TECHNOLOGY AND THE GLOBAL ECONOMY

By

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Abstract

Interpreting individual heterogeneity in terms of probability theory has proved powerful in connecting behaviour at the individual and aggregate levels. Returning to Ricardo's focus on comparative efficiency as a basis for international trade, much recent quantitative equilibrium modeling of the global economy builds on particular probabilistic assumptions about technology. We review these assumptions and how they deliver a unified framework underlying a wide range of static and dynamic equilibrium models.

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1 Introduction

Access to firm and household level data has transformed research in economics across a broad range of fields. Such data provide a window on the granular interactions of individual agents that prior work could only speculate on. But exploiting such data poses a particular challenge for fields such as international trade in which general equilibrium outcomes are at the essence of most research questions: How do we incorporate the rich behavior we observe at the individual level into a framework that has something to say about aggregates?

Probability theory has provided a tool for making progress: By treating outcomes at the individual level as realizations of random variables generated by particular probability distributions, individual heterogeneity can yield aggregates governed by a parsimonious set of parameters.

Research on the export participation of firms exemplifies this evolution. Prior to this century the theory of international trade relied on a representative firm at the sectoral level. But firm-level data revealed that only a small minority of firms export at all.[1] These observations led to a reconstruction of the theory of international trade to give producer heterogeneity an essential role.

Somewhat ironically, this research returned the field to Ricardo's original emphasis on technology. For Ricardo, differences in technology were the source of comparative advantage and the gains from trade. For a long time pursuing Ricardo's insight quantitatively was stymied by the difficulty of extending differences in technology to dimensions beyond Ricardo's two. But a broad range of recent research has provided quantitative analyses of technological heterogeneity with multiple locations in general equilibrium.

Similar observations apply to research on productivity growth, for which, of course, technology is basic. Traditional growth theory again relied on an assumption of a representative agent. But indicators of innovative activity, such as the employment of research scientists and patenting, reveal enormous heterogeneity across individual firms. How can the modeling of economic growth account for this diversity?

Our purpose here is less to review this research itself but to review the building blocks it stands on. A wide range of work builds on a very similar set of probabilistic assumptions. By reviewing these building blocks we hope to reveal what we think is an underlying unity that

[1] Bernard and Jensen (1995), Roberts and Tybout (1997), and Clerides et al. (1998), among others, were influential in revealing heterogeneity in export activity.

may not be immediately evident from a reading of individual papers. Our goal is to provide an accessible guide to these assumptions and how they're employed in a wide range of contexts.

Section 2 introduces the probabilistic foundations of all that follows. It characterizes the arrival of ideas to an economy and how the use of these ideas in production generates particular distributions of costs.

Section 3 embeds these cost distributions into general equilibrium in a single location. We first consider economies with various market structures standard in the literature. In these market economies agents face a common set of prices, purchasing from a continuum of suppliers and selling to a continuum of buyers. Recent quantitative work on networks and trade has revealed the granular nature of firm interactions. Most firms buy from only a handful of suppliers and sell to only a handful of buyers.[2] We show how the cost distributions derived in Section 2 can be used as a foundation for a general equilibrium matching economy in which buyers and sellers interact with few partners.

Section 4 introduces location into the various economies considered in Section 3. With iceberg trade costs, the underlying cost structure introduced in Section 2 gives rise to a formulation of trade that holds across various market structures, applying as well to economies with bilateral matching frictions.

The analysis in the first four sections makes no specific reference to time. Section 5 considers the arrival of ideas in different locations over time. As ideas arrive they give birth to new potential producers. Extending the matching framework from the previous section, we allow age to enhance the ability to make contacts. As it ages, a producer lowers its cost through encounters with better suppliers and expands its sales as it meets more buyers. The framework generates a stochastic life cycle of firms in dynamic general equilibrium. On a balanced growth path the measures of active producers in different locations and their age structure is constant as better ideas displace worse ones.

Section 5 treats the arrival of ideas as exogenous. Section 6 ties their arrival to endogenous research activity in each location. New ideas are initially available only locally but diffuse to other locations over time. On a balanced growth path the stock of ideas in each location grows endogenously at a common rate determined by research activity and patterns of diffusion.

The research we review ranges from very old to brand new. The building blocks in Section 2 go back to the origins of probability theory, while the relationship to cost distributions

[2] Chaney (2014) is a seminal contribution. Bernard and Moxnes (2018) review the literature.

is more recent. How these distributions connect with different market structures in either closed or open market economies relates to several prominent papers from the first decade of this century. We see our contribution as showing how these various papers fit into a broad framework. Our review of matching relates to much more recent work on networks in international trade, showing how tools used in earlier work remain relevant. Our model of dynamic matching in general equilibrium is, as far as we know, new, but we include it to show the close connections to earlier work. Finally, our review of growth in open economies relates to work from over the last half century, but a contribution here is to show how it connects to recent work on dynamic matching. While much of the work we review is established, our own understanding of how it fits together continues to evolve.

A note on mathematics: We haven't shied away from showing the steps we took to get an answer. Our goal is to convey our reasoning in the hope of inspiring new research. We've been using these tools over many years, sometimes with confusion and frustration but then with a sense of awe at what can be achieved. The work that has been done only touches the surface of a vast set of critical research questions out there that these tools can address.

2 Technology: Basic Building Blocks

We begin reviewing the basic probabilistic assumptions that underlie all of what follows. The state of technology, and the cost structure it gives rise to, derive from the accumulation of individual technological advances.[3]

2.1 An Idea

The basic unit of our analysis is an idea about how to produce a good. We characterize an idea by a single number Z, its efficiency, indicating the quantity of that good a unit of input can produce using that idea. For simplicity, and in the Ricardian tradition, we'll sometimes call this input labor, while incorporating additional inputs in examples below.

We treat the efficiency of an idea as a random variable with a Pareto distribution, that is:

$$F(z) = \Pr[Z \leq z] = 1 - \left(\frac{z}{\underline{z}}\right)^{-\theta} \quad z \geq \underline{z}. \tag{1}$$

[3]The literature relating the state of technology to research output goes way back. Muth (1986) provides an overview of contributions as of that time. Evenson and Kislev (1976) is an early application. The characterization here builds on Bental and Peled (1996) and particularly Kortum (1997).

The distribution has two parameters. The lower bound $\underline{z} > 0$ determines its scale. Our assumptions below drive $\underline{z}$ to zero. The parameter $\theta > 0$, which is the star in all that follows, determines the shape of the distribution. A higher value implies that Z hugs the lower bound $\underline{z}$ more tightly while a lower value implies a distribution with a fatter upper tail.[4]

A unique feature of the Pareto, that its shape is preserved as the truncation point rises, renders it extremely convenient:

$$\Pr[Z \geq z | Z \geq z'] = \frac{\Pr[Z \geq z]}{\Pr[Z \geq z']} = \left(\frac{z}{z'}\right)^{-\theta} \qquad z \geq z' \geq \underline{z}.$$

Hence, truncating the Pareto distribution from below just leaves another Pareto distribution with the same shape parameter θ and higher lower bound.

2.2 The Accumulation of Ideas

Since ideas are durable they accumulate over time. Say that by time t, an integer number N have arrived each with an efficiency that's a realization of Z drawn independently from (1). The probability that any one of these ideas has efficiency greater than some level $z \geq \underline{z}$ is $p_z = 1 - F(z)$. With N ideas having arrived, the number N_z with efficiency greater than z is distributed binomially:

$$\Pr[N_z = n] = \begin{pmatrix} N \\ n \end{pmatrix} p_z^n (1 - p_z)^{N-n}, \tag{2}$$

where n can be any integer between 0 and N. Its expectation is:

$$E\left[N_z\right] = p_z N = T z^{-\theta}; \qquad T = \underline{z}^{\theta} N.$$

The term T, like θ, will stay with us for the rest of this review. It summarizes the state of technology taking into account the number of ideas N that have arrived and the efficiency of those ideas, as reflected in $\underline{z}^{\theta}$.

A convenient limit holds the expectation $E\left[N_z\right]$, and hence T, fixed while letting the number of ideas get arbitrarily large ($N \to \infty$), forcing the efficiency of the typical idea to become arbitrarily small ($\underline{z}^{\theta} \to 0$). In this limit the binomial distribution converges to a

[4] The Pareto's mean, $(\theta/(\theta-1))\underline{z}$, is defined only for $\theta > 1$. For $\theta \in (0, 1]$ the tail is so fat that the mean is infinite. The median $z^m = 2^{1/\theta}\underline{z}$ is defined for all θ. Both the mean, when it exists, and the median are proportional to the lower bound $\underline{z}$. Given θ, a higher value of $\underline{z}$ means that ideas tend to be better while $\underline{z}$ approaching 0, the case we consider later, means that most ideas are useless.

Poisson distribution with the same expectation, the Poisson parameter $Tz^{-\theta}$:[5]

$$\Pr[N_z = n] = \frac{\left(Tz^{-\theta}\right)^n}{n!} e^{-Tz^{-\theta}}. \tag{3}$$

Since we've taken the limit as $\underline{z}^\theta \to 0$, the distribution applies to all $z > 0$. Hence the number of ideas with efficiency $Z > z$ is distributed Poisson with parameter:

$$\mu^Z(z) = Tz^{-\theta}, \tag{4}$$

approaching infinity as $z \to 0$ (so there's no shortage of very bad ideas). This formulation is the building block for all that follows.

Consider the ideas that have accumulated, as determined by T, and rank them according to their efficiency, so that the k'th most efficient idea has efficiency $Z^{(k)}$. Hence $Z^{(1)} > Z^{(2)} > Z^{(3)} > ....$ The probability that the k'th most efficient idea has efficiency below z is the probability that at most $k-1$ ideas have efficiency greater than z:

$$F^{(k)}(z) = \Pr[Z^{(k)} \leq z] = e^{-Tz^{-\theta}} \sum_{l=0}^{k-1} \frac{\left(Tz^{-\theta}\right)^l}{l!}. \tag{5}$$

Of particular interest is the distribution of the efficiency of the best idea, $Z^{(1)}$. It's given by the probability that there is no idea more efficient than z, or:

$$F^{(1)}(z) = \Pr[Z^{(1)} \leq z] = e^{-Tz^{-\theta}}$$

which is the Fréchet or type II extreme value distribution.

2.3 Costs

We now connect ideas with the costs they imply for production. With inputs costing v, an idea of efficiency Z delivers at unit cost $C = v/Z$. (Since returns to scale are constant, henceforth we drop "unit" from "unit cost".) From equation (4), the number of ideas that can deliver at

[5]It emerges by rewriting (2) as $\Pr[N_z = n] = \frac{\lambda^n}{n!}\left(1 - \frac{\lambda}{N}\right)^N \frac{N!}{(N-n)!N^n}\left(1 - \frac{\lambda}{N}\right)^{-n}$, where $\lambda = p_z N$. Taking the limit as $N \to \infty$, fixing n and λ, gives (3).

cost below c is distributed Poisson with parameter:

$$\mu(c) = \mu^Z\left(\frac{v}{c}\right) = \Phi c^\theta; \qquad \Phi = Tv^{-\theta} \tag{6}$$

Like θ and T, the term Φ, or a variant, will remain with us throughout. It summarizes how the state of technology and input costs combine to govern the distribution of the cost of goods.

