Scoring and Cartel Discipline in Procurement Auctions*

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Abstract

We study theoretically and empirically how scoring auctions affect bidding behavior in the presence of tacit bidder collusion. Because assessments of quality may be subjective, scoring rules can give rise to imperfect monitoring: a high quality score may reflect a bids' intrinsically high quality, or may reflect a subjectively high evaluation. As a result, changing the auction format from first-price to scoring can lead to lower winning bids. The predictions of our model are borne out in procurement data from Japan, where a change in the procurement mechanism from first-price auctions to scoring auctions lead to a significant drop in winning bids. We show that the change in bidding behavior was likely driven by imperfect monitoring.

Keywords: procurement, scoring, cartel discipline, imperfect monitoring.

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1 Introduction

This paper studies theoretically and empirically how scoring auctions affect bidding behavior in the presence of tacit bidder collusion. Scoring auctions are commonly used by procurement agencies to allocate contracts when quality considerations are important. Under a scoring auction, the auctioneer assigns to each bid a quality score, and the allocation is determined by a combination of price and quality. Because assessments of quality may be subjective, scoring rules can introduce imperfect monitoring: a high quality score may reflect a bids' intrinsically high quality, or may reflect a subjectively high evaluation. Since imperfect monitoring is one of the key frictions to cooperation emphasized by the repeated games literature (Green and Porter, 1984, Abreu et al., 1986, 1990), sustaining tacit collusion can be harder under scoring auctions than under price-only auctions.

Our analysis is motivated by a change in the procurement mechanism used by Japan's Ministry of Land, Infrastructure and Transportation (MLIT). Prior to 2005, Japan's MLIT used price-only auctions to allocate contracts. Starting in 2005, the MLIT gradually shifted towards scoring auctions. As we document in the Section 2, winning bids decreased markedly following the introduction of scoring auctions. We argue that the observed changes in bidding behavior are difficult to reconcile with competitive models, but instead can be rationalized by a model of collusive bidding.

We analyze the effect that scoring rules have on auction outcomes through a repeated game model. A group of firms repeatedly participates in procurement auctions. Firms' types are drawn i.i.d. across periods, and are publicly observed among firms. At the end of each period, after the outcome of the auction is realized, firms are able to make costly transfers. The auction format can be either first-price or scoring. Under both auction formats, each firm submits a cash bid $b_i \in [0, r]$ and an intended quality $q_i \geq \underline{q} > 0$, where r is the reserve price and $\underline{q}$ is the minimum quality requirement. Under a first-price auction, the firm with the lowest cash bid wins the contract, and receives a payment equal to its bid. Under scoring auction, each firm receives a random recorded quality $\widehat{q}_i$, with the distribution of recorded quality $\widehat{q}_i$ increasing (in terms of FOSD) in intended quality q_i. The firm with highest score $s_i = \widehat{q}_i/b_i$ wins the contract, and receives a payment equal to its bid.[1] At the end of each auction, the auctioneer makes public the cash bids and recorded quality scores of each firm. Importantly, firms don't observe their competitors' intended qualities, so monitoring is imperfect under scoring auctions.

In previous work, Kandori (1992) shows that worsening the monitoring structure by garbling public signals reduces the value of cooperation in repeated interactions. This comparative static result requires keeping stage-game payoffs unchanged. In contrast, changing the auction format from first-price to scoring changes both the monitoring structure and the stage game. Hence, it does not immediately follow that scoring always reduces a cartel's ability to collude. Our theory highlights the conditions under which sustaining cooperation is harder under scoring auctions than under first-price auctions.

Our first set of results highlight that the observed bidding patterns in the MLIT auctions are difficult to reconcile with a competitive model. We show that, when firms compete and there is no noise in the evaluation of quality (i.e., $\widehat{q}_i = q_i$), scoring auctions lead to higher winning bids relative to first-price auctions. Intuitively, firms compete by providing higher quality under scoring auctions, leading to higher procurement costs and higher bids. Moreover, even if we account for noise in the evaluation of quality, the bidding patterns in MLIT auctions are hard to rationalize with a competitive model. Indeed, winning bids in these auctions tend to be isolated. As our prior work shows (Chassang et al., 2022), isolated winning bids are inconsistent with competitive bidding.

Next, we study how changes in the auction format affect bidding behavior under collusion, and draw inferences about which incentive constraints were likely binding to firms participating in MLIT auctions. For this, we leverage the fact that the introduction of scor-

[1] We use the scoring rule $s = \widehat{q}/b$, since this was the rule used by Japan's MLIT. However, our analysis and results don't depend on this particular scoring rule.

ing auctions was staggered. This allows us to compare first-price auctions held in the pre period (i.e., prior to 2005) with first-price auctions held in the post period; and to compare first-price auctions held in the post period with scoring auctions held in the post period.

We draw inferences in two steps. First, we argue that incentive constraints with respect to price deviations were binding in first-price auctions. If these constraints were not binding, bidding patterns in first-price auctions held across the two periods would be similar. In the data, however, the winning bid distribution in first-price auctions held in the pre period first-order stochastically dominates the winning bid distribution in first-price auctions held in the post period. This implies that incentive constraints for price deviations were binding, and that the introduction of scoring auctions reduced the value of cooperation for the cartel.

In the second step, we argue that scoring auctions introduced imperfect monitoring, and that incentive constraints with respect to quality deviations were also binding. We show that, if either firms' intended quality was observable (i.e. monitoring was perfect), or if incentive constraints with respect to quality deviations were not binding, then a cartel would be able to sustain higher winning bids under scoring auctions relative to first-price auctions. In data, however, the winning bid distribution in first-price auctions held in the post period first-order stochastically dominates the winning bid distribution in scoring auctions held in the post period. This implies that scoring auctions introduced imperfect monitoring, and that incentive constraints with respect to quality deviations were binding.

Our results have implications for the design of effective procurement mechanisms. Procurement agencies are sometimes hesitant to use scoring auctions to allocate contracts, fearing that this may open the door to corruption between bidders and the auctioneer. Our results highlight that the underlying environment is important at the time of assessing the pros and cons of introducing scoring rules. In particular, our results imply that scoring auctions may be an appropriate choice when collusion among bidders, rather than corruption between bidders and auctioneer, is the main concern. In addition, the way in which scoring auctions are implemented may also be important. In the case of Japan's MLIT, the task of assigning quality scores to bids was performed by a committee of engineering university professors; and the identity of the people in charge of assigning scores was unknown to bidders. This likely limited the scope for corruption in these auctions.

Related literature. To be added.

2 Motivating Facts

Our dataset consists of auctions for construction projects held by the Ministry of Land, Infrastructure and Transportation (MLIT) in Japan. Prior to 2005, the MLIT allocated contracts using first-price auctions, with a secret reserve price. Starting in 2005, the MLIT gradually shifted to scoring auctions.

In Chassang et al. (2022), we studied the set of auctions held by the MLIT prior to the introduction of scoring. One key finding of that previous work is that winning bids were isolated in these auctions, and that this is a marker of non-competitive bidding. To see this, for each each bidder i participating in auction a, define $\Delta_{i,a} \equiv \frac{b_{i,a} - \min_{j \neq i} b_{j,a}}{r_a}$, where $b_{i,a}$ is i's cash bid, $\min_{j \neq i} b_{j,a}$ is the lowest cash bid among i's opponents, and r_a is the auction's reserve price. Hence, $\Delta_{i,a}$ measures the difference between bidder i's own bid and the most competitive bid among i's rivals. When $\Delta_{i,a} < 0$, bidder i won the auction; when $\Delta_{i,a} > 0$, bidder i lost the auction. Figure 1 plots the distribution of Δ for first-price auctions held by the MLIT in the nine regions of Japan. Across all regions, there is a noticeable missing mass of Δ around 0. As we argue in Chassang et al. (2022), these bidding patterns are inconsistent with competitive bidding: when winning bids are isolated, it is a profitable deviation for bidders to increase their bids.

Figure 2 illustrates the change in the procurement mechanism used by the MLIT. For auctions held in each of Japan's nine regions, the figure plots the evolution over time of the fraction of auctions that had a scoring format (in blue, right axis), along with the winning

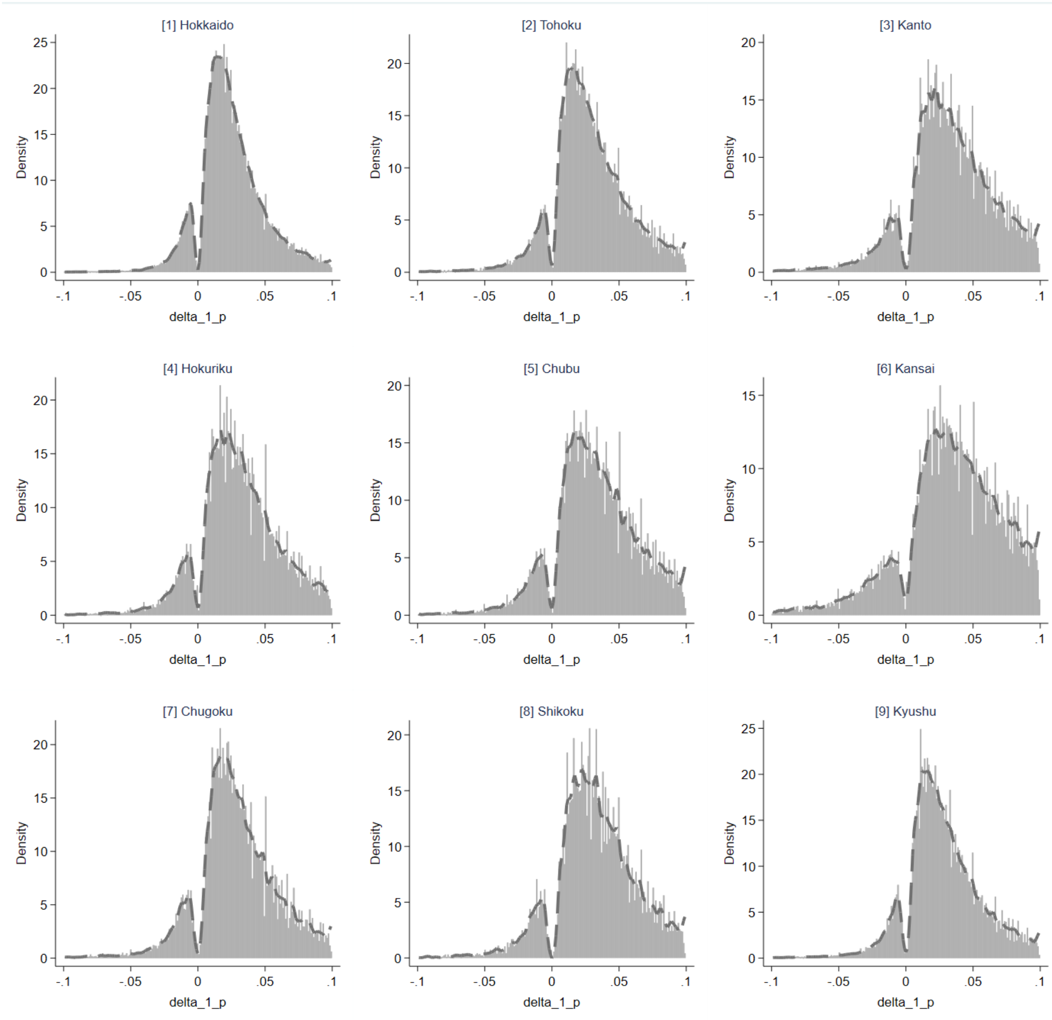

Figure 1: Distribution of bid differences Δ over (bidder, auction) pairs.

bid as a fraction of the reserve price (in red, left axis). As the figure shows, the introduction of scoring auctions coincided with a sizable drop in winning bids.

