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WITH CAPACITY CONSTRAINTS

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Dynamic Price Competition with Capacity Constraints*

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Abstract

We study dynamic price competition between sellers offering differentiated products with limited capacity and a common sales deadline. In every period, firms simultaneously set prices, and a randomly arriving buyer decides whether to purchase a product or leave the market. Given remaining capacities, firms trade off selling today against shifting demand to competitors to obtain future market power. We provide conditions for the existence and uniqueness of pure-strategy Markov perfect equilibria. In the continuous-time limit, prices solve a system of ordinary differential equations. We derive properties of equilibrium dynamics and show that prices increase the most when the product with the lowest remaining capacity sells. Because firms do not fully internalize the social option value of future sales, equilibrium prices can be inefficiently low such that both firms and consumers would benefit if firms could commit to higher prices. We term this new welfare effect the *Bertrand scarcity trap*.

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1 Introduction

The use of dynamic pricing permeates markets with limited capacity and perishable goods. Examples include seats on airplanes and trains, tickets for entertainment events, reservations for cruises and hotels, inventory in retailing, and units in housing rental. In the absence of competition, two main forces drive price dynamics. First, prices reflect changing opportunity costs. This is because the value a firm places on selling a unit of capacity today depends on that firm's ability to sell it in the future. Second, demand may change over time in predictable ways. For example, if consumers with high willingness to pay tend to arrive later on, a firm has an incentive to save capacity for future consumers. These forces have been extensively studied for a monopolist in the theoretical and empirical literature. With competition, a third force emerges that is less well understood: Firms also have an incentive to adjust prices to affect their rivals' remaining capacities and, hence, future competition.

In this paper, we provide a theoretical analysis of dynamic pricing that incorporates all three forces simultaneously.[1] We make three contributions. First, we develop a theory of dynamic pricing with competition for differentiated products in perishable goods markets. We start with an analysis of stage games and establish how general forms of scarcity affect equilibrium prices. Using these results, we provide sufficient conditions for the existence and uniqueness of pure-strategy Markov perfect equilibria in discrete time. Second, we use the continuous-time limit of the equilibrium to derive properties of equilibrium dynamics and to illustrate key economic forces. We show that much of the intuition from monopoly dynamic pricing models does not apply to the oligopoly case. For example, competing firms' strategies are aligned with encouraging the firm with the lowest capacity to sell out first, whereas a monopolist attempts to preserve product variety. Third, we study the welfare effects of dynamic price competition. We find that while competition reduces prices, it can also exacerbate misallocation by causing firms to sell capacity too quickly. This can be harmful to both consumers and firms. We formalize this new economic force, which we call the *Bertrand scarcity trap*, and contrast the outcomes with those of alternative pricing mechanisms, including the social planner's solution, monopoly pricing, and algorithmic pricing.

In our model, firms are exogenously endowed with limited initial capacity and face a common

[1] We use the term *dynamic pricing* throughout the paper for consistency. The term *revenue management* is also appropriate.

sales deadline. Capacity could represent seats on an airplane or capacity in one of the other aforementioned examples.[2] We consider an arbitrary number of firms, each offering an arbitrary number of differentiated products. Consumers arrive randomly according to a time-varying Poisson process with preferences that depend on the consumers' arrival time. Upon arrival, each short-lived consumer makes a discrete-choice decision, electing to purchase an available product or to exit the market forever. We consider regularity conditions on demand that can accommodate models frequently studied in theoretical and empirical work, including logit and nested logit demand. Firms are forward-looking and maximize expected payoffs. They internalize the uncertainty in aggregate demand, the presence of heterogeneous consumers, and the effects of competition. Within a period, after observing all remaining capacities, firms simultaneously choose prices.[3] Then, an arriving customer decides which product to purchase, and remaining capacities are updated. This process repeats until the deadline or until all products are sold out. Unsold capacity is scrapped after the deadline.

As our first contribution, we characterize Markov-perfect equilibria (MPE) in pure strategies, where the payoff-relevant state is the vector of remaining capacities and time. The Markovian structure allows us to summarize the impact of today's prices on the continuation game in a scarcity matrix that depends on the current state. We call the marginal impact on a firm's continuation profit of selling its own product an *own-scarcity effect*. Similarly, we define the impact on a firm's continuation payoff of a competitor selling a *competitor-scarcity effect*. Together, all scarcity effects define the scarcity matrix, which has dimensions equal to the number of firms and the number of products in the game. Matrix entries dictate how firms price their products in the stage game.

Analyzing stage games for a given scarcity matrix and demand system is challenging because each firm's payoff is affected not only by the firm's residual demands but also by its competitors' demands through the competitor-scarcity effects. Stage-game payoffs are generally not (log) supermodular (Milgrom and Roberts, 1990), nor are they of the form in Caplin and Nalebuff (1991)

[2] We use the word *capacity* throughout the paper in lieu of the word *inventory*. Sometimes we add the word *remaining* before *capacity* to improve readability. The meaning is the same regardless. Capacity is treated as a state variable in our setting unlike in Kreps and Scheinkman (1983), who incorporate capacity choice.

[3] For example, airlines can observe their rival remaining capacities. If capacity is hidden, firms can infer about the capacity distribution from observed prices, their own demand realizations, and elapsed time. While this complicates the analysis, we conjecture that the forces identified under perfect information still apply in a private information context.

and Nocke and Schutz (2018). To make progress, we employ a fixed-point theorem in Kellogg (1976). We derive sufficient conditions for both the existence and uniqueness of stage-game equilibria in pure strategies. These conditions essentially bound the absolute size of competitor-scarcity effects.[4] We then discuss strategic properties of the stage game. An equilibrium exists for arbitrary scarcity matrices if demand satisfies the independence of irrelevant alternatives (IIA) condition. This is because with IIA, the stage game admits a markup formula. This Lerner index, while of the familiar form, is more complex, because other product prices enter the marginal cost of selling. For non-IIA demand, there is no Lerner representation, because the product's price itself enters the marginal cost of selling.

We provide sufficient conditions for the uniqueness of the dynamic equilibrium. We first show that if discrete-time equilibria are unique for sufficiently small periods, their continuous-time limit solves a system of first-order ordinary differential equations (ODEs).[5] Solutions to this ODE system exist when the limiting scarcity matrices remain within a bounded open set where unique stage game equilibria exist. Discrete-time games with sufficiently small periods must have a unique equilibrium whenever all ODE solutions remain in this set. This allows us to study the discrete game via the limiting ODE system. Since the bounded open set contains the zero scarcity matrix at the deadline, uniqueness holds close to the deadline.

As our second contribution, we establish a number of qualitative properties of the game using our ODE characterization, including formalizing links between scarcity and pricing dynamics. We show that if the firm with the lowest capacity sells, for a given level of product differentiation, this softens competition the most. That is, rival firms' strategies are aligned with encouraging the firm with the lowest capacity to sell out first. Rivals charge relatively high prices, and the firm with the lowest capacity relatively low prices, in an attempt to raise future prices. Intuitively, the firm with the lowest capacity is closest to selling out, so this firm selling out first is the most efficient way to soften competition. This result generalizes Dudey (1992), who considers deterministic, single-valued demand for undifferentiated products, and Martínez-de Albéniz and Talluri (2011), who in addition allow for stochastic arrivals. These papers involve a deterministic order of sale, whereas

[4] We provide some guidance on when multiplicity of equilibria in the stage game may arise. Multiplicity does not require extreme assumptions on demand. For example, it can arise under simple logit demand, unlike in standard oligopoly games (Nocke and Schutz, 2018).

[5] If the discrete games admit multiple equilibria, then price policies can contain jumps as we illustrate in an example.

in our model, it is possible and probable that the firm with the lowest capacity does not sell. When this occurs, the firm that sells may lower its prices, and its rivals increase their prices.

Firms generally benefit from asymmetries because competition is fiercest when firms have the same capacities.If firms reach these competitive states, firms may offer fire sales (offer low prices) in order to return to an asymmetric state. Hence, when competing firms use dynamic pricing, they attempt to soften price competition by promoting scarcity through reducing product variety. In contrast, a monopolist price discriminates while trying to preserve product variety. This strategic difference can have important consequences for welfare.

Most intuition from monopoly dynamic pricing models (e.g., Gallego and Van Ryzin, 1994; Zhao and Zheng, 2000) does not generalize to the oligopoly case. We show that price policy functions are non-monotonic in time and capacity, value functions are non-monotonic in capacity, and scarcity effects can be positive or negative and non-monotonic in time and capacity. These results are economically significant for our understanding of competition in perishable goods markets. For example, markets can be more competitive if firms have lower, not greater, initial capacities.

In addition to allowing us to establish theoretical results, our characterization serves as a powerful tool for conducting empirical research and complements frameworks used to study industry dynamics (e.g., Ericson and Pakes, 1995). Although (Doraszelski and Judd, 2012) caution against applying continuous-time methods in settings where observations (data) are naturally discrete, we exactly address these concerns by showing when the continuous-time solution approximates discrete-time equilibria. To address the curse of dimensionality, it is common in empirical work to impose sequential-move (Maskin and Tirole, 1988) or random–sequential-move assumptions (e.g., Baron and Ferejohn, 1989; Doraszelski and Judd, 2019). However, the possibility that a firm's action can affect other firms' states (i.e., a firm's pricing decision can cause a competitor to sell) precludes the ability to use this approach: It is not possible to decompose the game into independent separate problems for each firm. Our approach can trace out equilibrium prices without explicitly solving all stage-game equilibria. We only need to solve stage games at the deadline where all scarcity effects are zero. From there, the system of ODEs can be solved using standard numerical methods. For large-dimensional problems, we discuss how our ODE structure can be coupled with reinforcement learning.

As our third contribution, we explore the welfare implications of dynamic price competition.

We show that the incentive to promote scarcity can be so strong that both firms and consumers could be made better off if firms could commit to restricting their pricing strategies to higher prices. We call this new welfare effect the Bertrand scarcity trap (BST). Using examples, we show that competition can worsen misallocation due to two forces. First, firms capture only a fraction of total surplus and do not internalize the social option value of holding capacity for the future. Second, competitive pressure causes prices to drop to inefficiently low levels, leading to early sellouts. The consequence is over-provision of capacity early on and under-provision (due to sellouts) close to the deadline. In one example, we show that a monopolist could increase consumer surplus relative to the competitive equilibrium outcome with dynamic pricing. We also demonstrate that the use of algorithmic pricing can increase consumer surplus and revenues relative to the competitive equilibrium outcome. Lastly, we show that uncertain demand is critical for the existence of the BST.

1.1 Related Literature

Although there is a large literature on dynamic pricing, much of it focuses on monopolies. As in the classic revenue management literature (e.g., Gallego and Van Ryzin, 1994), we consider short-lived consumers for tractability reasons and based on recent empirical evidence (Hortaçsu et al., 2024), but our work also relates to models with long-lived buyers.[6] In Board and Skrzypacz (2016), Gershkov et al. (2018) and Dilme and Li (2019), a key tension with forward-looking buyers is that the monopolist is essentially competing with its future self. While Board and Skrzypacz (2016) and Gershkov et al. (2018) assume the firm can fully commit to a selling mechanism and hence resist the temptation to fire-sale,[7] Dilme and Li (2019) find that a firm has fire sales in order to create future scarcity. Fire sales are possible in our model, but they stem from competitive interactions.

We assume consumers know their preferences, which is another restriction that has been relaxed in work studying monopolist pricing. For example, Akan et al. (2015) demonstrates that if consumers learn their demand over time, a monopolist increases profits by offering advance-

[6] Hortaçsu et al. (2024) find little evidence of consumers delaying airline ticket purchases, using two years of clickstream data for a large international airline company based in the US. This does not imply that consumers are not forward-looking. If demand becomes increasing inelastic, the incentives to delay purchase decrease.

[7] Hörner and Samuelson (2011) consider the corresponding setting without commitment when the seller has only one unit of capacity to sell.

purchase contracts with varying return policies. Ely et al. (2017) show that a firm can benefit from overbooking, i.e., selling more than its capacity constraint. Garrett (2016) finds that a monopolist without capacity constraints may offer discounts if consumer preferences change over time.[8]

It is known from the literature that in the presence of demand uncertainty, competing firms may want to offer a menu of prices and restrict the number of units to be sold at any given price (Dana, 1999a,b). In our model, firms adjust prices continuously as demand uncertainty is resolved. Our relatively mild demand assumptions can cause firms to implement advance purchase discounts, as studied in other settings (e.g., Gale and Holmes, 1993; Nocke et al., 2011).

Another key force in our model is that firms may want to shift sales to rivals. Dana and Williams (2022) consider a related sequential capacity-price model absent demand uncertainty. They show that when products are undifferentiated, the unique pricing equilibrium typically involves a uniform price over time, unless one firm has more capacity than all other firms and sells part of its capacity first. In contrast, Dudey (1992) and Martínez-de Albéniz and Talluri (2011) show that firms offering undifferentiated products to homogeneous consumers sell their inventory in a deterministic order.[9] We reconcile these forces by contrasting our rich dynamic model with one in which demand uncertainty and product differentiation are not present.[10]

Technically, our work leverages continuous-time techniques as common in dynamic pricing work (Deb, 2014; Bergemann and Strack, 2015; Board and Skrzypacz, 2016; Dilme and Li, 2019; Bonatti et al., 2017). Unlike most work in this field, we use the continuous-time structure to derive not only the dynamic properties of equilibria but also the conditions for the existence and uniqueness of Markov equilibria of the discrete-time game.

Finally, our work connects to an emerging literature on algorithmic pricing (e.g., Calvano et al., 2020; Banchio and Mantegazza, 2022; Lamba and Zhuk, 2022). These papers explore how algorithms can significantly soften price competition. Our findings show that the opposite can also be true—the competitive equilibrium outcome of our dynamic game can result in lower welfare than allocation rules involving algorithmic pricing. The key driver of this result is that algorithmic pricing reduces over-provision early on and shifts sales to later periods.

[8] Other examples analyzing sequential screening include Courty and Li (2000) and Deb and Said (2015).

[9] In both papers, if demand increases over time, all equilibria must be in mixed strategies, whereas in our model, we can show both the existence and uniqueness of equilibria in pure strategies.

[10] Bergemann and Välimäki (2006) also study a competitive model with a single long-lived buyer and unit capacity.

2 Model

Firms, products, and timing. We consider a set $\mathscr{F} := \{1, \ldots, F\}$ of firms and a set $\mathscr{J} := \{1, \ldots, J\}$ of products. Products in $\mathscr{J}_f$ are owned by firm f, where $\left(\mathscr{J}_f\right)_{f \in \mathscr{F}}$ is a partition of $\mathscr{J}$ so that each product is sold by exactly one firm.[11] Each firm f is endowed with discrete initial capacities $K_{j,0} \in \mathbb{N}$, $j \in \mathscr{J}_f$. Any remaining capacity at the deadline $T > 0$ is scrapped with zero value. We study a discrete-time environment with periods $t \in \{0, \Delta, \ldots, T - \Delta\}$, $\Delta > 0$, and later consider the continuous-time limit as $\Delta \to 0$. In every period t, firms simultaneously set the prices of their products $\mathbf{p}_{f,t} := (p_{j,t})_{j \in \mathscr{J}_f}$. Then, a single consumer arrives with probability $\Delta \lambda_t$, where λ_t is analytic in t. We index each consumer by her arrival time t. Consumer t makes a discrete choice by either buying a single unit of an available product upon arrival or leaving the market by choosing her outside option. We denote the outside option by $j = 0$. Because capacity is scarce, it may be that consumer t can purchase only from a subset of $\mathscr{J}$. We denote the capacity vector in period t by $\mathbf{K}_t := (K_{j,t})_{j \in \mathscr{J}}$ and the set of available products by $\mathscr{A}(\mathbf{K}) = \{j \in \mathscr{J} \mid K_j > 0\} \subseteq \mathscr{J}$.

Demand. In order to focus on the dynamic game between firms, we introduce consumer demand functions as primitives. We discuss micro-foundations for these demand functions and welfare results in Section 6. If all products are available, then consumer t, facing a price vector $\mathbf{p} := \left(p_j\right)_{j \in \mathscr{J}}$, buys product j with probability $s_j(\mathbf{p}; \boldsymbol{\theta}_t, \mathscr{J})$, where $\boldsymbol{\theta}_t \in \mathscr{T} \subset \mathbb{R}^n$ is a vector of $n \geq 1$ demand parameters that are analytic in t, $\mathscr{T}$ is compact, and s_j is smooth in $\mathbf{p}$ and $\boldsymbol{\theta}$. Importantly, $\boldsymbol{\theta}$ can capture changes in preferences over time, as documented in the airline context (e.g., Williams, 2022; Hortaçsu et al., 2024). We impose the following regularity conditions on demand, s_j.

Assumption 1. *For all $\boldsymbol{\theta} \in \mathscr{T}$ and $\mathbf{p} \in \mathbb{R}^{\mathscr{J}}$, the following hold:*

i) For all $j \in \mathscr{J}$, $\lim_{p_j \to \infty} s_j(\mathbf{p}; \boldsymbol{\theta}, \mathscr{J}) p_j = 0$. For any subset $\mathscr{A} \subset \mathscr{J}$ and $j \in \mathscr{A}$, the limit[12]

$$s_j(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A}) := \lim_{\substack{p_{j'} \to \infty \\ j' \notin \mathscr{A}}} s_j(\mathbf{p}; \boldsymbol{\theta}, \mathscr{J}) \in [0, 1]$$

exists and is smooth in $\boldsymbol{\theta}$ and $\mathbf{p}^{\mathscr{A}} \in \mathbb{R}^{\mathscr{A}}$, where $p^{\mathscr{A}}_{j'} = p_{j'}$ for all $j' \in \mathscr{A}$;

[11] That is, $\mathscr{J} = \bigcup_{f \in \mathscr{F}} \mathscr{J}_f$ and $\mathscr{J}_f \cap \mathscr{J}_{f'} = \emptyset$ for $f \neq f'$.

[12] The limit takes all prices of products $j' \notin \mathscr{A}$ to infinity. The order of limits does not matter.

ii) $\frac{\partial s_j}{\partial p_j} < 0$, *i.e.,* $s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})$ *is strictly decreasing in* p_j, *and* $\frac{\partial s_j}{\partial p_{j'}} < 0$, *i.e.,* $s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})$ *is strictly increasing in* $p_{j'}$, *for all* $j' \neq j$*;*

iii) letting $s_0(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) := 1 - \sum_{j'\in\mathscr{A}} s_{j'}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ *be the probability of choosing the outside good,* $\frac{\partial s_0}{\partial p_j} > 0$*, and* $\lim_{p_j\to-\infty} s_0(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) = 0$*, for all* $j \in \mathscr{A}$*;*

iv) for all $\mathscr{A} \subset \mathscr{J}$*,* $j \in \mathscr{A}$ *and* $\underline{\mathbf{p}} \in \mathbb{R}^{\mathscr{A}}$*, there exists a* $\overline{p}_j$ *so that*

$$\inf_{\substack{p_j^{\mathscr{A}} \geq \overline{p}_j,\\ \mathbf{p}^{\mathscr{A}} \geq \underline{\mathbf{p}}}} \frac{\frac{\partial s_0}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})}{s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})} p_j > 1.$$

Assumption 1 puts relatively mild assumptions on demand. Condition (i) ensures that demand is well-defined when products sell out, i.e., when these products' prices are equal to infinity. Condition (ii) states that all products are substitutes. Condition (iii) states that the outside option is also a substitute to all other products. Finally, condition (iv) ensures that the profit-maximizing prices of available products are never infinite (see the Online Appendix). This condition is reminiscent of Assumption 1 in Nocke and Schutz (2018). In the single-product monopoly setting, the expression inside the infimum corresponds to the negative elasticity of demand; thus, condition (iv) simply states that demand remains strictly elastic for large prices.

Because profit-maximizing prices are interior, by condition (iv), they solve a system of first-order conditions (FOCs). To write these FOCs in matrix form, denote the vector of choice probabilities for firm f's available products by $\mathbf{s}^f(\cdot;\boldsymbol{\theta},\mathscr{A}) := \left(s_j(\cdot;\boldsymbol{\theta},\mathscr{A})\right)_{j\in\mathscr{A}\cap\mathscr{J}_f}$. Condition (iii) in Assumption 1 implies that $D_{\mathbf{p}}\mathbf{s}^f$ is diagonally dominant and, hence, non-singular. Then, the vector of inverse quasi own-price elasticities for a firm f with $\mathscr{A} \cap \mathscr{J}_f \neq \emptyset$ is given by

$$\hat{\boldsymbol{\epsilon}}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) := \left(\left(D_{\mathbf{p}^f}\boldsymbol{s}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A})\right)^{\intercal}\right)^{-1}\boldsymbol{s}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) \in \mathbb{R}^{\mathscr{A}\cap\mathscr{J}_f}.$$

Therefore, if firms solve a static profit-maximization problem of the form

$$\max_{\mathbf{p}^f\in\mathbb{R}^{\mathscr{A}\cap\mathscr{J}_f}} \mathbf{s}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A})^{\intercal}(\mathbf{p}^f - \mathbf{c}^f), \tag{1}$$

for an arbitrary marginal cost vector $\mathbf{c}^f \in \mathbb{R}^{\mathscr{A}\cap\mathscr{J}_f}$, then the solution to the system of FOCs can be written as a markup formula equal to $\mathbf{p}^f - \mathbf{c}^f = -\hat{\boldsymbol{\epsilon}}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A})$. We denote the vector of all own-price elasticities by $\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) := \left(\hat{\boldsymbol{\epsilon}}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) : f \in \mathscr{F} \text{ such that } \mathscr{A} \cup \mathscr{J}_f \neq \emptyset\right) \in \mathbb{R}^{\mathscr{A}}$.

In Section 6 we show that Assumption 1 is satisfied by a rich set of discrete-choice models where firms sell differentiated products. Our demand assumptions cover products that are imperfect substitutes and products that are nearly perfect substitutes. Note that with completely undifferentiated products, Bertrand competition is the strongest and the demand functions s_j are not continuous. Nonetheless, such demand functions can be approximated by demand functions satisfying Assumption 1. An important class of functions that satisfies Assumption 1 is multinomial logit and nested logit demand. We demonstrate some of our theoretical results using these demand systems, which we define formally later on.

We additionally impose Assumption 2, which allows us to invoke Konovalov and Sándor (2010) to establish that there exists a unique pure-strategy equilibrium given marginal costs.

Assumption 2. *The vector of inverse quasi own-price elasticities $\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A})$ satisfies the following two conditions:*

i) $\det\left(-D_{\mathbf{p}^f}\hat{\boldsymbol{\epsilon}}^f(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) - I_{\mathscr{A}\cap\mathscr{J}_f}\right) \neq 0$ *for all* $\mathbf{p} \in \mathbb{R}^{\mathscr{A}}$, $\boldsymbol{\theta} \in \mathscr{T}$, *and* $\mathscr{A} \subset \mathscr{J}$, $f \in \mathscr{F}$ *;*

ii) $\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) - I_{\mathscr{A}}\right) \neq 0$ *for all* $\mathbf{p} \in \mathbb{R}^{\mathscr{A}}$, $\boldsymbol{\theta} \in \mathscr{T}$, *and* $\mathscr{A} \subset \mathscr{J}$.

Here, $I_{\mathscr{S}} \in \mathbb{R}^{\mathscr{S}\times\mathscr{S}}$ *denotes an identity matrix.*

Condition (i) in Assumption 2 is reminiscent of the commonly made assumption of quasi-concavity or log-concavity in a single-product setting. However, quasi-concavity is not sufficient when considering multi-product demand (see Hanson and Martin, 1996). Condition (i) is exactly the assumption required in Kellogg (1976) guaranteeing that each firm's optimization problem of the form in equation (1) admits a unique solution $\mathbf{p}^{f,\mathrm{BR}}(\mathbf{c}^f,\boldsymbol{\theta},\mathscr{A})$. Condition (ii) is exactly the assumption required in Kellogg (1976) guaranteeing that there is a unique solution to the system of FOCs of all firms. The solution of this system $\mathbf{p}^*((\mathbf{c}^f : f \in \mathscr{F}),\boldsymbol{\theta},\mathscr{A})$ corresponds to profit-maximizing prices and hence a competitive pricing equilibrium given marginal costs $(\mathbf{c}^f : f \in \mathscr{F})$.

To streamline our exposition, we omit the conditioning arguments $\boldsymbol{\theta}$, $\mathscr{A}$, or both in all expressions whenever the meaning is unambiguous. We sometimes also write $s_{j,t}(\mathbf{p},\mathbf{K}) := s_j(\mathbf{p};\boldsymbol{\theta}_t,\mathscr{A}(\mathbf{K}))$ and $\mathbf{s}_t(\mathbf{p},\mathbf{K}) := (s_{j,t}(\mathbf{p},\mathbf{K}))_{j\in\mathscr{A}(\mathbf{K})}$.

Markov perfect equilibrium. The payoff-relevant state in this pricing game is given by the vector of capacities $\mathbf{K} := (K_j)_{j\in\mathcal{J}}$ and time t. We study Markov perfect equilibria (MPE) in pure strategies in which each firm's strategy is measurable with respect to $(\mathbf{K}, t)$. We denote a Markov pricing strategy of firm f by $\mathbf{p}_{f,t}(\mathbf{K}) = \left(p_{j,t}(\mathbf{K})\right)_{j\in\mathcal{A}(\mathbf{K})\cap\mathcal{J}_f}$.

3 The Monopoly Benchmark

Before analyzing the dynamic game, we first consider the special case where a single firm owns all products. We show that the monopoly solution exhibits several desirable properties, which we contrast with those of the oligopoly case. We also introduce notation relevant in the oligopoly case.

