Interbank trade: why it's good and how to get it*

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September 21, 2023

Abstract

We provide a theory for the collapse in bank-to-bank trade experienced in the United States since the 2007-2009 financial crisis and since the onset of a floor system for monetary policy. We show that the choice of framework to implement monetary policy affects welfare, and that interbank trade is not feasible under a floor system. Our theory incorporates a key feature of the Federal Funds market: credit is unsecured and, thus, subject to an endogenous borrowing limit. When the punishment for failing to repay loans is permanent exclusion from interbank markets, if interest rates are too low, then the opportunity cost of holding money is also low, and so is the punishment for defaulting on loans. Hence borrowing banks have no incentive to repay their loans, the endogenous borrowing limit is zero and bank-to-bank lending collapses, despite banks still having a need to borrow. We propose a framework of Voluntary Reserve Targets to implement monetary policy that restores an active interbank market, and we provide conditions for it to be welfare improving over a simple reserve remuneration framework, even away from the floor.

*Previous versions of this paper circulated with the title "Limited Commitment and the Implementation of Monetary Policy".

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1 Introduction

Central banks' have radically altered their playbooks for intervening in money markets since the financial crisis. For example, the Federal Reserve now affects interbank markets not through daily changes in the supply of reserves, but instead through the rate it pays on deposits, operating a so-called floor system for its implementation of monetary policy. Under this floor system, banks hold large excess reserves at the central bank instead of lending to each other, the central bank's administered rate supplants the market rate as the relevant marginal rate, and the previously robust federal funds market has become moribund. As the federal funds market once played a key role in reallocating funds among banks, ultimately supporting overall investment, one wonders whether floor systems have an effect not just on money markets, but on the broader economy, and whether alternative frameworks could revive money markets, potentially improving welfare.

In addressing these questions, this paper seeks to understand the impact of a floor system on the structure of money markets, banks' portfolio choices, and ultimately aggregate investment, in a general equilibrium, microfounded model. In the model, agents seek to borrow and lend to reallocate funds to those with the most valuable opportunities. Crucially, we incorporate a distinguishing feature of the Federal Funds market: these loans are unsecured and subject to borrowing constraints. Hence, we adopt the standard model of endogenous credit limits for unsecured markets, the limited commitment model of Kehoe and Levine (1993) and Alvarez and Jermann (2000) where it has to be in borrowers' interest to repay loans, as they cannot be forced to do so. In this model, when punishment for failing to repay loans is permanent exclusion from money markets, a lender is more willing to extend credit to a borrower who will more often need to borrow in the future. In fact, such borrower values future access to credit more than a borrower with fewer needs to borrow again, and, as a consequence, has more incentives to repay current loans. In the interbank market, when banks are flush with reserves they are not likely to need to borrow often. This makes them untrustworthy borrowers to the eyes of a lender, causing credit limits in the interbank market to collapse despite banks would want to borrow were they not constrained.

Our analysis delivers three key insights. First, we characterize the set of parameters which lead to a floor system in the long run and to a collapse in money market activity. However, that lack of borrowing does not imply satiation in real balances – a floor system is not sufficient to be at the Friedman rule. Second, among the set of parameters that lead to a floor system, the Federal Reserve's current policy – a so-called "ample reserve regime" where banks hold the least possible reserves consistent with the floor – is the locally *worst* policy, and lifting off of the floor is the preferred local perturbation of policy. Third, we introduce a monetary policy implementation framework that restores money market activity and characterize the welfare properties of the unique equilibrium associated with it.

In policy circles, authors define a floor in terms of an endogenous object, the market rate, equaling an exogenous object, the rate the central bank pays in interest on reserves (IOR). Our first result shows that, in the context of our model, this can be stated purely in terms of exogenous parameters: The market rate equals IOR when IOR equals or exceeds the rate of inflation. That is, a floor obtains when IOR is weakly positive in real terms.

The intuition is as follows. Consider first frictionless markets: from the perspective of a lender, if the market rate is at the floor then lending to another bank and depositing at the central bank are payoff equivalent strategies. From the perspective of a borrower, however, borrowing at a rate above IOR is more profitable than having to carry money balances which depreciate in real terms overnight. Therefore, it is necessary that borrowers demand no loans for the market rate to fall to the floor. In other words, with frictionless credit markets, borrowers' demand for loans disappears only at the Friedman rule, when IOR compensates both for inflation and the rate of time preference. A similar intuition applies to markets with exogenous credit limits. With a given upper bound on borrowing capacity, a potential borrower would always be willing to pay a rate above IOR to avoid the inflation tax. Hence the market rate falls to the floor only if borrowers' demand for loans disappears. In equilibrium this can happen generically only at the Friedman rule, if the exogenous credit limits are strictly positive. In contrast, with endogenous credit limits, borrowing capacity collapses when a borrower is indifferent between repaying and defaulting on his loans. Without access to the money market, which is the punishment for defaulting, agents with few investment opportunities will carry balances between periods, earning IOR. When inflation exceeds IOR, these balances depreciate in real terms, a cost. Agents with market access, however, will lend to one another, and, as a group, carry no excess balances, avoiding this cost. Hence, endogenous credit limits that reflect the value of market access will depend on the difference between inflation and IOR. When IOR exceeds inflation, there is no value of future market access, so credit limits collapse, addressable loan demand evaporates, and the market rate falls to the floor, even if the economy is away from the Friedman rule. Endogenous credit frictions imply that the market hits the floor before IOR even begins to compensate for time preference. Therefore, a floor system does not imply that agents are satiated with real balances.

