposition 1]. Suppose that

$$\sup_{\mu \in \mathcal{M}^0} \mathbb{E}_0 \sum_{t \in T} \mu_t \sum_{i \in I} \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon} \leq 0, \tag{7.3}$$

where the space $\mathcal{M}^0$ contains all stochastic processes for the Lagrange multiplier μ_t in $\mathbb{R}_+$ satisfying condition (7.1). Condition (7.3) can be interpreted as a government intertemporal budget constraint, evaluated at implicit state prices subject to no arbitrage, when the government is unable to issue contingent debt.[7] It ensures that, at the margin, an increase in interest rates would generate consumption variations that are intertemporally balanced. The KKT conditions cannot be satisfied in this circumstance, and our characterization delivers the existence of a policy intervention sustaining a robust welfare improvement. In addition, our characterization clarifies that condition (7.3) (or, equivalently, condition (7.4)) is also necessary for a robust Pareto improvement in our as well as in their meaning.[8]

7.4. **Optimal fiscal policy with borrowing limits.** Following the framework of Azzimonti and Yared [2], we consider a simple two-period economy with government debt, where agents are subject to borrowing constraints. The government finances initial public spending through lump-sum taxes and debt. Tax financing exacerbates private borrowing constraints, while debt financing alleviates them. We examine the optimal fiscal policy in this setting.

Each agent is subject to budget constraints of the form

$$qb_i + c_{i0} \leq e_{i0} - \tau_0 \text{ and } c_{i1} \leq e_{i1} + b_i - \tau_1,$$

where $b_i \geq 0$ is the holding of government debt. The *per-capita* budget constraints for the government are

$$qb + \tau_0 = g \text{ and } \tau_1 = b,$$

where g in $\mathbb{R}_+$ is the given level of government spending. We assume that a unique competitive equilibrium exists for all b in $\mathbb{R}_+$.[9]

The welfare of agent i in I, as determined by their indirect utility, is given by

$$f_i(b) = u_i\left(c_{i0}(b)\right) + \beta u_i\left(c_{i1}(b)\right).$$

By the Envelope Theorem, the first-order effect on welfare can be estimated as

$$\nabla f_i(b) = \nabla u_i\left(c_{i0}\right)\left(q + \frac{\partial q}{\partial b}(b - b_i)\right) - \beta \nabla u_i\left(c_{i1}\right).$$

Assuming that no agent is actually borrowing constrained, the condition for robust local efficiency becomes

$$\sum_{i \in I} \lambda_i \nabla u_{i0}\left(c_{i0}\right)\left(b - b_i\right)\frac{\partial q}{\partial b} = 0.$$

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7 Equivalently, condition (7.3) could be expressed as

$$\sum_{t \in T}\left(\prod_{s=0}^{t} \frac{1}{f'\left(k_t^0\right)}\right)\sum_{i \in I} \frac{\partial c_{i,t}\left(0\right)}{\partial \epsilon} \leq 0, \tag{7.4}$$

where the inequality holds true almost surely. This basically corresponds to equation (11) in Aguiar et al. [1]. The difference is due to the fact that aggregate uncertainty is absent in their framework because of the *continuum* of agents. We instead consider an economy with finitely many agents subject to Markov uncertainty, so that in general a competitive equilibrium will retain this source of stochasticity.

8 Indeed, their idea of a robust Pareto improvement at the end coincides with our notion of robust local inefficiency, as an increase in interest rates yields first-order welfare gains (Proposition 4.3).

9 Acknowledging that $\tau_0 < 0$ is permitted in Azzimonti and Yared [2], a competitive equilibrium without binding borrowing limits always exists for any sufficiently large amount of government debt b in $\mathbb{R}_+$, that is, $b > \max_{i \in I} e_{i1}$.

Even if we remain agnostic about the effect of government debt on the interest rate, this condition can still be satisfied because

$$\sum_{i \in I} (b - b_i) = 0.$$

In fact, assuming the interest rate increases with government debt, $\partial q/\partial b < 0$, postponing taxation via debt redistributes welfare from those holding less or even no debt to those holding more debt. As an increase in government debt affects differently agents with different accumulation plans, no Pareto improvement is locally feasible.

A Pareto improvement may occur only when the welfare losses due to binding borrowing limits are rather large. Assuming that the interest rate rises with government debt, $\partial q/\partial b < 0$, an increase in government debt results in a redistribution of wealth from individuals holding more debt, $b - b_i < 0$, to those holding less debt, $b - b_i > 0$. However, as long as an agent is borrowing-constrained, government debt also mitigates the effects of the borrowing constraint by reducing current taxation and, so, increasing consumption. A rise in government debt leads to a Pareto improvement only when the positive effect on the borrowing limit is sufficiently large to outweigh the welfare loss caused by the implicit redistribution, that is,

$$\nabla u_i (c_{i0}) q - \beta \nabla u_i (c_{i1}) > -\nabla u_i (c_{i0}) \frac{\partial q}{\partial b} b.$$

To help a comparison, we reproduce the experiment in Azzimonti and Yared [2] in which Pareto weights are assumed to be equal across agents. With a rich (H) and a poor (L) agent, and log-utilities, the first-order conditions reduce to

$$\nabla f_H(b) = \frac{1}{c_{H0}} q - \frac{1}{c_{H0}} \frac{\partial q}{\partial b} b - \frac{1}{c_{H1}}$$

and

$$\nabla f_L(b) = \frac{1}{c_{L0}} q + \frac{1}{c_{L0}} \frac{\partial q}{\partial b} b - \frac{1}{c_{L1}}$$