Consider the monopolist M that offers J products for sale. The firm maximizes expected revenues $\mathbb{E}\left[\sum_{t=0}^{T-\Delta} \Delta\lambda_t\, \mathbf{p}_t \cdot \mathbf{s}_t(\mathbf{p}_t, \mathbf{K}_t)\right]$ subject to $\mathbf{K}_t \geq \mathbf{0}$ by choosing a price process $\{\mathbf{p}_t\}_{t=0}^{T-\Delta}$ that is adapted with respect to the arrival and demand process.[13] If we express this as an optimal control problem, the monopolist at time $t \leq T-\Delta$, given capacity vector $\mathbf{K}$, solves

$$\Pi_{M,t}(\mathbf{K};\Delta) = \max_{\mathbf{p}} \Delta\lambda_t \sum_{j\in\mathcal{J}} \underbrace{s_{j,t}(\mathbf{p},\mathbf{K})\left(p_j + \Pi_{M,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)\right)}_{\substack{\text{payoff from selling product } j\\ +\text{ continuation value with}\\ \text{one fewer unit of } j}} + \underbrace{\left(1-\Delta\lambda_t \sum_{j\in\mathcal{J}} s_{j,t}(\mathbf{p},\mathbf{K})\right)}_{\substack{\text{probability of}\\ \text{no purchase}}} \underbrace{\Pi_{M,t+\Delta}(\mathbf{K};\Delta)}_{\substack{\text{continuation value}\\ \text{with same}\\ \text{capacities}}},$$

where $\mathbf{e}_j \in \mathbb{N}^{\mathcal{J}}$ is the unit vector with 1 in the jth position and 0 everywhere else. The firm receives a revenue of p_j and a continuation value in period $t+\Delta$ with one fewer unit of j if it sells j. If the firm does not sell at all, the capacity vector remains unchanged, and time moves forward by Δ. The key tension is obtaining revenue versus the option value of holding capacity for the future.

The firm faces two boundary conditions: (i) $\Pi_{M,T}(\mathbf{K};\Delta) = 0$ for all $\mathbf{K}$ and (ii) $\Pi_{M,t}(\mathbf{K};\Delta) = 0$ if $K_j = 0$ for all $j \in \mathcal{J}$. The boundary conditions ensure that capacity is scrapped with zero value after the deadline T and that the firm cannot oversell.

Critically, the optimal price vector at each state $(\mathbf{K}, t)$ can be found by considering a static maximization problem parameterized by $\boldsymbol{\theta}$ and $\boldsymbol{\omega} = (\omega_j)_{j\in\mathcal{A}(\mathbf{K})}$, where $\omega_j = \Pi_{M,t}(\mathbf{K};\Delta) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j;\Delta)$

[13] By *adapted*, we mean that it is measurable with respect to the filtration generated by demand.

is commonly referred to as the opportunity cost of selling product j.[14] The static maximization problem is

$$\max_{\mathbf{p}} \sum_{j\in\mathscr{A}} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{A})\big(p_j-\omega_j\big),$$

which is simply demand times a markup. We denote the profit-maximizing prices by $\mathbf{p}^M(\boldsymbol{\omega},\boldsymbol{\theta};\mathbf{K}) := (p_j^M(\boldsymbol{\omega},\boldsymbol{\theta};\mathbf{K}))_{j\in\mathscr{A}(\mathbf{K})}$. This expression is key because it conveys how scarcity, via opportunity costs, affects a firm's pricing decision. We will derive an analogous expression for the oligopoly game in order to characterize equilibrium prices that encompass all firms' scarcity.

By Kellogg (1976), Assumption 2 implies that there is a unique optimal price vector for any time period t, which is continuous in $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. As $\Delta \to 0$, the solution of the optimal control problem converges to the solution of a system of ODEs. We state this result in Lemma 1.

Lemma 1. $\Pi_{M,t}(\mathbf{K}) := \lim_{\Delta\to 0}\Pi_{M,t}(\mathbf{K};\Delta)$ *solves a system of ordinary differential equations*

$$\dot{\Pi}_{M,t}(\mathbf{K}) = -\lambda_t \sum_{j\in\mathscr{A}(\mathbf{K})} s_{j,t}\left(\mathbf{p}^M(\boldsymbol{\omega}_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K}),\mathbf{K}\right)\Big(p_j^M(\boldsymbol{\omega}_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})-\omega_{j,t}(\mathbf{K})\Big),\ \forall \mathbf{K}\le\mathbf{K}_0, \tag{2}$$

where $\boldsymbol{\omega}_t(\mathbf{K}) := \big(\Pi_{M,t}(\mathbf{K})-\Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\big)_{j\in\mathscr{A}(\mathbf{K})}$ *with boundary conditions (i)* $\Pi_{M,T}(\mathbf{K}) = 0$ *for all* $\mathbf{K}\le\mathbf{K}_0$, *and (ii)* $\Pi_{M,t}(\mathbf{K}) = 0$ *if* $K_j = 0$ *for all* $j\in\mathscr{J}$.

Lemma 1, which generalizes Gallego and Van Ryzin (1994) to an arbitrary number of products and richer demand functions, provides a convenient characterization of the monopoly pricing problem for each state $(\mathbf{K},t)$. Here $\boldsymbol{\omega}_t(\mathbf{K}) := \big(\Pi_{M,t}(\mathbf{K})-\Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\big)_{j\in\mathscr{A}(\mathbf{K})}$ defines the vector of opportunity costs. The profit-maximizing pricing policy $\mathbf{p}_t^M(\mathbf{K})\in\mathbb{R}^{\mathscr{A}(\mathbf{K})}$ solves

$$\mathbf{p} = \underbrace{\boldsymbol{\omega}_t(\mathbf{K})}_{\text{opportunity costs}} - \underbrace{\hat{\epsilon}(\mathbf{p};\boldsymbol{\theta}_t,\mathscr{A}(\mathbf{K}))}_{\substack{\text{inverse quasi}\\ \text{own-price elasticities}}}. \tag{3}$$

Hence, the optimal pricing strategy $\mathbf{p}_t^M(\mathbf{K})$ is continuous in time, and its dynamics are governed by the evolution of quasi own-price elasticities and opportunity costs. The opportunity costs in turn

[14] Strictly speaking, the opportunity cost of selling product j is represented by $\omega_j - \sum_{j'\neq j}\frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})}\omega_{j'}$. This formula indicates that by choosing to sell product $j\in\mathscr{A}(\mathbf{K})$, the firm forgoes the opportunity to sell any other product j' in a given period.

depend on the stochastic process of remaining capacity $\mathbf{K}_t$.[15]

Proposition 1 (Properties of Monopoly Dynamic Pricing)**.** *The solution to the monopoly revenue maximization problem in Lemma 1 satisfies the following:*

(i) $\Pi_{M,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and increasing in* K_j*, for all* $j \in \mathscr{J}$ *and* $t < T$*;*

(ii) $\omega_{j,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and decreasing in* K_j*, for all* $j \in \mathscr{J}$ *and* $t < T$*;*

(iii) the stochastic process $\omega_{j,t\wedge\tau}(\mathbf{K}_t)$*,* $\tau := \inf\{t \geq 0 | K_{j,t} \leq 1\}$*, is a submartingale.*

Proposition 1 summarizes the properties of the solution to the monopolist's dynamic pricing problem. Statements (i) and (ii) of Proposition 1 generalize the findings in Gallego and Van Ryzin (1994). The properties imply that more capacity and more time remaining increase continuation profits, continuation profits are concave in capacity, and opportunity costs are decreasing toward the deadline if $\mathbf{K}$ is held fixed. Note that $\omega_{j,t}(\mathbf{K})$ captures the marginal cost in the period-t problem of the monopolist. Hence, (ii) implies that if demand is constant over time ($\dot{\boldsymbol{\theta}}_t = 0$), then the optimal price policy $\mathbf{p}_t^M(\mathbf{K})$ is decreasing in t. However, if demand becomes more inelastic over time, then the optimal price policy may be increasing or decreasing over time, depending on whether the cost or demand effect dominates.

Statement (iii) asserts that, on average, opportunity costs are increasing. This implies that if demand is weakly increasing over time, demand uncertainty alone can cause price paths to increase on average. This result can rationalize the inverted U-shape in prices documented in the literature (e.g., for airlines in McAfee and Te Velde, 2006). The average observed prices decline closer to the deadline because once a product sells out, its price is excluded from the average.

Remark 1. The analysis generalizes to firms that have material marginal costs for each unit of their products. The marginal cost is simply added to the corresponding opportunity costs.

4 Dynamic Pricing with Competition

With the additional notation provided in the previous section, we now turn to the dynamic oligopoly game. With more than one firm, there are additional strategic forces. First, each firm's residual de-

[15]We abuse notation slightly by denoting the optimal pricing strategy $\mathbf{p}_t^M(\mathbf{K})$, while also denoting the static optimal price parameterized by $(\boldsymbol{\omega}, \boldsymbol{\theta})$ by $\mathbf{p}^M(\boldsymbol{\omega}, \boldsymbol{\theta}; \mathbf{K})$.

mands increase with competitors' prices, as in static oligopoly settings. Second, a firm cares about whether a competitor sells or not because competitor capacity affects future competition. As a result, the existence and uniqueness of pure-strategy equilibria are not straightforward, because we cannot appeal to the insights from previous oligopoly frameworks, including Caplin and Nalebuff (1991) and Nocke and Schutz (2018).

A pure-strategy Markov perfect equilibrium in the dynamic game, if it exists, can be constructed recursively. At the deadline T, the payoffs of all firms are zero for all capacity vectors. In earlier periods, we can write firm f's value function given pricing policies $\mathbf{p}_t(\mathbf{K}) := \big(p_{j,t}(\mathbf{K})\big)_{j\in\mathscr{A}(\mathbf{K})}$ recursively as[16]

$$\Pi_{f,t}(\mathbf{K};\Delta) = \Delta\lambda_t \Bigg(\underbrace{\sum_{j\in\mathscr{A}(\mathbf{K})\cap\mathscr{J}_f} s_{j,t}(\mathbf{p}_t(\mathbf{K}),\mathbf{K})\,(p_{j,t}(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)) +}_{\text{payoff from own sale}}$$
$$\underbrace{\sum_{j'\in\mathscr{A}(\mathbf{K})\cap(\mathscr{J}\setminus\mathscr{J}_f)} s_{j',t}(\mathbf{p}_t(\mathbf{K}),\mathbf{K})\,\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{j'};\Delta)}_{\text{payoff if competitor sells } j'} \Bigg) + \underbrace{\Bigg(1-\Delta\lambda_t \sum_{j'\in\mathscr{A}(\mathbf{K})} s_{j',t}(\mathbf{p}_t(\mathbf{K}),\mathbf{K})\Bigg)\Pi_{f,t+\Delta}(\mathbf{K};\Delta)}_{\text{payoff if no firm sells}}.$$

Similar to the monopoly problem, subtracting $\Pi_{f,t+\Delta}(\mathbf{K};\Delta)$ does not change pricing incentives in state $(\mathbf{K},t)$. Hence, letting

$$\omega^f_{j,t}(\mathbf{K};\Delta) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta),$$

each firm f chooses its prices $\mathbf{p}^f_t(\mathbf{K})$ to maximize

$$\begin{aligned}&\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta) = \\ &\Delta\lambda_t\Bigg(\sum_{j\in\mathscr{A}(\mathbf{K})\cap\mathscr{J}_f} s_{j,t}\,(\mathbf{p}_t(\mathbf{K}),\mathbf{K})\Big(p_{j,t}(\mathbf{K}) - \omega^f_{j,t}(\mathbf{K};\Delta)\Big) - \sum_{j'\in\mathscr{A}(\mathbf{K})\cap(\mathscr{J}\setminus\mathscr{J}_f)} s_{j',t}\,(\mathbf{p}_t(\mathbf{K}),\mathbf{K})\,\omega^f_{j',t}(\mathbf{K};\Delta)\Bigg).\end{aligned} \tag{4}$$

The first part of equation (4) is analogous to the monopoly setting—expected demand times a markup. The second part of the equation measures how a firm is affected by competitor scarcity, weighted by competitor residual demand. The ωs in equation (4) are critical to studying dynamic pricing with competition and are defined as follows.

[16] We omit Δ in the pricing policies, for readability.

Definition 1. For $j \in \mathscr{A}(\mathbf{K}) \cap \mathscr{J}_f$, we call $\omega^f_{j,t}(\mathbf{K};\Delta)$ an *own-scarcity effect*, and for $j \in \mathscr{A}(\mathbf{K}) \cap (\mathscr{J} \setminus \mathscr{J}_f)$, we call $\omega^f_{j,t}(\mathbf{K};\Delta)$ a *competitor-scarcity effect*.

The own-scarcity effects capture the impact of a sale of a firm's own product on own continuation profits. The competitor-scarcity effects capture the impact of a sale of a competitor's product on own continuation profits.

Importantly, prices in state $(\mathbf{K}, t)$ do not affect these scarcity effects but rather the probability that they are realized. Therefore, for any pure-strategy equilibrium Markov pricing policy $(\mathbf{p}^*_t(\cdot))_{t=0,\dots T-\Delta}$ of the dynamic game, $\mathbf{p}^*_t(\mathbf{K})$ is an equilibrium of a stage game where each firm f maximizes equation (4) given these scarcity effects. To make progress, we start by analyzing generic stage games parameterized by a scarcity matrix defined as

$$\Omega_t(\mathbf{K};\Delta) = \left(\omega^f_{j,t}(\mathbf{K};\Delta)\right)_{f,j}.$$

Example 1 contains a scarcity matrix.

Example 1. Suppose

$$\Omega_t(\mathbf{K};\Delta) = \begin{bmatrix} 3 & 4 & -1 \\ -.3 & -.2 & 1 \end{bmatrix}.$$

In this example, there are two firms and three products. Products one and two are owned by firm 1; product three is owned by firm 2. In this example, $\omega^1_1 = 3$ and $\omega^1_2 = 4$ are the own-scarcity effects for firm 1. The entry in the first row and third column, $\omega^1_3 = -1$, is the competitor-scarcity effect. The second row can be similarly read from the perspective of firm 2. $\lozenge$

A challenge arises because the signs of the scarcity effects are unknown, and a pure-strategy Markov equilibrium can only exist if all stage games attain a pure-strategy equilibrium. Therefore, in the next section, we analyze the stage game before returning to the dynamic game.

4.1 Analysis of the Stage Game

Consider a static game with a set $\mathscr{F}$ of players and a set $\mathscr{J}$ of products with ownership structure $(\mathscr{J}_f)_{f\in\mathscr{F}}$ that is parameterized by a matrix $\Omega = \left(\omega^f_j\right)_{f,j} \in \mathbb{R}^{\mathscr{F}\times\mathscr{J}}$. When we analyze the stage game, we omit the parameters $\boldsymbol{\theta}$ and assume that all products are available, without loss of generality. All players f simultaneously choose their prices $\mathbf{p}^f = (p_j)_{j \in \mathscr{J}_f}$ to maximize

$$\sum_{j \in \mathscr{J}_f} s_j(\mathbf{p})\left(p_j - \omega_j^f\right) - \sum_{j' \notin \mathscr{J}_f} s_{j'}(\mathbf{p})\,\omega_{j'}^f.$$

We are interested in the pure-strategy equilibria of this game and proceed as follows. We first show that equilibrium prices must be interior, using Assumption 1. Thus, in a pure-strategy equilibrium, each firm f's best-response prices $\mathbf{p}^f$ must solve

$$\mathbf{p}^f = \underbrace{\left(\left(D_{\mathbf{p}^f}\mathbf{s}_f(\mathbf{p})\right)^{\intercal}\right)^{-1} D_{\mathbf{p}^f}\left(\mathbf{s}(\mathbf{p})^{\intercal}\boldsymbol{\omega}^f\right)^{\intercal}}_{\substack{\text{net opportunity costs}\\ \text{of selling}}} - \underbrace{\left(\left(D_{\mathbf{p}^f}\mathbf{s}_f(\mathbf{p})\right)^{\intercal}\right)^{-1}\mathbf{s}_f(\mathbf{p})}_{\substack{\text{inverse quasi}\\ \text{own-price elasticities}}} =: \mathbf{g}_f(\mathbf{p};\boldsymbol{\omega}^f), \tag{5}$$

where $\boldsymbol{\omega}^f := \left(\omega_j^f\right)_j$ are firm f's own- and competitor-scarcity effects. To show the existence and uniqueness of equilibria of this game, we impose conditions such that each firm's systems of FOCs have a unique solution $\mathbf{p}^f$ and that all firms' FOCs also jointly has a unique solution.

In the absence of competitor-scarcity effects (i.e., $\omega_j^f = 0$ for $j \notin \mathscr{J}_f$), the game corresponds to a classic oligopoly price competition game. In that case, Assumptions 1 and 2(i) guarantee that all players have a unique best response. It follows from Brouwer's fixed-point theorem that there exists an equilibrium.[17] If there are multiple best-response prices, the best-response set is not convex anymore and pure-strategy equilibria may not exist. In the presence of competitor-scarcity effects, such multiplicities of best responses can arise unless we impose additional conditions.

4.1.1 Stage-game Equilibria for IIA Demand Systems

The key complexity in studying a model with competitor-scarcity effects is that there may be multiple best-response price vectors for each firm that do not form a convex set. Hence, we cannot apply classic fixed point theorems, such as by Brouwer or Kakutani, to establish the existence of pure-strategy equilibria. We offer two paths forward. First, we show the existence of equilibria, regardless of Ω, if demand satisfies the independence of irrelevant alternatives property. Second, in Section 4.1.2, we add restrictions on Ω to establish the existence (and uniqueness) of pure-strategy

[17] As pointed out by Vives (2018) and Nocke and Schutz (2018), this game is generally not supermodular.

equilibria for general demand systems.

Assumption 3 (Independence of Irrelevant Alternatives)**.** *A demand system* $\mathbf{s} = (s_j)_{j \in \mathscr{J}}$ *satisfies the independence of irrelevant alternatives property (IIA) if* $\frac{\partial}{\partial p_j} \frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})} = 0$ *for* $j \neq j_1, j_2 \in \mathscr{J} \cup \{0\}$.

Assumption 3 states that the relative market share of two products does not change if the price of a third product changes.[18] We use this assumption to derive a markup formula in Proposition 2, which allows us to establish the existence of equilibria in the stage game.

Proposition 2 (Markup Formula Under IIA)**.** *Let Assumptions 1, 2, and 3 hold. Then, there exists an equilibrium of the stage game for any scarcity matrix* Ω*. All equilibrium prices* $\mathbf{p}^*(\Omega)$ *coincide with the equilibrium prices of a game with a set* $\mathscr{J}$ *of players who each simultaneously choose a price* p_j *maximizing* $s_j(\mathbf{p})\big(p_j - c_j\big(\mathbf{p}_{-j};\Omega\big)\big)$*, with a cost function equal to*

$$c_j\left(\mathbf{p}_{-j};\Omega\right) := \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j})\omega_{j'}^f, \text{ and } \tilde{s}_{j,j'}(\mathbf{p}_{-j}) := \frac{s_{j'}(\mathbf{p})}{1 - s_j(\mathbf{p})}. \quad (6)$$

Proposition 2 implies that even with multiple firms and products, the FOC that implicitly define firms' best-response functions can be written in a markup formulation for each product, where $\epsilon_j(\mathbf{p}) = \frac{\partial s_j(\mathbf{p})}{\partial p_j} \frac{p_j}{s_j(\mathbf{p})}$ is the elasticity of demand for product j. This provides us a Lerner expression,

$$\frac{p_j^*(\Omega) - c_j(\mathbf{p}_{-j};\Omega)}{p_j^*(\Omega)} = -\frac{1}{\epsilon_j(\mathbf{p}^*(\Omega))}.$$

The marginal cost c_j is independent of firm j's price. This marginal cost property, along with Assumptions 1 and 2(i), guarantees that best-response prices are unique, holding other products' prices fixed. Hence, pure-strategy equilibria exist by Brouwer's fixed-point theorem.

Other demand systems satisfying our demand assumptions but not the IIA assumption cannot be solved in this way because the own-product price can enter the marginal cost term.

4.1.2 Stage-game Equilibria for General Demand Systems

With general demand functions satisfying Assumptions 1 and 2, we can immediately establish uniqueness using the fixed-point theorem in Kellogg (1976) if there are no competitor-scarcity

[18] A special case of such a demand system is multinomial logit demand analyzed in Lin and Sibdari (2009), where each firm owns only a single product.

effects (Lemma 2).[19] When there are competitor scarcity effects, we impose Assumption 4 in order to invoke the fixed-point theorem in Kellogg (1976) and Konovalov and Sándor (2010) to ensure uniqueness of best-response price vectors and uniqueness of equilibria.

Lemma 2. *Given Assumptions 1 and 2, any stage game without competitor scarcity effects (i.e., $\omega_j^f = 0$ for all $j \notin \mathscr{J}_f$) admits a unique pure-strategy equilibrium.*

Assumption 4. *We assume the following two conditions hold for a fixed* Ω*:*

i) $\det\left(D_{\mathbf{p}_f}\mathbf{g}_f(\mathbf{p},\boldsymbol{\omega}^f) - I_{\mathscr{J}_f}\right) \neq 0$ *for all* $\mathbf{p}$ *and* f*;*

ii) $\det\left(D_{\mathbf{p}}\mathbf{g}(\mathbf{p},\Omega) - I_{\mathscr{J}}\right) \neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p},\Omega) := \left(\mathbf{g}_f(\mathbf{p},\boldsymbol{\omega}^f) : f \in \mathscr{F}\right) \in \mathbb{R}^{\mathscr{J}}$.

Assumption 4(i) ensures that best-response price vectors are unique everywhere. In the single-dimensional case, this assumption is akin to requiring that the second-order condition never changes sign. Assumption 4(ii) guarantees that the system of all firms' FOCs admits a unique solution. We can then prove existence and uniqueness for general demands, in Proposition 3.

Proposition 3 (Stage-game Equilibrium Uniqueness)**.** *Consider a stage game parameterized by* Ω*. It follows that*

i) there exists a pure-strategy equilibrium if Assumptions 1 and 4(i) are satisfied;

ii) there exists a unique pure-strategy equilibrium if Assumptions 1 and 4 are satisfied.

Not all scarcity matrices Ω satisfy Assumption 4. Indeed, it is possible that extreme competitor scarcity effects may either cause non-existence in pure strategies or multiplicity of equilibria in the stage game. Although we have not encountered non-existence in simulations, we have encountered multiplicity of equilibria. Multiplicity of equilibria does not require extreme assumptions on demand. In Example 2, we demonstrate that multiplicities can arise even with a demand system that satisfies the IIA assumption.[20] Essentially, when both competitor-scarcity effects are very negative, Assumption 4(ii) can be violated, and we obtain a region with multiple pure-strategy equilibria. When we analyze the dynamic game in Section 5, we construct Markov perfect equilibria where Ω stays within the set of Ωs that satisfy Assumption 4.

[19] When all competitor-scarcity effects are zero, Assumption 4 coincides with Assumption 2. Nocke and Schutz (2018) show that the pricing game without competitor scarcity effects and IIA demand admits a unique pure-strategy equilibrium. Indeed, absent competitor scarcity effects, IIA implies Assumption 4.

[20] Note that if all competitor-effects were zero, uniqueness follows from Nocke and Schutz (2018) or Lemma 2.

Example 2. Consider a duopoly example with two products and logit demand of the form $s_f(\mathbf{p}) = \frac{\exp(1-p_f)}{1+\sum_{f'\in\{1,2\}}\exp(1-p_{f'})}$ for $f \in \{1,2\}$. Logit demand systems belong to the class of IIA demand systems. For logit demand, Assumption 4(i) is equivalent to

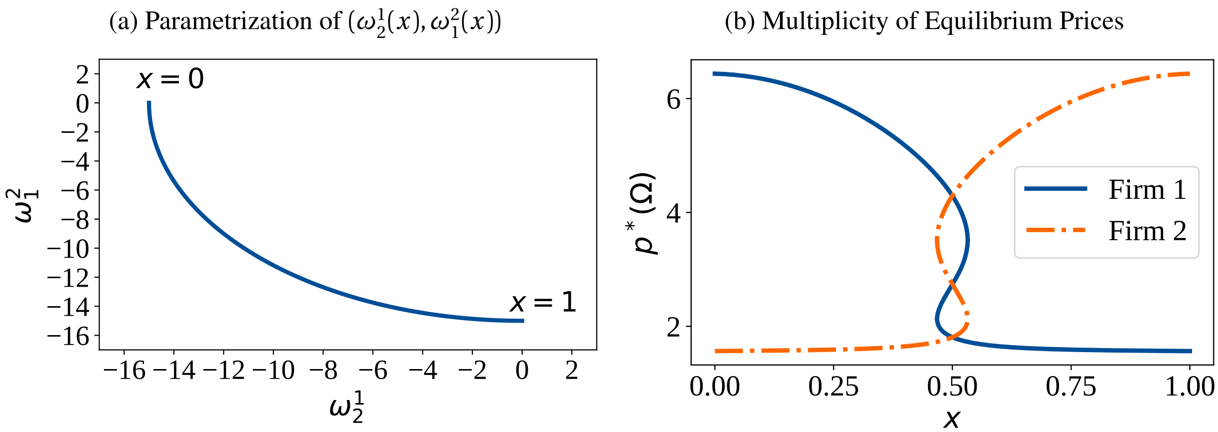
$$-\frac{s_f(\mathbf{p})}{1-s_f(\mathbf{p})} - 1 \neq 0 \text{ for all } \mathbf{p} \text{ and } f \neq f',$$

which is satisfied for all Ω, and Assumption 4(ii) is equivalent to

$$\left(s_1(\mathbf{p}) + \omega_2^1 s_0(\mathbf{p})\right)\left(s_2(\mathbf{p}) + \omega_1^2 s_0(\mathbf{p})\right) \neq 1 + \frac{1-s_1(\mathbf{p})-s_2(\mathbf{p})}{s_1(\mathbf{p})s_2(\mathbf{p})} \text{ for all } \mathbf{p}.$$

We can find ω_2^1 and ω_1^2 such that this condition is violated. Figure 1(a) shows a particular parameterization of competitor-scarcity effects. Figure 1(b) depicts prices of both firms where their systems of FOCs are satisfied given the competitor-scarcity effects in panel (a). For example, if $\omega_2^1 = \omega_1^2 = -\frac{15\sqrt{2}}{2}$, corresponding to $x = \frac{1}{2}$ in the parameterized curve, then there are three equilibria: two equilibria where one firm sets a low price (≈ 1.8) and the other sets a high price (≈ 4.3), and one equilibrium where both firms set the same price (≈ 2.7). It is easy to verify that both firms play global best-responses at all fixed points. $\Diamond$

Figure 1: Multiplicity in Stage-Game Equilibria



Note: In these graphics we parameterize (ω_1^2, ω_2^1) with a curve $(\omega_2^1(x), \omega_1^2(x)) = \left(-15\cos\left(\frac{\pi}{2}x\right), -15\sin\left(\frac{\pi}{2}x\right)\right), x \in [0,1]$, where we set $(\omega_1^1, \omega_2^2) = (0,0)$. Panel (a) depicts the parameterized curve, and panel (b), the prices of firm 1 and 2, where both firms' FOCs are satisfied given (ω_1^2, ω_2^1) at varying values of x.