A floor system may not achieve efficiency because credit frictions prevent the efficient reallocation of reserves and because real balances are too low for agents to self-fund all opportunities. Hence, among different floor systems, higher real balances coincide with higher welfare. Alternately, setting IOR just less than inflation, reducing real balances but lifting off of the floor, revives the money market, and allows for reallocation of liquidity, discretely improving welfare. Hence, our second result: a policy where balances are just large enough to keep rates on the floor and kill the money market is dominated by both lower and higher balances, and the locally preferred policy features lower balances with an active market.

Overall, our findings are consistent with several pieces of empirical evidence: Afonso *et al.* (2019a), Afonso *et al.* (2013a) and Afonso *et al.* (2013b) report trading volume in the federal funds market collapsing from $250 billion per day prior to 2008, with the majority of trades occurring between banks to $75 billion or less per day and rare activity between banks during the decade following the 2009 financial crisis. Eisenbach *et al.* (2019) document that in the

US since the Federal Reserve moved to a floor system for monetary policy implementation "total payments have been co-moving with the total stock of reserves" whereas "payments grew roughly in line with GDP" before the 2008 crisis. The authors attribute these findings to the IOR arbitrage activity, by which they mean that banks borrow at a rate below IOR from other financial market participants who do not have access to IOR, hold the funds as reserves, and earn the interest rate difference. Indeed IOR arbitrage activity has dominated the federal funds market since the 2008 financial crisis: "With no bank-to-bank lending, the overall market volume dropped precipitously."[1] Copeland *et al.* (2021) document stress in the intraday management of reserve balances held at the Federal Reserve by large bank holding companies between 2015 and 2020, contrasting it with the experience preceding the failure of Lehman Brothers in September 2008, when "a small aggregate supply of federal reserve balances was sufficient for large U.S. banks to manage daily payments and for wholesale overnight funding markets to function efficiently". Anbil *et al.* (2020) report how banks' reluctance to increase their lending of excess reserves contributed to amplifying stress events in the repo and federal funds markets in September 2019, and Afonso *et al.* (2021) find suggestive evidence that frictions in the interbank market, related to banks' risk management framework, may have prevented the efficient allocation of reserves across institutions.

In addition, the experience with floor systems at other central banks is consistent with our story. A dynamic similar to the one described in our model has led the Norges Bank in 2010 to switch from a system with abundant reserves (i.e. a floor system) to a system with scarce reserves: "When Norges Bank keeps reserves relatively high for a period, it appears that banks gradually adjust to this level. The incentive to lend is already weak and appears to be further undermined if banks do not need to borrow for a period to meet their needs" (Norges Bank, 2010).

Finally, we advance a concrete proposal for implementing monetary policy that maintains an active interbank market regardless of the level of aggregate reserves, our third result. Within our proposed framework, central banks' assessment of whether reserves are abundant or ample is not necessary to guarantee that money markets effectively reallocate funds without episodes of stress linked to the level of reserves, as experienced in the United States in the last quarter of 2019.[2] In our proposed framework, called a voluntary reserve target (VRT) framework, banks commit in advance to a target for reserve balances at the central bank, and then the central bank pays high interest on balances up to the target, low interest on balances exceeding the target, and charges a fee on shortages relative to the target. With

[1] See Lester (2019), Keating and Macchiavelli (2017), and Federal Reserve Bank of New York (https://www.newyorkfed.org/fed-funds-lending/index.html)

[2] See Lester (2019), Afonso *et al.* (2019c) and Afonso *et al.* (2019b). Also see Nelson (2019) for a discussion of the current implementation framework with abundant reserves and the September 2019 "events in money markets ... [which] ... demonstrated that ... the Fed will have to actively manage the supply of reserves through open market operations and will have to run a significantly larger balance sheet than it ever projected; thus the advantage of the framework seemingly has been lost" and Governor Jefferson for a discussion of the technical adjustment to the supply of reserves related to the money .

