Aggregation, Liquidity, and Asset Pricing

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January 29, 2024

Preliminary and Incomplete.

Abstract

We provide analytical results on asset-pricing and consumption behavior for a standard two-account heterogeneous agent model with aggregate risk. The model can simultaneously explain (1) household-level consumption behavior such as high marginal propensities to consume, (2) a high and volatile zero-beta rate on equities that satisfies an aggregate consumption Euler equation, and (3) a low and stable return on liquid safe assets that does not. The large and volatile spread between the expected return of equities and safe assets does not represent compensation for risk, but rather an endogenous liquidity premium.

*Acknowledgements: The authors would like to thank Adrien Auclert for encouraging us to work on these issues. We would also like to thank Michael Yip for excellent research assistance. Di Tella: sditella@stanford.edu; Hébert: bhebert@stanford.edu; Kurlat: kurlat@usc.edu
1 Introduction

In this paper we propose a theory linking asset pricing and consumption behavior based on
liquidity frictions. The starting point are a set of empirical facts in apparent tension with each other:

1. At the household level, consumption behavior is not well described by a simple Euler
equation. Households face uninsurable risk and borrowing constraints and have high
marginal propensities to consume, even for households with significant wealth.

2. At the aggregate level, consumption satisfies a simple Euler equation with the zero-beta
rate, the interest rate recovered from equity returns after controlling for risk premia
[Di Tella et al., 2023].

3. The return of liquid assets such as Treasury bills does not satisfy an aggregate con-
sumption Euler equation. There is a large and volatile spread between the zero-beta
and more liquid rates (which is zero in standard models).

The contribution of this paper is to show that all these facts may have a common source: liq-
uid assets enjoy a substantial and time-varying liquidity premium relative to illiquid assets.
What we have in mind is that safe bonds can be used by banks and other financial intermedi-
aries to back deposits and other payment systems, whereas equities cannot. It is well known
that trading frictions are important to explain the high marginal propensity to consume of
households with significant stock holdings [Kaplan and Violante, 2022]. We show that these
same trading frictions can also explain why the aggregate consumption Euler equation holds
for the return of illiquid assets (equities) after accounting for risk premia, but not for the
return of liquid assets (safe bonds and deposits). In this view, the large and volatile spread
between zero-beta and safe rates represents an endogenous liquidity premium.

We work with a canonical “two account” heterogeneous-agent model, designed to explain
household level consumption and asset-holding behavior,¹ and we extend it to incorporate
aggregate shocks. The main friction is that only a fraction of capital income can be used to
back liquid assets. The rest must be held in an illiquid account that can only be rebalanced
when a trading opportunity arises with Poisson intensity. The spread between liquid and
illiquid assets compensates households for the risk of running out of liquid assets and having
to live hand-to-mouth.

An important technical contribution of the paper is that we solve this heterogeneous-agent
model with aggregate shocks in closed form. This is important when studying asset-pricing

¹e.g. Kaplan and Violante [2014], Kaplan et al. [2018], and Auclert et al. [2023]
features that would be eliminated by linearization. Our results build on the work of Krueger and Lustig [2010] and Werning [2015], but we account for many features essential to the question we tackle: trading frictions, assets in positive supply (both liquid and illiquid), intertemporal elasticity below one, and general stochastic processes for idiosyncratic labor income and aggregate output. To obtain exact aggregation we incorporate a form of counter-cyclical labor income risk (e.g. it is harder to find a job during recessions). This combination of assumptions allows us to analytically characterize asset prices in a heterogenous-agent economy with aggregate risk, without imposing any restrictions on the details of the income process and frictions that govern the counterfactual steady-state economy.² Our solution method involves introducing a stochastic time change that allows us to represent the economy with aggregate shocks in terms of the economy without aggregate shocks but where the flow of time may be time-varying.

We start with a setting with log preferences and show that while there is a spread between the return of the liquid and illiquid assets, this spread is invariant to aggregate shocks. The return of both assets satisfies an aggregate consumption Euler equation (with different intercepts) and consumption-wealth ratios are constant. Empirically the spreads are time-varying, intertemporal elasticity is significantly below one, and consumption-wealth ratios are not constant. This motivates us to consider the economy with CRRA preferences.

We then consider the CRRA case with intertemporal elasticity below one. We derive an aggregate Euler equation for both the liquid and illiquid asset that accounts for uninsured idiosyncratic risk (reminiscent of results in Constantinides and Duffie [1996]). The illiquid asset is not very good at insuring idiosyncratic labor income risk because households may not be able to access it when they really need it, so the aggregate consumption Euler equation holds as a good approximation. Liquid assets, on the other hand, allow agents to self insure against idiosyncratic labor income risk, so they carry a liquidity premium. This premium is time-varying because consumption-wealth ratios are not constant. With low intertemporal elasticity, when low growth is expected, discount rates fall and asset prices are high relative to output. As a result, the supply of liquidity is high relative to demand and the liquidity premium on liquid asset shrinks. The result is that when expected consumption growth is low the return of illiquid assets falls consistent with an aggregate Euler equation with low intertemporal elasticity, while the return of liquid assets moves much less. This is qualitatively in line with facts 2 and 3 above.

We obtain a quantitative expression for the effect of this cyclical uninsured idiosyncratic

²That is, our results apply to the steady-state economies studied by e.g. Kaplan and Violante [2014], Kaplan et al. [2018], and Auclert et al. [2023]. They also cover, as a special case, steady-state heterogenous agent models with a single account (as in Aiyagari [1994] and the literature that followed).
risk that depends on the difference between agents’ rate of time preference and the interest rates that would prevail in a steady-state economy without growth (i.e. the kind of model studied by Kaplan and Violante [2022]). Using these aggregation results, we provide a quantitative example in which the standard aggregate Euler equation holds approximately for illiquid assets, but not for liquid assets, consistent with the findings of Di Tella et al. [2023]. We then argue that this conclusion is not specific to our quantification, but rather a general feature of any parameterization in which some agents are wealthy and other agents are borrowing-constrained.

We provide our aforementioned results under the assumption of exogenous individual labor endowments and a fixed capital stock. In our last extension (TBD), we demonstrate how our techniques can be applied to the case of endogenous labor supply and an endogenous capital stock. The key additional assumptions in these cases are balanced-growth preferences (for endogenous labor supply) and cyclical capital adjustment costs (for endogenous capital with CRRA preferences). This last extension illustrates how close our results are to quantitative heterogenous agents models. Of course, our exact aggregation results, while important to properly characterize asset prices, hinge on an absence of redistribution in response to aggregate shocks. In this sense, we may be accused of missing the most important feature that heterogenous-agent models aim to incorporate. We view our analytical characterization of asset prices in two-account heterogenous-agent models as a step towards a unified model that can simultaneously match empirical facts about the distribution of wealth and income, rationalize evidence on individuals’ marginal propensities to consume, and match facts about the relationship between aggregate consumption and asset prices.

**Related Literature.** Our work is most closely related to Krueger and Lustig [2010] and Werning [2015]. Krueger and Lustig [2010] study asset prices in a “one-account” heterogeneous agent model in which there are shocks to the level of aggregate output, but not to its growth rate (aggregate output and consumption growth are I.I.D.). They consider the case of CRRA preferences and show that the standard “consumption CAPM” holds despite the presence of uninsurable idiosyncratic income risk and borrowing constraints. The key proof technique involves constructing a steady-state equilibrium without aggregate risk, and then constructing the equilibrium with aggregate risk using that steady-state equilibrium. The Krueger and Lustig [2010] model has little to say about the consumption Euler equation directly, as interest rates are constant as a consequence of the I.I.D. growth shocks. We share with Krueger and Lustig [2010] an interest in the asset pricing implications of heterogenous agents model. However, our main analysis focuses on the polar opposite case in which there are no shocks to output growth itself, and instead only shocks to the rate of output growth,
which then affect the level output over time. These shocks, interestingly enough, do not generate any risk premium at all. That is, there is no difference in our model between the expected return of a safe illiquid asset and a risky illiquid asset. However, our approach can be readily adapted to accommodate shocks that affect the level of output directly, and in this case we believe that the Krueger and Lustig [2010] on risk premia would hold in our model.

Our paper and Werning [2015] adapt the technique of Krueger and Lustig [2010] to consider the case in which consumption growth is not constant over time. Werning [2015] studies three different versions of the log utility case in which the standard aggregate Euler equation holds: (i) a case with zero asset supply, (ii) a case in which initial bond holdings are zero, and (iii) an RBC model with fully-depreciating capital (as in the Brock and Mirman [1972] model). The first two of these cases involve models with no aggregate risk (the aggregate shocks is a one-time, unanticipated “MIT” shock), and all of these cases are one-account models in the sense used by Kaplan and Violante [2022]. Our results hold in a two-account model in which assets are positive supply and there is aggregate risk. In this sense, they extend the results of Werning [2015]. They are closest, conceptually, to case (ii) mentioned above, in that the aggregate shocks in our model do not redistribute wealth across agents.