4.1.3 Strategic Forces in the Stage Game

One of the reasons why our model yields such rich predictions is that the stage game can be one of either strategic complements or strategic substitutes (Bulow et al., 1985), depending on the direction and magnitude of competitor-scarcity effects. That is, the model yields both Bertrand-like and Cournot-like strategic interactions. We show in Section 5 that equilibrium dynamics are shaped by switching between these different incentives.

We illustrate these effects using the same setup as in Example 2. With two firms, the FOC in equation (5) is given by

$$p_f = \frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})\left(\frac{\partial s_f}{\partial p_f}(\mathbf{p})\right)^{-1}\omega^f_{f'} + \omega^f_f - s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1} =: g_f(\mathbf{p};\boldsymbol{\omega}^f).$$

The competitor's price is a strategic complement of firm f's price if an increase in the competitor price increases firm f's best-response price, i.e., $\frac{\partial g_f}{\partial p_{f'}} > 0$. If $\frac{\partial g_f}{\partial p_{f'}} < 0$, then the competitor's price is a strategic substitute. Now, consider the cross derivative of a firm's FOC,

$$\frac{\partial}{\partial p_{f'}} g_f(\mathbf{p}) = \frac{\partial}{\partial p_{f'}}\left(\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})\left(\frac{\partial s_f}{\partial p_f}(\mathbf{p})\right)^{-1}\right)\omega^f_{f'} - \frac{\partial}{\partial p_{f'}}\left(s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1}\right). \tag{7}$$

The inverse-quasi own-price elasticity under logit demand is defined as $s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1} = -(1-s_f(\mathbf{p}))^{-1}$, which is decreasing in the competitor's price $p_{f'}$. Hence, if the competitor-scarcity effect $\omega^f_{f'}$ is zero, then the competitor price is a strategic complement, as in Caplin and Nalebuff (1991) and Nocke and Schutz (2018). If the competitor-scarcity effect $\omega^f_{f'}$ is positive, the competitor price remains a strategic complement because $\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})\left(\frac{\partial s_f}{\partial p_f}(\mathbf{p})\right)^{-1} = -\frac{\exp(1-p_{f'})}{1+\exp(1-p_{f'})}$ is increasing in $p_{f'}$. However, when $\omega^f_{f'}$ is large and negative, the competitor price is a strategic substitute.

Recall that in the dynamic game, the competitor-scarcity effect $\omega^f_{f'}$ measures the impact on a firm's profit if a rival sells. The classic intuition is that scarcity increases future prices. This is consistent with $\omega^f_{f'}$ being negative because it implies that a firm benefits in the future when a rival sells. Large, negative competitor-scarcity effects dissuade a firm from selling today and can flip the incentives in the stage game. We demonstrate these incentives in Figure 8 in the Online Appendix, and in Section 5.1 we investigate which sources of scarcity dominate.

4.2 Continuous-Time Approximation of Dynamic Markov Equilibria

To study equilibrium pricing dynamics, we appeal to continuous-time approximations so that we can work with time-derivatives and a system of ODEs that describe continuation profits and pricing dynamics. We proceed by first showing that Markov equilibria of the discrete game, if they are unique, converge to a solution of the ODE system. We then show that one of the solutions of the ODE system must correspond to the limit of discrete-time Markov equilibria as $\Delta \to 0$, provided the scarcity matrices Ω remain within a bounded open set where Assumption 4 holds. Finally, we show that all solutions of the system of ODEs always stay in this set close to the deadline. We thus establish both existence and uniqueness.

Assume that there exists a time interval length $\bar{\Delta}$ and initial capacity vector $\mathbf{K}_0$ such that for all $\Delta < \bar{\Delta}$, a unique Markov equilibrium exists. Under this premise, we show in Theorem 1 that a sequence of unique equilibria of the dynamic game converges as $\Delta \to 0$ to a system of ODEs. The argument follows analogously to the monopoly optimal control problem in Lemma 1.[21] Let $\mathbf{p}^*(\Omega, \boldsymbol{\theta}; \mathbf{K})$ be the vector of stage-game equilibrium prices given Ω and $\mathscr{A}(\mathbf{K})$.

Theorem 1 (Continuous-Time Limit)**.** *Assume that there exists a $\bar{\Delta} > 0$ so that for all $\Delta < \bar{\Delta}$, there is a unique pure-strategy equilibrium of the dynamic game, and for all t, $\mathbf{K} \leq \mathbf{K}_0$, $\Delta < \bar{\Delta}$, $(\Omega_t(\mathbf{K};\Delta), \boldsymbol{\theta}_t) \in \mathscr{M}$ for an open bounded $\mathscr{M}$ such that any stage game with $\mathscr{A}(\mathbf{K})$ available products and parameterized by $(\Omega, \boldsymbol{\theta}) \in \mathscr{M}$ admits a unique equilibrium. Then, each equilibrium value function $\Pi^*_{f,t}(\mathbf{K};\Delta)$ converges as $\Delta \to 0$ to a limit $\Pi^*_{f,t}(\mathbf{K})$ that solves the following system of ordinary differential equations:*

$$\begin{aligned}
&\dot{\Pi}_{f,t}(\mathbf{K}) = \\
&-\lambda_t\Bigg(\sum_{j \in \mathscr{A}(\mathbf{K}) \cap \mathscr{J}_f} s_j\big(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}); \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})\big) \cdot \big(p^*_j(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}) - (\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_j))\big) \\
&- \sum_{j' \in \mathscr{A}(\mathbf{K}) \cap (\mathscr{J} \setminus \mathscr{J}_f)} s_{j'}\big(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}); \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})\big) \cdot \big(\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_{j'})\big)\Bigg),
\end{aligned}$$

for $\mathbf{K} \leq \mathbf{K}_0$ and f such that $\mathscr{A}(\mathbf{K}) \cap \mathscr{J}_f \neq \emptyset$, where $\Omega_t(\mathbf{K}) = (\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} -$

[21] Gallego and Hu (2014) state that for any solution to these Hamilton-Jacobi-Bellman equations, the corresponding optimal prices are a Markov equilibrium of the continuous-time, stochastic game. They do not, however, show equilibrium existence. We establish (i) conditions that guarantee that equilibria of the continuous-time game are the limit of equilibria of the discrete-time game, (ii) that a unique solution of the ODE system exists close to the deadline, and (iii) that a Markov equilibrium exists for the discrete-time game if Δ is sufficiently small.

$\mathbf{e}_j))_{j\in\mathscr{A}(\mathbf{K}),f\in\{f|\mathscr{A}(\mathbf{K})\cap\mathscr{J}_f\neq\emptyset\}}$, *with boundary conditions (i)* $\Pi_{f,T}(\mathbf{K})=0$ *for all* $\mathbf{K}$*, (ii)* $\Pi_{f,t}(\mathbf{K})=0$ *if* $K_j=0$ *for all* $j\in\mathscr{J}_f$.

Although Theorem 1 shows that a continuous-time approximation is possible if Markov equilibria are unique, the price dynamics are described implicitly only through the equilibrium prices $\mathbf{p}^*$ of the stage games. Moreover, these equilibrium prices are the result of rich strategic interactions, as we discussed in Section 4.1. To overcome the challenge of calculating equilibrium prices $\mathbf{p}^*$ for all Ω and $\boldsymbol{\theta}$, we derive a differential equation that characterizes equilibrium price paths. Together, Theorem 1 and the differential equation for $\mathbf{p}^*$ define a system of first-order differential equations that can be solved with standard techniques.

To derive this system of equations, note that if a unique equilibrium exists for all $\Delta<\overline{\Delta}$, then all equilibrium prices $\mathbf{p}_t^*(\mathbf{K};\Delta)$ must satisfy the system of FOCs, $\mathbf{g}(\mathbf{p}_t^*(\mathbf{K};\Delta),\Omega_t(\mathbf{K},\Delta),\boldsymbol{\theta}_t;\mathbf{K})=0$.[22] By continuity, the limit price policy $\mathbf{p}_t^*(\mathbf{K}):=\lim_{\Delta\to 0}\mathbf{p}_t^*(\mathbf{K};\Delta)$ must satisfy the system of FOCs $\mathbf{g}(\mathbf{p}_t^*(\mathbf{K}),\Omega_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})=0$ for all $t\in[0,T]$. This allows us to derive differential equations for any capacity vector $\mathbf{K}\le\mathbf{K}_0$ that must be satisfied if $\mathbf{p}_t^*(\mathbf{K})$ is differentiable in t. As long as $\det\big(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}_t^*(\mathbf{K}),\Omega_t^*(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})-I_{\mathscr{A}(\mathbf{K})}\big)\neq 0$ for all $\mathbf{K}\le\mathbf{K}_0$ so that the matrix entering $\dot{\mathbf{p}}_t$ is invertible, $((\Pi^*_{f,t}(\mathbf{K}))_f,\mathbf{p}_t^*(\mathbf{K}))_{\mathbf{K}\le\mathbf{K}_0}$ must be a solution to the following system of ODEs:

$$\left.\begin{aligned}\dot{\Pi}_{f,t}(\mathbf{K})=\;&-\lambda_t\Bigg(\sum_{j\in\mathscr{A}(\mathbf{K})\cap\mathscr{J}_f}s_j\big(\mathbf{p}_t(\mathbf{K});\boldsymbol{\theta}_t,\mathscr{A}(\mathbf{K})\big)\cdot\big(p_{j,t}(\mathbf{K})-(\Pi_{f,t}(\mathbf{K})-\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j))\big)\\&-\sum_{j'\in\mathscr{A}(\mathbf{K})\cap(\mathscr{J}\setminus\mathscr{J}_f)}s_{j'}\big(\mathbf{p}_t(\mathbf{K});\boldsymbol{\theta}_t,\mathscr{A}(\mathbf{K})\big)\cdot\big(\Pi_{f,t}(\mathbf{K})-\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{j'})\big)\Bigg),\\\dot{\mathbf{p}}_t(\mathbf{K})=\;&-\big(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}_t(\mathbf{K}),\Omega_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})-I_{\mathscr{A}(\mathbf{K})}\big)^{-1}\\&\Bigg(\sum_{j,f}D_{\omega_j^f}\mathbf{g}(\mathbf{p}_t(\mathbf{K}),\Omega_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})\dot{\omega}^f_{j,t}(\mathbf{K})+D_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{p}_t(\mathbf{K}),\Omega_t(\mathbf{K}),\boldsymbol{\theta}_t;\mathbf{K})\dot{\boldsymbol{\theta}}_t\Bigg),\end{aligned}\right\}\quad(8)$$

with two boundary conditions, namely (i) and (ii) from Theorem 1, and equilibrium prices at the deadline defined as $\mathbf{p}_T(\mathbf{K})=\mathbf{p}^*(\mathbf{O}_{\mathscr{A}(\mathbf{K})},\boldsymbol{\theta}_T;\mathbf{K})$, $\mathbf{O}_{\mathscr{S}}\in\mathbb{R}^{\mathscr{F}\times\mathscr{S}}$ being a matrix of zeros.

We next prove the existence of solutions of this system of first-order ODEs.[23] First, note that

[22]Note that $\mathbf{g}$ and $\mathbf{g}^f$ are as defined in Section 4.1, but with additional parameters $\boldsymbol{\theta}$ and $\mathbf{K}$, which were omitted in the analysis of the stage game.

[23]This system has a unique solution by the Picard-Lindelöf theorem (Lindelöf (1894)) if the right-hand sides of the equations are Lipschitz-continuous in $\Omega_t(\mathbf{K})$ and $\mathbf{p}_t$. With (nested) logit demand, this property is satisfied in an open bounded set where $\Gamma_f(\cdot)>\epsilon$ $(\epsilon>0)$, with Γ_f as defined next, because the first three demand derivatives are bounded uniformly in $\boldsymbol{\theta}$.

Peano's existence theorem implies that there exists a solution on an interval $(\iota, T]$, $\iota < T$, as long as $\det\big(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}_t(\mathbf{K}), \Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}) - I_{\mathscr{A}(\mathbf{K})}\big) \neq 0$ for all $\mathbf{K} \leq \mathbf{K}_0$. However, in order to derive conclusions about the discrete-time game, we need to ensure that each price vector indeed corresponds to an equilibrium of the stage game. To this end, we construct open bounded sets containing $\Omega = \mathbf{O}_{\mathscr{A}(\mathbf{K})}$ such that Assumption 4 is satisfied.

Consider for every $\mathbf{K} \leq \mathbf{K}_0$ an arbitrary bounded open set $\mathscr{O}^{\mathbf{K}}$ containing $\Omega = \mathbf{O}_{\mathscr{A}(\mathbf{K})}$. In the proof of stage-game uniqueness in Proposition 3, we show that in any equilibrium of a stage game parameterized by $(\Omega, \boldsymbol{\theta})$, prices must lie in a compact box such that $\underline{p} < p_j + \omega_j^f < \overline{p}$, where $\underline{p}, \overline{p}$ can depend on $\boldsymbol{\theta}$. Consequently, there must be an open box $\mathscr{P}_{\mathbf{K}}(\boldsymbol{\theta})$ of price vectors that must contain all equilibrium stage-game prices given $\boldsymbol{\theta}$ as long as $\Omega \in \mathscr{O}^{\mathbf{K}}$.

It is useful to define the following expressions from Assumption 4:

$$\begin{aligned} \Gamma_f(\mathbf{p}, \boldsymbol{\omega}^f, \boldsymbol{\theta}; \mathbf{K}) &:= \det\big(D_{\mathbf{p}_f}\mathbf{g}_f(\mathbf{p}, \boldsymbol{\omega}^f, \boldsymbol{\theta}; \mathbf{K}) - I_{\mathscr{A}(\mathbf{K}) \cap \mathscr{J}_f}\big), \\ \Gamma(\mathbf{p}, \Omega, \boldsymbol{\theta}; \mathbf{K}) &:= \det\big(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}, \Omega, \boldsymbol{\theta}; \mathbf{K}) - I_{\mathscr{A}(\mathbf{K})}\big). \end{aligned}$$

Then, for any $\epsilon < \min_{\mathbf{K} \leq \mathbf{K}_0} |\Gamma(\mathbf{p}_T(\mathbf{K}), \mathbf{O}_{\mathscr{A}(\mathbf{K})}, \boldsymbol{\theta}_T; \mathbf{K})|$, there exist bounded open sets $\mathscr{O}_t^{\mathbf{K}}$ containing $\Omega = \mathbf{O}_{\mathscr{A}(\mathbf{K})}$ such that[24]

$$\mathscr{O}_t^{\mathbf{K}} \subset \big\{\Omega \in \mathscr{O}^{\mathbf{K}} \,\big|\, |\Gamma(\mathbf{p}, \Omega, \boldsymbol{\theta}_t; \mathbf{K})| > \epsilon \text{ for all } f \text{ and } \Gamma_f(\mathbf{p}, \boldsymbol{\omega}^f, \boldsymbol{\theta}_t; \mathbf{K}) \neq 0 \text{ for all } \mathbf{p} \in \mathscr{P}(\boldsymbol{\theta}_t)\big\}$$

and $\{(t, \Omega) | \Omega \in \mathscr{O}_t^{\mathbf{K}}\}$ is open. Fix an arbitrarily small $0 < \epsilon < \min_{\mathbf{K} \leq \mathbf{K}_0} |\Gamma(\mathbf{p}_T(\mathbf{K}), \mathbf{O}, \boldsymbol{\theta}_T; \mathbf{K})|$ to define this set. Note that all $\Omega \in \mathscr{O}_t^{\mathbf{K}}$ satisfy Assumption 4. The following lemma shows that a global solution to the system of ODEs exists on an interval away from the deadline such that $\Omega_t(\mathbf{K})$ remains in this bounded open neighborhood $\mathscr{O}_t^{\mathbf{K}}$.

Lemma 3. *There exists a $\tau \in [0, T)$ such that the system of ODEs in equation* (8) *admits a solution on $(\tau, T]$, where $\tau := \inf\{t \geq 0 \,|\, \Omega_t(\mathbf{K}) \in \mathscr{O}_t^{\mathbf{K}}$ for all $\mathbf{K} \leq \mathbf{K}_0\}$. There also exists a $\bar{\tau} \in [\tau, T)$ such that for all solutions evolving from T, $\Omega_t(\mathbf{K}) \in \mathscr{O}_t^{\mathbf{K}}$ for all $\mathbf{K} \leq \mathbf{K}_0$, $t \in (\bar{\tau}, T]$.*

Importantly, Lemma 3 also shows that there is an open interval of time away from the deadline $(\bar{\tau}, T]$ where all solutions of the ODEs remain in the sets $\mathscr{O}_t^{\mathbf{K}}$. In addition, $\bar{\tau}$ can also be 0, which

[24] This follows by continuity of Γ in all parameters.

would imply that all solutions of the ODEs on $(0, T]$ remain in the sets $\mathcal{O}_t^{\mathbf{K}}$.[25]

To complete the argument for why we can analyze the system of ODEs in equation (8) to derive properties of the discrete-time game, we show for sufficiently small Δ that there exists a unique continuation equilibrium of the discrete game on $t \in (\bar{\tau}, T]$ for all $\mathbf{K} \leq \mathbf{K}_0$. Recall that for a given time horizon T and initial capacity vector $\mathbf{K}_0$, we can construct any pure-strategy equilibrium by backward induction in t.[26] In the last period, $\Omega_T(\mathbf{K}; \Delta) = \mathbf{O}_{\mathscr{A}(\mathbf{K})}$, so the stage game admits a unique solution by Lemma 2. Going backwards in time, there exists a unique continuation equilibrium in period t as long as stage games corresponding to $\Omega_t(\mathbf{K}; \Delta)$ satisfy Assumption 4. Analogously to τ in Lemma 3, define for any Δ,

$$\tau^{\Delta} := \min\left\{t \in \mathbb{T}^{\Delta} \,\middle|\, \Omega_s(\mathbf{K}; \Delta) \in \mathcal{O}_s^{\mathbf{K}} \text{ for all } t \leq s \leq T \text{ and } \mathbf{K} \leq \mathbf{K}_0\right\},$$

where $\mathbb{T}^{\Delta} := \{0, \Delta, \ldots, T - \Delta\}$ and let $\tau^* := \limsup_{\Delta \to 0} \tau^{\Delta}$. We show that $\tau^* \leq \bar{\tau}$, so by Theorem 1, the solution of the discrete-time game must remain close to the continuous-time limit on $(\bar{\tau}, T]$.

Theorem 2 (Uniqueness of Dynamic Equilibrium)**.** *Consider a set of parameters $T, \mathbf{K}_0$, and deterministic parameter processes $(\boldsymbol{\theta}_t)_{t \in [0,T]}$, $(\lambda_t)_{t \in [0,T]}$. Then, $\tau^* \leq \bar{\tau}$. In particular, on $(\bar{\tau}, T]$ there exists a unique Markov equilibrium in the discrete-time game with sufficiently small Δ. If the Markov equilibrium is unique, it is also the unique subgame perfect equilibrium.*

All in all, we have established that close to the deadline (i.e., for $t \in (\bar{\tau}, T]$), there is always a unique equilibrium for sufficiently small Δ. Therefore, we can use equation (8) to study the dynamics of the equilibrium by Theorem 1. One caveat is that it is difficult to explicitly calculate $\bar{\tau}$, because Assumption 4 is a global condition on all $\mathbf{p}$ that is oftentimes difficult to check. However, as we argue in the following remark, we can easily check whether all firms are playing local best responses at the candidate price path derived from the ODEs.

Remark 2. Assumption 4 is a global condition that is difficult to check for all $\mathbf{p}$. However, as long as the condition is satisfied locally around the price vector that solves the ODEs, we know that

[25] Generally, we can construct solutions of the system of ODEs backwards in time until $\Gamma_f(\mathbf{p}_t(\mathbf{K}), \boldsymbol{\omega}_t^f(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}) \neq 0$ for all $\mathbf{K} \leq \mathbf{K}_0$. Once Ω hits this singularity, price policy functions may jump.

[26] Unlike Bonatti et al. (2017) who study a dynamic Cournot game where firms signal their costs, our game has a simple boundary condition at T that only depends on the capacity vector at the deadline. Capacity vectors $\mathbf{K}$ are in a finite ordered set which allows us to proceed recursively. Their game is one of imperfect information where the end game depends non-trivially on the states.

all firms' FOCs are satisfied simultaneously. To ensure that we have a local (but not necessarily global) best response of all firms, we can check that $\frac{\partial}{\partial p_j}\left(\mathbf{g}_f(\mathbf{p}, \boldsymbol{\Omega}_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}) - p_j\right)|_{\mathbf{p}=\mathbf{p}_t^*(\mathbf{K})} < 0 \; \forall j$.

4.3 Implementing Our Model as an Empirical Framework

Our equilibrium characterization offers a powerful tool for calculating market outcomes. It entails significant computational benefits over full-solution, backward induction approaches and complements existing approaches that characterize MPE for infinite-horizon, stationary games (e.g., Ericson and Pakes, 1995) with different actions (e.g., investment choice). To simulate equilibrium outcomes, we do not need to calculate stage-game equilibria in every state $(\mathbf{K}, t)$ but only at the deadline where $\Omega = \mathbf{O}_{\mathscr{A}(\mathbf{K})}$. Once the stage game is solved at T, standard methods to compute solutions to first-order ODEs can be employed. The following examples use this approach.

There are two key issues to address. The first is that data are always available in discrete time, and Doraszelski and Judd (2012) caution against using estimates from a discrete-time model to simulate continuous-time equilibria because "the fact that the continuous-time Bellman equation...is the limit of [a discrete-time] equation...does not imply the equilibria of a sequence of discrete-time games converge to the equilibria of the continuous-time game." Our theory addresses this issue; therefore, our model can be applied to discrete data. Note that we consider a single arrival per period, but we can approximate the same Poisson arrival process as $\Delta \to 0$ with a discrete arrival distribution that allows for multiple arrivals in every period that are Poisson distributed with intensity $\Delta\lambda_t$. As we let $\Delta \to 0$, the arrival process converges to the one in our model.

The second issue is that our characterization does not immediately address the curse of dimensionality of the state space, which has dimension equal to $T \times \frac{1}{\Delta} \times F \times J \times \prod_{j=1}^{J}(1+K_{j,0})$. For a few products with initial capacities in the hundreds, solving for the equilibrium exactly is trivial.[27] For larger problems, such as three firms, each offering three products with 150 units of capacity over 1000 periods, there are over a septillion (10^{24}) states. Of course, the curse of dimensionality is a common problem, and one solution to addressing it is to assume random sequential moves. The reason that this is beneficial is that it can allow for separating the computation of each firm's value function, yielding significant speed boosts and potentially similar outcomes to the simultaneous-move game (Doraszelski and Judd, 2019). However, this alternative timing approach cannot be

[27] It can be done on a 2022 MacBook Air using `numba` in Python in just a few minutes.

used in our setting because a firm's pricing decision impacts any of its rivals' remaining capacities. That is, it is not possible to separate the computation of each firm's value functions.

For large problems, our characterization can be combined with reinforcement learning. As before, stage games at $t = T$ are solved exactly. For $t < T$, the ODEs can be solved over a subset of capacities, and the value functions can be approximated as $\Pi_{f,t}(\mathbf{K}) \approx \sum_i w_{i,f,t}\phi_{i,f,t}(\mathbf{K})$, where $\phi(\cdot)$ are basis functions and $w_{i,f,t}$ are estimable parameters. By storing pricing policies as lower-dimensional functions, our model can be used to study policy-relevant questions in a wide range of perishable goods markets, such as hospitality, services, and retailing.

5 Properties of Dynamic Pricing Equilibria

5.1 Strategic Equilibrium Forces

In the previous section, we provided the conditions under which the dynamic equilibrium outcome is unique. The continuous-time approximation allows us to study equilibrium pricing dynamics for small Δ. Essentially, from a firm's perspective, firms trade off selling today against shifting demand to competitors, to obtain future market power. Although these two forces oppose one another, we show that the equilibrium structure reconciles them through a particular order of sale. Pricing policies are aligned to promote the sale of the product that yields the largest price increase.

We formalize this idea by comparing the evolution of equilibrium prices $\mathbf{p}^*_t(\mathbf{K})$ over time across different capacity vectors $\mathbf{K}$ near the deadline T. At the deadline, price levels correspond to the equilibrium price vector in a static Bertrand game with zero marginal cost given available products. Hence, we compare the order of change of $\mathbf{p}^*_t(\mathbf{K})$ for different $\mathbf{K}$ with the same available products near the deadline. We show that the order of change of prices over time for all products is determined by the smallest remaining capacity. Specifically, price changes are largest if the capacity of this product changes. We formally state the proposition as follows.

Proposition 4 (Order of Sale)**.** *For any solution of the system of ODEs in equation* (8)*, the following holds close to the deadline T:*

$$p_{j,t}(\mathbf{K}) = p^*_j(\mathbf{O}_{\mathscr{A}(\mathbf{K})}, \boldsymbol{\theta}_t; \mathbf{K}) + \mathscr{O}(|T-t|^{\min_j K_j}),\ t \to T \textit{ for all } j;$$

i.e., price deviations from the static benchmark close to the deadline are at most of order $\min_j K_j$. *If* $\lim_{t\to T}(\Pi_{f,t})^{(\min_j K_j)}(\mathbf{K}-\mathbf{e}_{j'}) \neq 0$ *for all* f *and* j' *with* $K_{j'} \in \arg\min_j K_j$, *then*

$$p_{j,t}(\mathbf{K}) = p_j^*(\mathbf{O}_{\mathscr{A}(\mathbf{K})}, \boldsymbol{\theta}_t; \mathbf{K}) + \Theta(|T-t|^{\min_j K_j}),\ t \to T \text{ for all } j;$$

i.e., price changes are exactly of order $\min_j K_j$.