banks earning a low rate when overshooting and paying a penalty when undershooting their targets, banks are incentivized to set targets which reflect their expected end of day holdings. This intuition is preserved when banks' choice of targets is complicated by the risk introduced by customers' withdrawals and deposits throughout the day. For example, by raising targets, banks absorb any expected abundance of reserves. A bank experiencing larger withdrawals than expected will need to borrow in the money market, while a bank experiencing smaller withdrawals than expected will want to lend. Hence, we say that VRT generates endogenous scarcity in the interbank market and, in doing so, drives trade in the money market.[3] By endogenously forcing banks to trade on a regular basis (i.e. every time they have to meet their targets) a VRT framework makes every bank a potential future borrower, induces banks to value future access to credit, and, thus, relaxes borrowing constraints.

The paper is organized as follows: section 2 describes the model and characterizes the equilibrium under a floor system; section 3 introduces the VRT framework and characterizes its properties while section 4 provides sufficient conditions for it to relax borrowing constraints in money markets and improve welfare; section 5 concludes.

1.1 Literature

Much of monetary economics abstracts from policy implementation and simply assumes that the central bank can effortlessly establish any policy stance it desires. Yet, not only the practical challenges of implementing policy can constrain the set of feasible policy stances but, most importantly, a central bank's choice of implementation framework can affect welfare.[4] Our analysis contributes to this literature.

Our model builds on Berentsen *et al.* (2014), who study general equilibrium effects of monetary policy implementation via a channel system and a floor system. In their model, borrowing in the money market and at the central bank lending facility is constrained by the amount of government bonds posted as collateral. When the economy is away from the Friedman rule, the general equilibrium effect of running a channel system (i.e. the lending rate exceeds the deposit rate) is higher demand and real value of money, which benefits agents who are borrowing constrained more than hurting agents who are not constrained.

In our model general equilibrium effects play an important role as well, but they interact in non trivial ways with the endogenous borrowing limit. If access to money market is valuable and the borrowing limit is greater than zero, then money demand decreases, as agents use the market to reallocate liquidity. In equilibrium, the interest rate in money market increases, thus raising the opportunity cost of not having access to the market as a lender. Since agents who default on their loans and are excluded from the market carry larger money holdings, because they're unable to borrow, the final effect is a reduction in their payoff. As a result, the endogenous borrowing limit increases.

[3] See Baughman and Carapella (2018).

[4] See Afonso *et al.* (2020), Bech and Keister (2017) and Boutros and Witmer (2020) for the former and Martin and Monnet (2011) for the latter.

Our result that a floor system leads to a collapse in interbank activity is related to the result that inflation is welfare improving when frictions in credit markets give rise to endogenous borrowing constraints. In Berentsen *et al.* (2007) inflation acts as a tax on defaulting agents due to the larger money balances they hold to fund consumption. Our model allows to link this intuition to the choice of implementation framework for monetary policy and to observed borrowing and lending activity in interbank markets. When the central bank runs a floor system there is no opportunity cost of being excluded from credit markets because the interest on reserves deposited at the central bank equals the interest rate earned on loans in private credit markets. As a result, the endogenous borrowing limit collapses.

By comparing the welfare properties of channel systems and open market operations, Martin and Monnet (2011) is very close to our work in spirit. Theirs is a monetary model where banks are risk averse agents facing liquidity shocks, thus lending and deposit facilities offered by the central bank are valuable in facilitating a more efficient allocation of liquidity and of money growth rates.[5] In our economy, a reallocation of liquidity among banks occurs in the interbank market, and different implementation frameworks for monetary policy affect banks' ability to borrow by tightening or relaxing their endogenous borrowing limit. As a result, banks' ability to achieve a more efficient allocation of liquidity depends on the central bank choice of implementation framework. Also, our evaluation of monetary policy implementation frameworks does not rely on swapping assets with different liquidity properties, as it would be for open market operations. In our economy the central bank simply adopts different rules to tender reserves. An implementation framework is the set of such rules, which, we show, matter for welfare.

Finally, but importantly, our results are closely connected to results from the sovereign debt literature. Bulow and Rogoff (1989) show that, when defaulted countries can save at the same rate they borrow at, positive borrowing can not be sustained by reputation alone. Hence, commonly in the literature, in addition to being banned from borrowing, defaulted countries are also assumed excluded from saving on world capital markets. We follow suit, assuming exclusion from both borrowing and saving in the market. Here, it is the central bank, by making private savings too attractive, that delivers the same market collapse in Bulow and Rogoff (1989). Some further differences: Ours is a model of intra-temporal liquidity reallocation, not one of inter-temporal consumption smoothing. And, ours is a monetary model instead of a real one, so we can deliver conditions for the collapse purely in terms of monetary policy.