The extension of the Werning [2015] results on the log utility case to the two-account, positive-asset supply case with aggregate risk brings those results closer to the quantitative HANK models, and is for this reason itself a contribution. However, the log utility case is a non-starter for the purposes of asset pricing, primarily because it leads to constant price-dividend ratios. In fact, it is exactly this property that ensures that the assumption of acyclical labor income risk that Werning [2015] emphasizes leads to acyclical human capital values, which is the key to aggregation in the log case. A second contribution of our paper is to provide conditions for the CRRA case involving cyclical labor income risk that lead to acyclical human capital values, and hence aggregation. This step is what allows us to provide analytical results for the CRRA case in our model; these results have no parallel in Werning [2015].

More importantly, our results for the CRRA case allow us to study a case in which the spread between liquid and illiquid assets is not constant, and in fact quite volatile. The third key contribution of our paper is to show that two-account heterogenous-agent models with CRRA preferences can explain why the aggregate Euler equation holds for illiquid assets but not liquid assets, consistent with the evidence of Di Tella et al. [2023]. The model can also generate a high (on average) equity returns with moderate levels of risk-aversion, and

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3Werning [2015] has a few results for the zero asset supply case that apply in the CRRA as well.
at the same time explain why illiquid assets that are more exposed to aggregate risk (high “beta” assets) do not offer higher returns than illiquid assets that are less exposed (lower “beta” assets). That is, the model generates a flat “security market line,” consistent with empirical evidence (Black et al. [1972], Frazzini and Pedersen [2014], Hong and Sraer [2016]). The key to these results is the large spread between liquid and illiquid assets generated by the two-account model. That is, the two-account model leads naturally to a theory of asset pricing that emphasizes liquidity instead of risk, in the spirit of Bansal and Coleman [1996]. That said, we view our model as having nothing to say about bonds, and hence nothing to say about the equity premium over bonds. In our view, bonds are not themselves liquid, but can be used by banks to back deposits, and for this reason will inherit some amount of the liquidity premium enjoyed by deposits. Diamond [2020], in particular, models this point explicitly, and his approach could potentially be integrated into the framework of our paper.

Our work, which delivers analytical asset pricing results in heterogenous-agent models, is part of a literature that attempts to study heterogenous-agent models with aggregate shocks analytically. Acharya and Dogra [2020] and Acharya et al. [2023] provide results for the CARA-normal setting without borrowing constraints. Bilbiie [2008] derives analytical results for economies in which agents have two types (“TANK” models). Ravn and Sterk [2021] and Challe [2020] provide results that (like many of the results in Werning [2015]) apply to an economy with zero-liquidity. We innovate, relative to this literature, by providing results for the canonical two-account setting, with standard preferences, general forms of heterogeneity, and positive liquidity. That said, many of the aforementioned papers incorporate standard New Keynesian features (e.g. price stickiness) that have not yet been incorporated into our framework.

Our work complements recent efforts to develop numerical techniques capable to studying asset pricing in these models (Bhandari et al. [2023], Bilal [2023]). Relative to these efforts, our approach has the clear advantage of sharp and intuitive analytical results, and the clear disadvantage of requiring particular assumptions that limit the redistributive effects of aggregate shocks in the model. However, we view the two approaches as complementary: our results show the assumptions required to generate redistributive effects, and the approaches of Bhandari et al. [2023] and Bilal [2023] show how quantitative results can be obtained in the presence of these effects.

The structure of the paper is as follows. We begin in Section 2 with the log utility case. We then extend our main results to the CRRA case (Section 3). We show via a back-of-the-envelope quantification that the CRRA case can generate volatile price-dividend rations and an aggregate Euler equation that holds approximately for the illiquid but not the liquid assets (Section 4). We then extend our results to the case of transition dynamics (Section
5) and endogenous capital and labor (Section 6). In Section 7 we conclude.

2 The Log Economy

We first introduce the benchmark model with log preferences. Our analysis in this section follows closely the arguments of Krueger and Lustig [2010] and in particular Werning [2015], applied to the “two-account” models of Kaplan and Violante [2014] and the literature that followed.

2.1 Preferences and Technology

There is a continuum of households \( i \in [0, 1] \) with identical preferences

\[
U(C_i) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \log(C_{it}) dt \right].
\]

(1)

Capital and labor are in fixed aggregate supply and are used to produced output

\[
Y_t = Z_t K^\alpha L^{1-\alpha}.
\]

(2)

With competitive factor markets, the labor and capital shares are constant, \( R_t K = \alpha Y_t \) and \( W_t L = (1 - \alpha) Y_t \). \( Z_t \) is total factor productivity, and we can normalize capital and labor so \( Y_t = Z_t \) and \( Y_0 = Z_0 = 1 \). We assume the following process for aggregate output

\[
dY_t = g_t Y_t dt, \tag{3}
\]

\[
dg_t = \mu_g(g_t, Y_t) dt + \sigma_g(g_t, Y_t) dM_t, \tag{4}
\]

where \( M \) is a standard Brownian motion. We model output \( Y_t \) as locally deterministic, to focus our analysis on the relationship between expected growth and interest rates (i.e. the Euler equation). This assumption can be viewed as the polar opposite of the Krueger and Lustig [2010] assumption of I.I.D. aggregate shocks. That said, a local diffusion term can be added to (3) without major difficulty.

Idiosyncratic labor endowments are \( \exp(e_{it}^0) L \), where \( e_{it} = (e_{it}^0, e_{it}^1, ...) \) encodes household \( i \)’s idiosyncratic state. It’s evolution can be summarized by an infinitesimal generator \( \mathcal{L}_e \). For example, \( e_{it}^0 \) could follow a finite Markov chain describing employment and unemployment. At this point we assume \( \mathcal{L}_e \) is invariant to aggregate shocks, which implies that labor market dynamics do not depend on the business cycle. We will relax this assumption later.
Aggregate resource constraints are\(^4\)
\[
\int_0^1 C_{it}di = C_t = Y_t, \tag{5}
\]
\[
\int_0^1 \exp(e_{it}^0)Ldi = L. \tag{6}
\]

2.2 Market Incompleteness and Trading Frictions

Households have two accounts with dynamic budget constraints
\[
dA_{it} = r_{at}A_{it}dt + D_{it}dN_{it} + \sigma_{at}A_{it}dM_{t}, \tag{7}
\]
\[
 dB_{it} = (r_{bt}B_{it} + \exp(z_{it}^0)W_tL - C_{it})dt - D_{it}dN_{it} + \sigma_{bt}B_{it}dM_{t}, \tag{8}
\]

and borrowing constraints \(A_{it}, B_{it} \geq 0\). Each account has an expected return, \(r_{at}\) and \(r_{bt}\), and may also be exposed to aggregate risk through \(\sigma_{at}\) and \(\sigma_{bt}\). Account \(B\) is liquid: the household can use it to pay for consumption and receives their labor income there. Account \(B\) is illiquid and we model the trading friction a la Calvo.\(^5\) Households can only move funds to or from the illiquid account \(A\) at an idiosyncratic trading opportunity that arrives with Poisson probability \(\chi\). \(D_{it}\) are the funds moved into the liquid account and \(N_{it}\) the Poisson counting process associated with household \(i\)’s trading opportunities.

*Household \(i\)’s problem* is to pick processes \((C_{it}, D_{it})\) to maximize utility (1) subject to budget constraints ((7) and (8)) and the no-borrowing constraints, taking processes \((W, r_a, r_b, \sigma_a, \sigma_b)\) as given.

2.3 Supply of Liquid and Illiquid Assets

Both liquid and illiquid assets are ultimately claims on capital. A fraction \(\theta\) of capital income can be used to back liquid assets. The value of each asset is then
\[
\bar{A}_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^s r_{aa}du}(1 - \theta)\alpha Y_s ds \right], \tag{9}
\]
\[
\bar{B}_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^s r_{ba}du}\theta\alpha Y_s ds \right]. \tag{10}
\]

\(^4\)Equation (6) involves only exogenous objects, so it is really a consistency condition on \(\mathcal{L}_c\).
\(^5\)In this respect, our version of the two-account model is closest to Auclert et al. [2023], although alternative approaches involving rebalancing costs (Kaplan and Violante [2022]) or adjustment costs (Kaplan et al. [2018]) can be readily accommodated.
Although the cash flows are proportional to each other, the discount rates \( r_{at} \) and \( r_{bt} \) are different and can move differently in response to aggregate shocks. Consistent with the budget constraints (7) and (8), the expected return from holding each asset is \( r_{at} \) and \( r_{bt} \), and the volatility of \( \bar{A}_t \) and \( \bar{B}_t \) pin down \( \sigma_{at} \) and \( \sigma_{bt} \).