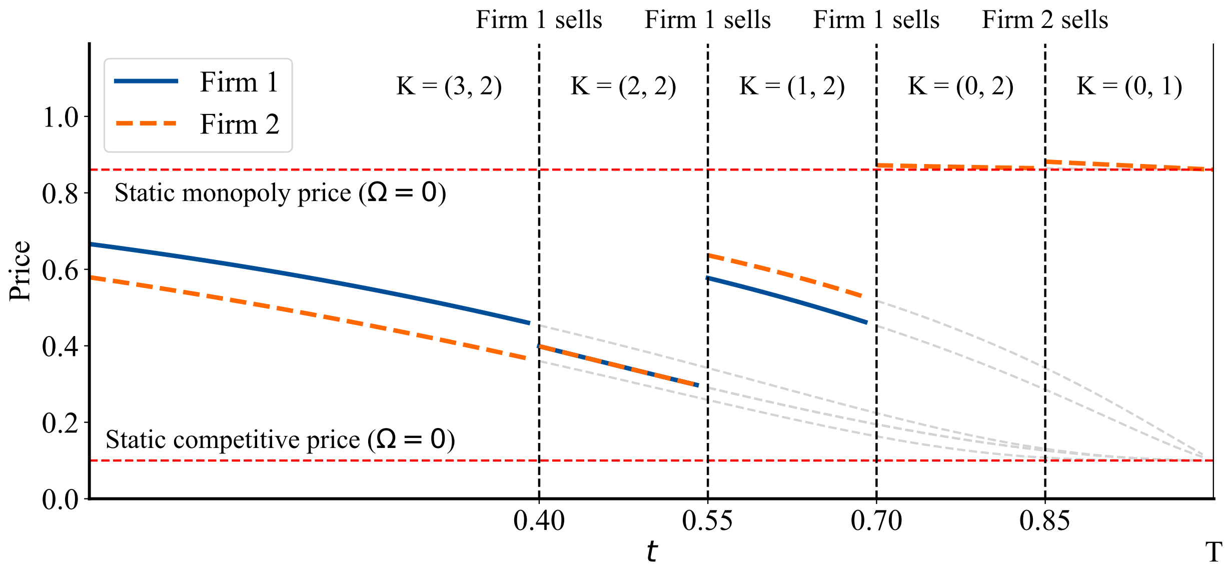
Proposition 4 is a generalization of the equilibrium property in Dudey (1992), who considers two undifferentiated products and single-valued demand, and Martínez-de Albéniz and Talluri (2011), who additionally consider stochastic arrivals.[28] In both cases, the firm with the least capacity deterministically sells out first before the second product is offered at an acceptable price. We show that this order of sale is preserved stochastically, regardless of the number of firms, products, and level of differentiation. Firms still agree that, holding all else equal, promoting the sale by the firm with the lowest capacity will result in the largest price jump.

When products are differentiated, the firm with the lowest capacity remaining will not necessarily sell. It is possible and probable that a different firm sells instead. Our second new insight is that firms adjust their prices in response to sales realizations. For example, if a firm with more capacity sells, it may then drop its prices because it is closer to being the firm with the lowest level of capacity. The fact that firms cannot guarantee that the order of sale will be preserved implies that if a firm could dispose of capacity for free, there are times in which it would do so. It also suggests that prices can in fact become negative, a possibility we verify in simulations. This is the opposite of Proposition 1 in which a monopolist has no incentive to reduce its own capacity.

To illustrate the core economic forces behind Proposition 4, in Figure 2 we plot a sample equilibrium price path using simple logit demand. The horizontal lines mark the competitive and monopoly prices when all scarcity effects are set equal to zero. We consider certain sales realizations for each firm over time. At $t = 0.40$, the firm with initially higher capacity ($K_1 = 3$) sells. The result is that Firm 1 decreases its price while Firm 2 increases its price (the gray dashed lines show what prices would have been absent the sale). This realization cannot happen in equilibrium in both Dudey (1992) and Martínez-de Albéniz and Talluri (2011), but is possible in our model.

[28] In both papers, the dynamic equilibrium may involve mixed pricing strategies and multiplicities if demand is not constant over time due to assuming single-valued demand. By adding heterogeneous consumers and product differentiation, we can establish uniqueness of the dynamic equilibrium (close to the deadline).

Figure 2: Example Equilibrium Path



Notes: This plot shows an example equilibrium realization assuming two firms, each offering one product. The time horizon is defined as $t \in [0,1]$ such that $\Delta = .01$, and the arrival rate is constant per period equal to $\Delta\lambda = .03$. We assume a logit demand system such that the probability of purchase of product f, conditional on arrival, is equal to $s_f = \frac{\exp(20-20 \cdot p_f)}{1+\sum_{f'=1,2}\exp(20-20 \cdot p_{f'})}$. The horizontal (red) lines denote the static competitive and monopoly price. The vertical dashed lines denote the sale of a unit as marked. The blue and orange lines denote equilibrium prices. Finally, the gray dashed lines denote what prices would have been absent the sale.

Prices equalize when firms have the same remaining capacity because the firms offer symmetrically differentiated products. Prices again decrease when no firm sells. At $t = 0.55$, Firm 1 sells an additional unit. Both firms' prices increase substantially, reflecting the insight of Proposition 4 that competition is fiercest when firms are in symmetric states and firms benefit from being in asymmetric states. Finally, beyond $t = 0.70$, Firm 2 becomes a monopolist. Just as before, selling units of capacity results in price jumps—in this case, always in the positive direction. Monopolist prices stay at or above the static monopoly price, reflecting strictly positive own-scarcity effects.[29]

Note that the order of sale in the competitive equilibrium outcome is the opposite of what occurs in the monopoly problem. Because firms benefit the most from getting the firm with the lowest capacity to sell out, this coincides with competition being softened the most through firms' aligned incentives to reduce product variety.[30] A monopolist instead attempts to preserve variety,

[29]In Figure 9 in the Online Appendix, we demonstrate these forces when consumer willingness to pay increases over time. The same demand realizations shown in Figure 2 are depicted.

[30]In Figure 11 in the Online Appendix, we show that total revenues are increasing in the number of sales that go to the firm with the lowest capacity remaining, i.e., correctly follow the order of sale. For example, if 10 units are sold in total, and the lowest initial capacity is 10, total revenues are highest if all 10 units are sold by the firm with the lowest initial capacity. If all 10 units are sold by the firm with the highest initial capacity, total revenues are lowest.

as this can be a tool to increase all product prices.

5.2 Additional Properties of the Dynamic Game

Although the previous section established that equilibrium dynamics involve firms attempting to soften competition by coordinating on the order of sales, the example used to illustrate this result also highlighted that prices may go up or down as scarcity increases. What else can be said about properties of the dynamic game? Proposition 5 establishes a number of qualitative properties of the equilibrium pricing policies, scarcity effect dynamics, and value functions.

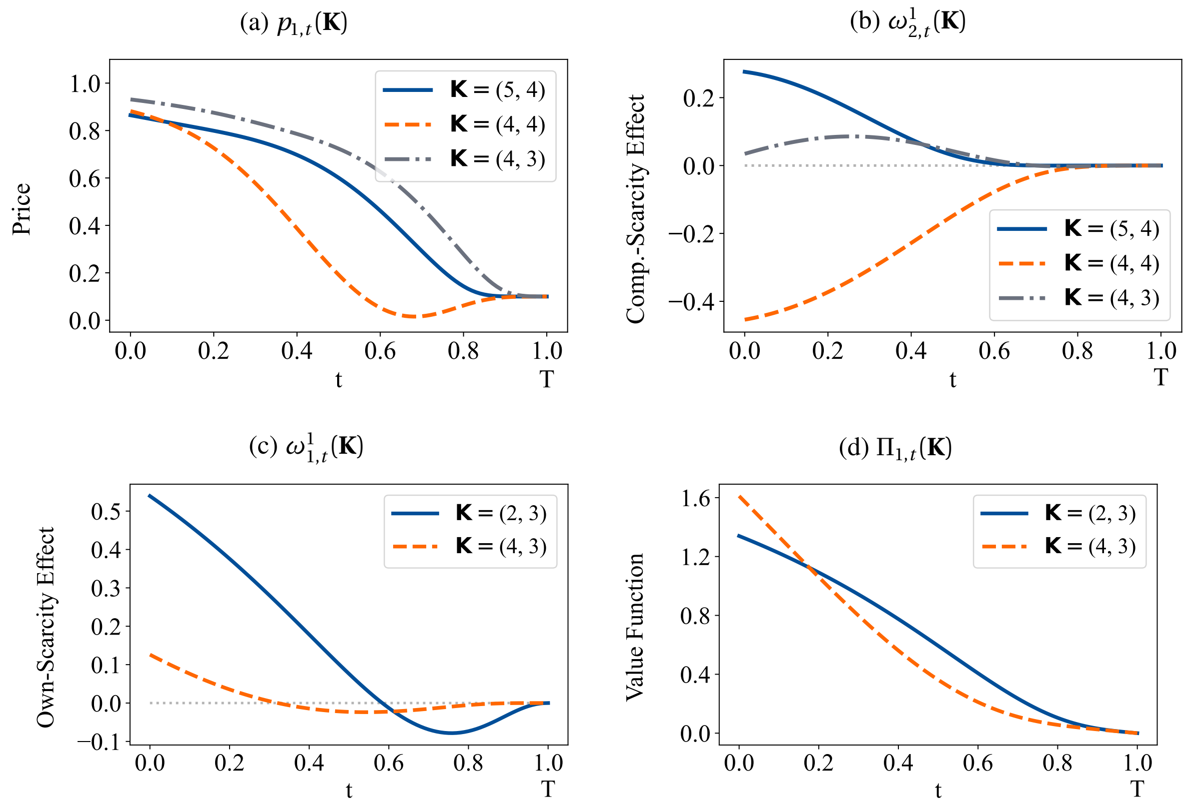
Proposition 5. *There exist parameters of the game and sufficiently small Δ such that the Markov equilibrium of the dynamic game has the following properties:*

(a) price policy functions are non-monotonic in time and capacity;
(b) own-scarcity effects are non-monotonic in time and capacity;
(c) cross-scarcity effects are non-monotonic in time and capacity;
(d) value functions are non-monotonic in capacity.

Proposition 5 states that most of the general properties of the monopoly dynamic pricing solution (Proposition 1) do not extend to oligopoly. We establish each claim in Proposition 5 using simple examples, i.e., each sub-plot in Figure 3 corresponds to each claim in Proposition 5. The properties in Proposition 5 are economically important. For example, typically, capacity constraints are viewed as a force that inflates market prices; however, when competing firms use dynamic pricing, having less capacity can lead to more competitive outcomes and lower market prices. This is shown in panel (a). Firm 1 charges substantially lower prices under (4,4) than under (5,4). Panels (b) and (c) highlight that all scarcity effects can be positive or negative and non-monotonic in time and capacity. In panel (d), we similarly show that value functions are non-monotonic in capacity. Firm 1 prefers to have only two units instead of four units early on.

We also investigate how equilibrium prices are affected by product differentiation. In Figure 10 in the Online Appendix, we plot equilibrium prices for Firm 1 (panel a) and Firm 2 (panel b) as a function of Firm 2's remaining capacity at a single point in time. We find that Firm 1's price is usually higher when Firm 2's product quality is lower. This is the opposite of what occurs in pricing games with upward-sloping reaction functions where Firm 1 would benefit (charge higher

Figure 3: Simulated Prices, Scarcity Effects, and Value Functions



Notes: This figure shows an example equilibrium realization assuming two firms, each offering one product. The time horizon is defined as $t \in [0,1]$ such that $\Delta = .01$, and the arrival rate is constant per period equal to $\Delta\lambda = .1$ in figure (a) and $\Delta\lambda = .05$ in the remaining figures. We assume a logit demand system such that the probability of purchase of product f, conditional on arrival, is equal to $s_f = \frac{\exp(20-20\cdot p_f)}{1+\sum_{f'=1,2}\exp(20-20\cdot p_{f'})}$.

prices) if Firm 2's product quality increased. Here, the opposite occurs because Firm 1 finds it increasingly beneficial to shift demand to its rival when its rival has low product quality.

6 Welfare and the Bertrand Scarcity Trap

In this section, we discuss welfare consequences of dynamic price competition. To this end, we formulate a discrete-choice micro-foundation of demand, as follows. A period-t consumer's utility of consuming product j is $v_{j,t} = \bar{v}_{j,t} + \epsilon_{j,t}$, $\bar{v}_{j,t} \in \mathbb{R}$, and the utility of consuming the outside option is normalized to $v_{0,t} = \epsilon_{0,t}$, where $\epsilon_t = (\epsilon_{j,t})_{j \in \mathcal{J} \cup \{0\}}$ are drawn from a distribution μ_t, and ϵ_t are independent across time. Products are substitutes, and each consumer can at most consume one product. Utilities are quasi-linear, and the utility from buying product j at price $p_{j,t}$ is $u_{j,t} = v_{j,t} - p_{j,t}$, $j \in \mathcal{J}$. Thus, a consumer buys the product that maximizes $\max_j u_{j,t}$. For example, if $\epsilon_{j,t}$ are independently type-1 extreme value distributed, then the probability $s_{j,t}(\mathbf{p};\mathbf{K})$ of buying product j corresponds to a logit demand system. Nested logit demand (see Definition 2) also satisfies the demand assumptions made in Section 2. We use this demand system for simulations.

Definition 2 (Nested Logit Demand)**.** Consider a nested logit demand system such that the probability of purchase of product f in time t, conditional on arrival, is equal to

$$s_{f,t} = \left(\frac{D^{1-\sigma}}{1+D^{1-\sigma}}\right)\left(\frac{\exp\left(\frac{\delta_f - \alpha_t \cdot p_f}{1-\sigma}\right)}{D}\right), \quad \text{where} \quad D = \sum_{f'=1,2} \exp\left(\frac{\delta_{f'} - \alpha_t \cdot p_{f'}}{1-\sigma}\right).$$

The nesting parameter $\sigma \in [0,1]$ affects substitution patterns. As $\sigma \to 0$, the model collapses to a logit demand system. As $\sigma \to 1$, there is little substitution to the outside good.

If $\epsilon_{j,t} = \epsilon_{j',t}$ and $\bar{v}_{j,t} = \bar{v}_{j',t}$ for all $j, j' \in \mathcal{J}$, products are undifferentiated. In that case,

$$s_{j,t}(\mathbf{p};\mathbf{K}) = \mathbb{1}\big(p_{j,t} > \max_{j' \neq j} p_{j',t}\big) D_t(p_{j,t}) + \mathbb{1}\big(p_{j,t} = \max_{j' \neq j} p_{j',t}\big) \alpha_j D_t(p_{j,t})$$

for some $\alpha_j \in [0,1]$ and a period-t demand function D_t. This demand function is not differentiable everywhere and fails to satisfy the assumptions in Section 2. As a result, our equilibrium analysis in Sections 4 and 5.1 does not apply, and pure-strategy equilibria generally do not exist. Dudey (1992) and Martínez-de Albéniz and Talluri (2011) examine a scenario where all consumers have the same fixed known valuation for the product. They note that if valuations increase towards the deadline, only mixed-strategy equilibria prevail. However, note that one can approximate undifferentiated demand systems with differentiated demand systems that satisfy our demand assumptions.

6.1 Welfare-Maximizing Prices

A natural benchmark for our analysis is the social planner's solution subject to capacity constraints. For any discrete-choice model that also satisfies our demand assumptions, we can define a per-period welfare function in period t given prices $\mathbf{p}$ as

$$w_t(\mathbf{p};\mathbf{K}) = \mathbb{E}\Bigg[\sum_{j\in\mathscr{A}(\mathbf{K})\cup\{0\}} v_{j,t}\,\mathbb{1}\big(v_{j,t}-p_j = \max_{j'} v_{j',t}-p_{j'}\big)\Bigg].$$

setting the "price" of the outside option $p_0 = 0$. The cumulative continuation welfare is then

$$W_t^*(\mathbf{K};\Delta) = -\lambda_t \max_{\mathbf{p}} \Big(w_t(\mathbf{p};\mathbf{K}) - \sum_{j\in\mathscr{A}(\mathbf{K})} s_j(\mathbf{p};\boldsymbol{\theta}_t,\mathscr{A}(\mathbf{K}))\,(W_{t+\Delta}^*(\mathbf{K};\Delta) - W_{t+\Delta}^*(\mathbf{K}-\mathbf{e}_j;\Delta))\Big),$$

where $\nu_{j,t}^*(\mathbf{K};\Delta) := W_{t+\Delta}^*(\mathbf{K};\Delta) - W_{t+\Delta}^*(\mathbf{K}-\mathbf{e}_j;\Delta)$ is the scarcity effect on welfare of a unit of good j. Next, note that for arbitrary vectors $\boldsymbol{\nu} = (\nu_j)_{j\in\mathscr{J}}$,

$$\begin{aligned}\boldsymbol{\nu} \;&\in\; \arg\max_{\mathbf{p}} w_t(\mathbf{p}) - \sum_{j\in\mathscr{A}(\mathbf{K})} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{A}(\mathbf{K}))\,\nu_j \\ &=\; \arg\max_{\mathbf{p}} \mathbb{E}\Bigg[\sum_{j\in\mathscr{A}(\mathbf{K})\cup\{0\}} (v_{j,t}-\nu_j)\mathbb{1}\big(v_{j,t}-p_j = \max_{j'} v_{j',t}-p_{j'}\big)\Bigg].\end{aligned}$$

Thus, we can show an analogous result to Lemma 1 for the optimal welfare function, as follows.

Proposition 6. $W_t^*(\mathbf{K}) := \lim_{\Delta\to 0} W_t^*(\mathbf{K};\Delta)$ *solves the ordinary differential equation*

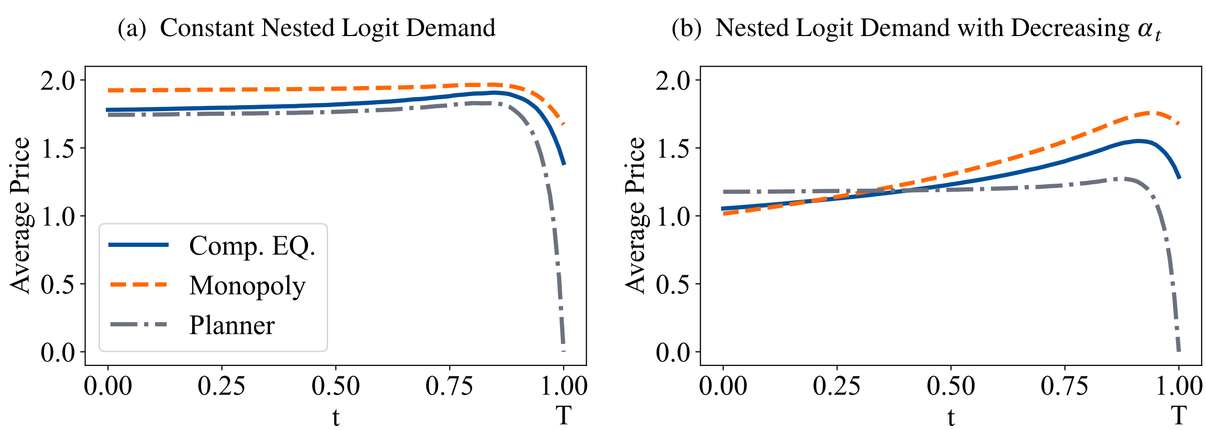
$$\dot{W}_t^*(\mathbf{K}) = -\lambda_t\Bigg(w_t(\boldsymbol{\nu}_t^*(\mathbf{K});\mathbf{K}) - \sum_{j\in\mathscr{J}} s_j(\boldsymbol{\nu}_t^*(\mathbf{K});\boldsymbol{\theta},\mathscr{A}(\mathbf{K}))\;\nu_{j,t}^*(\mathbf{K})\Bigg), \tag{9}$$

with boundary conditions (i) $W_T(\mathbf{K}) = 0\ \forall\mathbf{K}$ *and (ii)* $W_t(\mathbf{K}) = 0$ *if* $K_j = 0$ *for all* $j\in\mathscr{J}$*, where* $\boldsymbol{\nu}_t^*(\mathbf{K}) = (\nu_{j,t}^*(\mathbf{K}))_{j\in\mathscr{J}}$ *with* $\nu_{j,t}^*(\mathbf{K}) := W_t^*(\mathbf{K}) - W_t^*(\mathbf{K}-\mathbf{e}_j)$.

The proof mirrors Lemma 1. This formulation highlights two types of distortions that occur if firms maximize payoffs, by comparing it with the ODEs in Theorem 1 and Lemma 1. The first is that firms do not internalize today's welfare benefits of selling a unit, w_t. The second is that firms do not fully internalize the social opportunity cost of selling, which is the value of keeping the unit for the future, $\nu_{j,t}^*$. As a result, prices can be higher or lower than the welfare-optimal price.

Lemma 6 also allows us to simulate socially optimal prices. Figure 4 shows average prices over time for a monopolist, in a competitive market, and for a social planner. Panel (a) considers a nested logit demand specification where demand is constant over time. The figure validates the classic intuition of price competition with differentiated products. Consider prices at the deadline. Since the value of capacity is zero after the deadline, the socially optimal price is zero. The competitive equilibrium price is higher due to market power from product differentiation. The monopoly price is the highest. This order is on average maintained for $t < T$. The order is not necessarily preserved state-by-state, as we will soon discuss. Conversely, panel (b) shows a reversal of the order of prices far from the deadline when demand becomes more inelastic over time. Here, the monopolist sets on average lower prices than socially optimal. Prices are even lower than the average competitive price. The reason for the reversal is that firms do not internalize the social opportunity cost of selling. Essentially, firms offer relatively low prices early on to create scarcity and avoid unsold capacity. This notion will be critical for our welfare analysis that follows.

Figure 4: Equilibrium, Monopolist, and Welfare-Optimal Prices



Notes: We assume that initial capacity is $\mathbf{K} = (20, 20)$, and that the per-period arrival rate is constant, equal to $\Delta\lambda_t = 1$. In panel (a), we use the nested logit demand system described in Definition 2, and assign $\delta_1 = \delta_2 = 1$, $\sigma = .5$, and $\alpha = 1$. In panel (b), we use the same model but assign α_t to decrease over time at a constant rate such that $\alpha_0 = 2$ and $\alpha_T = 1$.

6.2 Definition of Bertrand Scarcity Trap

The rich equilibrium dynamics discussed in Section 2 and the distortions discussed in Section 6.1 suggest that firms may promote scarcity inefficiently. We call this new welfare effect the *Bertrand scarcity trap (BST)*.

Definition 3. A dynamic pricing game is subject to the *Bertrand scarcity trap* if a price floor in some states increases both consumer surplus and all firms' profits.

Note that a single firm is never subject to the BST because a firm is always worse off if it faces a price floor restriction. The following are the main forces affecting whether a competitive game is subject to the BST. Consider a state $(\mathbf{K}, t)$. Slightly increasing all prices excludes some lower-valuation consumers from buying today, potentially reallocating capacity to higher-valuation consumers in the future. This strategy can improve welfare if the product is likely to sell out, preventing high-valuation customers from remaining empty-handed. However, this strategy may also reduce welfare if the product does not sell out, causing firms to lose revenue.

To understand the economics of BST, we define a state-$(\mathbf{K}, t)$ outcome as a tuple $(j, v_{j,t})$, where j is an available product and $v_{j,t}$ is the realized utility level.[31] An allocation rule a maps each state $(\mathbf{K}, t)$ and utility vector $(v_{j,t})_j$ to a probability distribution on available products $\mathscr{A} \subset \mathscr{J}$, where $a_j(t, \mathbf{K}, (v_{j,t})_j)$ is the probability with which a period-t consumer with utility vector $(v_{j,t})_j$ is allocated a unit of product j. An allocation rule can be induced by Bertrand price competition, a monopolist's pricing decision, a social planner, or alternative pricing mechanisms. We formalize how an allocation rule can add inefficiencies over time. We denote the continuation welfare in state $(\mathbf{K}, t)$, given an allocation rule a, by $W^a(\mathbf{K}, t)$. This continuation welfare is the expected sum (over t) of utility outcomes $v_{j,t}$ from purchased available products given the allocation rule a. Then, selling a unit of product j in state $(\mathbf{K}, t)$ results in a future cost on welfare of $W^a_{t+\Delta}(\mathbf{K}) - W^a_{t+\Delta}(\mathbf{K} - \mathbf{e}_j)$ and a gain in total welfare of $\mathbb{E}[\sum_j a_j(\mathbf{K}, t, (v_{i,t})_i) v_{j,t}]$. Holding fixed the allocation rule in period $t + \Delta$ and onwards, it is efficient to allocate product j in period t if and only if

$$v_{j,t} - (W^a_{t+\Delta}(\mathbf{K}) - W^a_{t+\Delta}(\mathbf{K} - \mathbf{e}_i)) \geq \max\left\{v_{0,t},\, \max_i v_{i,t} - (W^a_{t+\Delta}(\mathbf{K}) - W^a_{t+\Delta}(\mathbf{K} - \mathbf{e}_i))\right\}.$$

This leads us to define the notion of a constrained-efficient price.

Definition 4. We call

$$\bar{p}^a_{j,t}(\mathbf{K}) = W^a_{t+\Delta}(\mathbf{K}) - W^a_{t+\Delta}(\mathbf{K} - \mathbf{e}_j)$$

the *constrained-efficient price* of product j at time t, given allocation rule a.

[31] An allocation is feasible if the product is available given $\mathbf{K}$ and a period-t consumer (if she arrives) has the corresponding valuation with positive probability.

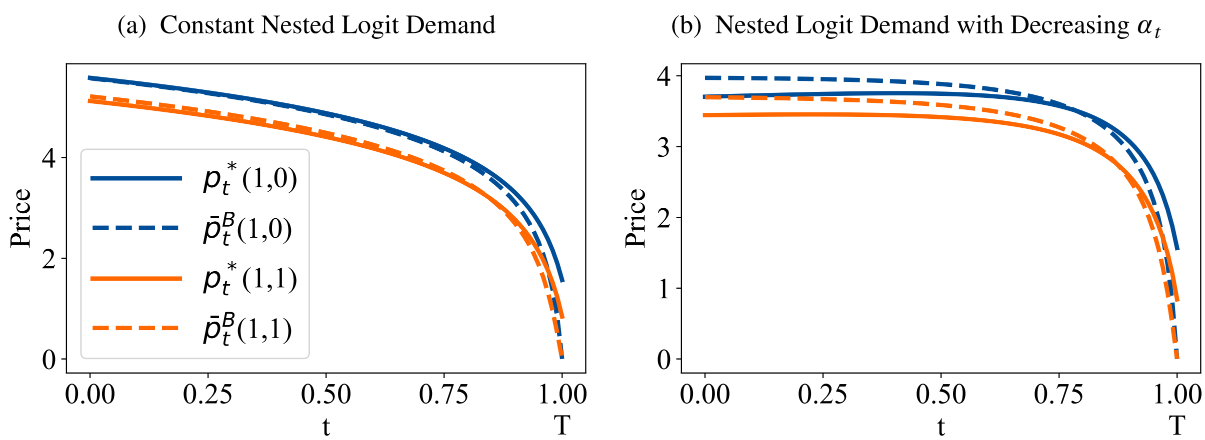
A social planner imposing constrained-efficient prices in period t can induce a constrained-efficient allocation in period t given that the future allocation rule remains a. With this notion, we can distinguish between two types of inefficiencies that are possible in each stage game.