A social optimum for this utilitarian criterion would require

$$\sum_{i \in I} \nabla u_{i0} (c_{i0}) (b - b_i) \frac{\partial q}{\partial b} = -\frac{1}{c_{H0}} \frac{\partial q}{\partial b} b + \frac{1}{c_{L0}} \frac{\partial q}{\partial b} b \geq 0,$$

where we assumed that no agent in constrained in borrowing. The interest rate increases with government debt in their framework, $\partial q/\partial b < 0$, so that optimality requires

$$\sum_{i \in I} \nabla u_{i0} (c_{i0}) (b - b_i) = -\frac{1}{c_{H0}} b + \frac{1}{c_{L0}} b \leq 0,$$

which cannot be satisfied as $c_{H0} > c_{L0}$.[10]

[10] It is worth noticing that this failure of social optimality for the utilitarian criterion occurs even though the underlying competitive equilibrium is Pareto efficient in a fully canonical sense.

8. CONCLUSION

We have provided a rigorous foundation to a Ramsey-type approach to constrained efficiency under nonconvexity. Locally efficient allocation can be in general characterized by means of first-order conditions involving Pareto weights and Lagrange multipliers. This is mostly relevant for macroprundential policies, as market frictions and price rigidities are natural sources of nonconvexity even under conventional preferences and technologies.

APPENDIX A. PROOFS

Proof of Lemma 1. We argue by contradiction and suppose that condition (4.1) fails. As condition (4.1) can only be violated because constraints are binding, for heuristic purposes, it simplifies to assume that *all* constraints are binding, that is, $g_j(x^*) = 0$ for every j in J. Consider the following convex sets: V is the interior of the negative cone of $\mathbb{R}^J$ (*i.e.*, the set of all $v \ll 0$ in $\mathbb{R}^J$) and

$$W = \left\{ w \in \mathbb{R}^J : w = Dg(x^*)(h) \text{ for some } h \in \mathbb{R}^N \right\}.$$

As condition (4.1) is violated by hypothesis, these sets do not intersect and, by a simple application of the Separation Theorem, there exists $\mu > 0$ in $\mathbb{R}^J$ such that, for every h in $\mathbb{R}^N$,

$$\sum_{j \in J(x^*)} \mu_j Dg_j(x^*)(h) \geq 0.$$

As the derivatives $Dg_j(x^*) : \mathbb{R}^N \to \mathbb{R}$ are linear functionals, this yields

$$\sum_{j \in J(x^*)} \mu_j Dg_j(x^*) = 0,$$

thus contradicting the regularity condition and proving our claim. □

Proof of Proposition 4.1 (necessity). Let $P^- \subset \mathbb{R}^I \times \mathbb{R}^J$ be defined by all (v, z) in $\mathbb{R}^I \times \mathbb{R}^J$ such that, for some h in $\mathbb{R}^N$,

$$v \leq Df(x^*)(h) \text{ and } g(x^*) + Dg(x^*)(h) \leq z.$$

Also, let $P^+ \subset \mathbb{R}^I \times \mathbb{R}^J$ be given by $0 \leq v$ and $z \leq 0$. These two sets are obviously convex. We now argue that P^- cannot contain an interior point of P^+.

Suppose that (v, z) lies in P^- with $0 \ll v$ and $z \ll 0$, so that (v, z) is also in the interior of P^+. This implies the existence of some h in $\mathbb{R}^N$ such that

$$0 \ll Df(x^*)(h) \text{ and } g(x^*) + Dg(x^*)(h) \ll 0.$$

We now prove that, for some sufficiently small neighborhood of economies $\mathcal{N}(x^*)$, given any sufficient small α in $(0, 1) \subset \mathbb{R}$, for all $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}(x^*)$,

$$\hat{f}(x^* + \alpha h) \gg \hat{f}(x^*) \text{ and } \hat{g}(x^* + \alpha h) \ll (1 - \alpha)\hat{g}(x^*) \leq 0,$$

thus contradicting the hypothesis that the allocation is not robustly locally inefficient.

Consider the compact set $K \subset X$ given by

$$\{x^* + \alpha h \in X : \alpha \in [0,1] \subset \mathbb{R}\} \subset K.$$

For any arbitrarily small $\epsilon > 0$ in $\mathbb{R}$, we can assume that the neighborhood $\mathcal{N}(x^*)$ is such that, for any $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}(x^*)$,

$$\sup_{x \in K} \sup \left\{ \left\| f(x) - \hat{f}(x) \right\|, \left\| Df(x) - D\hat{f}(x) \right\| \right\} < \epsilon$$

and

$$\sup_{x \in K} \sup \left\{ \| g(x) - \hat{g}(x) \|, \| Dg(x) - D\hat{g}(x) \| \right\} < \epsilon.$$

We then preliminarily observe that, for every economy in $\mathcal{N}(x^*)$, given any α in $(0,1) \subset \mathbb{R}$,

$$(\dagger) \qquad \hat{f}(x^* + \alpha h) - \hat{f}(x^*) \geq f(x^* + \alpha h) - f(x^*) - \epsilon\alpha \|h\| \mathbf{1}$$

and

$$(\ddagger) \qquad \hat{g}(x^* + \alpha h) - \hat{g}(x^*) \leq g(x^* + \alpha h) - g(x^*) + \epsilon\alpha \|h\| \mathbf{1}.$$

We explicitly prove condition ($\dagger$), while condition ($\ddagger$) is implied by similar arguments. Indeed, by the Mean Value Inequality Theorem (Ok [24, Chapter K]),

$$\begin{aligned} \left\| \left(\hat{f} - f\right)(x^* + \alpha h) - \left(\hat{f} - f\right)(x^*) \right\| &\leq \\ \alpha \sup_{\hat{\alpha} \in (0,1) \subset \mathbb{R}} \left\| \left(D\hat{f} - Df\right)(x^* + \hat{\alpha} h) \right\| \|h\| &\leq \epsilon\alpha \|h\|. \end{aligned}$$

Observing that $-\|v\| \mathbf{1} \leq v \leq \|v\| \mathbf{1}$, this is sufficient to establish the property.

