MULTIDIMENSIONAL MONOTONICITY AND ECONOMIC APPLICATIONS

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Multidimensional Monotonicity and Economic Applications[*]

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Abstract

We characterize the extreme points of multidimensional monotone functions from $[0,1]^n$ to $[0,1]$, as well as the extreme points of the set of one-dimensional marginals of these functions. These characterizations lead to new results in various mechanism design and information design problems, including public good provision with interdependent values; interim efficient bilateral trade mechanisms; asymmetric reduced form auctions; and optimal private private information structure. As another application, we also present a mechanism anti-equivalence theorem for two-agent, two-alternative social choice problems: A mechanism is payoff-equivalent to a deterministic DIC mechanism if and only if they are ex-post equivalent.

Keywords: Multidimensional monotone functions, extreme points, mechanism design, information design.

1 Introduction

Many important economic problems involve the design of the allocation or information among multiple agents. In this paper, we show that a common structure behind these problems involves selecting a multidimensional monotone function that satisfies certain properties. We systematically study the extreme points of the set of multidimensional monotone functions from $[0,1]^n$ to $[0,1]$, as well as the set of their one-dimensional marginals. We show that our results immediately lead to various new results in well-known mechanism design problems — including public good provision, bilateral trade, reduced form auctions, and mechanism equivalence — as well as in recent information design problems.

Abstract Results. We start by describing our abstract results before discussing the economic applications in detail. Recall that a function $f : [0,1]^n \to [0,1]$ is ***monotone*** if $f(x) \leqslant f(y)$ for all $x \leqslant y$ where the partial order $\leqslant$ is defined in the usual sense. The set of monotone functions is a convex set and hence admits extreme points. An elegant result of Choquet (1954) identifies the extreme points of this set: They are given by $\mathbb{1}_A$ where $A \subseteq [0,1]^n$ is an ***upward-closed set*** (henceforth, an ***up-set***), i.e., for any $x \leqslant y$, if $x \in A$, then $y \in A$ (see Figure 1a). In the one-dimensional case, these extreme points are exactly the one-jump step functions $\{\mathbb{1}_{[k,1]}\}_{k \in [0,1]}$. Combined with Choquet's integral representation theorem, this implies that any monotone function can be represented as a mixture over such up-set functions $\{\mathbb{1}_A : A \text{ is an up-set}\}$.

Our first result (Proposition 1) refines Choquet's representation of monotone functions: We show that any monotone function can be in fact represented as a mixture over a collection of ***nested up-set*** functions, i.e., these up-sets are totally ordered by set inclusion. This nested up-set representation is very useful as it allows us to equivalently view a multidimensional monotone function as a totally ordered chain of up-sets coupled with a one-dimensional probability distribution. Our second result (Theorem 1) exploits this structure and well-known results on one-dimensional distributions to characterize the extreme points of multidimensional monotone functions subject to m linear constraints: they can always be represented as a mixture of $m + 1$ nested up-sets (see Figure 1b).

Building on these characterizations, our next set of results studies what we call ***rationalizable monotone functions***. For any integrable function $f : [0,1]^n \to [0,1]$, recall that its marginal in dimension i is simply $x_i \mapsto f_i(x_i) := \int f(x) \, dx_{-i}$. We say that a collection of one-dimensional monotone functions $q := (q_i)_{i=1,\dots n}$ is ***rationalizable*** if there exists some function $f : [0,1]^n \to [0,1]$ such that q_i's are exactly the marginals of f. Note

(a) An up-set function $\mathbb{1}_A$

(b) Mixture of two nested up-set functions

(c) Nested up-sets differ by a rectangle
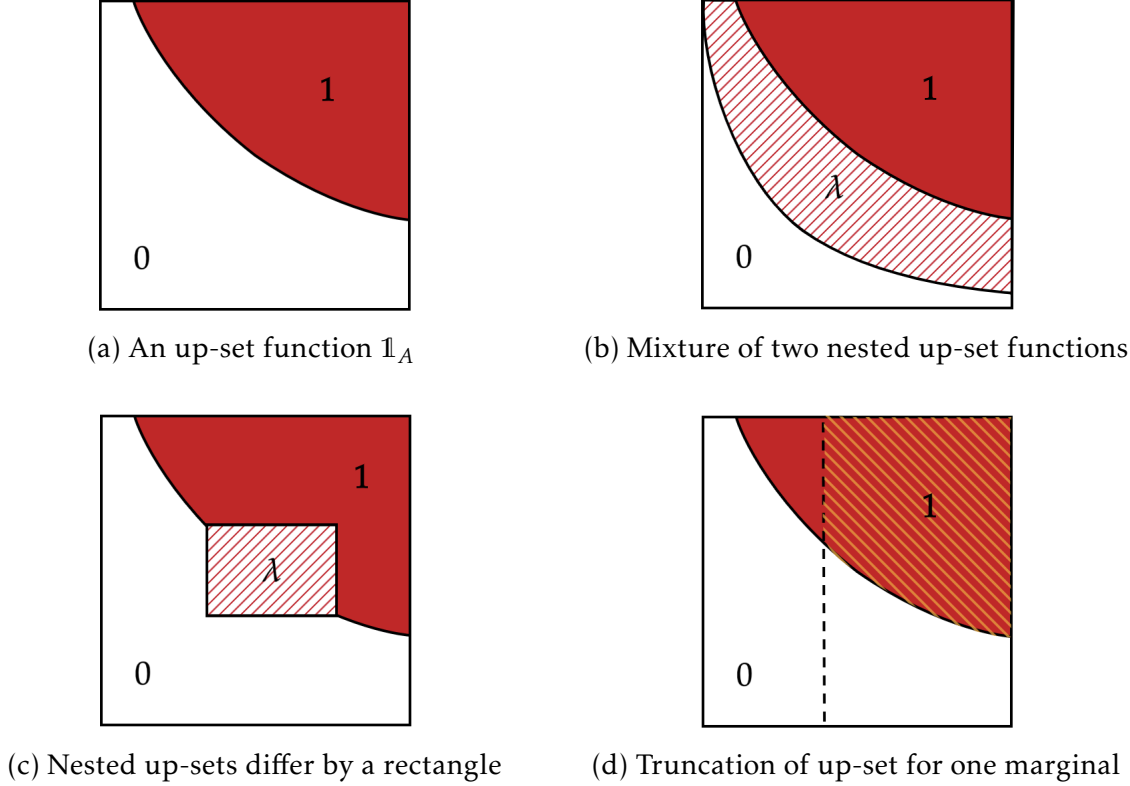
(d) Truncation of up-set for one marginal

Figure 1: Illustration of extreme points for different sets of functions

that this definition does not require the function f to be monotone. However, an elegant result by Gutmann et al. (1991) shows that any such q that is rationalizable by some function must also be rationalizable by a *monotone* function. We exploit this connection and our characterizations of multidimensional monotone functions to also characterize the extreme points of the set of rationalizable monotone functions q. In particular, we show that (Theorem 2) every extreme point of the set of rationalizable monotone functions must be the one-dimensional marginals of an up-set function $\mathbb{1}_A$, and every extreme point of rationalizable monotone functions subject to m linear constraints must also be the one-dimensional marginals of $m + 1$ mixture of $\mathbb{1}_{A_k}$ where A_k's are nested up-sets.

In the case of rationalizable pairs $q := (q_1, q_2)$, we sharpen Theorem 2 to a necessary and sufficient characterization of the extreme points. Without any linear constraint, we show that (Theorem 3) the one-dimensional marginals of the up-set functions $\mathbb{1}_A$ are exactly the extreme points of the rationalizable pairs, and these extreme points must be *uniquely* rationalized. With one linear constraint, we show that (Theorem 4) the extreme points of rationalizable pairs are characterized by a mixture of $\mathbb{1}_{A_1}$ and $\mathbb{1}_{A_2}$ where A_1 and A_2 are nested up-sets that differ by a single ***rectangle*** (see Figure 1c), and they must be *uniquely* rationalized among all monotone functions.

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In the case of rationalizable pairs, it is also known that a pair of monotone functions (q_1, q_2) is rationalizable if and only if q_1 is majorized by the conjugate of q_2 (Gutmann et al. 1991).[1] Building on this connection, our characterization immediately gives the extreme points of this joint majorization set — they are exactly the pairs where the majorization constraint binds everywhere (Proposition 4). This structure is very different from the one-dimensional case (Kleiner, Moldovanu, and Strack 2021), precisely because these majorization constraints are imposed on a pair of monotone functions, linking them between the original space q_1 and the inverse space q_2^{-1}. Using this logic, we also characterize the extreme points of the joint majorization set under a linear constraint (Proposition 5), and the extreme points of the weak joint majorization set (Proposition 6) where the majorization relation is replaced by weak majorization. For weak joint majorization, any extreme point (q_1, q_2) has the structure that q_2 is the marginal of an up-set function $\mathbb{1}_A$ and q_1 is the marginal of another up-set function $\mathbb{1}_{\tilde{A}}$ where $\tilde{A}$ is a ***truncation*** of A (see Figure 1d).

Economic Applications. We now discuss the economic applications of our results (see Section 4 for details). We apply our abstract results to obtain new results in four well-known mechanism design problems and one recent information design problem.

Our first application studies the classical problem of public good provision, where a designer decides whether to implement a public project subject to an ex-ante budget constraint for financing the project. Almost all existing studies of this problem focus on the case of independent private values (see e.g. Güth and Hellwig 1986). In practice, the values of the agents for a public good seem likely to be interdependent and correlated. However, under Bayesian incentive compatibility, it is well known that the designer can generically achieve the first best by constructing bets á la Crémer and McLean (1988) as long as there are correlated signals. This may not be satisfying because of the fragility of such mechanisms and calls for other approaches to this problem (Brooks and Du 2023). A natural candidate is to study the design of ***ex-post*** incentive-compatible and individually rational mechanisms. By a standard argument, an allocation rule is ex-post IC if and only if it is monotone in each agent's signal. Because the objective is a linear functional, our extreme point results (recall Figure 1b) immediately imply that the optimal ex-post IC mechanism is simply a ***two-threshold policy***, where the designer aggregates the signals of each agent into a score and if the score is higher than the high threshold, then the project is implemented for sure, and if it is lower than the high threshold but higher than the low threshold, the project is implemented with some positive probability (and otherwise

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[1] That is, $q_1(z)$ is majorized by $\hat{q}_2(z) := 1 - q_2^{-1}(1 - z)$.

abandoned). The specifics of the scoring rule and the two thresholds will depend on the correlation structure of the agents' signals, but the optimal mechanism turns out to always have this simple structure regardless of the correlation structure.

Our second application studies another well-known mechanism design problem, bilateral trade á la Myerson and Satterthwaite (1983). An important question in bilateral trade is to understand its *interim-efficient frontier* (Myerson 1984). It has been shown that in some special cases depending on the welfare weights and type distributions, the interim-efficient mechanism takes a familiar form of trading if and only if the suitably defined virtual value is above the virtual cost (Ledyard and Palfrey 1999, 2007). However, to understand the frontier, we have to characterize the optimal mechanism for arbitrary welfare weights and type distributions. Because any interim-efficient mechanism must maximize a linear functional over the pair of interim allocation rules for the buyer and seller, and the ex-post budget constraint can be represented as a linear constraint, our extreme point results (recall Figure 1c) immediately yield a new class of trading mechanisms that can attain the optimal welfare for any welfare weights and type distributions. We call these trading mechanisms *markup-pooling mechanisms*: In such a mechanism, trade is implemented if and only if the value v is above a monotone transformation of the cost $\phi(c)$ (the *markup function*), with the exception that when c falls into a single interval I (the *pooling interval*), we *resample* the cost from the two ends of the interval and execute the trade if the buyer's value v is above the marked up, resampled cost $\phi(\tilde{c})$.

Our third application studies the question of interim equivalence between mechanisms initiated by Manelli and Vincent (2010). Two mechanisms are *payoff-equivalent* if they yield the same interim expected utilities for all agents and the same ex-ante expected surplus. From a series of fundamental contributions (Manelli and Vincent 2010; Gershkov et al. 2013; Chen et al. 2019), it is known that *(i)* for any Bayesian incentive compatible (BIC) mechanism, there exists a payoff-equivalent dominant strategy incentive compatible (DIC) mechanism (*BIC-DIC equivalence*); and *(ii)* for any stochastic mechanism, there exists a payoff-equivalent deterministic mechanism (*stochastic-deterministic equivalence*). These mechanism equivalence results are very helpful, because DIC mechanisms are robust to changes in the agents' beliefs and deterministic mechanisms do not require a credible randomization device. However, as another application of our results, we show that there is a strong conflict between asking for DIC mechanisms and asking for deterministic mechanisms. In particular, we present a *mechanism anti-equivalence theorem* for social choice problems with two agents and two alternatives: A BIC mechanism is payoff-equivalent to a deterministic DIC mechanism if and only if they are ex-post equivalent. In other words, although every BIC mechanism has a payoff-equivalent

DIC counterpart and every stochastic mechanism has a payoff-equivalent deterministic counterpart, the interim payoffs under a mechanism that is both deterministic and DIC cannot be replicated by *any* other mechanism. Consequently, among DIC mechanisms, a stochastic mechanism and a deterministic mechanism must yield different interim payoffs; among deterministic mechanisms, a BIC mechanism and a DIC mechanism must yield different interim payoffs, unless they are ex-post equivalent. Thus, our result implies that the BIC-DIC equivalence necessarily relies on randomization, and the stochastic-deterministic equivalence necessarily relies on non-DIC mechanisms.

Our fourth application studies asymmetric reduced form auctions. A ***reduced form auction*** is a collection of interim allocation probabilities that is implementable by an ex-post auction allocation rule (Matthews 1984; Border 1991). In an important contribution, Kleiner, Moldovanu, and Strack (2021) characterize the extreme points of symmetric reduced form auctions, by representing the implementability constraint as a majorization constraint on a one-dimensional monotone function. Now, suppose we have two bidders and allow for asymmetric mechanisms. Let q_i be the allocation probability to bidder i as a function of bidder i's type in the quantile space. Exploiting the extreme-point structure of the weak joint majorization set, we show that (q_1, q_2) is implementable by an auction if and only if q_1 is weakly majorized by the inverse of q_2, with every extreme point of that set characterized by q_1 being pointwise equal to $q_2^{-1} \mathbb{1}_{[k,1]}$ for some k. These extreme points correspond to exactly Figure 1d (up to relabeling), where the complement of the up-set A is the region of types that we allocate to bidder 2, and the left truncation of the up-set is the region where the seller keeps the object. An immediate consequence of this extreme point characterization is that for any linear or convex objective, the optimal auction can always be implemented by a ***deterministic and DIC*** mechanism as in Figure 1d. We apply this extreme point characterization to auctions with endogenous values introduced in Gershkov et al. (2021), where revenue maximization becomes a convex objective, and show how our characterization helps understand their examples of optimal asymmetric mechanisms.

Our last application studies private private information structures introduced in He, Sandomirskiy, and Tamuz (2024). Consider a binary state and n agents with a common prior about the state. A ***private private*** information structure sends an independent signal to each agent so that they update their belief about the state but not about other agents' signals. He, Sandomirskiy, and Tamuz (2024) characterize the Blackwell-Pareto undominated information structures and use a garbling argument to characterize the set of ***feasible belief distributions*** that can be induced by some private private information structure. Complementary to their analysis, we use our extreme point results to provide

a convex-hull characterization of the ***feasible belief quantiles*** under private private information. Unlike the feasible belief distributions, the set of feasible belief quantiles is convex and admits extreme points, which we also characterize using our abstract results. In particular, we show that any extreme feasible belief quantiles can be implemented by a ***nested bi-upset*** signal structure, where we randomize between two nested up-sets in $[0,1]^n$, and uniformly draw signals from there. This is because a collection of belief quantile functions is feasible if and only if it is rationalizable in our sense and satisfies one linear constraint, the Bayes plausibility constraint. As an immediate consequence, for many information design problems subject to the privacy constraint, a nested bi-upset signal would be optimal. Moreover, as we show, a pair (q_1, q_2) is an extreme point of the feasible belief quantiles if and only if it can be implemented using a nested bi-upset signal where the nested up-sets differ by at most a rectangle. Combining with the characterization of Blackwell-Pareto frontier from He, Sandomirskiy, and Tamuz (2024), our result then implies that in the case of two agents, any such optimal private private information — even though it may not be Blackwell-Pareto undominated — can be attained by *(i)* selecting a Blackwell-Pareto undominated signal and then *(ii)* applying a single ***interval-pooling*** for one of the two agents.

1.1 Related Literature

We study the convex set of multidimensional monotone functions and their marginals. In particular, we characterize the extreme points of the set of multidimensional monotone functions subject to finitely many linear constraints, as well as the extreme points of the rationalizable monotone functions. Choquet (1954) provides a well-known characterization of extreme points of monotone functions, which states that a monotone function is an extreme point if and only if it is an indicator function defined on an up-set. We use Choquet's characterization to further characterize the extreme points of monotone functions subject to finitely many linear constraints. Our characterization sharpens the set of candidate extreme points derived from Winkler (1988) to finite mixtures of indicator functions defined on *nested* up-sets.

Our results on rationalizable monotone functions build on and contribute to the mathematics literature on the existence of a joint density for given marginals (Lorentz 1949; Kellerer 1961; Strassen 1965). In particular, Gutmann et al. (1991) provide conditions on when a collection of monotone functions is rationalizable and show the equivalence between the rationalizability by an arbitrary function, by a monotone function, and by

an indicator function.[2] Exploiting this equivalence and our results on multidimensional monotone functions, we characterize the extreme points of the set of rationalizable monotone functions, with and without linear constraints. The unique rationalizability property of our extreme points in the two-dimensional case also connects to and builds on results from the mathematical tomography literature on "sets of uniqueness" (Lorentz 1949; Fishburn et al. 1990; Kemperman 1991; Kellerer 1993), which studies necessary and sufficient conditions for a function to be uniquely determined by its marginals.

In economics, several recent papers characterize extreme points of one-dimensional monotone functions under various constraints and derive economic applications. Kleiner, Moldovanu, and Strack (2021) characterize the extreme points of the set of nondecreasing functions that majorize, or are majorized by, some fixed nondecreasing function. They then derive applications on reduced-form implementation, BIC-DIC equivalence, delegation, and persuasion.[3] Nikzad (2023) and Candogan and Strack (2023) further characterize the extreme points of these convex sets subject to finitely many linear constraints. Meanwhile, Yang and Zentefis (2024) characterize the extreme points of the set of nondecreasing functions that are pointwise bounded by two other nondecreasing functions, and derive applications to gerrymandering, quantile-based persuasion, overconfidence, and security design. Our results complement these recent characterizations and we provide new applications in various settings as well.

A one-dimensional monotone function can be equivalently viewed as a probability distribution. However, this equivalence breaks down for multidimensional monotone functions.[4] Complementary to our analysis of multidimensional monotone functions, Kleiner et al. (2024) characterize the extreme points of multidimensional probability distributions that are dominated by a fixed distribution in the convex stochastic order.

Our analysis also connects to the literature on multidimensional screening. Manelli and Vincent (2007) introduce the extreme point approach to study multidimensional screening. They show that the well-known result of a posted price being optimal for selling one good (Myerson 1981; Riley and Zeckhauser 1983) is an immediate consequence of the extreme points of one-dimensional monotone functions. They then show

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[2]Specifically, Gutmann et al. (1991) show these three results using complementary but separate approaches: the rationalizability characterization follows from Strassen (1965), which in turn is built upon supporting hyperplanes, instead of extreme points, of convex sets; rationalizability by monotone functions follows from a convex minimization program, which is later generalized by Gershkov et al. (2013); whereas the proof of rationalizability by indicator functions relies on the existence and the characterization of extreme points of a different convex set compared to our focus — the set of (not necessarily monotone) functions on $[0,1]^n$ with *fixed* marginals.

[3]See also Arieli et al. (2023).

[4]Indeed, recall that a bivariate CDF F must also satisfy $F(x_1', x_2') - F(x_1, x_2') - F(x_1', x_2) + F(x_1, x_2) \geqslant 0$ for all $x_1 < x_1'$ and $x_2 < x_2'$.

that with multiple goods, the optimal mechanism is drastically more complex — it generally involves many lotteries. The key difference between our setting and theirs is that with multiple dimensions, monotonicity no longer characterizes the implementable allocations.[5] Indeed, the indirect utility functions in multidimensional screening are monotone *convex* functions, which turn out to have drastically different structures and are known to admit a dense set of extreme points (Johansen 1974; Bronshtein 1978; Lahr and Niemeyer 2024).[6] In comparison, we relax the convexity constraint and use multidimensional monotone functions to study design problems with multiple agents.

Several recent economics papers also build on results from mathematical tomography including Gershkov et al. (2013) who study BIC-DIC equivalence, Chen et al. (2019) who study stochastic-deterministic equivalence, and He, Sandomirskiy, and Tamuz (2024) who study information structures with independent signals.[7] As our applications to these settings show, our extreme point characterizations of rationalizable monotone functions lead to a variety of new results.

The remainder of the paper proceeds as follows. Section 2 presents our results on multidimensional monotone functions. Section 3 presents our results on rationalizable monotone functions. Section 4 presents our economic applications. Section 5 concludes. Appendix A provides the omitted proofs.

2 Multidimensional Monotone Functions

Endow the set of integrable functions from $[0,1]^n \to [0,1]$ with the L^1 norm. Let $\mathcal{F}$ be the set of monotone functions from $[0,1]^n$ to $[0,1]$. That is, a function $f : [0,1]^n \to [0,1]$ is in $\mathcal{F}$ if $f(x) \leqslant f(y)$ for any $x, y \in [0,1]^n$ such that $x \leqslant y$. Clearly, $\mathcal{F} \subseteq L^1([0,1]^n)$ is a convex set. Moreover, by a suitable version of Helly's selection theorem and the dominated convergence theorem, $\mathcal{F}$ is compact.[8]

We say that a set $A \subseteq [0,1]^n$ is an ***upward-closed set*** (or simply an ***up-set***) if, for any $x \in A$, $y \geqslant x$ implies $y \in A$. Clearly, an indicator function $\mathbb{1}_A$ is in $\mathcal{F}$ if and only if A is an

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[5]Indeed, the implementable allocations are characterized by ***cyclic monotonicity*** (Rochet 1987). However, in certain screening problems without monetary transfers, the implementable allocations can be characterized by multidimensional monotonicity; see e.g. Vravosinos (2024).

[6]These results are also the reason why recent studies of multidimensional screening do not adopt an extreme point approach but provide conditions on primitives such that a simple mechanism is optimal (see e.g. Haghpanah and Hartline 2021; Yang 2022).

[7]See also Arieli et al. (2021) who study feasible posterior distributions when signals need *not* be independent. Their question corresponds to the study of coherent distributions (see Dawid et al. 1995), for which the characterization of extreme points is an open question (Burdzy and Pitman 2020).

[8]See, e.g., Leonov (1996) for a multidimensional version of Helly's selection theorem.

up-set. Given a totally ordered index set $\mathcal{I}$, a family of up-sets $\{A_i\}_{i\in\mathcal{I}}$ is **nested** if $A_i \subseteq A_{i'}$ for all $i, i' \in \mathcal{I}$ such that $i < i'$. Our results pertain to the properties of the extreme points of sets related to $\mathcal{F}$. To begin with, we first note the following well-known observation.

Lemma 1 (Choquet 1954, Theorem 40.1). *$f \in \mathcal{F}$ is an extreme point of $\mathcal{F}$ if and only if $f = \mathbb{1}_A$ for some up-set $A \subseteq [0,1]^n$.*

An immediate consequence of Lemma 1 is that *any* monotone function $f \in \mathcal{F}$ can be written as a mixture of indicator functions defined by up-sets (by Choquet's integral representation theorem). Nonetheless, since up-sets in $[0,1]^n$ are only partially ordered under set-inclusion when $n \geqslant 2$, these indicator functions may not be ordered. As a result, mixtures implied by Lemma 1 could potentially involve mixing up-sets that are not nested. However, as Proposition 1 below shows, there is always a way to represent a monotone function $f \in \mathcal{F}$ as a mixture of indicator functions defined by a family of *nested* up-sets.

Proposition 1 (Nesting representation). *A function $f : [0,1]^n \to [0,1]$ is monotone if and only if there exist a collection of nested up-sets $\{A_r\}_{r\in[0,1]}$ and a probability measure $\mu \in \Delta([0,1])$ such that for all $x \in [0,1]^n$,*

$$f(x) = \int_0^1 \mathbb{1}_{A_r}(x)\mu(\mathrm{d}r).$$

Moreover, if $\left(\{A_r\}_{r\in[0,1]}, \mu\right)$ and $\left(\{A'_r\}_{r\in[0,1]}, \mu'\right)$ represent the same monotone function in the above sense, then they must induce the same distribution over the same family of nested up-sets.