Definition 5. We say j is *over-provided* if $p_{j,t} < \bar{p}^a_{j,t}(\mathbf{K})$ and is *under-provided* if $p_{j,t} > \bar{p}^a_{j,t}(\mathbf{K})$.

Under static Bertrand price competition with differentiated products, we expect under-provision of products given that firms maintain some market power, and any restriction on competition typically exacerbates this inefficiency. However, in dynamic settings subject to scarcity, inefficient rationing can naturally occur due to over-provision of a product early on. Bertrand competition tends to exacerbate over-provision, and restricting competition can be welfare improving, as we will demonstrate next. The following example illustrates these effects.

Example 3. Consider two symmetric firms, each offering one product, with initial capacities $(1,1)$. In Figure 5(a), we plot equilibrium (solid curves) and constrained-efficient (dashed curves) prices over time for capacity vectors (1,1) and (1,0) using a nested logit demand system. The figure depicts under-provision as $\bar{p}^B_t < p^*_t$ in all states. In panel (b), we plot the same policy functions assuming logit demand with increasing willingness to pay over time. There is over-provision early on ($\bar{p}^B_t > p^*_t$) and under-provision ($\bar{p}^B_t < p^*_t$) closer to the deadline. ◊

Figure 5: Equilibrium and Constrained-Efficient Prices



Notes: We assume that initial capacity is $\mathbf{K} = (1,1)$, and that the per-period arrival rate is constant, equal to $\Delta\lambda_t = 1$. In panel (a), we use a nested logit demand system described in Definition 2, and assign $\delta_1 = \delta_2 = 1$, $\sigma = .5$, and $\alpha = 1$. In panel (b), we use the same model but assign α_t to decrease over time at a constant rate such that $\alpha_0 = 2$ and $\alpha_T = 1$.

Dilme and Li (2019) highlight that flash sales early on can result in inefficiently low prices early on. Indeed, if a firm faces forward-looking buyers and is therefore competing with its future self, then it may benefit form committing to higher prices early on, and consumer surplus may also be higher because capacity is allocated to higher-valuation buyers.[32]

6.3 An Analytic Example of the Bertrand Scarcity Trap

We present an analytic example of the BST with undifferentiated products, uncertain demand, and discrete time. In this example, the BST is so severe that a monopolist prices more efficiently than what is achieved under dynamic price competition.

Consider two undifferentiated products that are available for sale over three sequential markets, $t = 1,2,3$. In every period t, a single short-lived consumer with i.i.d. unit demand arrives. If p is the lowest available price, a consumer in period t buys with probability $s_t(p) = 2(1-p)\mathbb{1}(p > 0.5) + \mathbb{1}(p \leq 0.5)$, as illustrated in Figure 6. We compare two market structures. In one market structure, each product is sold by competing firms; in the other, a single firm sells both products. We denote per-period welfare given a price p by $w_t(p)$, continuation welfare for a single merged firm with remaining capacity K by $W_t^M(K)$, and continuation welfare for two firms with capacity vector $(1,1)$, by $W_t^c(1,1)$. Prices are denoted by $p_t^M(K)$ and $p_t^c(1,1)$, analogously. In Figure 6, the per-period welfare is illustrated by the filled regions under the demand curves.

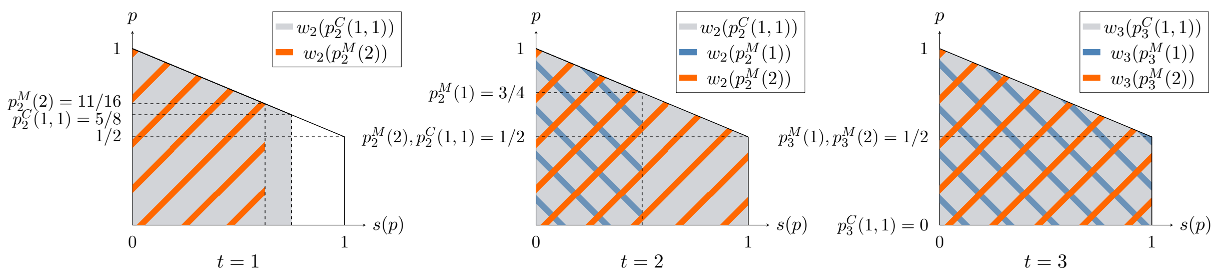
In the last period ($t = 3$), the monopoly price is $p_3^M(1) = \frac{1}{2}$, and monopoly profits are $\frac{1}{2}$. The equilibrium price and profits with Bertrand competition are 0. Total welfare is 0.75 in both settings. Thus, welfare in the last period is unaffected by market structure and maximized. If two firms compete, a firm can gain $\frac{1}{2}$ in profits if the other firm sells in period 1 or 2.

Period-2 demand is identical to period-3 demand. However, if a single firm has only one unit remaining, the firm sets a higher price equal to $p_2^M(1) = 0.75$ because it knows that there is another chance to sell this unit in period 3, yielding expected profits of $\frac{1}{2}$. The constrained-optimal price is also 0.75, so the single firm is pricing efficiently.[33] With $K = 2$, the price is $p_2^M(2) = \frac{1}{2}$, which is also constrained optimal, because only one unit can be sold in period 3. With competition,

[32]In a similar vein, in a quantity choice game, Bonatti et al. (2017) show that firms may engage in excess production to encourage rivals to scale back. However, this leads to a stronger signaling effect that improves allocative efficiency.

[33]This is because $p = 0.75$ maximizes $2p(1-p)+(1-p)^2+(1-2(1-p))0.75$.

Figure 6: Illustrative Example of the Bertrand Scarcity Trap



Notes: The graph depicts demand curves, single-firm optimal prices, and competitive prices in the three periods. The orange and blue regions represent per-period welfare given a single firm with two and one unit left, respectively. The gray region represents per-period welfare if two competing firms are active.

prices are the same, i.e., $p_2^c(1,1) = \frac{1}{2}$.[34] Finally, continuation welfare is given by $W_2^M(1) = 0.8125$, $W_2^M(2) = 1.5$, and $W_2^c(1,1) = 1.5$, respectively.

Moving to the first period, one can show that $p_1^M(2) = 0.6875 > p_1^c(1,1) = 0.625$. The welfare-maximizing price is 0.6875. Hence, the single firm is exactly solving a social planner's problem as all prices are constrained-efficient, while a competitive market is over-providing the product at $t = 1$. Therefore, the equilibrium is subject to the Bertrand scarcity trap.

Intuitively, competing firms do not internalize the entire option value of keeping a seat because their continuation payoffs are half of the social planner's. Hence, while in a static Bertrand pricing game, firms can never set prices that are lower than efficient, in a dynamic pricing game with scarcity, competitive prices can be so low that firms sell out inefficiently early. As a result, both consumers and firms are worse off relative to pricing regimes that restrict competition.

6.4 Alleviating the Bertrand Scarcity Trap with Algorithmic Pricing

Our second example of the BST uses a nested logit demand system (see Definition 2). We choose this specification because it showcases that BST can occur in flexible demand systems. This is in contrast with our first example, which relied on very stylized consumer preferences. The nested logit demand system is widely used in empirical work (e.g., Berry, 1994). A promising area for future research is to quantify the extent to which the BST occurs in data.

[34] Firms do not have an incentive to deviate to higher or lower prices. In general, competition with undifferentiated products can lead to multiplicities and non-existence of symmetric pure-strategy equilibria, as also shown in Talluri and Van Ryzin (2004) and Dudey (1992).

We compute the equilibrium of the dynamic game using our ODE characterization and compare its market outcomes with those under a pricing algorithm. We design the pricing algorithm based on key features of revenue management systems used in several markets with perishable inventory (Hortaçsu et al., 2024). The two most important features are the use of discrete prices and a simplified objective function. We formally define the algorithm as follows.

Definition 6 (Algorithmic Pricing)**.** Firms operate in discrete time, such that $t \in 0, \Delta, ..., T-\Delta$. Prices are discrete. The set of potential prices of firm f at time t is given by $\mathscr{P}_{f,t} = (p^n_{f,t})^N_{n=1}$, such that $p^m_{j,t} < p^n_{j,t}$ if $m < n$. The parameter N controls how many prices there are. The price corresponding to the n-th entry in the menu can vary over time (as observed in car rentals, airlines, etc.). Therefore, the pricing behavior of firm f is characterized by the index in the pricing menu that it chooses over time. This is defined as $(n_{f,t})^{T-\Delta}_{t=0}$.

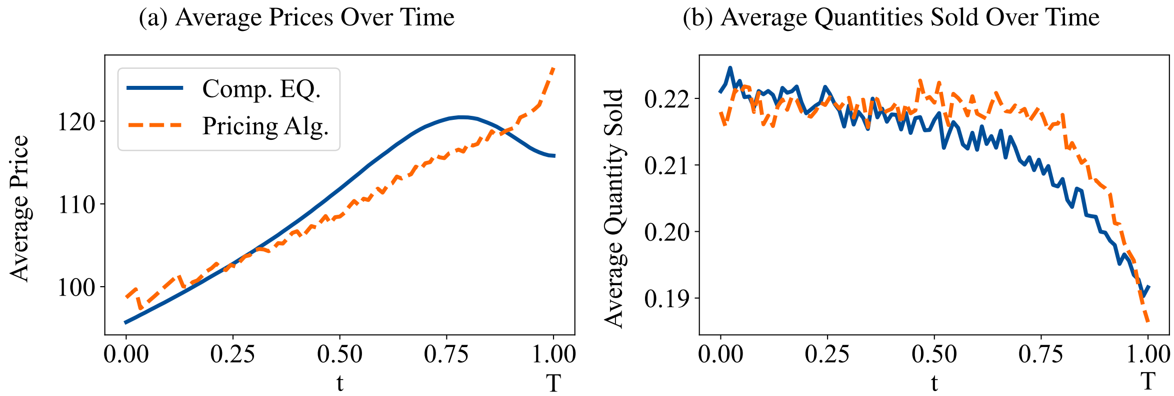
As is typical for algorithms, we assume each firm does not internalize its competitor as a strategic player. At time t, each firm f, having observed its competitor's last-period index choice $n_{f',t-\Delta}$, assumes that this price will also be charged in the current and all future periods. Therefore, firm f's beliefs on all future prices of firm f' is a Dirac distribution, positioned at $p^{n_{f',t-\Delta}}_{f',\tau}$ for all $\tau \in \{t, \dots, T-\Delta\}$. With this simplification, each firm simply calculates its residual demand curves in all remaining periods and solves the monopoly problem as defined in Section 3, i.e.,

$$\Pi_{f,t}(K;\Delta) = \max_{p \in \mathscr{P}_{f,t}} \Delta\lambda_t s_{f,t}(p)\Big(p_f + \Pi_{M,t+\Delta}(K-1;\Delta)\Big) + \Big(1 - \Delta\lambda_t s_{f,t}(p)\Big)\Pi_{M,t+\Delta}(K;\Delta).$$

Using the equilibrium outcome and firms' pricing algorithm solutions, we simulate market outcomes 10,000 times. The exact specification used appears in the notes of Figure 7, which shows average prices (panel a) and quantity sold (panel b) over time. Average prices increase as our specification assumes a slight decrease in consumer price sensitivity over time. Average equilibrium prices fall close to the deadline due to decreasing opportunity costs and the exclusion of sold-out products. In contrast, algorithmic prices steadily increase, starting higher than competitive equilibrium prices early on. Average quantities sold remain relatively stable over time with algorithmic pricing, while they drop faster in the competitive equilibrium as products sell out.

Table 1 summarizes revenues, consumer surplus, and social welfare for the two allocation mechanisms. Both firms' revenues and consumer surplus are about one percent higher with algo-

Figure 7: Price and Quantity Paths for Full Information Benchmark and Algorithmic Pricing



Notes: These plots show average prices and quantities sold across firms under dynamic pricing versus algorithmic pricing. We assume that initial capacity is $\mathbf{K} = (10, 20)$. The time horizon is defined as $t \in [0, 1]$, such that $\Delta = .01$, and the arrival rate is constant per period, equal to $\Delta\lambda = 0.58$. We assume the demand system described in Definition 2 and assign $\delta_1 = 0.28$, $\delta_2 = 0.18$, and $\sigma = 0.65$, and α_t decreases at a constant rate over time, starting at 1 in period $t = 0$ and ending at 0.8022 in period $t = T$.

rithmic pricing relative to the competitive equilibrium outcome. As indicated in Figure 7, products sell out more quickly in a competitive equilibrium. The welfare numbers indicate that products would be allocated more efficiently if a price floor was imposed at the beginning of the game. That is, this example is subject to the BST, which is alleviated by algorithmic pricing. These competing algorithms result in higher prices early on and less steep price increases over time.

Table 1: Example of Bertrand Scarcity Trap Using Nested Logit Demand

	Firm 1 Revenue	Firm 2 Revenue	Consumer Surplus	Welfare
Dynamic Pricing	995.72	1095.12	2711.26	4802.11
Algorithmic Pricing	1004.83	1106.91	2743.45	4855.20
Difference	0.91%	1.08%	1.19%	1.11%
Test Statistic	10.13	6.89	19.69	18.15

Notes: This table shows market outcomes for 10,000 simulations using our ODE characterization and comparing it with algorithmic pricing. The time horizon is defined as $t \in [0, 1]$, such that $\Delta = .01$, and the arrival rate is constant per period, equal to $\Delta\lambda = 0.58$. We assume the demand system described in Example 3. The test statistic reported is a t−test for equal means.

6.5 Demand Uncertainty Is Necessary for the BST to Exist

We examine two alternative models of dynamic price competition with capacity constraints where uncertainty in demand is shut down. The first example considers two periods and homogeneous demands. The second example corresponds to our baseline model without demand uncertainty. In both cases, the inefficiency of not accounting for the social option value of a sale today is not present. As a result, we establish that the BST cannot occur absent demand uncertainty.

6.5.1 A Two-period Example Without Uncertainty

We begin with a two-period environment with undifferentiated products based on the model of Dana and Williams (2022), who focus on quantity-price choices but solve for pricing equilibria as part of their analysis. In every period, a unit mass of consumers arrives, with valuations governed by a demand function $D_t(p)$. We allow firms to sell infinitesimal quantities and assume that the demand functions are such that the inverse demand functions $P_t(q)$ satisfy $P_t''(q)q + 2P_t'(q) < 0$, i.e., revenues are concave in quantities. The analysis allows for an arbitrary number of firms.

First, note that welfare-maximizing prices are constant across periods; otherwise, consumption from a lower-valuation customer in a period with a low price could be shifted to a higher-valuation customer in a period with a high price. The efficient price p^w must ensure that the entire initial capacity is sold, i.e.,

$$D_1(p) + D_2(p) = \sum_f K_{f,0}. \tag{10}$$

This is because demand is predictable and deterministic. Wasting capacity reduces welfare. Then, the following proposition follows from the analysis in Dana and Williams (2022).

Proposition 7. *The following properties hold:*

i) *The welfare-optimal price is constant over time and given by equation* (10).

ii) *It can be optimal for a single firm to set a higher or lower price in the first period, and it may optimally not sell out its capacity. The profit-maximizing price is constant over time if demand is constant over time but may be higher than p^w.*

iii) *With competition, the capacity constraint is always binding, i.e., equation* (10) *holds, and*

   (a) *if firms are symmetric, i.e., $K_f = 1$ for all f, and demand is constant over time ($D_1 = D_2$), then $p_1 = p_2 = p^w$ in equilibrium;*

(b) if the welfare-optimal price is not an equilibrium, then only one firm sells in the first period in equilibrium and at a higher price than in the second period.

The proposition highlights several economic forces. First, with a single-firm, prices can be higher or lower in the first period, and the capacity constraint may or may not bind. Hence, there are two types of inefficiencies. The firm may waste capacity to extract more surplus, or the firm may misallocate capacity by not charging the same price across periods. Second, in a competitive equilibrium, the market must always clear in the last period because there is no demand uncertainty. However, the prices in the first period may be higher than in the second period if one firm has sufficient market power. The reason is that a firm can unilaterally control scarcity in the second period and, therefore, increase prices. Finally, if demand is constant over time, equilibrium prices coincide with the socially efficient price.

All in all, this analysis establishes that competition reduces inefficiencies relative to a monopolist. However, products may still be misallocated if a dominant firm creates scarcity in the last competitive period. This can only occur if demand becomes more inelastic over time. Importantly, lowering or increasing the price in the first period always harms the dominant firm. Therefore, the game is not subject to inefficiencies stemming from the BST.

6.5.2 Differentiated Products Without Uncertainty

We extend the insights of the two-period example with undifferentiated products to a fully dynamic model with differentiated products, as studied in Gallego and Hu (2014). Using the same notation and demand assumptions as in our baseline model, we assume that $\boldsymbol{\theta}_t$ is constant over time, for tractability, omitting the time index in the demand function henceforth. Instead of assuming Poisson arrivals, we assume that consumers flow in continuously at a constant rate of 1. Capacity is arbitrarily divisible. We use the same discrete-choice micro-foundation to calculate consumer surplus and welfare as before. However, unlike in the baseline model, uncertainty is absent, as consumers aggregate continuously over time, and capacity is divisible. We work directly in continuous time and assume that $\mathbf{K}_0 < T\mathbf{s}(\mathbf{0})$ to ensure the capacity constraint meaningful.

A social planner, monopolist, and each firm in an oligopoly solve the following maximization problems, respectively:

$$\text{Social planner:} \qquad \max_{\mathbf{p}_t \text{ measurable}} \int_0^T w(\mathbf{p}_t)dt \qquad \text{subject to} \int_0^T \mathbf{s}(\mathbf{p}_t)dt \le \mathbf{K}_0;$$

$$\text{Monopolist:} \qquad \max_{\mathbf{p}_t \text{ measurable}} \int_0^T \mathbf{s}(\mathbf{p}_t)\cdot \mathbf{p}_t dt \qquad \text{subject to} \int_0^T \mathbf{s}(\mathbf{p}_t)dt \le \mathbf{K}_0;$$

$$\text{Oligopoly firm:} \qquad \max_{\mathbf{p}_t^f \text{ measurable}} \int_0^T \mathbf{s}^f(\mathbf{p}_t)\cdot \mathbf{p}_t^f dt \qquad \text{subject to} \int_0^T \mathbf{s}^f(\mathbf{p}_t)dt \le \mathbf{K}_0^f.$$

Given that demand does not change over time and there is no uncertainty, a natural solution to any of these problems is a constant price path.[35] Proposition 8 shows that such solutions exist for all three problems and characterizes the corresponding price level.

Proposition 8. *The following properties hold:*

i) The unique welfare-maximizing price path is constant. The price $\mathbf{p}^w$ *is given by* $\mathbf{s}(\mathbf{p}^w) = \frac{1}{T}\mathbf{K}_0$.

ii) The unique profit-maximizing price path of a monopolist is constant. The price $\mathbf{p}^M$ *is the unique solution to the static problem* $\max_{\mathbf{p}} \mathbf{s}(\mathbf{p})\mathbf{p}$ *subject to* $\mathbf{s}(\mathbf{p}) \le \frac{1}{T}\mathbf{K}_0$.

iii) All competitive equilibrium prices paths are constant, where the equilibrium price vector $\mathbf{p}^O$ *is the unique equilibrium* $\mathbf{p}^*(\boldsymbol{\lambda}^{SP})$ *of a game where firms faces costs given by* $\boldsymbol{\lambda}_f^{SP} \ge \mathbf{0} \in \mathbb{R}^{\mathscr{J}}$, *where*

$$\boldsymbol{\lambda}^{SP} \in \arg\min \left\{ \boldsymbol{\lambda} \ge \mathbf{0} \;\middle|\; \mathbf{s}(\mathbf{p}^*(\boldsymbol{\lambda})) \le \frac{1}{T}\mathbf{K}_0 \right\}.$$

Note that $\mathbf{s}(\mathbf{p}) = \frac{1}{T}\mathbf{K}_0$ has a unique solution $\mathbf{p}$ for any $\mathbf{K}_0$ because $\mathbf{s}$ is a bijection (see Online Appendix). Proposition 8 shows that firms deviate from efficient pricing only to exercise market power, raising prices even if doing so leaves some capacity unsold. Increasing the price further in those instances makes consumers strictly worse off by leaving even more capacity unsold. Hence, just as in the two-period example, the competitive constant equilibrium is not subject to the BST.

[35] We showed in Proposition 1 that with uncertainty, the average price paths of a monopolist are non-monotonic exactly because of the option value of waiting for high-valuation consumers to arrive in the future. A similar result holds for the social planner and an equilibrium of the dynamic pricing game.

7 Conclusion

We introduce a framework to study dynamic price competition in complete information. We provide conditions for the existence and uniqueness of pure-strategy Markov perfect equilibria and show that in the continuous-time limit, prices solve a system of ordinary differential equations. This convenient structure allows us to explore new strategic interactions that arise in this setting, with relevant applications to hospitality, transportation, retailing, and housing.

We provide three sets of theoretical findings. The first set illustrates how intuition from monopoly dynamic pricing models does not carry over to the oligopoly case. For example, prices can be lower if firms had fewer units of capacity to sell. Our second set of findings concerns the drivers of dynamic prices. We show that firms' strategies are aligned with softening future price competition through a particular order of sale. Prices jump the most if the firm with the most scarce capacity sells. As a result, price increases are driven primarily by reducing product variety. Our last set of theoretical insights concerns welfare. We show that while competition lowers prices, as in most static models with competition, in the dynamic setting, this can also facilitate misallocation of capacity. Early prices can be too low, leading to over-provision early on and under-provision close to the perishability date, and harming both consumers and firms. We call this phenomenon the Bertrand scarcity trap. It can occur because firms pricing in perishable goods markets do not internalize the social option value of keeping capacity for the future. Hence, uncertain demand is critical to the occurrence of the Bertrand scarcity trap. We find that the Bertrand scarcity trap can be alleviated when firms commit to limiting competitive interactions; for example, through algorithmic pricing, it is possible that both firms and consumers are made better off.

We see several promising directions for future research. Our framework provides a computationally tractable tool to characterize equilibrium outcomes under dynamic pricing and alternative allocation rules, including the social planner's problem. We believe it can be used for policy-relevant research to quantify market power and conduct counterfactuals in perishable goods markets broadly. In addition, our simulations using algorithmic pricing suggest that examining dynamic price competition with commitment, such as through algorithms, holds great potential because it may alleviate the Bertrand scarcity trap more broadly. Finally, note that a key driver of all our results is scarcity. Further research should explore endogenizing capacity decisions.

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Online Appendix

Dynamic Price Competition with Capacity Constraints

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A Proofs

A.1 Technical results

A.1.1 $\mathbf{s}^f_{\mathscr{A}}(\cdot;\boldsymbol{\theta},\mathscr{A})$ is bijective

We show, using the Implicit Function Theorem in Chapter 6 in Krantz and Parks (2002), that the demand function $\mathbf{s}^f_{\mathscr{A}}(\cdot;\boldsymbol{\theta},\mathscr{A})$ is a bijection. We omit $\boldsymbol{\theta},\mathscr{A}$ in this proof. According to the theorem, a continuously differentiable function $\mathbf{s}:\mathbb{R}^{\mathscr{A}}\to\left\{\mathbf{s}\in(0,1)^{\mathscr{A}}\,\middle|\,\sum_{j\in\mathscr{A}}s_j<1\right\}$ is a homeomorphism, if the following hold:

i) For any compact set $K\subset\left\{\mathbf{s}\in(0,1)^{\mathscr{A}}\,\middle|\,\sum_{j\in\mathscr{A}}s_j<1\right\}$, $\mathbf{s}^{-1}(K)$ is compact, i.e., $\mathbf{s}$ is proper;

ii) the matrix $D_{\mathbf{p}^f}\mathbf{s}^f_{\mathscr{A}}$ is invertible

First, recall that by Assumption 1-i) and iii), $s_0(\mathbf{p})=0$ if $p_j=-\infty$ for some $j\in\mathscr{A}$, and $s_j(\mathbf{p})=0$ if $p_j=\infty$. Furthermore, each s_j is continuous. Thus, for a compact set K as in i), $\mathbf{s}^{-1}(K)$ is compact. Furthermore,$D_{\mathbf{p}^f}\mathbf{s}^f_{\mathscr{A}}$ is invertable for all $\mathbf{p}^{\mathscr{A}}\in\mathbb{R}^{\mathscr{A}}$ by the Levy-Desplanques Theorem (see, e.g., Theorem 6.1.10. in Horn and Johnson (2012)) because it is diagonally dominant by Assumption 1-ii) and iii): $\frac{\partial s_0}{\partial p_j}(\mathbf{p})=\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p})\right|-\sum_{j'\in\mathscr{A}\setminus\{j\}}\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p})>0.$[36] This concludes the proof.

A.1.2 Continuous time limit

We use the following result for the proofs of Lemma 1 and Theorem 2. We denote the set of active firms given capacity vector $\mathbf{K}$ by

$$\mathscr{F}(\mathbf{K}):=\left\{f\in\mathscr{F}\middle|\exists j\in\mathscr{J}_f\text{ such that }K_j>0\right\}.$$

[36] Consistent with the common convention, the Jacobi matrix of a vector-valued function $f(\mathbf{x})\in\mathbb{R}^n$, $\mathbf{x}\in\mathbb{R}^n$ is $D_{\mathbf{x}}f(\mathbf{x}):=\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$, i denoting rows and j columns, and bold vectors $\mathbf{x}$ are column vectors.