Considering the benchmark economy, notice that $\epsilon > 0$ in $\mathbb{R}$ can be chosen so small that any sufficiently small α in $(0,1) \subset \mathbb{R}$ implies

$$\frac{g(x^* + \alpha h) - g(x^*)}{\alpha} \leq Dg(x^*)(h) + \epsilon\mathbf{1} \ll -g(x^*) - \epsilon(1 + \|h\|)\mathbf{1}.$$

The left inequality holds true for any sufficiently small α in $(0,1) \subset \mathbb{R}$ by the definition of (Fréchet) derivative. For the right inequality, just notice that, if $\epsilon > 0$ in $\mathbb{R}$ is sufficiently small,

$$g(x^*) + Dg(x^*)(h) \ll 0 \text{ implies } g(x^*) + Dg(x^*)(h) \ll -\epsilon(2 + \|h\|)\mathbf{1}.$$

Therefore, we have that

$$\frac{g(x^* + \alpha h) - g(x^*)}{\alpha} + \epsilon \|h\| \mathbf{1} \ll -g(x^*) - \epsilon\mathbf{1}.$$

Furthermore, it can be assumed that, for any sufficiently small α in $(0,1) \subset \mathbb{R}$,

$$\frac{f(x^* + \alpha h) - f(x^*)}{\alpha} \geq Df(x^*)(h) - \epsilon\mathbf{1} \gg \epsilon \|h\| \mathbf{1}.$$

By conditions ($\dagger$)-($\ddagger$), and noticing that

$$\| g(x^*) - \hat{g}(x^*) \| < \epsilon \text{ implies } -g(x^*) - \epsilon\mathbf{1} \leq -\hat{g}(x^*),$$

we conclude that the allocation is robustly locally inefficient, a contradiction.

By the Separation Theorem, there exists a nontrivial (λ, μ) in $\mathbb{R}^I \times \mathbb{R}^J$ such that

$$\lambda \cdot v^- - \mu \cdot z^- \leq \lambda \cdot v^+ - \mu \cdot z^+,$$

where (v^-, z^-) lies in P^- and (v^+, z^+) lies in P^+. By the nature of P^+, we conclude that $(\lambda, \mu) \geq 0$ in $\mathbb{R}^I \times \mathbb{R}^J$. And, assuming that $\lambda = 0$, choosing $(0, 0)$ in P^+ and any h in $\mathbb{R}^N$ satisfying the regularity condition, we obtain $\mu \cdot z^- \geq 0$, where

$$z^- = g(x^*) + Dg(x^*)(h) \ll 0,$$

a contradiction. Therefore, at no loss of generality, we take $\sum_{i \in I} \lambda_i = 1$.

From the separation property, we have

$$\lambda \cdot Df(x^*)(h) - \mu \cdot (g(x^*) + Dg(x^*)(h)) \leq 0.$$

Setting $h = 0$, we see that $\mu \cdot g(x^*) = 0$. As this conditions holds for any h in $\mathbb{R}^N$, we conclude that

$$\lambda \cdot Df(x^*)(h) - \mu \cdot (Dg(x^*)(h)) = 0,$$

thus establishing the claim. □

Proof of Proposition 4.1 (sufficiency). Supposing the allocation is robustly locally inefficient, there exists a sequence of locally improving reallocations and, at no loss of generality, we can assume that

$$\lim_{n \to \infty} \frac{x_n - x^*}{\|x_n - x^*\|} = h \in \mathbb{R}^N.$$

This implies that, for every j in J,

$$\mu_j Dg_j(x^*)(x_n - x^*) + \mu_j o(\|x_n - x^*\|) = \mu_j g_j(x_n) - \mu_j g_j(x^*) \leq 0,$$

where we use condition (4.3). Going to the limit, this yields, for every j in J,

$$\mu_j Dg_j(x^*)(h) \leq 0.$$

Analogously, for every i in I,

$$\lambda_i Df_i(x^*)(x_n - x^*) + \mu_i o(\|x_n - x^*\|) = \lambda_i f_i(x_n) - \lambda_i f_i(x^*) \geq 0.$$

The inequality holds true by Pareto domination along the sequence. In the limit, we obtain, for every i in I,

$$\lambda_i Df_i(x^*)(h) \geq 0.$$

By condition (4.2), we conclude that, for every i in I and every j in J,

$$\lambda_i Df_i(x^*)(h) = 0 \text{ and } \mu_j Dg_j(x^*)(h) = 0.$$

Consider a smooth map $\hat{f}_i : X \to \mathbb{R}$ that is locally given by $\hat{f}_i(x) = f_i(x) + \Phi_i(x - x^*)$, where $\Phi_i : \mathbb{R}^N \to \mathbb{R}$ is a linear map with $\|\Phi_i\| < \epsilon$ for some sufficiently small $\epsilon > 0$ in $\mathbb{R}$. This ensures that economy $\left(\hat{f}, g\right)$ lies in neighborhood $\mathcal{N}(x^*)$. Notice that, for every sufficiently large n in $\mathbb{N}$,