Although Choquet's extreme point theorem (Lemma 1) implies that any monotone function $f \in \mathcal{F}$ can be represented as a mixture of indicator functions defined by up-sets, Proposition 1 ensures that there must be such a mixture that is supported on a family of nested up-sets. Moreover, Proposition 1 asserts that such a nesting representation must be unique, even though Choquet's integral representation is in general not unique. This shows that a multidimensional monotone function can be equivalently viewed as a *one-dimensional* probability distribution on a family of nested up-sets.

An immediate consequence of this single-dimensional representation of a multidimensional monotone function is a characterization of extreme points of multidimensional monotone functions subject to finitely many affine constraints. Fix any finite collection $\{\phi^j\}_{j=1}^m$ of essentially bounded functions on $[0,1]^n$, and a finite collection $\{\eta^j\}_{j=1}^m$ of real

numbers. Consider the following subset $\overline{\mathcal{F}}$ of $\mathcal{F}$:

$$\overline{\mathcal{F}} := \left\{ f \in \mathcal{F} : \int_{[0,1]^n} f(x)\phi^j(x)\,\mathrm{d}x \leqslant \eta^j, \forall j \in \{1,\ldots,m\} \right\}.$$

From Lemma 1, it is well known that, by Proposition 2.1 of Winkler (1988), any extreme point of $\mathcal{F}$ is a mixture of at most $m+1$ indicator functions on up-sets. However, there are many different up-sets in $[0,1]^n$, and mixtures of them, even if there are only finitely many, could be complex. Nonetheless, Proposition 1 allows us to further sharpen the characterization, as stated by Theorem 1 below.

Theorem 1. *Every extreme point of $\overline{\mathcal{F}}$ is a mixture of at most $m+1$ indicator functions $\{\mathbb{1}_{A_j}\}_{j=1}^{m+1}$ where $\{A_j\}_{j=1}^{m+1}$ are nested up-sets.*

Theorem 1 further simplifies the structure of extreme points of $\overline{\mathcal{F}}$ implied by Lemma 1 and the result of Winkler (1988). In addition to being a mixture of at most $m+1$ indicator functions defined on up-sets, Theorem 1 states that these up-sets must be nested. An immediate consequence is that any extreme point f of $\overline{\mathcal{F}}$ can be written as

$$f(x) = \sum_{j=1}^{m+1} p_j \mathbb{1}\{g(x) \in [k_{j-1}, k_j]\},$$

where $0 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_{m+1} = 1$, $0 \leqslant k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{m+1} = 1$, and $g : [0,1]^n \to [0,1]$ is a monotone function. That is, f increases from 0, through $\{p_1,\ldots,p_m\}$, to 1 as a monotone aggregator $g \in \mathcal{F}$ increases from 0 to 1. Note that without the nesting structure identified by Theorem 1, $m+1$ many mixtures of arbitrary up-sets would in general induce a monotone function that has 2^{m+1} many distinct values. Thus, Theorem 1 provides an exponential reduction in the complexity of the extreme points by exploiting the nesting structure from Proposition 1.

Clearly, not all mixtures of $m+1$ indicator functions defined by nested up-sets are extreme points of $\overline{\mathcal{F}}$. Indeed, if $\phi^j \equiv 0$ and $\eta^j = 0$ for all j, then the only extreme points of $\overline{\mathcal{F}}$ are single indicator functions defined on up-sets, according to Lemma 1. However, if the constraints are tight and have enough independence, then every mixture of at most $m+1$ indicator functions defined on nested up-sets is an extreme point of $\overline{\mathcal{F}}$. Specifically, consider any family $\{\phi^j\}_{j=1}^m$ of essentially bounded functions on $[0,1]^n$ such that for any
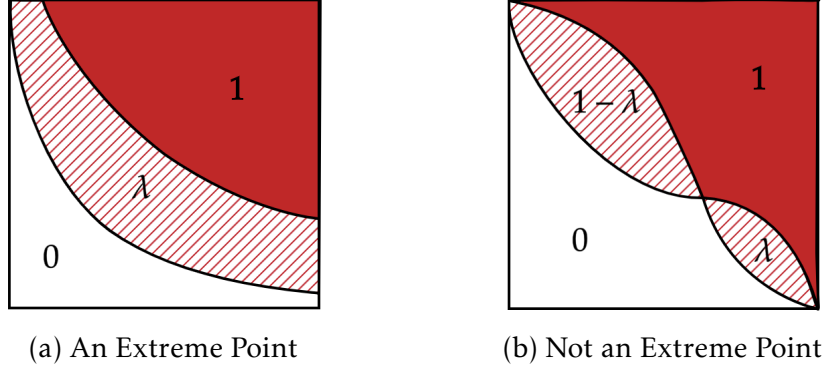
(a) An Extreme Point   (b) Not an Extreme Point

Figure 2: Extreme Points of $\overline{\mathcal{F}}$

disjoint subsets $\{B_j\}_{j=1}^m$, $B_j \subseteq [0,1]^n$ the vectors

$$
\left\{
\begin{pmatrix} \int_{B_1} \phi^1(x)\,dx \\ \vdots \\ \int_{B_1} \phi^m(x)\,dx \end{pmatrix}, \cdots, \begin{pmatrix} \int_{B_m} \phi^1(x)\,dx \\ \vdots \\ \int_{B_m} \phi^m(x)\,dx \end{pmatrix}
\right\}
$$

are linearly independent. Let

$$
\overline{\mathcal{F}}^\star := \left\{ f \in \mathcal{F} : \int_{[0,1]^n} f(x)\phi^j(x)\,dx = \eta^j, \forall j \in \{1,\ldots,m\} \right\}.
$$

We then have the following converse of Theorem 1

Proposition 2. $f \in \overline{\mathcal{F}}^\star$ *is an extreme point if and only if* $f = \sum_{j=1}^k \lambda_j \mathbb{1}_{A_j}$ *for some nested up-sets* $\{A_j\}_{j=1}^k$, $k \leqslant m$, *and for some* $\{\lambda_j\}_{j=1}^k$ *such that* $\lambda_j \geqslant 0$ *and* $\sum_{j=1}^k \lambda_j = 1$.

Figure 2 illustrates the structure of extreme points of $\overline{\mathcal{F}}$, in the case of $m = 1$. Figure 2a depicts a monotone function $f = \lambda \mathbb{1}_A + (1 - \lambda)\mathbb{1}_{A'}$, where $A' \subseteq A$ are nested up-sets. According to Theorem 1, any extreme point of $\overline{\mathcal{F}}$ must exhibit the structure given by Figure 2a. Proposition 2 further ensures that any such function must be an extreme point of $\overline{\mathcal{F}}^\star$ when the constraints have enough independence. On the other hand, Figure 2b depicts a monotone function $f = \lambda \mathbb{1}_A + (1 - \lambda)\mathbb{1}_{A'}$, where A and A' are up-sets but are not nested. According to Theorem 1, such functions can never be an extreme point of $\overline{\mathcal{F}}$, even if they are mixtures of two extreme points of $\mathcal{F}$.

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3   Rationalizable Monotone Functions

Next, we characterize extreme points of monotone functions that can be "rationalized" by some (not necessarily monotone) function $f : [0,1]^n \to [0,1]$. For any tuple $q = (q_1, \ldots, q_n)$ of nondecreasing, left-continuous functions from $[0,1]$ to $[0,1]$, we say that q is **rationalizable** if there exists a function $f : [0,1]^n \to [0,1]$ (not necessarily monotone) such that

$$q_i(x_i) = \int_{[0,1]^{n-1}} f(x) \, \mathrm{d}x_{-i},$$

for all i and all $x_i \in [0,1]$. For any such f and q, we also say that q is **rationalized** by f. Let $\mathcal{Q}$ be the set of all rationalizable tuples of nondecreasing left-continuous functions. Fix a family $\{\psi_i^j\}_{1 \leq i \leq n}^{1 \leq j \leq m}$ of essentially bounded functions on $[0,1]$ and a family $\{\eta^j\}_{j=1}^m$ of real numbers, let $\overline{\mathcal{Q}}$ be the set of all tuples of rationalizable monotone functions subject to $m+1$ affine constraints:

$$\overline{\mathcal{Q}} := \left\{ q \in \mathcal{Q} : \sum_{i=1}^n \int_0^1 q_i(x_i) \psi_i^j(x_i) \, \mathrm{d}x_i \leq \eta^j, \forall j \in \{1, \ldots, m\} \right\}.$$

While we do not impose monotonicity on the joint function f, an elegant result of Gutmann et al. (1991) shows that any rationalizable monotone tuples q can be rationalized by a multidimensional monotone $f \in \mathcal{F}$:

Lemma 2 (Gutmann et al. 1991, Theorem 5; Gershkov et al. 2013, Theorem 1). *A tuple of one-dimensional monotone functions $q = (q_1, \ldots, q_n)$ is rationalizable if and only if it can be rationalized by a multidimensional monotone function $f \in \mathcal{F}$.*

As a consequence of Lemma 2, the set $\mathcal{Q}$ is exactly a linear projection of the set $\mathcal{F}$. Lemma 3 below connects the extreme points of a convex set to the extreme points of its linear projection.[9]

Lemma 3 (Affine mapping lemma). *Let X be a compact convex subset of a locally convex topological vector space, and let Y be a topological vector space. For any continuous affine map $L : X \to Y$, we have*

$$\mathrm{ext}(L(X)) \subseteq L(\mathrm{ext}(X)).$$

Now, using Lemma 2 and Lemma 3, we exploit Lemma 1 and Theorem 1 to characterize the extreme points of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$:

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[9]We are not aware of any existing proofs of Lemma 3 in the literature, and thus include a proof of it in the Appendix. We thank Elliot Lipnowski for providing a simple proof.

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Theorem 2.

  (i) *Every extreme point of $\mathcal{Q}$ is rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [0,1]^n$.*

  (ii) *Every extreme point of $\overline{\mathcal{Q}}$ is rationalized by a mixture of $\{\mathbb{1}_{A_j}\}_{j=1}^{m+1}$ for some nested up-sets $\{A_j\}_{j=1}^{m+1}$.*

While Theorem 2 provides necessary conditions for the extreme points of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, they may not be sufficient in general. There might be (mixtures of) indicator functions defined on (nested) up-sets that are not extreme points of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$. Nonetheless, when $n = 2$, these necessary conditions can be greatly sharpened. Below, we explore the cases where $n = 2$ and $m \in \{0, 1\}$.

3.1   Rationalizable Monotone Pairs

Our next set of results characterizes the extreme points of rationalizable functions in the case of $n = 2$ with and without a linear constraint. Namely, $\mathcal{Q}$ is the set of pairs $q = (q_1, q_2)$ of rationalizable monotone functions on $[0,1]$; and

$$\overline{\mathcal{Q}} = \left\{ q \in \mathcal{Q} : \int_0^1 q_1(x_1)\psi_1(x_1)\,\mathrm{d}x_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\,\mathrm{d}x_2 \leqslant \eta \right\},$$

for some essentially bounded functions ψ_1 and ψ_2 and some real number η.

When $n = 2$, part (i) of Theorem 2 becomes both necessary and sufficient, and the sufficient condition also exhibits a notion of uniqueness, as Theorem 3 below shows. To state the theorem, we say that $q \in \mathcal{Q}$ is ***uniquely rationalized*** by a function $f : [0,1]^2 \to [0,1]$ if for any function $\tilde{f} : [0,1]^2 \to [0,1]$ that also rationalizes q, it must be that $f \equiv \tilde{f}$ (up to measure zero).

Theorem 3. *$q = (q_1, q_2)$ is an extreme point of $\mathcal{Q}$ if and only if q is rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [0,1]^2$. Moreover, every extreme point of $\mathcal{Q}$ is uniquely rationalized.*

For a general $n \in \mathbb{N}$, Theorem 2 states that any extreme point q of $\mathcal{Q}$ can be rationalized by an indicator function $\mathbb{1}_A$ defined on an up-set. It does not imply that any $q \in \mathcal{Q}$ rationalized by indicator functions defined on an up-set must be an extreme point. However, when $n = 2$, being rationalized by an indicator function defined on an up-set is both necessary and sufficient for q to be an extreme point of $\mathcal{Q}$, according to Theorem 3. Furthermore, any extreme point must be *uniquely* rationalized by such a function — thus, any $q \in \mathcal{Q}$ rationalized by some function f that is *not* an indicator function $\mathbb{1}_A$ defined on an up-set *cannot* be an extreme point.

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The proof of Theorem 3 proceeds as follows. It actually shows that if q is rationalized by an indicator function defined on an up-set, then q must be an **exposed point** (the unique maximizer for some linear functional). Since every exposed point is an extreme point, the sufficiency part of Theorem 3 follows.[10] The argument exploits that when $n = 2$, an up-set is an **additive set** in the sense of Fishburn et al. (1990). Using this property, we can define an additively separable function $\phi(x_1, x_2)$ on $[0,1]^2$ such that $\mathbb{1}_A$ is the unique maximizer for the linear functional $\Phi : f \mapsto \int \phi f \, dx$. But then it implies that its one-dimensional marginals q must be an exposed point of $\mathcal{Q}$ by the additive separability of ϕ.[11] Now, for the unique rationalizability part of Theorem 3, we observe that $\mathbb{1}_A$ is actually the unique maximizer for Φ among *all* functions (not necessarily monotone). But then any function f sharing the same marginals with $\mathbb{1}_A$ must also maximize Φ, and hence is identical to $\mathbb{1}_A$.[12]

Next, we further characterize the extreme point of $\overline{\mathcal{Q}}$ in the case of $n = 2$ and $m = 1$. To this end, first recall that for two nondecreasing functions $g_1 : [0,1] \to \mathbb{R}$ and $g_2 : [0,1] \to \mathbb{R}$, g_2 is said to **weakly majorize** g_1 (denoted by $g_1 \preceq_w g_2$) if

$$\int_x^1 g_1(z) \, dz \leq \int_x^1 g_2(z) \, dz$$

for all $x \in [0,1]$. Moreover, g_2 is said to **majorize** g_1 (denoted by $g_1 \preceq g_2$) if, in addition to the above inequalities, the equality holds at $x = 0$. Note that if g_1 and g_2 are CDFs, then g_2 majorizes g_1 if and only if g_1 is a mean-preserving spread of g_2, whereas g_2 weakly majorizes g_1 if and only if g_2 second-order stochastic dominates g_1.

Another elegant result of Gutmann et al. (1991) shows that rationalizable monotone functions can be characterized by a joint majorization relation:

Lemma 4 (Gutmann et al. 1991, Theorem 4). *For any pair $q = (q_1, q_2)$ of nondecreasing functions, $q \in \mathcal{Q}$ if and only if $q_1 \preceq \hat{q}_2$, where $\hat{q}_2(z) := 1 - q_2^{-1}(1 - z)$ is the conjugate of q_2.*[13]

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[10]In fact, as a consequence of the proof, we have also shown that every extreme point in $\mathcal{Q}$ is exposed when $n = 2$, and every extreme point in $\mathcal{F}$ is exposed for any n.

[11]The proof can be readily extended to show that any q rationalizable by an indicator function $\mathbb{1}_Z$ defined on an additive set Z is an exposed point of $\mathcal{Q}$ for any dimension n, but it is known that up-sets do not coincide with additive sets when $n > 2$ and hence there would still be a gap between our necessary and sufficient conditions for $n > 2$.

[12]Alternatively, we can also prove the unique rationalizability of our extreme points by exploiting that every additive set is a "strong set of uniqueness" (Lorentz 1949; Fishburn et al. 1990; Kemperman 1991). However, that argument would not generalize to prove the unique rationalizability of the extreme points of $\overline{\mathcal{Q}}$ in Theorem 4.

[13]For any nondecreasing left-continuous function $g : [0,1] \to [0,1]$, g^{-1} is defined as $g^{-1}(z) := \inf\{x \in [0,1] : g(x) > z\}$. Note that g^{-1} is right-continuous, and hence $\hat{g}$ is left-continuous.
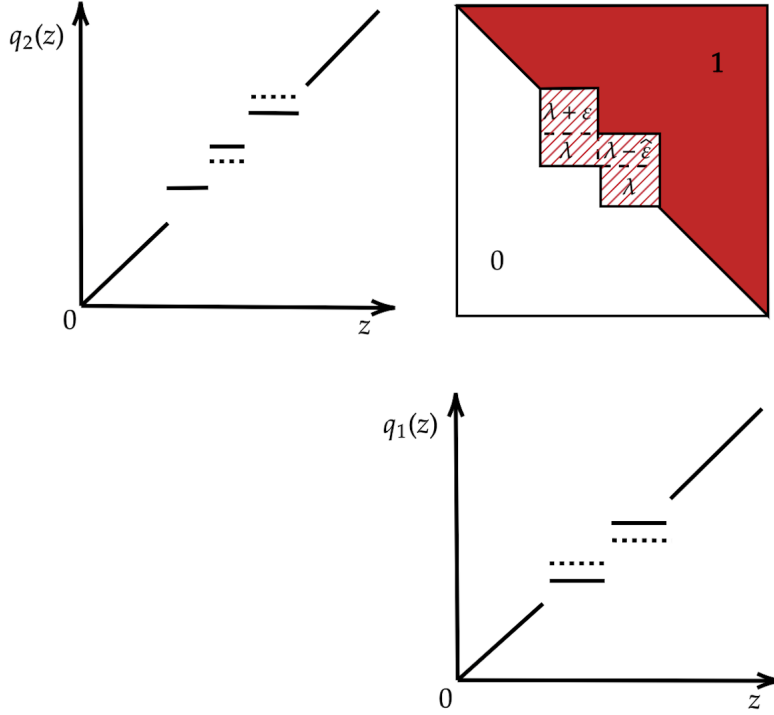
Figure 3: Perturbation of Non-Rectangular Regions, with Original Marginals (Solid) and Perturbed Marginals (Dashed)

We say that a subset $B \subseteq [0,1]^2$ is a **rectangle** if $\mathrm{cl}(B) = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$, where $\underline{x}_1 \leqslant \overline{x}_1$ and $\underline{x}_2 \leqslant \overline{x}_2$. Two nested up-sets $A' \subseteq A$ are said to **differ by** $B \subseteq [0,1]^2$ if $A \backslash A' = B$.

Theorem 4. *Every extreme point of $\overline{Q}$ is rationalized by a mixture of $\mathbb{1}_A$ and $\mathbb{1}_{A'}$, where $A' \subseteq A \subseteq [0,1]^2$ are nested up-sets that differ by at most a rectangle. Moreover, every extreme point of $\overline{Q}$ is uniquely rationalized among all monotone functions.*

Notably, Theorem 4 greatly reduces the set of possible extreme points of $\overline{Q}$ when $n = 2$. Indeed, part (ii) of Theorem 2 indicates that, even when $m = 1$, an extreme point of $\overline{Q}$ is a mixture of two indicator functions defined on nested up-sets. When $n = 2$, Theorem 4 states that these two up-sets can only differ by a single rectangle, which sharpens the characterization of extreme points of $\overline{Q}$.

The proof of Theorem 4 can be found in the Appendix. The argument is involved, so we sketch the intuition here. We first sketch the intuition behind the first part of Theorem 4. By part (ii) of Theorem 2, we know that any extreme point $q \in \overline{Q}$ must be rationalized by a mixture of indicator functions defined on two nested up-sets $A_1 \subseteq A_2$. Suppose for contradiction that the two nested up-sets differ not by a rectangle but by a "flipped Z" as in the dashed region of Figure 3. A key observation is that by choosing the correct ratio of (small enough) $\varepsilon, \hat{\varepsilon} > 0$ and perturbing the mixture in the dashed region

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using the way illustrated by Figure 3, there exist two distinct monotone $q', q'' \in \overline{Q}$ such that $1/2q' + 1/2q'' = q$, as illustrated by the marginals in Figure 3, contradicting to that q is an extreme point of $\overline{Q}$. The actual proof shows that for two arbitrary nested up-sets, if they differ by *any* non-rectangular region, then there must exist such a perturbation. It consists of three steps. First, we combine Gutmann et al. (1991)'s characterization of Q (Lemma 4) and Theorem 1 of Nikzad (2023) to conclude that for any extreme point q of $\overline{Q}$, q_1 must be an extreme point of a convex set of nondecreasing functions majorized by $\hat{q}_2$, which in turn implies that there exist countably many disjoint intervals $\{(\underline{z}_k, \overline{z}_k]\}_{k=1}^{\infty}$ such that $q_1 \equiv \hat{q}_2$ on $[0,1] \setminus \cup_{k=1}^{\infty} (\underline{z}_k, \overline{z}_k]$, and equals a step function with at most two steps on each $(\underline{z}_k, \overline{z}_k]$. We then switch the role of q_1 and q_2 and apply the symmetric argument to q_2 fixing $\hat{q}_1$. Second, we use these properties and the fact that (q_1, q_2) must be rationalizable by a mixture of two nested up-sets according to Theorem 2 to derive a set of necessary conditions that (q_1, q_2) must satisfy. In particular, we show that on each interval $(\underline{z}_k, \overline{z}_k]$, q_1 must be a constant and $\hat{q}_2$ must be a step function with only one jump (symmetrically, q_2 and $\hat{q}_1$ have the same feature). Moreover, we show that the intervals on which q_1 and q_2 are constants form a countable collection of disjoint "off-diagonal" rectangles on $[0,1]^2$. Third, for any such (q_1, q_2), we explicitly construct a monotone f that rationalizes it, and show that the monotone function f allows for a perturbation similar to the one we construct in Figure 3, unless it coincides with a mixture of two nested up-sets differing by at most a rectangle as depicted in Figure 4a.

For the unique rationalizability in Theorem 4, we first show that any extreme point (q_1, q_2) must be uniquely rationalized by the mixture of two nested up-sets that differ by a rectangle among all mixtures of two nested up-sets, using a similar argument as before, by constructing an additively separable function $\phi(x_1, x_2)$ such that our rationalization using a rectangle is the unique maximizer for the linear functional $\Phi : f \mapsto \int \phi f \, dx$ among the mixtures of two nested up-sets fixing the marginals. Now, for any monotone f that rationalizes (q_1, q_2), we can apply Theorem 1 and Choquet's integral representation theorem to represent f as an integral over the mixtures of two nested up-sets. Projecting that representation into marginals also gives an integral representation of (q_1, q_2). But since (q_1, q_2) is an extreme point, and hence has a unique integral representation, all such mixtures of two nested up-sets must have the same marginals and hence they all rationalize (q_1, q_2). But then all such mixtures, and hence f, must be identical to our construction using the rectangle.

Similar to Theorem 1, the conditions identified by Theorem 4 are also sufficient when the affine constraints have enough independence. Specifically, consider any pair ψ_1, ψ_2 of essentially bounded functions such that for any rectangle $D \subseteq [0,1]^2$ with a positive

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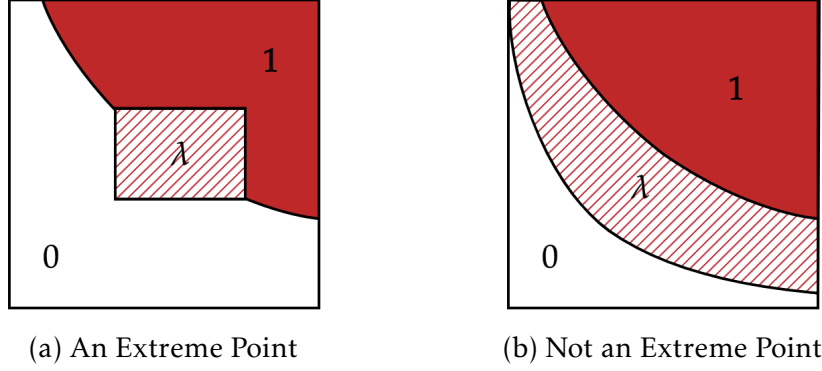(a) An Extreme Point   (b) Not an Extreme Point

Figure 4: Extreme Points of $\overline{\mathcal{Q}}$

measure,

$$\int_D (\psi_1(x_1) + \psi_2(x_2))\, dx \neq 0.$$

Fix any $\eta \in \mathbb{R}$. Let

$$\overline{\mathcal{Q}}^\star := \left\{ q \in \mathcal{Q} : \int_0^1 q_1(x_1)\psi_1(x_1)\, dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\, dx_2 = \eta \right\}.$$

We have the following characterization:

Proposition 3. $q = (q_1, q_2) \in \overline{\mathcal{Q}}^\star$ *is an extreme point if and only if q is rationalized by a mixture of $\mathbb{1}_A$ and $\mathbb{1}_{A'}$, where $A' \subseteq A \subseteq [0,1]^2$ are nested up-sets that differ by at most a rectangle. Moreover, every extreme point of $\overline{\mathcal{Q}}^\star$ is uniquely rationalized among all monotone functions.*

Figure 4 illustrates the extreme points of $\overline{\mathcal{Q}}$. Figure 4a depicts a monotone function $f = \lambda \mathbb{1}_A + (1 - \lambda)\mathbb{1}_{A'}$, where $A' \subseteq A$ are nested up-sets that differ by a rectangle. According to Theorem 4, any extreme point of $\overline{\mathcal{Q}}$ must be rationalized by some $f \in \mathcal{F}$ of this shape. Moreover, Proposition 3 states that any $q \in \overline{\mathcal{Q}}^\star$ rationalized by such a monotone function must be an extreme point of $\overline{\mathcal{Q}}^\star$. Meanwhile, Figure 4b depicts a monotone function $f = \lambda \mathbb{1}_A + (1 - \lambda)\mathbb{1}_{A'}$, where $A' \subseteq A$ are nested up-sets but do not differ by a rectangle. According to Theorem 4, because any extreme point of $\overline{\mathcal{Q}}$ is uniquely rationalized among monotone functions, this function can never rationalize an extreme point of $\overline{\mathcal{Q}}$, even though it might be an extreme point of $\overline{\mathcal{F}}$.