Lemma 4. *For each* $\mathbf{K} \leq \mathbf{K}_0$ *consider continuous price functions* $(\Omega, \boldsymbol{\theta}) \mapsto \mathbf{p}^*(\Omega, \boldsymbol{\theta}; \mathbf{K}) = (p_j^*(\Omega, \boldsymbol{\theta}; \mathbf{K}))_{j \in \mathscr{A}(\mathbf{K})}$, *and bounded and continuous functions* $\mathbf{A_K} : \mathbb{R}^{\mathscr{A}(\mathbf{K})} \times \mathbb{R}^{\mathscr{F}(\mathbf{K}) \times \mathscr{A}(\mathbf{K})} \times \mathscr{T} \to \mathbb{R}^{\mathscr{F}(\mathbf{K})}$. *Let* $(\Pi_{f,t}(\mathbf{K}; \Delta))_{f \in \mathscr{F}, \mathbf{K} \leq \mathbf{K}_0}$, *be a solution to the system of difference equations*

$$\left(\frac{\Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t}(\mathbf{K};\Delta)}{\Delta}\right)_f = -\lambda_t \mathbf{A_K}\Big(\mathbf{p}^*\big(\Omega_t(\mathbf{K};\Delta)), \boldsymbol{\theta}_t; \mathbf{K}\big), \Omega_t(\mathbf{K};\Delta), \boldsymbol{\theta}_t\Big) \text{ for all } \mathbf{K} \leq \mathbf{K}_0,$$

where $\Omega_t(\mathbf{K};\Delta) = (\omega^f_{j,t}(\mathbf{K};\Delta))_{f,j}$, $\omega^f_{j,t}(\mathbf{K};\Delta) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K} - \mathbf{e}_j;\Delta)$, *with boundary conditions (i)* $\Pi_{f,T}(\mathbf{K};\Delta) = 0$, *(ii)* $\Pi_{f,t}(\mathbf{K};\Delta) = 0$ *if* $K_j = 0$ *for all* $j \in \mathscr{J}_f$. *Then,* $(\Pi_{f,t}(\mathbf{K};\Delta))_{f,\mathbf{K}}$ *converges and any limit* $(\Pi_{f,t}(\mathbf{K}))_{f,\mathbf{K}}$ *satisfies*

$$\left(\dot{\Pi}_{f,t}(\mathbf{K})\right)_f = -\lambda_t \mathbf{A_K}\Big(\mathbf{p}^*\big(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K}\big), \Omega_t(\mathbf{K}), \boldsymbol{\theta}_t\Big) \text{ for all } \mathbf{K} \leq \mathbf{K}_0,$$

where $\Omega_t(\mathbf{K}) = (\omega^f_{j,t}(\mathbf{K}))_{f,j}$, $\omega^f_{j,t}(\mathbf{K}) := \Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_j)$, *with boundary conditions (i)* $\Pi_{f,T}(\mathbf{K}) = 0$, *(ii)* $\Pi_{f,t}(\mathbf{K}) = 0$ *for all* $j \in \mathscr{J}_f$.

Proof. Since $\mathbf{A_K}$ is bounded for all $\mathbf{K} \leq \mathbf{K}_0$, the difference equations show that $(\Pi_f(\mathbf{K};\Delta))_{f \in \mathscr{F}, \mathbf{K} \leq \mathbf{K}_0}$ is equicontinuous and equibounded in t as $\Delta \to 0$. Hence, by the Arzela-Ascoli Theorem, there exist limit points $(\Pi_f(\mathbf{K}))_{f \in \mathscr{F}, \mathbf{K} \leq \mathbf{K}_0}$. We claim that

$$\left(\Pi_{f,t}(\mathbf{K})\right)_f = \int_t^T \lambda_u \mathbf{A_K}\Big(\mathbf{p}^*\big(\Omega_u(\mathbf{K}), \boldsymbol{\theta}_u; \mathbf{K}\big), \Omega_u(\mathbf{K}), \boldsymbol{\theta}_u\Big) du. \tag{11}$$

To this end, we note that if we let $\lceil u \rceil_\Delta$ to be the smallest number that is divisible by Δ and larger or equal than u

$$\left(\Pi_{f,t}(\mathbf{K};\Delta)\right)_f = \int_t^T \lambda_{\lceil u \rceil_\Delta} \mathbf{A_K}\Big(\mathbf{p}^*\big(\Omega_{\lceil u \rceil_\Delta}(\mathbf{K};\Delta), \boldsymbol{\theta}_{\lceil u \rceil_\Delta}; \mathbf{K}\big), \Omega_{\lceil u \rceil_\Delta}(\mathbf{K};\Delta), \boldsymbol{\theta}_{\lceil u \rceil_\Delta}\Big) du. \tag{12}$$

We take the limit $\Delta \to 0$ on both sides. The left-hand side of (12) converges to the left-hand side of (11). On the right-hand side, $\Omega_{\lceil u \rceil_\Delta}(\mathbf{K};\Delta)$ converges to $\Omega_u(\mathbf{K})$. Hence, by continuity of $\mathbf{p}^*$ and $\mathbf{A_K}$ the integrand in (12) converges to the integrand in (11). By the dominated convergence theorem the right-hand side of (12) converges to the right-hand side of (11). Thus, any limiting value function exists and must satisfy (11). ■

A.1.3 Continuity of stage game prices

Consider the stage game introduced in Subsection 4.1. In the following we fix the vector of remaining capacity $\mathbf{K}$, and therefore the set of available products $\mathscr{A}$, and omit it in the notation.

Lemma 5. *Let $\mathscr{P} \subset \mathbb{R}^{\mathscr{J}}$ be compact and $\mathscr{M}$ a compact set of $(\Omega, \boldsymbol{\theta})$. Further, let $\mathbf{g} : \mathscr{P} \times \mathscr{M} \to \mathscr{P}, (\mathbf{q}; \Omega, \boldsymbol{\theta}) \mapsto \mathbf{p}$ be (i) continuous in $\mathbf{q}$, (ii) continuous in Ω and $\boldsymbol{\theta}$, (iii) such that it implicitly defines a unique $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ satisfying $\mathbf{g}(\mathbf{p}^*(\Omega, \boldsymbol{\theta}); \Omega, \boldsymbol{\theta}) = \mathbf{p}^*(\Omega, \boldsymbol{\theta})$ for all $(\Omega, \boldsymbol{\theta}) \in \mathscr{M}$. Then, $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ depends continuously on Ω and $\boldsymbol{\theta}$.*

Proof. Consider the graph of $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$: $G = \{(\mathbf{p}, \Omega, \boldsymbol{\theta}) : g(\mathbf{p}; \Omega, \boldsymbol{\theta}) = \mathbf{p}\}$. By continuity of $\mathbf{g}$, G is closed in $\mathscr{P} \times \mathscr{M}$. Since $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ stays in the compact set $\mathscr{P}$ and is single-valued, it is upper hemicontinuous as a correspondence. Hence, $\mathbf{p}^*$ is continuous as a function.[37] ■

A.2 Proofs for Monopoly Benchmark

A.2.1 Proof of Lemma 1

In steps 1 and 2 of this proof we omit the conditioning argument $\mathscr{A}$ and $\boldsymbol{\theta}$.

Step 1: All profit-maximizing prices $\mathbf{p}^M$ are interior. First, we show that given $\boldsymbol{\omega}$,

$$\mathbf{p}^M \in \arg\max_{\mathbf{q}} \sum_{j \in \mathscr{A}} s_j(\mathbf{q})(q_j - \omega_j)$$

is bounded from below by a vector $\underline{\mathbf{p}} = (\underline{p} + \omega_1, \dots, \underline{p} + \omega_J)$, $\underline{p} \in \mathbb{R}$. We proceed with a proof by contradiction. Suppose such a $\underline{\mathbf{p}}$ did not exist. Then, for any $\underline{p} \in \mathbb{R}$ there exists an optimal price vector $\mathbf{p}^M$ and a j such that $p_j^M - \omega_j = \min_{j'}(p_{j'}^M - \omega_{j'}) < \underline{p}$. At this optimal price $\mathbf{p}^M$ (which can include (minus) infinite prices), the derivative of the stage game profit with respect to any price dimension has to be smaller than or equal to zero by optimality. Otherwise, the firm would have

[37] We thank Satoru Takahashi for helping us to simplify this proof.

an incentive to increase that price. The derivative of the profit with respect to p_j is

$$\sum_{k\neq j}\underbrace{\frac{\partial s_k}{\partial p_j}(\mathbf{p})}_{>0}\underbrace{(p_k-\omega_k)}_{\geq p_j-\omega_j}+s_j(\mathbf{p})+\frac{\partial s_j}{\partial p_j}(\mathbf{p})(p_j-\omega_j) \geq$$

$$\underbrace{(p_j-\omega_j)}_{<\underline{p}}\underbrace{\sum_k\frac{\partial s_k}{\partial p_j}(\mathbf{p})}_{=-\frac{\partial s_0}{\partial p_j}(\mathbf{p})<0 \text{ by Assumption 1-iii)}}+s_j(\mathbf{p})$$

This expression is stirctly positive if we choose $\underline{p}<0$ sufficiently negative. This yields a contradiction.

So, any optimal price vector is bounded by a vector $\underline{\mathbf{p}}$ from below. Take such a lower bound $\underline{\mathbf{p}}$ and let for each $j\in\mathscr{A}$, $\tilde{p}_j$ be such that

$$C:=\inf_{\mathbf{p}\geq\underline{\mathbf{p}},p_j\geq\tilde{p}_j}\frac{\frac{\partial s_0}{\partial p_j}(\mathbf{p})}{s_j(\mathbf{p})}p_j>1.$$

Such a $\tilde{p}_j$ exists by Assumption 1-iv).

Next, we show that given $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$, any profit maximizing price vector $\mathbf{p}^M$ is bounded from above by a vector $\overline{\mathbf{p}}=(\overline{p}+\omega_1,\ldots,\overline{p}+\omega_J)$, $\overline{p}\in\mathbb{R}$. We again proceed with a proof by contradiction. Suppose such a $\overline{\mathbf{p}}$ did not exist. Then, for any $\overline{p}\in\mathbb{R}$, there exists an optimal price vector $\mathbf{p}^M$ and j such that $p_j^M-\omega_j=\max_{j'}\left(p_{j'}^M-\omega_{j'}\right)>\overline{p}$. At the optimal price $\mathbf{p}^M$ (which could include infinite prices), the derivative of the profit with respect to any price dimension has to be greater than or equal to zero by optimality. The derivative of the firm's profit with respect to p_j at $p_j>\bar{p}_j>\tilde{p}_j$, $\mathbf{p}\geq\underline{\mathbf{p}}$ is

$$\underbrace{\sum_{k\neq j}\frac{\partial s_k}{\partial p_j}(\mathbf{p})}_{\geq 0}(p_k-\omega_k)+s_j(\mathbf{p})+\frac{\partial s_j}{\partial p_j}(\mathbf{p})(p_j-\omega_j) \leq$$

$$\underbrace{\sum_k\frac{\partial s_k}{\partial p_j}(\mathbf{p})}_{=-\frac{\partial s_0}{\partial p_j}}(p_j-\omega_j)+C^{-1}\frac{\partial s_0}{\partial p_j}(\mathbf{p})p_j \leq$$

$$\frac{\partial s_0}{\partial p_j}(\mathbf{p})(\underbrace{(C^{-1}-1)}_{<0}\underbrace{(p_j-\omega_j)}_{\geq\overline{p}}+C^{-1}\omega_j).$$

This expression is negative if $p_j - \omega_j \geq \overline{p} > \frac{C^{-1}\bar{\omega}}{1-C^{-1}}$, where $\bar{\omega} := \max_j \omega_j$ is the maximum opportunity cost. This yields a contradiction. Hence, any optimal price vector $\mathbf{p}^M$ is bounded by a vector $\overline{\mathbf{p}}$ from above.

Step 2: Uniqueness of profit-maximizing price $\mathbf{p}^M$. It follows from Step 1 that any profit-maximizing price $\mathbf{p}^M$ of the stage game must satisfy the FOCs of the firm. $D_{\mathbf{p}}\mathbf{s}(\mathbf{p})$ is non-singular by Section A.1.1. Hence, the FOCs can be written as equation (3). Because of Assumption 2 there is a unique solution to this system of equations by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010).

Step 3: Convergence. We can apply the Implicit Function Theorem to equation (3) by Assumption 2 and it follows that the unique optimal price $\mathbf{p}^M(\boldsymbol{\omega}, \boldsymbol{\theta}; \mathbf{K})$ is continuous in $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. Convergence to equation (2) follows by Lemma 4.

A.2.2 Proof of Proposition 1

Proof. i) To see that $\Pi_{M,t}(\mathbf{K})$ is decreasing in t, note that in equation (2), setting $p_j > \Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K} - \mathbf{e}_j)$ results in a positive stage-game payoff, so $\dot{\Pi}_{M,t}(\mathbf{K}) < 0$.

Next, we show that $\Pi_{M,t}(\mathbf{K}) > \Pi_{M,t}(\mathbf{K} - \mathbf{e}_j)$ for all j by induction in $\sum_j K_j$.

Induction start: It is immediate that $\Pi_{M,t}(\mathbf{e}_j) \geq \Pi_{M,t}(\mathbf{0}) = 0$ for all j and $t \leq T$.

Induction hypothesis: Assume that $\Pi_{M,t}(\mathbf{K}) > \Pi_{M,t}(\mathbf{K} - \mathbf{e}_j)$ for all $\mathbf{K}$ with $\sum_j K_j = \bar{K}$ and $j \in \mathscr{J}$.

Induction step: Now, consider a capacity vector $\mathbf{K}$ with $\sum_j K_j = \bar{K} + 1$. The solution of the differential equation for the profits is

$$\Pi_{M,t}(\mathbf{K}) = \int_t^T \lambda_z \sum_j s_{j,z}(\mathbf{p}_{M,z}(\mathbf{K}))\,(p_{M,j,z}(\mathbf{K}) + \Pi_{M,z}(\mathbf{K} - \mathbf{e}_j)) \cdot e^{-\int_t^z \lambda_u \sum_{j'} s_{j'}(\mathbf{p}_{M,u}(\mathbf{K}))du} dz.$$

By sub-optimality of the prices $\mathbf{p}_{M,t}(\mathbf{K}-\mathbf{e}_k)$ given capacity vector $\mathbf{K}$, we have for all k

$$\begin{aligned}
&\Pi_{M,t}\big(\mathbf{K}\big) \geq \\
&\int_t^T \lambda_z \sum_j s_{j,z}\big(\mathbf{p}_{M,z}(\mathbf{K}-\mathbf{e}_k)\big)\left(p_{M,j,z}(\mathbf{K}-\mathbf{e}_k)+\underbrace{\Pi_{M,z}(\mathbf{K}-\mathbf{e}_j)}_{\substack{>\Pi_{M,z}\left(\mathbf{K}-\mathbf{e}_k-\mathbf{e}_j\right)\\ \text{by induction hypothesis}}}\right)\cdot e^{-\int_t^z \lambda_u \sum_{j'} s_{j'}(\mathbf{p}_{M,u}(\mathbf{K}-\mathbf{e}_k))du} dz \\
&>\Pi_{M,t}\big(\mathbf{K}-\mathbf{e}_k\big).
\end{aligned}$$

ii) Next, we show that $\Pi_{M,t}(\mathbf{K})-\Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\leq \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)-\Pi_{M,t}(\mathbf{K}-2\mathbf{e}_j)$ for all j. To this end, let

$$H(\mathbf{x};\boldsymbol{\theta})=-\max_{\mathbf{p}}\sum_j s_j(\mathbf{p};\boldsymbol{\theta})(p_j-x_j).$$

Note that H is concave as a minimum of affine functions, strictly increasing in $\mathbf{x}$. Since H is concave and continuous, by the Fenchel-Moreau Theorem, it admits the representation

$$H(\mathbf{x};\boldsymbol{\theta})=\inf_{\mathbf{s}}(\mathbf{s}\cdot\mathbf{x}-H^*(\mathbf{s};\boldsymbol{\theta}))$$

where $H^*(\mathbf{s};\boldsymbol{\theta})=\inf_{\mathbf{x}}(\mathbf{x}\cdot\mathbf{s}-H(\mathbf{x};\boldsymbol{\theta}))$ is the concave conjugate of H. Moreover,

$$\dot{\Pi}_{M,t}(\mathbf{K})=\lambda_t H(\nabla\Pi_t(\mathbf{K});\boldsymbol{\theta}_t)$$

where $\nabla\Pi_{M,t}(\mathbf{K})=\big(\Pi_{M,t}(\mathbf{K})-\Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\big)_j$. Thus, $\Pi_{M,t}(\mathbf{K})$ is the value function for the optimal control problem

$$\Pi_{M,t}(\mathbf{K})=\sup_{\mathbf{s}\in\mathscr{A}}\mathbb{E}\left[\int_t^T \lambda_u H^*(\mathbf{s}_u;\boldsymbol{\theta}_u)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}}=\mathbf{K}\right]=:\sup_{\mathbf{s}} J_t(\mathbf{K},\mathbf{s})$$

where $\mathbf{X}_t^{\mathbf{a}}$ is the process which jumps by $-\mathbf{e}_j$ at rate $\lambda_t s_{j,t}$ and $\mathbf{s}\in\mathscr{A}$ are processes adapted with respect to the filtration on the probability space supporting $\mathbf{X}^{\mathbf{s}}$, with the property $s_{j,t}=0$ if $X_{j,t}^{\mathbf{s}}=0$ (Theorem 8.1 in Fleming and Soner (2006)). Let $\mathbf{s}_{\mathbf{K}}^*$ be the optimal control in the previous equation and $\mathbf{s}_{K-2}^*$ be the optimal control when $\mathbf{K}$ is replaced by $\mathbf{K}-2\mathbf{e}_j$. Then, note that since $\mathbf{s}_{\mathbf{K}}^*,\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*\in\mathscr{A}$,

$\frac{\mathbf{s}^*_{\mathbf{K}}+\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_j}}{2} \in \mathscr{A}$ because the process $\left(\mathbf{X}_s^{\frac{\mathbf{s}^*_{\mathbf{K}}+\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_f}}{2}}\right)_s$ can be chosen as $\left(\frac{\mathbf{X}_s^{\mathbf{s}^*_{\mathbf{K}}}+\mathbf{X}_s^{\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_f}}}{2}\right)_s$ ("coupling argument"). Hence,

$$
\begin{aligned}
&\Pi_{M,t}(\mathbf{K})+\Pi_{M,t}(\mathbf{K}-2\mathbf{e}_j)-2\Pi_{M,t}\big(\mathbf{K}-\mathbf{e}_j\big) && \leq \\
&J_t(\mathbf{K},s^*_{\mathbf{K}})+J_t(\mathbf{K}-2\mathbf{e}_j,s^*_{\mathbf{K}-2\mathbf{e}_j})-2\,J_t\left(\mathbf{K}-\mathbf{e}_j,\frac{\mathbf{s}^*_{\mathbf{K}}+\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_j}}{2}\right) && \leq \\
&\mathbb{E}\left[\int_t^T \lambda_u\left(H^*(\mathbf{s}^*_{\mathbf{K},u})+H^*(\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_j,u})-2H^*\left(\frac{\mathbf{s}^*_{\mathbf{K},u}+\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_f,u}}{2}\right)\right)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}^*_{\mathbf{K}}}=\mathbf{K},\,\mathbf{X}_t^{\mathbf{s}^*_{\mathbf{K}-2\mathbf{e}_j}}=\mathbf{K}-2\mathbf{e}_j,\right] && \leq 0.
\end{aligned}
$$

iii) To show that $\omega^M_{j,t\wedge\tau}(\mathbf{K}_t)$ is a submartingale, we show that for any capacity vector $\bar{\mathbf{K}}$ with $\bar{K}_j \geq 2$:

$$
\lim_{\Delta\to 0}\frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)\big|\mathbf{K}_t=\bar{\mathbf{K}}\big]}{\Delta}\geq 0.
$$

To this end, first, note that $\mathbf{K}_t$ is right-continuous in t. Consider $\bar{\mathbf{K}}$ with $\bar{K}_j \geq 2$. Then, we have that

$$
\begin{aligned}
&\lim_{\Delta\to 0}\frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\big]}{\Delta}= \\
&\lim_{\Delta\to 0}\frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_{t+\Delta})|\mathbf{K}_t=\bar{\mathbf{K}}\big]}{\Delta}+\lim_{\Delta\to 0}\frac{\mathbb{E}_0\big[\omega^M_{j,t}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\big]}{\Delta}= \\
&\dot{\omega}^M_{j,t}(\bar{\mathbf{K}})+\lambda_t\sum_{j'}s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\big(\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\omega^M_{j,t}(\bar{\mathbf{K}})\big)
\end{aligned}
$$

by right-continuity of the process $\mathbf{K}_t$. By (2), we can write

$$
\begin{aligned}
&\dot{\omega}^M_{j,t}(\bar{\mathbf{K}})= \\
&-\lambda_t\left[\sum_{j'}s_{j',t}(\mathbf{p}_{M,t}(\bar{\mathbf{K}}))\big(p_{M,j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}})\big)-s_{j',t}(p^M_t(\bar{\mathbf{K}}-\mathbf{e}_j))\big(p_{M,j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\right].
\end{aligned}
$$

and we know that

$$
\begin{aligned}
-\omega^M_{j',t}(\bar{\mathbf{K}})+\omega^M_{j,t}(\bar{\mathbf{K}})-\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'}) &= \Pi_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\Pi_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\Pi_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})+\Pi_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'}-\mathbf{e}_j) \\
&= -\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)
\end{aligned}
$$

Hence, $\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}$ is equal to

$$-\lambda_t\Big[\sum_{j'}s_{j',t}\big(\mathbf{p}_{M,t}(\bar{\mathbf{K}})\big)\big(p_{M,j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)-s_{j',t}\big(\mathbf{p}_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\big(p_{M,j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\Big]$$

Then, note that by optimality of $\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)$,

$$\sum_{j'}s_{j',t}\big(\mathbf{p}_{M,t}(\bar{\mathbf{K}})\big)\big(p_{M,j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\le\sum_{j'}s_{j',t}\big(\mathbf{p}_{M,t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\big(p_{M,j',t}(\bar{\mathbf{K}}-\mathbf{e}_j))-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})\big).$$

Hence, $\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}\ge 0.$ ■

A.3 Proofs of Dyanamic Pricing Model with Competition

A.3.1 Proof of Proposition 2

Recall that we omit the parameters $\boldsymbol{\theta}$ and $\mathscr{A}$ for the analysis of the stage game.

Proof. Let Assumptions 1, 2 and 3 hold. First, note that Assumption 3 implies that for $j_1, j_2 \neq k$

$$\frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})}=\frac{\frac{\partial s_{j_1}}{\partial p_k}(\mathbf{p})}{\frac{\partial s_{j_2}}{\partial p_k}(\mathbf{p})}.$$

By Step 1 in the proof of Proposition 3, any equilibrium price vector $\mathbf{p}^*(\Omega)$ of the stage game must satisfy the FOCs of firm f's payoff for all $j\in\mathscr{J}_f$ given by:

$$p_j-\omega^f_j+\sum_{j'\in\mathscr{J}_f\setminus\{j\}}\frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}(p_{j'}-\omega^f_{j'})-\sum_{j'\notin\mathscr{J}_f}\frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}\omega^f_{j'}=-\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

Since $\frac{\partial s_j}{\partial p_j}(\mathbf{p}) = -\sum_{k\in\mathscr{J}\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p}) - \frac{\partial s_0}{\partial p_j}$, this can be rewritten as

$$p_j - \omega_j^f - \sum_{j'\in\mathscr{J}_f\setminus\{j\}} \frac{1}{\sum_{k\in\mathscr{J}\setminus\{j\}} \frac{s_k(\mathbf{p})}{s_{j'}(\mathbf{p})} + \frac{s_0(\mathbf{p})}{s_{j'}(\mathbf{p})}} (p_{j'} - \omega_{j'}^f) + \sum_{j'\notin\mathscr{J}_f} \frac{1}{\sum_{k\in\mathscr{J}\setminus\{j\}} \frac{s_k(\mathbf{p})}{s_{j'}(\mathbf{p})} + \frac{s_0(\mathbf{p})}{s_{j'}(\mathbf{p})}} \omega_{j'}^f = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}$$

$$\Leftrightarrow p_j - \omega_j^f - \sum_{j'\in\mathscr{J}_f\setminus\{j\}} \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})}(p_{j'} - \omega_{j'}^f) + \sum_{j'\notin\mathscr{J}_f} \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})} \omega_{j'}^f = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

By Assumption 3, for $j' \neq j$, $\frac{\partial}{\partial p_j}\frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})} = 0$, we can define $\tilde{s}_{j,j'}((p_{j'})_{j'\neq j}) := \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})}$ and

$$c((p_{j'})_{j'\neq j};\Omega) := \omega_j^f + \sum_{j'\in\mathscr{J}_f\setminus\{j\}} \tilde{s}_{j,j'}((p_{j'})_{j'\neq j})(p_{j'} - \omega_j^f) - \sum_{j'\notin\mathscr{J}_f} \tilde{s}_{j,j'}((p_{j'})_{j'\neq j})\omega_{j'}^f.$$

Thus, the FOCs of the stage game are equivalent to the first order conditions of a game with $\mathscr{J}$ players where each player j's payoff is given by

$$s_j(\mathbf{p})\big(p_j - c((p_{j'})_{j'\neq j};\Omega)\big).$$

We call this game the "auxiliary game with J players." Note that the derivative of player j's payoff is greater or equal than zero if and only if

$$\frac{\partial s_j(\mathbf{p})}{\partial p_j}\left(p_j - c\left((p_{j'})_{j'\neq j};\Omega\right)\right) + s_j(\mathbf{p}) \geq 0.$$

Hence any equilibrium of the stage game is an equilibrium of a game with J players with the above payoffs and vice versa.

Given Assumption 2 i), the first-order condition of each player in the auxiliary game has a unique solution which must be a maximizer of player j's payoff function. Hence, the best response function of player j, $\mathscr{R}_j$, maps a compact convex set of prices $\mathbf{q}$ into a compact convex set of prices $\mathbf{p}$. For $\epsilon > 0$, consider the mapping

$$\Phi : (\mathbf{p},\mathbf{q}) \mapsto \left(p_j - \epsilon\Big(p_j - c_j(\mathbf{q}_{-j};\Omega) + \frac{s_j(\mathbf{q}_{-j}, p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}\Big)\right)_{j\in\mathscr{J}}$$

Then $D_{\mathbf{p}}\Phi$ is a diagonal matrix with diagonal entries

$$\phi_j := 1-\epsilon\Bigg(\underbrace{1+\frac{\partial}{\partial p_j}\frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{>0 \text{ by Assumption } 2i)}\Bigg)$$

Let $\epsilon > 0$ be so that $\phi_j > 0$ for all j. Then all diagonal entries are in $(0,1-\epsilon)$ and Φ is Lipschitz continuous with Lipschitz constant $\max_j \phi_j$. Further $D_{\mathbf{q}}\Phi$ is bounded because it is continuous. Then, the implicit function theorem in the form of Theorem 1.A.4 in Dontchev and Rockafellar (2009) implies continuity of $\mathscr{R} = ((\mathscr{R}_j)_j)$. Hence, by Brouwer's fixed-point theorem $\mathscr{R} = ((\mathscr{R}_j)_j)$ has a fixed point. ■

A.3.2 Proof of Lemma 2

Follows directly from Kellogg (1976) and Konovalov and Sándor (2010), and by Step 1 in the proof of Proposition 3 that shows that any equilibrium price vector $\mathbf{p}^*(\Omega)$ does not lie in the boundaries, i.e., is bounded in any component.