One confounding factor in Figure 2 is that, together with the introduction of scoring, the MLIT introduced other changes in their procurement auctions. In particular, prior to 2005 most auctions were by invitation: only invited bidders could participate. Starting in 2005, the MLIT gradually shifted to an "open auction" format, under which any firm was allowed to participate. The green line (right axis) in Figure 2 tracks the share of open auctions in

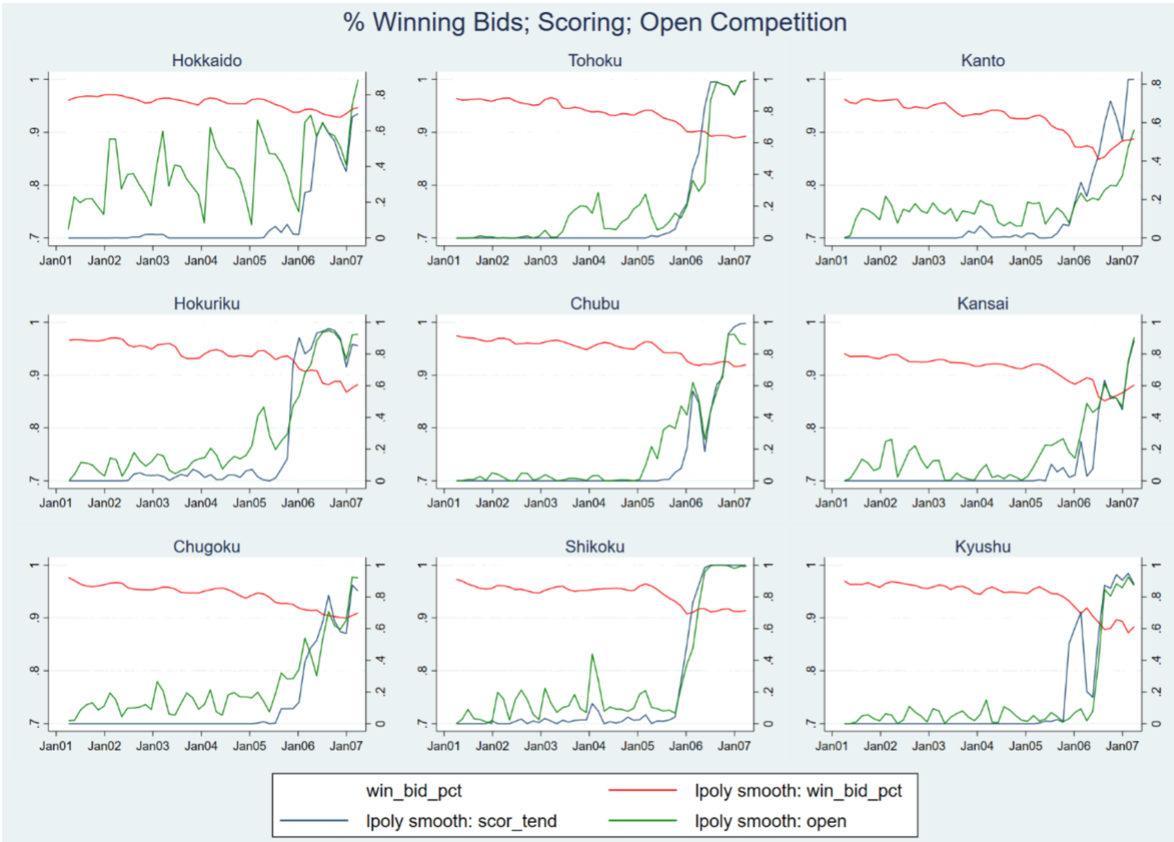

Figure 2: Evolution of winning bids as a fraction of reserve price (red, left axis), fraction of scoring auctions (blue, right axis) and fraction of open auctions (green, right axis).

the data.

To deal with this confounding factor, Figure 3 focuses on auctions with a reserve price above 200 million yens. Because regulation typically requires large projects to be auctioned off through an open auction, by focusing on this subset of auctions we can control for other changes in procurement practices. Indeed, as the green line (right axis) in Figure 3 shows, the vast majority of auctions with a reserve price above 200 million yens was open prior to 2005. Moreover, the introduction of scoring also coincided with a significant decline in the winning bids for these larger auctions.

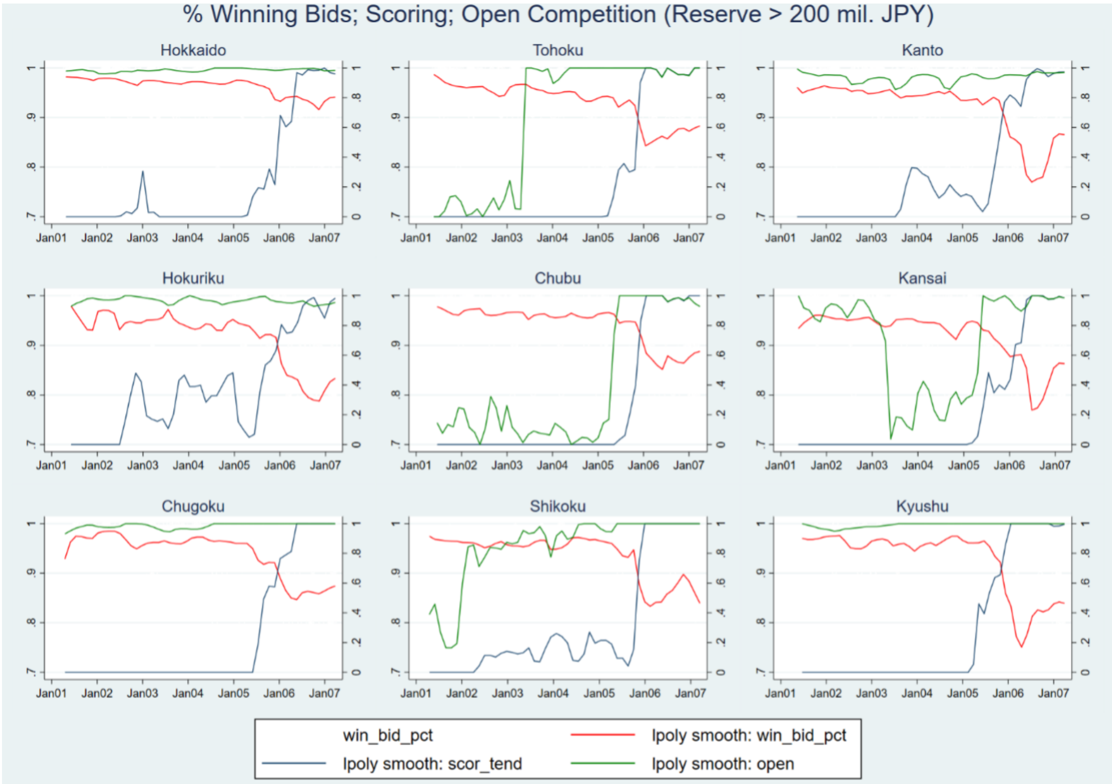

Figure 3: Evolution of winning bids as a fraction of reserve price (red, left axis), fraction of scoring auctions (blue, right axis) and fraction of open auctions (green, right axis), for auctions with a reserve price larger that 200 million yens.

3 Framework

At each period $t \in \mathbb{N}$, a buyer procures a single item through an auction from a group of firms $I = \{1, ..., n\}$. Each firm $i \in I$ can produce a good of quality $q_i \in Q \subset [\underline{q}, \overline{q}]$ (with $\overline{q} > \underline{q} > 0$) at a cost $c(q_i, \theta_i) \geq 0$ that depends on each firm i's type $\theta_i \in [0, 1]$ and is strictly increasing in both q and θ and continuous in q. Cost function $c(q, \theta)$ is such that, for all $\theta' > \theta, q' > q$, $c(q', \theta) - c(q, \theta) > c(q', \theta') - c(q, \theta')$. The set of qualities Q is finite, and satisfies $\underline{q}, \overline{q} \in Q$.

We assume that in each period t, the profile of firms' types $\theta_t \equiv (\theta_{i,t})_{i \in N}$ is drawn independently from past outcomes from a symmetric joint c.d.f. F with support $\mathsf{supp}\, F \subset [0, 1]^N$.

Auction formats. The procurement contract is allocated using one of two auction formats: a first price auction (FPA) or a scoring auction (SA).

Under both auction formats, each firm i submits a cash bid $b_i \in [0, r]$ (where r is the reserve price, normalized to $r = 1$) to be paid to i if they win, as well as a proposed quality $q_i \in [\underline{q}, \overline{q}]$, which is noisily evaluated as $\widehat{q}_i \in Q$ by the auctioneer. We assume that, for each $i \in I$, the distribution of $\widehat{q}_i$ depends only on q_i. We further assume that, conditional on $q = (q_i)_{i\in I}$, recorded qualities are independently distributed across bidders, and that the associated distribution $\gamma(\widehat{q}_i|q_i) \in [0, 1]$ is increasing in q_i in the sense of first-order stochastic dominance. Lastly, we assume that for all $q_i \in Q$, $\gamma(\cdot|q_i)$ has full support over Q. For any $\widehat{q} = (\widehat{q}_i)_{i\in I}$ and any $q = (q_i)_{i\in I}$, we use $\mu(\widehat{q}|q) = \prod_i \gamma(\widehat{q}_i|q_i)$ to denote the probability of observing recorded qualities $\widehat{q}$ when firms' proposed qualities are q.

The two auction formats differ only in their allocation rule. Let us denote by $x_i \in \{0, 1\}$ whether bidder i wins the procurement contract or not. Under FPA, the bidder with the lowest cash bid wins the auction:

$$x_i \equiv \begin{cases} 1 & \text{if} \quad b_i < \wedge b_{-i} \\ 0 & \text{if} \quad b_i > \wedge b_{-i} \end{cases} .$$

where $\wedge b_{-i} \equiv \min\{b_j, j \in I \setminus i\}$ denotes the most competitive bid from bidders other than i.

Under SA, the bidder with the highest evaluated score s_i defined as $\frac{\widehat{q}_i}{b_i}$ wins the auction:

$$x_i \equiv \begin{cases} 1 & \text{if} \quad s_i > \vee s_{-i} \\ 0 & \text{if} \quad s_i < \vee s_{-i} \end{cases} .$$

where $\vee s_{-i} \equiv \max\{s_j, j \in I \setminus i\}$ denotes the most competitive score associated with bidders other than i.

In both cases ties are broken using a uniform distribution over tied bidders.

<# comment on the fact that constraint on q is the same across FPA, and SA. #>

Information. In any period t, after bidding and allocation take place, both bids $b_t \equiv (b_{i,t})_{i\in I}$ and *evaluated* qualities $\widehat{q}_t \equiv (\widehat{q}_{i,t})$ are publicly observed. Actual quality submissions $q_t \equiv (q_{i,t})_{i\in I}$ are not observed.

We assume that the type profile $\theta_t = (\theta_{i,t})_{i\in I}$ is publicly observable at the beginning of period t, before bidding occurs: there is complete information about costs. Consistent with Chassang and Ortner (2019), assuming complete information about costs is sensible since bidders are local firms well informed about one another's businesses. Uncertainty over future realizations of type profile θ reflects variation in the firms' circumstances, as well as uncertainty over auction characteristics and fit with individual firms' capabilities.

Payoffs and costly transfers between firms. Firm i's payoff from the auction allocation is

$$u_{i,t} = x_{i,t}(b_{i,t} - c(q_{i,t}, \theta_{i,t}))$$

In addition, firms can make costly monetary transfers between one another at the end of each period, after the outcome of the auction is realized. Let $T_{i,t}$ denote the aggregate net transfer received by firm i in period t. Transfers must be budget-balanced: for all $t \in \mathbb{N}$, $\sum_{i\in I} T_{i,t} = 0$. Finally sending transfers is costly, with a loss rate $\lambda \geq 0$, so that, including transfers, firm i's overall payoff in period t is

$$u^T_{i,t} \equiv x_{i,t}(b_{i,t} - c(q_{i,t}, \theta_{i,t})) + T_{i,t}(1 + \lambda \mathbf{1}_{T_{i,t}<0}).$$

Firms discount future payoffs with common discount factor $\delta \in [0, 1)$.

Policy experiments. Let us denote by $f_t \in \{\text{FPA}, \text{SA}\}$ the auction format selected in period t. We idealize the variation in auction format in the data and assume that

- In all periods t before the policy change (pre periods), the auction format f_t was $f_t = \text{FPA}$. Firms believe that a regime switch corresponding to the introduction of scoring auctions takes place with fixed probability $\alpha \in (0,1)$.

- In all periods t after the regime switch (post periods), the auction format f_t is selected from $\{\text{FPA}, \text{SA}\}$ with a full support distribution that depends only the vector $\theta = (\theta_i)_{i\in I}$ summarizing bidder types given auction characteristics.

This allows us to perform the following comparisons: (i) comparisons between FPA auction in the pre vs. post periods; (ii) comparisons between FPA vs. SA auctions in the pre vs. post periods; and (iii) comparisons between FPA vs. SA auctions in the post periods. The latter comparisons turn out to be especially helpful for inference since they keep fixed the set of possible continuation values that can be used to enforce bidding behavior within each auction format.

Strategies and solution concepts. Let us denote by $h_{t^-} \equiv (\theta_s, f_s, b_s, \widehat{q}_s, T_s, \theta_t, f_t)_{s<t}$ and $h_{t^+} \equiv (\theta_s, f_s, b_s, \widehat{q}_s, T_s, \theta_t, f_t, b_t, \widehat{q}_t)_{s<t}$ the public history of play in period t before and after auction outcomes are realized.

A strategy σ_i for firm i is a mapping from public histories to bidding and transfer actions:

$$\begin{aligned} \sigma_i : \quad & h_{t^-} \mapsto (b_{i,t}, q_{i,t}) \\ & h_{t^+} \mapsto T_{i,t} \end{aligned}$$

Throughout, we say that a strategy profile $\sigma = (\sigma_i)_{i\in I}$ is competitive if firms play a Nash equilibrium in weakly undominated strategies of the stage game in every period. In particular no transfers are made under a competitive strategy profile.

In contrast, we say that firms behave like a unitary cartel if at every history h_{t^-}, firms follow a strategy profile $\sigma(h_{t^-})$ solving

$$\max \mathbb{E}_{F,\mu}\left[\sum_{i\in I} u_i \middle| h_{t^-}\right].$$

Note that no enforcement constraints are imposed on a unitary cartel. We model *constrained-optimal* cartel behavior in Section 5 using on-path stationary perfect public equilibria as our solution concept.

4 Cartel Discipline Is Binding

In this section we argue that the bidding data described in Section 2 cannot be explained as the outcome of either competitive equilibrium, or optimal bidding behavior of a unitary cartel, i.e. a cartel that is not bound by enforcement constraints and implements bids maximizing cartel surplus. The implication is that patterns described in Section 2 reflect bidding behavior by a cartel for whom enforcement constraints are binding.

4.1 Preliminary: Precise Quality Evaluations

To help build intuition around the facts presented in Section 2, we show that scoring increases competitive prices when quality is precisely evaluated. Within this subsection, we assume that that quality is precisely evaluated: $\mu(\cdot|q)$ is a unit mass at q.