What we have in mind is that the illiquid asset represents equities and the liquid asset deposits. Concretely, \( B \) could be nominally safe (perhaps because it is backed by taxes on output), in which case its real value will satisfy (10). But liquid assets could also include nominally nearly-safe liabilities of the corporate or financial sector. The parameter \( \theta \) represents the fraction of capital income that can ultimately be used to back liquid assets, either issued by the government or the private sector.

The price-dividend ratios for each asset are

\[
P_{at} = \frac{\bar{A}_t}{(1-\theta)\alpha Y_t}, \quad P_{bt} = \frac{\bar{B}_t}{\theta \alpha Y_t}.
\]  

(11)

Market clearing for assets requires

\[
\int_0^1 A_{it} di = \bar{A}_t, \quad \int_0^1 B_{it} di = \bar{B}_t.
\]

(12)

2.4 State Space and Generator

It is useful to normalize variables as follows

\[
c_{it} = C_{it}/Y_t, \quad w_t = W_t/Y_t, \quad a_{it} = A_{it}/\bar{A}_t, \quad b_{it} = B_{it}/\bar{B}_t, \quad d_{it} = D_{it}/\bar{B}_t.
\]

An advantage of this normalization is that \( a_{it} \) and \( b_{it} \) are the shares of the assets that each household owns, and are therefore not affected by aggregate shocks on impact (in contrast to the value of those asset positions, \( A_{it} \) and \( B_{it} \)). There are no diffusion terms in the evolution of \( a_{it} \) and \( b_{it} \). In the absence of rebalancing (\( dN_{it} = 0 \)),

\[
da_{it} = \frac{a_{it}}{P_{at}} dt, \quad db_{it} = \frac{1}{\alpha \theta P_{bt}} (\alpha \theta b + \exp(e^0)(1 - \alpha) - c_t) dt,
\]

where we have also used that \( w_t L = 1 - \alpha \). Households’ optimal policies depend on the history of aggregate shocks and on their idiosyncratic state, \( c_{it} = c_t(a_{it}, b_{it}, e_{it}) \) and \( d_{it} = d_t(a_{it}, b_{it}, e_{it}) \). Given these policy functions, the evolution of \( (a_{it}, b_{it}, e_{it}) \) can be described by the infinitesimal generator \( \mathcal{L}_{abe}(c_t, d_t; P_{at}, P_{bt}) \), defined for arbitrary test function \( f(a, b, e) \)

\[
\mathcal{L}_{abe}(c_t, d_t; P_{at}, P_{bt}) f(\cdot) = \mathcal{L}_c f(\cdot) + \frac{a}{P_{at}} f_a(\cdot) + \frac{1}{\alpha \theta P_{bt}} (\alpha \theta b + \exp(e^0)(1 - \alpha) - c_t(\cdot)) f_b(\cdot)
\]

(13)
\[ + \chi \left( f \left( a + \frac{\theta P_{bt}}{(1 - \theta) P_{at}} d_t(\cdot), b - d_t(\cdot), e \right) - f(\cdot) \right). \]

The first term captures the evolution of \( c_{it} \), the second and third the evolution of \( a_{it} \) and \( b_{it} \) in the absence of a trading opportunity, and the last term captures rebalancing across accounts.

The generator \( L_{abe}(c_t, d_t; P_{at}, P_{bt}) \) depends on the history of aggregate shocks through \( P_{at} \) and \( P_{bt} \) and the policy functions \( c_t(\cdot) \) and \( d_t(\cdot) \).

Let \( \mu_t(a, b, e) \) be the population measure of households’ idiosyncratic state. Its evolution can be described with a KFE

\[ d \mu_t(\cdot) = L^\dagger_{abe}(c_t, d_t; P_{at}, P_{bt}) \mu_t(\cdot) dt, \tag{14} \]

where \( L^\dagger_{abe}(c_t, d_t; P_{at}, P_{bt}) \) is the adjoint operator of \( L_{abe}(c_t, d_t; P_{at}, P_{bt}) \). \(^6\) Aggregate shocks affect the evolution of \( \mu_t \) through the operator \( L^\dagger_{abe}(c_t, d_t; P_{at}, P_{bt}) \), but they don’t “shake” the population measure on impact.

We will also use the generator for \( Y \) and \( g \):

\[ L_{Yg} f(\cdot) = f_Y(\cdot) g_Y + f_g(\cdot) \mu_g(Y, g) + \frac{1}{2} f_{gg}(\cdot) \sigma_g(Y, g)^2. \]

### 2.5 Competitive Equilibrium

For an initial measure \( \mu_0 \), a competitive equilibrium is a set of adapted policy functions \((c^*_t(\cdot), d^*_t(\cdot))\), price processes \((r^*_a, \sigma^*_a, P^*_a, r^*_b, \sigma^*_b, P^*_b)\), and a measure \( \mu^*_t \) such that (1) policies are optimal in households’ problem, (2) \( \mu^*_t \) satisfies KFE (14) with initial condition \( \mu^*_0 = \mu_0 \), (3) prices and returns satisfy (9) and (10), and (4) markets clear:

\[ \int_{\text{supp}(\mu^*_t)} b d\mu^*_t(a, b, e) = 1, \]

\[ \int_{\text{supp}(\mu^*_t)} a d\mu^*_t(a, b, e) = 1, \tag{15} \]

\[ \int_{\text{supp}(\mu^*_t)} c^*_t(a, b, e) d\mu^*_t(a, b, e) = 1. \]

In the absence of aggregate shocks, \( Y_t = 1 \), \( g_t = \mu_g = \sigma_g = 0 \), we can define a steady state competitive equilibrium as a set of policy function \((\bar{c}(\cdot), \bar{d}(\cdot))\), price processes \((\bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b)\) and a measure \( \bar{\mu} \) such that (1) policies are optimal in households’ problem, (2) \( \bar{\mu} \) satisfies the KFE (14) with initial condition \( \mu_0 = \bar{\mu} \), (3) \( \bar{P}_a = \bar{r}_a^{-1} \) and \( \bar{P}_b = \bar{r}_b^{-1} \) and (4) markets clear.

\(^6\) The adjoint operator and the “KFE” equation should be understood in the weak*-sense; in two-account models, \( \mu_t \) will often not be absolutely continuous with respect to the Lebesgue measure.
analogously to (15).

2.6 Steady State Equilibrium

We start with the steady state equilibrium. The value function for a household is \( \bar{V}(a, b, e) \) and the HJB is

\[
\rho \bar{V}(a, b, e) = \max_{c \geq 0, \; d \in [-\frac{a}{1-\theta} \bar{P}_a/\bar{P}_b, b]} \ln c + \mathcal{L}_{abe}(c, d; \bar{P}_a, \bar{P}_b) \bar{V}(a, b, e). \tag{16}
\]

The policy functions \( \bar{c} \) and \( \bar{d} \) are derived from the the HJB, and the KFE is

\[
\bar{L}_{abe}(\bar{c}, \bar{d}; \bar{P}_a, \bar{P}_b) \bar{\mu} = 0. \tag{17}
\]

Finding a steady state equilibrium requires finding \( \bar{P}_a, \bar{P}_b \) such that the policy functions \( \bar{c} \) and \( \bar{d} \) derived from (16) and the measure \( \bar{\mu} \) derived from (17) satisfy market clearing. In what follows, we will simply assume that a steady state equilibrium can be found.

2.7 Competitive Equilibrium with Aggregate Shocks

We conjecture a competitive equilibrium with aggregate shocks that is Markov in \( (Y, g) \) and where \( P_{at}^* = \bar{P}_a \) and \( P_{bt}^* = \bar{P}_b \). Given this conjecture, the generator is the same as in the steady state for any policy functions,

\[
\mathcal{L}_{abe}(c, d; P_{at}^*, P_{bt}^*) = \mathcal{L}_{abe}(c, d; \bar{P}_a, \bar{P}_b).
\]

The value function is \( V(a, b, e; Y, g) \), and the HJB is

\[
\rho V(a, b, e; Y, g) = \max_{c \geq 0, \; d \in [-\frac{a}{1-\theta} P_a/P_b, b]} \ln(Y) + \ln c + \mathcal{L}_{abe}(c, d; \bar{P}_a, \bar{P}_b) V(a, b, e; Y, g) + \mathcal{L}_{Yg}(Y, g) V(a, b, e; Y, g), \tag{18}
\]

We can then guess and verify that

\[
V(a, b, e; Y, g) = \bar{V}(a, b, e) + \phi(Y, g).
\]