Consider ideas for a good with efficiencies $Z^{(1)} > Z^{(2)} > Z^{(3)} > ....$ The associated costs are $C^{(1)} < C^{(2)} < C^{(3)} < ...$, where $C^{(k)} = v/Z^{(k)}$. We can derive the distribution of the $C^{(k)}$'s from equation (5):

$$G^{(k)}(c) = \Pr[C^{(k)} \leq c] = 1 - \Pr\left[Z^{(k)} \leq \frac{v}{c}\right] = 1 - F^{(k)}\left(\frac{v}{c}\right). \tag{7}$$

Which costs are of interest depend on preferences and market structure. If the ideas are all about producing a homogeneous good, with perfect competition all that matters is the distribution of the cheapest:

$$G^{(1)}(c) = \Pr[C^{(1)} \leq c] = 1 - e^{-\Phi c^\theta}, \tag{8}$$

the Weibull or reversed type III extreme value distribution. At the other extreme, if each idea is about a different variety of a good, as with monopolistic competition, we're interested in all the costs.

In between are more complex cases of Bertrand competition and oligopoly, which require saying more about the joint distribution of costs. Before considering these cases explicitly, we show some basic results on the joint distribution of costs which we use below.

In dealing with the joint distribution of costs, it's useful to transform them as $U = \Phi C^\theta$. The number of ideas with transformed cost U less than u is then Poisson with parameter u.

Ordering $U^{(k)}$'s in line with their respective $C^{(k)}$'s, the distribution of $U^{(1)}$ is:

$$\Pr\left[U^{(1)} \leq u\right] = 1 - e^{-u}, \tag{9}$$

or simply the unit exponential distribution. In words, the probability that $U^{(1)}$ is less than u is just 1 minus the probability that there's no transformed cost below u.

To deal with higher costs, consider the probability that $U^{(k+1)}$ is less than u given that $U^{(k)} = u_k$. It's just 1 minus the probability that there's no transformed cost U between u_k

and u, or

$$\Pr\left[U^{(k+1)} \leq u | U^{(k)} = u_k\right] = 1 - e^{-(u-u_k)} \qquad u \geq u_k. \tag{10}$$

Hence, the distribution of the gap between $U^{(k)}$ and $U^{(k+1)}$ is unit exponential.[6] Knowing Φ and θ, one can retrieve the associated costs from the $U^{(k)}$ as:

$$C^{(k)} = \left(U^{(k)}/\Phi\right)^{1/\theta}. \tag{11}$$

A useful property of the $U^{(k)}$'s is that they're distributed Erlang with shape parameter k and rate parameter 1:

$$\Pr[U^{(k)} \leq u] = \int_0^u \frac{x^{k-1}e^{-x}}{(k-1)!}dx, \tag{12}$$

a special case of the gamma distribution.[7] That the Erlang distribution emerges isn't a surprise. It's complementary to the Poisson distribution we've been exploiting heavily. The Poisson tells us about the number of ideas delivering costs along a given interval. The Erlang tells us about the length of the interval to deliver k ideas.

We showed above how to characterize a higher cost given a lower cost. For many purposes we're interested in flipping the question to characterize a lower cost given a higher cost: e.g., how big an advance can we expect when the next advance occurs? Starting with (10):

$$\Pr\left[U^{(k)} \leq u | U^{(k+1)} = u_{k+1}\right] = \left(\frac{u}{u_{k+1}}\right)^k \qquad u \leq u_{k+1}, \tag{13}$$

which for $k = 1$ is simply the uniform distribution on $[0, u_2]$.[8] For higher k the distribution

[6] The set of equations (9) and (10) provide a means of drawing a set of ordered costs. One simply needs to draw a series of independent realizations of unit exponentials V_k and set:

$$U^{(1)} = V_1; \qquad U^{(2)} = U^{(1)} + V_2; \qquad U^{(3)} = U^{(2)} + V_3; \ \ldots \qquad .$$

This procedure provides a simple way of simulating costs in numerical analysis.

[7] The proof is by induction. Observe first that $U^{(1)}$ has the unit exponential distribution. Then consider the sum $U^{(k+1)} = U^{(k)} + V^{(k+1)}$, where $U^{(k)}$ is distributed Erlang with parameter k and $V^{(k+1)}$ is unit exponential:

$$\begin{aligned}\Pr[U^{(k+1)} \leq u] &= \int_0^u \left(1 - e^{-(u-x)}\right)\frac{x^{k-1}e^{-x}}{(k-1)!}dx = \int_0^u \frac{x^{k-1}e^{-x}}{(k-1)!}dx - \int_0^u \frac{x^{k-1}e^{-u}}{(k-1)!}dx \\ &= \int_0^u \frac{x^{k-1}e^{-x}}{(k-1)!}dx - \frac{u^k e^{-u}}{k!} = \int_0^u \frac{x^k e^{-x}}{k!}dx,\end{aligned}$$

which is the Erlang with parameter $k+1$. (The last step uses integration by parts in reverse. The Erlang can be obtained more laboriously by inserting the transformation of cost in (11) into equation (7).)

[8] Differentiating (12) with respect to u we get $h^{(k)}(u_k) = u_k^{k-1}e^{-u_k}/(k-1)!$, the density of $U^{(k)}$. Multiplying

tilts away from 0 relative to the uniform.

The key application of this result is to derive the distribution of the ratio of the second lowest to the lowest cost:

$$G^{(2)/(1)}(m) = \Pr\left[\frac{C^{(2)}}{C^{(1)}} \leq m | C^{(2)} = c_2\right] = 1 - m^{-\theta} \qquad m \geq 1, \tag{14}$$

which is simply the Pareto distribution, independent of $C^{(2)}$.[9] This result is very useful in modeling Bertrand competition among producers each with a competing idea for producing a homogeneous good with inelastic demand. The seller has cost $C^{(1)}$, price $C^{(2)}$, and markup $C^{(2)}/C^{(1)}$. The markup distribution is the same whatever the price.[10]

3 The Closed Economy

Up to this point we've considered the ideas for producing a particular good. We now apply our results to an economy that has a unit continuum of goods, indexed by ω.[11] We refer to ideas that apply to the same ω as delivering different varieties of good ω. In some applications these different varieties are perfect substitutes; in others they're not.

Equations (9) and (10) and the parameters θ and Φ now govern the distributions of the costs of the varieties of each good. With a unit continuum of goods we can invoke the law

(10) by this density and differentiating with respect to u_{k+1} yields the joint density of $U^{(k)}$ and $U^{(k+1)}$:

$$h^{(k,k+1)}(u_k, u_{k+1}) = \frac{u_k^{k-1}}{(k-1)!} e^{-u_{k+1}} \qquad u_k \leq u_{k+1}.$$

We get (13) by integrating along the first dimension of the joint density and conditioning on $U^{(k+1)} = u_{k+1}$:

$$\Pr\left[U^{(k)} \leq u | U^{(k+1)} = u_{k+1}\right] = \int_0^u \frac{h^{(k,k+1)}(x, u_{k+1})}{h^{(k+1)}(u_{k+1})} dx = \frac{u^k e^{-u_{k+1}}/k!}{u_{k+1}^k e^{-u_{k+1}}/k!} = \left(\frac{u}{u_{k+1}}\right)^k.$$

[9] This result is from:

$$\begin{aligned}
\Pr\left[\frac{C^{(2)}}{C^{(1)}} \leq m | C^{(2)} = c_2\right] &= \Pr\left[\frac{U^{(2)}}{U^{(1)}} \leq m^\theta | U^{(2)} = u_2\right] = \Pr\left[U^{(1)} \geq u_2 m^{-\theta} | U^{(2)} = u_2\right] \\
&= 1 - \left(\frac{m^{-\theta} u_2}{u_2}\right) = 1 - m^{-\theta},
\end{aligned}$$

[10] By contrast, from expressions (10) and (11), the distribution of the markup conditional on the lowest cost is $\Pr[C^{(2)}/C^{(1)} \leq m | C^{(1)} = c_1] = 1 - \exp\left(-(m^{1/\theta} - 1)\Phi c_1^\theta\right)$. A more efficient seller, with lower c_1, tends to charge a higher markup. Even though its markup tends to be higher, however, its price tends to be lower.

[11] Dornbusch et al. (1977) provide an early specification of technological heterogeneity across a unit continuum of goods, an approach that's been used extensively since.

of large numbers and treat the distributions from Section 2.3 as the realization across the continuum. For example: $\mu^Z(z)$ in equation (4) becomes the measure of ideas better than z across goods (as well as the Poisson parameter of the number of ideas better than z for a given good); $\mu(c)$ in equation (6) gives the measure of ideas (as well as the Poisson parameter of the number of ideas) that deliver cost below c; $G^{(1)}(c)$ in equation (8) becomes the fraction of goods (as well as the probability of a particular good) having lowest cost below c.[12]

3.1 Aggregation

In the first part of this section we assume agents interact through markets. Households as consumers and firms as users of intermediates have equal access to all producers, who charge all buyers the same price. Aggregation across goods and varieties is the same nested CES function for any buyer, with elasticity of substitution $\sigma > 0$ across goods and $\sigma' \geq \sigma$ across varieties of any good:

$$Q = \left[\int_0^1 Q(\omega)^{(\sigma-1)/\sigma} d\omega\right]^{\sigma/(\sigma-1)}; \qquad Q(\omega) = \left[\sum_{k=1}^{\infty} Q^{(k)}(\omega)^{(\sigma'-1)/\sigma'}\right]^{\sigma'/(\sigma'-1)}, \tag{15}$$

where $Q^{(k)}(\omega)$ is the quantity of the k'th lowest cost variety of good ω. One important special case has $\sigma' \to \infty$, in which case all of the varieties are perfect substitutes. Another is $\sigma' = \sigma$, in which case buyers distinguish varieties of a given good as much as varieties of different goods (so that the assignment of varieties to goods doesn't matter).

The associated price indices are:

$$P = \left[\int_0^1 P(\omega)^{1-\sigma} d\omega\right]^{1/(1-\sigma)}; \qquad P(\omega) = \left[\sum_{k=1}^{\infty} P^{(k)}(\omega)^{1-\sigma'}\right]^{1/(1-\sigma')},$$

where $P^{(k)}(\omega)$ is the price of the k'th lowest cost variety of ω. Spending on varieties and goods, given aggregate spending of $X = PQ$, is simply:

$$X^{(k)}(\omega) = \left(\frac{P^{(k)}(\omega)}{P(\omega)}\right)^{1-\sigma'} X(\omega); \qquad X(\omega) = \left(\frac{P(\omega)}{P}\right)^{1-\sigma} X. \tag{16}$$

[12] While most of the literature assumes a continuum, Eaton et al. (2013) apply the results in Section 2 to an economy with an integer number of goods.

3.2 Prices and Welfare

A feature of all the market structures we consider is that prices are homogeneous of degree 1 in costs. Hence for each good ω there is a homogeneous of degree 1 function p such that:

$$P(\omega) = p(C^{(1)}(\omega), C^{(2)}(\omega), ...).$$

Using (11) and the linear homogeneity property of p, this expression becomes:

$$P(\omega) = p(U^{(1)}(\omega)^{1/\theta}, U^{(2)}(\omega)^{1/\theta}, ...)\Phi^{-1/\theta}.$$

and the overall price index is:

$$P = \Gamma\Phi^{-1/\theta}, \tag{17}$$

where Γ doesn't depend on the state of technology T or the input cost v.[13] As we show in Appendix A, if we restrict $\sigma' > \theta + 1$ and $\sigma < \theta + 1$ we can guarantee a strictly positive Γ in any market structure in which prices weakly exceed costs. We impose these restrictions in what follows unless noted otherwise.

We can write the real input cost, using (6), as $v/P = \Gamma^{-1}T^{1/\theta}$, which is the real wage if labor is the only input. We can easily generalize the production structure to one in which inputs are a Cobb-Douglas combination of labor and intermediates, with labor having a share $\beta \in (0, 1]$. If (15) applies to intermediates then their price is also P and we can write $v = w^{\beta}P^{1-\beta}$, where w is the wage. The real wage becomes:

$$\frac{w}{P} = \Gamma^{-1/\beta}T^{1/(\theta\beta)}. \tag{18}$$

As T rises so does the real wage, with elasticity $1/(\theta\beta)$. A low θ means that ideas are drawn from a fatter right tail. A low β means that an idea contributes more since it's used not only in what consumers ultimately buy but also in intermediates embodied in what they buy.