Proposition 1. *Under both FPA and SA auctions, the stage game admits a unique Nash equilibrium in weakly undominated strategies. The contract is allocated to the most efficient bidder, i.e. the bidder with minimum cost type* $\min\{\theta_i, i \in I\}$.

For $f \in \{FPA, SA\}$*, let* $b_f^{\mathsf{comp}}(\theta)$ *denote the stage game Nash winning bid at type profile* θ *under auction format* f*. We have that*

$$\forall\theta, b_{SA}^{\mathsf{comp}}(\theta) \geq b_{FPA}^{\mathsf{comp}}(\theta).$$

In words, fixing the bidders' types, winning bids under scoring auctions are weakly larger than winning bids under first-price auctions. The intuition behind this result is straightforward: scoring auctions induce bidders to provide higher quality, increasing procurement costs and winning bids. An immediate Corollary of Proposition 1 is that, under competition, the policy change in Japan should have lead to higher winning bids. The data in Section 2 is at odds with this prediction.

Corollary 1. *Under the assumption that quality is precisely evaluated and that firms are competitive:*

(i) the distribution of winning bids in the post *period first-order stochastically dominates (FOSD) the distribution of winning bids in the* pre *period;*

(ii) in the post *period, the distribution of winning bids from SA first-order stochastically dominates the distribution of winning bids from FPA.*

4.2 Imprecise Quality Evaluations

We now allow for noisy quality evaluations. In contrast to the case of precise evaluation, it is no longer the case that competitive behavior will lead to efficient allocation under SA: noise in quality evaluation may lead the contract to be allocated away from the lowest cost firm. As a result, firms, whether competitive or cartelized, may adjust their bidding to improve the odds of correct allocation, potentially reducing prices under SA. We show that price reductions motivated by a pure allocation motive are inconsistent with bidding patterns described in Section 2.

Given a bidding profile $b = (b_i)_{i \in I}$, $q = (q_{i \in I})$, let

$$D_i(b, q) = \text{prob}_\mu(s_i > \vee s_{-i} | b, q)$$

denote firm i's likelihood of winning the contract. Flow profits $u_i(b, q, \theta)$ (excluding transfers) are

$$u_i(b, q, \theta) = D_i(b, q)(b_i - c(q_i, \theta_i)).$$

Lemma 1 (bounds on profits)**.** *Consider a bidding profile* b, q *such that for all* i*,* $b_i \geq c(q_i, \theta_i)$*, and an alternative profile* b', q*. For any bidder* i *the change in flow profits satisfies*

$$u_i(b', q, \theta) - u_i(b, q, \theta) \geq D_i(b', q)b'_i - D_i(b, q)b_i - [D_i(b', q) - D_i(b, q)]^+ b_i. \tag{1}$$

$$\geq \mathbb{E}_\mu \left[\mathbf{1}_{i \ wins \ |b',q} b'_i - \mathbf{1}_{i \ wins \ |b,q} b_i - \left[\mathbf{1}_{i \ wins \ |b',q} - \mathbf{1}_{i \ wins \ |b,q} \right]^+ b_i \right]. \tag{2}$$

In words, the change in flow profits is bounded below by the change in revenue, minus potential cost increases related to increasing demand, where costs are bounded above by initial bids. Condition (2) provides a version of this bound for which an unbiased estimate can be computed using observable data.

Corollary 2. *Under competitive bidding* b, q*, it must be that for any individual deviation* $b'_i > b_i$*,*

$$D_i(b'_i, b_{-i}, q)b'_i - D_i(b, q)b_i \leq 0. \tag{3}$$

Under unitary cartel bidding b, q*, it must be that for any profile of increased bids* $b' \geq b$*,*

$$\sum_{i \in I} D_i(b', q)b'_i - D_i(b, q)b_i - [D_i(b', q) - D_i(b, q)]^+ b_i \leq 0. \tag{4}$$

Importantly, condition (2) implies that conditions (3) and (4) admit testable counterparts, where expectations are replaced with sample averages.

Bidding patterns described in Section 2 are neither explained by competitive behavior or behavior of a unitary cartel. The implication is our bidding data must be generated by a cartel limited by enforcement constraints: cartel discipline is binding.

We now seek to disentangle how the introduction of scoring affects enforcement constraints. We first draw qualitative inferences about which incentive compatibility constraints must be binding. We then calibrate a model and explore the impact of scoring under plausible parameter specifications.

5 Modeling Cartel Discipline

5.1 Characterization

Solution concept. We focus on strategy profiles σ mapping public histories to price and quality bids (b, q) and transfers T such that

(i) Profile σ is a perfect public equilibrium (PPE, Fudenberg et al., 1994);

(ii) Behavior is stationary and symmetric on path: for any on-path histories h_t^- and h_t^+, bidding behavior $\sigma(h_{t^-})$ and transfer behavior $\sigma(h_{t^+})$ are independent of previous observables $(\theta_s, b_s, \widehat{q}_s, T_s)_{s<t}$ and invariant to a relabeling of players;

(iii) Off-path punishment is achieved through reversion to stage game Nash.[2]

We refer to such equilibria as Stationary PPEs.

Let us denote by $\underline{V}$ the discounted cartel profits $\frac{1}{1-\delta}\mathbb{E}\left[\sum_{i\in i} u_i\right]$ under stage game Nash and by $\overline{V}$ the highest discounted cartel profits $\frac{1}{1-\delta}\mathbb{E}\left[\sum_{i\in i} u_i^T\right]$ sustainable under a Stationary PPE. Note that gross payoffs u_i can be used to evaluate surplus under stage game Nash since no transfers are used. Payoffs net of transfer costs must be used to evaluate more general Stationary PPEs since transfers may happen on path.

Let $[x]^- = \max\{-x, 0\}$. Recall that, for any $\widehat{q} = (\widehat{q}_i)_{i\in I}$ and any $q = (q_i)_{i\in I}$, $\mu(\widehat{q}|q) = \prod_i \gamma(\widehat{q}_i|q_i)$ is the probability of observing recorded qualities $\widehat{q}$ when firms' proposed qualities are q.

Lemma 2 (cartel optimal behavior)**.** *At any profile of types θ, cartel optimal bidding solves*

$$\max_{b,q} \sum_{i\in I} u_i(b, q, \theta) - \lambda K(b, q, \theta) \tag{P}$$

[2] On path behavior may be preserved following inoffensive deviations that do not affect bidding outcomes or incentive constraints.

with

$$K(b,q,\theta) \equiv \min_{T:Q^I \mapsto [\underline{T},+\infty)^I} \mathbb{E}_\mu \left[\sum_{i\in I} [T_i(\widehat{q})]^- \right] \; s.t. \tag{K-IC}$$

$$\forall i \in I, \forall b_i' \neq b_i, q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta) - u_i(b,q,\theta) \leq \frac{\delta}{n}(\overline{V} - \underline{V}) + \sum_{\widehat{q}\in Q^I} \mu(\widehat{q}|q) T_i(\widehat{q})(1 + \lambda \mathbf{1}_{T_i(\widehat{q})<0}) \tag{IC-p}$$

$$\forall i \in I, \forall q_i' \neq q_i, \quad u_i(b, q_i', q_{-i}, \theta) - u_i(b,q,\theta) \leq \sum_{\widehat{q}\in Q^I} T_i(\widehat{q})(1 + \lambda \mathbf{1}_{T_i(\widehat{q})<0})(\mu(\widehat{q}|q) - \mu(\widehat{q}|q_i', q_{-i})) \tag{IC-q}$$

$$\forall \widehat{q} \in Q^I, \quad \sum_{i\in I} T_i(\widehat{q}) = 0. \tag{BB}$$

where $\underline{T} = -\frac{1}{1+\lambda}\frac{\delta}{n}(\overline{V} - \underline{V})$ *and, by convention,* $K(b,q,\theta) = +\infty$ *if* b, q *is such that there are no transfers* $T : Q^I \mapsto [\underline{T}, +\infty)^I$ *satisfying* (IC-p), (IC-q) *and* (BB).

Lemma 2 clarifies the mechanics of enforcement. Because bidding deviations are observable, they happen off-the-equilibrium-path and can be at least partly deterred using-off-the-equilibrium-path punishment. In contrast, deviations in quality are not directly observable since evaluated quality has full support. Hence, quality deviations must be deterred using costly on path transfers, resulting in direct enforcement cost $K(b,q,\theta)$. Note that program (K-IC) is computationally tractable: it minimizes a convex piece-wise linear function over convex piece-wise linear constraints.

For any on-path continuation value $V \geq \underline{V}$, let $\beta_V(\theta)$ denote the solution to program (P) when $\overline{V}$ is replaced by V. Define the mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\Phi(V) = \mathbb{E}_F \left[\sum_{i\in I} u_i(\beta_V(\theta), \theta) - K(\beta_V(\theta), \theta) \right] + \delta V.$$

Then, $\overline{V}$ is the largest fixed-point of Φ.

5.2 How scoring affects IC constraints

Misallocation and pledgeable surplus. As we highlighted in Section 4, even when IC constraints (IC-p) and (IC-q) are slack, scoring can result in misallocation of the project to higher cost firms that reduce flow payoffs. This mechanically reduces cartel surplus $\overline{V}$.

Noisy evaluation also affect stage game Nash payoffs. When firms' costs are close, so that profits under the FPA are close to 0, noisy evaluations increase cartel profits under stage game Nash.

In this particular configuration (i.e., slack constraints, and low FPA profits) scoring reduces the pledgeable surplus $\overline{V} - \underline{V}$.

Price deviation temptation. When IC constraints are binding, scoring affect the left hand side of (IC-p): undercutting another bidder's bid becomes less attractive since the change in allocation now has probability less than 1.

Quality deviation temptation. Under FPA there is no deviation temptation associated with quality deviations: evaluated quality does not affect the allocation. Under SA, quality choices can change the allocation creating a deviation temptation with respect to quality.

Imperfect monitoring. Quality deviations are different from price deviations because quality choices are unobserved and evaluated quality has full support. As a result, quality deviations are deterred by costly transfers T on path. These reduce the surplus available to the cartel.

In the next section we sign the impact of these different mechanisms on enforceable surplus.

Inference. We assess the impact of different mechanisms by comparing two different sets of auctions:

- Comparing pre and post FPAs allows us to identify the impact of scoring through its effect on pledgeable surplus $\overline{V} - \underline{V}$ alone.

- Comparing post FPAs and post SAs allows us to identify the impact of quality deviations and imperfect monitoring, keeping fixed pledgeable surplus.

5.3 An Example

We now illustrate the impact of scoring on cartel discipline with a simple example. The example highlights the role of imperfect monitoring in limiting a cartel's ability to sustain high prices.

Suppose there are two firms that share the same procurement costs: i.e., $\theta_1 = \theta_2 = \theta$. Firms can produce two possible qualities, $Q = \{\underline{q}, \overline{q}\}$. The set of possible recorded qualities is also $\{\underline{q}, \overline{q}\}$. Assume that

$$\text{prob}(\widehat{q}_i = \overline{q} | q_i = \overline{q}) = \text{prob}(\widehat{q}_i = \underline{q} | q_i = \underline{q}) = \alpha \in (1/2, 1).$$

Parameter $\alpha \in (1/2, 1)$ measures how noisy monitoring is under scoring. As α approaches 1 (resp., approaches 1/2), recorded quality becomes a perfect signal (resp., a fully uninformative signal) of a firm's intended quality.

Consider first auction format FPA. Let $b_{\text{FPA}}(\theta)$ denote the largest winning bid that firms can sustain under FPA. Then, for $i = 1, 2$, we must have

$$(1 - x_i)(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta)) \leq \frac{\delta}{2}(\overline{V} - \underline{V}) + T_i(1 + \lambda \mathbf{1}_{T_i < 0}),$$

where $x_i \in [0, 1]$ is the probability with which i wins the auction. Indeed, if the above inequality does not hold, firm i has an incentive to undercut the winning bid. Summing this constraint across $i = 1, 2$, and using $x_1 + x_2 \le 1$ and $T_1 + T_2 = 0$ we get that

$$b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta) \le \delta(\overline{V} - \underline{V}). \tag{5}$$

Hence, $b_{\mathrm{FPA}}(\theta) \le \min\{r, c(\underline{q}, \theta) + \delta(\overline{V} - \underline{V})\}$. Moreover, a winning bid of $\min\{r, c(\underline{q}, \theta) + \delta(\overline{V} - \underline{V})\}$ is enforceable under FPA, by a bidding scheme under which both firms submit bid $\min\{r, c(\underline{q}, \theta) + \delta(\overline{V} - \underline{V})\}$ and quality $\underline{q}$, and with no transfers. Therefore, $b_{\mathrm{FPA}}(\theta) = \min\{r, c(\underline{q}, \theta) + \delta(\overline{V} - \underline{V})\}$.

Consider next bidding under SA. Let us focus on bidding schemes under which both firms submit the same bid $b_{\mathrm{SA}}(\theta)$ and quality $\underline{q}$. For this bidding profile to be enforceable under SA, there must exist budget-balanced transfers $(T_1(\widehat{q}), T_2(\widehat{q}))_{\widehat{q} \in Q^2}$, with $T_i(\widehat{q}) \ge \underline{T}$ for all $\widehat{q}$, such that (IC-q) holds.