$$\begin{aligned}\lambda_i Df_i\left(x^*\right)\left(x_n - x^*\right) + \lambda_i \Phi_i\left(x_n - x^*\right) + \lambda_i o\left(\left\|x_n - x^*\right\|\right) &= \\ \lambda_i D\hat{f}_i\left(x^*\right)\left(x_n - x^*\right) + \lambda_i o\left(\left\|x_n - x^*\right\|\right) &= \\ \lambda_i \hat{f}_i\left(x_n\right) - \lambda_i \hat{f}_i\left(x^*\right) &\geq 0,\end{aligned}$$

where the inequality is due to robust local inefficiency. Taking again the limit, we obtain, for every i in I,

$$\lambda_i Df_i\left(x^*\right)(h) + \lambda_i \Phi_i(h) = \lim_{n\to\infty} \frac{\lambda_i \hat{f}_i\left(x_n\right) - \lambda_i \hat{f}_i\left(x^*\right)}{\left\|x_n - x^*\right\|} \geq 0.$$

Our previous argument in turn yields $\lambda_i \Phi_i(h) \geq 0$. As the linear map $\Phi_i : \mathbb{R}^N \to \mathbb{R}$ can be chosen arbitrarily, this reveals a contradiction. □

Proof of Proposition 4.2. Arguing exactly as in the proof of Proposition 4.1 (sufficiency), we obtain the existence of a direction h in $\mathbb{R}^N$ such that, for every economy $\left(\hat{f}, \hat{g}\right)$ in $\mathcal{N}_{\text{KKT}}\left(x^*\right)$,

$$\hat{\lambda}_i D\hat{f}_i\left(x^*\right)(h) = 0 \text{ and } \hat{\mu}_j D\hat{g}_j\left(x^*\right)(h) = 0.$$

For a linear map $\Phi : \mathbb{R}^N \to \mathbb{R}$, we can consider the local perturbations

$$\hat{f}_{i_1}(x) = f_{i_1}(x) + \frac{1}{\lambda_{i_1}}\Phi\left(x - x^*\right) \text{ and } \hat{f}_{i_2}(x) = f_{i_2}(x) - \frac{1}{\lambda_{i_2}}\Phi\left(x - x^*\right),$$

where $\lambda_{i_1} > 0$ and $\lambda_{i_2} > 0$. The KKT conditions are preserved by these perturbations and, hence, $\Phi(h) = 0$. As the linear map $\Phi : \mathbb{R}^N \to \mathbb{R}$ can be arbitrarily chosen, this uncovers a contradiction, thus establishing our claim. □

Proof of Proposition 4.3. Assuming the limit condition is satisfied and the allocation is *not* robustly locally inefficient. By Proposition 4.1, KKT conditions are satisfied. At no loss of generality, we can assume that, for some h in $\mathbb{R}^N$,

$$\lim_{n\to\infty} \frac{x_n - x^*}{\left\|x_n - x^*\right\|} = h.$$

Arguing as in the proof of Proposition 4.1 (sufficiency),

$$\sum_{j\in J} \mu_j Dg_j\left(x^*\right)(h) \leq 0.$$

Furthermore, for every agent i in I,

$$Df_i\left(x^*\right)\left(x_n - x^*\right) + o\left(\left\|x_n - x^*\right\|\right) = f_i\left(x_n\right) - f_i\left(x^*\right),$$

which going to the limit yields

$$Df_i\left(x^*\right)(h) = \lim_{n\to\infty} \frac{f_i\left(x_n\right) - f_i\left(x^*\right)}{\left\|x_n - x^*\right\|} > 0.$$

This in turn implies

$$\sum_{i\in I} \lambda_i Df_i\left(x^*\right)(h) > 0,$$

thus contradicting condition (4.2).

Assume now that the allocation is robustly locally inefficient. We claim that, for some h in $\mathbb{R}^N$,

$$0 \ll Df(x^*)(h) \text{ and } g(x^*) + Dg(x^*)(h) \ll 0.$$

Indeed, if not, by the arguments for the proof of Proposition 4.1 (necessity), KKT conditions are satisfied, thus violating robust local inefficiency by the equivalence established in Proposition 4.1. Reproducing the logic for the proof of Proposition 4.1 (sufficiency), for any sufficiently small α in $(0,1) \subset \mathbb{R}$,

$$f(x^* + \alpha h) \gg f(x^*) \text{ and } g(x^* + \alpha h) \ll (1-\alpha) g(x^*) \leq 0.$$

We thus conclude that

$$\lim_{\alpha \to 0} \frac{f(x^* + \alpha h) - f(x^*)}{\alpha \|h\|} = \frac{Df(x^*)(h)}{\|h\|} \gg 0,$$

which proves our claim. □

Proof of Proposition 4.4 (necessity). Assume that the allocation is not trembling-hand locally inefficient and, yet, KKT conditions are not satisfied. Arguing as in the proof of Proposition 4.1 (necessity), there exists h in $\mathbb{R}^N$ satisfying

$$0 \ll Df(x^*)(h) \text{ and } g(x^*) + Dg(x^*)(h) \ll 0.$$

Choose any sufficiently small ϵ in $(0,1) \subset \mathbb{R}$ such that, for any h' in $\mathbb{R}^N$ such that $\|h' - h\| \leq \epsilon \|h\|$,

$$0 \ll Df(x^*)(h') \text{ and } g(x^*) + Dg(x^*)(h') \ll 0. \tag{*}$$

This is possible as the derivative is a continuous linear map. Considering any monotonically vanishing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1) \subset \mathbb{R}$, $x_n = x^* + \alpha_n h$ is an element of X for every sufficiently large n in $\mathbb{N}$. We now prove that the allocation is trembling-hand locally inefficient, thus uncovering a contradiction and proving our claim.