[13] Specifically, $\Gamma = E\left[p(U^{(1)}(\omega)^{1/\theta}, U^{(2)}(\omega)^{1/\theta}, ...)^{1-\sigma}\right]^{1/(1-\sigma)}$, which depends on the parameters θ, σ, and σ'. An explicit expression for Γ requires specific assumptions about market structure, which we turn to next.

3.3 Market Structures

How do our assumptions about costs and aggregation determine market outcomes? We turn to three specific market structures that have been used in the literature. In the first, perfect competition, ideas are freely available so varieties are sold by producers at cost. In the others, Bertrand and monopolistic competition, ideas are proprietary. The producer that owns the idea for a variety has a monopoly on its use.

3.3.1 Perfect Competition

Under perfect competition, free entry guarantees that prices correspond to costs. Hence $P^{(k)}(\omega) = C^{(k)}(\omega)$ for variety k of good ω. The special case $\sigma' \to \infty$ is the standard one in the literature. Only the cheapest variety of a good is sold in equilibrium at its cost of production. The distribution of prices across goods is given by (8) above and the price index is:

$$P = \left[\int_0^\infty c^{1-\sigma} dG^{(1)}(c)\right]^{1/(1-\sigma)} = \Gamma^{PC}\Phi^{-1/\theta},$$

which is equation (17) with:

$$\Gamma^{PC} = \Gamma\left(\frac{\theta - (\sigma - 1)}{\theta}\right)^{1/(1-\sigma)},$$

where $\Gamma(z) = \int_0^\infty y^{z-1}e^{-z}dz$ is the gamma function. Restricting $\sigma < \theta + 1$ guarantees that the argument of the gamma function, and hence Γ^{PC}, is strictly positive.[14]

3.3.2 Bertrand Competition

Consider now the case in which the ideas for producing different varieties of a good are proprietary, each owned by a separate firm. We continue to send $\sigma' \to \infty$, so that consumers regard all varieties of a good as perfect substitutes. The Bertrand equilibrium leaves the market to the owner of the lowest-cost technology, but constrained not to sell at a price above the cost of the second-best technology. From (7), the second lowest-cost has the distribution:

$$G^{(2)}(c) = \Pr[C^{(2)} \leq c] = 1 - e^{-\Phi c^\theta}(1 + \Phi c^\theta).$$

[14] A high σ means that the consumer regards different goods as close substitutes. A low θ means that there's a fat tail of goods with arbitrarily low prices. If $\sigma \geq \theta + 1$, the buyer would shift spending toward these low-priced goods to the point of getting something for nothing.

With $\sigma > 1$ the seller, unfettered by competing varieties, would set the price to $\bar{m}C^{(1)}$, where:

$$\bar{m} = \frac{\sigma}{\sigma - 1}. \tag{19}$$

But competition prevents the seller from pricing above $C^{(2)}$. Hence the Bertrand price is:

$$P(\omega) = \min\left\{C^{(2)}, \bar{m}C^{(1)}\right\} = \min\left\{1, \frac{\bar{m}}{C^{(2)}/C^{(1)}}\right\} \times C^{(2)}.$$

From (14), the distribution of $C^{(2)}/C^{(1)}$ is independent of the distribution of $C^{(2)}$, letting us write the Bertrand price index as:

$$P = \left[\left(G^{(2)/(1)}(\bar{m}) + \int_{\bar{m}}^{\infty} \left(\frac{\bar{m}}{m}\right)^{1-\sigma} dG^{(2)/(1)}(m)\right) \int_0^{\infty} c^{1-\sigma} dG^{(2)}(c)\right]^{1/(1-\sigma)} = \Gamma^{BC}\Phi^{-1/\theta},$$

which is equation (17) with:

$$\Gamma^{BC} = \left[\left(1 + \frac{(\sigma-1)\bar{m}^{-\theta}}{\theta - (\sigma - 1)}\right)\Gamma\left(\frac{2\theta - (\sigma - 1)}{\theta}\right)\right]^{1/(1-\sigma)}.$$

If $\sigma \leq 1$, this expression still holds by letting $\bar{m} \to \infty$.[15]

The share of profits Π in total expenditure X is:

$$\Delta^{BC} = \frac{1}{\theta + 1}.$$

We return to this case in Section 6.2 where we study endogenous innovation.

3.3.3 Monopolistic Competition

Consider finally the case in which $1 < \sigma' = \sigma < \theta + 1$, so that buyers regard different varieties of the same good to be as distinct as different goods. The idea for each variety is again proprietary, with no owner controlling more than a measure zero of ideas. The owner of the technology will set a markup over cost of $\bar{m}$ given by (19). Since we're now violating our restriction above that $\sigma' > \theta + 1$, to insure a strictly positive price index we need to impose a maximum cost $\bar{c}$.

[15]With some effort, the reader can verify that $\Gamma^{BC} > \Gamma^{PC}$: Prices are higher under Bertrand competition than under perfect competition.

The measure of varieties with cost below c is given by (6). The price index is:

$$P = \left[\bar{m}^{1-\sigma}\int_0^{\bar{c}} c^{1-\sigma} d\mu(c)\right]^{1/(1-\sigma)} = \bar{m}\left[\frac{\theta\Phi}{\theta-(\sigma-1)}\bar{c}^{\theta-(\sigma-1)}\right]^{1/(1-\sigma)}. \tag{20}$$

Since the price index reflects both the availability of goods as well as their average price, it's decreasing in $\bar{c}$, going to zero as $\bar{c} \to \infty$.

Melitz (2003) and Chaney (2008) relate $\bar{c}$ to a fixed cost of producing a variety. Introducing such a cost, E, leaves a firm with unit cost c earning a net profit:

$$\Pi(c) = \left(1 - \frac{c}{p}\right)\left(\frac{p}{P}\right)^{1-\sigma} X - E = \frac{1}{\sigma}\left(\frac{\bar{m}c}{P}\right)^{1-\sigma} X - E, \tag{21}$$

which, since $\sigma > 1$, is decreasing in c.

If only profitable varieties are produced, (20) implies a price index:

$$P = \Gamma^{MC}\Phi^{-1/\theta},$$

which is equation (17) with:

$$\Gamma^{MC} = \bar{m}\left[\left(\frac{\theta}{\theta-(\sigma-1)}\right)\left(\frac{X}{\sigma E}\right)^{[\theta-(\sigma-1)]/(\sigma-1)}\right]^{-1/\theta}.$$

The measure of active varieties is:

$$\mu(\bar{c}) = \frac{\theta-(\sigma-1)}{\theta}\cdot\frac{X}{\sigma E}.$$

Unlike the other market structures we've considered, market size X matters. The reason is the fixed cost of producing a variety. A larger market has more variety and a lower price index (even though the average price of a variety is higher).

Profits net of fixed costs Π have a share in expenditure X of $\Delta^{MC} = 1/(\theta\bar{m})$. Note that a higher markup $\bar{m}$ implies a lower profit share Δ^{MC}. The reason is that, with a higher markup, there's more entry, eating up profit through entry costs.

3.3.4 Cournot and Bertrand Competition with Imperfect Substitutes

Moving away from the extremes of $\sigma' \to \infty$ and $\sigma' = \sigma$, we lose any closed-form solutions. Atkeson and Burstein (2008) provide expressions for the markup in this more general setting with both Bertrand and Cournot competition. In either case it lies between $\sigma'/(\sigma' - 1)$ and $\sigma/(\sigma - 1)$, moving closer to the second as the firm's market share within a good rises. Their results deliver a neat algorithm for numerical solution.

3.3.5 Discussion

We began in Section 2 with a characterization of the ideas available to an economy for production. We've now examined three specific formulations of how those ideas can be employed in a market economy. If varieties of a good are perfect substitutes, only the best gets used. If varieties are imperfect substitutes, all ideas are useful. To bound utility, we need to restrict $\sigma' > \theta + 1$, so that bad ideas have little presence, or to impose an entry cost, so that only ideas crossing an entry cutoff see the light of day. We now turn to a matching economy: whether an idea gets used now involves an element of luck.

3.4 Matching

So far we've treated markets as Walrasian: Firms buy the representative continuum of goods as intermediates and households buy them as final goods at common prices. The matching literature has recognized that individual buyers may have idiosyncratic needs that only certain sellers can satisfy, and that any particular buyer-seller pair may have trouble hooking up. In fact, data on purchases indicate that most buyers purchase from only a handful of sellers and most firms sell to only a small number of buyers. A reformulation of the structure considered so far is able to capture the sparseness of buyer-seller interactions.

We develop a very stylized model to illustrate equilibrium in an economy with idiosyncratic buyer-seller matching.[16] As in the models of imperfect competition above, there's a one-to-one mapping between an idea and the firm that produces with it. In order to make something, a producer needs to perform two tasks. The first uses labor (with Cobb-Douglas share β) and the second an intermediate. With a wage w and an intermediate costing C, a producer with

[16] The formulation here simplifies Eaton et al. (2022), which captures features of cross-country buyer-seller interactions in the European Union. Lenoir et al. (2022) extend the model to capture the product dimension.

efficiency Z has unit cost:

$$c = \frac{1}{Z} w^{\beta} C^{1-\beta}. \tag{22}$$

The labor market remains Walrasian: Any producer can hire homogeneous labor at the same wage w. But procuring an intermediate occurs by random matching. A buyer encounters only a handful of potential suppliers, choosing the cheapest among them.

In our matching economy households buy goods through retailers. A retailer is just like a producer in buying one variety from the cheapest supplier it's encountered. It sells to households at cost. Households have symmetric Cobb-Douglas scale-independent preferences over the continuum of retailers.[17] For parsimony the measure of retailers equals the measure of households L.

3.4.1 The Matching Function

Individual buyers and sellers meet through random matches.[18] A seller is a potential producer while a buyer can be an active firm seeking an intermediate or a retailer. The measure B of buyers is thus the sum of the exogenous measure L of retailers and the endogenous measure F of firms, derived below.

The matching intensity between a buyer and a seller with cost c is:

$$\lambda(c) = \lambda B^{-\varphi} S(c)^{-\nu}, \tag{23}$$

where λ is a parameter capturing how easy it is for buyers and sellers to meet. As is common in the matching literature, the matching intensity between any given buyer and seller may fall as the pool of participants on either side gets more crowded. We capture the presence of buyers simply with their measure B and the presence of sellers with $S(c) = \lambda\mu(c)$, the measure

[17] Specifically, with a measure R of retailers indexed by j, the utility function is:

$$U = R \ \exp\left(\frac{1}{R}\int_0^R \ln Q(j)dj\right).$$

With prices drawn independently from a common distribution $G(p)$, the price index is:

$$P = \exp\left(\frac{1}{R}\int_0^R \ln P(j)dj\right) = \exp\left(\int_0^\infty \ln p \ dG(p)\right).$$

[18] Our analysis builds on the classic matching model of the labor market in Mortensen and Pissarides (1994). Petrongolo and Pissarides (2001) review approaches to matching.

with cost below c, weighted by λ. The parameters φ and ν are the elasticities of congestion with respect to buyer and seller presence. No congestion is $\varphi = \nu = 0$: the intensity with which any given buyer and seller meet is independent of scale, implying that a given buyer or seller encounters proportionately more potential partners in a larger economy.