The optimal transfer scheme to sustain the proposed bidding profile takes the following intuitive form. When the recorded qualities of both firms are the same, there are no transfers: i.e., $T_i(\widehat{q}, \widehat{q}) = 0$ for all $\widehat{q}$. When firm i's recorded quality is $\underline{q}$ and firm $-i$'s recorded quality is $\overline{q}$, firm i obtains a transfer $T \ge 0$, which is paid by firm $-i$.

Given this bidding and transfer profile, (IC-q) becomes

$$\alpha(b_{\mathrm{SA}}(\theta) - c(\overline{q}, \theta)) - \frac{1}{2}(b_{\mathrm{SA}}(\theta) - c(\underline{q}, \theta)) \le (2\alpha - 1)((1 - \alpha)T - \alpha T(1 + \lambda \mathbf{1}_{T<0}))$$

The left-hand side of the inequality above is the payoff gain that firm i obtains from submitting quality $\overline{q}$ instead of quality $\underline{q}$: by submitting quality $\overline{q}$, firm i wins with probability $\alpha > 1/2$ instead of $1/2$. The right-hand side corresponds to the change in firm i's expected transfers following the deviation. Indeed, $-(2\alpha - 1)(1 - \alpha)$ is the change in the probability that recorded qualities are $(\widehat{q}_i, \widehat{q}_{-i}) = (\underline{q}, \overline{q})$ when i deviates to $q_i = \overline{q}$, and $(2\alpha - 1)\alpha$ is the change in the probability that recorded qualities are $(\widehat{q}_i, \widehat{q}_{-i}) = (\overline{q}, \underline{q})$ following this same deviation.

Using $-T \geq \underline{T} = -\frac{1}{1+\lambda}\frac{\delta}{2}(\overline{V} - \underline{V})$ in the inequality above we get

$$b_{\mathrm{SA}}(\theta) - c(\underline{q}, \theta) \leq \frac{1+\lambda\alpha}{1+\lambda}\delta(\overline{V} - \underline{V}) + \frac{2\alpha}{2\alpha - 1}(c(\overline{q}, \theta) - c(\underline{q}, \theta)). \tag{6}$$

Comparing (6) with (5), the winning bid under SA is lower than under FPA if

$$\frac{1+\lambda\alpha}{1+\lambda}\delta(\overline{V} - \underline{V}) + \frac{2\alpha}{2\alpha - 1}(c(\overline{q}, \theta) - c(\underline{q}, \theta)) < \delta(\overline{V} - \underline{V})$$
$$\iff \frac{2\alpha}{2\alpha - 1}(c(\overline{q}, \tau) - c(\underline{q}, \tau)) < \frac{\lambda(1-\alpha)}{1+\lambda}\delta(\overline{V} - \underline{V})$$

We note that the above inequality is more likely to hold when λ is large (i.e., transfers are costly) and α is neither close to 1/2 nor close to 1 (i.e., monitoring is imperfect but not too much so).

Lastly, when the inequality above holds, and

$$\delta(\overline{V} - \underline{V}) \geq r - c(\underline{q}, \theta)) > \frac{1+\lambda\alpha}{1+\lambda}\delta(\overline{V} - \underline{V}) + \frac{2\alpha}{2\alpha - 1}(c(\overline{q}, \theta) - c(\underline{q}, \theta)),$$

the cartel's payoffs are strictly lower under SA than under FPA. Indeed, when these inequalities hold, the cartel cannot sustain winning bid of r and quality $\underline{q}$ under SA, but can do so under FPA.

6 How Scoring Affects Cartel Discipline

We now study how the introduction of scoring affects a cartel's ability to sustain collusion. We start by noting that if either incentive constraints for price deviations (IC-p) did not bind in the pre and post periods, or if values $\overline{V}$ and $\underline{V}$ remained unchanged across the two periods, then optimal collusive bidding under FPA would be the same in the pre and post periods. Hence, under either of these conditions, the winning bid distribution of FPA held in the pre and post periods should be identical. Our next result summarizes this.

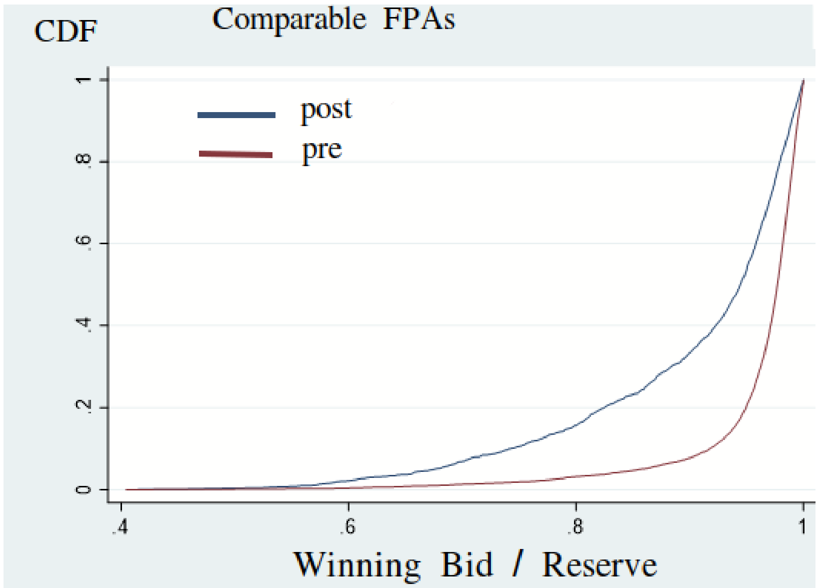

Figure 4: Distribution of winning bids in first-price auctions, pre and post periods.

Proposition 2. *If* (IC-p) *is not binding for FPA auctions, or if*

$$\overline{V}_{pre} - \underline{V}_{pre} = \overline{V}_{post} - \underline{V}_{post}$$

then winning bids in the pre *and* post *FPA periods have the same distribution.*

Figure 4 plots the distribution of normalized winning bids (i.e., winning bids divided by the reserve price) of comparable FPA's held in the two periods. The c.d.f. of winning bids in the pre period first-order stochastically dominates the c.d.f. of winning bids in the post period. This implies that incentive constraints for price deviations were binding, and that the introduction of scoring reduced the cartel's ability to enforce collusive bids (i.e., $\overline{V}_{\text{pre}} - \underline{V}_{\text{pre}} > V_{\text{post}} - \underline{V}_{\text{post}}$).

Our next result highlights the conditions under which the introduction of scoring can lead to lower winning bids.

Proposition 3. *Suppose* $\delta(\overline{V} - \underline{V}) > 0$. *There exists* $\eta > 0$ *such that, if* $prob_F(\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$, *and if either of the following hold:*

(i) (IC-q) *is not binding (e.g. quality is fixed);*

(ii) quality is noisily evaluated but perfectly monitored;[3]

(iii) transfers are costless (i.e., $\lambda = 0$);

then, there exists $\underline{b} < r$ such that enforceability of a winning bid of $b \in [\underline{b}, r]$ under FPA implies enforceability of a winning bid of b under SA.

To understand Proposition 3, note that, while scoring auctions may introduce imperfect monitoring, they also change the stage game that firms play. An important observation is that price defections by cartel members are less profitable under scoring auctions than under first-price auctions. Indeed, under FPA, a bidder who undercuts the winning bid wins the auction with probability 1. In contrast, under SA, undercutting the winning bid leads to a less than certain probability of winning when quality is noisily measured.[4] Hence, for scoring to reduce equilibrium winning bids under collusion, it must be that monitoring is imperfect under scoring, and that incentive constraints for quality deviations are binding. This explains points (i) and (ii) in Proposition 3.

To understand Proposition 3(iii), note that if transfers are costless, the cartel can costlessly deter quality deviations by making firms pay a large transfer whenever they receive a high quality score $\widehat{q}$: even if these transfers end up being paid on the equilibrium path, they don't destroy aggregate surplus.

Figure 5 plots the distribution of normalized winning bids for comparable first-price and scoring auctions held in the post period. The winning bid distribution of first-price auctions first-order stochastically dominates the winning bid distribution of scoring auctions. This implies that there is imperfect monitoring, and that incentive constraints for quality deviations were binding.

[3]A stronger prediction holds if quality is perfectly evaluated.

[4]Building on this observation, Kawai et al. (2022) and Ortner et al. (2023) study how mediation can help cartels sustain higher equilibrium profits.

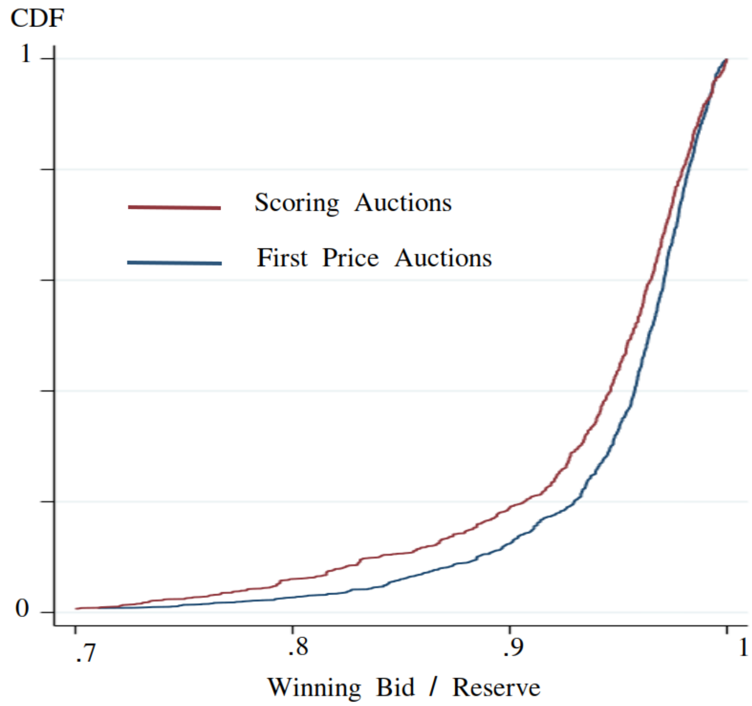

Figure 5: Distribution of winning bids in first-price and scoring auctions, post period.

Our last result highlights conditions on the cost for quality and the monitoring structure under which scoring can limit a cartel's ability to collude.

Proposition 4. *Suppose* $\delta(\overline{V} - \underline{V}) > 0$. *There exists* $\eta > 0$ *and* $\epsilon > 0$ *such that, if* $prob_F(\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| \leq \eta) = 1$*, and if*

$$\max_{q \neq q', \theta} |c(q, \theta) - c(q', \theta)| < \epsilon \text{ or } \min_{q \neq q', \theta} |c(q, \theta) - c(q', \theta)| > \frac{1}{\epsilon},$$

or if

$$\inf_{q \neq q', \widehat{q}} |\ln \gamma(\widehat{q}|q) - \ln \gamma(\widehat{q}|q')| > \frac{1}{\epsilon} \text{ or } \sup_{q \neq q', \widehat{q}} |\ln \gamma(\widehat{q}|q) - \ln \gamma(\widehat{q}|q')| < \epsilon,$$

then, there exists $\underline{b} < r$ *such that enforceability of a winning bid of* $b \in [\underline{b}, r]$ *under FPA implies enforceability of a winning bid of* b *under SA.*

Proposition 4 shows that, for scoring to limit collusion, two things must be true: (i) firms' cost of providing higher quality should be larger, but not too large; and (ii) monitoring should be imperfect, but not too much so. To see why, note that if the cost of providing higher quality is very large, or if monitoring is too noisy, then quality incentive constraints will not bind; and so by Proposition 3, scoring would lead to higher equilibrium bids. Similarly, if the cost of providing higher quality is negligible, colluding firms can almost costlessly provide higher quality, and hence quality incentive constraints will not bind. Lastly, if monitoring is almost perfect, quality deviations can be deter almost costlessly by having firms pay a large transfer whenever they get a high recorded quality.

7 Discussion

To be added.

Appendix

A Proofs

Proof of Proposition 1. Consider auction format FPA. Note that, since quality does not affect the allocation, in any Nash equilibrium in weakly undominated strategies all firms choose the lowest quality $q = \underline{q}$. In addition, the bidder with the lowest cost wins with probability 1, and the winning bid is equal to the second lowest cost. Hence, for any type vector $\theta = (\theta_i)_{i \in N}$, the winning bid under FPA is $b^{\mathsf{comp}}_{\mathrm{FPA}}(\theta) = c(\underline{q}, \theta_{(2)})$, where $\theta_{(2)}$ is the second lowest type in θ.