The HJB then simplifies to

\[
\rho \bar{V}(a, b, e) + \rho \phi(Y, g) = \max_{c \geq 0, \; d \in [-\frac{a}{1-\theta} \bar{P}_a/\bar{P}_b, b]} \ln(Y) + \ln c + \mathcal{L}_{abe}(c, d; \bar{P}_a, \bar{P}_b) \bar{V}(a, b, e) + \mathcal{L}_{Yg}(Y, g) \phi(Y, g).
\]
We use the steady state HJB (16) to cancel out terms and we get
\[ \rho \phi(Y, g) = \ln(Y) + L_{Yg}(Y, g)\phi(Y, g), \tag{19} \]
with solution
\[ \phi(Y, g) = E \left[ \int_t^\infty e^{-\rho s} \ln(Y_s)ds|Y_t = Y, g_t = g \right]. \]
The policy functions \( c \) and \( d \) are therefore the same as in the steady state equilibrium. This means that, if we start at \( \mu_0 = \bar{\mu} \),
\[ L^\dagger_{abe}(c^*_t, d^*_t; P^*_a, P^*_b)\mu_t = L^\dagger_{abe}(\bar{c}, \bar{d}; \bar{P}_a, \bar{P}_b)\bar{\mu} = 0, \]
so the measure remains at \( \mu_t = \bar{\mu} \). Since both policy functions and the measure \( \mu_t \) are as in the steady state equilibrium, we have market clearing. It only remains to pin down the returns \( r_{at}, r_{bt}, \sigma_{at}, \) and \( \sigma_{bt} \). For the illiquid asset,
\[ E_t \left[ \int_t^\infty e^{-\int_t^s r_{a_{au}}(1-\theta)\alpha Y_sds} \right] = \bar{P}_a(1-\theta)\alpha Y_t. \]
Taking time derivatives on both sides we get
\[ (r_{at}\bar{P}_a(1-\theta)\alpha Y_t - (1-\theta)\alpha Y_t)dt + \sigma_{at}\bar{A}_tdM_t = \bar{P}_a(1-\theta)\alpha g Y_t dt \]
We immediately get \( \sigma_{at} = 0 \) and dividing throughout by \( \bar{P}_a(1-\theta)\alpha Y_t \) and using \( \bar{P}_a = \bar{r}_a^{-1} \), we obtain \( r_{at} = \bar{r}_a + g_t \). The same argument works for the liquid asset, \( r_{bt} = \bar{r}_b + g_t \).

**Proposition 1.** Assume there exists a steady state equilibrium \((\bar{c}, \bar{d}, \bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b, \bar{\mu})\) with \( C^1 \) value function \( \bar{V}(a, b, e) \) satisfying the HJB equation (16). Then if \( \mu_0 = \bar{\mu} \), there exists a competitive equilibrium with
\[ c^*_t(\cdot) = \bar{c}(\cdot), \quad d^*_t(\cdot) = \bar{d}(\cdot), \quad P^*_a = \bar{P}_a, \quad P^*_b = \bar{P}_b, \quad \mu^*_t = \bar{\mu}. \]
There is an aggregate consumption Euler equation for both assets \( j \in \{a, b\} \),
\[ r_{jt} = \bar{r}_j + g_t \tag{20} \]
and assets are locally safe in real terms, \( \sigma_{at} = \sigma_{bt} = 0 \).

**Proof.** (TBD) For additional proof details (such as the verification step), see the appendix, section A.1. \( \square \)
This result allows us to construct the competitive equilibrium with aggregate shocks assuming the initial distribution \( \mu_0 = \bar{\mu} \). In Section 5 we extend this result to construct the competitive equilibrium with aggregate shocks for any initial distribution \( \mu_0 \).

**Discussion of the Log Economy**

Exact aggregation is possible because the aggregate shocks do not redistribute wealth. There is no redistribution between capital and labor (fixed capital share \( \alpha \)) or between liquid and illiquid assets (fixed \( \theta \)). In addition, there is no redistribution between labor-endowment types \( e_{it} \). This is because of log preferences and the invariance of the labor dynamic generator \( L_e \) to the aggregate state. To fix ideas, consider two households, one who is currently employed and one who is not. The unemployed one has a backloaded labor income profile relative to the currently employed one. When a recession starts and \( g_t \) falls, all households expect lower labor income in the future, but the interest rates fall one to one with \( g_t \), so the present value of that labor income is unchanged. In addition, the distribution of that labor income between the two households is unchanged because the generator that governs labor market dynamics, \( L_e \), is independent of the aggregate state of the economy. The currently unemployed household is just as likely to find a job during a boom or a recession. This last condition— that the process governing idiosyncratic income shares is unaffected by aggregate risk— is the same condition used by Werning [2015] to derive the standard aggregate Euler equation in his analysis of the log utility case.

The log economy generates a spread between liquid and illiquid assets and an aggregate consumption Euler equation for illiquid assets. This spread reflects the convenience of liquid assets, which are necessary to sustain consumption after periods of low labor income. To see this, consider a household with a trading opportunity who can decide to increase their consumption and reduce their holding of liquid or illiquid assets. For the illiquid asset, the cost is the return until the next trading opportunity, \( \tau_a \). For the liquid asset, the return until either the next trading opportunity \( \tau_a \) or the point at which the household runs out of liquid assets, \( \tau_b \). They could reduce their consumption before that, but they cannot guarantee postponing the reduction in consumption beyond that point. We can then write the following household-level Euler equations:

\[
C_{it}^{-\gamma} = \mathbb{E}_t \left[ e^{\int_t^{\tau_a} (r_{as} - \frac{1}{2} \sigma_{as}^2) dt + \int_t^{\tau_a} \sigma_{as} dM_s} C_{it}^{-\gamma} \right],
\]

\[
C_{it}^{-\gamma} = \mathbb{E}_t \left[ e^{\int_t^{\tau_a \wedge \tau_b} (r_{bs} - \frac{1}{2} \sigma_{bs}^2) dt + \int_t^{\tau_a \wedge \tau_b} \sigma_{bs} dM_s} C_{it}^{-\gamma} \right].
\]

Relative to the illiquid asset, the liquid one provides insurance against the event \( \tau_b < \tau_a \).
that the household runs out of liquidity before it can access their illiquid assets, which is a high marginal-utility state. If this is guaranteed not to happen, \( P(\tau_b < \tau_a) = 0 \), then the household will never hold liquid assets (at the margin) as long as there is a positive spread. Households reduce their liquid holdings up to the point that a positive probability of running out of liquid funds compensates for the higher return of illiquid assets. In equilibrium the spread is constant because the supply of liquid assets is in constant relation to the liquidity needs derived from idiosyncratic labor income shocks and trading opportunities.

The log economy has several shortcomings:

1. Price-dividend ratios are constant, whereas in reality they are quite volatile.

2. The spread between liquid and illiquid assets is constant, and as a result an aggregate Euler equation holds for both the liquid and illiquid assets. In the data the aggregate consumption Euler equation works for the illiquid asset (Di Tella et al. [2023]), but it fails for the liquid one (Hansen and Singleton [1982]). In the log model it works for both.

3. In the data, estimates of the intertemporal elasticity using the aggregate Euler equation fall well below one. An intertemporal elasticity of 1 is rejected.

4. In the data labor-market dynamics are not invariant to the cycle.

These failures motivate us to extend the model to CRRA preferences with intertemporal elasticity below 1.

### 3 The CRRA Economy

We modify the economy of the previous section in two ways. First, households have CRRA preferences

\[
U(C_t) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right].
\]

We are interested in the \( \gamma > 1 \) case. Second, the generator for idiosyncratic labor endowment and the trading friction will now depend on the aggregate state of the economy. The generator is now\(^7\)

\[
x_t^{-1} \mathcal{L}_e,
\]

and the arrival rate of a trading opportunity is

\[
x_t^{-1} \chi,
\]

\(^7\)This is still consistent with the aggregate resource constraint (6).
where
\[ x_t = x(Y_t, g_t) = \rho \mathbb{E} \left[ \int_t^{\infty} \exp \left( \int_t^s (1 - \gamma) g_u - \rho \right) du \right] ds | Y_t, g_t]. \]  
(23)

The object \( x(Y, g) \) happens to be the price-dividend ratio in a representative-agent economy, normalized by \( \rho \). With \( \gamma = 1 \) (log), \( x(Y, g) = 1 \) and we have the setting in the previous section. With \( \gamma > 1 \), \( x(Y, g) \) is high when growth is expected to be low (e.g. during the contraction phase of the cycle). The intuition is that interest rates fall more than one-to-one with expected growth \( g_t \).

Adjusting the generator and trading frictions by this precise amount is a mathematical trick to obtain exact aggregation, essential for asset pricing. But it has an economic meaning. What expression (21) says is that, while during recessions everyone may expect their labor income to contract, recessions are disproportionally bad for those who currently have low labor income. For example, currently unemployed households are disproportionally less likely to find a job during a recession than during a boom.\(^8\) Likewise, expression (22) says that during recessions liquidity frictions become tighter.