3.4.2 The Cost Distribution

The number of potential sellers with cost below c a buyer encounters is distributed Poisson with parameter:

$$\rho(c) = \int_0^c \lambda(c')d\mu(c') = \frac{1}{1-\nu}B^{-\varphi}S(c)^{1-\nu}, \tag{24}$$

the matching intensity integrated over the measure of sellers. The probability of encountering no seller with cost below c is $e^{-\rho(c)}$. Hence the distribution of cost for the low-cost seller is:

$$G^{(1)}(c) = 1 - e^{-\rho(c)}. \tag{25}$$

Here we assume the buyer purchases the good at that cost, mimicking perfect competition by giving the buyer all the bargaining power when matched with a seller. Hence, $G^{(1)}$ is the price distribution faced by the buyer.[19]

A buyer with efficiency Z and an input cost C itself has a cost given by (22). Given C, the measure of sellers who have cost below c is, from (4), $T\left(w^{\beta}C^{1-\beta}\right)^{-\theta}c^{\theta}$. Integrating over the distribution of C, the measure of producers who can deliver at cost below c is:

$$\mu(c) = Tc^{\theta}w^{-\beta\theta}\int_0^{\infty}(c')^{-\theta(1-\beta)}dG^{(1)}(c') = Tv^{-\theta}c^{\theta}, \tag{26}$$

where:

$$v^{-\theta} = w^{-\theta\beta}\int_0^{\infty}(c')^{-\theta(1-\beta)}dG^{(1)}(c')$$

reflects the common cost of labor and the cost of intermediates as averaged over the experience of individual producers.

Installing $S(c) = \lambda\mu(c) = \lambda\Phi c^{\theta}$ into the expression for $\rho(c)$ above, gives:

$$G^{(1)}(c) = 1 - \exp\left(-\frac{B^{-\varphi}}{1-\nu}\left(\tilde{\Phi}c^{\theta}\right)^{1-\nu}\right)$$

[19] Fontaine et al. (2023) consider Bertrand competition, which we introduce Section 6.2 to generate an incentive to innovate. The price distribution is then $G^{(2)}(c) = 1 - (1+\rho(c))e^{-\rho(c)}$.

where:

$$\tilde{\Phi} = \lambda \Phi = \lambda T v^{-\theta}.$$

As in the market economies considered above, Φ captures how the accumulation of ideas, as well as input costs, affect prices. But in this matching economy, the ability to access these ideas, as reflected in λ, matters as well.[20]

Using the expression for $G^{(1)}(c)$, the input cost term v solves:

$$v^{-\theta} = \Gamma\left(\frac{\beta-\nu}{1-\nu}\right) w^{-\theta\beta}\left(\left(\frac{B^{-\varphi}}{1-\nu}\right)^{1/(1-\nu)} \lambda T v^{-\theta}\right)^{1-\beta}. \tag{27}$$

Households also face the distribution $G^{(1)}(c)$, so the price index, from footnote 17, is:

$$P = \exp\left(\int_0^\infty \ln c \, dG^{(1)}(c)\right) = \Gamma^M \tilde{\Phi}^{-1/\theta}; \qquad \Gamma^M = \left(\frac{e^\gamma B^{-\varphi}}{1-\nu}\right)^{-1/[(1-\nu)\theta]},$$

where γ is the Euler-Mascheroni constant.[21]

3.4.3 The Measure of Firms

How many ideas turn into active producers? Consider a producer with cost c. Its number of buyers is distributed Poisson with parameter:

$$\eta(c) = \lambda(c) B\left(1 - G^{(1)}(c)\right), \tag{28}$$

where the first two terms govern the number of matches and the last the probability that a match results in a sale.

To be an active firm a producer must make a sale. Hence the measure of active firms is:

$$F = \int_0^\infty \left(1 - e^{-\eta(c)}\right) d\mu(c), \tag{29}$$

that is, potential producers with at least one customer. With the change of variable $x =$

[20] This matching economy resembles Oberfield (2018). Prices depend not only on firms' efficiencies but on their ability to buy from one another.

[21] This last result uses the Laplace transform, $F(s) = \int_0^\infty \ln t e^{-st} dt = -\frac{\ln s + \gamma}{s}$, evaluated at $s = 1$.

$B^{-\varphi/(1-\nu)}\tilde{\Phi}c^{\theta}$, we can write $\eta(c) = \lambda\eta^*(x)$, where:

$$\eta^*(x) = B^{1-\varphi/(1-\nu)}x^{-\nu}e^{-x^{1-\nu}/(1-\nu)}.$$

Substituting $\lambda\eta^*(x)$ into (29) gives a simple expression for the ratio of firms to the total measure of buyers $B = F + L$:

$$\frac{F}{B} = \frac{1}{\lambda B^{1-\varphi/(1-\nu)}} \int_0^\infty \left(1 - e^{-\lambda\eta^*(x)}\right) dx.$$

How does this ratio respond to changes in matching efficiency λ and to the scale of the economy? Differentiating with respect to λ shows that the elasticity of F/B with respect to λ is:

$$\frac{d\ln F/B}{d\ln\lambda} = \frac{F(1)}{F} - 1 \leq 0,$$

where:

$$\frac{F(1)}{B} = \frac{1}{B^{1-\varphi/(1-\nu)}} \int_0^\infty \eta^*(x)e^{-\lambda\eta^*(x)}dx$$

is the ratio of firms with *exactly* one buyer to total buyers. The measure of firms declines monotonically in λ as buyers converge on lower cost producers. With constant returns to scale in matching, $\varphi + \nu = 1$, the ratio of firms to buyers depends only on the parameters λ and ν, so is independent of scale L. Increasing returns, $\varphi + \nu < 1$, implies fewer firms per retailer in a larger economy: A buyer meets more potential sellers, so sales become more concentrated among more efficient producers.

4 The Open Economy

Building on the analysis in Section 3, we introduce $\mathcal{N}$ locations and trade between them. We characterize a location i by its state of technology T_i, its labor endowment L_i, and the cost of shipping goods to other locations. Adopting Samuelson's classic iceberg formulation, delivering one unit of a good to destination n requires shipping d_{ni} units from i (with $d_{ii} = 1$).

Say that an input bundle in i costs v_i. Focusing on a particular good ω, an idea from i can deliver the good to n at cost below c if its efficiency Z exceeds $v_i d_{ni}/c$. From (4), the number of ideas from i that do so is distributed Poisson with parameter:

$$\mu_{ni}(c) = \mu_i^Z\left(\frac{v_i d_{ni}}{c}\right) = T_i(v_i d_{ni})^{-\theta}c^{\theta}. \tag{30}$$

The number of ideas from **anywhere** delivering the good at cost below c is distributed Poisson with parameter:

$$\mu_n(c) = \Phi_n c^{\theta}; \qquad \Phi_n = \sum_{i=1}^{\mathcal{N}} T_i(v_i d_{ni})^{-\theta}, \tag{31}$$

the open-economy analog to equation (6). From the properties of the Poisson, the probability that the idea comes from source i is just i's contribution to $\mu_n(c)$, or

$$\frac{\mu_{ni}(c)}{\mu_n(c)} = \frac{T_i(v_i d_{ni})^{-\theta}}{\Phi_n} = \frac{T_i(v_i d_{ni})^{-\theta}}{\sum_{i'} T_{i'}(v_{i'} d_{ni'})^{-\theta}}. \tag{32}$$

Note that c drops out. The implication is that, whatever c an idea delivers, the probability that it's from i is $T_i(v_i d_{ni})^{-\theta}/\Phi_n$.

Consider a source i with ideas for a good of efficiencies $Z_i^{(1)} > Z_i^{(2)} > ....$ These ideas can deliver to destination n at costs $C_{ni}^{(1)} < C_{ni}^{(2)} < ...$, where $C_{ni}^{(k)} = v_i d_{ni}/Z_i^{(k)}$. We can combine all the ideas from any source delivering the good to destination n and reorder them by cost there as $C_n^{(1)} < C_n^{(2)} < ...$, starting with $C_n^{(1)} = \min_i\{C_{ni}^{(1)}\}$. Replacing Φ with Φ_n, equation (31) allows us to apply the results from Section 2 to ordered costs in destination n. The probability that $C_n^{(k)}$ is delivered by source i is $T_i(v_i d_{ni})^{-\theta}/\Phi_n$. Since this probability doesn't depend on c, the joint distribution of the ordered costs in n does not depend on which source countries deliver at those costs.

4.1 Market Economies

Following Section 3.2, if the price index for each good ω in destination n is homogeneous of degree 1 in costs there, irrespective of the origin of the varieties of that good, then:

$$P_n(\omega) = p(C_n^{(1)}(\omega), C_n^{(2)}(\omega), ...) = p(U_n^{(1)}(\omega)^{1/\theta}, U_n^{(2)}(\omega)^{1/\theta}, ...)\Phi_n^{-1/\theta},$$

where the U's follow the distributions in equations (9) and (10). The cases of perfect competition, monopolistic competition, and Bertrand competition from Section 3.3 each deliver analytic solutions for aggregate outcomes.

With these market structures, the law of large numbers implies that the probability that a variety of any good available in n comes from i, $T_i(v_i d_{ni})^{-\theta}/\Phi_n$ in equation (32), corresponds to the share π_{ni} of source i in n's purchases across goods.

We can connect the value of absorption X_n in destination n with the value of production Y_i in source i with the equations:

$$Y_i = \sum_{n=1}^{N} \pi_{ni} X_n. \tag{33}$$

With balanced trade and assuming that we're accounting for the universe of goods in the economy, we can write:

$$Y_i = X_i = w_i L_i + \Pi_i = \frac{1}{1 - \Delta^j} w_i L_i,$$

where w_i is the wage in location i, L_i its labor force, Π_i profits both earned and generated there, and Δ^j; $j \in \{PC, BC, MC\}$, one of the profit shares given in Section 3.3 (with $\Delta^{PC} = 0$).[22] Continuing with the Cobb-Douglas specification for how labor and intermediates combine, so that $v_i = w_i^{\beta} P_i^{1-\beta}$, world equilibrium implies the two sets of conditions:

$$w_i L_i = \sum_{n=1}^{\mathcal{N}} \frac{T_i(w_i^{\beta} P_i^{1-\beta} d_{ni})^{-\theta}}{\sum_{i'} T_{i'}(w_{i'}^{\beta} P_{i'}^{1-\beta} d_{ni'})^{-\theta}} w_n L_n; \qquad P_n = \Gamma^j \left(\sum_{i=1}^{\mathcal{N}} T_i(w_i^{\beta} P_i^{1-\beta} d_{ni})^{-\theta} \right)^{-1/\theta},$$

with Γ^j from Section 3.3. Notice that Δ^j drops out. These equations deliver wages w_i and price indices P_i around the world (subject to a choice of numéraire), given the states of technologies T_i, iceberg costs d_{ni}, and parameters θ and β.[23]

4.2 Matching Economies

We can extend the matching framework introduced in Section 3.4 to an open economy in which a buyer's frequency of encounters with potential sellers may depend on the origin of the seller. We generalize the closed economy matching function (23), specifying the intensity

[22] Deficits and nontraded goods complicate the accounting. We refer the interested reader to Dekle et al. (2007) for a treatment.