A.3.3 Proof of Proposition 3

Recall that we omit the parameters $\boldsymbol{\theta}$ and $\mathscr{A}$ for the analysis of the stage game.

Proof. **Step 1: All equilibrium prices $\mathbf{p}^*$ are interior.**

First, we show that for fixed Ω, any equilibrium price vector $\mathbf{p}^*$ is bounded from below by a vector $\underline{\mathbf{p}} = ((\underline{p}+\omega_j^f)_{j\in\mathscr{J}_f} : f \in \mathscr{F})$, $\underline{p} \in \mathbb{R}$. Let $\bar{\omega}_c := \max_{f,k\notin\mathscr{J}_c} \omega_k^f$ be the larges competitor scarcity effect in Ω. We proceed with a proof by contradiction. Suppose such a $\underline{\mathbf{p}}$ did not exist. Then, for any $\underline{p}$ there exists an equilibrium price vector $\mathbf{p}^*$ and a j such that $p_j^* - \omega_j^f = \min_{f'} \min_{k\in\mathscr{J}_{f'}} p_k^* - \omega_k^{f'} < \underline{p}$, where $j \in \mathscr{J}_f$. At this equilibrium price vector $\mathbf{p}^*$ (which can include (minus) infinite prices), the derivative of firm f's stage game profit with respect to all its prices has to be smaller or equal to zero by optimality. The derivative of firm f's profits with respect to p_j is

$$\frac{\partial s_j}{\partial p_j}(\mathbf{p})(p_j-\omega_j^f)+\sum_{k\in\mathscr{J}_f\setminus\{j\}}\frac{\partial s_k}{\partial p_j}(\mathbf{p})\underbrace{(p_k-\omega_k^f)}_{\geq p_j-\omega_j^f}-\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{p})\omega_k^f+s_j(\mathbf{p}) \quad \geq$$

$$-\underbrace{\frac{\partial s_0}{\partial p_j}}_{>0}(\mathbf{p})(p_j-\omega_j^f)-\underbrace{\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}}_{>0}(\mathbf{p})(\bar{\omega}_c+p_j-\omega_j^f)+s_j(\mathbf{p})$$

which is strictly positive if $\underline{p}<\min\{-\bar{\omega}_c,0\}$ by Assumption 1-iii) and because $p_j-\omega_j^f<\underline{p}$. This yields a contradiction.

So, any equilibrium price vector $\mathbf{p}^*$ is bounded by a vector $\underline{\mathbf{p}}$ from below. Take such a lower bound $\underline{\mathbf{p}}$ and elt for each $j\in\mathscr{J}$, $\tilde{p}_j$ be such that

$$C:=\inf_{\mathbf{p}\geq\bar{\mathbf{p}},p_j\geq\tilde{p}_j}\frac{\frac{\partial s_0}{\partial p_j}(\mathbf{p})}{s_j(\mathbf{p})}p_j>1.$$

Such a $\tilde{p}_j$ exists by Assumption 1-iv).

Next, we show that for fixed Ω, any equilibrium price vector $\mathbf{p}^*$ is bounded from above by a vector $\overline{\mathbf{p}}=((\overline{p}+\omega_j^f)_{j\in\mathscr{J}_f}:f\in\mathscr{F})$, $\overline{p}\in\mathbb{R}$, by contradiction. Let $\underline{\omega}_c:=\min_{f,k\notin\mathscr{J}_c}\omega_k^f$ be the smallest competitor scarcity effect in Ω. Suppose such a $\overline{\mathbf{p}}$ did not exist. Then, for any $\overline{p}$, there exists an equilibrium price vector $\mathbf{p}^*$ and a j such that $p_j^*-\omega_j^f=\max_{f'}\max_{k\in\mathscr{J}_{f'}}p_k^*-\omega_k^{f'}>\overline{p}$, $j\in\mathscr{J}_f$. At the equilibrium price $\mathbf{p}^*$ (which can include infinite prices), the derivative of firm f's stage game profit with respect to all its prices has to be greater or equal to zero by optimality.

The derivative of firm f's payoff with respect to p_j at $p_j>\bar{p}_j$ is

$$\frac{\partial s_j}{\partial p_j}(\mathbf{p})(p_j-\omega_j^f)+\sum_{k\in\mathscr{J}_f\setminus\{j\}}\frac{\partial s_k}{\partial p_j}(\mathbf{p})(p_k-\omega_k^f)-\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{p})\omega_k^f+s_j(\mathbf{p}) \quad \leq$$

$$-\frac{\partial s_0}{\partial p_j}(\mathbf{p})(p_j-\omega_j^f)-\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{p})(\underline{\omega}_c+p_j-\omega_j^f)+C^{-1}\frac{\partial s_0}{\partial p_j}(\mathbf{p})p_j \quad =$$

$$-\frac{\partial s_0}{\partial p_j}(\mathbf{p})\Big((1-C^{-1})(p_j-\omega_j^f)-C^{-1}\omega_j^f\Big)-\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{p})(p_j-\omega_j^f+\underline{\omega}_c).$$

This expression is strictly negative if $p_j-\omega_j^f>\overline{p}>\max\{\frac{C^{-1}\bar{\omega}_o}{1-C^{-1}},\ -\underline{\omega}_c\}$, where $\bar{\omega}_o :=$

$\max_f \max_{j\in\mathscr{J}_f} \omega_j^f$ is the maximum own-scarcity effect. This yields a contradiction. Hence, any equilibrium price vector $\mathbf{p}^*$ is bounded by a vector $\overline{\mathbf{p}} = ((\overline{p} + \omega_j^f)_{j\in\mathscr{J}_f} : f \in \mathscr{F})$ from above.

All in all, it follows that the best response of each firm must be within a box with extreme points $\overline{\mathbf{p}}$ and $\underline{\mathbf{p}}$.

Step 2: Uniqueness of equilibrium price $\mathbf{p}^*(\Omega)$.

It follows from Step 1 that any equilibrium price $\mathbf{p}^*(\Omega)$ of the stage game is a solution to the system of FOCs. $D_{\mathbf{p}^f} s(\mathbf{p}^f)$ is non-singular by Section A.1.1. Hence, the FOCs can be written as $\mathbf{g}(\mathbf{p}, \Omega) = \mathbf{p}$ where $\mathbf{g}$ is as defined in Assumption 4. There is a unique solution to this system of equations by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010) and Assumption 4-ii). Further, by Assumption 4-i) and Kellogg (1976), there is a unique solution of the first order condition of each firm's optimization problem, given by $\mathbf{g_f}(\mathbf{p}, \boldsymbol{\omega}^f) = \mathbf{p}^f$. Thus, for any competitor prices, there exists a unique best response of each firm f, which solves $\mathbf{g}_f(\mathbf{p}, \boldsymbol{\omega}^f) = \mathbf{p}^f$, and the unique solution to $\mathbf{g}(\mathbf{p}, \Omega) = \mathbf{p}$ must be an equilibrium. ■

A.3.4 Proof of Theorem 1

Let us assume that for all $\Delta < \bar{\Delta}$, there is a unique pure-strategy equilibrium of the dynamic game and $(\Omega_t(\mathbf{K}, \Delta), \theta_t) \in \mathscr{M}$ for an open bounded set $\mathscr{M}$ such stage games parameterized by $(\Omega, \boldsymbol{\theta}) \in \mathscr{M}$ admit a unique equilibrium. For every $\Delta < \bar{\Delta}$ the discrete-time game can be written as in Lemma 4 in Appendix A.1 where $\mathbf{A_K} : \mathbb{R}^{\mathscr{J}} \times \mathscr{M} \to \mathbb{R}^{\mathscr{F}}$ is given by

$$\mathbf{A_K}(\mathbf{p}, \Omega, \boldsymbol{\theta}) \quad := \left(\sum_{j\in\mathscr{A}(\mathbf{K})\cap\mathscr{J}_f} s_j(\mathbf{p}, \mathbf{K})\left(p_j - \omega_j^f\right) - \sum_{j'\in\mathscr{A}(\mathbf{K})\setminus(\mathscr{J}\setminus\mathscr{J}_f)} s_{j'}(\mathbf{p}, \mathbf{K})\,\omega_{j'}^f \right)_f .$$

$\mathbf{A_K}$ is bounded because demand functions s_j are bounded, ω-s are bounded and $s_j(\mathbf{p})p_j$ is bounded by Assumption 1-i). Thus, the discrete time derivatives $\frac{\Pi_{f,t+\Delta}(\mathbf{K};\boldsymbol{\Delta}) - \Pi_{f,t}(\mathbf{K};\Delta)}{\Delta}$ are uniformly bounded for $\Delta < \bar{\Delta}$, $\mathbf{K} \le \mathbf{K}_0$, $f \in \mathscr{F}$, and value functions are equicontinuous in t and equibounded as $\Delta \to 0$. As a result, $\omega_{j,t}^f(\mathbf{K}; \Delta)$ is equicontinuous and equibounded. Furthermore, equilibrium prices $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ of the stage game parameterized by $(\Omega, \boldsymbol{\theta})$ in Section 2 are continuous in $(\Omega, \boldsymbol{\theta})$ by Lemma 5. Then, convergence follows by Lemma 4 in Appendix A.1.

A.3.5 Proof of Lemma 3

Consider the system of first-order ODEs in equation (8). In order to show that this system of ODEs has a unique solution while given a capacity vector $\mathbf{K}$ Ω stays in a compact set $\mathscr{O}^{\mathbf{K}}$ in which Assumption 4 is satisfied, we use the Peano existence theorem (see for example Teschl (2012)).

Theorem 3 (Peano existence theorem)**.** *Consider a system of first-order ODEs:*

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(T) = \mathbf{x}_T \tag{13}$$

where $\mathbf{f} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a vector-valued function in* $t, \mathbf{x}$*. Consider subsets* $\mathscr{A}_t \subset \mathbb{R}^n$, $t \in [0,T]$ *such that* $\mathscr{A} = \{(t, \mathbf{x}) | \mathbf{x} \in \mathscr{A}_t\}$ *is open and* $\mathbf{x}_T \in \mathscr{A}_T$*. If* $\mathbf{f}$ *is continuous on* $\mathscr{A}$*, then there exist a* $\tau > 0$ *and a continuous function* $\mathbf{x}(t)$ *that satisfies equation* (13) *on* $(\tau, T]$*, with*

$$\tau = \inf\{t \in [0,T] | x(t) \in \mathscr{A}_t\}.$$

To apply this theorem, for every $\mathbf{K} \leq \mathbf{K}_0$ we define the set $\mathscr{C} := \{(t, \Omega, \mathbf{p}) | \Omega \in \mathscr{O}_t^{\mathbf{K}}, \mathbf{p} \in \mathscr{P}(\boldsymbol{\theta}_t)\}$, containing an open neighborhood of $(T, \mathbf{O}, \mathbf{p}_T(\mathbf{K}))$, and consider the functions $A_{f,\mathbf{K}} : \mathscr{C} \to \mathbb{R}$ and $\mathbf{B}_{\mathbf{K}} : \mathscr{C} \to \mathbb{R}$ given by

$$\begin{aligned}
A_{f,\mathbf{K}}(t, \Omega, \mathbf{p}) &= \sum_{j \in \mathscr{J}_f} s_j(\mathbf{p}; \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})) \left(p_j - \omega_j^f\right) - \sum_{j' \notin \mathscr{J}_f} s_{j'}(\mathbf{p}; \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})) \, \omega_{j'}^f, \; f \in \mathscr{F}(\mathbf{K}), \\
\mathbf{B}_{\mathbf{K}}(t, \Omega, \mathbf{p}) &= \left(D_{\mathbf{p}} \mathbf{g}(\mathbf{p}, \Omega, \boldsymbol{\theta}_t; \mathbf{K}) - I\right)^{-1} \cdot \\
&\quad \left(\sum_{j,f} D_{\omega_j^f} \mathbf{g}(\mathbf{p}, \Omega, \boldsymbol{\theta}_t; \mathbf{K})(-\lambda_t) \left(A_{f,\mathbf{K}}(t, \Omega, \mathbf{p}) - A_{f,\mathbf{K}-\mathbf{e}_j}(t, \Omega, \mathbf{p})\right)\right. \\
&\quad \left. + D_{\boldsymbol{\theta}} \mathbf{g}(\mathbf{p}, \Omega, \boldsymbol{\theta}_t; \mathbf{K}) \dot{\boldsymbol{\theta}}_t \right).
\end{aligned}$$

First, note that $A_{f,\mathbf{K}}$ and $\mathbf{B}_{\mathbf{K}}$ are continuous in $t, \Omega, \mathbf{p}$ on $\mathscr{C}$ because s_j are smooth in all parameters and $D_{\mathbf{p}} \mathbf{g}(\mathbf{p}, \Omega; \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})) - I$ is non-singular and uniformly bounded on $\mathscr{C}$ by assumption. In particular, $A_{f,\mathbf{K}}$ and $\mathbf{B}_{\mathbf{K}}$ are also bounded on $\mathscr{C}$. Then, by the Peano existence theorem, a solution of the ODEs exists on $(\tau, T]$, for a $\tau < T$ as described therein.

A.3.6 Proof of Theorem 2

We have established in Lemma 3 that there exists a solution to the system of ODEs given by equation (8) on $(\bar{\tau}, T]$ and that all solutions of the ODEs are such that $\Omega_t(\mathbf{K}) \in \mathcal{O}_t^{\mathbf{K}}$ for all $\mathbf{K} \leq \mathbf{K}_0$, $\in (\bar{\tau}, T]$, i.e., all solutions remain for all $\mathbf{K} \leq \mathbf{K}_0$ in the set $\mathcal{O}^{\mathbf{K}}$ in which $\Omega_t(\mathbf{K})$ satisfy Assumption 4.

Since the function $\mathbf{A}_{\mathbf{K}}$ as defined in the proof of Theorem 1 is uniformly bounded for $\Omega \in \mathcal{O}^{\mathbf{K}}$ by some vector $\bar{\mathbf{A}}_{\mathbf{K}}$ and

$$\left(\Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t}(\mathbf{K};\Delta)\right)_f = -\lambda_t \mathbf{A}_{\mathbf{K}}\Big(\mathbf{p}^*\big(\Omega(\mathbf{K};\Delta)), \boldsymbol{\theta}_t\big), \Omega(\mathbf{K};\Delta), \boldsymbol{\theta}_t\Big)\Delta < \bar{\lambda} \cdot |\bar{\mathbf{A}}| \cdot \Delta.$$

where $\bar{\lambda} := \max_{t\in[0,T]} \lambda_t$ and $|\bar{\mathbf{A}}_{\mathbf{K}}|$ is a vector of absolute values of all entries in the vector $\bar{\mathbf{A}}_{\mathbf{K}}$. Let $||\cdot||$ be the infinity norm on $\mathbb{R}^{\mathcal{F}(\mathbf{K})\times\mathcal{A}(\mathbf{K})}$, i.e., the maximum of the absolute values of entries in the matrix. Then, for $C := ||\left(\max_{t\in[0,T]} \lambda_t\right) \cdot 2 \cdot (|\bar{\mathbf{A}}_{\mathbf{K}}|, \ldots, |\bar{\mathbf{A}}_{\mathbf{K}}|)||$:

$$||\Omega_{t+\Delta}(\mathbf{K};\Delta) - \Omega_t(\mathbf{K};\Delta)|| < C\Delta.$$

We show by contradiction that $\tau^* = \limsup \tau^{\Delta} \leq \bar{\tau}$.

Assume that $\tau^* > \bar{\tau}$. Take $\tau^* > \delta > \delta' > 0$ such that

- $\delta' < T - \tau^*$ and
- for all $t' \in [0, \delta]$ and all $V \in \mathbb{R}^{\mathcal{A}(\mathbf{K})\times\mathcal{A}(\mathbf{K})}$ with $||V|| = 1$ and all $\tilde{\Omega}$ such that $||\tilde{\Omega} - \Omega_{\tau^*+\delta'}(\mathbf{K})|| < \delta'$:

$$\tilde{\Omega} + C \cdot V \cdot t' \in \mathcal{O}^{\mathbf{K}}_{\tau^*+\delta'-t'}.$$

Next, consider a decreasing sequence Δ_n such that $\lim_{n\to\infty} \Delta_n = 0$ and $\tau_{\Delta_n} < \tau^* + \delta'$ for all n. By Theorem 1, $\lim_{n\to\infty} \Omega_{\tau^*+\delta'}(\mathbf{K};\Delta_n) = \Omega_{\tau^*+\delta'}(\mathbf{K})$. Hence, by removing finitely many first terms of the sequence (Δ_n), we can ensure $||\Omega_{\tau^*+\delta'}(\mathbf{K};\Delta_n) - \Omega_{\tau^*+\delta'}(\mathbf{K})|| < \delta'$ for all n. Thus, for all $t' \in [0, \delta]$ and all $V \in \mathbb{R}^{\mathcal{F}(\mathbf{K})\times\mathcal{A}(\mathbf{K})}$ with $||V|| = 1$, it must be that $\Omega_{\tau^*+\delta'}(\mathbf{K};\Delta_n) + C \cdot V \cdot t' \in \mathcal{O}^{\mathbf{K}}_{\tau^*+\delta'-t'}$ for all n, implying $\Omega_t(\mathbf{K};\Delta_n) \in \mathcal{O}_t^{\mathbf{K}}$ for all $t \in [\tau^* + \delta' - \delta, \tau^* + \delta']$ and all n. Thus, $\tau_{\Delta_n} \leq \tau^* + \delta' - \delta < \tau^*$ for all n, a contradiction.

A.3.7 Proof of Proposition 4

First, recall that we know from Theorem 2 that for sufficiently small Δ, a unique Markov continuation equilibrium exists close to the deadline T, on an interval $(\bar{\tau}, T]$, $\bar{\tau} < T$, and it converges to the solution of a system of ODEs as $\Delta \to 0$. Furthermore, period-t equilibrium prices given $\Omega_t(\mathbf{K})$, and remaining capacity $\mathscr{A}(\mathbf{K})$ are given by $\mathbf{p}^*(\Omega_t(\mathbf{K}); \theta_t, \mathscr{A}(\mathbf{K}))$ where $\mathbf{p}^*$ is continuous is Ω and θ. Letting

$$\pi_f(\Omega, \boldsymbol{\theta}, \mathscr{A}) := \sum_{j \in \mathscr{J}} s_j(\mathbf{p}^*(\Omega; \boldsymbol{\theta}, \mathscr{A}); \boldsymbol{\theta}, \mathscr{A}) \left[p_j^*(\Omega; \boldsymbol{\theta}, \mathscr{A}) - \omega_j^f \right],$$

we can write

$$\dot{\Pi}_{f,t}(\mathbf{K}) \equiv -\lambda_t \pi_f(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})).$$

This yields the following differential equation for the scarcity effects:

$$\begin{aligned}
\dot{\omega}_{j,t}^f(\mathbf{K}) &= \dot{\Pi}_{f,t}(\mathbf{K}) - \dot{\Pi}_{f,t}(\mathbf{K} - \mathbf{e}_j) \\
&= -\lambda_t [\pi_f(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t \mathscr{A}(\mathbf{K})) - \pi_f(\Omega_t(\mathbf{K} - \mathbf{e}_j), \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K}))].
\end{aligned}$$

Finally, let us denote the minimum capacity in a capacity vector $\mathbf{K}$ by $\min(\mathbf{K}) := \min_j K_j$. We proceed to prove that for any capacity vector $\mathbf{K}$, the n-th time derivatives $\Omega_T^{(n)}(\mathbf{K}) = \lim_{t \to T} \frac{\partial}{\partial t} \Omega_t(\mathbf{K})$ vanish for all $n < \min(\mathbf{K})$, by doing induction on n.

Lemma 6. *For any* $\mathbf{K}$, $\Omega_T^{(n)}(\mathbf{K}) = 0$ *for all* $n < \min(\mathbf{K})$.

Proof. **Induction start:** Consider a $\mathbf{K}$ with $\min(\mathbf{K}) > 0$. For $n = 0$, we have that $\Omega_t^{(n)}(\mathbf{K}) = \Omega_t(\mathbf{K})$. It is immediate that $\Omega_T(\mathbf{K}) = \mathbf{O}$ for all $\mathbf{K}$ such that $\min(\mathbf{K}) > 0$.

Induction assumption: For a given n and all $\mathbf{K}$ with $n < \min(\mathbf{K})$, we assume that $\Omega_T^{(m)}(\mathbf{K}) = 0$ for all $m \leq n$.

Induction step: Let $\min(\mathbf{K}) > n + 1$ and suppose the induction assumption holds for n. Using the differential equation for Π and differentiating it n times, we can write the $(n+1)$-th time derivative $\Pi_{f,t}^{(n+1)}(\mathbf{K})$ as

$$\Pi_{f,t}^{(n+1)}(\mathbf{K}) = G_n^{\Omega}\left(\left(\Omega_t(\mathbf{K})^{(m)}\right)_{m=0}^{n-1}, \left(\lambda_t^{(m)}\right)_{m=0}^{n-1}, \left(\boldsymbol{\theta}_t^{(m)}\right)_{m=0}^{n-1}\right) + G_n^{\boldsymbol{\theta}}\left(\Omega_t(\mathbf{K}), \left(\lambda_t^{(m)}\right)_{m=0}^{n}, \left(\boldsymbol{\theta}_t^{(m)}\right)_{m=0}^{n}\right)$$
$$-\lambda_t \sum_{f'\in\mathscr{F}} \sum_{j'\in\mathscr{J}} \frac{\partial^n \pi_f\left(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K})\right)}{\left(\partial \omega_{j'}^{f'}\right)^{(n)}} \left(\omega_{j',t}^{f'}(\mathbf{K})\right)^{(n)}.$$

where G_n^{Ω} is defined to incorporate all the terms that depend on at least one derivative of $\Omega_t(\mathbf{K})$ of order less than n, $G_n^{\boldsymbol{\theta}}$ captures the terms with derivatives only on parameters and the third term is the only one that depends on $\Omega_t^{(n)}(\mathbf{K})$.

Since $\min(\mathbf{K}) > n+1$, we have that $\min(\mathbf{K}-\mathbf{e}_j) > n$ for all $j \in \mathscr{J}$. This implies that $\Omega_T^{(m)}(\mathbf{K}-\mathbf{e}_j) = 0$ for all $m \leq n$ and all $j \in \mathscr{J}$ by the induction assumption. Additionally, we have that $\Omega_T^{(m)}(\mathbf{K}) = \mathbf{O}$ for all $m \leq n$ by the induction assumption. Since the Ω derivatives in G_n^{Ω} enter multiplicatively, these terms vanish as $t \to T$. Additionally, the n-th order derivatives also vanish. This yields

$$\Pi_{f,T}^{(n+1)}(\mathbf{K}) = G_n^{\boldsymbol{\theta}}\left(\mathbf{O}, \left(\lambda_T^{(m)}\right)_{m=0}^{n}, \left(\boldsymbol{\theta}_T^{(m)}\right)_{m=0}^{n}\right),$$
$$\Pi_{f,T}^{(n+1)}(\mathbf{K}-\mathbf{e}_j) = G_n^{\boldsymbol{\theta}}\left(\mathbf{O}, \left(\lambda_T^{(m)}\right)_{m=0}^{n}, \left(\boldsymbol{\theta}_T^{(m)}\right)_{m=0}^{n}\right),$$

for all $j \in \mathscr{J}$. Therefore, we obtain

$$\left(\omega_{j,T}^{f}(\mathbf{K})\right)^{(n+1)} = \Pi_{f,T}^{(n+1)}(\mathbf{K}) - \Pi_{f,T}^{(n+1)}(\mathbf{K}-\mathbf{e}_j) = 0.$$

This concludes the proof that the derivatives $\Omega_T^{(n)}(\mathbf{K})$ vanish for all $n < \min(\mathbf{K})$. ■

For t close to T, we have established in Theorem 2 that the equilibrium of the stage game is unique. Recall that $\mathbf{p}_t^*(\mathbf{K}) := \mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t; \mathbf{K})$ and let $\mathbf{p}_t^{\mathbf{O}}(\mathbf{K}) := \mathbf{p}^*(\mathbf{O}; \boldsymbol{\theta}_t, \mathscr{A}(\mathbf{K}))$ denote the price vectors of the benchmark if there are no scarcity effects. We omit $\mathbf{K}$ for readability since we hold $\mathbf{K}$ fixed in the following analysis.

$$\dot{\mathbf{p}}_t^* = -\left(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}_t^*, \Omega_t, \boldsymbol{\theta}_t) - I\right)^{-1} \left(\sum_{j,f} D_{\omega_j^f}\mathbf{g}(\mathbf{p}_t^*, \Omega_t, \boldsymbol{\theta}_t)\dot{\omega}_{j,t}^f + D_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{p}_t^*, \Omega_t, \boldsymbol{\theta}_t)\dot{\boldsymbol{\theta}}_t\right),$$
$$\dot{\mathbf{p}}_t^{\mathbf{O}} = -\left(D_{\mathbf{p}}\mathbf{g}(\mathbf{p}_t^{\mathbf{O}}, \mathbf{O}, \boldsymbol{\theta}_t) - I\right)^{-1} \left(D_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{p}_t^{\mathbf{O}}, \mathbf{O}, \boldsymbol{\theta}_t)\dot{\boldsymbol{\theta}}_t\right).$$

To simplify notation, we write these equations as

$$\dot{\mathbf{p}}^*_t = \sum_{j,f} Q^{\Omega}_{j,f}(\mathbf{p}^*_t, \Omega_t, \boldsymbol{\theta}_t)\dot{\omega}^f_{j,t} + Q^{\boldsymbol{\theta}}(\mathbf{p}^*_t, \Omega_t, \boldsymbol{\theta}_t)\dot{\boldsymbol{\theta}}_t,$$
$$\dot{\mathbf{p}}^{\mathbf{O}}_t = Q^{\boldsymbol{\theta}}(\mathbf{p}^{\mathbf{O}}_t, \mathbf{O}, \boldsymbol{\theta}_t)\dot{\boldsymbol{\theta}}_t.$$

Define $\tilde{\mathbf{p}}_t \equiv \mathbf{p}^*_t - \mathbf{p}^{\mathbf{O}}_t$. This measures the deviation of prices from the benchmark without scarcity effects. Let

$$\underline{n} := \min\{n | \exists j : (\omega^f_{j,T})^{(n)} \neq 0\}.$$

We will prove that $\tilde{\mathbf{p}}^{(n)}_T = 0$ for all $n < \underline{n}$ by induction on n.