### 3.1 State Space and Generator

We have the same state space as in the log economy, but the generator is now modified to
\[
\mathcal{L}_{\text{abe}}(c_t, d_t; P_{at}, P_{bt}, x_t) f(\cdot) = x_t^{-1} \mathcal{L}_c f(\cdot) + \frac{\alpha}{P_{at}} f_a(\cdot) + \frac{1}{\alpha \theta P_{bt}} (\alpha \theta b + \exp(e^0)(1 - \alpha) - c_t(\cdot)) f_b(\cdot) 
\]
\[ + x_t^{-1} \chi \left( f \left( a + \frac{\theta P_{bt}}{(1 - \theta) P_{at}} d_t(\cdot), b - d_t(\cdot), e \right) - f(\cdot) \right). \]
(24)

It now depends on the history of aggregate shocks not only through policy functions and \( P_{at} \) and \( P_{bt} \), but also through \( x_t \).

An important property of the generator is that it is homogeneous of degree \(-1\) in the last three arguments:
\[
\mathcal{L}_{\text{abe}}(c, d; xP_a, xP_b, x) = x^{-1} \mathcal{L}_{\text{abe}}(c, d; P_a, P_b, 1), \]
(25)

and so is its adjoint
\[
\mathcal{L}_{\text{abe}}^\dagger(c, d; xP_a, xP_b, x) = x^{-1} \mathcal{L}_{\text{abe}}^\dagger(c, d; P_a, P_b, 1). \]
(26)

\(^8\)The baseline model does not have aggregate unemployment: fluctuations are caused by TFP \( Z_t \). But we can reinterpret TFP as fluctuation in unemployment \((1 - u_t) = Z_t^\frac{1}{\gamma}\), so that during recessions everyone is less likely to be employed by the end.
3.2 Steady State Equilibrium

The steady state equilibrium can be characterized by the HJB equation

$$\rho \bar V(a, b, e) = \max_{c \geq 0, \ d \in [-a \bar P_a / \bar P_b, b]} c^{1-\gamma} \frac{1}{1 - \gamma} + \mathcal{L}_{abe}(c, d; \bar P_a, \bar P_b, 1)\bar V(a, b, e).$$  \hspace{1cm} (27)

The policy functions $\bar c$ and $\bar d$ are derived from the the HJB, and the KFE is

$$\mathcal{L}_{abe}^\dagger(\bar c, \bar d; \bar P_a, \bar P_b, 1)\bar \mu = 0.$$  \hspace{1cm} (28)

As before, we assume that a steady state equilibrium can be found.

3.3 Competitive Equilibrium with Aggregate Shocks

We conjecture a competitive equilibrium with aggregate shocks that is Markov in $(Y, g)$ and where $P_{at}^* = x_t \bar P_a$ and $P_{bt}^* = x_t \bar P_b$. Given this conjecture, we can use the properties of the generator $\mathcal{L}_{abe}$ in (25) and (26) to write

$$\mathcal{L}_{abe}(c, d; P_{at}^*, P_{bt}^*, x_t) = x_t^{-1} \mathcal{L}_{abe}(c, d; \bar P_a, \bar P_b, 1),$$

and the adjoint

$$\mathcal{L}_{abe}^\dagger(c, d; P_{at}^*, P_{bt}^*, x_t) = x_t^{-1} \mathcal{L}_{abe}^\dagger(c, d; \bar P_a, \bar P_b, 1).$$

The HJB equation is

$$\rho V(a, b, e; Y, g) = \max_{c \geq 0, \ d \in [-a \bar P_a / \bar P_b, b]} Y^{1-\gamma} \frac{c^{1-\gamma}}{1 - \gamma} + x(Y, g)^{-1} \mathcal{L}_{abe}(c, d; \bar P_a, \bar P_b, 1)V(a, b, e; Y, g) + \mathcal{L}_{Y g}(Y, g)V(a, b, e; Y, g).$$  \hspace{1cm} (29)

We guess and verify that

$$V(a, b, e; Y, g) = x(Y, g)Y^{1-\gamma}V(a, b, e).$$

Plugging into the HJB we get

$$\rho x(Y, g)Y^{1-\gamma}V(a, b, e) = Y^{1-\gamma} \max_{c \geq 0, \ d \in [-a \bar P_a / \bar P_b, b]} \frac{c^{1-\gamma}}{1 - \gamma} + \mathcal{L}_{abe}(c, d; \bar P_a, \bar P_b, 1)V(a, b, e) + V(a, b, e)\mathcal{L}_{Y g}(Y, g)(x(Y, g).$$
Using the steady state HJB (27), we see the policy functions are unchanged and we divide throughout by $\bar{V}(a, b, e)$ to obtain

$$\rho x(Y, g)Y^{1-\gamma} = \rho Y^{1-\gamma} + \mathcal{L}_{Yg}(Y, g)(x(Y, g)Y^{1-\gamma}).$$

(30)

We can integrate this expression to obtain

$$x(Y_t, g_t)Y_t^{1-\gamma} = \mathbb{E}_t \left[ \int_t^\infty e^{-(\rho(s-t))} Y_s^{1-\gamma} ds | Y_t, g_t \right] = \rho Y_t^{1-\gamma} \mathbb{E}[\int_t^\infty \exp(\int_t^s \gamma g_u - \rho) du) ds | Y_t, g_t],$$

which verifies the guess.

The policy functions are the same as in the steady state, $c^*_t = \bar{c}$ and $d^*_t = \bar{d}$, so if we start at $\mu_0 = \bar{\mu}$ the KFE is

$$d\mu^*_t = \mathcal{L}_{abe}(c^*_t, d^*_t, P^*_a, P^*_b, x_t)\mu^*_t dt = x_t^{-1} \mathcal{L}_{abe}^s (\bar{c}, \bar{d}, \bar{P}_a, \bar{P}_b, 1) \bar{\mu} dt = 0,$$

so $\mu^*_t = \bar{\mu}$. Since the policy functions and the measure are both as in the steady state, we have market clearing.

All that remains is to pin down the returns. For the illiquid asset

$$\mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^s r_{au} du} (1 - \theta)a Y_s ds \right] = x_t \bar{P}_a (1 - \theta) \alpha Y_t.$$

Taking time derivatives and matching terms we obtain

$$r_{at} = \rho + \gamma g_t - \frac{\rho - \bar{r}_a}{x_t},$$

(31)

$$\sigma_{at} = \frac{x_g(Y, g)}{x(Y, g)} \sigma_g(Y, g).$$

(32)

The same procedure works for the liquid asset.

**Proposition 2.** Assume there exists a steady state equilibrium of the CRRA economy $\bar{c}, \bar{d}, \bar{r}_a, \bar{r}_b, \bar{P}_a, \bar{P}_b, \bar{\mu}$ with $C^1$ value function $\bar{V}(a, b, e)$ satisfying the HJB equation (27). Then if $\mu_0 = \bar{\mu}$, there exists a competitive equilibrium with

$$c^*_t(\cdot) = \bar{c}(\cdot), \quad d^*_t(\cdot) = \bar{d}(\cdot), \quad P^*_a = x_t \bar{P}_a, \quad P^*_b = x_t \bar{P}_b, \quad \mu_t^* = \bar{\mu}.$$

**Expected asset returns are**

$$r_{jt} = \rho + \gamma g_t - \frac{\rho - \bar{r}_j}{x_t}, \quad j \in \{a, b\}$$

(33)
and asset volatility is \( \sigma_{at} = \sigma_{bt} = \frac{x_a(Y,g)}{x(Y,g)} \sigma_g(Y,g) \).

As in the log case, this result allows us to construct the competitive equilibrium with aggregate shocks assuming the initial distribution \( \mu_0 = \bar{\mu} \). In Section 5 we extend this result to construct the competitive equilibrium with aggregate shocks for any initial distribution \( \mu_0 \).

### 3.4 Risk Premia

Even though output and dividend claims are locally deterministic, asset values are locally risky in real terms, \( \sigma_{at} = \sigma_{bt} > 0 \), because of fluctuations in price-dividend ratios.\(^9\) However, there is no risk premium. The expected return of any illiquid asset is \( r_{at} \). To show this, introduce a zero-net-supply derivative into the illiquid account, with Sharpe ratio \( \pi_t \) and unit loading on \( M_t \). This creates a dynamically complete financial market within the illiquid account, allowing us to price any illiquid asset. We will show that \( \pi_t = 0 \) and there is no trading in the derivative. The competitive equilibrium is therefore unchanged by the completion of the market.