[23] Eaton and Kortum (2002) develop and quantify a version of this specification with perfect competition. Costinot et al. (2012) and Caliendo and Parro (2015) extend it to multiple sectors, the first to analyze sectoral comparative advantage and the second to quantify the welfare effects of NAFTA. Bernard et al. (2003) assume Bertrand competition, relating the analysis to the export participation of U.S. manufacturing plants. Melitz (2003) introduces monopolistic competition, later quantified by Chaney (2008) and Eaton et al. (2011), the second using firm-level data.

of matching between a buyer in n and seller from i with cost c in n as:

$$\lambda_{ni}(c) = \lambda_{ni} B_n^{-\varphi} S_n(c)^{-\nu}; \qquad S_n(c) = \sum_i \lambda_{ni} \mu_{ni}(c), \tag{34}$$

where λ_{ni} is a parameter reflecting the intensity of bilateral matching. As in the closed economy, buyer congestion depends on $B_n = F_n + L_n$, active producers plus retailers in the destination. Local supplier congestion depends on the presence of potential producers with cost below c from each source, weighted by their source-specific matching intensity.

Parallel to (24), the number of encounters with sellers from anywhere with cost below c is distributed Poisson with parameter:

$$\rho_n(c) = \frac{1}{1-\nu} B_n^{-\varphi} S_n(c)^{1-\nu}.$$

Replacing $\rho(c)$ with $\rho_n(c)$ in (25) gives the distribution of the lowest cost in n, $G_n^{(1)}(c)$.

Solving as in (26) and what follows:

$$G_n^{(1)}(c) = 1 - \exp\left(-\frac{B_n^{-\varphi}}{1-\nu} \left(\tilde{\Phi}_n c^{\theta} \right)^{1-\nu} \right),$$

where:

$$\tilde{\Phi}_n = \sum_i \lambda_{ni} T_i (v_i d_{ni})^{-\theta}. \tag{35}$$

The measure of suppliers from i with cost below c in n, remains (30). Input costs, parallel to (27), solve:

$$v_n^{-\theta} = \Gamma\left(\frac{\beta-\nu}{1-\nu}\right) w_n^{-\theta\beta} \left(\left(\frac{B_n^{-\varphi}}{1-\nu} \right)^{1/(1-\nu)} \sum_i \lambda_{ni} T_i (v_i d_{ni})^{-\theta} \right)^{1-\beta}. \tag{36}$$

This expression delivers the v's given the B's and w's, which are endogenous. The share of sales that go to sellers from i in destination n is i's contribution to $\tilde{\Phi}_n$:

$$\pi_{ni} = \frac{\lambda_{ni} T_i (v_i d_{ni})^{-\theta}}{\tilde{\Phi}_n} = \frac{\lambda_{ni} d_{ni}^{-\theta} T_i v_i^{-\theta}}{\sum_{i'} \lambda_{ni'} d_{ni'}^{-\theta} T_{i'} v_{i'}^{-\theta}}. \tag{37}$$

As before, c drops out. Note how bilateral matching intensities λ_{ni} joins iceberg costs $d_{ni}^{-\theta}$ in governing trade shares. The price index is the same as in the closed economy, Section 3.4.2, with $\tilde{\Phi}_n$ replacing $\tilde{\Phi}$.

To solve for B's, consider first a producer in i with a cost c in destination n. Generalizing (28), its number of buyers there is distributed Poisson with parameter:

$$\eta_{ni}(c) = \lambda_{ni}(c) B_n \left(1 - G_n^{(1)}(c)\right).$$

If its cost at home is c, its cost in n is cd_{ni} Hence its number of buyers anywhere is distributed Poisson with parameter:

$$\eta_i(c) = \sum_n \eta_{ni}(cd_{ni}).$$

To be active, a producer needs at least one customer somewhere, so that the measure of active producers, parallel to (29), is:

$$F_i = \int_0^\infty \left(1 - e^{-\eta_i(c)}\right) d\mu_{ii}(c), \tag{38}$$

where, from (30), $\mu_{ii}(c)$ is the measure of potential producers in i with cost below c. Since $B_i = F_i + L_i$, the system (38) determines the measure of active producers around the world, and hence buyers, given the w's and v's. Finally, the w's solve the labor market equilibrium conditions which, imposing balanced trade, are:

$$w_i L_i = \sum_{n=1}^{\mathcal{N}} \frac{\lambda_{ni} T_i (v_i d_{ni})^{-\theta}}{\sum_{i'} \lambda_{ni'} T_{i'} (v_{i'} d_{ni'})^{-\theta}} w_n L_n, \tag{39}$$

given the v's. In summary, (36), (38), and (39) jointly determine the v's, B's, and w's.

The framework also delivers the measure of exporters from i to n, F_{ni}, which is simply the producers in i who have at least one customer in n, or $F_{ni} = \int_0^\infty \left(1 - e^{-\eta_{ni}(c)}\right) d\mu_{ni}(c)$. Proceeding as in Section 3.4, $\eta_{ni}(c) = \lambda_{ni} \eta_n^*(x)$, where x and $\eta_n^*(x)$ are the same as for the closed economy after replacing B with B_n and $\tilde{\Phi}$ with $\tilde{\Phi}_n$. The fraction of buyers in n served by sellers from i, F_{ni}/B_n, relative to the fraction of spending in n on goods from i, π_{ni}, is:

$$\frac{F_{ni}/B_n}{\pi_{ni}} = \frac{1}{\lambda_{ni} B_n^{1-\varphi/(1-\nu)}} \int_0^\infty \left(1 - e^{-\lambda_{ni} \eta_n^*(x)}\right) dx. \tag{40}$$

The ratio falls as bilateral matching intensity λ_{ni} rises, and is invariant to iceberg costs. We saw in (37) that λ_{ni} and $d_{ni}^{-\theta}$ both contribute to trade shares. The result above shows that when the trade share is larger because of greater matching intensity, the effect is disproportionately due to more sales per exporter rather than to more exporters.

5 Dynamic Matching

Section 2 introduced our primitives: ideas and their accumulation. So far we've taken the presence of these ideas as given, ignoring the timing of their arrival. To make this timing explicit, we now assume a constant common growth rate of ideas in each location, $\dot{T}_i/T_i = g$. We make g endogenous in Section 6. Hence the measure of ideas with efficiency greater than z aged a at date t in location i is:

$$\mu_i^Z(z; t, a) = ge^{-ga}T_i(t)z^{-\theta},$$

the dynamic analog of (4). Each idea is associated with a potential producer.

Into this environment we embed the open-economy matching framework developed in Section 4.2. To focus on dynamics, we eliminate buyer and seller congestion in matching. Instead, a buyer's or seller's ability to match improves with age. The intensity of matching between a buyer in location n aged a_b and a seller from location i aged a_s is $a_b a_s \lambda_{ni}$.[24]

Producers differ not only by their efficiency z and origin i, but also by their age a. We denote the measure of potential producers from i with unit cost below c in n aged a at date t as $\mu_{ni}(c; t, a)$. The number of encounters with suppliers from any location of any age with cost below c by a buyer aged a_b in location n is thus distributed Poisson with parameter:

$$a_b \rho_n(c; t) = a_b \sum_{i=1}^{\mathcal{N}} \lambda_{ni} \int_0^\infty a_s \mu_{ni}(c; t, a_s) da_s.$$

An older buyer encounters more suppliers, so its cheapest option tends to be lower. The distribution of the lowest cost that a buyer aged a in location i at date t encounters is thus:

$$G_i^{(1)}(c; t, a) = 1 - e^{-a\rho_i(c;t)}.$$

A producer in this cohort with input cost C and efficiency Z has cost itself given by (22), so

[24] Contacts with better suppliers and more buyers drive firm growth, while firms' technology is heterogeneous but unchanging. In contrast, Luttmer (2007) models firms' heterogeneous evolution of technology. Luttmer (2010) surveys the literature on firm dynamics.

that the measure of potential producers in i aged a with cost below c is:

$$\mu_{ii}(c;t,a) = \int_0^\infty \mu_i^Z\left(w_i^\beta (c')^{1-\beta}/c;t,a\right) dG_i^{(1)}(c';t,a) = T_i(t)v_i(t,a)^{-\theta}c^\theta, \tag{41}$$

where $v_i(t,a)$ summarizes wage and intermediate costs for a producer aged a in location i, which falls as the producer ages. It follows that $\mu_{ni}(c;t,a) = d_{ni}^{-\theta}T_i(t)v_i(t,a)^{-\theta}c^\theta$.

We can now write $\rho_n(c;t) = \tilde{\Phi}_n(t)c^\theta$ where, parallel to (35):

$$\tilde{\Phi}_n(t) = \sum_{i=1}^{\mathcal{N}} \lambda_{ni}d_{ni}^{-\theta}T_i(t)v_i(t)^{-\theta}; \qquad v_i(t)^{-\theta} = \int_0^\infty a v_i(t,a)^{-\theta}da, \tag{42}$$

where $v_i(t)$ aggregates across suppliers of different ages to capture overall input costs in i.

Using $G_n^{(1)}(c;t,a)$ to solve (41), we get:

$$v_i(t,a)^{-\theta} = \Gamma(\beta)ge^{-ga}w_i^{-\theta\beta}\left(a\tilde{\Phi}_i(t)\right)^{1-\beta} = \frac{g^{3-\beta}a^{1-\beta}e^{-ga}}{\Gamma(3-\beta)}v_i(t)^{-\theta},$$

where the $v(t)$'s solve:

$$v_n(t)^{-\theta} = \Gamma(\beta)\Gamma(3-\beta)g^{\beta-2}w_n^{-\theta\beta}\left(\sum_{i=1}^{\mathcal{N}}\lambda_{ni}d_{ni}^{-\theta}T_i(t)v_i(t)^{-\theta}\right)^{1-\beta} \tag{43}$$

conditional on the w's, which we treat as constant over time. From (42) and (43) we can infer that the $v^{-\theta}$'s grow at rate $(1-\beta)g/\beta$ and that the $\tilde{\Phi}$'s grow at rate g/β. The price index is $\Gamma^M\tilde{\Phi}_n(t)^{-1/\theta}$, as in the static matching economy of Section 3.4, but falling at rate $g/(\theta\beta)$.

Together (39) and (43) determine the v's, the w's, and a stationary bilateral trade share:

$$\pi_{ni} = \frac{\lambda_{ni}d_{ni}^{-\theta}T_i(t)v_i(t)^{-\theta}}{\tilde{\Phi}_n(t)}, \tag{44}$$

the equivalent of (37) above. Hence on a balanced growth path this dynamic economy closely resembles the static one in the previous section.

5.1 Firms and Their Customers

Solving for the measures of active firms is challenging since a producer's cost in a market interacts with a buyer's age. Older buyers typically have more options to choose among than younger buyers so are less likely to buy from a high-cost supplier.

Denote the measure of active producers aged a in destination n at date t as $F_n(t, a)$ and the measure of retailers aged a as $L_n(t, a)$. The number of customers of any age in destination n of a seller with cost c there, aged a_s from source i, is distributed Poisson with parameter:

$$a_s \eta_{ni}(c; t) = a_s \lambda_{ni} \int_0^\infty a_b \left[F_n(t, a_b) + L_n(t, a_b)\right] \left(1 - G_n^{(1)}(c; t, a_b)\right) da_b. \quad (45)$$

To be active as a buyer a producer aged a_b needs at least one buyer itself. Its number of buyers is distributed Poisson with parameter $a_b \eta_n(c; t)$ where $\eta_n(c; t) = \sum_m \eta_{mn}(cd_{mn}; t)$. The measure of active producers aged a_b in n is thus:

$$F_n(t, a_b) = \int_0^\infty \left(1 - e^{-a_b \eta_n(c;t)}\right) d\mu_{nn}(c; t, a_b).$$

We assume a fixed population of retailers L_n, which turn over with hazard δ. Hence the measure aged a_b is $L_n(t, a_b) = \delta e^{-\delta a_b} L_n$.