Consider next auction format SA. For each score $s > 0$ and each type θ_i, let $k(\theta_i, s) \equiv \max_{q \in Q} \frac{q}{s} - c(q, \theta_i)$, and let $q^*(\theta_i, s) \in \arg\max_{q \in Q} \frac{q}{s} - c(q, \theta_i)$. Note that $k(\theta_i, s)$ is the largest payoff that firm i with type θ_i can obtain by winning an auction with a score of s. Note further that $k(\theta_i, s)$ is continuous in θ_i and s, and is increasing in θ_i and decreasing in s. Moreover, since, for all $q' > q$, $c(q', \theta_i) - c(q, \theta_i)$ is strictly decreasing in θ_i, it follows that

$q^*(\theta_i', s) \geq q^*(\theta_i, s)$ for all $s > 0$ and all $\theta_i < \theta_i'$.[5]

Then, in any equilibrium in weakly undominated strategies, for any type vector $\theta = (\theta_i)_{i \in N}$, the bidder with the lowest type $\theta_{(1)}$ wins, and the winning score $s_{\text{SA}}^{\mathsf{comp}}(\theta)$ is such that the bidder with the second lowest type would make zero profits winning with that score: $s_{\text{SA}}^{\mathsf{comp}}(\theta) = \inf\{s : k(\theta_{(2)}, s) \leq 0\}$.[6] The winning bid $b_{\text{SA}}^{\mathsf{comp}}(\theta)$ and winning quality $q_{\text{SA}}^{\mathsf{comp}}(\theta)$ satisfy $b_{\text{SA}}^{\mathsf{comp}}(\theta) = \frac{q_{\text{SA}}^{\mathsf{comp}}(\theta)}{s_{\text{SA}}^{\mathsf{comp}}(\theta)}$, and the winning quality is optimal to the winning firm; i.e., $q_{\text{SA}}^{\mathsf{comp}}(\theta) = q^*(\theta_{(1)}, s_{\text{SA}}^{\mathsf{comp}}(\theta)) \in \arg\max_q \frac{q}{s_{\text{SA}}^{\mathsf{comp}}(\theta)} - c(q, \theta_{(1)})$.

Lastly, we show that, for all θ, $b_{\text{SA}}^{\mathsf{comp}}(\theta) \geq b_{\text{FPA}}^{\mathsf{comp}}(\theta)$. The winning score $s_{\text{SA}}^{\mathsf{comp}}(\theta) = q_{\text{SA}}^{\mathsf{comp}}(\theta)/b_{\text{SA}}^{\mathsf{comp}}(\theta)$ is such that $\forall q \in [\underline{q}, \overline{q}]$,

$$\frac{q}{s_{\text{SA}}^{\mathsf{comp}}(\theta)} - c(q, \theta_{(2)}) \leq 0,$$

with equality at $q = q^*(\theta_{(2)}, s_{\text{SA}}^{\mathsf{comp}}(\theta)) \in \arg\max_q \frac{q}{s_{\text{SA}}^{\mathsf{comp}}(\theta)} - c(q, \theta_{(2)})$. We then have that,

$$\begin{aligned} b_{\text{SA}}^{\mathsf{comp}}(\theta) &= \frac{q_{\text{SA}}^{\mathsf{comp}}(\theta)}{s_{\text{SA}}^{\mathsf{comp}}(\theta)} \\ &= \frac{c(q^*(\theta_{(2)}, s_{\text{SA}}^{\mathsf{comp}}(\theta)), \theta_{(2)})}{q^*(\theta_{(2)}, s_{\text{SA}}^{\mathsf{comp}}(\theta))} q_{\text{SA}}^{\mathsf{comp}}(\theta) \geq c(\underline{q}, \theta_{(2)}) = b_{\text{FPA}}^{\mathsf{comp}}(\theta), \end{aligned}$$

where the inequality follows since $q_{\text{SA}}^{\mathsf{comp}}(\theta) = q^*(\theta_{(1)}, s_{\text{SA}}^{\mathsf{comp}}(\theta)) \geq q^*(\theta_{(2)}, s_{\text{SA}}^{\mathsf{comp}}(\theta))$, and since $q^*(\theta_{(2)}, s_{\text{SA}}^{\mathsf{comp}}(\theta)) \geq \underline{q}$. ■

Proof of Lemma 1. For all bidding profiles b, q and b', q satisfying $b_i \geq c(q_i, \theta_i)$, and all

[5]Proof: Pick $\theta_i' > \theta_i$. Note that, for any $s > 0$, $\frac{q^*(\theta_i,s)}{s} - c(q^*(\theta_i, s), \theta_i) \geq \frac{q^*(\theta_i',s)}{s} - c(q^*(\theta_i', s), \theta_i)$ and $\frac{q^*(\theta_i',s)}{s} - c(q^*(\theta_i', s), \theta_i') \geq \frac{q^*(\theta_i,s)}{s} - c(q^*(\theta_i, s), \theta_i')$. These two inequalities imply

$$c(q^*(\theta_i', s), \theta_i) - c(q^*(\theta_i, s), \theta_i) \geq c(q^*(\theta_i', s), \theta_i') - c(q^*(\theta_i, s), \theta_i').$$

Since, for all $q' > q$, $c(q', \hat{\theta}_i) - c(q, \hat{\theta}_i)$ is strictly decreasing in $\hat{\theta}_i$, it follows that $q^*(\theta_i', s) \geq q^*(\theta_i, s)$.

[6]To see why, note that in any equilibrium in weakly undominated strategies, bidders with type $\theta_i \geq \theta_{(2)}$ must submit a price-quality bid (b, q) with score weakly smaller than $s_{\text{SA}}^{\mathsf{comp}}(\theta)$. Indeed, submitting a price-quality bid (b, q) with a score strictly larger than $s_{\text{SA}}^{\mathsf{comp}}(\theta)$ is dominated for bidders with type $\theta_i \geq \theta_{(2)}$, as they would lose money by winning the auction at such a score. This implies that the bidder with the highest type $\theta_{(1)}$ must win the auction with probability 1: otherwise, it would be a strictly profitable deviation for this bidder to 'overcut' the highest score among its rivals and win the auction with probability 1. Note next that the highest type's score must be weakly smaller than $s_{\text{SA}}^{\mathsf{comp}}(\theta)$, since all other bidders submit scores weakly smaller than this number. Lastly, if the winning score was strictly smaller than $s_{\text{SA}}^{\mathsf{comp}}(\theta)$, the bidder with the second highest type $\theta_{(2)}$ would find it strictly profitable to overcut the winning score.

type profiles θ,

$$u_i(b', q, \theta) - u_i(b, q, \theta) = D_i(b', q)b'_i - D_i(b, q)b_i - c(q_i, \theta_i)[D_i(b', q) - D_i(b, q)]$$
$$\geq D_i(b', q)b'_i - D_i(b, q)b_i - b_i[D_i(b', q) - D_i(b, q)]^+,$$

where the inequality follows since $b_i \geq c(q_i, \theta_i)$. This establishes (1).

Using $D_i(b', q)b'_i - D_i(b, q)b_i = \mathbb{E}_\mu\left[\mathbf{1}_{i \text{ wins } |b',q} b'_i - \mathbf{1}_{i \text{ wins } |b,q} b_i\right]$, and $D_i(b', q) - D_i(b, q) = \mathbb{E}_\mu\left[\mathbf{1}_{i \text{ wins } |b',q} - \mathbf{1}_{i \text{ wins } |b,q}\right]$, together with the fact that $[\cdot]^+$ is convex, we have that

$$D_i(b', q)b'_i - D_i(b, q)b_i - b_i[D_i(b', q) - D_i(b, q)]^+$$
$$\geq \mathbb{E}_\mu\left[\mathbf{1}_{i \text{ wins } |b',q} b'_i - \mathbf{1}_{i \text{ wins } |b,q} b_i - b_i[\mathbf{1}_{i \text{ wins } |b',q} - \mathbf{1}_{i \text{ wins } |b,q}]^+\right].$$

This establishes (2). ■

Proof of Corollary 2. Let $b = (b_i)_{i\in I}, q = (q_i)_{i\in I}$ be the bids under competitive bidding for some type profile θ. Fix $i \in I$, and let $b' = (b'_j)_{j\in I}$ be such that $b'_j = b_j$ for all $j \neq i$, and $b'_i > b_i$. Then, it must be that

$$\begin{aligned}0 &\geq u_i(b', q, \theta) - u_i(b, q, \theta)\\ &\geq D_i(b', q)b'_i - D_i(b, q)b_i - b_i[D_i(b', q) - D_i(b, q)]^+\\ &= D_i(b', q)b'_i - D_i(b, q)b_i,\end{aligned}$$

where the first inequality follows since b, q are competitive bids, the second inequality uses (1), and the equality follows since $b'_i > b_i$ implies $D_i(b', q) \leq D_i(b, q)$. This establishes (3).

Next, let $b = (b_i)_{i\in I}, q = (q_i)_{i\in I}$ be the bids under a unitary cartel for some type profile θ, and fix $b' = (b'_i)_{i\in I} \geq b$. Then, it must be that

$$\begin{aligned}0 &\geq \sum_{i\in I} u_i(b', q, \theta) - u_i(b, q, \theta)\\ &\geq \sum_{i\in I} D_i(b', q)b'_i - D_i(b, q)b_i - b_i[D_i(b', q) - D_i(b, q)]^+,\end{aligned}$$

where the first inequality follows since b, q is optimal for the cartel, and the second inequality uses (1). This establishes (4). ■

Proof of Lemma 2. Fix a type profile θ. Then, for any any bidding profile $b = (b_i)_{i\in I}, q =$

$(q_i)_{i\in I}$ and on-path transfer profile $T : Q^I \to \mathbb{R}^n$, the cartel's flow payoff is

$$\sum_i u_i(b,q,\theta) - \lambda \mathbb{E}_\mu \left[\sum_i [T_i(\widehat{q})]^- \right]. \tag{7}$$

For the bidding and transfer profile to be sustainable in a Stationary PPE, the following constraints must hold:

$$\forall i \in I, \forall b_i' \neq b_i, \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta) - u_i(b,q,\theta) \tag{8}$$
$$\leq \sum_{\widehat{q}\in Q^I} T_i(\widehat{q})(1+\lambda_{T_i(\widehat{q})<0})\mu(\widehat{q}|q) + \frac{\delta}{n}(\overline{V} - \underline{V})$$
$$\forall i \in I, \forall q_i' \neq q_i, \quad u_i(b, q_i', q_{-i}, \theta) - u_i(b,q,\theta) \tag{9}$$
$$\leq \sum_{\widehat{q}\in Q^I} T_i(\widehat{q})(1+\lambda_{T_i(\widehat{q})<0})(\mu(\widehat{q}|q) - \mu(\widehat{q}|q_i', q_{-i}))$$
$$\forall i \in I, \forall \widehat{q} \in Q^I, -T_i(\widehat{q})(1+\lambda_{T_i(\widehat{q})<0}) \leq \frac{\delta}{n}(\overline{V} - \underline{V}) \tag{10}$$
$$\forall \widehat{q} \in Q^I, \sum_i T_i(\widehat{q}) = 0. \tag{11}$$

Constraint (8) states that no bidder i can gain by defecting and placing a bid $b_i' \neq b_i$: since such deviations are detectable, they are punished with Nash reversion. Constraint (9) states that no bidder i can gain by defecting and placing a bid with intended quality $q_i' \neq q_i$ (without changing the bid b_i): since such deviations are not detectable (because $\mu(\cdot|q)$ has full support), and since the equilibrium is on-path stationary, such deviations can only be deterred using transfers on the equilibrium path. Constraint (10) guarantees that bidders have an incentive to pay their corresponding transfers. Lastly, constraint (11) says that transfers must be budget-balance.

The cartel-optimal bidding and transfer profile b, q, T when bidders' types are θ maximizes (7) subject to (8), (9), (10) and (11). This program can be decomposed as follows. For each b, q, find transfers $T : Q^I \to [\underline{T}, \infty)^I$ (with $\underline{T} \equiv -\frac{1}{1+\lambda}\frac{\delta}{n}(\overline{V} - \underline{V})$) that solve

$$K(b,q,\theta) = \min_{T:Q^I \to [\underline{T},\infty)^I} \mathbb{E}_\mu \left[\sum_i [T_i(\widehat{q})]^- \right],$$

subject to (8), (9) and (11).[7] Then, the cartel's problem is to find b, q that maximize $\sum_i u_i(b, q, \theta) - \lambda K(b, q, \theta)$. This completes the proof. ■

Proof of Proposition 2. Follows from the characterization of optimal bidding behavior in Lemma 2, and from the fact that (IC-q) is not relevant under FPA. ■

For any vector of types $\theta = (\theta_i)_{i \in I}$, we let $\underline{c}(\theta) = \min_i c(\underline{q}, \theta_i)$ and $\overline{c}(\theta) = \max_i c(\underline{q}, \theta_i)$. We also let $b_{\text{FPA}}(\theta)$ denote the winning bid under that solves program (P) under FPA when firms' types are θ.

Lemma A.1. *Suppose $\delta(\overline{V} - \underline{V}) > 0$, and let θ be such that $\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta$ for $\eta \in (0, \frac{\delta}{n-1}(\overline{V} - \underline{V}))$. Then, $b_{FPA}(\theta) \geq \min\{r, \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta\}$.*

Proof. Suppose not, so that $b_{\text{FPA}}(\theta) < \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta$. Note then that cartel flow profits (including transfers) when types are θ are bounded above by

$$b_{\text{FPA}}(\theta) - \underline{c}(\theta) < \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta.$$

Let $\hat{b} = \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V})$, and consider the following bidding and transfer profile: all firms bid $\hat{b}$ and quality $\underline{q}$, and there are no transfers. Note that this bidding profile is enforceable under FPA when firms' types are θ. Indeed, each i gets a payoff of $\frac{1}{n}(\hat{b} - c(\underline{q}, \theta_i)) + \frac{\delta}{n}\overline{V}$ by following this strategy profile, and gets $\hat{b} - c(\underline{q}, \theta_i) + \frac{\delta}{n}\underline{V}$ by undercutting bid $\hat{b}$. For each $i \in I$, the deviation is not profitable if and only if

$$\hat{b} \leq c(\underline{q}, \theta_i) + \frac{\delta}{n-1}(\overline{V} - \underline{V}),$$

which holds since $\underline{c}(\theta) = \min_i c(\underline{q}, \theta_i)$. Hence, this bidding and transfer profile is enforceable when bidders' types are θ. Moreover, note that cartel flow profits under this profile are weakly larger than

$$\hat{b} - \overline{c}(\theta) = \frac{\delta}{n-1}(\overline{V} - \underline{V}) - (\overline{c}(\theta) - \underline{c}(\theta)) \geq \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta.$$

Hence, this bidding and transfer profile lead to strictly larger cartel profits than the optimal one (with a winning bid $b_{\text{FPA}}(\theta) < \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta$), a contradiction. ■

[7] By convention, we set $K(b, q, \theta) = +\infty$ if b, q are such that there are no transfers $T : Q^I \to [\underline{T}, \infty)^I$ satisfying (8), (9) and (11).