Given any sequence of perturbed allocations, define

$$y_n = x^* + \alpha_n h_n.$$

Notice that, by construction,

$$\|h_n - h\| = \frac{\|y_n - x_n\|}{\alpha_n} \leq \frac{\epsilon \|x_n - x^*\|}{\alpha_n} = \epsilon \|h\|.$$

This implies that the sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ is bounded and any subsequence converging to some h' in $\mathbb{R}^N$ is such that $\|h' - h\| \leq \epsilon \|h\|$, thus ensuring that condition (*) holds true in the limit. To derive a contradiction, suppose that, for some j in $J(x^*)$, $g_j(y_n) > 0$ for infinitely many n in $\mathbb{N}$. Extracting a converging subsequence if necessary, by the Mean

Value Theorem, we obtain

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \frac{g_j(y_n) - g_j(x^*)}{\alpha_n} \\ &= Dg_j(x^*)\left(\lim_{n\to\infty} \frac{y_n - x^*}{\alpha_n}\right) \\ &= Dg_j(x^*)\left(\lim_{n\to\infty} h_n\right) \\ &= Dg_j(x^*)(h'), \end{aligned}$$

a contradiction of the strict inequality in condition (*). A similar argument applies if, for some agent i in I, $f_i(y_n) \leq f_i(x^*)$ for infinitely many n in $\mathbb{N}$. This establishes the claim. □

Proof of Proposition 4.4 (sufficiency). Fix any allocation satisfying the KKT conditions and assume, by way of contradiction, that it is trembling-hand locally inefficient. Passing to a subsequence if necessary, we can assume that there exists h in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty} \frac{x_n - x^*}{\|x_n - x^*\|} = h.$$

Arguing as in the proof of Proposition 4.1 (necessity), for every h' in $\mathbb{R}^N$ such that $\|h' - h\| < \epsilon$, we obtain that $Df_i(x^*)(h') = 0$ for every agent i in I such that $\lambda_i > 0$. Indeed, to this end, for any sufficiently large n in $\mathbb{N}$, consider the allocation y_n in X given by

$$y_n = x_n + \|x_n - x^*\|(h' - h),$$

which necessarily satisfies $g(y_n) \leq 0$ and $f(y_n) > f(x^*)$ by trembling-hand local inefficiency. It follows that, for every j in J,

$$\mu_j Dg_j(x^*)(y_n - x^*) + \mu_j o(\|y_n - x^*\|) = \mu_j g_j(y_n) - \mu_j g_j(x^*) \leq 0,$$

where we use condition (4.3). Going to the limit, this yields $\mu_j Dg_j(x^*)(h') \leq 0$. Similarly, for every i in I,

$$\lambda_i Df_i(x^*)(y_n - x^*) + \lambda_i o(\|y_n - x^*\|) = \lambda_i f_i(y_n) - \lambda_i f_i(x^*) \geq 0.$$

Passing to the limit, this implies $\lambda_i Df_i(x^*)(h') \geq 0$. Furthermore, by condition (4.2),

$$\sum_{i\in I} \lambda_i Df_i(x^*)(h') = \sum_{j\in J} \mu_j Dg_j(x^*)(h'),$$

thus delivering $Df_i(x^*)(h') = 0$ for every agent i in I such that $\lambda_i > 0$. We now derive a contradiction.

By the Second-Order Mean Value Theorem (for instance, Ok [24, Section K.2.1]),

$$f_i(x_n) - f_i(x^*) = Df_i(x^*)(x_n - x^*) + \frac{1}{2}D^2 f_i(z_n)(x_n - x^*, x_n - x^*),$$

where allocation z_n lies in the convex hull of $\{x_n, x^*\} \subset X$. Observe that, for all sufficiently large n in $\mathbb{N}$,

$$\left\| \frac{x_n - x^*}{\|x_n - x^*\|} - h \right\| < \epsilon.$$

By our previous arguments, this in turn implies

$$Df_i(x^*)\left(\frac{x_n - x^*}{\|x_n - x^*\|}\right) = 0 \text{ and, so, } Df_i(x^*)(x_n - x^*) = 0.$$

Therefore, for all sufficiently large n in $\mathbb{N}$,

$$0 \leq \frac{f_i(x_n) - f_i(x^*)}{\|x_n - x^*\|^2} = \frac{1}{2} D^2 f_i(z_n)\left(\frac{x_n - x^*}{\|x_n - x^*\|}, \frac{x_n - x^*}{\|x_n - x^*\|}\right) < 0,$$

where the right hand-side inequality is due to smooth strict concavity. By contradiction, this establishes our claim. $\square$

Proof of Proposition 4.5. We first argue that, if x^* in $\mathcal{F}$ is regular, then

$$\liminf_{\epsilon \to 0} \frac{\lambda^*(\mathcal{F} \cap B(x^*, \epsilon))}{\lambda^*(B(x^*, \epsilon))} > 0. \tag{A.1}$$