Applying these results on buyers, the term $\eta_{ni}(c; t)$ in (45) becomes:

$$\eta_{ni}(c; t) = \lambda_{ni} \int_0^\infty a \left[T_n(t) v_n(t; a)^{-\theta} \int_0^\infty \left(1 - e^{-a\eta_n(c';t)}\right) \theta (c')^{\theta-1} dc' + \delta L_n e^{-\delta a}\right] e^{-a\tilde{\Phi}_n(t)c^\theta} da.$$

As we show in Appendix B, integrating over buyer age, a, gives:

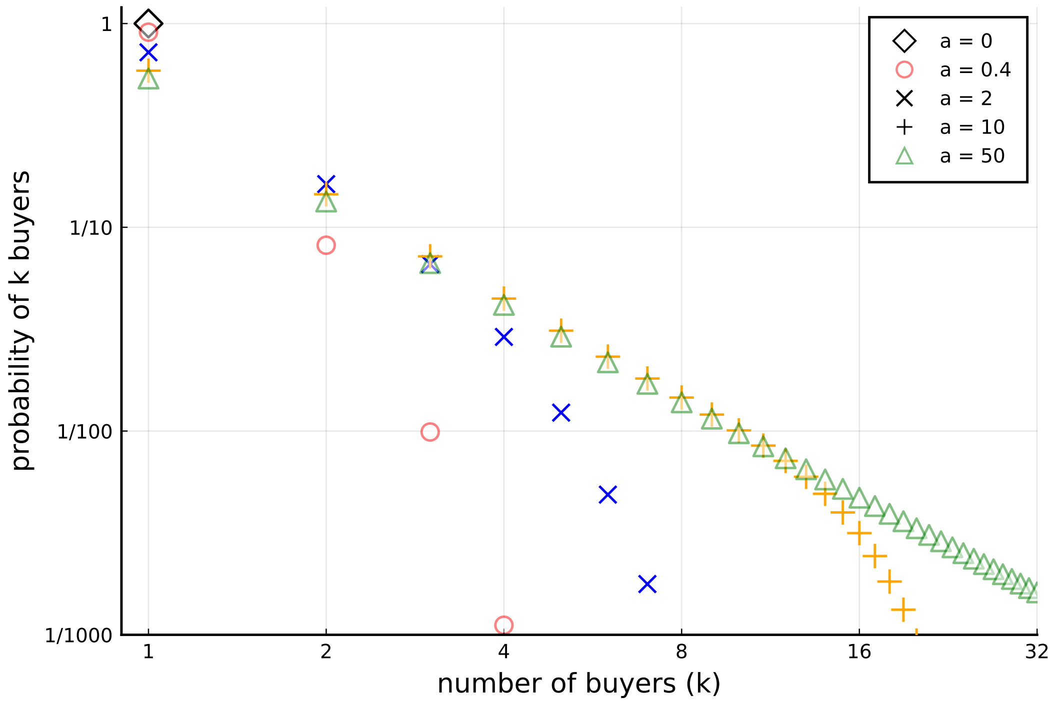
$$\begin{aligned} \eta_{ni}(c; t) &= \lambda_{ni} T_n(t) v_n(t)^{-\theta} \int_0^\infty \left[\left(\frac{g}{g + \tilde{\Phi}_n(t)c^\theta}\right)^{3-\beta} - \left(\frac{g}{g + \eta_n(c'; t) + \tilde{\Phi}_n(t)c^\theta}\right)^{3-\beta}\right] \theta (c')^{\theta-1} dc' \\ &\quad + \frac{\delta}{\left(\delta + \tilde{\Phi}_n(t)c^\theta\right)^2} \lambda_{ni} L_n. \end{aligned}$$

To simplify these functions, while also removing their dependence on time, we employ the changes of variables $x = \tilde{\Phi}_n(t)c^\theta$ and $y = T_n(t) v_n(t)^{-\theta} (c')^\theta$. We get $\eta_{ni}(c; t) = \lambda_{ni} \eta_n^*(x)$ and $\eta_n(c; t) = \bar{\eta}_n(y)$, where:

$$\eta_n^*(x) = \int_0^\infty \left[\left(\frac{g}{g + x}\right)^{3-\beta} - \left(\frac{g}{g + \bar{\eta}_n(y) + x}\right)^{3-\beta}\right] dy + \frac{\delta}{(\delta + x)^2} L_n. \quad (46)$$

and $\bar{\eta}_n(y) = \sum_m \lambda_{mn} \eta_m^* \left(\lambda_{mn} y / \pi_{mn}\right)$. We can solve the $\mathcal{N}$ equations numerically for $\eta_n^*(x)$ given g, δ, β, L_n, λ_{ni}, and π_{ni}. The solution gives us the Poisson parameter for the number of

Figure 1: Size Distribution of Firms, by Age



The figure shows $q_{ii}(k;a)$ with $\mathcal{N} = 3$, $g = 0.08$, $\delta = 0.1$, $\beta = 1/3$, $L_n = 1$ for all n, $\lambda_{ni} = 0.05$ for $n \neq i$, $\lambda_{ii} = 0.1$, $\pi_{ni} = 0.2$ for $n \neq i$, and $\pi_{ii} = 0.6$.

buyers in n of a producer aged a from i with cost $c = (x/\tilde{\Phi}_n(t))^{1/\theta}$ there, $a\eta_{ni}(c;t) = a\lambda_{ni}\eta_n^*(x)$.

We can use this result to derive the age-specific size distribution of firms, which is time-invariant. Dividing by the measure of all firms aged a from i selling in n irrespective of cost, the fraction having exactly k customers is:

$$q_{ni}(k;a) = \frac{1}{k!}\frac{\int (a\lambda_{ni}\eta_n^*(x))^k e^{-a\lambda_{ni}\eta_n^*(x)}dx}{\int (1 - e^{-a\lambda_{ni}\eta_n^*(x)})\, dx}.$$

Since $q_{ni}(k;a)$ depends on age only via $a\lambda_{ni}$, a higher matching intensity with buyers in n means sellers from i effectively age faster in that market. Figure 1 shows this distribution in the home market (for an example with three symmetric locations) as it evolves from a spike at 1 for the youngest firms toward the shape of a Pareto as they age.

5.2 The Age of Producers

A number of results about the age distribution of potential producers emerge.

1. Our specification of evolving technology directly implies that the age of potential pro-

ducers with efficiency above z in country i has exponential density:

$$f_1(a) = \frac{\mu_i^Z(z;t,a)}{\int \mu_i^Z(z;t,a')da'} = \frac{ge^{-ga}T_i(t)z^{-\theta}}{T_i(t)z^{-\theta}} = ge^{-ga}.$$

The values of z and i don't matter. The mean and variance are thus $1/g$ and $1/g^2$. The density falls with age.

2. But newly-born producers have little access to inputs so tend to be high cost. Counting only potential producers from i with cost below c in n, the age density becomes:

$$f_2(a) = \frac{\mu_{ni}(c;t,a)}{\int \mu_{ni}(c;t,a')da'} = \frac{T_i(t)d_{ni}^{-\theta}v_i(t,a)^{-\theta}}{T_i(t)d_{ni}^{-\theta}\int v_i(t,a'))^{-\theta}da'} = \frac{g^{2-\beta}a^{1-\beta}e^{-ga}}{\Gamma(2-\beta)},$$

which is gamma with parameters $2-\beta$ and $1/g$. The values of c, n, and i don't matter. The mean and variance are $(2-\beta)/g$ and $(2-\beta)/g^2$. While a potential producer's efficiency z is set at birth, its cost falls as it ages and finds better suppliers. The density peaks at $a = (1-\beta)/g$, proportional to the share of intermediates in production.

3. Since to be a firm a potential producer must have at least one buyer, the age density of firms from i actively selling in n is:

$$f_{ni}(a) = \frac{f_2(a)\int \left(1-e^{-a\lambda_{ni}\eta_n^*(x)}\right)dx}{\int\int f_2(a')\left(1-e^{-a'\lambda_{ni}\eta_n^*(x)}\right)dxda'}.$$

As we show in Appendix C, this density peaks where $q_{ni}(1;a) = ga-(1-\beta)$, which is above $(1-\beta)/g$ since $q_{ni}(1;a) > 0$.

4. Weighting by number of buyers, the age density of firms from i selling in n is:

$$f_3(a) = \frac{\int a\eta_{ni}(c;t)d\mu_{ni}(c;t,a)}{\int\int a'\eta_{ni}(c';t)d\mu_{ni}(c';t,a')da'} = \frac{av_i(t,a)^{-\theta}}{v_i(t)^{-\theta}} = \frac{g^{3-\beta}a^{2-\beta}e^{-ga}}{\Gamma(3-\beta)},$$

which is gamma with parameters $3-\beta$ and $1/g$. Again, the values of c, n, and i don't matter. The mean and variance are $(3-\beta)/g$ and $(3-\beta)/g^2$. The density peaks at $a = (2-\beta)/g$. Even when a cohort begins to decline as a proportion of all sellers, its market share is still rising. We use this density in calculating the value of invention in Section 6.2.3.[25]

[25] The measure of buyers in n procuring a good from a seller aged a_s in i is $f_3(a_s)\pi_{ni}(F_n+L_n)$. Dividing

6 Growth

The previous section posited that ideas everywhere grow at a common rate g. In this section we first tie g to innovation in each location and to the diffusion of ideas across locations. We then relate innovation in each location to the incentive to do research there, as in the literature on endogenous growth.[26]

6.1 Technology Dynamics

We now need to distinguish the state of technology available in each location n at date t, $T_n(t)$, and the technology *generated* there, which we denote $T_n^*(t)$. As in Krugman (1979), new technology is generated in location i in proportion to the state of technology available there, with an innovation rate ι_i:[27]

$$\dot{T}_i^*(t) = \iota_i T_i(t). \tag{47}$$

Denote by $T_{ni}(t)$ the technology available in n that originated from i. As in Nelson and Phelps (1966) and Krugman (1979), it evolves in proportion to the gap between $T_i^*(t)$ and $T_{ni}(t)$ at rate ϵ_{ni} (the inverse of the diffusion lag):

$$\dot{T}_{ni}(t) = \epsilon_{ni}\left(T_i^*(t) - T_{ni}(t)\right). \tag{48}$$

With balanced growth, all states of technology grow at rate g. We can solve for:

$$\frac{T_{ni}(t)}{T_i^*(t)} = \frac{\epsilon_{ni}}{g+\epsilon_{ni}}; \qquad \frac{T_i^*(t)}{T_i(t)} = \frac{\iota_i}{g},$$

by the dynamic analogue of (40), buyers per seller is:

$$\frac{f_3(a_s)\pi_{ni}(F_n+L_n)}{(f_3(a_s)/a_s)(\pi_{ni}/\lambda_{ni})\int\left(1-e^{-a_s\lambda_{ni}\eta_n^*(x)}\right)dx} = \frac{a_s\lambda_{ni}(F_n+L_n)}{\int\left(1-e^{-a_s\lambda_{ni}\eta_n^*(x)}\right)dx}.$$

Note that buyers per seller aged a_s from source i in n doesn't depend on d_{ni} and rises with the product $a_s\lambda_{ni}$.

[26]Shell (1966) is an early contribution with Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) at the forefront of a later wave of research. Rivera-Batiz and Romer (1991) and Grossman and Helpman (1993) introduced open economy endogenous growth models. The analysis here relates closely to Eaton and Kortum (1999), Eaton and Kortum (2001), Buera and Oberfield (2020), and Cai et al. (2022a). While we model growth in a matching economy, growth is the consequence of explicit research activity, not luck in bumping into better-informed agents, as in the idea flow literature reviewed by Buera and Lucas (2018).