Lemma 7. $\tilde{\mathbf{p}}^{(n)}_T = 0$ *for all* $n < \underline{n}$.

Proof. **Induction start:** Let $\underline{n} > 1$ and take $n = 1$. This means that we have $\dot{\omega}^f_{j,T} = 0$. Additionally, since $\Omega_T = 0$, we have that, at $t = T$, the first derivatives of prices satisfy

$$\dot{\mathbf{p}}^*_T = Q^{\boldsymbol{\theta}}\left(\mathbf{p}^*_T, \mathbf{O}, \boldsymbol{\theta}_t\right)\dot{\boldsymbol{\theta}}_T,$$
$$\dot{\mathbf{p}}^{\mathbf{O}}_T = Q^{\boldsymbol{\theta}}\left(\mathbf{p}^{\mathbf{O}}_T, \mathbf{O}, \boldsymbol{\theta}_t\right)\dot{\boldsymbol{\theta}}_T.$$

Since the equations that define $\mathbf{p}^*_T$ and $\mathbf{p}^{\mathbf{O}}_T$ are identical (because $\Omega_T = 0$), we have that $\mathbf{p}_T = \mathbf{p}^*_T$, which implies $\dot{\mathbf{p}}_T = \dot{\mathbf{p}}^*_T$ from the equation above. This implies $\dot{\tilde{\mathbf{p}}}_T = 0$.

Induction assumption: For a given n, we assume that for all $m \leq n$ the following holds:

$$(\mathbf{p}^{\mathbf{O}}_T)^{(m)} = (\mathbf{p}^*_T)^{(m)}.$$

Induction step: Let $n+1 < \underline{n}$ and suppose the induction assumption holds for n. By differentiating the equations for $\dot{p}^*_t$ and $\dot{p}^{\mathbf{O}}_t$ n times, we obtain equations of the form

$$(\mathbf{p}^*_t)^{(n+1)} = Q^{\Omega}_n\left(((\mathbf{p}^*_t)^{(m)})^n_{m=0}, (\Omega^{(m)}_t)^{n+1}_{m=0}, (\boldsymbol{\theta}^{(m)}_t)^n_{m=0}\right) + Q^{\boldsymbol{\theta}}_n\left(((\mathbf{p}^*_t)^{(m)})^n_{m=0}, \Omega_t, (\boldsymbol{\theta}^{(m)}_t)^{n+1}_{m=0}\right),$$
$$(\mathbf{p}^{\mathbf{O}}_t)^{(n+1)} = Q^{\boldsymbol{\theta}}_n\left(((\mathbf{p}^{\mathbf{O}}_t)^{(m)})^n_{m=0}, \Omega_t, (\boldsymbol{\theta}^{(m)}_t)^{n+1}_{m=0}\right),$$

where Q^{Ω}_n is defined to incorporate all the terms that depend on at least one derivative of Ω_t. Intuitively, these functions are isolating the effects from the derivatives of Ω_t from the effects of derivatives of prices and $\boldsymbol{\theta}_t$, which is why $Q_n^{\boldsymbol{\theta}}$ is the same in both equations. It is also important to note that these functions depend on derivatives of the prices of at most order n. Now, since the derivatives of Ω_t enter Q_n^{Ω} multiplicatively and, since $n+1<\underline{n}$, all these derivatives are 0 at $t=T$, we have that

$$Q_n^{\Omega}\left(((\mathbf{p}_T^*)^{(m)})_{m=0}^n,(\Omega_T^{(m)})_{m=0}^{n+1},(\boldsymbol{\theta}_T^{(m)})_{m=0}^n\right)=0.$$

Therefore, at $t=T$, the price derivatives satisfy

$$(\mathbf{p}_t^*)^{(n+1)}=Q_n^{\boldsymbol{\theta}}\left(((\mathbf{p}_T^*)^{(m)})_{m=0}^n,\mathbf{O},(\boldsymbol{\theta}_T^{(m)})_{m=0}^{n+1}\right)=(\mathbf{p}_T^{\mathbf{O}})^{(n+1)}=Q_n^{\boldsymbol{\theta}}\left(((\mathbf{p}_T^{\mathbf{O}})^{(m)})_{m=0}^n,\mathbf{O},(\boldsymbol{\theta}_T^{(m)})_{m=0}^{n+1}\right).$$

■

Using Taylor's Theorem, Lemmas 6 and 7 imply the first statement of Proposition 4. Additionally, if $\lim_{t\to T}(\Pi_{f,t})^{(\min_j K_j)}(\mathbf{K}-\mathbf{e}_{j'})\neq 0$ for all f and j' with $K_{j'}\in\arg\min_j K_j$, we see by following the proof of Lemma 6 that $\lim_{t\to T}\Omega_t^{(\min_j K_j)}(\mathbf{K})\neq\mathbf{O}$. Following the argument in the proof of Lemma 7, this implies that, generally, $\tilde{\mathbf{p}}_T^{(\min_j K_j)}\neq 0$. Again using Taylor's Theorem, we obtain the second statement of Proposition 4.

A.3.8 Proof of Proposition 5

We prove Proposition 5 using examples. Figure 3 plots price paths, value functions and scarcity effects over time for a logit demand system and various capacity vectors in the continuous-time limit. Each sub-plot corresponds to a statement in the proposition. We know from Theorem 2 that these price paths and value funcitons are close to the solutions of a discrete-time game with sufficiently small Δ as long as the scarcity matrix remains in an open bounded set in which Assumption 4 is satisfied. For the specific logit demand example where demand remains constant over time, the ODEs have a unique solution and we can verify that competitor scarcity effects are such that Assumption 4 is satisfied everywhere. This also implies that there is a unique equilibrium for the discrete-time game for sufficiently small Δ.

A.3.9 Proof of Proposition 6

The proof follows analogously to the proof of Lemma 1 from Lemma 4. The proof is even easier because the the welfare-maximizing price in each state is simply $\nu^*_{j,t}(\mathbf{K}) = W^*_t(\mathbf{K}) - W^*_t(\mathbf{K}-\mathbf{e}_j)$, so

$$\mathbf{A}_{\mathbf{K}}(\boldsymbol{\nu}, \boldsymbol{\theta}) = w_t(\boldsymbol{\nu}; \mathbf{K}) - \sum_j s_j(\boldsymbol{\nu}; \boldsymbol{\theta}, \mathscr{A}(\mathbf{K}))\boldsymbol{\nu}$$

which is continuous and bounded.

A.3.10 Proof of Proposition 7

i) **Claim:** The welfare-optimal price is constant over time and given by equation (10).

We first argue by contradiction that a social planner always sells exactly the available capacity $\mathbf{K}_0$ (without rationing). If not, the planner can lower the price in the last period to sell to more customers and increase welfare—a contradiction.

We next argue by contradiction that the welfare-optimal price is constant over time. Assume there are prices $p > p'$ that are offered at different points in time while selling out exactly (without rationing). Then, a social planner can increase p' and decrease p such that equation 10 remains satisfied. Then, the capacity is allocated to higher-valuation customers than with the initial prices, so total welfare is higher—a contradiction.

ii) **Claim:** It can be optimal for a single firm to set a higher or lower price in the first period, and it may optimally not sell out its capacity. The profit-maximizing price is constant over time if demand is constant over time but may be higher than p^w.

The first part of the claim is straight-forward to see with the following examples:

Consider $D_1(p) = 0.2 - p$ and $D_2(p) = 1 - p$, $K_0 = 1$. Then the profit maximizing prices are the static profit-maximizing prices $p_1 = 0.1 < p_2 = 0.5$. Total quantity sold is 0.6, so the capacity constraint is not binding.

Analogously, we can consider $D_1(p) = 1 - p$ and $D_2(p) = 0.1 - p$ and $K_0 = 1$ to obtain $p_1 = 0.5 > p_2 = 0.1$.

Finally, assume $D_1(p) \equiv D_2(p) = D(p)$. It is useful to write the problem in terms of inverse demand functions $P(q)$. The monopolist solves $\max_{q_1,q_2} P(q_1)q_1 + P(q_2)q_2$ subject to $q_1+q_2 \leq K_0$. The FOCs of the Lagrangian for this problem are

$$P'(q_1)q_1 + P(q_1) = \lambda$$
$$P'(q_2)q_2 + P(q_2) = \lambda,$$

so $q_1 = q_2$.

iii) **Claim:** With competition, the capacity constraint is always binding, i.e., equation (10) holds, and

(a) if firms are symmetric, i.e., $K_f = 1$ for all f, and demand is constant over time ($D_1 = D_2$), then $p_1 = p_2 = p^w$ in equilibrium;

(b) if the welfare-optimal price is not an equilibrium, then only one firm sells in the first period in equilibrium and at a higher price than in the second period.

This follows directly from Dana and Williams (2022).

A.3.11 Proof of Proposition 8

Let us consider the point-wise FOCs of the Lagrangian for all three optimization problems at time t, where $\boldsymbol{\lambda}^x = (\lambda_j^x)_{j\in\mathcal{J}}$, $x \in \{SP, M, O\}$, are the vectors of Lagrange multipliers:

$$\text{Social planner:} \quad D_{\mathbf{p}}\mathbf{s}(\mathbf{p}_t)\cdot\mathbf{p}_t + \mathbf{s}(\mathbf{p}_t) + \nabla\mathbb{E}[\max_j(v_{j,t} - p_{j,t})] = D_p\mathbf{s}(\mathbf{p}_t)\cdot\boldsymbol{\lambda}^{SP}$$

$$\text{Monopolist:} \quad D_{\mathbf{p}}\mathbf{s}(\mathbf{p}_t)\cdot\mathbf{p}_t + \mathbf{s}(\mathbf{p}_t) = D_{\mathbf{p}}\mathbf{s}(\mathbf{p}_t)\cdot\boldsymbol{\lambda}^{M}$$

$$\text{Oligopoly:} \quad D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p}_t)\cdot\mathbf{p}_t^f + \mathbf{s}^f(\mathbf{p}_t) = D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p}_t)\cdot\boldsymbol{\lambda}_f^{O}$$

We start with the social planner's problem and note that

$$\begin{aligned}
\frac{\partial}{\partial p_j}\mathbb{E}[\max_{j'}(v_{j'}-p_{j'})] &= \frac{\partial}{\partial p_j}\mathbb{E}\big[\max\{v_j-p_j,\max_{j'\neq j}(v_{j'}-p_{j'})\}\big] \\
&= \frac{1}{2}\frac{\partial}{\partial p_j}\mathbb{E}\Big[v_j-p_j+\max_{j'\neq j}(v_{j'}-p_{j'})+\big|\max_{j'\neq j}(v_{j'}-p_{j'})-(v_j-p_j)\big|\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[-1+\mathrm{sgn}\big(\max_{j'\neq j}(v_{j'}-p_{j'})-(v_j-p_j)\big)\Big] \\
&= -\frac{1}{2}+\frac{1-s_j(\mathbf{p})}{2}-\frac{s_j(\mathbf{p})}{2}=-s_j(\mathbf{p})
\end{aligned}$$

Hence, we can write the social planner's FOC as:

$$\mathbf{p}_t=\boldsymbol{\lambda}^{SP},$$

so the social planner's price must be constant. Next, note that

$$\begin{aligned}
D_{\mathbf{s}}w=(D_{\mathbf{p}}\mathbf{s})^{-1}D_{\mathbf{p}}w &= (D_{\mathbf{p}}\mathbf{s})^{-1}D_p\big(\mathbf{s}\cdot\mathbf{p}+\mathbb{E}[\max_j(v_{j,t}-p_{j,t})]\big) \\
&= (D_{\mathbf{p}}\mathbf{s})^{-1}((D_{\mathbf{p}}\mathbf{s})\mathbf{p}+\mathbf{s}-\mathbf{s})=\mathbf{p}\geq\mathbf{0}.
\end{aligned}$$

Thus, the constraint $\int_0^T \mathbf{s}_t\,dt\leq\mathbf{K}_0$ is binding and the constant welfare-maximizing price must solve $\mathbf{s}(\mathbf{p}^w)=\frac{1}{T}\mathbf{K}_0$.

Next, consider the monopolist and oligopoly problem. Given a vector $\boldsymbol{\lambda}$, the monopolist's problem has a unique fixed point by Assumption 2 and Kellogg's fixed point theorem. Similarly, the oligopoly game has a unique equilibrium by Assumption 2 and Konovalov and Sándor (2010). Hence, the unique price paths must involve constant prices.

Having established that the price path must be constant, the monopoly and oligopoly problem boil down to a static optimization and game, respectively. Hence, it is immediate that a monopolist optimally sets a price $\mathbf{p}^M$ solving $\max_{\mathbf{p}}\mathbf{s}(\mathbf{p})\mathbf{p}$ subject to $\mathbf{s}(\mathbf{p}^M)\leq\frac{1}{T}\mathbf{K}_0$. The competitive equilibrium price vector $\mathbf{p}^O$ is the unique equilibrium $\mathbf{p}^*((\boldsymbol{\lambda}_f^{SP})_f)$ of a game where each firm faces a cost vector $\boldsymbol{\lambda}_f^{SP}\geq 0$, where

$$\boldsymbol{\lambda}^{SP}\in\arg\min\Big\{\boldsymbol{\lambda}\geq\mathbf{0}\,\Big|\,\mathbf{s}(\mathbf{p}^*(\boldsymbol{\lambda}))\leq\frac{1}{T}\mathbf{K}_0\Big\}.$$

B Nested Logit Calculations

Since our examples use a nested logit specification, we verify in the following that Assumptions 1 and 2 are satisfied for a nested logit demand model given by

$$s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) = \underbrace{\frac{e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}{\sum_{j\in\mathscr{J}} e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}}_{=:s_{j|\mathscr{J}}(\mathbf{p})} \frac{\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}} \qquad s_0(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) = \frac{1}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}},$$

where $\boldsymbol{\theta} = (\alpha,\sigma,(\delta_j)_{j\in\mathscr{J}})$. We omit $\mathscr{J}$ for readability. Note that the same properties follow for regular logit by setting $\sigma = 0$. To simplify notation, let $D_{\mathscr{J}} := \sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}$ and $G := \sigma\frac{1+D_{\mathscr{J}}^{1-\sigma}}{D_{\mathscr{J}}^{1-\sigma}} + 1-\sigma$. Then,

$$\begin{aligned}
\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{1-\sigma} s_j\left(1-\left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right)\right) &= \frac{\alpha}{1-\sigma}(G s_j^2 - s_j) \\
\frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{1-\sigma} s_{j'}\left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right) &= \frac{\alpha}{1-\sigma} G s_{j'} s_j.
\end{aligned}$$

It is easy to check that Assumptions 1-i) and ii) are satisfied. We show that Assumption 1-iii) is satisfied. Letting $\underline{s}_0 \equiv s_0\left(\underline{\mathbf{p}};\boldsymbol{\theta}\right)$, we can set $C = \alpha\underline{s}_0 > 0$ since then

$$\frac{\partial s_0}{\partial p_j} = \alpha s_j s_0 > C s_j.$$

Then,

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = -\frac{1}{\alpha s_0}\cdot\begin{pmatrix}
1+\sigma D_{\mathscr{J}}^{\sigma-1} + (1-\sigma)\frac{s_0}{s_1} & 1+\sigma D_{\mathscr{J}}^{\sigma-1} & \dots & 1+\sigma D_{\mathscr{J}}^{\sigma-1} \\
1+\sigma D_{\mathscr{J}}^{\sigma-1} & \ddots & & 1+\sigma D_{\mathscr{J}}^{\sigma-1} \\
\vdots & & \ddots & \vdots \\
1+\sigma D_{\mathscr{J}}^{\sigma-1} & \dots & \dots & 1+\sigma D_{\mathscr{J}}^{\sigma-1} + (1-\sigma)\frac{s_0}{s_J}
\end{pmatrix}$$

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = -\frac{1-\sigma}{\alpha} \cdot \begin{pmatrix} \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} + \frac{1}{s_1} & \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} & \cdots & \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} \\ \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} & \ddots & & \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} \\ \vdots & & \ddots & \vdots \\ \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} & \cdots & \cdots & \frac{G+\sigma+D_{\mathscr{J}}^{1-\sigma}}{1-\sigma} + \frac{1}{s_J} \end{pmatrix}$$

Hence, $\hat{\boldsymbol{\epsilon}} = ((D_p\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{\intercal})^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\frac{1}{\alpha s_0}\mathbf{1}$ and noting that $\frac{\partial}{\partial p_j}\left(\frac{1}{s_0}\right) = -\alpha\frac{s_j}{s_0}$,

$$D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} \frac{s_1}{s_0} & \cdots & \frac{s_J}{s_0} \\ & \ddots & \\ \frac{s_1}{s_0} & \cdots & \frac{s_J}{s_0} \end{pmatrix}.$$

It follows that Assumption 2 is satisfied:

$$\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}} - I\right) = (-1)^J \frac{1}{s_0} \neq 0.$$

For our welfare analysis, we can separate the contributions form consumer surplus and firm revenues. To this end, note that we can write the static consumer surplus for our demand specification as

$$CS = \frac{1}{\alpha_t}\log\left(1 + D_{\mathscr{J}}^{1-\sigma}\right) = \frac{1}{\alpha_t}\log\left(1 + \left(\sum_{j\in\mathscr{J}}\exp\left(\frac{\delta_j - \alpha_t p_j}{1-\sigma}\right)\right)^{1-\sigma}\right).$$

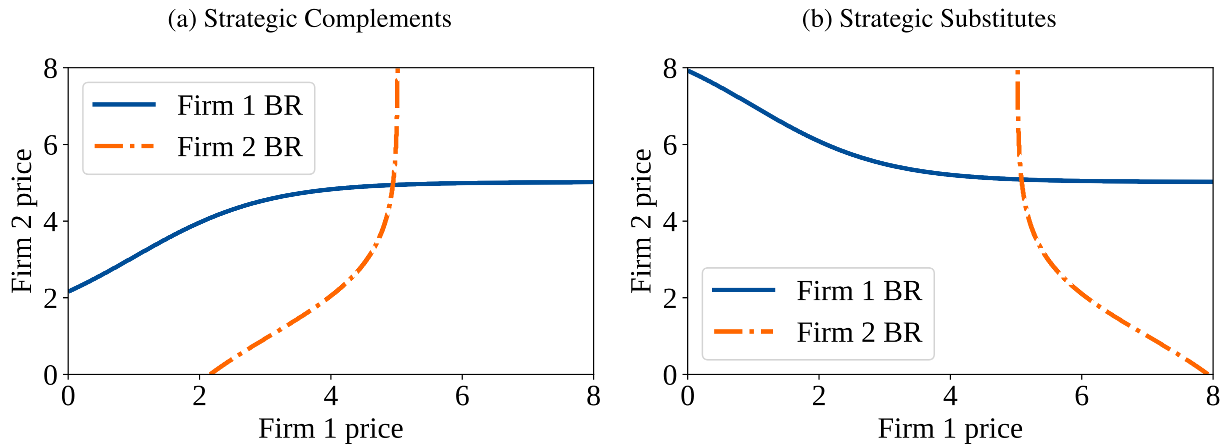
We can, then, write the per-period welfare as

$$w_t(\mathbf{p}) = CS + \sum_{j\in\mathscr{J}} s_j(\mathbf{p};\boldsymbol{\theta})p_j.$$

It follows immediately that all properties are satisfied for all subsets $\mathscr{A} \subset \mathscr{J}$.

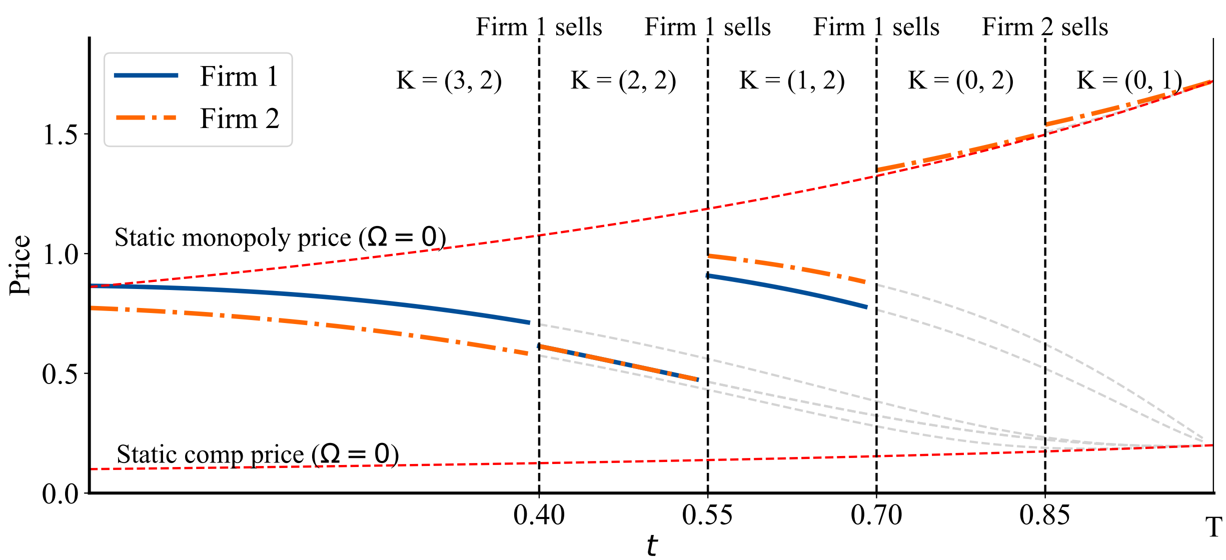
C Additional Figures

Figure 8: Strategic complements and substitutes in the stage game



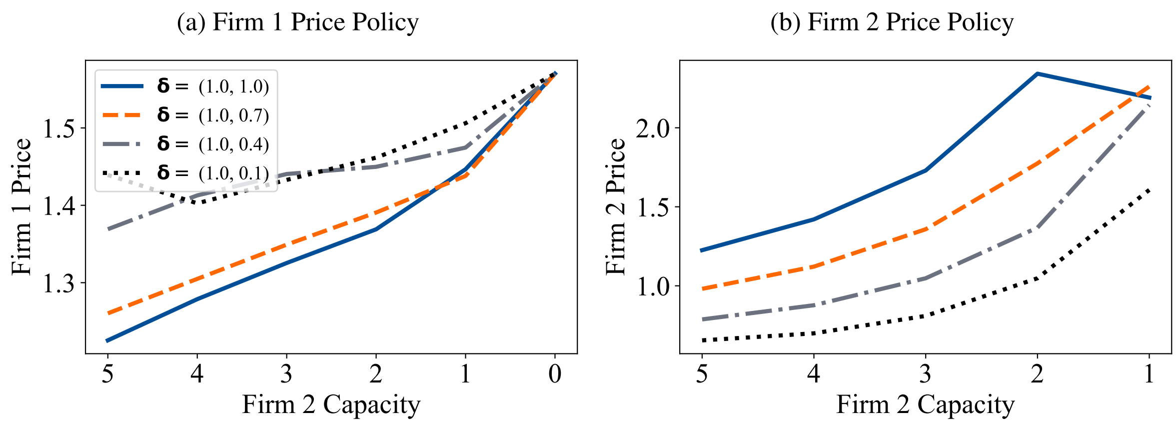
Notes: The simulations assume the logit demand system in Example 2 where $\omega_1^1 = \omega_2^2 = 4$. Panel (a) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = 4$. Panel (b) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = -4$.

Figure 9: Example Equilibrium Path with Decreasing Price Sensitivity



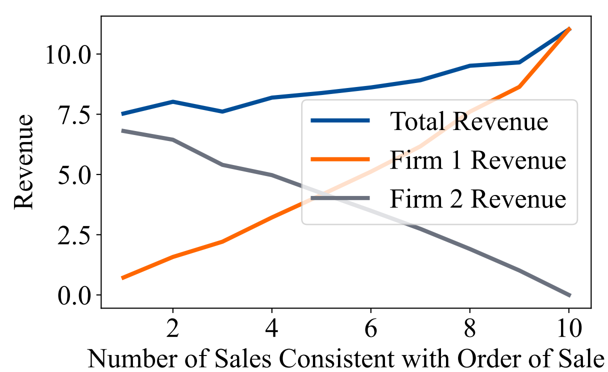
Notes: This plot shows an example equilibrium realization assuming two firms, each offering one product each. The time horizon is defined as $t \in [0,1]$, such that $\Delta = .01$, and the arrival rate is constant per period equal to $\Delta\lambda = .03$. We assume a logit demand system such that the probability of purchase of product j, conditional on arrival, is equal to $s_f = \frac{\exp(20-20\cdot p_f)}{1+\sum_{f'=1,2}\exp(20-20\cdot p'_f)}$. The horizontal (red) lines denote the static competitive and monopoly price. The vertical dashed lines denote the sale of a unit as marked. The blue and orange lines denote equilibrium prices. Finally, the grey dashed lines denote what prices would have been absent the sale.

Figure 10: Prices Across Firm 2 Capacity Levels



Notes: In this figure, we fix $K_1 = 5$, $t = .05$ and consider prices as K_2 varies. We assume $\alpha_t \in [0,1]$ is decreasing in time, and $T = 1$ with a step size of $\Delta = .01$ and a per-period arrival rate of $\Delta\lambda = .1$. Panel (a) shows Firm 1's prices for different values of δ_2, while panel (b) shows Firm 2's prices for the same δ_2 values. We assume a logit demand system such that the probability of purchase of product j, conditional on arrival, is equal to $s_f = \frac{\exp(20-20\cdot p_f)}{1+\sum_{f'=1,2}\exp(20-20\cdot p'_f)}$.

Figure 11: Revenues According to Order of Sale



Notes: This plot illustrates the average revenues from 100,000 simulations grouped by the number of sales by the firm with the lowest capacity (x-axis) when a total of 10 sales are made. The vertical axis depicts firm-specific and total revenues. When $x = 10$, all sales are made by the firm with the lowest initial capacity. When $x = 0$, all sales are made by the firm with the highest initial capacity. For the simulations, we set initial capacity to $\mathbf{K} = (10, 20)$. We define the time horizon as $t \in [0,1]$, such that $\Delta = .01$, with a constant per period arrival rate of $\Delta\lambda = 0.58$. Finally, the simulations assume the demand system described in Definition 2 and assign $\delta_1 = 0.28$, $\delta_2 = 0.18$, $\sigma = 0.65$, and α_t decreases at a constant rate over time, starting at 1 in period $t = 0$ and ending at 0.8022 in period $t = T$.