Let \( F_{it} \) be household \( i \)'s derivative position and \( f_{it} = F_{it}/A_{it} \). With \( \pi_t = 0 \), their illiquid budget constraint is

\[
dA_{it} = r_{at}A_{it}dt + D_{it}dN_{it} + (\sigma_{at} + f_{it})A_{it}dM_t.
\]

The law of motion of \( a_{it} \), in the absence of a trading opportunity, is now

\[
a_{it} = (\frac{a_{it}}{P_{at}} - a_{it}f_{it}\sigma_{at})dt + a_{it}f_{it}dM_t.
\]

The HJB (29) gets the following extra terms on the right hand side:

\[
x(Y,g)Y^{1-\gamma}\hat{V}_a(a, b, e)af\sigma_{at} + \frac{1}{2}x(Y,g)Y^{1-\gamma}\hat{V}_{aa}(a, b, e)(af)^2 + \frac{x_g(Y,g)}{x(Y,g)} \sigma_g(Y,g)afx(Y,g)Y^{1-\gamma}\hat{V}(a, b, e).
\]

Using that \( \sigma_{at} = \frac{x_a(Y,g)}{x(Y,g)} \sigma_g(Y,g) \), the expression simplifies to

\[
\frac{1}{2}x(Y,g)Y^{1-\gamma}\hat{V}_{aa}(a, b, e)(af)^2.
\]

As long as \( \hat{V}(a, b, e) \) is concave in \( a \), the optimal policy is \( f = 0 \), for any \( (a, b, e; Y, g) \). As a result, the value function and policy functions are unchanged, as well as the evolution of \( \mu \).

---

\(^9\)The model is cast in real terms, but a valid interpretation of liquid asset is nominally safe money or debt (public or private).
The competitive equilibrium is unchanged.

**Proposition 3.** Assume $\tilde{V}(a, b, e)$ is concave in $a$. Then there is no risk premium. The price of aggregate risk is $\pi_t = 0$ and any illiquid asset has expected return $r_{at}$.

**Discussion of the CRRA Economy**

The CRRA economy allows for exact aggregation in spite of the time-varying price-dividend ratios because the state-dependent idiosyncratic labor income process and trading opportunities prevent redistribution in response to aggregate shocks. With $\gamma > 1$ interest rates fall more than one-to-one with growth $g_t$. If the labor income generator was invariant to the cycle, as in the log economy, the onset of a recession (low $g_t$) would redistribute from those whose labor income is front-loaded to those whose labor income is backloaded. Unemployed households would gain, in present value terms, with the start of a recession (relative to employed households). Normalizing the generator by $x_t^{-1}$ as in (21) prevents this redistribution: currently unemployed households bear a disproportionate share of the reduction in future labor income which compensates for the lower discounting on future labor income.

But the time-varying price-dividend ratios do change the supply of liquidity and therefore affect the spread between liquid and illiquid assets. Since with $\gamma > 1$ interest rates fall more than one to one with growth, the onset of a recession raises the supply of liquidity relative to the needs derived from labor income and trading risk. As a result, the spread $s_t = (\tilde{r}_a - \tilde{r}_b)/x_t$ shrinks during recessions, when $x_t$ is large. Exact aggregation with aggregate shocks depends on three elements adjusting in the same way: price-dividend ratios, the generator for idiosyncratic labor income, and trading frictions. This can be seen in the inverse homogeneity property of the $L_{aeb}$ generator in (25).\(^{10}\)

The returns of the liquid and illiquid assets don’t satisfy exact aggregate consumption Euler equations. The last term in (33) captures the idiosyncratic consumption fluctuations, including mean reversion and precautionary saving. But if trading frictions are large, in the steady state $\bar{r}_a \approx \rho$ and the spread $\bar{s} = \bar{r}_a - \bar{r}_b$ will be large, and this term will vanish for the illiquid asset. Because of trading frictions, the illiquid asset is not very good at insuring the agent against transitory risks. We then get an aggregate consumption Euler equation for the illiquid asset, $r_{at} \approx \rho + \gamma g_t$, and a time-varying spread $s_t = (\bar{r}_a - \bar{r}_b)/x_t$ that shrinks during recessions, potentially consistent with a low and stable return on liquid assets. In addition, while the volatility of liquid and illiquid assets is locally the same, $\sigma_{at} = \sigma_{bt}$, the cumulative return over a finite interval could be more volatile for illiquid assets because their expected

\(^{10}\)Trading frictions become worse during recessions, $x_t^{-1} \chi$, which raises the spread. With constant $\chi$ the spread would fall even more.
returns are stochastic, \( r_{at} \) and \( r_{bt} \). In the next Section we provide a back-of-the-envelope quantification illustrating this point.

4 Back-of-the-Envelope Quantification

Proposition 2 delivers a sharp result that characterizes the expected returns \( r_{at} \) and \( r_{bt} \) in terms of the preference parameters \((\rho, \gamma)\), the steady state interest rates \( \bar{r}_a \) and \( \bar{r}_b \), and the exogenous growth process \( g_t \) (which determines the value of \( x_t \)). In this subsection, we provide a back-of-the-envelope quantification of these results. Specifically, we show that if consumption growth has a small but persistent component (as in the literature on long-run risks), this can generate substantial variation in \( x_t \). With \( \gamma > 1 \) and a persistent growth process, \( x_t \) will decrease when \( g_t \) increases.

Meanwhile, in steady state, \( \rho \geq \bar{r}_a \) as a result of the agents’ transversality conditions, and \( \bar{r}_a > \bar{r}_b \) as a result of the value agents place on self-insuring against binding borrowing constraints (as in Aiyagari [1994]). It follows that variation in \( x_t \) will have a larger effect on \( r_{bt} \) than on \( r_{at} \) (because \( \rho - \bar{r}_a \) is smaller than \( \rho - \bar{r}_b \)). Because \( x_t \) declines when \( g_t \) increases, this has the effect of reducing the sensitivity of \( r_{bt} \) to growth shocks, while at the same time increasing its sensitivity to shocks to \( x_t \) that are not related to growth rates. Put another way, a representative agent Euler equation will work better for \( r_{at} \) than \( r_{bt} \), consistent with the empirical evidence of Di Tella et al. [2023]. We explore this idea in a back-of-the-envelope quantification in the remainder of this section.

To proceed, we must first determine the empirical counterparts of the liquid and illiquid assets in our model. We adopt the view, as in Di Tella et al. [2023], that stocks are illiquid, and therefore identify \( r^a_t \) with the expected return on a stock portfolio that has no risk premium—what we have called the “zero-beta rate.” With regards to liquid assets, the empirical counterpart is less clear. Money market fund shares, savings deposits, checking deposits, and physical cash are all at least somewhat liquid, but offer different interest rates (presumably because they are not perfect substitutes, which is not captured by the two-account model). Following Kaplan and Violante [2022], we will use cash and checking deposits as the empirical counterpart for the liquid asset. As these assets have a nominal return that is close to zero, we will identify \( r^b_t \) with the negative of expected inflation. Note that our model has nothing to say about bond yields.

We must also choose an empirical counterpart for \( g_t \). In the version of the model presented thus far, consumption is equal to output, and the model offers no guidance about whether \( g_t \) should be the growth of consumption or of output. In what follows, we will assume that \( g_t \) is expected per-capita consumption growth, and return to this issue when we study a model.
with capital.

Armed with empirical counterparts for $r_{at}$, $r_{bt}$, and $g_t$, we can now consider the implications of Proposition 2. To do so, we need to determine preference parameters ($\gamma$ and $\rho$) as well as the interest rates $\bar{r}_a$ and $\bar{r}_b$ that would prevail in a counterfactual economy without aggregate risk.

We start by choosing preference parameters; we assume $\gamma = 6$ and $\rho = .05$. The former is consistent with the estimates of Di Tella et al. [2023]. The latter is an ad-hoc choice, but will end up playing only a small role in our calibration.\(^\text{11}\)

We next select a structure for our assumed growth process. Specifically, we define

$$
v_t = \begin{bmatrix} g_t - \mu \\ y_t - y_0 - \mu t \end{bmatrix}
$$

where $y_t = \ln(Y_t)$ and assume that

$$
dv_t = \begin{bmatrix} -(\kappa_g + \kappa_y) & -\kappa_g \kappa_y \\ 1 & 0 \end{bmatrix} v_t + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dM_t.
$$

This structure is equivalent to assuming that growth is the sum of an Ornstein-Uhlenbeck (OU) process with parameter $\kappa_g$ (the continuous time analog of an AR(1) process) and a reversion of output towards trend at rate $\kappa_y$. The latter force generates a trend-stationary output distribution.

We set $\mu = 1.5\%$ equal to the average growth rate of consumption. We choose the parameters $\sigma, \kappa_g, \kappa_y$ to match a standard deviation of expected consumption growth equal to 1.2%, a one-year auto-correlation of expected consumption growth equal to about 0.89, and a standard deviation of the log price-dividend ratio of a consumption claim (with $\gamma = 6$) to 26%. The first two of these are similar to the dynamics of expected consumption growth in the long-run risks model of Bansal and Yaron [2004], and hence are not obviously inconsistent with consumption data, while the last is close to the VAR estimate of Di Tella et al. [2023].