Projections under other measures. All of our results on rationalizable monotone functions can be readily extended to settings where rationalizability is defined under different (product) measures. For example, consider Theorem 3 and Theorem 4. Fix any pair of

CDFs G_1, G_2 that are continuous and have full supports on some compact intervals $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ respectively. We say that a pair of nondecreasing left-continuous functions $q = (q_1, q_2)$ is (G_1, G_2)-**rationalized** by a function $f : [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \to [0,1]$ if

$$q_1(x_1) = \int_{\underline{x}_2}^{\overline{x}_2} f(x_1, x_2) G_2(dx_2), \quad \text{and} \quad q_2(x_2) = \int_{\underline{x}_1}^{\overline{x}_1} f(x_1, x_2) G_1(dx_1),$$

for all $x_1 \in [\underline{x}_1, \overline{x}_1]$ and $x_2 \in [\underline{x}_2, \overline{x}_2]$. Let $\mathcal{Q}^{(G_1, G_2)}$ be the set of pairs of nondecreasing functions that are (G_1, G_2)-rationalizable. Furthermore, for any essentially bounded functions ψ_1, ψ_2, and for any $\eta \in \mathbb{R}$, let

$$\overline{\mathcal{Q}}^{(G_1, G_2)} := \left\{ q \in \mathcal{Q}^{(G_1, G_2)} : \int_0^1 q_1(x_1) \psi_1(x_1) G_1(dx_1) + \int_0^1 q_2(x_2) \psi_2(x_2) G_2(dx_2) \leqslant \eta \right\}.$$

By simply changing variables and letting $t_1 := G_1(x_1)$ and $t_2 := G_2(x_2)$, it follows immediately that, for any $f : [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \to [0,1]$,

$$\int_{\underline{x}_2}^{\overline{x}_2} f(x_1, x_2) G_2(dx_2) = \int_0^1 f(x_1, G_2^{-1}(t_2)) dt_2$$

and

$$\int_{\underline{x}_1}^{\overline{x}_1} f(x_1, x_2) G_1(dx_1) = \int_0^1 f(G_1^{-1}(t_1), x_2) dt_1.$$

Therefore, any $f : [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \to [0,1]$ uniquely corresponds to some $\tilde{f} : [0,1]^2 \to [0,1]$, where

$$\tilde{f}(t_1, t_2) = f(G_1^{-1}(t_1), G_2^{-1}(t_2)),$$

and any $q \in \mathcal{Q}^{(G_1, G_2)}$ uniquely corresponds to some $\tilde{q} \in \mathcal{Q}$, where

$$\tilde{q}_1(t_1) = q_1(G_1^{-1}(t_1)), \quad \text{and} \quad \tilde{q}_2(t_2) = q_2(G_2^{-1}(t_2)),$$

for all $(t_1, t_2) \in [0,1]^2$. By Theorem 3 and Theorem 4, we then immediately have the following corollaries.

Corollary 1. $q = (q_1, q_2)$ *is an extreme point of* $\mathcal{Q}^{(G_1, G_2)}$ *if and only if q is (G_1, G_2)-rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [\underline{x}_1, \overline{x}_2] \times [\underline{x}_1, \overline{x}_2]$. Moreover, every extreme point of* $\mathcal{Q}^{(G_1, G_2)}$ *is uniquely (G_1, G_2)-rationalized.*

Corollary 2. *Every extreme point of* $\overline{\mathcal{Q}}^{(G_1, G_2)}$ *is a mixture of $\mathbb{1}_A$ and $\mathbb{1}_{A'}$, where $A' \subseteq A \subseteq [\underline{x}_1, \overline{x}_1] \times [\underline{x}_1, \overline{x}_2]$ are nested up-sets that differ by at most a rectangle. Moreover, every extreme*

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point of $\overline{\mathcal{Q}}^{(G_1,G_2)}$ is uniquely (G_1,G_2)-rationalized among all monotone functions.

3.2 Extreme Points of Joint Majorization Sets

Theorem 3 and Theorem 4, together with the connection between rationalizability and majorization (Lemma 4), provide a characterization of extreme points of the set of pairs of monotone functions $q = (q_1, q_2)$ where q_1 is majorized by the conjugate of q_2.

Proposition 4. *q is an extreme point of the convex set*

$$\left\{ q : q_1 \preceq \hat{q}_2 \, ; q_1, q_2 \text{ are nondecreasing, left-continuous} \right\}$$

if and only if $q_1 = \hat{q}_2$.

The structure of the extreme points characterized by Proposition 5 differs considerably from the ones characterized by Kleiner, Moldovanu, and Strack (2021), who consider the set of nondecreasing functions that majorize, or are majorized by, a *fixed* nondecreasing function. In particular, when the majorization upper and lower bounds are endogenous and linked via the conjugate/inverse relation, the extreme points must be the ones where the majorization constraints bind everywhere, rather than those given by a combination of binding monotonicity and majorization constraints.

Proposition 5. *For any pair of essentially bounded functions $\{\psi_1, \psi_2\}$ on $[0,1]$ and any $\eta \in \mathbb{R}$, if q is an extreme point of the convex set*

$$\left\{ q : q_1 \preceq \hat{q}_2 \, ; q_1, q_2 \text{ are nondecreasing, left-continuous}; \right.$$
$$\left. \int_0^1 q_1(x_1)\psi_1(x_1)\,\mathrm{d}x_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\,\mathrm{d}x_2 \leqslant \eta \right\},$$

then there exist $\underline{z} \leqslant \overline{z}$, and $q_1(\underline{z}) \leqslant \underline{\gamma} \leqslant \overline{\gamma} \leqslant q_1(\overline{z}^+)$, and $\lambda \in (0,1)$, such that $q_1(z) = \hat{q}_2(z)$ for all $z \in [0,1] \backslash (\underline{z}, \overline{z}]$ and otherwise

$$q_1(z) = \lambda \overline{\gamma} + (1-\lambda)\underline{\gamma}, \quad \text{and} \quad \hat{q}_2(z) = \begin{cases} \underline{\gamma}, & \text{if } z \in (\underline{z}, \ (1-\lambda)\overline{z} + \lambda\underline{z}] \\ \overline{\gamma}, & \text{if } z \in ((1-\lambda)\overline{z} + \lambda\underline{z}, \ \overline{z}] \end{cases}.$$

Similar to Proposition 4, the structure of the extreme points given by Proposition 5 is considerably different from the ones identified by Nikzad (2023) and Candogan and Strack (2023), who consider the set of nondecreasing functions that majorize, or are majorized by, a *fixed* nondecreasing function, subject to finitely many linear constraints.

In particular, rather than possibly having multiple intervals on which the monotonicity constraints bind, the extreme points given by Proposition 5 have only a single interval on which the monotonicity constraints bind, whereas the majorization constraints bind everywhere else.

From Proposition 4, another immediate consequence is a necessary condition for the extreme points of the weak joint majorization set where q_1 is weakly majorized by the conjugate of q_2.

Proposition 6. *Every extreme point of the convex set*

$$\left\{ q : q_1 \preceq_w \hat{q}_2 \, ; q_1, q_2 \text{ are nondecreasing, left-continuous} \right\}$$

satisfies $q_1 = \hat{q}_2 \mathbb{1}_{[k,1]}$ *for some* $k \in [0, 1]$.

4 Economic Applications

In this section, we apply our abstract results in Section 2 and Section 3 to a series of economic applications. Section 4.1 studies public good provision; Section 4.2 studies bilateral trade; Section 4.3 studies the limits of mechanism equivalence; Section 4.4 studies reduced form auctions; Section 4.5 studies private private information.

4.1 Public Good with Interdependent Values and Correlated Types

Consider a classic public good provision problem. There are n agents. A designer wants to decide whether to implement a public project.

We allow for interdependent values and correlated types. Each agent i has a willingness-to-pay $v_i(s_i, s_{-i})$ for the project being implemented, where $s_i \in S_i \subset \mathbb{R}$ is the private signal received by the agent i, and s_{-i} are the private signals received by the other agents. We assume that S_i is a compact interval. We allow the joint distribution of signals $s := (s_1, \ldots, s_n)$ to be correlated and only assume that it admits a positive density g on $S_1 \times \cdots \times S_n$. We assume that the agent i's value function $v_i(s_i, s_{-i})$ is bounded, and also continuously differentiable and increasing in the agent i's signal s_i. Monetary transfers are allowed, and the agents have quasilinear preferences. Thus, given a probability α for implementing the project, and transfer t_i, agent i's ex-post payoff is

$$\alpha \cdot v_i(s_i, s_{-i}) - t_i.$$

The designer wants to maximize the expected surplus subject to ex-ante budget balance. In particular, implementing the project would cost the designer $c \in \mathbb{R}$, and hence the designer wants to raise enough money to cover the cost in expectation:

$$\mathbb{E}\Big[\sum_i t_i(s)\Big] \geq \mathbb{E}\Big[c \cdot \alpha(s)\Big]. \qquad \text{(Budget Balance)}$$

The designer can design any mechanism subject to ex-post incentive compatibility (ex-post IC) and ex-post individual rationality (ex-post IR) constraints. By the revelation principle, it is without loss of generality to focus on *(direct, incentive-compatible) mechanisms* $(\alpha, t) : S_1 \times \cdots \times S_n \to [0,1] \times \mathbb{R}^n$ that satisfy: for all i

$$\alpha(s)v_i(s) - t_i(s) \geq \alpha(\hat{s}_i, s_{-i})v_i(s_i, s_{-i}) - t_i(\hat{s}_i, s_{-i}) \qquad \text{for all } s_i, \hat{s}_i \text{ and } s_{-i}; \qquad \text{(ex-post IC)}$$

$$\alpha(s)v_i(s) - t_i(s) \geq 0 \qquad \text{for all } s. \qquad \text{(ex-post IR)}$$

Therefore, the designer's problem is given by

$$\max_{(\alpha,t)} \mathbb{E}\left[\alpha(s)\Big(\sum_i v_i(s) - c\Big)\right]$$

subject to the ex-ante budget balance, and ex-post IC and IR constraints. By the logic of Crémer and McLean (1985), it is well known that if we relax the ex-post incentive constraints to be interim incentive constraints, then the first best is generically implementable via full surplus extraction.[14] However, such mechanisms are generally fragile and depend on the exact knowledge of the designer (Brooks and Du 2023).

A direct application of our main results shows that with ex-post IC and IR constraints, the optimal mechanism has a simple structure. We say that a mechanism (α, t) is a *two-threshold policy* if there exist a monotone function $\phi : [0,1]^n \to [0,1]$ and three constants $p, k_L, k_H \in [0,1]$ where $k_L < k_H$ such that

$$\alpha(s) = \begin{cases} 1 & \text{if } \phi(s) \geq k_H, \\ p & \text{if } k_H > \phi(s) \geq k_L, \\ 0 & \text{otherwise}. \end{cases}$$

Such a mechanism can be implemented by aggregating the signals of each agent into a

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[14]In fact, Crémer and McLean (1985) shows that under stronger conditions, full surplus extraction is possible even with ex-post IC and interim IR mechanisms.

score $\phi(s)$ and comparing the score to two thresholds k_L and k_H: if the score is higher than the high threshold, then the project is implemented for sure; if it is lower than the high threshold but higher than the low threshold, the project is implemented with some probability p; otherwise, the project is abandoned.

Proposition 7. *There exists an optimal mechanism that is a two-threshold policy.*

Proof of Proposition 7. For any agent i, since $v_i(s_i, s_{-i})$ is increasing in s_i, by a standard argument, we know that $\alpha(s_i, s_{-i})$ is ex-post IC implementable for agent i if and only if $\alpha(\cdot, s_{-i})$ is nondecreasing in s_i. Moreover, it is without loss of generality to normalize $S_i = [0,1]$. Therefore, $\alpha : [0,1]^n \to [0,1]$ is ex-post IC implementable if and only if it is a multidimensional monotone function. Moreover, by ex-post IC and the Envelope theorem, we also have that the indirect utility functions $U_i(s) := \alpha(s)v_i(s) - t_i(s)$ must satisfy:

$$U_i(s) = U_i(0, s_{-i}) + \int_0^{s_i} \alpha(z, s_{-i}) \frac{\partial}{\partial s_i} v_i(z, s_{-i}) \, dz.$$

Thus,

$$t_i(s) = \alpha(s)v_i(s) - \int_0^{s_i} \alpha(z, s_{-i}) \frac{\partial}{\partial s_i} v_i(z, s_{-i}) \, dz - U_i(0, s_{-i}).$$

By ex-post IR, we have $U_i(0, s_{-i}) \geqslant 0$. We may assume without loss of generality that $U_i(0, s_{-i}) = 0$, since we can always define $\tilde{t}_i(s) = t_i(s) + U_i(0, s_{-i})$ and the resulting mechanism $(\alpha, \tilde{t})$ would continue to satisfy budget balance, ex-post IR and IC. Therefore, the designer's problem is equivalent to

$$\max_{\alpha \in \overline{\mathcal{F}}} \mathbb{E}\left[\alpha(s) \left(\sum_i v_i(s) - c \right) \right]$$

where $\overline{\mathcal{F}}$ is the set of monotone functions $\alpha : [0,1]^n \to [0,1]$ satisfying the constraint

$$\mathbb{E}\left[\sum_i \alpha(s)v_i(s) - \int_0^{s_i} \alpha(z, s_{-i}) \frac{\partial}{\partial s_i} v_i(z, s_{-i}) \, dz - c \cdot \alpha(s) \right] \geqslant 0.$$

By Bauer's maximum principle, we immediately know that the above problem admits a solution that is an extreme point of $\overline{\mathcal{F}}$. Let α^* be such a solution. By Theorem 1, we know that $\alpha^* = \mathbb{1}_{A_1} + p\mathbb{1}_{A_2 \setminus A_1}$ for two nested up-sets $A_1 \subseteq A_2$. Clearly, this mechanism can be represented via a two-threshold policy (e.g. set $\phi = \alpha^*$, $k_L = p$, and $k_H = 1$). $\qquad\square$

The specifics of the two-threshold policy in Proposition 7 of course depend on the details of the environment, but Proposition 7 shows the optimal mechanism always has
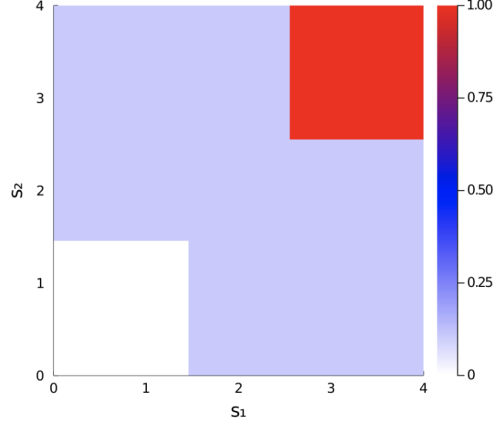
Figure 5: Illustration of optimal allocation rule $\alpha(s_1, s_2)$ for Example 1

this simple feature regardless of how the signals are correlated and how the preferences are interdependent.

To illustrate Proposition 7, consider the following example:

Example 1. There are two agents. There is a public good. There are positive externalities of their consumption, captured by the value functions: for all i,

$$v_i(s_i, s_{-i}) = s_i + w \cdot s_{-i},$$

where $w > 0$. Their signals (s_1, s_2) are also positively correlated and distributed according to a truncated, joint log-normal distribution. Proposition 7 immediately applies and shows that a two-threshold policy would be optimal. To illustrate, suppose the parameters are such that $w = 0.1$, and the log-normal distribution is truncated on $[0, 4]^2$, centered at $(2, 2)$, with $(\log(s_1), \log(s_2))$ having a standard deviation 0.4 and correlation 0.5. Figure 5 illustrates the optimal mechanism computed numerically (by discretizing the type space into a 30-by-30 grid). In this example, the optimal mechanism asks each agent to submit their signals s_i, and implements the project for sure if both signals are very high, abandons the project for sure if both signals are very low, and otherwise implements the project with a constant, interior probability. □

4.2   Interim Efficient Frontier of Bilateral Trade

Consider a classic bilateral trade problem as in Myerson and Satterthwaite (1983). There is a single good. A buyer privately observes their value $v \in [\underline{v}, \overline{v}]$, drawn from a continuous distribution G_B. A seller privately observes their cost $c \in [\underline{c}, \overline{c}]$, drawn from a continuous distribution G_S. For this application, we assume that v and c are independent.
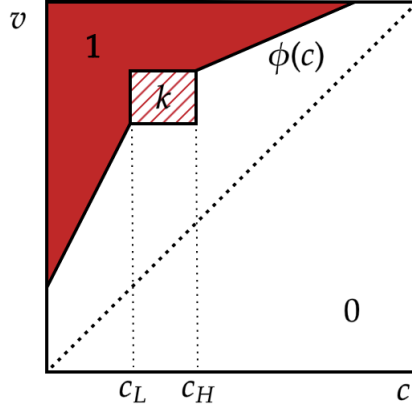
Figure 6: A Markup-Pooling Mechanism

By the revelation principle, it is without loss of generality to focus on *(direct, incentive-compatible) mechanisms* $(p,t) : [\underline{v}, \overline{v}] \times [\underline{c}, \overline{c}] \to [0,1] \times \mathbb{R}$, specifying the probabilities of trade and the expected payments made from buyer to the seller, that satisfy the Bayesian incentive compatibility and Bayesian individual rationality constraints:

$$\mathbb{E}_c \left[p(v,c)v - t(v,c) \right] \geq \mathbb{E}_c \left[p(\hat{v},c)v - t(\hat{v},c) \right] \qquad \text{for all } v, \hat{v}\,; \qquad \text{(Buyer IC)}$$

$$\mathbb{E}_v \left[t(v,c) - p(v,c)c \right] \geq \mathbb{E}_v \left[t(v,\hat{c}) - p(v,\hat{c})c \right] \qquad \text{for all } c, \hat{c}\,; \qquad \text{(Seller IC)}$$

$$\mathbb{E}_c \left[p(v,c)v - t(v,c) \right] \geq 0 \qquad \text{for all } v\,; \qquad \text{(Buyer IR)}$$

$$\mathbb{E}_v \left[t(v,c) - p(v,c)c \right] \geq 0 \qquad \text{for all } c\,. \qquad \text{(Seller IR)}$$

For a mechanism M, let $U_B(v)$ and $U_S(c)$ be the interim utility functions for the buyer and the seller. We are interested in characterizing the *interim efficient frontier* (Holmström and Myerson 1983), which is equivalent to finding the solutions to weighted welfare maximization

$$\max_{(p,t)} \left[\int_0^1 U_B(v) \Lambda_B(\mathrm{d}v) + \int_0^1 U_S(c) \Lambda_S(\mathrm{d}c) \right]$$

for all possible *welfare weights* Λ_B and Λ_S, which are Borel measures on $[0,1]$, subject to the IC and IR constraints.

We say that a mechanism (p,t) is a *markup-pooling* mechanism if there exists a non-decreasing function ϕ, an interval $I = [c_L, c_H]$, and a constant $k \in [0,1]$ such that

- If $c \notin I$, then trade happens if and only if $v \geq \phi(c)$.

- Otherwise, let $\tilde{c} = c_L$ with probability k and $\tilde{c} = c_H$ with probability $1 - k$. Trade happens if and only if $v \geq \phi(\tilde{c})$.

A markup-pooling mechanism is illustrated by Figure 6. In such a mechanism, trade is implemented if and only if the value v is above a monotone transformation of the cost $\phi(c)$ (the ***markup function***), with the exception that when c falls into a specific interval I (the ***pooling interval***), we simply *resample* the cost from the two ends of the interval and execute the trade if the buyer's value v is above the marked up, resampled cost $\phi(\tilde{c})$.

Proposition 8. *For any welfare weights Λ_B and Λ_S, there exists a markup-pooling mechanism that maximizes the expected welfare.*

Proof of Proposition 8. Let $\mathcal{M}$ be the set of IC and IR mechanisms. Let

$$\mathcal{U} := \left\{ (U_B^M, U_S^M) : M \in \mathcal{M} \right\}$$

be the set of implementable interim utility functions. We claim that any extreme point of $\mathcal{U}$ must be implementable by a markup-pooling mechanism. The result follows immediately from this claim and Bauer's maximum principle, since the expected welfare is a continuous linear functional on $\mathcal{U}$ by definition.

Clearly, any extreme point in $\mathcal{U}$ must be an extreme point in $\mathcal{U}_z := \big\{ (U_B^M, U_S^M) : M \in \mathcal{M}$ and $U_B^M(\underline{v}) = z \big\}$ for some $z \in \mathbb{R}_+$. We show that for any $z \in \mathbb{R}_+$, the extreme points of $\mathcal{U}_z$ can be implemented with a markup-pooling mechanism. By Myerson and Satterthwaite (1983), we may relax the ex-post budget constraint to be ex-ante budget constraint without affecting $\mathcal{U}_z$. In particular, let

$$q_1(v) = \int p(v,c) G_S(dc), \quad q_2(c) = \int p(v,c) G_B(dv).$$

The ex-ante budget constraint is simply that

$$\Pi(q_1, q_2) := \int \mathrm{MR}(v) q_1(v) G_B(dv) - \int \mathrm{MC}(c) q_2(c) G_S(dc) = U_B(\underline{v}) + U_S(\overline{c}),$$

where $\mathrm{MR}(v) := v - \frac{1 - G_B(v)}{g_B(v)}$ and $\mathrm{MC}(c) := c + \frac{G_S(c)}{g_S(c)}$, with g_i being the density of G_i. By Myerson and Satterthwaite (1983), we can write

$$U_B(v) = \int_{\underline{v}}^{v} q_1(s) \, ds + \underbrace{U_B(\underline{v})}_{=z} \qquad U_S(c) = \int_{c}^{\overline{c}} q_2(s) \, ds + \underbrace{U_S(\overline{c})}_{=\Pi(q_1,q_2)-z} .$$

Now consider

$$\mathcal{Q} := \Big\{ (q_1, q_2) : (G_B, G_S)\text{-rationalizable}; \ q_1 \text{ is nondecreasing and } q_2 \text{ is nonincreasing} \Big\}$$

and

$$\overline{\mathcal{Q}}_z := \big\{(q_1, q_2) \in \mathcal{Q} : \Pi(q_1, q_2) \geqslant z\big\}.$$

Note that $\Pi(q_1, q_2)$ is a continuous linear functional of (q_1, q_2). Therefore, for any $z \in \mathbb{R}_+$, we have that the map between $\overline{\mathcal{Q}}_z$ and $\mathcal{U}_z$ defined by

$$L_z[q_1, q_2](v, c) := \left(\int_{\underline{v}}^{v} q_1(s)\, ds + z, \int_c^{\overline{c}} q_2(s)\, ds + \Pi(q_1, q_2) - z \right)$$

is a continuous linear map. Moreover, $\mathcal{U}_z = L_z(\overline{\mathcal{Q}}_z)$. It follows by Lemma 3 that any extreme point of $\mathcal{U}_z$ must be implementable by an extreme point of $\overline{\mathcal{Q}}_z$. By Corollary 2, the extreme points must be

$$\mathbb{1}_{A_1} + \lambda \mathbb{1}_{A_2 \backslash A_1}$$

where $A_1 \subseteq A_2$ are two nested up-sets in the space of $(v, -c)$, and $A_2 \backslash A_1$ is a rectangle. Now, note that any such mechanism can be implemented by a markup-pooling mechanism, concluding the proof. $\square$

As the proof of Proposition 8 shows, the class of markup-pooling mechanisms can in fact attain any extreme point of the set of interim utility functions. An immediate consequence is that *any* trading mechanism is payoff-equivalent to a randomization over markup-pooling mechanisms. Moreover, by our unique rationalizability result (see Theorem 4), it also follows that if a ***dominant-strategy incentive compatible (DIC)*** mechanism can attain any such extreme point, then it must be a markup-pooling mechanism.