[27]Jones (1995) and Kortum (1997) provide an alternative specification in which current knowledge contributes to growth with diminishing returns, with growth driven by an exogenously expanding labor force.

with technology available in each location evolving according to:

$$\dot{T}_n(t) = \sum_i \dot{T}_{ni}(t) = \sum_i \left(\frac{\epsilon_{ni}}{g + \epsilon_{ni}} \right) \iota_i T_i(t).$$

In matrix form:

$$g\mathbf{T} = \mathbf{\Delta}(g)\mathbf{T}, \tag{49}$$

where $\mathbf{T}$ is an $\mathcal{N} \times 1$ vector with representative element T_i and $\mathbf{\Delta}(g)$ is an $\mathcal{N} \times \mathcal{N}$ matrix with representative element:

$$\Delta_{ni}(g) = \left(\frac{\epsilon_{ni}}{g + \epsilon_{ni}} \right) \iota_i.$$

The growth rate g is the Perron-Frobenius root of (49), with relative levels of technology $\mathbf{T}$ corresponding to the Perron-Frobenius eigenvector (defined up to a scalar multiple). This formulation delivers parallel growth despite arbitrary rates of innovation in each location.

The share of technology in n that originated in i is:

$$\omega_{ni} = \frac{T_{ni}(t)}{T_n(t)} = \frac{\epsilon_{ni}}{g + \epsilon_{ni}} \cdot \frac{\iota_i}{g} \cdot \frac{T_i}{T_n}, \tag{50}$$

where g and the ratio T_i/T_n emerge from the solution to (49). The ω_{ni} reflect how much locations benefit from each other's technologies. In what follows they also determine profit flows between locations.

6.2 Endogenous Innovation

We now relate the ι's to innovative effort. For workers to have an incentive to innovate, their inventions must generate income for them. To allow inventors to appropriate value from their ideas, we switch from a world of perfect competition to one of Bertrand competition. By charging the cost of the next best alternative, the owner of an idea can appropriate the cost saving from its increment in efficiency.

We introduce endogenous innovation into the dynamic matching economy from Section 5. Switching to Bertrand competition requires only a minimal modification. A buyer now pays the second lowest cost for an input among the suppliers it's encountered, so that the distribution $G_n^{(1)}(c; t, a)$, wherever it appears, gets replaced by the distribution of the second lowest cost:

$$G_n^{(2)}(c;t,a) = 1 - (1 + a\tilde{\Phi}_n(t)c^\theta)e^{-a\tilde{\Phi}_n(t)c^\theta}, \tag{51}$$

(that is, one minus the probability of only zero or only one cost below c). The only change to the results in the previous section is that the smaller term $\beta\Gamma(\beta)$ replaces $\Gamma(\beta)$ in the expressions for $v_i(t;a)^{-\theta}$ and, in equation (43), $v_n(t)^{-\theta}$.

6.2.1 Profits

The value of an innovation derives from the profits it earns around the world. As a benchmark we consider a world with perfect intellectual property protection, so that an innovator earns all the profits her idea earns globally.

Our framework forces us to distinguish three notions of profit for a location i: Profits earned from sales there, denoted Π_i, profits earned from production there, denoted Π_i^*, and profits earned from innovation there, denoted $\overline{\Pi}_i$.

For the first, from Section 3.3.2, total profits earned selling in i are just a fraction Δ^{BC} of total spending on goods there, X_i, which consists of spending on intermediates by producers X_i^I and final spending by consumers X_i^F.

With location i allocating a fraction r_i of its workers to research, spending on production workers is $w_i(1-r_i)L_i$. Our Cobb-Douglas assumption implies that intermediate spending is then $X_i^I = [(1-\beta)/\beta]w_i(1-r_i)L_i$. Final spending is income earned in location i less spending on research, so that $X_i^F = w_i(1-r_i)L_i + \overline{\Pi}_i$. Hence profits earned selling in i are:

$$\Pi_i = \Delta^{BC}X_i = \Delta^{BC}\left(\frac{1}{\beta}w_i(1-r_i)L_i + \overline{\Pi}_i\right).$$

Since exporters earn profits on sales abroad, $\Pi_i^* = \sum_n \pi_{ni}\Pi_n$, with π_{ni} as given in (44). Since innovators earn profits on the use of their inventions abroad, $\overline{\Pi}_i = \sum_n \omega_{ni}\Pi_n^*$, with ω_{ni} as given in (50). With our choice of numéraire profits are stationary.

We continue to assume overall trade balance in goods and technology services. But since earnings on technology don't necessarily balance, trade in goods isn't balanced either. We need to replace the labor-market clearing condition (39) with the condition:

$$\frac{1}{\beta}w_i(1-r_i)L_i + \Pi_i^* = \sum_n \pi_{ni}\left(\frac{1}{\beta}w_n(1-r_n)L_n + \overline{\Pi}_n\right), \tag{52}$$

equating what's spent on goods from i to what the rest of the world purchases.

6.2.2 The Price Index

An inventor earns profit from her invention over time. To assess the value of a current invention requires translating the future income it generates into today's prices, requiring a price index.

To construct the price index, first consider purchases from retailers aged a, who offer the distribution of prices (51). With scale-free symmetric Cobb-Douglas preferences, the price index for these retailers, intermediaries with no markups, is:

$$P_n(t,a) = \exp\left(\int_0^\infty \ln c \, dG_n^{(2)}(c;t,a)\right) = e^{(1-\gamma)/\theta}(a\tilde{\Phi}_n(t))^{-1/\theta},$$

where, again, γ is the Euler-Mascheroni constant.[28] Aggregating across cohorts of retailers, weighting by cohort size, the overall price index is:

$$P_n(t) = \exp\left(\int_0^\infty \ln P_n(t,a)\delta e^{-\delta a} da\right) = \Gamma^G \tilde{\Phi}_n(t)^{-1/\theta},$$

where $\Gamma^G = (e\delta)^{1/\theta}$. More retailer turnover (a higher δ) implies higher prices. From our result in Section 5, that the $\tilde{\Phi}$'s grows at rate g/β, prices fall everywhere at rate $g/(\theta\beta)$.

6.2.3 The Value of Innovation

We calculate the value of innovation in i in two parts. First consider the discounted profits earned by a unit of technology from i in a particular location n at the moment t_A it arrives at n, at which point it's available for production. Upon arrival, the technology is aged 0 from n's perspective.

Technology aged a earns a flow profit in n of $f_3(a)\Pi_n^*$ shared among a cohort of size $gT_n(t_A)$. Integrating the expected lifetime returns, taking into account discounting at rate ρ and inflation at rate $-g/(\theta\beta)$, yields an expected value of an idea arriving in n at date t_A:

$$V_n(t_A) = \frac{\Pi_n^*}{gT_n(t_A)} \int_0^\infty e^{-(\rho - g/(\theta\beta))a} f_3(a) da = \frac{\Pi_n^*}{gT_n(t_A)} \left(\frac{g}{\rho + g - g/(\theta\beta)}\right)^{3-\beta}.$$

But an inventor needs to wait for the technology to arrive at location n before receiving this value. The expected discounted value of a unit of new technology at the moment t_I of

[28] This last result uses the Laplace transform, $F(s) = \int_0^\infty t \ln t e^{-st} dt = -\frac{\ln s + \gamma - 1}{s^2}$, evaluated at $s = 1$.

invention in location i, accounting for its random diffusion to different destinations, is:

$$\begin{aligned}\overline{V}_i(t_I) &= \int_0^\infty e^{-(\rho-g/(\theta\beta))s}\sum_n \epsilon_{ni}e^{-\epsilon_{ni}s}V_n(t_I+s)ds \\ &= \left(\frac{g}{\rho+g-g/\theta\beta}\right)^{3-\beta}\sum_n \frac{1}{\rho+\epsilon_{ni}+g-g/\theta\beta}\cdot\frac{\epsilon_{ni}\Pi_n^*}{gT_n(t_I)}. \end{aligned} \tag{53}$$

Since the T's grow at rate g, the value of new technology falls over time as it faces a more crowded field.

6.2.4 Equilibrium Research

Endogenizing innovation, we set the innovation parameter in (47) equal to a research production function of the form:

$$\iota_i = \alpha_i r_i^{\varepsilon} L_i.$$

Here α_i is a parameter reflecting research productivity in location i, r_i is the share of the labor force doing research, and $\varepsilon \leq 1$ captures diminishing returns to research effort.[29]

Occupational choice within each location equates the value of the marginal product of research to the wage of production workers, where the return to research depends on the value of new technology from there. Specifically:

$$\varepsilon\alpha_i r_i^{\varepsilon-1}T_i(t)\overline{V}_i(t) = w_i. \tag{54}$$

6.2.5 Balanced Growth

A balanced growth path consists of vectors across locations of relative wages w, input costs v, values of technology V, research intensity r, and relative technology levels T, together with a common growth rate g, satisfying (43), (49), (52), (53), and (54). The parameter values required for solution are the labor endowments L and research productivities α in each location, the iceberg trade costs d, the matching intensities λ, and diffusion rates ϵ between each pair of locations, and the general parameters θ, governing the distribution of ideas (the parameter that launched the analysis), β, the labor share, ρ, the discount factor, and ε, governing diminishing returns to research.

[29] Phelps (1966) derives such a specification assuming a Pareto distribution of research talent.

7 Conclusion

To concentrate on core ideas we've ignored some significant extensions. We've treated distributions of ideas as independent in different locations, which oversimplifies the modelling of trade and diffusion and correlation in comparative advantage.[30] We've left out capital accumulation and the endogenous formation of transportation links.[31] We've adopted the Ricardian treatment of a location as an endowment of labor, ignoring migration and commuting. What's now a very rich literature has incorporated labor mobility into the framework here to address myriad issues in spatial outcomes.[32] Incorporating these extensions is technically demanding, but the basic tools we've reviewed here remain relevant.

[30]Ramondo and Rodríguez-Clare (2013), Arkolakis et al. (2018), Cai et al. (2022b), and Lind and Ramondo (2022) model how diffusion gives rise to correlation across locations in efficiencies, relating diffusion to multinational production.

[31]Contributions here are Eaton et al. (2016), Fajgelbaum and Schaal (2020), and Allen and Arkolakis (2022).

[32]Allen and Arkolakis (2014), Caliendo et al. (2019), and Kleinman et al. (2023) introduce migration between regions, the third with capital accumulation as well. Ahlfeldt et al. (2015) model commuting within a city. Redding and Rossi-Hansberg (2017) review the earlier literature.

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A Prices and Welfare

This appendix concerns the general form of the price index, equation (17) from Section 3.2. We show that if prices weakly exceed costs and $\sigma < \theta + 1 < \sigma'$ then:

$$\Gamma = E\left[p\left(U^{(1)}(\omega)^{1/\theta}, U^{(2)}(\omega)^{1/\theta}, \ldots\right)^{1-\sigma}\right]^{1/(1-\sigma)} > 0,$$

so that the general form of the price index is well defined. Prices weakly exceeding costs implies:

$$\begin{aligned}
\Gamma &\geq E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}(\omega)\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\right]^{1/(1-\sigma)} \\
&= \left(\int_0^{\infty} E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\middle| U^{(1)} = u\right] e^{-u} du\right)^{1/(1-\sigma)},
\end{aligned} \tag{55}$$

where the second line employs the law of iterated expectations and our result on the distribution of $U^{(1)}$.

In the case of $\sigma' \to \infty$ the problem simplifies to:

$$E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\middle| U^{(1)} = u\right] = u^{(1-\sigma)/\theta}.$$

Hence:

$$\Gamma \geq \left[\int_0^{\infty} u^{(1-\sigma)/\theta} e^{-u} du\right]^{1/(1-\sigma)} = \Gamma\left(\frac{\theta + 1 - \sigma}{\theta}\right)^{1/(1-\sigma)} > 0.$$

The restriction that $\sigma < \theta + 1$ keeps the gamma function finite by limiting substitution into goods whose cost of production approaches zero.[33]

For σ' finite we need to consider higher cost varieties together with the lowest-cost variety. In this case σ' must remain high enough so that the presence of a countably infinite number of varieties does not drive the price index to zero.