The calibrated parameters are $(\kappa_g = .1, \kappa_y = 0.01, \sigma = 3 \times 10^{-5})$, which is to say that growth rate shocks have a half-life of about 7 years, and there is a very slow reversion towards trend output. The resulting consumption process is trend stationary, but would be difficult to distinguish from a unit root process in the data (consistent with empirical evidence).

\(^\text{11}\)That is, whether the annual discount rate is 4, 5, or 6% matters very little in what follows. However, very high discount rates would have the effect of eliminating most of the volatility in $x_t$.
in the consumption-wealth ratio. We conjecture that other approaches from the asset pricing literature (such as persistent time-varying disaster probabilities, Wachter [2013], or persistent shocks to the volatility of consumption growth, which is the key mechanism in Bansal and Yaron [2004]) could deliver similar results. Note, however, that unlike these models our model has no risk premia, and so in this sense is quite different.

Armed with a description of the process for \( v_t \), we can compute \( x_t = x(v_t) \) using (23). For the purpose of computing \( r_{at} \) and \( r_{bt} \) using Proposition 2, it is more useful express our calibration results in terms of \( x_t^{-1} \). Table 1 below shows some of the summary statistics that result from our calibration.

Let us now apply the results of Proposition 2. We will begin by considering a fictitious economy without growth or risk. Proposition 2 illustrates the relationship between the interest rate in this economy and the average interest rate and value of \( x_t^{-1} \) in the stochastic economy:

\[
E[r_{jt}] = \rho + \gamma E[g_t] - (\rho - \bar{r}_j) E[x_t^{-1}].
\] (34)

In the main sample of Di Tella et al. [2023], the average real zero-beta rate (\( E[r_{at}] \)) is about 9% annualized, and the negative of inflation (\( E[r_{bt}] \)) is about -3.8% annualized. Using these values, the means of \( g_t \) and \( x_t^{-1} \) from Table 1, and the equation above, we calibrate

\[
\bar{r}_a \approx 2.9\%, \quad \bar{r}_b \approx -2.5\%.
\] (35)

Note that these rates, particularly for the illiquid asset, are quite different from the average rates observed in the data. With \( \gamma = 6 \), the effect of growth on the level of rates is substantial, and a counter-factual economy without growth would have very different interest rates than the actual economy.

Armed with this calibration, we can study the properties of \( r_{at} \) and \( r_{bt} \) generated from Proposition 2 in our model. The following table shows the volatility and correlation of \( r_{a,t} \) and \( r_{b,t} \) with growth, as well as the “beta” of a regression of \( r_{jt} \) on growth.

These result reproduce, qualitatively, several features of the empirical findings of Di Tella et al. [2023]. First, an aggregate Euler equation holds for the illiquid asset, in the sense that
Table 2: Properties of Expected Returns in the Stochastic Economy

<table>
<thead>
<tr>
<th></th>
<th>$r_{bt}$</th>
<th>$r_{at}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$std(r_{jt})$</td>
<td>3.0%</td>
<td>5.8%</td>
</tr>
<tr>
<td>$corr(r_{jt}, g_{t})$</td>
<td>0.88</td>
<td>1</td>
</tr>
<tr>
<td>$\beta : r_{jt} = \beta g_{jt} + \epsilon_{t}$</td>
<td>2.2</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Notes: The statistics in this table are derived from Proposition 2, the statistics in Table 1, and our calibration of the steady state interest rates, (35).

that $r_{at}$ and $g_{t}$ co-move almost perfectly in the model, with an IES of about 0.2. The same is not true of $r_{bt}$, which is only imperfectly correlated with $g_{t}$. Intriguingly, the aggregate Euler in this calibration holds approximation for the illiquid asset but not the liquid asset, despite the fact that in the model many agents (all of those who are not borrowing constrained) are “on their Euler equation” with respect to the liquid asset but not the illiquid one.

The specific numbers in this table depend, of course, on our calibration. However, we expect that the same results will hold qualitatively in any calibration in which (i) $\gamma > 1$ and $g_{t}$ is persistent, so that $g_{t}$ and $x_{t}$ are negatively correlated, (ii) $x_{t}$ is not perfectly correlated with $g_{t}$, and (iii) $\bar{r}_{a}$ is substantially closer to $\rho$ than $\bar{r}_{b}$. The last of these is a feature of almost all of the calibrations in Kaplan and Violante [2022], and arises as a consequence of the fact that wealthy agents hold most of the wealth (which ensures that $\bar{r}_{a}$ cannot be too far below $\rho$) and that borrowing constraints bind for many agents (which is necessary to match MPC data and ensures that $\bar{r}_{b}$ is substantially below $\bar{r}_{a}$).

5 Transition Dynamics

In Sections 2 and 3 we constructed competitive equilibria with aggregate shocks for the case where the cross-sectional distribution starts at its steady state value, $\mu_{0} = \bar{\mu}$. Here we extend those results for an arbitrary initial distribution, $\mu_{0}$, which is to that we construct competitive equilibria that feature both aggregate shocks and transition dynamics.

5.1 Deterministic path

Consider a aggregate-deterministic economy starting with an arbitrary $(Y_{0}, g_{0})$ and $\mu_{0}$ but no aggregate shocks $M_{t} = 0$, and associated paths $\bar{Y}(t)$, $\bar{g}(t)$, and $\bar{x}(t)$. Assume there exists a competitive equilibrium,

\[ \bar{c}(\cdot, t), \bar{d}(\cdot, t), \bar{P}_{a}(t), \bar{P}_{b}(t), \bar{r}_{a}(t), \bar{r}_{b}(t), \bar{\mu}(t), \]
and $\bar{\sigma}_a(t) = \bar{\sigma}_b(t) = 0$. The generator $L_{abe}$ is as in (24). The value function is $\bar{V}(a, b, e, t)$, with HJB equation

$$
\rho \bar{V}(a, b, e, t) = \max_{c \geq 0, d \in [-a \bar{\nu} / \bar{P}_a / \bar{P}_b, a]} Y(t)^{1-\gamma} \frac{c^{1-\gamma}}{1-\gamma} + L_{abe}(c, d; \bar{P}_a(t), \bar{P}_b(t), \bar{x}(t)) \bar{V}(a, b, e, t) + \bar{V}_t'(a, b, e, t).
$$

(36)

The policy functions $\bar{c}(\cdot, t)$ and $\bar{d}(\cdot, t)$ are derived from the HJB equation, and the KFE is

$$
d\bar{\mu}(t) = L_{abe}(\bar{c}(\cdot, t), \bar{d}(\cdot, t); \bar{P}_a(t), \bar{P}_b(t), \bar{x}(t)) \bar{\mu}(t) dt,
$$

with $\bar{\mu}(0) = \mu_0$.

The deterministic path is a generalization of the steady state equilibrium we used in Sections 2 and 3. If we start with the initial distribution $\mu_0$ corresponding to the steady state, then the deterministic path coincides with the steady state equilibrium.

### 5.2 Stochastic Time Change

We will construct the competitive equilibrium using the deterministic path with a stochastic time change. Define the stochastic process $\tau_t$ as the solution to the SDE

$$
\tau_t = \int_0^t \bar{x}(\tau_s)x_s^{-1} ds,
$$

where $x_t = x(Y, g)$ is as in (23). $\tau$ is a strictly increasing stochastic process, but the speed is not uniform. Time goes always forward, but at varying speed. It encodes the history of aggregate shocks in terms of deviations from the deterministic path. If shocks actually realize all to zero, $M_t = 0$, then $x_t = \bar{x}(t)$ and $\tau_t = t$. Aggregate shocks that raise $x_t$ relative to the deterministic path (with $\gamma > 1$, a period of low growth relative to the path, such as a recession), are encoded as a slowdown of time $\tau$.