Proposition 9. *Any mechanism is payoff-equivalent to a randomization over markup-pooling mechanisms. Moreover, if a DIC mechanism attains an extreme point of the set of feasible interim payoffs, then it must be a markup-pooling mechanism.*

Proof of Proposition 9. By the proof of Proposition 8, we know that any extreme point of the set of interim utility functions $\mathcal{U}$ must be implementable with a markup-pooling mechanism. Now fix any mechanism M. Let $V \in \mathcal{U}$ be the interim utility pairs generated by M. By Choquet's theorem, there exists $\mu \in \Delta(\text{ext}(\mathcal{U}))$ such that

$$V = \int_{\text{ext}(\mathcal{U})} U\, \mu(dU).$$

But then randomizing over the markup-pooling mechanisms that implement $\text{ext}(\mathcal{U})$ according to μ generates a payoff-equivalent mechanism.

Now we prove the second part. Fix any DIC mechanism (p, t) that attains any extreme point of $\mathcal{U}$. Let $z = U_B(\underline{v})$. Clearly, the mechanism (p, t) must also attain an extreme
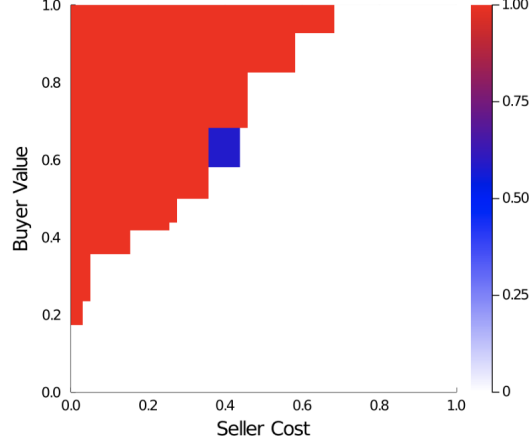
Figure 7: Illustration of interim-efficient allocation rule $p(v,c)$ for Example 2

point of $\mathcal{U}_z$ where $\mathcal{U}_z$ is defined as the set of interim payoffs where the buyer of type $\underline{v}$ obtains payoff z, as in the proof of Proposition 8. Now, by the proof of Proposition 8, any extreme point of $\mathcal{U}_z$ must be implementable by some extreme point (q_1, q_2) of the set $\overline{\mathcal{Q}}_z$ defined there. Since (q_1, q_2) are pinned down by (U_B, U_S) as their derivatives, the interim allocation rules of the mechanism (p, t) must be an extreme point of $\overline{\mathcal{Q}}_z$. Moreover, we know that they must be (G_B, G_S)-rationalized by $p(v,c)$ which is monotone in $(v, -c)$ since (p, t) is DIC. By Theorem 4 (and Corollary 2), any extreme point of $\overline{\mathcal{Q}}_z$ must be uniquely (G_B, G_S)-rationalized among all monotone functions, and hence $p(v,c)$ must be identical to a mixture of two indicator functions defined on two nested up-sets that differ by a rectangle. Thus, (p, t) is a markup-pooling mechanism. $\qquad\square$

Example 2. To illustrate the markup-pooling mechanisms and Proposition 8, we provide a numerical example. Consider the case where $v, c \in [0, 1]$. Figure 7 depicts a numerically computed optimal mechanism in a 50-by-50 discretized grid, where we uniformly sample a generic probability density and a generic welfare weight for both the buyer and the seller. The blue rectangle in Figure 7 is the region where the trade happens with a constant, interior probability. The allocation rule can be implemented using a markup-pooling mechanism, where the markup function is the lower boundary of the larger up-set (the union of the red up-set and the blue rectangle). The pooling interval is the interval of the seller's types that fall into the blue rectangle. $\qquad\square$

An alternative approach to solve for the optimal mechanism is to dualize the ex-ante budget constraint and iron two "virtual type functions" that depend on the dual variable. In some special cases depending on the welfare weights and type distributions, ironing is not needed and the optimal allocation rule is an indicator function defined on an up-set

(i.e. only consists of the markup function), as in Myerson and Satterthwaite (1983) and Ledyard and Palfrey (1999). However, for generic welfare weights and type distributions, ironing is needed which, without our results, could only deduce that an optimal allocation rule is a monotone function that equals one on some up-set but may take different interior values on a countable collection of rectangles.[15] Our extreme point characterization refines the ironing approach by showing that the trading probabilities are all equal to one *constant* on these rectangles and in fact, there is only a *single* rectangle needed in the optimal allocation rule. Together, these two features characterize our markup-pooling mechanisms.

4.3 Mechanism Anti-Equivalence

Consider a standard social choice environment with linear utilities and independent, one-dimensional, private types as in Gershkov et al. (2013). Suppose that we have two agents $\mathcal{I} = \{1,2\}$ and two alternatives $\mathcal{K} = \{1,2\}$. Each agent i privately observes their type $\theta_i \in \Theta_i \subset \mathbb{R}$, where Θ_i is a compact interval and θ_i follows some continuous distribution G_i. Agent i's utility for alternative k is given by

$$u_i^k(\theta_i, t_i) = a_i^k \theta_i + c_i^k - t_i$$

where $a_i^k, c_i^k \in \mathbb{R}$ are constants with $a_i^1 \neq a_i^2$, and $t_i \in \mathbb{R}$ is a monetary transfer. Examples of such an environment include the classic public good provision and bilateral trade problems. By the revelation principle, it is without loss of generality to consider direct mechanisms

$$(p,t) : \Theta_1 \times \Theta_2 \to \Delta(\mathcal{K}) \times \mathbb{R}^2$$

that specify the probabilities of implementing each alternative and the transfers from each agent for a given profile of type report.

A mechanism (p,t) is ***Bayesian incentive compatible (BIC)*** if truthful reporting by all agents is a Bayes-Nash equilibrium. A mechanism (p,t) is ***dominant strategy incentive compatible (DIC)*** if truthful reporting is a dominant strategy equilibrium. A mechanism

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[15]This is because, given the endogenous dual variable, there would generally be many optimal solutions to the ironed Lagrangian and one needs to carefully select an optimal solution that is *feasible* with respect to the budget constraint and may assign arbitrary values on a countable collection of rectangles. In a related but different problem where there are no such constraints, Loertscher and Wasser (2019) apply an ironing approach to a partnership model where each agent has an initial endowment and can endogenously be either a buyer or a seller. Their ironing approach irons around interior worst-off types which, under regularity conditions, implies the optimal mechanism can have one pooling rectangle due to the countervailing incentives.

(p, t) is *deterministic* if $p(\theta)$ is a deterministic allocation for all θ.

We study the question of interim equivalence between mechanisms initiated by Manelli and Vincent (2010). Two BIC mechanisms (p, t) and $(\tilde{p}, \tilde{t})$ are *payoff equivalent* if they yield the same interim expected utilities for all agents and same ex-ante expected surplus.[16] From a series of fundamental contributions (Manelli and Vincent 2010; Gershkov et al. 2013; Chen et al. 2019), we now know that for any BIC mechanism, there exists a payoff-equivalent DIC mechanism (*BIC-DIC equivalence*); for any stochastic mechanism, there exists a payoff-equivalent deterministic mechanism (*stochastic-deterministic equivalence*). These two equivalence results suggest for a wide range of objectives, it is without loss to consider DIC mechanisms which are strategically simpler and robust to changes in the agents' beliefs; similarly, it is without loss to consider deterministic mechanisms which do not require a verifiable randomization device.

However, as an application of our abstract results, we show that there is a strong conflict between asking for strategically robust implementations and asking for verifiable deterministic mechanisms. Say that two payoff-equivalent mechanisms are *ex-post equivalent* if they implement the same ex-post allocation probabilities for all type profiles (up to measure zero).

Proposition 10. *A BIC mechanism is payoff-equivalent to a deterministic DIC mechanism if and only if they are ex-post equivalent.*

Proof of Proposition 10. For any BIC mechanism (p, t), by the Envelope theorem,

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \sum_k a_i^k q_i^k(s_i) \, \mathrm{d}s_i,$$

where $U_i(\theta_i)$ denotes the interim payoff for type θ_i, and $q_i^k(\theta_i)$ denotes the interim allocation probability of alternative k for type θ_i. Thus, the payoff equivalence between two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ implies that for all i and θ_i

$$\sum_k a_i^k q_i^k(\theta_i) = \sum_k a_i^k \tilde{q}_i^k(\theta_i).$$

Since $\sum_k q_i^k(\theta_i) = 1$ and we have two alternatives, the above implies that

$$q_i^k(\theta_i) = \tilde{q}_i^k(\theta_i),$$

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[16]Because we have two alternatives, this equivalence notion is the same as asking the two mechanisms to have the same interim expected allocation probabilities.

for all k, i, and θ_i.

Now, note that for any BIC mechanism (p, t), by a standard argument, we also have $\sum_k a_i^k q_i^k(\theta_i)$ is nondcreasing in θ_i. Because we have two alternatives, this is equivalent to $a_i^2 + (a_i^1 - a_i^2) q_i^1(\theta_i)$ being nondecreasing in θ_i. Thus, for $a_i^1 > a_i^2$, this is equivalent to $q_i^1(\theta_i)$ being nondecreasing, and otherwise this is equivalent to $q_i^1(\theta_i)$ being nonincreasing. If $a_i^1 > a_i^2$, let

$$\widehat{q}_i^1(s_i) := q_i^1(G_i^{-1}(s_i)),$$

and otherwise let

$$\widehat{q}_i^1(s_i) := q_i^1(G_i^{-1}(1 - s_i)).$$

Then $\widehat{q}_i^1$ must be nondecreasing in s_i. Define analogously $\widehat{p}^1(s_1, s_2)$, where we apply the transformation $G_i^{-1}(\cdot)$ or $G_i^{-1}(1 - \cdot)$ to each argument of $p^1(\theta_1, \theta_2)$ depending on whether $a_i^1 > a_i^2$. For any BIC mechanism (p, t), clearly the monotone pair $(\widehat{q}_1^1, \widehat{q}_2^1)$ must be rationalized by $\widehat{p}^1$. Moreover, for any DIC mechanism (p, t), by a similar argument, we also have that $\widehat{p}^1$ must be monotone.

Fix two payoff-equivalent mechanisms (p, t) and $(\tilde{p}, \tilde{t})$. Their corresponding transformed interim allocation rules $(\widehat{q}_1^1, \widehat{q}_2^1)$ and $(\widehat{\tilde{q}}_1^1, \widehat{\tilde{q}}_2^1)$ must coincide. Now, if (p, t) is deterministic *and* DIC, then by the above argument $\widehat{p}^1$ must be an indicator function defined on an up-set. Thus, by Theorem 3, $(\widehat{q}_1^1, \widehat{q}_2^1)$ is an extreme point of the set of monotone rationalizable pairs $\mathcal{Q}$, and hence must be uniquely rationalized — thus, we have $\widehat{\tilde{p}}^1 \equiv \widehat{p}^1$. Thus, the two mechanisms are ex-post equivalent. $\square$

Another way to state Proposition 10 is that every deterministic DIC mechanism must yield interim payoffs that differ from those of every mechanism that is non-deterministic or non-DIC (in the sense of ex-post allocations). Proposition 10 shows that, in a strong sense, the BIC-DIC equivalence results of Manelli and Vincent (2010) and Gershkov et al. (2013) necessarily rely on non-deterministic mechanisms. Similarly, the stochastic-deterministic equivalence result of Chen et al. (2019) necessarily relies on non-DIC mechanisms.[17]

4.4 Asymmetric Reduced Form Auctions

Consider a classic auction setting á la Myerson (1981) with two bidders. There is one seller and one good. Each bidder i privately observes their value v_i for the object, drawn

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[17] Chen et al. (2019) provide an example to show that not every stochastic BIC mechanism is payoff-equivalent to a deterministic DIC mechanism. Proposition 10 shows that this is in fact never possible in the two-agent, two-alternative setting.

from a distribution G_i. We assume that v_1 and v_2 are independent, but they can be drawn from two different distributions.

By the revelation principle, it is without loss of generality to focus on *(direct, incentive-compatible) mechanisms* $(p_1, p_2, t_1, t_2) : [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2] \rightarrow [0,1]^2 \times \mathbb{R}^2$, specifying the probabilities of allocating the good to bidder 1 and bidder 2 as well as the monetary transfers they make to the seller, that satisfy the Bayesian incentive compatibility, Bayesian individual rationality constraints, and feasibility constraint:

$$\mathbb{E}_{v_{-i}} \left[p_i(v) v_i - t_i(v) \right] \geqslant \mathbb{E}_{v_{-i}} \left[p_i(\hat{v}_i, v_{-i}) v_i - t_i(\hat{v}_i, v_{-i}) \right] \qquad \text{for all } i, v_i, \hat{v}_i; \qquad \text{(BIC)}$$

$$\mathbb{E}_{v_{-i}} \left[p_i(v) v_i - t_i(v) \right] \geqslant 0 \qquad \text{for all } i, v_i; \qquad \text{(BIR)}$$

$$\sum_i p_i(v) \leqslant 1 \qquad \text{for all } v. \qquad \text{(Feasibility)}$$

A pair of monotone functions (q_1, q_2), where $q_i : [\underline{v}_i, \overline{v}_i] \rightarrow [0,1]$, is a **reduced-form auction** if there exists a mechanism (p, t) such that for all i

$$q_i(v_i) = \int p_i(v_i, v_{-i}) G_{-i}(\mathrm{d}v_{-i}).$$

An immediate consequence of our main result is a characterization of the reduced-form auctions with two bidders and their extreme points:

Proposition 11. (q_1, q_2) *is a reduced-form auction if and only if the associated quantile allocations* $\tilde{q}_i(s) := q_i(G_i^{-1}(s))$ *satisfy*

$$\tilde{q}_1 \preceq_w \tilde{q}_2^{-1}$$

with every extreme point satisfying

$$\tilde{q}_1 \equiv \tilde{q}_2^{-1} \mathbb{1}_{[k,1]}$$

for some $k \in [0,1]$. Moreover, every extreme reduced-form auction can be implemented with a deterministic and DIC mechanism.

Proof of Proposition 11. We start by showing that any reduced form auction (q_1, q_2) must satisfy $\tilde{q}_1 \preceq_w \tilde{q}_2^{-1}$. Let

$$\overline{q}_1(x_1) := \int_0^1 [1 - p_1(G_1^{-1}(x_1), G_2^{-1}(x_2))] \, \mathrm{d}x_2, \qquad \overline{q}_2(x_2) := \int_0^1 [1 - p_1(G_1^{-1}(x_1), G_2^{-1}(x_2))] \, \mathrm{d}x_1.$$

Let $\overline{q}_i^{\uparrow}$ be the monotone rearrangement of $\overline{q}_i$. By construction, $(\overline{q}_1^{\uparrow}, \overline{q}_2^{\uparrow})$ is rationalizable

and hence by Lemma 4, we have

$$\int_s^1 \overline{q}_1^{\uparrow}(z)\,dz \leqslant \int_s^1 1 - (\overline{q}_2^{\uparrow})^{-1}(1-z)\,dz$$

with equality when $s = 0$. Thus,

$$\int_0^s \overline{q}_1^{\uparrow}(z)\,dz \geqslant \int_0^s 1 - (\overline{q}_2^{\uparrow})^{-1}(1-z)\,dz$$

By definition of $\overline{q}_1^{\uparrow}$, and the monotonicity of $\tilde{q}_1$, we have $\overline{q}_1^{\uparrow}(z) = 1 - \tilde{q}_1(1-z)$ and hence

$$\int_0^s [1 - \tilde{q}_1(1-z)]\,dz \geqslant \int_0^s [1 - (\overline{q}_2^{\uparrow})^{-1}(1-z)]\,dz$$

which is equivalent to

$$\int_{1-s}^1 \tilde{q}_1(z)\,dz \leqslant \int_{1-s}^1 (\overline{q}_2^{\uparrow})^{-1}(z)\,dz.$$

But since $p_1 + p_2 \leqslant 1$, we have $\overline{q}_2 \geqslant \tilde{q}_2$ and hence $\overline{q}_2^{\uparrow} \geqslant \tilde{q}_2^{\uparrow} = \tilde{q}_2$, where the equality is due to the monotonicity of $\tilde{q}_2$. Thus, $(\overline{q}_2^{\uparrow})^{-1} \leqslant (\tilde{q}_2)^{-1}$ and hence

$$\int_{1-s}^1 \tilde{q}_1(z)\,dz \leqslant \int_{1-s}^1 (\tilde{q}_2)^{-1}(z)\,dz.$$

Since this holds for all s, we have $\tilde{q}_1 \preceq_w \tilde{q}_2^{-1}$.

Now, for the other direction, consider the set

$$\tilde{\mathcal{Q}} := \left\{ (\tilde{q}_1, \tilde{q}_2) : \tilde{q}_1 \preceq_w \tilde{q}_2^{-1} ; \tilde{q}_1, \tilde{q}_2 \text{ are nondecreasing and left-continuous} \right\}.$$

By Proposition 6, we know that $\tilde{\mathcal{Q}}$ is convex and every extreme point of $\tilde{\mathcal{Q}}$ satisfies

$$\tilde{q}_1 \equiv \tilde{q}_2^{-1} \mathbb{1}_{[k,1]}$$

for some $k \in [0,1]$.[18] We show that any (q_1, q_2) satisfying the above is a reduced-form

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[18]Indeed, consider the convex set $\mathcal{Q}^w := \{(q_1, q_2) : q_1 \preceq_w \hat{q}_2 ; q_1, q_2 \text{ are nondecreasing, left-continuous}\}$ in Proposition 6, and the linear map L defined by $(q_1, q_2) \mapsto (q_1, 1 - q_2(1 - \cdot))$. Let $\check{q}_2(x_2) := 1 - q_2(1 - x_2)$. Note that $(\check{q}_2)^{-1} = \hat{q}_2$. Thus, $q_1 \preceq_w \hat{q}_2$ if and only if $q_1 \preceq_w \check{q}_2^{-1}$. Therefore, $L(\mathcal{Q}^w) = \{(q_1, q_2) : q_1 \preceq_w q_2^{-1} ; q_1, q_2 \text{ are nondecreasing, left-continuous}\} = \tilde{\mathcal{Q}}$. By Lemma 3, we have $\text{ext}(L(\mathcal{Q}^w)) \subseteq L(\text{ext}(\mathcal{Q}^w))$. Thus, $\tilde{\mathcal{Q}}$ is convex and every extreme point of $\tilde{\mathcal{Q}}$ satisfies $q_1 \equiv q_2^{-1} \mathbb{1}_{[k,1]}$ for some $k \in [0,1]$ by Proposition 6.

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auction. In particular, let

$$\tilde{p}_1(x_1, x_2) := \begin{cases} 1 & x_1 \geqslant \max\{k, \tilde{q}_2(x_2)\} \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\tilde{p}_2(x_1, x_2) := \begin{cases} 1 & x_1 < \tilde{q}_2(x_2) \\ 0 & \text{otherwise}. \end{cases}$$

Let $p_i(v_1, v_2) := \tilde{p}_i(G_1(v_1), G_2(v_2))$ be the allocation rules in the value space. One may verify that (p_1, p_2) induces (q_1, q_2) and hence (q_1, q_2) is a reduced-form auction. Moreover, note that (p_1, p_2) is deterministic and can be implemented as a DIC mechanism.

Now, take any (q_1, q_2) such that $(\tilde{q}_1, \tilde{q}_2) \in \tilde{\mathcal{Q}}$. By Choquet's theorem, we know that there exists $\mu \in \Delta(\text{ext}(\tilde{\mathcal{Q}}))$ such that

$$\tilde{q} = \int_{\text{ext}(\tilde{\mathcal{Q}})} q^\dagger \mu(\mathrm{d}q^\dagger).$$

For any $q^\dagger \in \text{ext}(\tilde{\mathcal{Q}})$, by the above construction, there exists $\tilde{p}(q^\dagger)$ that induces $q^\dagger$. It follows immediately that we can induce (q_1, q_2) by setting $p(v_1, v_2) := \overline{p}(G(v_1), G(v_2))$ where

$$\overline{p} := \int_{\text{ext}(\tilde{\mathcal{Q}})} \tilde{p}(q^\dagger) \mu(\mathrm{d}q^\dagger),$$

proving that (q_1, q_2) is a reduced-form auction. $\square$

Note that the extreme points characterized by Proposition 11 are different from the ones resulting from symmetric reduced-form auctions. For example, Kleiner, Moldovanu, and Strack (2021) show that for symmetric auctions, the symmetric extreme reduced-form allocation rules can have a countable collection of intervals where the object is randomly allocated among the bidders. The difference is due to that, when restricting to symmetric mechanisms, we cannot perturb a given reduced-form allocation rule using *asymmetric* mechanisms. Proposition 11 shows that once asymmetric mechanisms are allowed, we can implement every extreme reduced-form auction with a deterministic *and* DIC mechanism.[19]

An immediate consequence of Proposition 11 is the following:[20]

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[19]Note that this is only a feature of the extreme points — in general there is no mechanism equivalence when we ask for both DIC and deterministic mechanisms, as we have just shown in our mechanism anti-equivalence theorem (Proposition 10).

[20]This follows from a suitable version of Bauer's maximum principle; see e.g. Ball (2023).

Corollary 3. *For any objective* $\Phi(q_1, q_2)$ *where* Φ *is an upper semicontinuous and quasiconvex functional, there exists a deterministic and DIC mechanism that is optimal.*

Standard objectives such as weighted welfare and revenue maximization are all linear in (q_1, q_2), and hence Corollary 3 immediately applies. Moreover, for optimal auctions with endogenous valuations such as in Gershkov et al. (2021), the objective is a convex functional because of the investment decisions by the bidders, and hence Corollary 3 also applies immediately to such settings. To illustrate, consider the following example from Gershkov et al. (2021):

Example 3 (Auction with investments). There are two symmetric bidders. Each bidder i can take an investment decision $a_i \in \mathbb{R}_+$ at a cost $c(a_i)$ prior to participating in the mechanism. After making investment a_i, a type-θ_i bidder would have value $v(a_i, \theta_i) = a_i + \theta_i$ for the object. As shown in Gershkov et al. (2021), we may treat each bidder as having a convex utility in reduced-form allocation probabilities:

$$h(q, \theta_i) = q\theta_i + w(q)$$

where $w(q) := \max_a\{qa - c(a)\}$ is a convex function. The seller's expected revenue is then

$$\sum_i \int q_i(\theta_i)\psi(\theta_i) + w(q_i(\theta_i))G(\mathrm{d}\theta_i),$$

where $\psi(\theta_i)$ is the usual virtual value function. The objective is a convex functional in the reduced-form allocation rules (q_1, q_2), and hence there always exists an optimal mechanism taking the form identified in Proposition 11. □

To derive optimal mechanisms, Gershkov et al. (2021) restrict attention to symmetric mechanisms, which as they show can be with loss even in a completely symmetric environment such as Example 3.[21] Gershkov et al. (2021) then provide additional conditions in the case of two bidders to ensure the optimal mechanism is symmetric. Here, we make no symmetry restriction and show that the optimum can always be achieved via a deterministic and DIC mechanism that is pinned down by a nondecreasing **cutoff function** q_2 and a single **threshold** k. Indeed, all examples of optimal asymmetric mechanisms in Gershkov et al. (2021) have this structure, which as Proposition 11 and Corollary 3 show is not a coincidence.

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[21]Indeed, for $c(a) = b \cdot \frac{a^2}{2}$ where $b < 0.5$, and uniform type distribution on $[0, 1]$, Gershkov et al. (2021) show that the optimal mechanism for Example 3 must be *asymmetric*.

4.5 Optimal Private Private Information

Consider a binary state $\omega \in \Omega := \{0,1\}$, with a full-support prior distribution. Let $p := \mathbb{P}(\omega = 1)$ be the prior of the state being 1. Consider a collection of n signals $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ about the state ω, where S_i's are some measurable spaces. Formally, $(\omega, s_1, \ldots, s_n)$ are random variables defined on a common probability space. Following He, Sandomirskiy, and Tamuz (2024), we call $\mathcal{I} := (\omega, s_1, \ldots, s_n)$ an **information structure**, and say that $\mathcal{I}$ is a **private private information structure** if $(s_1, \ldots, s_n)$ are independent random variables.