Proof of Proposition 3. Fix a profile of types θ such that $\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta$ for some $\eta > 0$ to be determined shortly. Recall that $b_{\mathrm{FPA}}(\theta)$ is the winning bid that solves program (P) under FPA when firms' types are θ, and that $\overline{c}(\theta) = \max_i c(\underline{q}, \theta_i)$ and $\underline{c}(\theta) = \min_i c(\underline{q}, \theta_i)$. Note that $b_{\mathrm{FPA}}(\theta)$ must be such that, for each $i \in I$,

$$(1 - x_i)(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i)) \leq \frac{\delta}{n}(\overline{V} - \underline{V}) + T_i(1 + \lambda \mathbf{1}_{T_i < 0}),$$

where $x_i \in [0, 1]$ is the probability that i wins the auction, and T_i is i's net transfer. Summing this inequality over all i, and using $\sum_i x_i \leq 1$, $\sum_i T_i = 0$ and $c(\underline{q}, \theta_i) \leq \overline{c}(\theta)$ for all i, we get

$$\frac{n-1}{n}(b_{\mathrm{FPA}}(\theta) - \overline{c}(\theta)) \leq \frac{\delta}{n}(\overline{V} - \underline{V}) \Longleftrightarrow b_{\mathrm{FPA}}(\theta) \leq \overline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V}). \tag{12}$$

Consider the following bidding and transfer profile under SA: all bidders place bid $b_{\mathrm{FPA}}(\theta)$ and intended quality $\underline{q}$, and there are no transfers (i.e., $T_i(\widehat{q}) = 0$ for all $\widehat{q}$ and all i). Clearly, these transfers are feasible (i.e., for all $\widehat{q}$, $\sum_i T_i(\widehat{q}) = 0$, and, for all i, $T_i(\widehat{q}) \geq \underline{T}$).

Let $(b, q) = (b_i, q_i)_{i \in I}$ be the bidding profile in which all bidders submit bid $b_{\mathrm{FPA}}(\theta)$ and intended quality $\underline{q}$. Bidder i's payoff under SA under this bidding profile is $u_i(b, q, \theta_i) = \frac{1}{n}(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i))$. Note that there exists $\alpha_1 \in (1/n, 1)$ independent of $b_{\mathrm{FPA}}(\theta)$ such that, for all $b'_i < b_{\mathrm{FPA}}(\theta)$ and all q'_i, $u_i(b'_i, b_{-i}, q'_i, q_{-i}, \theta_i) \leq \alpha_1(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i))$.[8] Hence,

$$\begin{aligned} \forall i, \forall b'_i < b_{\mathrm{FPA}}(\theta), \forall q'_i, \quad u_i(b'_i, b_{-i}, q'_i, q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\leq \left(\alpha_1 - \frac{1}{n}\right)(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i)) \\ &\leq \left(\alpha_1 - \frac{1}{n}\right)(b_{\mathrm{FPA}}(\theta) - \underline{c}(\theta)). \end{aligned} \tag{13}$$

[8]To see why the claim is true, note that if $b'_i \in (b_{\mathrm{FPA}}(\theta)\underline{q}/\overline{q}, b_{\mathrm{FPA}}(\theta))$, then by the full support assumption the probability that i wins the auction by bidding b'_i, q'_i when all others bid $b_{\mathrm{FPA}}(\theta), \underline{q}$ is bounded by some $\alpha < 1$. And so the payoff i gets from bidding the deviation is bounded by $\alpha(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i))$. On the other hand, the payoff that i obtains by bidding $b'_i \leq b_{\mathrm{FPA}}(\theta)\underline{q}/\overline{q}$ is bounded by $b_{\mathrm{FPA}}(\theta)(\underline{q}/\overline{q}) - c(\underline{q}, \theta_i) < (\underline{q}/\overline{q})(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i))$. Letting $\alpha_1 = \max\{\alpha, \underline{q}/\overline{q}\}$ establishes the claim.

Let $\eta > 0$ be such that $\eta \le \eta_1 \equiv \delta(\overline{V} - \underline{V})\left(\frac{1}{n\alpha_1 - 1} - \frac{1}{n-1}\right)$. Note then that

$$\begin{aligned}\left(\alpha_1 - \frac{1}{n}\right)(b_{\mathrm{FPA}}(\theta) - \underline{c}(\theta)) &\le \left(\alpha_1 - \frac{1}{n}\right)\left(\overline{c}(\theta) - \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V})\right) \\ &\le \left(\alpha_1 - \frac{1}{n}\right)\left(\eta + \frac{\delta}{n-1}(\overline{V} - \underline{V})\right) \le \frac{\delta}{n}(\overline{V} - \underline{V}),\end{aligned}$$

where the first inequality uses (12), the second inequality uses $\overline{c}(\theta) - \underline{c}(\theta) \le \eta$ and the last inequality uses $\eta \le \eta_1$. Combining this with (13), we get

$$\forall i, \forall b_i' < b_{\mathrm{FPA}}(\theta), \forall q_i' \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) \le \frac{\delta}{n}(\overline{V} - \underline{V}). \tag{14}$$

There are two cases to consider: (a) $b_{\mathrm{FPA}}(\theta) = r$, and (b) $b_{\mathrm{FPA}}(\theta) < r$. In case (a), the inequalities in (14) imply that the proposed bidding and transfer profile satisfy (IC-p) under auction format SA.

Consider next case (b). By Lemma A.1, we have that

$$r > b_{\mathrm{FPA}}(\theta) \ge \underline{c}(\theta) + \frac{\delta}{n-1}(\overline{V} - \underline{V}) - \eta \Longrightarrow \overline{c}(\theta) < \underline{c}(\theta) + \eta < r - \frac{\delta}{n-1}(\overline{V} - \underline{V}) + 2\eta. \tag{15}$$

Since $\gamma(\widehat{q}_i|q_i')$ has full support over Q for all q_i', there exists $\alpha_2 \in (1/n, 1)$ such that, for all $q_i' \ne \underline{q}$ and all $b_i' > b_{\mathrm{FPA}}(\theta)$, $D_i(b_i', b_{-i}, q_i', q_{-i}) \le \alpha_2$. Hence, for all $i \in I$, all $b_i' > b_{\mathrm{FPA}}(\theta)$ and all q_i', $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) \le \alpha_2(r - c(\underline{q}, \theta_i))$. Let $\eta > 0$ be such that

$$\eta < \eta_2 \equiv \frac{1 - \alpha_2}{2n(1-\alpha_2) + n - 1}\frac{n}{n-1}\delta(\overline{V} - \underline{V}).$$

Define

$$\underline{b} \equiv r - (1 - \alpha_2)\frac{\delta}{n-1}(\overline{V} - \underline{V}) + \eta\left(2(1-\alpha_2) + \frac{n-1}{n}\right).$$

Note that $\eta < \eta_2$ implies $\underline{b} < r$. Suppose $b_{\mathrm{FPA}}(\theta) \ge \underline{b}$. Then,

$$\begin{aligned}\forall i, \forall b_i' > b_{\mathrm{FPA}}(\theta), \forall q_i' \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\le \alpha_2(r - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i)) \\ &\le \frac{n-1}{n}(b_{\mathrm{FPA}}(\theta) - \overline{c}(\theta)) \le \frac{\delta}{n}(\overline{V} - \underline{V}).\end{aligned} \tag{16}$$

The first inequality follows from the bound on $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i)$ derived above, and the last inequality follows from (12). To see that the middle inequality holds, note that

$$\alpha_2(r - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) \leq \frac{n-1}{n}(b_{\text{FPA}}(\theta) - \overline{c}(\theta))$$

$$\iff b_{\text{FPA}}(\theta) \geq \alpha_2 r + (1 - \alpha_2)c(\underline{q}, \theta_i) + \frac{n-1}{n}(\overline{c}(\theta) - c(\underline{q}, \theta_i)),$$

which is always satisfied since $b_{\text{FPA}}(\theta) \geq \underline{b}$, $\overline{c}(\theta) - c(\underline{q}, \theta_i) \leq \eta$ and since, by (15), $c(\underline{q}, \theta_i) \leq r - \frac{\delta}{n-1}(\overline{V} - \underline{V}) + 2\eta$. Hence, if $\eta < \min\{\eta_1, \eta_2\}$, and if $b_{\text{FPA}}(\theta) \geq \underline{b}$, then (16) and (14) both hold, and so the proposed bidding and transfer scheme satisfies (IC-b).

Therefore, if $\text{prob}_F(\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$ for $\eta < \min\{\eta_1, \eta_2\}$, and if either (IC-q) is not binding, or if quality is noisily evaluated but perfectly monitored, then enforceability of $b \geq \underline{b}$ under FPA implies enforceability of b under SA.[9]

Lastly, we consider the case in which transfers are costless, so that $\lambda = 0$. Fix a vector of types θ such that $\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta$, and let again $b_{\text{FPA}}(\theta)$ be optimal winning bid under FPA. By the arguments above, $b_{\text{FPA}}(\theta)$ satisfies (12). Consider the following bidding and transfer profile under SA when bidders' types are θ. Each bidder i submits bid $b_i = b_{\text{FPA}}(\theta)$ and intended quality $q_i = \underline{q}$. If bidder i wins the auction, it pays transfer $T_i = -\frac{n-1}{n}(b_{\text{FPA}}(\theta) - \overline{c}(\theta))$, which is divided evenly among all losing bidders; i.e., each loser $j \neq i$ gets $T_j = -\frac{1}{n-1}T_i = \frac{1}{n}(b_{\text{FPA}}(\theta) - \overline{c})$. Since $b_{\text{FPA}}(\theta)$ satisfies (12), it follows that $T_j \geq \underline{T}$ for all i, so the transfer profile is feasible.

We now show that, if $\eta > 0$ is small and if $b_{\text{FPA}}(\theta) \geq \underline{b}$ (with $\underline{b} < r$ defined above), this bidding and transfer profile also satisfy (IC-p) and (IC-q), and so it's enforceable. We start by showing that, under these conditions, (IC-p) holds. Let $(b, q) = (b_i, q_i)_{i \in I}$ be such that, for all i, $b_i = b_{\text{FPA}}(\theta)$ and $q_i = \underline{q}$. Note that, for all i, $u_i(b, q, \theta_i) = \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i))$: under this bidding profile, each bidder wins with probability $1/n$, and pays cost $c(\underline{q}, \theta_i)$ when it wins. Note further that $\sum_{\widehat{q}} \mu(\widehat{q}|q)T_i(\widehat{q}) = 0$; i.e., on average, each bidder pays zero transfers.

In addition, and by the same arguments as above, there exists $\alpha_1 < 1$ such that, for all $b_i' < b_{\text{FPA}}(\theta)$ and for all q_i', $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) \leq \alpha_1(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i))$. Hence, the same arguments we used above imply that, if $\eta > 0$ is smaller than η_1, then (14) holds. In particular, if $b_{\text{FPA}}(\theta) = r$, the (IC-p) holds.

Suppose next that $b_{\text{FPA}}(\theta) < r$. The same arguments used above imply that there exists $\alpha_2 < 1$ such that, for all i, all $b_i' > b_{\text{FPA}}(\theta)$ and all q_i', $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) \leq \alpha_2(r - c(\underline{q}, \theta_i))$. Hence, the same arguments we used above imply that, if $\eta > 0$ is smaller than η_2, and if

[9] If quality is noisily evaluated but perfectly monitored, then the only relevant constraints for enforceability are (IC-p) and the constraints on transfers. Indeed, (IC-q) is not relevant, since quality deviations are detected and can be punished with Nash reversion.

$b_{\text{FPA}}(\theta) \geq \underline{b}$, then (16) also holds. Hence, if $\eta < \min\{\eta_1, \eta_2\}$ and $b_{\text{FPA}}(\theta) \geq \underline{b}$, then the proposed bidding and transfer profile satisfies (IC-p).