To this end, using Lemma 1, by the continuity of the mappings $h \mapsto Dg(x^*)(h)$, there exists a compact set H of $\mathbb{R}^N$ with a nonempty interior such that $g(x^*) + Dg(x^*)(h) \ll 0$ for every h in H. Furthermore, at no loss of generality, we can assume that $\|h\| \leq 1$ for all h in H. We claim that $x^* + \epsilon h$ lies in $\mathcal{F}$ for any h in H and any sufficiently small ϵ in $\mathbb{R}_{++}$. Indeed, assuming not, we can construct a sequence satisfying, for some j in J, at all n in $\mathbb{N}$,

$$g_j(x^* + \epsilon_n h_n) > 0,$$

where $\lim_{n\to\infty} \epsilon_n = 0$ and, at no loss of generality, $\lim_{n\to\infty} h_n = h$ with h in H. By the Mean Value Theorem, for every n in $\mathbb{N}$, there exists η_n in $[0, \epsilon_n] \subset \mathbb{R}$ such that

$$g_j(x^* + \epsilon_n h_n) - g_j(x^*) = \epsilon_n Dg_j(x^* + \eta_n h_n)(h_n),$$

which in turn implies

$$Dg_j(x^* + \eta_n h_n)(h_n) = \frac{g_j(x^* + \epsilon_n h_n) - g_j(x^*)}{\epsilon_n} > 0.$$

Taking the limit, we obtain $Dg_j(x^*)(h) \geq 0$ and, therefore, $g_j(x^*) < 0$, thus contradicting $g_j(x^* + \epsilon_n h_n) > 0$ for all n in $\mathbb{N}$. As a conclusion, for all sufficiently small ϵ in $\mathbb{R}_{++}$,

$$x^* + [0, \epsilon) H \subset \mathcal{F} \cap B(x^*, \epsilon),$$

where $[0, \epsilon) H \subset \mathbb{R}^N$ is short notation for

$$\left\{z \in \mathbb{R}^N : z = \eta h \text{ for some } h \in H \text{ and some } \eta \in [0, \epsilon) \subset \mathbb{R}\right\}.$$

Noticing that

$$\lambda^*(\mathcal{F} \cap B(x^*, \epsilon)) \geq \lambda^*(x^* + [0, \epsilon) H) \geq \epsilon^N \lambda^*(H)$$

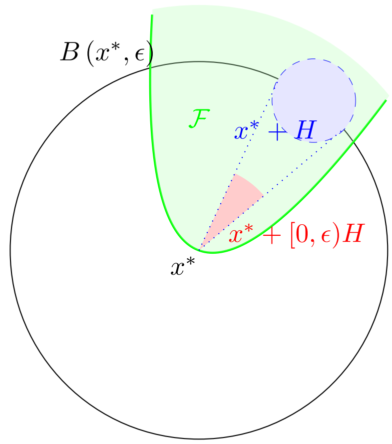

FIGURE 5. Regularity

and

$$\lambda^* \left(B\left(x^*, \epsilon\right)\right) = 2^N \epsilon^N,$$

this is sufficient to establish condition (A.1) as $\lambda^* (H) > 0$. Figure 5 provides a simple graphical illustration.

Assume that a regular allocation x^* in $\mathcal{F}$ satisfies the KKT conditions. For each n in $\mathbb{N}$, consider the set H_n in $\mathbb{R}^N$ given by

$$H_n = \left\{ \frac{x - x^*}{\|x - x^*\|} \in \mathbb{R}^N : x \in \mathcal{P} \cap B\left(x^*, \epsilon_n\right) \right\}.$$

Obviously, we have the inclusion

$$\mathcal{P} \cap B(x^*, \epsilon_n) \subset x^* + [0, \epsilon_n] \, H_n,$$

which implies that

$$\frac{\lambda^* \left(\mathcal{P} \cap B\left(x^*, \epsilon_n\right)\right)}{\lambda^* \left(B\left(x^*, \epsilon_n\right)\right)} \leq \frac{\lambda^* \left(x^* + [0, \epsilon_n] \, H_n\right)}{\lambda^* \left(B\left(x^*, \epsilon_n\right)\right)} = \frac{\lambda^* \left([0,1] \, H_n\right)}{2^N},$$

where we use the fact that

$$\lambda^* \left(x^* + [0, \epsilon_n] \, H_n\right) = \lambda^* \left([0, \epsilon_n] \, H_n\right) = \epsilon_n^N \lambda^* \left([0,1] \, H_n\right).$$

As the sequence $(H_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$ is decreasing for the inclusion, we have

$$\lim_{n \to \infty} \lambda^* \left([0,1] \, H_n\right) = \lambda^* \left([0,1] \, H_\infty\right), \quad \text{where} \quad H_\infty = \bigcap_{n \in \mathbb{N}} H_n.$$

To verify our claim, by condition (A.1), it is sufficient to show that $\lambda^*([0,1] \, H_\infty) = 0$.

For every h in H_∞, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, with each x_n lying in $\mathcal{P}\cap B(x^\star, \epsilon_n)$, such that

$$h = \frac{x_n - x^\star}{\|x_n - x^\star\|}.$$

Letting $\alpha_n = \|x_n - x^\star\|$, (Fréchet) differentiability implies that

$$Df_i(x^*)(h) = \lim_{n\to\infty} \frac{f_i(x_n) - f_i(x^*)}{\alpha_n} \text{ and } Dg_j(x^*)(h) = \lim_{n\to\infty} \frac{g_j(x_n) - g_j(x^\star)}{\alpha_n}.$$

As $f(x_n) > f(x^*)$, $g_j(x_n) \leq 0$ and $g_j(x^*) = 0$ for every (i, j) in $I \times J(x^*)$, we deduce that

$$Df_i(x^*)(h) \geq 0 \text{ and } Dg_j(x^*)(h) \leq 0.$$

Because the KKT conditions are satisfied, there must exist i in I such that $Df_i(x^*)(h) = 0$.