We are interested in providing a convex hull characterization of private private information structures. As shown in He, Sandomirskiy, and Tamuz (2024), it is without loss of generality to assume that each s_i is distributed uniformly on $[0, 1]$ and represent a private private information structure by a function $f : [0,1]^n \to [0,1]$ such that $\int f(s) \, ds = p$. In particular, $f(s)$ represents $\mathbb{P}(\omega = 1 \mid s)$.

Let $p_i(s_i) := \mathbb{P}(\omega = 1 \mid s_i)$ be the belief of the state being 1 given signal s_i. A **belief distribution** $(\mu_1, \ldots, \mu_n) \in \Delta([0,1]) \times \cdots \times \Delta([0,1])$ is **feasible under the constraint of privacy** if there exists a private private information structure such that μ_i is the distribution of $p_i(s_i)$ for all i.

Let $\mathcal{F}^p$ be the set of functions $f : [0,1]^n \to [0,1]$ such that $\int f(s) \, ds = p$. We say that a signal structure $f \in \mathcal{F}^p$ is a **nested bi-upset** signal if

$$f = \mathbb{1}_{A_1} + \lambda \mathbb{1}_{A_2 \setminus A_1} \text{ for two nested up-sets } A_1 \subset A_2 \text{ and some } \lambda \in [0,1].$$

Note that for any $A_1 \subset A_2$, there exists a unique λ^* such that $\int \mathbb{1}_{A_1}(s) + \lambda^* \mathbb{1}_{A_2 \setminus A_1}(s) \, ds = p$. Thus, a nested bi-upset signal structure is pinned down by the two nested upsets $A_1 \subset A_2$. Any such signal structure can be implemented by *(i)* randomizing between A_1 and A_2 with probabilities $1 - \lambda^*$ and λ^*, and *(ii)* drawing signals s uniformly from the up-set A_i if the state is 1, and drawing signals s uniformly from the complement A_i^c if the state is 0.

Let $\mathcal{F}^* \subset \mathcal{F}^p$ be the set of nested bi-upset signals. Let

$$\mathcal{Q}^* := \left\{ (q_1, \ldots, q_n) : \text{ there exists } f \in \mathcal{F}^* \text{ such that } q_i(s_i) = \int f(s_i, s_{-i}) \, ds_{-i} \text{ for all } i \right\}$$

be the set of one-dimensional marginals generated by such signals.

An immediate consequence of our results is the following convex hull characterization of the feasible belief quantile functions under private private information structures:

Proposition 12. *A belief distribution $(\mu_1, \ldots, \mu_n)$ is feasible under the constraint of privacy if*

and only if its associated quantile functions are in the closed convex hull of Q^*, i.e.,

$$(q_1, \ldots, q_n) \in \mathrm{cl}(\mathrm{co}(Q^*)),$$

with the set of extreme points in Q^*.

Proof of Proposition 12. Note that the set of feasible quantile functions $(q_1, \ldots, q_n)$ are exactly the following ones:[22]

$$\widehat{Q} := \left\{ (q_1, \ldots, q_n) : q_i\text{'s are monotone and rationalizable, and } \int q_1(s_1)\, ds_1 = p \right\},$$

where the additional constraint $\int q_1(s_1)\, ds_1 = p$ coupled with rationazibility ensures that any joint function f rationalizing q satisfies $\int f(s)\, ds = p$.

By Theorem 2, it follows immediately that

$$\mathrm{ext}(\widehat{Q}) \subseteq Q^*.$$

It remains to show $\widehat{Q} = \mathrm{cl}(\mathrm{co}(Q^*))$. By the Krein-Milman theorem, we immediately have

$$\widehat{Q} = \mathrm{cl}(\mathrm{co}(\mathrm{ext}(\widehat{Q}))) \subseteq \mathrm{cl}(\mathrm{co}(Q^*)) \subseteq \widehat{Q},$$

and hence they are all equal, concluding the proof. $\qquad\square$

Complementary to He, Sandomirskiy, and Tamuz (2024) who provide a garbling characterization of the feasible belief *distributions* under private private information, Proposition 12 provides a convex hull characterization of the feasible belief *quantiles*. Unlike the feasible belief distributions, the set of feasible belief quantiles is convex and admits extreme points, which Proposition 12 characterizes.

An immediate consequence of our convex hull characterization is that the optimal private private information structure can be attained using nested bi-upset signals, whenever the objective is a quasiconvex functional of the feasible belief quantiles:

Corollary 4. *For any objective $\Phi(q_1, \ldots, q_n)$ where Φ is an upper semi-continuous and quasiconvex functional, an optimal private private information structure is given by a nested bi-upset signal structure.*

An example of a convex objective in the quantile space is maximizing the weighted welfare of multiple receivers, each of whom has a decision problem to solve. Indeed,

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[22]To see this, note that the belief quantile function $q_i(s_i)$ is simply the monotone rearrangement of $p_i(s_i)$.

He, Sandomirskiy, and Tamuz (2024) show that in such cases, the optimal private private information would be on the Blackwell-Pareto frontier and can be implemented using a single up-set. Corollary 4 shows that more generally, for non-welfare-maximizing objectives, even though the optimal signal structure may not be on the Blackwell-Pareto frontier, they can still be implemented by randomizing over two nested up-sets.

In the special case of $n = 2$, recall that our extreme point characterization also shows that the two nested up-sets can only differ by a *rectangle* $[\underline{s}_1, \overline{s}_1] \times [\underline{s}_2, \overline{s}_2]$ (Theorem 4). This characterization turns out to provide an even simpler implementation of the optimal information structure. In particular, any extreme point of the feasible belief quantiles can be implemented by *(i)* drawing signals uniformly from some up-set A^* if the state is 1 and from its complement $[0, 1]^2 \backslash A^*$ if the state is 0, and *(ii)* for one of the two dimensions, pooling the signals in an interval into a single signal. For example, we may take A^* to be the "horizontal average" of the two nested up-sets and pool the signals $s_2 \in [\underline{s}_2, \overline{s}_2]$ into a single signal.

Combining with the characterization of the Blackwell-Pareto frontier from He, Sandomirskiy, and Tamuz (2024), our result shows that the extreme belief quantiles differ from the ones resulting from the Blackwell-Pareto undominated information structure only by a single pooling interval:

Proposition 13. *Every extreme belief quantile pair (q_1, q_2) under private private information can be implemented by (i) selecting a Blackwell-Pareto undominated information structure and (ii) for either $i = 1$ or $i = 2$, pooling the signals s_i in an interval I_i into a single signal. Thus, for any upper semi-continuous and quasiconvex objective $\Phi(q_1, q_2)$, an optimal information structure is given by a Blackwell-Pareto undominated information structure with a single pooling interval.*

Proof of Proposition 13. By the proof of Proposition 12, any extreme belief quantile pair can be rationalized by

$$f = \mathbb{1}_{A_1} + \lambda \mathbb{1}_{A_2 \backslash A_1}$$

where $A_1 \subset A_2$ are two nested up-sets and $\lambda \in [0, 1]$ is pinned down given A_1, A_2 and prior p. By Theorem 4, we also know that $A_2 \backslash A_1$ must be a rectangle $I_1 \times I_2$ where I_i is a closed interval on dimension i. Let g_i be the lower nonincreasing boundary of A_i. Note that $g_1 \geqslant g_2$. Let $g^* = (1 - \lambda) g_1 + \lambda g_2$. Let $A^* := \{(s_1, s_2) : s_2 \geqslant g^*(s_1)\}$ be the up-set defined by g^*. Note that

$$\int \mathbb{1}_{A^*}(s) \, \mathrm{d}s = \int \mathbb{1}_{A_1}(s) + \lambda \mathbb{1}_{A_2 \backslash A_1}(s) \, \mathrm{d}s = p,$$

and hence $\mathbb{1}_{A^*}$ defines a feasible private private information structure. Now consider the

information structure defined by *(i)* drawing signals uniformly from A^* if the state is 1 and uniformly from its complement $[0,1]^2 \backslash A^*$ otherwise, and *(ii)* pooling the signals s_2 in the interval I_2 into a single signal. (Alternatively, we can also define A^* by averaging g_i^{-1} and pool the signals in dimension 1 instead of dimension 2.)

Note that this information structure generates the same belief distribution in each dimension i. Moreover, since the pooling in dimension 2 is only a function of s_2, the resulting signals are still independent of each other. Thus, it is still a private private information structure. Finally, by Theorem 1 of He, Sandomirskiy, and Tamuz (2024), we know that a private private information structure is Blackwell-Pareto undominated if and only if it can be implemented using an up-set in the sense we describe above, concluding the proof. □

Note that the characterization given by Proposition 13 is also *sufficient* for a belief quantile pair to be an extreme point if, after selecting any Blackwell-Pareto undominated information structure, we pool an interval of signals in each dimension into a single signal in a consistent way that results in two nested up-sets differing by at most a rectangle, because the Bayes plausibility constraint clearly satisfies the condition in Proposition 3.

To illustrate Proposition 13, consider the following information design example adapted from He, Sandomirskiy, and Tamuz (2024):

Example 4 (Information design with privacy constraint)**.** Consider a platform market with two buyers and one seller. The seller has two units of the same object (with cost normalized to be 0). The buyers have unit demand and have a common value $v(\omega)$ for the object, where $\omega \in \{0,1\}$ is a random state with a common prior $p = \mathbb{P}(\omega = 1)$. Suppose without loss that $v(1) > v(0)$. The platform observes the realized state ω and wants to design an information structure that sends signals s_i about ω to each buyer i while keeping the signals independent of each other (e.g. to prevent full surplus extraction by the seller). Suppose that each unit is sold at some price k, where $v(0) < k < v(1)$. The platform wants to maximize the probability of trade (because e.g. a per-trade commission):

$$\sum_i \int \mathbb{1}_{\{v(1)p_i + v(0)(1-p_i) \geq k\}} \mu_i(\mathrm{d}p_i).$$

Note that $\mathbb{1}_{\{v(1)p_i + v(0)(1-p_i) \geq k\}}$ is not convex in p_i, and hence we cannot restrict attention to Blackwell-Pareto undominated information structures as in He, Sandomirskiy, and Tamuz (2024). But the objective is quasi-convex in the belief quantiles (q_1, q_2). Thus, Proposition 13 applies and shows that the optimal private private information can be implemented using an up-set and applying a single interval pooling. □

5 Conclusion

We characterize the extreme points of multidimensional monotone functions from $[0,1]^n$ to $[0,1]$, as well as the extreme points of the set of one-dimensional marginals of these functions. These characterizations lead to new results in various mechanism design and information design problems — including public good provision with interdependent values; interim efficient bilateral trade mechanisms; asymmetric reduced form auctions — as well as recent information design problems such as designing private private information. As another application, we also uncover a mechanism anti-equivalence theorem for two-agent, two-alternative social choice problems.

A Proofs

The proofs are presented in the same order in which they are referenced in the main text, except that we prove Lemma 3 first, since it will be used in the proofs of Theorem 1 and Theorem 2.

A.1 Proof of Lemma 3

It suffices to show that, for any $y \in \text{ext}(L(X))$, $L^{-1}(y) \cap \text{ext}(X) \neq \varnothing$. Clearly, $L^{-1}(y) \neq \varnothing$ since $y \in L(X)$. Moreover, $L^{-1}(y)$ is compact since X is compact and L is continuous. Since L is affine and X is convex, $L^{-1}(y)$ is also convex. By the Krein-Milman theorem, $\text{ext}(L^{-1}(y)) \neq \varnothing$. Take any $z \in \text{ext}(L^{-1}(y))$. We claim that $z \in \text{ext}(X)$. To see this, fix any $x, x' \in X$ and $\lambda \in (0,1)$ such that $z = \lambda x + (1-\lambda)x'$. Since L is affine, we have

$$y = L(z) = L(\lambda x + (1-\lambda)x') = \lambda L(x) + (1-\lambda)L(x').$$

Since $y \in \text{ext}(L(X))$, it must be that $L(x) = L(x') = y$, which in turn implies that $x, x' \in L^{-1}(y)$. Furthermore, since $z \in \text{ext}(L^{-1}(y))$, it must then be that $x = x' = z$. As a result, $z \in \text{ext}(X)$.

A.2 Proof of Proposition 1

The "if" part follows immediately as monotonicity is preserved under mixtures. For the "only if" part, consider any $f \in \mathcal{F}$. For any $x \in [0,1]^n$, write

$$f(x) = \int_0^1 \mathbb{1}\{f(x) \geq 1 - r\}\,dr.$$

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For any $r \in [0,1]$, let

$$A_r := \left\{ x \in [0,1]^n : f(x) \geq 1 - r \right\}.$$

Note that since f is monotone, A_r is an up-set for all $r \in [0,1]$. Moreover, $A_r \subseteq A_{r'}$ for all $r < r'$. Let μ be the uniform distribution on $[0,1]$. Then, for all $x \in [0,1]^n$, we have

$$\int_0^1 \mathbb{1}_{A_r}(x) \mu(dr) = \int_0^1 \mathbb{1}\{f(x) \geq 1 - r\} \, dr = f(x).$$

To prove uniqueness, suppose that there exist a family of nested up-sets $\{A_r\}_{r \in [0,1]}$ and a probability measure $\mu \in \Delta([0,1])$ such that

$$f(x) = \int_0^1 \mathbb{1}_{A_r}(x) \mu(dr)$$

for all $x \in [0,1]^n$. Let $M^{-1}(r) := \inf\{r' \in [0,1] : \mu([0,r']) \geq r\}$ for all $r \in [0,1]$. Since $\{A_r\}_{r \in [0,1]}$ is nested, $\{A_{M^{-1}(r)}\}_{r \in [0,1]}$ is also nested. Moreover, since $A_{M^{-1}(r)}$ is an up-set, there exists $g : [0,1]^n \to [0,1]$ such that

$$A_{M^{-1}(r)} = \left\{ x : x_1 \geq g(1 - r, x_{-1}) \right\}$$

for all $r \in [0,1]$. Since $\{A_{M^{-1}(r)}\}_{r \in [0,1]}$ is nested, $g(\cdot, x_{-1})$ is nondecreasing for all $x_{-1} \in [0,1]^{n-1}$. Let $g^{-1}(q, x_{-1}) := \inf\{x_1 \in [0,1] : g(x_1, x_{-1}) \geq q\}$ for all $q \in [0,1]$ and for all $x_{-1} \in [0,1]^{n-1}$. As a result, for all $x \in [0,1]^n$, we have

$$\begin{aligned}
f(x) = \int_0^1 \mathbb{1}_{A_r}(x) \mu(dr) &= \int_0^1 \mathbb{1}_{A_{M^{-1}(r)}}(x) \, dr \\
&= \int_0^1 \mathbb{1}\{x_1 \geq g(1 - r, x_{-1})\} \, dr \\
&= \int_0^1 \mathbb{1}\{g^{-1}(x_1, x_{-1}) \geq 1 - r\} \, dr \\
&= g^{-1}(x_1, x_{-1}).
\end{aligned}$$

Therefore, for all $r \in [0,1]$,

$$A_{M^{-1}(r)} = \left\{ x : x_1 \geq g(1 - r, x_{-1}) \right\} = \left\{ x : g^{-1}(x_1, x_{-1}) \geq 1 - r \right\} = \left\{ x : f(x) \geq 1 - r \right\},$$

equivalent to our construction. This completes the proof.

A.3 Proof of Theorem 1

Consider any extreme point $\overline{f}$ of $\overline{\mathcal{F}}$. By Proposition 1, there exist nested up-sets $\{A_r\}_{r\in[0,1]}$ and a probability distribution $\mu \in \Delta([0,1])$ such that

$$\overline{f}(x) = \int_0^1 \mathbb{1}_{A_r}(x)\mu(dr).$$

For each $j \in \{1,\dots,m\}$, let

$$\rho_j(r) := \int_{[0,1]^n} \phi^j(x)\mathbb{1}_{A_r}(x)\,dx$$

for all $r \in [0,1]$. Then, for each $j \in \{1,\dots,m\}$,

$$
\begin{aligned}
\int_0^1 \rho_j(r)\mu(dr) &= \int_0^1 \left(\int_{[0,1]^n} \phi^j(x)\mathbb{1}_{A_r}(x)\,dx \right) \mu(dr) \\
&= \int_{[0,1]^n} \phi^j(x) \left(\int_0^1 \mathbb{1}_{A_r}(x)\mu(dr) \right) dx \\
&= \int_{[0,1]^n} \phi^j(x)\overline{f}(x)\,dx \\
&\leqslant \eta^j.
\end{aligned}
$$

Let $\mathcal{M}$ be the set of probability measures $\tilde{\mu} \in \Delta([0,1])$ such that

$$\int_0^1 \rho_j(r)\tilde{\mu}(dr) \leqslant \eta^j$$

for all $j \in \{1,\dots,m\}$. Note that $\mathcal{M}$ is a compact (under the weak-* topology) and convex set.[23] Define a continuous affine map $L : \mathcal{M} \to \mathcal{F}$ by

$$L(\tilde{\mu})[x] := \int_0^1 \mathbb{1}_{A_r}(x)\tilde{\mu}(dr),$$

for all $x \in [0,1]^n$. Then, $\mu \in \mathcal{M}$, $\overline{f} = L(\mu)$, and

$$\int_{[0,1]^n} L(\tilde{\mu})[x]\phi^j(x)\,dx = \int_0^1 \left(\int_{[0,1]^n} \phi^j(x)\mathbb{1}_{A_r}(x)\,dx \right) \tilde{\mu}(dr) = \int_0^1 \rho_j(r)\tilde{\mu}(dr) \leqslant \eta^j,$$

---

[23]Convexity of $\mathcal{M}$ follows from the linearity of the constraints. To see that $\mathcal{M}$ is compact, note that since $\{A_r\}_{r\in[0,1]}$ are nested up-sets, for any sequence $\{r_k\} \to r \in [0,1]$, $\{\mathbb{1}_{A_{r_k}}\} \to \mathbb{1}_{A_r}$ pointwise almost everywhere. Therefore, by the dominated convergence theorem, ρ_j is continuous for all j. As a result, $\tilde{\mu} \mapsto \int_0^1 \rho_j(r)\tilde{\mu}(dr)$ is continuous under the weak-* topology. Therefore, $\mathcal{M}$ is a closed subset of the compact set of probability measures on $[0,1]$ under the weak-* topology, and hence is compact.

for all $\tilde{\mu} \in \mathcal{M}$ and all $j \in \{1,\ldots,m\}$. That is, $L(\mathcal{M})$ is a convex subset of $\overline{\mathcal{F}}$ that includes $\overline{f}$. Since $\overline{f}$ is an extreme point of $\overline{\mathcal{F}}$, it must also be an extreme point of $L(\mathcal{M})$. By Lemma 3, $\overline{f} = L(\mu^*)$, for some $\mu^* \in \mathcal{M}$ that is an extreme point of $\mathcal{M}$. Moreover, by Proposition 2.1 of Winkler (1988), $|\mathrm{supp}(\mu^*)| \leqslant m + 1$. Together, it then follows that $\overline{f}$ is a mixture of at most $m + 1$ indicator functions defined on nested up-sets. This completes the proof.

A.4   Proof of Proposition 2

Necessity follows immediately from the proof of Theorem 1. For sufficiency, consider any such $f \in \overline{\mathcal{F}}^\star$. If $k = 1$, then f must be an extreme point of $\overline{\mathcal{F}}^\star$ by Lemma 1 as A_1 is an up-set. Suppose that $k \geqslant 2$ and that there exists $u : [0,1]^n \to [0,1]$ such that $f + u$ and $f - u$ are in $\overline{\mathcal{F}}^\star$. For each $j \in \{1,\ldots,k-1\}$, let $A_j^\star := A_{j+1} \backslash A_j$. Since $\{A_j\}_{j=1}^k$ is nested, $\{A_j^\star\}_{j=1}^{k-1}$ is disjoint. Moreover, since $f = \sum_{j=1}^k \lambda_j \mathbb{1}_{A_j}$, f is constant on $A_j^\star$ for all $j \in \{1,\ldots,k-1\}$. Since $f + u$ and $f - u$ are monotone, and since $A_j \subseteq A_{j+1}$ are up-sets, u must also be constant on $A_j^\star$ for all $j \in \{1,\ldots,k-1\}$. Let $u_j := u(x)$ for all $x \in A_j^\star$. Then, for each $j \in \{1,\ldots,m\}$,

$$\sum_{l=1}^k u_l \int_{A_l^\star} \phi^j(x)\, \mathrm{d}x = 0.$$

However, since

$$\left\{ \begin{pmatrix} \int_{A_1^\star} \phi^1(x)\, \mathrm{d}x \\ \vdots \\ \int_{A_1^\star} \phi^m(x)\, \mathrm{d}x \end{pmatrix}, \cdots, \begin{pmatrix} \int_{A_m^\star} \phi^1(x)\, \mathrm{d}x \\ \vdots \\ \int_{A_m^\star} \phi^m(x)\, \mathrm{d}x \end{pmatrix} \right\}$$

are linearly independent, where $A_j^\star := \varnothing$ for all $j \in \{k,\ldots,m\}$, it must be that $u_j = 0$ for all $j \in \{1,\ldots,k\}$. Moreover, since $f \equiv 0$ on $[0,1]^n \backslash A_k$ and $f \equiv 1$ on A_1, and since $f + u$ and $f - u$ are in $\overline{\mathcal{F}}^\star$, $u \equiv 0$ on $[0,1]^n \backslash \cup_{j=1}^{k-1} A_j^\star$. Together, $u \equiv 0$. Therefore, f must be an extreme point of $\overline{\mathcal{F}}^\star$.

A.5   Proof of Theorem 2

To prove part (i), define an affine map $L : \mathcal{F} \to \mathcal{Q}$ by

$$L(f) := \left(\int_{[0,1]^{n-1}} f(x)\, \mathrm{d}x_{-1}, \ldots, \int_{[0,1]^{n-1}} f(x)\, \mathrm{d}x_{-n} \right).$$

By Lemma 2 (Theorem 5 of Gutmann et al. 1991), a tuple q of monotone functions is rationalizable if and only if it can be rationalized by a monotone function $f \in \mathcal{F}$.

Thus, $L(\mathcal{F}) = \mathcal{Q}$.[24] Moreover, by the dominated convergence theorem, L is continuous. Together, since $\mathcal{F}$ is compact and convex, part (i) follows by invoking Lemma 1 and Lemma 3.

To prove part (ii), let

$$\overline{\mathcal{F}} := \left\{ f \in \mathcal{F} : \int_{[0,1]^n} f(x) \sum_{i=1}^n \psi_i^j(x_i)\, dx \leqslant \eta^j \text{ for all } j \right\}.$$

By Lemma 2 (Theorem 5 of Gutmann et al. 1991) again, any $q \in \overline{\mathcal{Q}}$ is rationalized by some $f \in \overline{\mathcal{F}}$, and hence $L(\overline{\mathcal{F}}) = \overline{\mathcal{Q}}$. Moreover, by the dominated convergence theorem, L is continuous. Lastly, since $\overline{\mathcal{F}} \subseteq \mathcal{F}$ is closed and since $\mathcal{F}$ is compact, $\overline{\mathcal{F}}$ is compact as well. Part (ii) then follows by invoking Theorem 1 and Lemma 3. This completes the proof.

A.6 Proof of Theorem 3

Necessity follows immediately from Theorem 2. For sufficiency, fix any $q \in \mathcal{Q}$ that is rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [0,1]^2$.