We now show that there exists $\eta_3 > 0$ such that, if $\eta < \eta_3$, then the proposed bidding and transfer profile also satisfies (IC-q). To see why, note that, for all i and all q_i',

$$\begin{aligned} u_i(b, q_i', q_{-i}, \theta_i) + \sum_{\widehat{q}} \mu(\widehat{q}|q_i', q_{-i}) T_i(\widehat{q}) &= D_i(b, q_i', q_{-i})(b_{\text{FPA}}(\theta) - c(q_i', \theta_i)) + \sum_{\widehat{q}} \mu(\widehat{q}|q_i', q_{-i}) T_i(\widehat{q}) \\ &= \text{prob}(i \text{ wins}|q_i', q_{-i}) \left(b_{\text{FPA}}(\theta) - c(q_i', \theta_i) - \frac{n-1}{n}(b_{\text{FPA}}(\theta) - \overline{c}(\theta)) \right) \\ &\quad + \text{prob}(i \text{ loses}|q_i', q_{-i}) \frac{1}{n}(b_{\text{FPA}}(\theta) - \overline{c}(\theta)) \\ &= \frac{1}{n}(b_{\text{FPA}}(\theta) - \overline{c}(\theta)) + \text{prob}(i \text{ wins}|q_i', q_{-i})(\overline{c}(\theta) - c(q_i', \theta_i)). \end{aligned}$$

Hence, (IC-q) holds if and only if

$$\forall i, \forall q_i' \neq \underline{q}, \quad \text{prob}(i \text{ wins}|q_i', q_{-i})(\overline{c}(\theta) - c(q_i', \theta_i)) \leq \text{prob}(i \text{ wins}|q)(\overline{c}(\theta) - c(\underline{q}, \theta_i)),$$

which holds for all $\eta < \eta_3 \equiv \min_{\theta_i \in [\underline{\theta}, \overline{\theta}], q_i' \in Q \setminus \{\underline{q}\}} c_i(q_i', \theta_i) - c_i(\underline{q}, \theta_i)$.[10] Hence, for $\eta < \min\{\eta_1, \eta_2, \eta_3\}$, the proposed bidding and transfer profile satisfies (IC-q) and (IC-p) whenever $b_{\text{FPA}}(\theta) \geq \underline{b}$.

Therefore, if $\text{prob}_F(\max_{i,j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$ for $\eta < \min\{\eta_1, \eta_2, \eta_3\}$, and if $\lambda = 0$, then enforceability of $b \geq \underline{b}$ under FPA implies enforceability of b under SA. ■

Proof of Proposition 4. Fix $\theta = (\theta_i)_{i \in I}$ such that $\max_{i \neq j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta$, and let $b_{\text{FPA}}(\theta)$ denote the optimal enforceable winning bid under FPA when bidders' types are θ. From the proof of Proposition 3, we know that the following bidding and transfer profile under SA satisfies (IC-p) whenever $\eta < \min\{\eta_1, \eta_2\}$, and $b_{\text{FPA}}(\theta) \geq \underline{b}$: all bidders bid $b_i = b_{\text{FPA}}(\theta)$ and quality $q_i = \underline{q}$, and there are no transfers. We now show that, if $\min_{q' \neq q, \theta_i} |c(q', \theta_i) - c(q, \theta_i)| > 1/\epsilon$ for $\epsilon > 0$ small enough, or if $\sup_{q_i' \neq q_i'', \widehat{q}} |\ln \gamma(\widehat{q}|q_i') - \ln \gamma(\widehat{q}|q_i'')| < \epsilon$ for ϵ small enough, this bidding and transfer profile also satisfies (IC-q). Note that this implies that the bidding and transfer profile with winning bid $b_{\text{FPA}}(\theta)$ is enforceable under SA. Hence, if $\text{prob}_F(\max_{i \neq j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$, enforceability of $b \geq \underline{b}$ under FPA implies enforceability of b under SA.

Consider first the case in which $\min_{q_i' \neq q_i'', \theta_i'} |c(q_i', \theta_i') - c(q_i'', \theta_i)| > 1/\epsilon$, and assume $\epsilon \in$

[10] Since Q is discrete, $c_i(q_i', \theta_i) > c_i(\underline{q}, \theta_i)$ for all $q_i' \in Q \setminus \{\underline{q}\}$ and all θ_i. Since $c(q, \theta_i)$ is continuous in θ_i, we have that $\eta_3 > 0$.

$(0, 1/r)$. Note that this implies that, for all i and all $q_i' \neq \underline{q}$, $c(q_i', \theta_i) > c(\underline{q}, \theta_i) + r \geq r$. Then,

$$\forall i, \forall q_i' \neq \underline{q}, \quad u_i(b, q_i', \theta_i) - u_i(b, q, \theta_i) = D_i(b, q_i', q_{-i})(b_{\text{FPA}}(\theta) - c(q_i', \theta_i)) - \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) < 0,$$

where the strict inequality follows since $c(q_i', \theta_i) > r \geq b_{\text{FPA}}(\theta)$. Since there are no transfers, the bidding and transfer profile satisfies (IC-q).

Suppose next that $\sup_{q_i' \neq q_i'', \widehat{q}} |\ln \gamma(\widehat{q}|q_i') - \ln \gamma(\widehat{q}|q_i'')| < \epsilon$ for $\epsilon > 0$ small. Note that there exists $\varepsilon(\epsilon) > 0$, with $\varepsilon(\epsilon) \to 0$ as $\epsilon \to 0$, such that for all $q_i' \neq \underline{q}$, $|D_i(b, q_i', q_{-i}) - D_i(b, q)| = |D_i(b, q_i', q_{-i}) - \frac{1}{n}| \leq \varepsilon(\epsilon)$. Hence,

$$\begin{aligned}
\forall i, \forall q_i' \neq \underline{q}, \quad u_i(b, q_i', \theta_i) - u_i(b, q, \theta_i) &\leq (\frac{1}{n} + \varepsilon(\epsilon))(b_{\text{FPA}}(\theta) - c(q_i', \theta_i)) - \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) \\
&= \frac{1}{n}(c(\underline{q}, \theta_i) - c(q_i', \theta_i)) + \varepsilon(\epsilon)(b_{\text{FPA}}(\theta) - c(q_i', \theta_i))
\end{aligned}$$

Since $c(q_i', \theta_i) - c(\underline{q}, \theta_i) \geq \min_{q_i'' \neq \underline{q}, \theta_i'} c(q_i'', \theta_i') - c(\underline{q}, \theta_i') > 0$, it follows that $u_i(b, q_i', \theta_i) - u_i(b, q, \theta_i) < 0$ for all $\epsilon > 0$ smaller than some $\overline{\epsilon} > 0$. Since the proposed bidding and transfers profile has no transfers, the profile satisfies (IC-q).

Suppose next that $\max_{q' \neq q, \theta_i} |c(q', \theta_i) - c(q, \theta_i)| < \epsilon$ for ϵ small, and consider the following bidding and transfer profile under SA: all bidders bid $b_i = b_{\text{FPA}}(\theta)$ and $q_i = \overline{q}$, and there no transfers. We now show that, when $\eta < 0$ and $\epsilon > 0$ are both small enough, and when $b_{\text{FPA}}(\theta)$ is larger than some $\underline{b}' < r$, this bidding and transfer profile satisfies (IC-p) and (IC-q). Hence, if $\text{prob}_F(\max_{i \neq j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$, enforceability of $b \geq \underline{b}'$ under FPA implies enforceability of b under SA.

Each bidder's payoff under SA under bidding profile (b, q) is $u_i(b, q, \theta_i) = \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\overline{q}, \theta_i)) \geq \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) - \frac{\epsilon}{n}$. By the same arguments as in the proof of Proposition 3, there exists $\alpha_1 \in (1/n, 1)$ independent of $b_{\text{FPA}}(\theta)$ such that, for all $b_i' < b_{\text{FPA}}(\theta)$, and all q_i', $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) \leq \alpha_1(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i))$. Hence,

$$\begin{aligned}
\forall i, \forall b_i' < b_{\text{FPA}}(\theta), \forall q_i' \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\leq \left(\alpha_1 - \frac{1}{n}\right)(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) + \frac{\epsilon}{n} \\
&\leq \left(\alpha_1 - \frac{1}{n}\right)(b_{\text{FPA}}(\theta) - \underline{c}(\theta)) + \frac{\epsilon}{n} \\
&\leq \left(\alpha_1 - \frac{1}{n}\right)\left(\eta + \frac{\delta}{n-1}(\overline{V} - \underline{V})\right) + \frac{\epsilon}{n},
\end{aligned} \tag{17}$$

where the first inequality uses the bound on $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i)$ derived above, the second one uses $\underline{c}(\theta) \leq c(\underline{q}, \theta_i)$, and the third one uses (12) and $\overline{c}(\theta) - \underline{c}(\theta) \leq \eta$. Note that, for $\eta < \eta_1$ and for ϵ smaller than some $\overline{\epsilon}(\eta) > 0$, the right-hand side of (17) is smaller than $\frac{\delta}{n}\delta(\overline{V} - \underline{V})$. Hence, for η and ϵ small enough,

$$\forall i, \forall b_i' < b_{\text{FPA}}(\theta), \forall q_i' \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) \leq \frac{\delta}{n}(\overline{V} - \underline{V}), \tag{18}$$

If $b_{\text{FPA}}(\theta) = r$, then (18) implies that the bidding and transfer profile satisfies (IC-p).

Suppose next that $b_{\text{FPA}}(\theta) < r$. Note then that, for all $i \in I$, all $b_i' > b_{\text{FPA}}(\theta)$ and all q_i', $u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) \leq \frac{1}{n}(r - c(\underline{q}, \theta_i))$. Indeed, i's probability of winning when bidding $b_i' > b_{\text{FPA}}(\theta)$ and q_i' must be weakly lower than i's probability of winning when bidding $b_i = b_{\text{FPA}}(\theta)$ and $q_i = \overline{q}$. Define $\underline{b}' \equiv r - \delta(\overline{V} - \underline{V}) + \epsilon$ and suppose $b_{\text{FPA}}(\theta) \geq \underline{b}'$. Then,

$$\begin{aligned} \forall i, \forall b_i' > b_{\text{FPA}}(\theta), \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\leq \frac{1}{n}(r - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\overline{q}, \theta_i)) \\ &\leq \frac{1}{n}(r - b_{\text{FPA}}(\theta)) + \frac{1}{n}\epsilon \\ &\leq \frac{\delta}{n}(\overline{V} - \underline{V}), \end{aligned} \tag{19}$$

where the first inequality uses $c(\underline{q}, \theta_i) - c(\overline{q}, \theta_i) \leq \epsilon$, and the second inequality uses $b_{\text{FPA}}(\theta) \geq \underline{b}'$. Together with (18), this implies that when $b_{\text{FPA}}(\theta) \geq \underline{b}'$, and when $\eta > 0$ and $\epsilon > 0$ are small enough, the bidding and transfer profile satisfies (IC-p).

We now show that the proposed bidding and transfer profile also satisfies (IC-q) whenever $\epsilon > 0$ is small enough. Note that there exists $\alpha_3 < \frac{1}{n}$ such that, for all i and all $q_i' \neq q_i$, $D_i(b, q_i', q_{-i}) \leq \alpha_3$. Indeed, i's probability of winning the auction when bidding $b_i = b_{\text{FPA}}(\theta)$ and $q_i' \neq \overline{q}$ when all other bidders bid $b_{\text{FPA}}(\theta)$ and $\overline{q}$ is strictly lower than $1/n$. Hence,

$$\begin{aligned} \forall i, \forall q_i' \neq \underline{q}, \quad u_i(b, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\leq \alpha_3(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\text{FPA}}(\theta) - c(\overline{q}, \theta_i)) \\ &< \left(\alpha_3 - \frac{1}{n}\right)(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) + \frac{1}{n}\epsilon, \end{aligned}$$

where the first inequality uses $D_i(b, q_i', q_{-i}) \leq \alpha_3$ and the second uses $c(\overline{q}, \theta_i) - c(\underline{q}, \theta_i) < \epsilon$. Since $\alpha_3 < \frac{1}{n}$, the proposed bidding and transfer profile satisfies (IC-q) whenever ϵ is small enough.

Finally, suppose $\inf_{q_i' \neq q_i'', \widehat{q}} |\ln \gamma(\widehat{q}|q_i') - \ln \gamma(\widehat{q}|q_i'')| > \frac{1}{\epsilon}$ for $\epsilon > 0$ small. Define $\varepsilon(\epsilon) \equiv \frac{1}{\exp\left(\frac{1}{\epsilon}\right)}$, so that $\lim_{\epsilon\to 0}\varepsilon(\epsilon)=0$. Let

$$\widehat{Q}\equiv\left\{\widehat{q}\in Q:\forall q_i\neq\underline{q},\ln\left(\frac{\gamma(\widehat{q}|\underline{q})}{\gamma(\widehat{q}|q_i)}\right)>\frac{1}{\epsilon}\right\}$$

to be the set of signals that are more likely under $\underline{q}$ than under any $q_i\neq\underline{q}$.[11] Note that, for all $q_i\neq\underline{q}$, $\text{prob}(\widehat{q}_i\in\widehat{Q}|q_i)<|\widehat{Q}|\varepsilon(\epsilon)$, and that $\text{prob}(\widehat{q}_i\in\widehat{Q}|\underline{q})\geq 1-|\widehat{Q}|\varepsilon(\epsilon)$.[12]

Fix a type profile θ with $\max_{i,j}|c(\underline{q},\theta_i)-c(\underline{q},\theta_j)|<\eta$, and consider the following bidding and transfer profile $(b,q)=(b_i,q_i)_{i\in I}$ under SA. All bidders submit bid $b_i=b_{\text{FPA}}(\theta)$ and $q_i=\underline{q}$. If $\widehat{q}=(\widehat{q}_i)_{i\in I}$ is such that $I(\widehat{q})=\{i\in I:\widehat{q}_i\neq\widehat{Q}\}$ has only one element, bidder i with $\widehat{q}_i\notin\widehat{Q}$ pays $\frac{1}{1+\lambda}\frac{\delta}{n}(\overline{V}-\underline{V})$, which is divided even among all other bidders. Otherwise, if $I(\widehat{q})$ is either empty or has more than one element, there are no transfers. Note that this transfer profile is feasible; i.e. $T_i(\widehat{q})\geq\underline{T}$ for all i and $\widehat{q}$, and $\sum_i T_i(\widehat{q})=0$ for all $\widehat{q}$. We now how that, for $\eta>0$ and $\epsilon>0$ small enough, this bidding and transfer profile satisfy (IC-p) and (IC-q) as long as $b_{\text{FPA}}(\theta)$ is close enough to r.