We have proved that

$$[0,1]\, H_\infty \subset \bigcup_{i\in I} \{h \in \mathbb{R}^N : Df_i(x^*)(h) = 0\}.$$

This implies that

$$\lambda^*([0,1]\, H_\infty) \leq \sum_{i\in I} \lambda^* \left(\{h \in \mathbb{R}^N : Df_i(x^*)(h) = 0\}\right).$$

Because $Df_i(x^*) \neq 0$, the set $\{h \in \mathbb{R}^N : Df_i(x^*)(h) = 0\}$ is a hyperplane in $\mathbb{R}^N$. Therefore, $\lambda^*(\{h \in \mathbb{R}^N : Df_i(x^\star)(h) = 0\}) = 0$, thus establishing that $\lambda^*([0,1]\, H_\infty) = 0$.

To prove the reverse claim, we shall argue by contrapositive. Let x^* in $\mathcal{F}$ be a regular allocation that does not satisfy the KKT conditions, while condition (4.10) is fulfilled. By reproducing the logic in the previous part of the proof, there exists a compact set H in $\mathbb{R}^N$ with a nonempty interior such that, for every h in H,

$$0 \ll Df\,(x^*)\,(h) \text{ and } g\,(x^*) + Dg\,(x^*)\,(h) \ll 0.$$

Furthermore, at no loss of generality, $\|h\| \leq 1$ for all h in H. Simple adaptations of the previous arguments reveal that, for all sufficiently small ϵ in $\mathbb{R}_{++}$ and for all h in H, $x^* + \epsilon h$ lies in $\mathcal{P}$.

Notice that, for all sufficiently small ϵ in $\mathbb{R}_{++}$,

$$x^* + [0,\epsilon)H \subset \mathcal{P} \cap B(x^*, \epsilon).$$

This in turn implies that

$$\lambda^*\left(\mathcal{P} \cap B(x^*, \epsilon)\right) \geq \lambda^*(x^* + [0,\epsilon)H) \geq \epsilon^N \lambda^*\left(H\right).$$

Therefore, we obtain

$$\begin{aligned} \frac{\lambda^*(\mathcal{P} \cap B(x^*, \epsilon))}{\lambda^*(\mathcal{F} \cap B(x^*, \epsilon))} &\geq \frac{\lambda^*(x^* + [0,\epsilon)H)}{\lambda^*(B(x^*, \epsilon))} \\ &\geq \frac{\lambda^*\left(H\right)}{2^N}, \end{aligned}$$

thus contradicting condition (4.10) as $\lambda^* (H) > 0$. $\square$

Proof of Proposition 6.1. Assume the allocation is robustly locally efficient and, for the sake of contradiction, suppose there exists another allocation x in $\mathcal{F}$ such that $f(x) \gg f(x^*)$. Consider the alternative allocation x_n in $\mathcal{F}$ given by $x^* + n^{-1}(x - x^*)$ for n in $\mathbb{N}$. By concavity,

$$\frac{f(x_n) - f(x^*)}{\|x_n - x^*\|} \geq \frac{f(x) - f(x^*)}{\|x - x^*\|}.$$

This reveals a failure of robust local efficiency, thus establishing the claim. $\square$

Proof of Proposition 6.2. Supposing the allocation is not weakly efficient, there exists x in $\mathcal{F}$ such that $f(x) \gg f(x^*)$. For every n in $\mathbb{N}$, consider the alternative allocation $x_n = x^* + n^{-1}(x - x^*)$ in $\mathcal{F}$. This implies that, for every j in J,

$$\mu_j Dg_j(x^*)(x_n - x^*) + \mu_j o(\|x_n - x^*\|) = \mu_j g_j(x_n) - \mu_j g_j(x^*) \leq 0.$$

Going to the limit, this yields

$$\sum_{j \in J} \mu_j Dg_j(x^*) \left(\frac{x - x^*}{\|x - x^*\|} \right) \leq 0.$$

On the other side,

$$\sum_{i \in I} \lambda_i Df_i(x^*) \left(\frac{x - x^*}{\|x - x^*\|} \right) > 0.$$

The strict inequality in the limit reflects the assumption of strict quasiconcavity. This contradicts condition (4.2). $\square$

Appendix B. Equality constraints

We enlarge our framework by encompassing equality constraints. In particular, given a smooth map $h : X \to \mathbb{R}^K$, the feasible set is defined as

$$\mathcal{F} = \{x \in X : g(x) \leq 0 \text{ and } h(x) = 0\}.$$

A canonical regularity condition is imposed.

Smooth regularity condition. *An allocation x^* in $\mathcal{F}$ is* regular *if (a) it does not exist a nonzero ν in $\mathbb{R}^N$ such that*

$$\sum_{k \in K} \nu_k Dh_k(x^*) = 0;$$

(b) there exists d in $\mathbb{R}^N$ such that, for every k in K, $Dh_k(x^*)(d) = 0$ and, for every j in J,

$$g_j(x^*) + Dg_j(x^*)(d) < 0.$$

We say that an allocation x^* in $\mathcal{F}$ satisfies the *KKT conditions* if there exist an element $\lambda > 0$ of $\mathbb{R}^I$, an element $\mu \geq 0$ of $\mathbb{R}^J$ and an element ν of $\mathbb{R}^K$ such that

$$\sum_{i \in I} \lambda_i Df_i(x^*) = \sum_{j \in J} \mu_j Dg_j(x^*) + \sum_{k \in K} Dh_k(x^*) \tag{B.1}$$