We prove a stronger claim that q is an exposed point of $\mathcal{Q}$ and hence an extreme point. Since A is an up-set, there exists a nonincreasing function $g : [0,1] \to [0,1]$ such that $A = \{(x_1, x_2) : x_2 \geqslant g(x_1)\}$. Let $\phi_1(x_1) := -g(x_1)$ and $\phi(x_2) := x_2$. Then note that

$$\phi_1(x_1) + \phi(x_2) \begin{cases} > 0, & \text{if } (x_1, x_2) \in A \backslash \{(x_1, g(x_1))\}_{x_1 \in [0,1]} \\ = 0, & \text{if } (x_1, x_2) \in \{(x_1, g(x_1))\}_{x_1 \in [0,1]} \\ < 0, & \text{otherwise} \end{cases}.$$

We claim that the optimization problem

$$\max_{f \in \mathcal{F}} \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2)) f(x_1, x_2)\, dx \qquad (1)$$

has a unique solution given by $\mathbb{1}_A$. To see this, note that we can relax the monotonicity constraint imposed by $\mathcal{F}$, and show that $\mathbb{1}_A$ is still the unique maximizer. Indeed, $\mathbb{1}_A$ can clearly attain the upper bound given by $\int \max\{\phi_1(x_1) + \phi_2(x_2), 0\}\, dx$. Moreover, if any function $f : [0,1]^2 \to [0,1]$ differs from $\mathbb{1}_A$ on a positive measure of x, then f must have a strictly lower objective by the construction of ϕ_1 and ϕ_2. This proves that $\mathbb{1}_A$ must be the

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[24]Note that since $\mathcal{F} \subseteq L^1([0,1]^n)$, $f \in \mathcal{F}$ is identified by an equivalent class of functions $f' \equiv f$ almost everywhere. Thus, $L(\mathcal{F}) \subseteq \mathcal{Q}$, since for any monotone function f', there exists f whose marginals are left-continuous and $f \equiv f'$ almost everywhere. Together with $\mathcal{Q} \subseteq L(\mathcal{F})$, we have $L(\mathcal{F}) = \mathcal{Q}$.

unique maximizer in $\mathcal{F}$ for the above linear functional on $\mathcal{F}$.[25]

Now consider the linear functional $\Phi : \mathcal{Q} \to \mathbb{R}$ defined by

$$\Phi(\tilde{q}) := \int_0^1 \phi_1(x_1)\tilde{q}_1(x_1)\,dx_1 + \int_0^1 \phi_2(x_2)\tilde{q}_2(x_2)\,dx_2.$$

For any $\tilde{q} \in \mathcal{Q}$, let $\tilde{f} \in \mathcal{F}$ be a monotone function that rationalizes $\tilde{q}$ (recall Lemma 2). Then, we have

$$
\begin{aligned}
\int_0^1 \phi_1(x_1)\tilde{q}_1(x_1)\,dx_1 + \int_0^1 \phi(x_2)\tilde{q}_2(x_2)\,dx_2 &= \int_0^1 (\phi_1(x_1) + \phi_2(x_2))\tilde{f}(x_1,x_2)\,dx \\
&\leqslant \int_0^1 (\phi_1(x_1) + \phi_2(x_2))\mathbb{1}_A(x)\,dx \\
&= \int_0^1 \phi_1(x_1)q_1(x_1)\,dx_1 + \int_0^1 \phi(x_2)q_2(x_2)\,dx_2,
\end{aligned}
$$

where the inequality is due to the optimality of $\mathbb{1}_A$ in the previous claim. Moreover, for any $\tilde{q} \neq q$, the function $\tilde{f}$ cannot equal $\mathbb{1}_A$, which means that the above inequality is strict since $\mathbb{1}_A$ is the unique maximizer for the above objective by the previous claim. This proves that q is an exposed point and hence an extreme point of $\mathcal{Q}$.

For unique rationalizability, suppose that q is rationalized by $\mathbb{1}_A$ and some other (not necessarily monotone) function $\tilde{f}$. Suppose for contradiction that $\tilde{f} \neq \mathbb{1}_A$ on a positive measure of x. Then, since $\mathbb{1}_A$ is the unique solution of (1) even if we relax the monotonicity constraint imposed by $\mathcal{F}$, we must have

$$
\begin{aligned}
\Phi(q) &= \int_0^1 \phi_1(x_1)q_1(x_1)\,dx_1 + \int_0^1 \phi_2(x_2)q_2(x_2)\,dx_2 \\
&= \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2))\mathbb{1}_A(x)\,dx \\
&> \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2))\tilde{f}(x_1,x_2)\,dx \\
&= \int_0^1 \phi_1(x_1)q_1(x_1)\,dx_1 + \int_0^1 \phi_2(x_2)q_2(x_2)\,dx_2 = \Phi(q),
\end{aligned}
$$

a contradiction. Thus, it must be that $\tilde{f} \equiv \mathbb{1}_A$ almost everywhere. This completes the proof.

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[25]Note that this also shows that any extreme point of $\mathcal{F}$ is exposed.

A.7   Proof of Theorem 4

Appendix A.7.1 proves the first part of Theorem 4; Appendix A.7.2 proves the second part of Theorem 4.

A.7.1   Rationalizability by Nested Upsets Differing by a Rectangle

Consider any extreme point q of $\overline{\mathcal{Q}}$. Since $q \in \overline{\mathcal{Q}}$ is an extreme point of $\overline{\mathcal{Q}}$, by Theorem 2, there exist nested up-sets $A_1 \subseteq A_2$ such that q is rationalized by $f = \lambda \mathbb{1}_{A_1} + (1-\lambda)\mathbb{1}_{A_2}$, for some $\lambda \in (0,1)$. We now show that A_1 and A_2 can be chosen to differ by at most a rectangle (and in fact they must differ by at most a rectangle from our unique rationalizability result which we prove in Appendix A.7.2). For each $j \in \{1,2\}$, since A_j is an up-set, there exists a nonincreasing left-continuous function $g^j : [0,1] \rightarrow [0,1]$ such that $A_j = \{(x_1, x_2) \in [0,1]^2 : x_2 \geqslant g^j(x_1)\}$. Moreover, since $A_1 \subseteq A_2$, $g^1(z) \geqslant g^2(z)$ for all $z \in [0,1]$. Let $\gamma^j(z) := 1 - g^j(z)$ for all $z \in [0,1]$ and $j \in \{1,2\}$. Then, for all $z \in [0,1]$,

$$q_1(z) = \int_0^1 \left[\lambda \mathbb{1}_{A_1}(z, x_2) + (1-\lambda)\mathbb{1}_{A_2}(z, x_2)\right] dx_2 = \lambda \gamma^1(z) + (1-\lambda)\gamma^2(z) \tag{2}$$

and

$$q_2(z) = \int_0^1 \left[\lambda \mathbb{1}_{A_1}(x_1, z) + (1-\lambda)\mathbb{1}_{A_2}(x_1, z)\right] dx_1 = \lambda \hat{\gamma}^1(z) + (1-\lambda)\hat{\gamma}^2(z), \tag{3}$$

where $\hat{\gamma}^j(z) := 1 - (\gamma^j)^{-1}(1-z)$ is the conjugate of γ^j, for all $j \in \{1,2\}$.

  We first claim that q_1 must be an extreme point of the set of monotone functions that are majorized by $\hat{q}_2$ and satisfy the linear constraint (fixing q_2). Indeed, suppose not. Then there exist distinct monotone functions q_1^1, q_1^2 such that $q_1^l \preceq \hat{q}_2$ for $l \in \{1,2\}$, $\lambda' q_1^1 + (1-\lambda')q_1^2 = q_1$ for some $\lambda' \in (0,1)$, and that

$$\int_0^1 q_1^l(x_1)\psi_1(x_1)\,dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\,dx_2 \leqslant \eta,$$

for all $l \in \{1,2\}$. Since $q_1^l \preceq \hat{q}_2$ for $l \in \{1,2\}$, by Lemma 4, there exist $f^1, f^2 \in \mathcal{F}$ such that (q_1^l, q_2) is rationalized by f^l for $l \in \{1,2\}$. Therefore, there exist distinct $q^1 := (q_1^1, q_2)$, $q^2 := (q_1^2, q_2) \in \overline{\mathcal{Q}}$ such that $\lambda' q^1 + (1-\lambda')q^2 = q$, and hence q is not an extreme point of $\overline{\mathcal{Q}}$.

  As a result, by Theorem 1 of Nikzad (2023), there exists at most a countable collection of disjoint intervals $\{(\underline{z}_k, \overline{z}_k]\}_{k=1}^{\infty}$ such that $q_1 \equiv \hat{q}_2$ on $[0,1] \setminus \cup_{k=1}^{\infty} (\underline{z}_k, \overline{z}_k]$, and q_1 is a step function with at most two steps on $(\underline{z}_k, \overline{z}_k]$ for all $k \in \mathbb{N}$. Note that this collection can be empty but then our result would follow immediately since such a pair can be rationalized by a single up-set as shown in the proof of Proposition 4. From now on, we assume

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that the collection is not empty. We order the intervals from left to right indexed by k. Together with (2), it then follows that, since γ^1 and γ^2 are nondecreasing and $q_1 = \lambda \gamma^1 + (1-\lambda)\gamma^2$, we must have γ^1, γ^2 as step functions with at most two steps on $(\underline{z}_k, \overline{z}_k]$, for all $k \in \mathbb{N}$. In particular, for each $k \in \mathbb{N}$, there exists $y_k \in (\underline{z}_k, \overline{z}_k)$ such that

$$\gamma^j(z) = \begin{cases} \underline{\gamma}_k^j, & \text{if } z \in (\underline{z}_k, y_k] \\ \overline{\gamma}_k^j, & \text{if } z \in (y_k, \overline{z}_k] \end{cases},$$

for all $j \in \{1, 2\}$, where $\overline{\gamma}_k^j \geq \underline{\gamma}_k^j$ for $j \in \{1, 2\}$, $\overline{\gamma}_k^1 \leq \overline{\gamma}_k^2$, and $\underline{\gamma}_k^1 \leq \underline{\gamma}_k^2$. Clearly, we have

$$q_1(z) = \lambda \gamma^1(z) + (1-\lambda)\gamma^2(z) = \begin{cases} \lambda \underline{\gamma}_k^1 + (1-\lambda)\underline{\gamma}_k^2, & \text{if } z \in (\underline{z}_k, y_k] \\ \lambda \overline{\gamma}_k^1 + (1-\lambda)\overline{\gamma}_k^2, & \text{if } z \in (y_k, \overline{z}_k] \end{cases}.$$

Next, we show that, for each $k \in \mathbb{N}$, if $\overline{\gamma}_k^1 < \overline{\gamma}_k^2$ and $\underline{\gamma}_k^1 < \underline{\gamma}_k^2$, then $\overline{\gamma}_k^j = \underline{\gamma}_k^j$ for $j \in \{1, 2\}$. We first prove this claim in the case of $\underline{\gamma}_k^2 > \overline{\gamma}_k^1$. Suppose for contradiction that there exists $k \in \mathbb{N}$ such that $\overline{\gamma}_k^1 < \overline{\gamma}_k^2$, $\underline{\gamma}_k^1 < \underline{\gamma}_k^2$, and $\overline{\gamma}_k^j > \underline{\gamma}_k^j$ for some $j \in \{1, 2\}$. Suppose that $\overline{\gamma}_k^1 > \underline{\gamma}_k^1$. The case of $\overline{\gamma}_k^2 > \underline{\gamma}_k^2$ while $\overline{\gamma}_k^1 = \underline{\gamma}_k^1$ can be treated by a similar argument. Let $B := (\underline{z}_k, y_k] \times (1 - \overline{\gamma}_k^1, 1 - \underline{\gamma}_k^1]$, and $C := (y_k, \overline{z}_k] \times (1 - \underline{\gamma}_k^2, 1 - \overline{\gamma}_k^1]$. Note that by assumption, both regions B and C have positive measures. By Nikzad (2023), we also know that $q_1(\underline{z}_k) < q_1(\underline{z}_k^+)$ and $q_1(\overline{z}_k) < q_1(\overline{z}_k^+)$. Thus,

$$q_1(\underline{z}_k) < \lambda \underline{\gamma}_k^1 + (1-\lambda)\underline{\gamma}_k^2 < \lambda \overline{\gamma}_k^1 + (1-\lambda)\overline{\gamma}_k^2 < q_1(\overline{z}_k^+)$$

is a strictly increasing sequence of four numbers. Let $\delta_1 > 0$ be the minimum of the consecutive differences.

We claim that

$$q_2(1 - \underline{\gamma}_k^1) < q_2((1 - \underline{\gamma}_k^1)^+) \tag{4}$$

$$q_2(1 - \overline{\gamma}_k^1) < q_2((1 - \overline{\gamma}_k^1)^+) \tag{5}$$

$$q_2(1 - \underline{\gamma}_k^2) < q_2((1 - \underline{\gamma}_k^2)^+) \tag{6}$$

To see (4), note that if we let $\tilde{A}_1 := A_1 \cup B$, then $\tilde{A}_1 \subseteq A_2$ are still nested up-sets and projecting $\lambda \mathbb{1}_{\tilde{A}_1} + (1-\lambda)\mathbb{1}_{A_2}$ to dimension 2 would result in a monotone $\tilde{q}_2$ that coincides with q_2 on $(1 - \underline{\gamma}_k^1, 1]$ with $\tilde{q}_2(1 - \underline{\gamma}_k^1) > q_2(1 - \underline{\gamma}_k^1)$. Thus,

$$q_2((1 - \underline{\gamma}_k^1)^+) = \tilde{q}_2((1 - \underline{\gamma}_k^1)^+) \geq \tilde{q}_2(1 - \underline{\gamma}_k^1) > q_2(1 - \underline{\gamma}_k^1).$$

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To see (5), apply the same argument with $\tilde{A}_1 := A_1 \cup (y_k, 1] \times (1 - \underline{\gamma}_k^2, 1 - \overline{\gamma}_k^1]$. To see (6), apply the same argument but now keep A_1 as before and let $\tilde{A}_2 := A_2 \cup (\underline{z}_k, 1] \times (0, 1 - \underline{\gamma}_k^2]$. In particular, note that $A_1 \subseteq \tilde{A}_2$ are still two nested up-sets, and projecting $\lambda \mathbb{1}_{A_1} + (1 - \lambda) \mathbb{1}_{\tilde{A}_2}$ to dimension 2 would result in a monotone $\tilde{q}_2$ that coincides with q_2 on $(1 - \underline{\gamma}_k^2, 1]$ with $\tilde{q}_2(1 - \underline{\gamma}_k^2) > q_2(1 - \underline{\gamma}_k^2)$. Now, let

$$\delta_2 := \min\left\{ q_2((1-\underline{\gamma}_k^1)^+) - q_2(1-\underline{\gamma}_k^1), \; q_2((1-\overline{\gamma}_k^1)^+) - q_2(1-\overline{\gamma}_k^1), \; q_2((1-\underline{\gamma}_k^2)^+) - q_2(1-\underline{\gamma}_k^2) \right\}.$$

By (4), (5), (6), we have $\delta_2 > 0$.

Take $\varepsilon_B, \varepsilon_C \in \mathbb{R}$ with at least one of $\varepsilon_B, \varepsilon_C$ being non-zero such that

$$\varepsilon_B \int_B (\psi_1(x_1) + \psi_2(x_2)) \, dx = \varepsilon_C \int_C (\psi_1(x_1) + \psi_2(x_2)) \, dx, \tag{7}$$

and

$$\max\left\{ |\varepsilon_B|, |\varepsilon_C| \right\} < \frac{1}{2} \cdot \min\left\{ \delta_1, \delta_2 \right\}. \tag{8}$$

Now, let

$$f'(x) := \begin{cases} f(x) + \varepsilon_B, & \text{if } x \in B \\ f(x) - \varepsilon_C, & \text{if } x \in C \\ f(x), & \text{otherwise} \end{cases} \quad ; \quad \text{and} \quad f''(x) := \begin{cases} f(x) - \varepsilon_B, & \text{if } x \in B \\ f(x) + \varepsilon_C, & \text{if } x \in C \\ f(x), & \text{otherwise.} \end{cases}$$

By construction, $\frac{1}{2} f' + \frac{1}{2} f'' = f$, and hence $q = \frac{1}{2} q' + \frac{1}{2} q''$, where q' and q'' are the one-dimensional marginals of f' and f'', respectively.

Since the rectangles $B, C \subseteq [0,1]^2$ both have positive measures and do not overlap when projecting to either dimension 1 or dimension 2, we must have $q' \neq q''$. Moreover, by (7) and (8), q'_i and q''_i are nondecreasing for $i \in \{1, 2\}$, and

$$\int_0^1 q'_1(x_1)\psi_1(x_1) \, dx_1 + \int_0^1 q'_2 \psi_2(x_2) \, dx_2$$
$$= \int_{[0,1]^2} f'(x_1, x_2)(\psi_1(x_1) + \psi_2(x_2)) \, dx$$
$$= \int_{[0,1]^2} f(x_1, x_2)(\psi_1(x_1) + \psi_2(x_2)) \, dx + \varepsilon_B \int_B (\psi_1(x_1) + \psi_2(x_2)) \, dx - \varepsilon_C \int_C (\psi_1(x_1) + \psi_2(x_2)) \, dx$$
$$= \int_{[0,1]^2} f(x_1, x_2)(\psi_1(x_1) + \psi_2(x_2)) \, dx$$

$$= \int_0^1 q_1(x_1)\psi_1(x_1)\,dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\,dx_2$$

$$\leq \eta.$$

and

$$\int_0^1 q_1''(x_1)\psi_1(x_1)\,dx_1 + \int_0^1 q_2''(x_2)\psi_2(x_2)\,dx_2$$

$$= \int_{[0,1]^2} f''(x_1,x_2)(\psi_1(x_1) + \psi_2(x_2))\,dx$$

$$= \int_{[0,1]^2} f(x_1,x_2)(\psi_1(x_1) + \psi_2(x_2))\,dx - \varepsilon_B \int_B (\psi_1(x_1) + \psi_2(x_2))\,dx + \varepsilon_C \int_C (\psi_1(x_1) + \psi_2(x_2))\,dx$$

$$= \int_{[0,1]^2} f(x_1,x_2)(\psi_1(x_1) + \psi_2(x_2))\,dx$$

$$= \int_0^1 q_1(x_1)\psi_1(x_1)\,dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\,dx_2$$

$$\leq \eta.$$

Together, $q', q'' \in \overline{\mathcal{Q}}$, $q' \neq q''$ and $\frac{1}{2}q' + \frac{1}{2}q'' = q$. Therefore, q is not an extreme point of $\overline{\mathcal{Q}}$, a contradiction. As a result, whenever $\overline{\gamma}_k^1 < \overline{\gamma}_k^2$ and $\underline{\gamma}_k^1 < \underline{\gamma}_k^2$, it must be that $\overline{\gamma}_k^j = \underline{\gamma}_k^j$ for $j \in \{1,2\}$.

Now, for the case of $\underline{\gamma}_k^2 \leq \overline{\gamma}_k^1$, apply the same argument as above, except now letting $B := (\underline{z}_k, y_k] \times (1 - \underline{\gamma}_k^2, 1 - \underline{\gamma}_k^1]$ and $C := (y_k, \overline{z}_k] \times (1 - \overline{\gamma}_k^2, 1 - \overline{\gamma}_k^1]$. In particular, by the same argument as before, we still have

$$q_1(\underline{z}_k) < \lambda \underline{\gamma}_k^1 + (1 - \lambda)\underline{\gamma}_k^2 < \lambda \overline{\gamma}_k^1 + (1 - \lambda)\overline{\gamma}_k^2 < q_1(\overline{z}_k^+).$$

Moreover, by the same arguments, we also have

$$q_2(1 - \underline{\gamma}_k^1) < q_2((1 - \underline{\gamma}_k^1)^+) \tag{9}$$

$$q_2(1 - \underline{\gamma}_k^2) < q_2((1 - \underline{\gamma}_k^2)^+) \tag{10}$$

$$q_2(1 - \overline{\gamma}_k^1) < q_2((1 - \overline{\gamma}_k^1)^+) \tag{11}$$

$$q_2(1 - \overline{\gamma}_k^2) < q_2((1 - \overline{\gamma}_k^2)^+) \tag{12}$$

Thus, we can repeat the previous perturbation argument and conclude that whenever $\overline{\gamma}_k^1 < \overline{\gamma}_k^2$ and $\underline{\gamma}_k^1 < \underline{\gamma}_k^2$, it must be that $\overline{\gamma}_k^j = \underline{\gamma}_k^j$ for $j \in \{1,2\}$.

Together, we know that there exists a countable collection of intervals $\{(\underline{z}_k, \overline{z}_k]\}_k$ such

that q_1 is a constant on each of the interval $(\underline{z}_k, \overline{z}_k]$, and $q_1 = \hat{q}_2$ otherwise. Moreover, for each interval $(\underline{z}_k, \overline{z}_k]$, we also know that the majorization inequality holds with strict inequality for all $s \in (\underline{z}_k, \overline{z}_k)$, and that it holds with equality at the two ends of the interval. In particular, the constant γ_k is pinned down by

$$\gamma_k = \frac{\int_{\underline{z}_k}^{\overline{z}_k} \hat{q}_2(s) \, ds}{\overline{z}_k - \underline{z}_k}.$$

In fact, by the same two perturbation arguments given above, we also know that the closure of the pooling intervals must be disjoint: for any $k \neq k' \in \mathbb{N}$,

$$\overline{z}_k \neq \underline{z}_{k'}.$$

This implies that we must have[26]

$$q_1(\underline{z}_k) = \hat{q}_2(\underline{z}_k), \ q_1(\overline{z}_k^+) = \hat{q}_2(\overline{z}_k^+).$$

Now, apply the symmetric argument to q_2. In particular, we can equivalently write $\overline{\mathcal{Q}}$ as the set of monotone pairs (q_1, q_2) that satisfy the linear constraint and satisfy

$$q_2 \leq \hat{q}_1.$$

Using this representation, we can apply exactly the symmetric argument to conclude that there exists a countable collection of disjoint intervals $\{(\underline{t}_j, \overline{t}_j]\}_j$ such that q_2 is a constant on each of the interval $(\underline{t}_j, \overline{t}_j]$, and $q_2 = \hat{q}_1$ otherwise.

Now, fix any pooling interval $(\underline{z}_k, \overline{z}_k]$ of q_1. Consider the interval $I_k^* := (1 - q_1(\overline{z}_k^+), 1 - q_1(\underline{z}_k)]$. Note that on this interval, $\hat{q}_1$ must be a step function with the two steps equal to $1 - \overline{z}_k$ and $1 - \underline{z}_k$, and a single jump at $1 - \gamma_k$. Note that there must exist at least one pooling interval $(\underline{t}_j, \overline{t}_j]$ of q_2 that overlaps with I_k^*, because otherwise $q_2 = \hat{q}_1$ everywhere on I_k^*, which, by definition, means that for all $z \in (\underline{z}_k, \overline{z}_k]$,

$$
\begin{aligned}
\hat{q}_2(z) = 1 - q_2^{-1}(1 - z) &= 1 - \inf\{y \in [0,1] : q_2(y) > 1 - z\} \\
&= 1 - \inf\{y \in (1 - q_1(\overline{z}_k^+), 1] : q_2(y) > 1 - z\} \\
&= 1 - \inf\{y \in (1 - q_1(\overline{z}_k^+), 1] : \hat{q}_1(y) > 1 - z\} \\
&= 1 - (1 - \gamma_k) = \gamma_k.
\end{aligned}
$$

---

[26]This is because, for any k, there exist sequences $\{\overline{z}_k^m\}$ and $\{\underline{z}_k^m\}$, along which the majorization constraint binds and $\{\overline{z}_k^m\} \downarrow \overline{z}_k$ and $\{\underline{z}_k^m\} \uparrow \underline{z}_k$.

Thus, $q_1 = \hat{q}_2$ everywhere on $(\underline{z}_k, \overline{z}_k]$, and hence the majorization constraint binds everywhere on $(\underline{z}_k, \overline{z}_k]$, contradicting to the construction of the interval. Moreover, note that any pooling interval $(\underline{t}_j, \overline{t}_j]$ that overlaps with I_k^* must satisfy $(\underline{t}_j, \overline{t}_j] \subseteq I_k^*$. To see this, suppose not. Then there exists some pooling interval $(\underline{t}_j, \overline{t}_j]$ such that $(\underline{t}_j, \overline{t}_j] \cap I_k^* \neq \varnothing$ and it includes $1 - q_1(\overline{z}_k^+)$ or $(1 - q_1(\underline{z}_k))^+$. Note that if it includes $1 - q_1(\overline{z}_k^+)$, then $\underline{t}_j < 1 - q_1(\overline{z}_k^+) < \overline{t}_j$ and hence there must be a jump discontinuity in $\hat{q}_2$ that jumps strictly across the value $q_1(\overline{z}_k^+)$. But then

$$\hat{q}_2(s) \neq q_1(\overline{z}_k^+)$$

for all $s \in [0,1]$ by monotonicity of $\hat{q}_2$, which contradicts that $q_1(\overline{z}_k^+) = \hat{q}_2(\overline{z}_k^+)$. Similarly, if it includes $(1 - q_1(\underline{z}_k))^+$, then $\overline{t}_j > 1 - q_1(\underline{z}_k) > \underline{t}_j$ and hence there must be a jump discontinuity in $\hat{q}_2$ that jumps strictly across the value $q_1(\underline{z}_k)$. But then

$$\hat{q}_2(s) \neq q_1(\underline{z}_k)$$

for all $s \in [0,1]$ by monotonicity of $\hat{q}_2$, which contradicts that $q_1(\underline{z}_k) = \hat{q}_2(\underline{z}_k)$. Therefore, we conclude that any pooling interval $(\underline{t}_j, \overline{t}_j]$ of q_2 that overlaps with I_k^* must satisfy $(\underline{t}_j, \overline{t}_j] \subseteq I_k^*$. But then, because $\hat{q}_1$ is a step function on the interval I_k^* with two steps, any such pooling interval $(\underline{t}_j, \overline{t}_j]$ must contain the jump point $1 - \gamma_k$ in its interior in order to have the majorization constraint hold with strict inequality in the interior of the interval. Since these intervals are disjoint, there can be at most one interval $(\underline{t}_j, \overline{t}_j]$ that contains $1 - \gamma_k$. Together, we conclude that there exists a unique pooling interval $(\underline{t}_j, \overline{t}_j]$ such that $(\underline{t}_j, \overline{t}_j] \cap I_k^* \neq \varnothing$. Moreover, the pooling interval $(\underline{t}_j, \overline{t}_j]$ is entirely contained in I_k^* and includes the jump point $1 - \gamma_k$. Let $(\underline{t}_k^*, \overline{t}_k^*]$ denote this unique interval associated with I_k^*. On this interval, q_2 is a constant κ_k that is pinned down by

$$\kappa_k = \frac{\int_{\underline{t}_k^*}^{\overline{t}_k^*} \hat{q}_1(s)\, \mathrm{d}s}{\overline{t}_k^* - \underline{t}_k^*}.$$

Note that the above conclusion also implies that on each interval $(\underline{z}_k, \overline{z}_k]$, $\hat{q}_2$ is a step function with exactly two steps, where one step is strictly below and one step is strict above the constant value of q_1, and a jump point at $1 - \kappa_k$.