### 5.3 Competitive Equilibrium with Shocks

We will look for a competitive equilibrium that is Markov in $(Y, g, \tau)$ with $P^*_a = \frac{\bar{x}_t}{x(\tau_t)} \bar{P}_a(\tau_t)$ and $P^*_b = \frac{\bar{x}_t}{x(\tau_t)} \bar{P}_b(\tau_t)$. Notice that the deterministic-path objects $\bar{x}(\cdot)$ and $\bar{P}_j(\cdot)$ are evaluated at the stochastic time $\tau_t$ instead of the usual $t$. Using (25), the generator is

$$
L_{abe}(c, d; P^*_a, P^*_b, x_t) = \bar{x}(\tau_t)x_t^{-1}L_{abe}(c, d; \bar{P}_a(\tau_t), \bar{P}_b(\tau_t), \bar{x}(\tau_t)).
$$

(37)
We will also use the infinitesimal generator for $Y$, $g$, and $\tau$,
\[ \mathcal{L}_{Yg}(Y, g, \tau)f(\cdot) = f_Y(\cdot)Yg + f_g(\cdot)\mu_g(Y, g) + \frac{1}{2}f_{gg}(\cdot)\sigma_g(Y, g)^2 + f_\tau(\cdot)\bar{x}(\tau)x(Y, g)^{-1}. \]

The value function is $V(a, b, c; Y, g, \tau)$ and the HJB equation (suppressing arguments to avoid clutter)
\[ \rho V = \max_{c \geq 0, \bar{d} \in [-a \frac{1}{\sigma_a}, a \frac{1}{\sigma_b}/b]} Y^{1-\gamma} \frac{c^{1-\gamma}}{1-\gamma} + \bar{x}(\tau)x(Y, g)^{-1} \mathcal{L}_{abe}(c, d; \bar{P}_a(\tau), \bar{P}_b(\tau), \bar{x}(\tau))V + \mathcal{L}_{Yg}\bar{V}. \]

We guess and verify that
\[ V(a, b, c, Y, g, \tau) = \frac{x(Y, g)}{\bar{x}(\tau)}(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} \bar{V}(a, b, c, \tau). \]

Plugging this into the HJB equation:
\[ \rho \frac{x(Y, g)}{\bar{x}(\tau)}(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} \bar{V} = (\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} \left\{ \max_{c \geq 0, \bar{d} \in [-a \frac{1}{\sigma_a}, a \frac{1}{\sigma_b}/b]} Y^{1-\gamma} \frac{c^{1-\gamma}}{1-\gamma} + \mathcal{L}_{abe}(c, d; \bar{P}_a(\tau), \bar{P}_b(\tau), \bar{x}(\tau))V + \bar{V} \right\} \]
\[ + \bar{V} \mathcal{L}_{Yg} \left[ \frac{x(Y, g)}{\bar{x}(\tau)}(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} \right]. \]

Use the deterministic-path HJB to notice the policy functions will be unchanged but evaluated at $\tau$, $c_\tau^* = \bar{c}(\tau)$ and $d_\tau^* = \bar{d}(\tau)$. Plug in (36) and divide by $\bar{V}$ to get
\[ \rho \frac{x(Y, g)}{\bar{x}(\tau)}(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} = \rho(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} + \mathcal{L}_{Yg} \left[ \frac{x(Y, g)}{\bar{x}(\tau)}(\frac{Y}{\bar{Y}(\tau)})^{1-\gamma} \right], \]

which expanding the last term is
\[ \rho \frac{x(Y, g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}} = \rho \frac{Y^{1-\gamma}}{\bar{Y}(\tau)^{1-\gamma}} + \mathcal{L}_{Yg} \frac{x(Y, g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}} - \frac{x(Y, g)Y^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}} \mathcal{L}_{Yg} \frac{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}}{\bar{x}(\tau)\bar{Y}(\tau)^{1-\gamma}}. \]

To verify our guess we need to make sure this expression is true. First, write $x(Y, g)$ and $\bar{x}(t)$ in the following form
\[ x(Y, g)^{1-\gamma} = \mathbb{E} \left[ \int_t^\infty e^{-\rho(s-t)} \rho Y_s^{-1} ds | Y_t = Y, g_t = g \right], \]
\[ \bar{x}(t)\bar{Y}(t)^{1-\gamma} = \mathbb{E} \left[ \int_t^\infty e^{-\rho(s-t)} \rho \bar{Y}(s)^{-1} ds \right]. \]
We can write the HJB equations for each,

\[
\mathcal{L}_{Ygt}x(Y, g)Y^{1-\gamma} = \rho x(Y, g)Y^{1-\gamma} - \rho Y^{1-\gamma},
\]

(40)

\[
\nabla_t \bar{x}(t) \bar{Y}(t)^{1-\gamma} = \rho \bar{x}(t) \bar{Y}(t)^{1-\gamma} - \rho \bar{Y}(t)^{1-\gamma},
\]

(41)

\[
\implies \mathcal{L}_{Ygt} \bar{x}(\tau) \bar{Y}(\tau)^{1-\gamma} = (\rho \bar{x}(\tau) \bar{Y}(\tau)^{1-\gamma} - \rho \bar{Y}(\tau)^{1-\gamma}) \times \bar{x}(\tau) x(Y, g)^{-1},
\]

(42)

where the last expression uses the time change. Plugging (40) and (42) into (39) and cancelling terms, we verify our guess.

Since the policy functions are unchanged, the measure \(\mu_t = \bar{\mu}(\tau_t)\). The KFE is

\[
d\mu_t = \mathcal{L}^\dagger_{abe}(c_t^*, d_t^*, P_a^*, P_b^*, x_t)\mu_t dt = \underbrace{\mathcal{L}_{abe}(\bar{c}(\tau_t), \bar{d}(\tau_t); \bar{P}_a(\tau_t), \bar{P}_b(\tau_t), \bar{x}(\tau_t)) \bar{\mu}(\tau_t) \bar{x}(\tau_t)x^{-1}_t dt}_{d\bar{\mu}(\tau)/dt}.
\]

(43)

Since both policy functions and \(\mu\) correspond to the deterministic path evaluated at \(\tau\), we also have market clearing.

All that remains is to pin down the returns. For the illiquid asset

\[
\mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^u r_u du} (1 - \theta) \alpha Y_s ds \right] = \frac{x_t}{\bar{x}(\tau_t)} \bar{P}_a(\tau_t) (1 - \theta) \alpha Y_t.
\]

Take time derivatives and match terms to obtain

\[
r_{at}^* = \rho + \gamma g_t - \frac{\bar{x}(\tau_t)}{x_t} (\rho + \gamma \tilde{g}(\tau_t) - \tilde{r}_a(\tau_t)),
\]

\[
\sigma_{at} = \frac{x'_{g}(Y, g)}{x(Y, g)} \sigma_g(Y, g).
\]

The same argument can be applied to liquid assets.

**Proposition 4.** Take initial conditions \(Y_0, g_0, \) and \(\mu_0\) as given, with associated deterministic path \(\bar{Y}(t), \tilde{g}(t), \) and \(\bar{x}(t)\). Assume there exists a deterministic equilibrium of the CRRA economy \((c(\cdot, t), d(\cdot, t), \tilde{r}_a(t), \tilde{r}_b(t), \bar{P}_a(t), \bar{P}_b(t), \bar{\mu}(t))\) with \(C^1\) value function \(\bar{V}(a, b, e, t)\) satisfying the HJB equation (36). Then there exists a competitive equilibrium with

\[
c_t^*(\cdot) = \bar{c}(\cdot, \tau_t), \quad d_t^*(\cdot) = \bar{d}(\cdot, \tau_t), \quad P_{at} = \frac{x_t}{\bar{x}(\tau_t)} \bar{P}_a(\tau_t), \quad P_{bt} = \frac{x_t}{\bar{x}(\tau_t)} \bar{P}_b(\tau_t), \quad \mu_t^* = \bar{\mu}(\tau_t).
\]

Asset returns are

\[
r_{jt}^* = \rho + \gamma g_t - \frac{\bar{x}(\tau_t)}{x_t} (\rho + \gamma \tilde{g}(\tau_t) - \tilde{r}_j(\tau_t)), \quad j \in \{a, b\}
\]

(44)
and asset volatility is \( \sigma_{at} = \sigma_{bt} = \frac{x'(Y,g)}{x(Y,g)} \sigma_g(Y,g) \).

**Proof.** (TBD) See the Appendix, section A.2.

Observe that Proposition 2 can be viewed as a special case of this result, in which \( \bar{g}(t) = 0 \), \( \bar{x}(t) = 1 \), and \( \bar{\mu}(t) \) is constant and equal to the ergodic population measure.

### 6 Endogenous Inputs and Level Shocks

In this section, we extend our results to a setting with endogenous labor and capital, in which there are both shocks to growth and shocks to the level of TFP. To reduce the need for additional notation, we will return to considering the steady-state economy, as opposed to building on the transition dynamics results of the previous section. That said, merging of this results of this section and of the previous one is notationally cumbersome but conceptually straightforward.

[REMAINDER OF SECTION TBD]

### 7 Conclusion

In this paper, we have analytically characterized asset prices in a two-account heterogenous agent model with uninsured idiosyncratic risk, borrowing constraints, and aggregate risk. We have shown that, in the log utility case and in the absence of redistributive effects, an aggregate Euler equation holds for both assets. In the CRRA case, a modified form of the aggregate Euler equation holds. In our back-of-the-envelope quantification, this modification matters very little for the illiquid assets but substantially affects the liquid assets, offering an explanation for the empirical evidence of Di Tella et al. [2023] and pointing towards to a liquidity-based, rather than risk-based, explanation for asset prices. Our methods can be extended to the case of transition dynamics and endogenous labor and capital, and thus provide analytical results in models that are close to the quantitative models of the existing literature.

### References


Iván Werning. Incomplete markets and aggregate demand. 2015.

A Appendix Starts Here (TBD)

A.1 Proof of 1

A.2 Proof of 4