By the arguments in the proof of Proposition 3, there exists $\alpha_1<1$ such that

$$\begin{aligned}\forall i,\forall b_i'<b_{\text{FPA}}(\theta),\forall q_i',\quad u_i(b_i',b_{-i},q_i',q_{-i},\theta_i)-u_i(b,q,\theta_i)&\leq\left(\alpha_1-\frac{1}{n}\right)(b_{\text{FPA}}(\theta)-c(\underline{q},\theta_i))\\&\leq\left(\alpha_1-\frac{1}{n}\right)(b_{\text{FPA}}(\theta)-\underline{c}(\theta)).\end{aligned}\tag{20}$$

Using (12) and $\overline{c}(\theta)-\underline{c}(\theta)<\eta$, we have that

$$\begin{aligned}\left(\alpha_1-\frac{1}{n}\right)(b_{\text{FPA}}(\theta)-\underline{c}(\theta))&\leq\left(\alpha_1-\frac{1}{n}\right)\left(\overline{c}(\theta)-\underline{c}(\theta)+\frac{\delta}{n-1}(\overline{V}-\underline{V})\right)\\&\leq\left(\alpha_1-\frac{1}{n}\right)\left(\eta+\frac{\delta}{n-1}(\overline{V}-\underline{V})\right).\end{aligned}$$

[11]Note that $\widehat{Q}$ is non empty. Indeed, since for every $q_i\neq\underline{q}$, $\gamma(\cdot|q_i)$ F.O.S.D. $\gamma(\cdot|\underline{q})$, and since $\inf_{q_i'\neq q_i'',\widehat{q}}|\ln\gamma(\widehat{q}|q_i')-\ln\gamma(\widehat{q}|q_i'')|>\frac{1}{\epsilon}$, it must be that $\forall q_i\neq\underline{q}$, $\ln\left(\frac{\gamma(\underline{q}|\underline{q})}{\gamma(\underline{q}|q_i)}\right)>\frac{1}{\epsilon}$; i.e., $\underline{q}\in\widehat{Q}$.

[12]Indeed, for all $\widehat{q}\in\widehat{Q}$, we have that $\gamma(\widehat{q}|q_i)<\varepsilon(\epsilon)\gamma(\widehat{q}|\underline{q})\leq\varepsilon(\epsilon)$, and so for all $q_i\neq\underline{q}$, $\text{prob}(\widehat{q}_i\in\widehat{Q}|q_i)<|\widehat{Q}|\varepsilon(\epsilon)$. Similarly, for all $\widehat{q}\notin\widehat{Q}$, there exists $q_i\neq\underline{q}$ such that $\ln\left(\frac{\gamma(\widehat{q}|q_i)}{\gamma(\widehat{q}|\underline{q})}\right)>\frac{1}{\epsilon}\iff\gamma(\widehat{q}|\underline{q})<\varepsilon(\epsilon)\gamma(\widehat{q}|q_i)\leq\varepsilon(\epsilon)$. Hence, $\text{prob}(\widehat{q}_i\in\widehat{Q}|\underline{q})\geq 1-|\widehat{Q}|\varepsilon(\epsilon)$.

Combining this with (20) we have that

$$\forall i, \forall b_i' < b_{\mathrm{FPA}}(\theta), \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) \leq \left(\alpha_1 - \frac{1}{n}\right)\left(\eta + \frac{\delta}{n-1}(\overline{V} - \underline{V})\right).$$

Note next that since the bidding and transfer profile are symmetric, we have that $\sum_{\widehat{q}} \mu(\widehat{q}|q) T_i(\widehat{q}|q) = 0$. Hence,

$$\begin{aligned}
\sum_{\widehat{q}} \mu(\widehat{q}|q) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q})<0}) &= -\lambda \mathrm{prob}(\widehat{q}_i \notin \widehat{Q}, \widehat{q}_{-i} \in \widehat{Q}^{n-1}|q) \frac{1}{1+\lambda} \frac{\delta}{n}(\overline{V} - \underline{V}) \\
&\geq -\frac{\lambda}{1+\lambda} \mathrm{prob}(\widehat{q}_i \notin \widehat{Q}|q) \frac{\delta}{n}(\overline{V} - \underline{V}) \geq -\frac{\lambda}{1+\lambda} |\widehat{Q}| \varepsilon(\epsilon) \frac{\delta}{n}(\overline{V} - \underline{V}),
\end{aligned}$$

Since $\alpha_1 < 1$ and since $\varepsilon(\epsilon) \to 0$ as $\epsilon \to 0$, for all $\eta > 0$ and $\epsilon > 0$ small enough we have that

$$\begin{aligned}
\forall i, \forall b_i' < b_{\mathrm{FPA}}(\theta), \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) &\leq \left(\alpha_1 - \frac{1}{n}\right)\left(\eta + \frac{\delta}{n-1}(\overline{V} - \underline{V})\right) \\
&\leq \frac{\delta}{n}(\overline{V} - \underline{V}) \\
&\quad + \sum_{\widehat{q}} \mu(\widehat{q}|q) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)).
\end{aligned}$$

Hence, if $\eta > 0$ and $\epsilon > 0$ are small enough and if $b_{\mathrm{FPA}}(\theta) = r$, the bidding and transfer profile satisfy (IC-p).

Suppose next that $b_{\mathrm{FPA}}(\theta) < r$. As in the proof of Proposition 3, there exists $\alpha_2 < 1$ such that

$$\forall i, \forall b_i' > b_{\mathrm{FPA}}(\theta), \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) \leq \alpha_2(r - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i)).$$

Assume $\eta < \eta_2$ and let $\underline{b}' = \underline{b} + \frac{\lambda}{1+\lambda} |\widehat{Q}| \varepsilon(\epsilon) \frac{\delta}{n}(\overline{V} - \underline{V})$, with $\eta_2 > 0$ and $\underline{b} < r$ as defined in the proof of Proposition 3. Since $\underline{b} < r$, as since $\lim_{\epsilon \to 0} \varepsilon(\epsilon) = 0$, $\underline{b}' < r$ for all ϵ small enough. Note that, for $b_{\mathrm{FPA}}(\theta) \geq \underline{b}'$, we have that

$$\begin{aligned}
\alpha_2(r - c(\underline{q}, \theta_i)) - \frac{1}{n}(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i)) &\leq \frac{n-1}{n}(b_{\mathrm{FPA}}(\theta) - \overline{c}(\theta)) - \frac{\lambda}{1+\lambda} |\widehat{Q}| \varepsilon(\epsilon) \frac{\delta}{n}(\overline{V} - \underline{V}) \\
\iff b_{\mathrm{FPA}}(\theta) &\geq \alpha_2 r + (1 - \alpha_2) c(\underline{q}, \theta_i) + \frac{n-1}{n}(\overline{c}(\theta) - c(\underline{q}, \theta_i)) \\
&\quad + \frac{\lambda}{1+\lambda} |\widehat{Q}| \varepsilon(\epsilon) \frac{\delta}{n}(\overline{V} - \underline{V}),
\end{aligned}$$

which is always satisfied since $b_{\mathrm{FPA}}(\theta) \geq \underline{b}'$, $\bar{c}(\theta) - c(\underline{q}, \theta_i) \leq \eta$ and since, by (15), $c(\underline{q}, \theta_i) \leq r - \frac{\delta}{n-1}(\overline{V} - \underline{V}) + 2\eta$. Since $\frac{n-1}{n}(b_{\mathrm{FPA}}(\theta) - \bar{c}(\theta)) \leq \frac{\delta}{n}(\overline{V} - \underline{V})$ (by (12)), and since

$$\sum_{\widehat{q}} \mu(\widehat{q}|q) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)) \geq -\frac{\lambda}{1+\lambda}|\widehat{Q}|\varepsilon(\epsilon)\frac{\delta}{n}(\overline{V} - \underline{V}), \tag{21}$$

it follows that, for $b_{\mathrm{FPA}}(\theta) \geq \underline{b}'$,

$$\begin{aligned}\forall i, \forall b_i' > b_{\mathrm{FPA}}(\theta), \forall q_i', \quad u_i(b_i', b_{-i}, q_i', q_{-i}, \theta_i) - u_i(b, q, \theta_i) \leq \frac{\delta}{n}(\overline{V} - \underline{V}) \\ + \sum_{\widehat{q}} \mu(\widehat{q}|q) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)).\end{aligned}$$

Hence, for $\eta > 0$ and $\epsilon > 0$ small enough, the bidding and transfer profile satisfies (IC-p) whenever $b_{\mathrm{FPA}}(\theta) \in [\underline{b}', r]$.

Lastly, we show that the bidding and transfer profile also satisfies (IC-q) whenever ϵ is small enough. Note that, for all $q_i' \neq \underline{q}$

$$\begin{aligned}&\sum_{\widehat{q}} \mu(\widehat{q}|q_i', q_{-i}) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)) \\ &\leq -\operatorname{prob}(\widehat{q}_i \notin \widehat{Q}, \widehat{q}_{-i} \in \widehat{Q}^{n-1}|q_i', q_{-i})(1+\lambda)\frac{1}{1+\lambda}\frac{\delta}{n}(\overline{V} - \underline{V}) \\ &\quad + (1 - \operatorname{prob}(\widehat{q}_i \notin \widehat{Q}, \widehat{q}_{-i} \in \widehat{Q}^{n-1}|q_i', q_{-i}))\frac{\lambda}{1+\lambda}\frac{\delta}{n(n-1)}(\overline{V} - \underline{V}) \to -\frac{\delta}{n}(\overline{V} - \underline{V}) \text{ as } \epsilon \to 0.\end{aligned}$$

Indeed, since for all $q_i' \neq \underline{q}$, $\operatorname{prob}(\widehat{q}_i \notin \widehat{Q}|q_i = q_i') > 1 - |\widehat{Q}|\varepsilon(\epsilon)$, and since $\operatorname{prob}(\widehat{q}_i \in \widehat{Q}|q_i = \underline{q}) \geq 1 - |\widehat{Q}|\varepsilon(\epsilon)$, it follows that

$$\operatorname{prob}(\widehat{q}_i \notin \widehat{Q}, \widehat{q}_{-i} \in \widehat{Q}|q_i', q_{-i})) > (1 - |\widehat{Q}|\varepsilon(\epsilon))^n \to 1 \text{ as } \epsilon \to 0.$$

Combining this with (21), we get that

$$\forall q_i \neq \underline{q}, \quad \sum_{\widehat{q}} (\mu(\widehat{q}|q_i', q_{-i}) - \mu(\widehat{q}|q)) T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)) \to -\frac{\delta}{n}(\overline{V} - \underline{V}) \text{ as } \epsilon \to 0.$$

Note further that, by the full support assumption, there exists $\alpha_2 < 1$ such that for all $q_i' \neq \underline{q}$, $u_i(b, q_i', q_{-i}, \theta_i) \leq \alpha_2(b_{\mathrm{FPA}}(\theta) - c(\underline{q}, \theta_i))$. Hence, for all $\eta > 0$ small enough, we have that

$$\begin{aligned}\forall i, \forall q_i' \neq \underline{q}, \quad u_i(b, q_i', q_{-i}, , \theta_i) - u_i(b, q, \theta_i) &\leq \left(\alpha_2 - \frac{1}{n}\right)(b_{\text{FPA}}(\theta) - c(\underline{q}, \theta_i)) \\ &\leq \left(\alpha_2 - \frac{1}{n}\right)(b_{\text{FPA}}(\theta) - \overline{c}(\theta)) + \eta\frac{n-1}{n} \\ &\leq \left(\frac{n\alpha_2 - 1}{n}\right)\frac{\delta}{n-1}(\overline{V} - \underline{V}) + \eta\frac{n-1}{n} \\ &< \frac{\delta}{n}(\overline{V} - \underline{V}),\end{aligned}$$

where the first inequality follows from the bound on $u_i(b, q_i', q_{-i}, \theta_i)$ derived above, the second inequality uses $\overline{c}(\theta) - c(\underline{q}, \theta_i) \leq \eta$, the third inequality uses (12), and the last inequality uses $\alpha_2 < 1$ and $\eta > 0$ small enough. Hence, for $\eta > 0$ and $\epsilon > 0$ small enough,

$$\forall i, \forall q_i' \neq \underline{q}, \quad u_i(b, q_i', q_{-i}, , \theta_i) - u_i(b, q, \theta_i) \leq \sum_{\widehat{q}}(\mu(\widehat{q}|q) - \mu(\widehat{q}|q_i', q_{-i}))T_i(\widehat{q}|q)(1 + \lambda \mathbf{1}_{T_i(\widehat{q}} < 0)).$$

Thus, the proposed bidding and transfer profile also satisfies (IC-q). This implies that the bidding and transfer profile with winning bid $b_{\text{FPA}}(\theta)$ is enforceable under SA. Hence, if $\text{prob}_F(\max_{i\neq j} |c(\underline{q}, \theta_i) - c(\underline{q}, \theta_j)| < \eta) = 1$, enforceability of $b \geq \underline{b}$ under FPA implies enforceability of b under SA. ■

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