We also claim that any pooling interval $(\underline{t}_j, \overline{t}_j]$ of q_2 must be contained in the interval I_k^* for some k. To see this, suppose not. Then there exists a pooling interval $(\underline{t}_j, \overline{t}_j]$ such that it is not contained in I_k^* for any k. But then by the previous argument, it cannot overlap with I_k^* for any k. Thus, $(\underline{t}_j, \overline{t}_j] \subseteq [0,1] \setminus \cup_k I_k^*$. Note that by symmetric arguments, we also know that on the pooling interval $(\underline{t}_j, \overline{t}_j]$ of q_2, we must have $\hat{q}_1$ being a step function with

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exactly two steps, one strictly below and one strictly above the constant value β_j of q_2, and a single jump point. Let $1 - w_j$ denote the jump point, and let $1 - \bar{s}_j < \beta_j$ be the lower step, $1 - \underline{s}_j > \beta_j$ be the higher step. Then $q_1(s) = 1 - (\hat{q}_1)^{-1}(1 - s) = w_j$ for all $s \in (\underline{s}_j, \bar{s}_j]$. Note that $(\underline{s}_j, \bar{s}_j] \cap (\underline{z}_k, \bar{z}_k] = \varnothing$ for all k, because otherwise there exist some k and some $z \in (\underline{z}_k, \bar{z}_k]$ such that

$$1 - q_1(z) = 1 - w_j \in (\underline{t}_j, \bar{t}_j],$$

contradicting that $(\underline{t}_j, \bar{t}_j] \cap I_k^* = \varnothing$. Now, since $(\underline{s}_j, \bar{s}_j] \subseteq [0,1] \setminus \cup_k (\underline{z}_k, \bar{z}_k]$, we know that $q_1(s) = \hat{q}_2(s)$ for all $s \in (\underline{s}_j, \bar{s}_j]$. However, since $1 - \bar{s}_j < \beta_j < 1 - \underline{s}_j$, for all s such that $1 - s \in (1 - \bar{s}_j, \beta_j)$, we have $\hat{q}_2(s) = 1 - q_2^{-1}(1 - s) \geqslant 1 - \underline{t}_j$. Likewise, for all s' such that $1 - s' \in (\beta_j, 1 - \underline{s}_j)$, we have $\hat{q}_2(s') = 1 - q_2^{-1}(1 - s') \leqslant 1 - \bar{t}_j$. Thus, we have $s, s' \in (\underline{s}_j, \bar{s}_j)$, such that

$$\hat{q}_2(s) \geqslant 1 - \underline{t}_j > 1 - \bar{t}_j \geqslant \hat{q}_2(s').$$

But we also know that

$$\hat{q}_2(s) = q_1(s) = w_j = q_1(s') = \hat{q}_2(s'),$$

a contradiction.

Combining the previous two paragraphs, we can conclude that the pooling intervals $(\underline{t}_j, \bar{t}_j]$ of q_2 where the majorization inequality holds with strict inequality in their interiors are exactly the pooling intervals $(\underline{t}_k^*, \bar{t}_k^*]$ that we identify. On each interval $(\underline{t}_k^*, \bar{t}_k^*]$, we know that q_2 is a constant, $\hat{q}_1$ is a step function with two steps where the jump point is exactly $1 - q_1(\bar{z}_k)$. Note that the two steps are exactly $1 - \bar{z}_k$ and $1 - \underline{z}_k$.

Now we construct a function f that rationalizes (q_1, q_2) and then argue that we can always perturb f to be f' and f'' such that their one-dimensional marginals q' and q'' satisfy $\frac{1}{2}q' + \frac{1}{2}q'' = q$, unless the function f is a mixture of two indicator functions defined on nested up-sets that differ by a rectangle.

Let

$$\tilde{g}^1(z) := 1 - q_1(z), \quad \tilde{g}^2(z) := 1 - q_1(z)$$

for all $z \notin \bigcup_k (\underline{z}_k, \bar{z}_k]$, and let

$$\tilde{g}^1(z) := \bar{t}_k^*, \quad \tilde{g}^2(z) := \underline{t}_k^*.$$

for all $z \in (\underline{z}_k, \bar{z}_k]$ and for all k. For each $j \in \{1, 2\}$, let

$$\tilde{A}_j := \left\{ x \in [0,1]^2 : x_2 \geqslant \tilde{g}^j(x_1) \right\}.$$

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For each k, let $\lambda_k^* \in (0,1)$ be the unique solution to

$$\lambda_k^*(1 - \bar{t}_k^*) + (1 - \lambda_k^*)(1 - \underline{t}_k^*) = q_1(\bar{z}_k).$$

There always exists such a solution because

$$1 - \bar{t}_k^* < q_1(\bar{z}_k) < 1 - \underline{t}_k^*,$$

where the inequalities are due to that the interval $(\bar{t}_k^*, \underline{t}_k^*]$ contains $1 - \gamma_k$, and γ_k is the constant value q_1 takes on $(\underline{z}_k, \bar{z}_k]$. For each k, let

$$D_k := (\underline{z}_k, \bar{z}_k] \times (\underline{t}_k^*, \bar{t}_k^*].$$

Let

$$f := \mathbb{1}_{\tilde{A}_1} + \sum_k (1 - \lambda_k^*)\mathbb{1}_{D_k}.$$

Clearly, we have $\tilde{g}^1 \geqslant \tilde{g}^2$ by construction. Moreover, by construction, we also have

$$1 - q_1(\underline{z}_k) \geqslant \bar{t}_k^* > \underline{t}_k^* \geqslant 1 - q_1(\bar{z}_k^+).$$

Combined with q_1 being nondecreasing, this shows that the constructed $\tilde{g}^j$ is nonincreasing for each $j \in \{1, 2\}$. Therefore, $\tilde{A}_1 \subset \tilde{A}_2$ are two nested up-sets that differ by

$$\bigcup_k (\underline{z}_k, \bar{z}_k] \times (\underline{t}_k^*, \bar{t}_k^*],$$

which is a countable collection of rectangles. Moreover, these rectangles $\{D_k\}_k$ satisfy that for any $k \neq k'$ and any $i \in \{1, 2\}$, we have

$$\left\{x_i : \int_0^1 \mathbb{1}_{D_k}(x_i, x_{-i}) \, dx_{-i} > 0\right\} \bigcap \left\{x_i : \int_0^1 \mathbb{1}_{D_{k'}}(x_i, x_{-i}) \, dx_{-i} > 0\right\} = \varnothing.$$

That is, any two different rectangles have non-overlapping projections on each dimension. Combined with that

$$\bigcup_k D_k = \tilde{A}_2 \backslash \tilde{A}_1$$

is the difference between two nested up-sets, we can conclude that the function f is a multidimensional monotone function for any collection of $\{\lambda_k^*\}_k$.

We verify that f rationalizes (q_1, q_2). Note that by construction q_1 is f's one-dimensional

marginal in dimension 1. It remains to verify that q_2 is equal to f's one-dimensional marginal in dimension 2. By the previous claims, we know that the majorization inequality

$$\int_s^1 q_2(z)\,dz \leqslant \int_s^1 \hat{q}_1(z)\,dz$$

hold with strict inequality if and only if $s \in (\underline{t}_k^*, \overline{t}_k^*)$ for some k. Since we also know that $\overline{t}_k^* < \underline{t}_{k+1}^*$ for all k, it follows that $q_2(s) = \hat{q}_1(s)$ for all $s \notin \cup_k (\underline{t}_k^*, \overline{t}_k^*]$ and

$$q_2(s) = \frac{\int_{\underline{t}_k^*}^{\overline{t}_k^*} \hat{q}_1(t)\,dt}{\overline{t}_k^* - \underline{t}_k^*}$$

for all $s \in (\underline{t}_k^*, \overline{t}_k^*]$ and all k. Now, for any $s \notin \cup_k (\underline{t}_k^*, \overline{t}_k^*]$, we can compute

$$\int_0^1 f(x_1, s)\,dx_1 = 1 - (g^1)^{-1}(s) = 1 - (g^2)^{-1}(s) = 1 - q_1^{-1}(1-s) = \hat{q}_1(s).$$

Moreover, for any k and any $s \in (\underline{t}_k^*, \overline{t}_k^*]$, we can compute

$$\int_0^1 f(x_1, s)\,dx_1 = (1 - \lambda_k^*)(1 - (g^2)^{-1}(s)) + \lambda_k^*(1 - (g^1)^{-1}(s))$$

$$= (1 - \lambda_k^*)(1 - \underline{z}_k) + \lambda_k^*(1 - \overline{z}_k).$$

Recall that we also have

$$\lambda_k^*(1 - \overline{t}_k^*) + (1 - \lambda_k^*)(1 - \underline{t}_k^*) = q_1(\overline{z}_k),$$

and hence

$$\lambda_k^* \overline{t}_k^* + (1 - \lambda_k^*)\underline{t}_k^* = 1 - q_1(\overline{z}_k).$$

Let $l_k := 1 - q_1(\overline{z}_k)$, which is the jump point of $\hat{q}_1$ on the interval $(\underline{t}_k^*, \overline{t}_k^*]$. Note that we can compute

$$q_2(s) = \frac{\int_{\underline{t}_k^*}^{\overline{t}_k^*} \hat{q}_1(t)\,dt}{\overline{t}_k^* - \underline{t}_k^*} = \frac{\hat{q}_1((\underline{t}_k^*)^+) \cdot (l_k - \underline{t}_k^*) + \hat{q}_1(\overline{t}_k^*) \cdot (\overline{t}_k^* - l_k)}{\overline{t}_k^* - \underline{t}_k^*} = \frac{(1 - \overline{z}_k)(l_k - \underline{t}_k^*) + (1 - \underline{z}_k)(\overline{t}_k^* - l_k)}{\overline{t}_k^* - \underline{t}_k^*}.$$

Using the construction of λ_k^*, we have

$$(1 - \overline{z}_k)(l_k - \underline{t}_k^*) + (1 - \underline{z}_k)(\overline{t}_k^* - l_k) = (1 - \overline{z}_k)\lambda_k^*(\overline{t}_k^* - \underline{t}_k^*) + (1 - \underline{z}_k)(1 - \lambda_k^*)(\overline{t}_k^* - \underline{t}_k^*),$$

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and hence
$$q_2(s) = \lambda_k^*(1 - \overline{z}_k) + (1 - \lambda_k^*)(1 - \underline{z}_k) = \int_0^1 f(x_1, s)\, dx_1,$$

proving the claim.

Therefore, f rationalizes (q_1, q_2). We first claim that λ_k^* must all coincide. Suppose for contradiction that $\lambda_k^* \neq \lambda_{k'}^*$ for some k, k'. Recall that $\lambda_k^*, \lambda_{k'}^* \in (0, 1)$. Then, by Theorem 1, f is not an extreme point of $\overline{\mathcal{F}}$ where the constraint is the linear constraint imposed on (q_1, q_2). Thus, there exists some $u : [0, 1]^2 \to \mathbb{R}$ such that $f' := f + u \in \overline{\mathcal{F}}$, $f'' := f - u \in \overline{\mathcal{F}}$ and $f = \frac{1}{2}f' + \frac{1}{2}f''$. But clearly, u must be equal to 0 on $\tilde{A}_1$ and also equal to 0 on $[0, 1]^2 \backslash \tilde{A}_2$, since f equals either 1 or 0 on these two regions, respectively. Thus, u takes non-zero values only on $\tilde{A}_2 \backslash \tilde{A}_1 = \bigcup_j D_j$. Let q' and q'' be the one-dimensional marginals of f' and f'', respectively. Clearly we have q' and q'' are both in $\overline{\mathcal{Q}}$ and $\frac{1}{2}q' + \frac{1}{2}q'' = q$. Since any two different rectangles $D_j, D_{j'}$ have non-overlapping projections on each dimension, note that $q' \neq q$ and $q'' \neq q$. But then q cannot be an extreme point. Contradiction.

Therefore, λ_k^* must be a constant λ^* across all k. Now, we claim that there can be at most one such rectangle. Indeed, if there are two such rectangles, simply following the same perturbation we used in the beginning for the case of $\underline{\gamma}_k^2 \leqslant \overline{\gamma}_k^1$ to perturb f would result in f' and f'' such that their one-dimensional marginals q' and q'' are both in $\overline{\mathcal{Q}}$, are different, and satisfy $\frac{1}{2}q' + \frac{1}{2}q'' = q$. Thus, it must be the case $\tilde{A}_2 \backslash \tilde{A}_1$ has only one rectangle, and hence (q_1, q_2) can be rationalized by a mixture of two indicator functions defined on two nested up-sets that differ by at most a rectangle.

A.7.2 Unique Rationalizability

Now we prove that every extreme point of $\overline{\mathcal{Q}}$ must be uniquely rationalized among all monotone functions. Fix any $q = (q_1, q_2) \in \text{ext}(\overline{\mathcal{Q}})$. We first prove that it is uniquely rationalized among all mixtures of two indicator functions defined on nested up-sets. Since q is an extreme point, by Appendix A.7.1, it is rationalized by $f^* := \lambda^* \mathbb{1}_{A_1^*} + (1 - \lambda^*) \mathbb{1}_{A_2^*}$ for some nested up-sets $A_1^* \subseteq A_2^*$ that differ by at most a rectangle and some λ^*. If $A_2^* \backslash A_1^*$ has measure 0, then q must be an extreme point of $\mathcal{Q}$ and uniquely rationalized among all functions by Theorem 3. Now suppose $A_2^* \backslash A_1^* = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$, for some $\underline{x}_1 < \overline{x}_1$ and $\underline{x}_2 < \overline{x}_2$. Since A_1^* and A_2^* are up-sets, there exist nonincreasing left-continuous functions $g_1^*, g_2^* : [0, 1] \to [0, 1]$ such that $g_1^* \geqslant g_2^*$, and that $A_j^* = \{(x_1, x_2) : x_2 \geqslant g_j^*(x_1)\}$ for $j \in \{1, 2\}$ (up to a measure zero set). Then, since A_1^* and A_2^* differ by a rectangle $[\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$, it must be that $g_1^* \equiv g_2^*$ on $[0, 1] \backslash (\underline{x}_1, \overline{x}_1]$, $g_1^* \equiv \overline{x}_2$ and $g_2^* \equiv \underline{x}_2$ on $(\underline{x}_1, \overline{x}_1]$.

Take any $\hat{x}_2 \in (\underline{x}_2, \overline{x}_2)$, and let

$$\phi_1(x_1) := \begin{cases} -g_1^*(x_1), & \text{if } x_1 \notin (\underline{x}_1, \overline{x}_1] \\ -\hat{x}_2, & \text{if } x_1 \in (\underline{x}_1, \overline{x}_1] \end{cases}, \quad \text{and} \quad \phi_2(x_2) := \begin{cases} x_2, & \text{if } x_2 \notin (\underline{x}_2, \overline{x}_2] \\ \hat{x}_2, & \text{if } x_2 \in (\underline{x}_2, \overline{x}_2] \end{cases},$$

for all $(x_1, x_2) \in [0,1]^2$. Then, note that

$$\phi_1(x_1) + \phi_2(x_2) = \begin{cases} x_2 - g_1^*(x_1), & \text{if } x_1 \notin (\underline{x}_1, \overline{x}_1], x_2 \notin (\underline{x}_2, \overline{x}_2] \\ x_2 - \hat{x}_2, & \text{if } x_1 \in (\underline{x}_1, \overline{x}_1], x_2 \notin (\underline{x}_2, \overline{x}_2] \\ \hat{x}_2 - g_1^*(x_1), & \text{if } x_1 \notin (\underline{x}_1, \overline{x}_1], x_2 \in (\underline{x}_2, \overline{x}_2] \\ 0, & \text{if } x_1 \in (\underline{x}_1, \overline{x}_1], x_2 \in (\underline{x}_2, \overline{x}_2] \end{cases}$$

In particular, since $g_1^* \geqslant g_2^*$ are nonincreasing and since $g_1^* \equiv \overline{x}_2$ and $g_2^* \equiv \underline{x}_2$ on $(\underline{x}_1, \overline{x}_1]$,

$$\phi_1(x_1) + \phi_2(x_2) \begin{cases} > 0, & \text{if}(x_1, x_2) \in \{(x_1, x_2) : x_2 > g_1^*(x_1), x_1 \notin (\underline{x}_1, \overline{x}_1]\} \cup (\underline{x}_1, \overline{x}_1] \times (\overline{x}_2, 1] \\ < 0, & \text{if } (x_1, x_2) \in \{(x_1, x_2) : x_2 < g_1^*(x_1), x_1 \notin (\underline{x}_1, \overline{x}_1]\} \cup (\underline{x}_1, \overline{x}_1] \times [0, \underline{x}_2] \\ = 0, & \text{if } (x_1, x_2) \in \{(x_1, x_2) : x_2 = g_1^*(x_1), x_1 \notin (\underline{x}_1, \overline{x}_1]\} \cup (\underline{x}_1, \overline{x}_1] \times (\underline{x}_2, \overline{x}_2]. \end{cases}$$

Let $\overline{\mathcal{F}} \subseteq \mathcal{F}$ be the set of monotone functions f such that

$$\int_{[0,1]^2} (\psi_1(x_1) + \psi_2(x_2)) f(x_1, x_2) \, \mathrm{d}x \leqslant \eta,$$

and consider the linear functional $\Phi : \overline{\mathcal{F}} \to \mathbb{R}$, where

$$\Phi(\tilde{f}) := \int_{[0,1]^2} [\phi_1(x_1) + \phi_2(x_2)] \tilde{f}(x_1, x_2) \, \mathrm{d}x.$$

Note that since $f^*(x_1, x_2) = 1$ for all (x_1, x_2) such that $\phi_1(x_1) + \phi_2(x_2) > 0$, and $f^*(x_1, x_2) = 0$ for all (x_1, x_2) such that $\phi_1(x_1) + \phi_2(x_2) < 0$, we must have that f^* solves

$$\max_{\tilde{f} \in \overline{\mathcal{F}}} \Phi(\tilde{f}). \tag{13}$$

Moreover, we claim that for any $f \in \overline{\mathcal{F}}$, if *(i)* f is a solution of (13), *(ii)* $f = \lambda \mathbb{1}_{A_1} + (1 - \lambda) \mathbb{1}_{A_2}$, where $A_1 \subseteq A_2$ are nested up-sets, and *(iii)* f rationalizes q, then $f \equiv f^*$ almost everywhere. Indeed, for $f = \lambda_{A_1} + (1 - \lambda) \mathbb{1}_{A_2} = \mathbb{1}_{A_1} + (1 - \lambda) \mathbb{1}_{A_2 \setminus A_1}$ to solve (13), it must be that $\Phi(f) = \Phi(f^*)$. As a result, $f(x_1, x_2)$ must equal 1 whenever $\phi_1(x_1) + \phi_2(x_2) >$

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0, and must equal 0 whenever $\phi_1(x_1) + \phi_2(x_2) < 0$. Hence, f and f^* could only differ on $(\underline{x}_1, \overline{x}_1] \times (\underline{x}_2, \overline{x}_2]$. Moreover, we may assume that $\lambda \in (0,1)$, because otherwise f is an indicator function defined on an up-set, which by Theorem 3 implies that $f^* \equiv f$. Meanwhile, since $A_1 \subseteq A_2$ are up-sets, there exist nonincreasing functions $g_1, g_2 : [0,1] \rightarrow [0,1]$ with $g_1 \geqslant g_2$ such that $\{(x_1, x_2) : x_2 \geqslant g_j(x_1)\} \backslash A_j$ has measure zero for each j. Since f also rationalizes q, we have

$$\lambda g_1(x_1) + (1 - \lambda) g_2(x_1) = 1 - q_1(x_1) = \lambda^* g_1^*(x_1) + (1 - \lambda^*) g_2^*(x_1),$$

for all $x_1 \in [0,1]$, and

$$\lambda g_1^{-1}(x_2) + (1 - \lambda) g_2^{-1}(x_2) = 1 - q_2(x_2) = \lambda^* (g_1^*)^{-1}(x_2) + (1 - \lambda^*)(g_2^*)^{-1}(x_2),$$

for all $x_2 \in [0,1]$. Since q_1 and q_2 are constant on $(\underline{x}_1, \overline{x}_1]$ and $(\underline{x}_2, \overline{x}_2]$, respectively, it must be that g_1 and g_2 are constant on $(\underline{x}_1, \overline{x}_1]$, while g_1^{-1} and g_2^{-1} are constant on $(\underline{x}_2, \overline{x}_2]$. Moreover, since $f \equiv f^*$ on $[0,1]^2 \backslash (\underline{x}_1, \overline{x}_1] \times (\underline{x}_2, \overline{x}_2]$, it must be that $\lambda = \lambda^*$, $g_1 \equiv \overline{x}_2$, and $g_2 \equiv \underline{x}_2$ on $(\underline{x}_1, \overline{x}_1]$. Indeed, if the constant value κ_1 of g_1 does not coincide with $\overline{x}_2$, then sine $\lambda \in (0,1)$, q_2 must take at least two distinct values on $(\underline{x}_2, \overline{x}_2]$. Thus, $\kappa_1 = \overline{x}_2$. Similarly the constant value κ_2 of g_2 must coincide with $\underline{x}_2$ in order for q_2 to be constant on $(\underline{x}_2, \overline{x}_2]$. But then the constant λ must coincide with λ^* for f to rationalize (q_1, q_2). Together, this implies that $f \equiv f^*$ almost everywhere, as desired.

Lastly, for any $f \in \overline{\mathcal{F}}$ that rationalizes q, since

$$\Phi(f^*) = \int_{[0,1]^2} [\phi_1(x_1) + \phi_2(x_2)] f^*(x_1, x_2) \, dx$$

$$= \int_0^1 \phi_1(x_1) q_1(x_1) \, dx_1 + \int_0^1 \phi_2(x_2) q_2(x_2) \, dx_2$$

$$= \int_{[0,1]^2} [\phi_1(x_1) + \phi_2(x_2)] f(x_1, x_2) \, dx$$

$$= \Phi(f),$$

we have that f solves (13), where the second and the third equations follow from rationalizability. As a result, if, in addition, $f = \lambda \mathbb{1}_{A_1} + (1 - \lambda) \mathbb{1}_{A_2}$ for some $\lambda \in (0,1)$ and for some nested up-sets $A_1 \subseteq A_2$ then it must be that $f \equiv f^*$ almost everywhere, as desired.