and

$$\sum_{j\in J}\mu_j g_j(x^*) = 0. \tag{B.2}$$

Proposition B.1 (KKT characterization)**.** *A regular allocation x^* in $\mathcal{F}$ satisfies the KKT conditions if and only if there exists a sequence of allocations $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, converging to x^* in $\mathcal{F}$, such that*

$$\lim_{n\to\infty}\frac{f(x_n)-f(x^*)}{\|x_n-x^*\|}\gg 0.$$

Proof of Proposition B.1 (necessity). Let $P^-\subset\mathbb{R}^I\times\mathbb{R}^J\times\mathbb{R}^K$ be defined by (v,z,y) in $\mathbb{R}^I\times\mathbb{R}^J\times\mathbb{R}^K$ such that, for some d in $\mathbb{R}^N$,

$$v\le Df(x^*)(d),\ g(x^*)+Dg(x^*)(d)\le z \text{ and } y=Dh(x^*)(d)=0,$$

where the smooth map $h:X\to\mathbb{R}^K$ corresponds to additional constraints of the form $h(x)=0$. Also, let $P^+\subset\mathbb{R}^I\times\mathbb{R}^J\times\mathbb{R}^K$ be given by $0\le v$, $z\le 0$ and $y=Dh(x^*)(d)$ for some d in $\mathbb{R}^N$. Notice that, by the regularity condition, $Dh(x^*):\mathbb{R}^N\to\mathbb{R}^K$ is surjective. These two sets are obviously convex. We now argue that P^- cannot contain an interior point of P^+.

Suppose that (v,z,y) lies in P^- with $0\ll v$, $z\ll 0$ and $y=0$, so that (v,z,y) is also in the interior of P^+. This implies the existence of some d in $\mathbb{R}^N$ such that

$$0\ll Df(x^*)(d),\ g(x^*)+Dg(x^*)(d)\ll 0 \text{ and } Dh(x^*)(d)=0,$$

Here the proof requires a major adjustment by means of the Implicit Function Theorem.

The Euclidean space $\mathbb{R}^N$ can be decomposed as $E_0\oplus E_1$, where E_0 is the null space of $Dh(x^*):\mathbb{R}^N\to\mathbb{R}^K$. By the regularity condition, the Implicit Function Theorem ensures the existence of neighborhoods $U_0\subset E_0$ and $U_1\subset E_1$ of (x_0^*,x_1^*) in X and of a smooth map $\varphi:U_0\to U_1$ such that $x_1^*=\varphi(x_0^*)$ and

$$h(x_0,x_1)=0 \text{ if and only if } x_1=\varphi(x_0).$$

Furthermore, by construction, $D\varphi(x_0^*)=0$. As d belongs to E_0, we have $x_0^*+\alpha d$ in U_0 and $H(x_0^*+\alpha d,\varphi(x_0^*+\alpha d))=0$ for every small α in $\mathbb{R}$. Computing the derivative of the planning objective,

$$\lim_{\alpha\to 0}\frac{f(x_0^*+\alpha d,\varphi(x_0^*+\alpha d))-f(x_0^*,x_1^*)}{\alpha}=Df(x^*)(d,D\varphi(x_0^*)(d))=Df(x^*)(d)\gg 0,$$

where we use the fact that $D\varphi(x_0^*)(d)=0$. Similarly,

$$\lim_{\alpha\to 0}\frac{g(x_0^*+\alpha d,\varphi(x_0^*+\alpha d))-g(x_0^*,x_1^*)}{\alpha}=Dg(x^*)(d)\ll -g(x^*).$$

The proof then proceeds as in the simplified framework.

By the Separation Theorem, there exists a nontrivial (λ,μ,ν) in $\mathbb{R}^I\times\mathbb{R}^J\times\mathbb{R}^K$ such that

$$\lambda\cdot v^- -\mu\cdot z^-\le\lambda\cdot v^+ -\mu\cdot z^+ +\nu\cdot y^+,$$

where $(v^{-}, z^{-}, 0)$ lies in P^{-} and (v^{+}, z^{+}, y^{+}) lies in P^{+}. By the nature of P^{+}, we conclude that (λ, μ) is in $\mathbb{R}^{I}_{+} \times \mathbb{R}^{J}_{+}$. And, assuming that $\lambda = 0$, choosing $(0, 0, 0)$ in P^{+} and any d in $\mathbb{R}^{N}$ satisfying the regularity condition with $Dh\left(x^{*}\right)(d) = 0$, we obtain $\mu \cdot z^{-} \geq 0$, where

$$z^{-} = g\left(x^{*}\right) + Dg\left(x^{*}\right)(d) \ll 0,$$

a contradiction unless $\mu = 0$, In this case, $\nu \cdot Dh\left(x^{*}\right)(h) = 0$ for every d in $\mathbb{R}^{K}$, thus violating the regularity condition. Therefore, at no loss of generality, we take $\sum_{i \in I} \lambda_i = 1$.

From the separation property, we have

$$\lambda \cdot Df\left(x^{*}\right)(d) - \mu \cdot \left(g\left(x^{*}\right) + Dg\left(x^{*}\right)(d)\right) - \nu \cdot Dh\left(x^{*}\right)(d) \leq 0.$$

Setting $d = 0$, we see that $\mu \cdot g\left(x^{*}\right) = 0$. As this conditions holds for any d in $\mathbb{R}^{N}$, we conclude that

$$\lambda \cdot Df\left(x^{*}\right)(d) - \mu \cdot \left(Dg\left(x^{*}\right)(d)\right) - \nu \cdot Dh\left(x^{*}\right)(d) = 0,$$

thus establishing the claim. □

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