First consider the case of $\sigma > 1$. Let $a = (\sigma' - 1)/(\sigma - 1)$ so that $a > 1$. With

$$X = \left(\sum_{k=1}^{\infty}\left(U^{(k)}(\omega)\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}, \tag{56}$$

[33] If we were to let σ approach $\theta + 1$ from below:

$$\lim_{\sigma \to \theta+1} \Gamma\left(\frac{\theta + 1 - \sigma}{\theta}\right) \to \infty \quad \Longrightarrow \quad \lim_{\sigma \to \theta+1} \Gamma\left(\frac{\theta + 1 - \sigma}{\theta}\right)^{1/(1-\sigma)} \to 0,$$

driving Γ, and hence the price index, to zero.

Jensen's inequality implies $E[X] \leq E[X^a]^{1/a}$, that is:

$$\begin{aligned}
E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\Big| U^{(1)}=u\right] &\leq E\left[\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)}=u\right]^{(1-\sigma)/(1-\sigma')} \\
&= \left(\sum_{k=1}^{\infty} E\left[\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)}=u\right]\right)^{(1-\sigma)/(1-\sigma')} \qquad (57)
\end{aligned}$$

Using results from Section 2.3, that the $U^{(k)}$ are distributed Erlang (with parameters k and 1) and that the gaps $U^{(k+1)} - U^{(k)}$ are distributed exponential (with parameter 1), we have:

$$\begin{aligned}
\sum_{k=1}^{\infty} E\left[\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)}=u\right] &= u^{(1-\sigma')/\theta} + \sum_{l=1}^{\infty}\int_0^{\infty}(u+x)^{(1-\sigma')/\theta}\frac{x^{l-1}e^{-x}}{(l-1)!}dx \\
&= u^{(1-\sigma')/\theta} + \int_0^{\infty}(u+x)^{(1-\sigma')/\theta}\left(\sum_{m=0}^{\infty}\frac{x^m e^{-x}}{m!}\right)dx \\
&= u^{(1-\sigma')/\theta} + \int_0^{\infty}(u+x)^{(1-\sigma')/\theta}dx \\
&= u^{(1-\sigma')/\theta}\left(1+\frac{\theta}{\sigma'-(\theta+1)}u\right), \qquad (58)
\end{aligned}$$

where the last integral was finite because $\sigma' > \theta + 1$.

Continuing with $\sigma > 1$, we can combine (55), (57), and (58) to get:

$$\begin{aligned}
\Gamma^{1-\sigma} &\leq \int_0^{\infty} E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\Big| U^{(1)}=u\right]e^{-u}du \\
&\leq \int_0^{\infty}\left(\sum_{k=1}^{\infty} E\left[\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)}=u\right]\right)^{(1-\sigma)/(1-\sigma')}e^{-u}du \\
&= \int_0^{\infty} u^{(1-\sigma)/\theta}\left(1+\frac{\theta}{\sigma'-(\theta+1)}u\right)^{(1-\sigma)/(1-\sigma')}e^{-u}du \\
&\leq \int_0^{\infty} u^{(1-\sigma)/\theta}\left(1+\frac{\theta}{\sigma'-(\theta+1)}u\right)e^{-u}du \\
&= \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)\frac{\sigma'-\sigma}{\sigma'-(\theta+1)}.
\end{aligned}$$

It follows that Γ is strictly positive:

$$\Gamma \geq \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)\left(\frac{\sigma'-\sigma}{\sigma'-(\theta+1)}\right)\right]^{1/(1-\sigma)} > 0. \qquad (59)$$

For the case of $\sigma < 1$, define σ_1 satisfying $1+\theta > \sigma_1 > 1$, so that $b = (1-\sigma_1)/(1-\sigma) < 0$. With X given in (56), Jensen's inequality implies $E[X]^b \leq E[X^b]$ so that raising both sides to the power of $1/(1-\sigma_1)$ gives

$E[X]^{1/(1-\sigma)} \geq E[X^b]^{1/(1-\sigma_1)}$. In combination with the first line of (55) we have:

$$\Gamma \geq E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}(\omega)\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma)/(1-\sigma')}\right]^{1/(1-\sigma)} \geq E\left[\left(\sum_{k=1}^{\infty}\left(U^{(k)}(\omega)\right)^{(1-\sigma')/\theta}\right)^{(1-\sigma_1)/(1-\sigma')}\right]^{1/(1-\sigma_1)}.$$

We showed above that the right-hand side is strictly positive, since $\sigma_1 > 1$. It follows that $\Gamma > 0$ for the case of $\sigma < 1$ as well.

We now consider the case in which $\lim \sigma \to 1$, so that (55) becomes:

$$\begin{aligned}
\Gamma &\geq \exp\left(\frac{1}{1-\sigma'} E\left[\ln\left(\sum_{k=1}^{\infty}\left(U^{(k)}(\omega)\right)^{(1-\sigma')/\theta}\right)\right]\right) \\
&= \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} E\left[\ln\left(\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\right)\Big| U^{(1)} = u\right] e^{-u} du\right) \\
&\geq \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} \ln E\left[\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)} = u\right] e^{-u} du\right),
\end{aligned}$$

where the last line follows from Jensen's inequality, noting that $1 - \sigma' < 0$. Further simplification, using (58) and $\ln(1 + x) \leq x$ for $x \geq 0$, delivers the result:

$$\begin{aligned}
\Gamma &\geq \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} \ln E\left[\sum_{k=1}^{\infty}\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)} = u\right] e^{-u} du\right) \\
&= \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} \ln\left(\sum_{k=1}^{\infty} E\left[\left(U^{(k)}\right)^{(1-\sigma')/\theta}\Big| U^{(1)} = u\right]\right) e^{-u} du\right) \\
&= \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} \ln\left(u^{(1-\sigma')/\theta}\left(1 + \frac{\theta}{\sigma' - (\theta+1)} u\right)\right) e^{-u} du\right) \\
&\geq \exp\left(\frac{1}{1-\sigma'}\int_0^{\infty} \left(\frac{1-\sigma'}{\theta}\ln u + \frac{\theta}{\sigma' - (\theta+1)} u\right) e^{-u} du\right) \\
&= \exp\left(-\frac{1}{\theta}\gamma - \frac{1}{\sigma' - 1}\frac{\theta}{\sigma' - (\theta+1)}\Gamma(2)\right) > 0,
\end{aligned}$$

where γ is the Euler-Mascheroni constant (which also appeared at the end of Section 3.4.2).

B Firms and Their Customers

This appendix concerns steps to simplify (from Section 5.1 of the paper):

$$\eta_{ni}(c;t) = \lambda_{ni} \int_0^\infty a \left[T_n(t) v_n(t;a)^{-\theta} \int_0^\infty \left(1 - e^{-a\eta_n(c';t)}\right) \theta (c')^{\theta-1} dc' + \delta L_n e^{-\delta a} \right] e^{-a\tilde{\Phi}_n(t)c^\theta} da.$$

Substituting in the expression for $v_n(t;a)^{-\theta}$ in terms of $v_n(t)^{-\theta}$ and reversing the order of integration:

$$\begin{aligned} \eta_{ni}(c;t) &= \lambda_{ni} T_n(t) v_n(t)^{-\theta} \int_0^\infty \frac{g^{3-\beta}}{\Gamma(3-\beta)} \left[\int_0^\infty a^{2-\beta} e^{-ga} \left(1 - e^{-a\eta_n(c';t)}\right) e^{-a\tilde{\Phi}_n(t)c^\theta} da \right] \theta(c')^{\theta-1} dc' \\ &+ \lambda_{ni} \int_0^\infty a\delta L_n e^{-\delta a} e^{-a\tilde{\Phi}_n(t)c^\theta} da. \end{aligned}$$

Grouping together related terms:

$$\begin{aligned} \eta_{ni}(c;t) &= \lambda_{ni} T_n(t) v_n(t)^{-\theta} \int_0^\infty \frac{g^{3-\beta}}{\Gamma(3-\beta)} \left[\int_0^\infty a^{2-\beta} \left(e^{-\left(g+\tilde{\Phi}_n(t)c^\theta\right)a} - e^{-\left(g+\eta_n(c';t)+\tilde{\Phi}_n(t)c^\theta\right)a} \right) da \right] \theta(c')^{\theta-1} dc' \\ &+ \lambda_{ni} \delta L_n \int_0^\infty a e^{-\left(\delta+\tilde{\Phi}_n(t)c^\theta\right)a} da. \end{aligned}$$

Integrating over buyer age, a, delivers the next expression for $\eta_{ni}(c;t)$ in the paper:

$$\begin{aligned} \eta_{ni}(c;t) &= \lambda_{ni} T_n(t) v_n(t)^{-\theta} \int_0^\infty \left[\left(\frac{g}{g+\tilde{\Phi}_n(t)c^\theta} \right)^{3-\beta} - \left(\frac{g}{g+\eta_n(c';t)+\tilde{\Phi}_n(t)c^\theta} \right)^{3-\beta} \right] \theta(c')^{\theta-1} dc' \\ &+ \frac{\delta}{\left(\delta+\tilde{\Phi}_n(t)c^\theta\right)^2} \lambda_{ni} L_n. \end{aligned}$$

C The Age of Producers

This appendix concerns properties of the age density, $f_{ni}(a)$, of firms from i that are actively selling (with $k \geq 1$ buyers) in location n. This density, the third one introduced in Section 5.2, is:

$$f_{ni}(a) = \frac{f_2(a) \int_0^\infty \left(1 - e^{-a\lambda_{ni}\eta_n^*(x)}\right) dx}{\int_0^\infty \int_0^\infty f_2(a') \left(1 - e^{-a'\lambda_{ni}\eta_n^*(x)}\right) dx da'}.$$

To find the modal age $\hat{a}_{ni}$ of a firm from i that is actively selling in n, we set the derivative of this density equal to zero to obtain:

$$-f_2'(\hat{a}_{ni}) \int_0^\infty \left(1 - e^{-\hat{a}_{ni}\lambda_{ni}\eta_n^*(x)}\right) dx = f_2(\hat{a}_{ni}) \int_0^\infty \lambda_{ni}\eta_n^*(x) e^{-\hat{a}_{ni}\lambda_{ni}\eta_n^*(x)} dx.$$

Rearranging this expression:

$$-\frac{\hat{a}_{ni} f_2'(\hat{a}_{ni})}{f_2(\hat{a}_{ni})} = \frac{\int_0^\infty \hat{a}_{ni}\lambda_{ni}\eta_n^*(x) e^{-\hat{a}_{ni}\lambda_{ni}\eta_n^*(x)} dx}{\int_0^\infty \left(1 - e^{-\hat{a}_{ni}\lambda_{ni}\eta_n^*(x)}\right) dx} = q_{ni}(1; \hat{a}_{ni}),$$

where $q_{ni}(1; a)$, from Section 5.1, is the fraction of firms with exactly one customer in n among firms age a from i that are actively selling in location n.

Since $q_{ni}(1; a) > 0$, the modal age $\hat{a}_{ni}$ is beyond the peak of $f_2(a)$, that is $\hat{a}_{ni} > (1 - \beta)/g$. To say more we calculate the elasticity of the second age density:

$$\frac{a f_2'(a)}{f_2(a)} = -ga + 1 - \beta.$$

The modal age for the third density therefore satisfies:

$$\hat{a}_{ni} = \frac{q_{ni}(1; \hat{a}_{ni}) + 1 - \beta}{g}.$$