It now remains to prove that q is uniquely rationalized among all monotone functions. To see this, note that any monotone $f \in \mathcal{F}$ that rationalizes q must be in $\overline{\mathcal{F}}$. Now, for any monotone $f \in \overline{\mathcal{F}}$ that rationalizes the extreme point $q \in \overline{\mathcal{Q}}$, by Theorem 1 and Choquet's

integral representation theorem,

$$f = \int f^r \mu(\mathrm{d}r)$$

where $r \in [0,1]$ is an index, $f^r = \lambda^r \mathbb{1}_{A_1^r} + (1 - \lambda^r)\mathbb{1}_{A_2^r}$ where $A_1^r \subseteq A_2^r$ are two nested up-sets, $\lambda^r \in (0,1)$, and $f^r \in \overline{\mathcal{F}}$. Let $q^r = (q_1^r, q_2^r) \in \overline{\mathcal{Q}}$ be the one-dimensional marginals of f^r. That is,

$$q_1^r(x_1) := \int_0^1 f^r(x_1, x_2)\,\mathrm{d}x_2 \quad \text{and} \quad q_2^r(x_2) := \int_0^1 f^r(x_1, x_2)\,\mathrm{d}x_1.$$

It follows immediately that, for $i \in \{1,2\}$,

$$q_i = \int q_i^r \mu(\mathrm{d}r).$$

Since (q_1, q_2) is an extreme point of $\overline{\mathcal{Q}}$, by Bauer (1961), there exists a unique probability measure on $\overline{\mathcal{Q}}$ that represents q. Thus,

$$q^r \equiv q$$

for almost all r with respect to measure μ. For all such r, because f^r is a mixture of two indicator functions defined on nested up-sets that also rationalizes q, by the previous claim, it must be that

$$f^r \equiv \lambda^* \mathbb{1}_{A_1^*} + (1 - \lambda^*)\mathbb{1}_{A_2^*},$$

almost everywhere. Therefore, we have

$$f \equiv \lambda^* \mathbb{1}_{A_1^*} + (1 - \lambda^*)\mathbb{1}_{A_2^*},$$

almost everywhere. This completes the proof.

A.8 Proof of Proposition 3

Necessity and uniqueness follow immediately from the proof of Theorem 4. For sufficiency, consider any such q. Let $f = \mathbb{1}_A + \lambda \mathbb{1}_D$ be a monotone function that rationalizes q, where $\lambda \in (0,1)$, $A \subseteq [0,1]^2$ is an up-set, and $D \subseteq [0,1]^2$ is a rectangle. If D has measure zero, then q is an extreme point of $\overline{\mathcal{Q}}^\star$ according to Theorem 3. Suppose now that D has a positive measure. Then, $\mathrm{cl}(D) = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$ with $\underline{x}_1 < \overline{x}_1$ and $\underline{x}_2 < \overline{x}_2$. In particular, $q_1(x_1) = 1 - (\lambda \underline{x}_2 + (1-\lambda)\overline{x}_2)$ for all $x_1 \in (\underline{x}_1, \overline{x}_1]$, whereas $q_2(x_2) = 1 - (\lambda \underline{x}_1 + (1-\lambda)\overline{x}_1)$

for all $x_2 \in (\underline{x}_2, \overline{x}_2]$. Let

$$g_1(z) := \begin{cases} 1 - q_1(z), & \text{if } z \notin (\underline{x}_1, \overline{x}_1] \\ \overline{x}_2, & \text{if } z \in (\underline{x}_1, \overline{x}_1] \end{cases} ; \quad \text{and} \quad g_2(z) := \begin{cases} 1 - q_1(z), & \text{if } z \notin (\underline{x}_1, \overline{x}_1] \\ \underline{x}_2, & \text{if } z \in (\underline{x}_1, \overline{x}_1] \end{cases} ,$$

for all $z \in [0, 1]$. Note that g_1, g_2 are nonincreasing and left-continuous, $g_1 \geqslant g_2$, and $A = \{(x_1, x_2) : x_2 \geqslant g_1(x_1)\}$ whereas $A \cup D = \{(x_1, x_2) : x_2 \geqslant g_2(x_1)\}$ almost everywhere.

In the meantime, by Choquet's representation theorem, there exist $\{q^r\}_{r \in [0,1]} \subseteq \text{ext}(\overline{\mathcal{Q}}^\star)$ and a probability measure $\mu \in \Delta([0, 1]])$ such that

$$q_i(x_i) = \int_0^1 q_i^r(x_i) \mu(\mathrm{d}r), \tag{14}$$

for all $x_i \in [0, 1]$ and $i \in \{1, 2\}$. We prove that q is an extreme point of $\overline{\mathcal{Q}}^\star$ by showing that $q^r \equiv q$ almost everywhere, for μ-almost all $r \in [0, 1]$. To this end, first note that since q_i is constant on $(\underline{x}_i, \overline{x}_i]$, q_i^r must also be constant on $(\underline{x}_i, \overline{x}_i]$ for μ-almost all $r \in [0, 1]$ and for all $i \in \{1, 2\}$. Furthermore, since q^r is an extreme point of $\overline{\mathcal{Q}}^\star$, by Theorem 4, q^r must be rationalized by some $f^r := \mathbb{1}_{A_r} + \lambda_r \mathbb{1}_{D_r}$, for all $r \in [0, 1]$, where $\lambda_r \in (0, 1)$, $A_r \subseteq [0, 1]^2$ is an up-set and $D_r \subseteq [0, 1]^2$ is a rectangle.

Similar to the proof of the uniqueness part of Theorem 4, take any $\hat{x}_2 \in (\underline{x}_2, \overline{x}_2)$, and let

$$\phi_1(x_1) := \begin{cases} -g_1(x_1), & \text{if } x_1 \notin (\underline{x}_1, \overline{x}_1] \\ -\hat{x}_2, & \text{if } x_1 \in (\underline{x}_1, \overline{x}_1] \end{cases} , \quad \text{and} \quad \phi_2(x_2) := \begin{cases} x_2, & \text{if } x_2 \notin (\underline{x}_2, \overline{x}_2] \\ \hat{x}_2, & \text{if } x_2 \in (\underline{x}_2, \overline{x}_2] \end{cases} ,$$

for all $(x_1, x_2) \in [0, 1]^2$. Then, as argued in the uniqueness proof of Theorem 4 (see Appendix A.7.2),

$$\phi_1(x_1) + \phi_2(x_2) \begin{cases} > 0, & \text{if}(x_1, x_2) \in A \\ < 0, & \text{if } (x_1, x_2) \in [0, 1]^2 \backslash A \cup D \\ = 0, & \text{if } (x_1, x_2) \in D , \end{cases}$$

for almost all $(x_1, x_2) \in [0, 1]^2$. Let $\overline{\mathcal{F}}^\star \subseteq \mathcal{F}$ be the set of monotone functions $\tilde{f}$ such that

$$\int_{[0,1]^2} (\psi_1(x_1) + \psi_2(x_2)) \tilde{f}(x_1, x_2) \, \mathrm{d}x = \eta. \tag{15}$$

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Note that, since $q^r \in \overline{\mathcal{Q}}^\star$ for all $r \in [0,1]$ and $q \in \overline{\mathcal{Q}}^\star$, and since f rationalizes q and f^r rationalized q^r for all $r \in [0,1]$, we have $f, f^r \in \overline{\mathcal{F}}^\star$ for all $r \in [0,1]$. Consider now the linear functional $\Phi : \overline{\mathcal{F}}^\star \to \mathbb{R}$, where

$$\Phi(\tilde{f}) := \int_{[0,1]^2} [\phi_1(x_1) + \phi_2(x_2)] \tilde{f}(x_1, x_2) \, dx.$$

Note that since $f(x_1, x_2) = 1$ for all (x_1, x_2) such that $\phi_1(x_1) + \phi_2(x_2) > 0$, and $f(x_1, x_2) = 0$ for all (x_1, x_2) such that $\phi_1(x_1) + \phi_2(x_2) < 0$, we must have that f solves

$$\max_{\tilde{f} \in \mathcal{F}^*} \Phi(\tilde{f}). \tag{16}$$

Moreover, any other solution of (16) can only differ from f with a positive measure on D.

Note that, by (14), we have

$$
\begin{aligned}
\Phi(f) &= \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2)) f(x_1, x_2) \, dx \\
&= \int_0^1 q_1(x_1) \phi_1(x_1) \, dx_1 + \int_0^1 q_2(x_2) \phi_2(x_2) \, dx_2 \\
&= \int_0^1 \left(\int_0^1 q_1^r(x_1) \mu(dr) \right) \phi_1(x_1) \, dx_1 + \int_0^1 \left(\int_0^1 q_2^r(x_2) \mu(dr) \right) \phi_2(x_2) \, dx_2 \\
&= \int_0^1 \left(\int_0^1 \int_0^1 f^r(x_1, x_2) \mu(dr) \, dx_2 \right) \phi_1(x_1) \, dx_1 + \int_0^1 \left(\int_0^1 \int_0^1 f^r(x_1, x_2) \mu(dr) \, dx_1 \right) \phi_2(x_2) \, dx_2 \\
&= \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2)) \left(\int_0^1 f^r(x_1, x_2) \mu(dr) \right) dx \\
&= \Phi \left(\int_0^1 f^r \mu(dr) \right) \\
&= \int_0^1 \Phi(f^r) \mu(dr),
\end{aligned}
$$

and therefore f^r must also solve (16) for μ-almost $r \in [0,1]$. As a result, for μ-almost all $r \in [0,1]$, $f^r \equiv f$ on $[0,1]^2 \backslash D$ almost everywhere. In particular, $D_r \subseteq D$ for μ-almost all $r \in [0,1]$. Furthermore, since q_1^r and q_2^r are constant on $(\underline{x}_1, \overline{x}_1]$ and $(\underline{x}_2, \overline{x}_2]$, respectively, and since $\mathrm{cl}(D) = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$, it must be that $D_r = D$ for μ-almost all $r \in [0,1]$ (by the same argument as in Appendix A.7.2), and thus, $q_1^r(x_1) = 1 - (\lambda_r \underline{x}_2 + (1 - \lambda_r) \overline{x}_2)$ for all $x_1 \in (\underline{x}_1, \overline{x}_1]$ and $q_2^r(x_2) = 1 - (\lambda_r \underline{x}_1 + (1 - \lambda_r) \overline{x}_1)$ for all $x_2 \in (\underline{x}_2, \overline{x}_2]$. Together, since $f^r \equiv f$

outside of D, (14) and (15) then imply that, for μ-almost $r \in [0,1]$,

$$
\int_{\underline{x}_1}^{\overline{x}_1} (1 - (\lambda \underline{x}_2 + (1-\lambda)\overline{x}_2)) \psi_1(x_1) \, dx_1 + \int_{\underline{x}_2}^{\overline{x}_2} (1 - (\lambda \underline{x}_1 + (1-\lambda)\overline{x}_1)) \psi_2(x_2) \, dx_2
$$

$$
= \int_{\underline{x}_1}^{\overline{x}_1} (1 - (\lambda_r \underline{x}_2 + (1-\lambda_r)\overline{x}_2)) \psi_1(x_1) \, dx_1 + \int_{\underline{x}_2}^{\overline{x}_2} (1 - (\lambda_r \underline{x}_1 + (1-\lambda_r)\overline{x}_1)) \psi_2(x_2) \, dx_2,
$$

which simplifies to

$$
\lambda \int_D (\psi_1(x_1) + \psi_2(x_2)) \, dx = \lambda_r \int_D (\psi_1(x_1) + \psi_2(x_2)) \, dx.
$$

Since

$$
\int_D (\psi_1(x_1) + \psi_2(x_2)) \, dx \neq 0,
$$

it must be that $\lambda = \lambda_r$. As a result, it must be that, for μ-almost $r \in [0,1]$, $f^r \equiv f$ almost everywhere, and hence $q^r \equiv q$ almost everywhere. This completes the proof.

A.9   Proof of Proposition 4

By Lemma 4, we have

$$
\left\{ q : q_1 \preceq \hat{q}_2 ; q_1, q_2 \text{ are nondecreasing and left-continuous} \right\} = \mathcal{Q},
$$

which is clearly convex. Now, by Theorem 3, $q \in \mathcal{Q}$ is an extreme point of $\mathcal{Q}$ if and only if it is rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [0,1]^2$.

Now fix any extreme point q of $\mathcal{Q}$. There exists $\mathbb{1}_A$ for some up-set A that rationalizes q. Since A is an up-set, there exists nonincreasing and left-continuous function $g : [0,1] \to [0,1]$ such that $A = \{(x_1, x_2) : x_2 \geqslant g(x_1)\}$. Thus,

$$
q_1(x_1) = \int_0^1 \mathbb{1}_A(x_1, x_2) \, dx_2 = \int_{g(x_1)}^1 1 \, dx_2 = 1 - g(x_1),
$$

and

$$
q_2(x_2) = \int_0^1 \mathbb{1}_A(x_1, x_2) \, dx_1 = \int_{g^{-1}(x_2)}^1 1 \, dx_1 = 1 - g^{-1}(x_2),
$$

and hence, $q_1 \equiv \hat{q}_2$.

Conversely, fix any q such that q_1, q_2 are nondecreasing and left-continuous, and $q_1 \equiv \hat{q}_2$. Let

$$
A := \{(x_1, x_2) : x_2 \geqslant 1 - q_1(x_1)\}.
$$

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Then, A is an up-set since q_1 is nondecreasing. Moreover, we have

$$q_1(x_1) = \int_0^1 \mathbb{1}_A(x_1, x_2)\, dx_2$$

by definition, and

$$q_2(x_2) = \hat{q}_1(x_2) = \int_0^1 \mathbb{1}_A(x_1, x_2)\, dx_1.$$

That is, q is rationalized by $\mathbb{1}_A$, and hence q is an extreme point of

$$\mathcal{Q} = \left\{ q : q_1 \preceq \hat{q}_2 \,;\, q_1, q_2 \text{ are nondecreasing and left-continuous} \right\}.$$

This completes the proof.

A.10   Proof of Proposition 5

By Lemma 4, we have

$$\left\{ q \,:\, q_1 \preceq \hat{q}_2 \,;\, q_1, q_2 \text{ are nondecreasing and left-continuous};\right.$$
$$\left. \int_0^1 q_1(x_1)\psi_1(x_1)\, dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\, dx_2 \leqslant \eta \right\} = \overline{\mathcal{Q}}.$$

Thus, by Theorem 4, any extreme point of

$$\left\{ q \,:\, q_1 \preceq \hat{q}_2 \,;\, q_1, q_2 \text{ are nondecreasing and left-continuous};\right.$$
$$\left. \int_0^1 q_1(x_1)\psi_1(x_1)\, dx_1 + \int_0^1 q_2(x_2)\psi_2(x_2)\, dx_2 \leqslant \eta \right\}$$

must be rationalized by $(1-\lambda)\mathbb{1}_{A_1} + \lambda\mathbb{1}_{A_2}$ for some $\lambda \in [0,1]$ and some nested up-sets $A_1 \subseteq A_2 \subseteq [0,1]^2$ that differ by at most a rectangle $[\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$.

Furthermore, for the up-sets A_1, A_2, there exist nonincreasing and left-continuous functions $g_1, g_2 : [0,1] \to [0,1]$ such that $A_j = \{(x_1, x_2) : x_2 \geqslant g_j(x_1)\}$ for $j \in \{1, 2\}$, where $g_1 \equiv g_2$ on $[0,1] \backslash [\underline{x}_1, \overline{x}_1]$, and $g_2(x_1) = \underline{x}_2$, $g_1(x_1) = \overline{x}_2$ for all $x_1 \in (\underline{x}_1, \overline{x}_1]$. Since q is rationalized by $(1-\lambda)\mathbb{1}_{A_1} + \lambda\mathbb{1}_{A_2}$, it follows that

$$q_1(x_1) = \int_0^1 (1-\lambda)\mathbb{1}_{A_1}(x_1, x_2) + \lambda\mathbb{1}_{A_2}(x_1, x_2)\, dx_2 = (1-\lambda)(1 - g_1(x_1)) + \lambda(1 - g_2(x_1)),$$

and

$$q_2(x_2) = \int_0^1 (1 - \lambda) \mathbb{1}_{A_1}(x_1, x_2) + \lambda \mathbb{1}_{A_2}(x_1, x_2) \, dx_1 = (1 - \lambda)(1 - g_1^{-1}(x_2)) + \lambda(1 - g_2^{-1}(x_2)).$$

Let $\underline{z} := \underline{x}_1$ and $\overline{z} := \overline{x}_1$. Let $\underline{\gamma} := 1 - \overline{x}_2$ and $\overline{\gamma} := 1 - \underline{x}_2$. This then completes the proof.

A.11   Proof of Proposition 6

We first show that the set

$$\mathcal{Q}^w := \left\{ q : q_1 \preceq_w \hat{q}_2 \, ; q_1, q_2 \text{ are nondecreasing and left-continuous} \right\}$$

is convex. By Theorem 4.A.6 of Shaked and Shanthikumar (2007), for any two nondecreasing left-continuous functions $g_1, g_2 : [0,1] \to [0,1]$, $g_1 \preceq_w g_2$ if and only if

$$g_1 \leqslant \tilde{g}_1 \preceq g_2$$

for some nondecreasing left-continuous function $\tilde{g}_1 : [0,1] \to [0,1]$.

Consider the set

$$\mathcal{K} := \left\{ (q_1, \tilde{q}_1, q_2) : q_1 \leqslant \tilde{q}_1 \preceq \hat{q}_2 \, ; q_1, \tilde{q}_1, q_2 \text{ are nondecreasing and left-continuous} \right\}.$$

Note that $\mathcal{K}$ is convex since for any $\lambda \in [0,1]$ and any $(q_1, \tilde{q}_1, q_2), (q_1', \tilde{q}_1', q_2') \in \mathcal{K}$, we have

$$\lambda q_1 + (1 - \lambda) q_1' \leqslant \lambda \tilde{q}_1 + (1 - \lambda) \tilde{q}_1' \preceq \widehat{\lambda q_2 + (1 - \lambda) q_2'},$$

where the majorization relation is due to the convexity of the set

$$\left\{ q : q_1 \preceq \hat{q}_2 \, ; q_1, q_2 \text{ are nondecreasing and left-continuous} \right\},$$

established in Proposition 4. Therefore, $\mathcal{K}$ is convex. As $\mathcal{Q}^w$ is a linear projection of $\mathcal{K}$, and $\mathcal{K}$ is convex, $\mathcal{Q}^w$ must also be convex.

Now, fix any extreme point q of the convex set $\mathcal{Q}^w$. Since $q_1 \preceq_w \hat{q}_2$, we know that q_1 must be an extreme point of the convex set of nondecreasing left-continuous functions that are weakly majorized by $\hat{q}_2$. By Corollary 2 of Kleiner, Moldovanu, and Strack (2021), there exists $k \in [0,1]$ such that $q_1 \equiv 0$ on $[0,k]$ almost everywhere and that q_1 is an extreme point of the convex set of nondecreasing left-continuous functions that are majorized by

$\hat{q}_2 \mathbb{1}_{[k,1]}$. In particular,

$$\int_s^1 q_1(z)\,dz \leqslant \int_s^1 \hat{q}_2 \mathbb{1}_{[k,1]}(z)\,dz = \int_s^1 \hat{q}_2(z)\,dz$$

for all $s \in [k,1]$, and

$$\int_k^1 q_1(z)\,dz = \int_0^1 q_1(z)\,dz = \int_0^1 \hat{q}_2 \mathbb{1}_{[k,1]}(z)\,dz = \int_k^1 \hat{q}_2(z)\,dz.$$

Now, consider the set

$$\mathcal{Q}^k := \left\{ q : q_1(s) = 0 \text{ for all } s \leqslant k \,;\, q_1 \leq \hat{q}_2 \,;\, q_1, q_2 \text{ are nondecreasing and left-continuous} \right\}.$$

We know that $\mathcal{Q}^k$ is convex since it is the intersection of two convex sets. We claim that the extreme point q of $\mathcal{Q}^w$ must satisfy

$$\left(q_1, \min\{q_2, 1-k\} \right) \in \text{ext}(\mathcal{Q}^k).$$

Indeed, note that for all $s \leqslant k$,

$$\min\widehat{\{q_2, 1-k\}}(s) = 0$$

and for all $s > k$,

$$\min\widehat{\{q_2, 1-k\}}(s) = \hat{q}_2(s),$$

by definition. Thus, by the construction of k and previous observation, we have

$$q_1 \leq \min\widehat{\{q_2, 1-k\}}$$

and $q_1(s) = 0$ for all $s \leqslant k$. Therefore, $(q_1, \min\{q_2, 1-k\}) \in \mathcal{Q}^k$. Moreover, note that by [Lemma 4](), we have

$$\mathcal{Q}^k = \mathcal{Q} \cap \{(q_1, q_2) : q_1(s) = 0 \text{ for all } s \leqslant k\}.$$

where $\mathcal{Q}$ is the set of rationalizable monotone pairs. Then, any $(\tilde{q}_1, \tilde{q}_2) \in \mathcal{Q}^k$ must be rationalized by some function f such that $f(x_1, x_2) = 0$ for all $x_1 \leqslant k$. Thus, $\tilde{q}_2(s) \leqslant 1-k$. Suppose for contradiction that $(q_1, \min\{q_2, 1-k\})$ is not an extreme point of $\mathcal{Q}^k$. Then there exist $u_1, u_2 : [0,1] \to \mathbb{R}$ such that at least one of u_1 and u_2 is not identically zero and

$$(q_1 + u_1, \min\{q_2, 1-k\} + u_2) \in \mathcal{Q}^k,$$

and
$$(q_1 - u_1, \min\{q_2, 1 - k\} - u_2) \in \mathcal{Q}^k.$$

We claim that
$$(q_1 + u_1, q_2 + u_2) \in \mathcal{Q}^w.$$
$$(q_1 - u_1, q_2 - u_2) \in \mathcal{Q}^w.$$

Indeed, note that $u_2(s) = 0$ for all s such that

$$q_2(s) \geqslant 1 - k,$$

since otherwise it would imply that at such s we have either

$$\min\{q_2(s), 1 - k\} + u_2(s) = 1 - k + u_2(s) > 1 - k$$

or

$$\min\{q_2(s), 1 - k\} - u_2(s) = 1 - k - u_2(s) > 1 - k,$$

but as argued before, any $(\tilde{q}_1, \tilde{q}_2) \in \mathcal{Q}^k$ must satisfy $\tilde{q}_2(s) \leqslant 1 - k$ for all $s \in [0, 1]$. Now, for all s such that $q_2(s) < 1 - k$, we can write

$$\min\{q_2(s), 1 - k\} \pm u_2(s) = q_2(s) \pm u_2(s).$$

Together, for all $s \in [0, 1]$, we have

$$\min\{q_2(s), 1 - k\} \pm u_2(s) = q_2(s) \pm u_2(s).$$

Therefore, $q_2 \pm u_2$ are both monotone. Clearly, $q_1 \pm u_1$ are both monotone. Moreover,

$$\int_s^1 \left(q_1(z) + u_1(z)\right) dz \leqslant \int_s^1 \text{conjugate}[\min\{q_2, 1 - k\} + u_2](z) \, dz = \int_s^1 \widehat{q_2 + u_2}(z) \, dz,$$

for all $s \in [0, 1]$. Similarly,

$$\int_s^1 \left(q_1(z) - u_1(z)\right) dz \leqslant \int_s^1 \text{conjugate}[\min\{q_2, 1 - k\} - u_2](z) \, dz = \int_s^1 \widehat{q_2 - u_2}(z) \, dz,$$

for all $s \in [0, 1]$. Therefore,
$$(q_1 + u_1, q_2 + u_2) \in \mathcal{Q}^w.$$
$$(q_1 - u_1, q_2 - u_2) \in \mathcal{Q}^w.$$

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But then, since at least one of u_1 and u_2 is not identically 0, the pair (q_1, q_2) cannot be an extreme point of $\mathcal{Q}^w$, a contradiction.

Therefore, the extreme point q of $\mathcal{Q}^w$ must satisfy

$$\left(q_1, \min\{q_2, 1-k\}\right) \in \text{ext}(\mathcal{Q}^k).$$

Note that by Lemma 2, the set

$$\mathcal{Q}^k = \mathcal{Q} \cap \{(q_1, q_2) : q_1(s) = 0 \text{ for all } s \leqslant k\}$$

is exactly the set of one-dimensional marginals of the following set

$$\mathcal{F}^k := \left\{f \in \mathcal{F} : f(x_1, x_2) = 0 \text{ for all } x_1 \leqslant k \text{ and all } x_2\right\},$$

where $\mathcal{F}$ is the set of monotone functions from $[0,1]^2$ to $[0,1]$. Clearly, $\mathcal{F}^k$ can be identified by the set of monotone functions from $[k,1] \times [0,1]$ to $[0,1]$. Therefore, by Lemma 3 and the proof of Theorem 2, we immediately have that any extreme point of $\mathcal{Q}^k$ must be rationalized by $\mathbb{1}_A$ for some up-set $A \subseteq [k,1] \times [0,1]$, which is also an up-set in $[0,1]^2$. Since $(q_1, \min\{q_2, 1-k\})$ is an extreme point of $\mathcal{Q}^k$ and hence is rationalized by such an up-set function $\mathbb{1}_A$, by the proof of Proposition 4, we have

$$q_1 = \widehat{\min\{q_2, 1-k\}} = \hat{q}_2 \mathbb{1}_{[k,1]},$$

proving the result.

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