[5] By setting different rates on its deposit and lending facility, a central bank running a channel system can effectively discriminate between borrower and lender banks, which is not feasible with open market operations where the price of assets in terms of money is unique.

2 Model

Time is discrete and continues forever. Each period is divided into three sub-periods. First, a settlement market where debts are repaid, money is issued by the central bank and agents rebalance their portfolios. Second, a money market with borrowing and lending. Third, an investment market where money can be exchanged to purchase inputs into an investment technology.

There are two non-storable goods. One is produced and consumed in the settlement market and will be termed the settlement good, denoted x. Another good is produced and consumed in the investment market and will be termed the investment good, denoted q. The economy is populated with a unit measure each of two kinds of agents, buyers and sellers. Both agents discount with factor $\beta \in (0, 1)$. Buyers produce the settlement good and consume the investment good and have preferences

$$\mathbb{E}\left[\sum_t \beta^t \left(x_t - h_t + \tilde{\varepsilon}_t \log(q_t)\right)\right]$$

where x_t is the utility from the consumption of the settlement good, $-h_t$ the disutility of production of the settlement good, $\tilde{\varepsilon}_t \log(q_t)$ is utility from consumption of the investment good, and $\tilde{\varepsilon}_t$ is an IID preference shock drawn from $F(\varepsilon)$ at the beginning of the money market each period. Sellers produce the investment good and consume the settlement good with preferences

$$\sum_t \beta^t \left(x_t - q_t\right)$$

where x_t is utility from consumption of the settlement good and $-q_t$ is disutility from production of the investment good.

In our model, buyers represent banks. They face investment opportunities with uncertain returns at the time they need to gather funds to be able to finance those investment. When information about the returns becomes available, banks may choose to borrow additional money to fund particularly productive investments, or to lend the money previously acquired. Hence, the market where buyers can trade money, described below,represents the interbank market in our model economy. Sellers represent firms, financial and non-financial. They produce goods and services, q_t, that banks can purchase. A bank's payoff from the services acquired, $\varepsilon_t \log(q_t)$, represents the return on a project or a portfolio to which those services add value (e.g. loans, repurchase agreements, various securities and so forth). At times, we refer to such payoff as a buyer's utility from consumption of investment goods.

2.1 Efficient Allocation

No goods are traded in the money market, so we are prepared to formulate the efficient allocation. Since the settlement market good is just a pure transfer, it drops from welfare and an egalitarian social planner would set the marginal utility of each buyer, ε/q, equal to the marginal cost faced by the sellers, which is unity. Hence, the efficient consumption of the investment good is $q_t^\varepsilon = \varepsilon$ for all t, ε, with aggregate supply equaling $\int \varepsilon dF(\varepsilon)$ and total welfare equal to

$$\frac{\int \varepsilon \left[\log(\varepsilon) - 1\right] dF(\varepsilon)}{1-\beta}.$$

2.2 Markets and Monetary Policy

To generate a role for a medium of exchange, assume that the investment market is anonymous. To produce, sellers require quid-pro-quo in the form of fiat money issued by a non-strategic central bank. In each period, t, the supply of money is M_t. The central bank operates a deposit facility paying a flat interest rate $r^E \geq 0$; a deposit of d units of fiat at the end of the investment market returns $(1 + r^E)d$ units at the beginning of the next settlement market. It will occasionally prove convenient to express interest in terms of a discount, so write $\rho_E = 1/(1 + r^E)$.

In addition to interest payments, the central bank operates lump-sum transfers of cash at the beginning of the settlement market. This allows the central bank to grow the money supply at a different rate than the interest it pays. For simplicity, the central bank makes such transfers only with buyers. Let γ be the growth of money, so that $M_{t+1} = \gamma M_t$. In steady state, prices will grow at the same rate as the money supply, and we can write the inflation rate as $\pi = \gamma - 1$. Note, if all money is deposited every period, total transfers equal $T_t = (\pi - r^E)M_t$. Money can then be traded against the settlement good in a Walrasian market before the settlement market closes.

Working backwards from the investment market, sellers can produce and buyers want to consume the investment good, hence they trade money for goods in a Walrasian market. Write p_t for the price of the investment good, q_t^S for the quantity supplied by sellers, and q_t^ε for the quantity demanded by a buyer with shock ε. Market clearing requires $\int q_t^\varepsilon dF(\varepsilon) = q_t^S$.

Some buyers will want to consume more than others in the investment market because of the preference shock at the beginning of the money market. Hence, there is an incentive for low-ε buyers to lend to high-ε buyers. They can do so in a Walrasian money market. Loans in the money market are repaid in the next settlement market. Let l_t^ε denote the borrowing of a buyer with shock ε in period t (negative if lending) and r_t^l the interest rate on such a loan. Trading in money market is subject to limited commitment frictions following Kehoe and Levine (1993) as explored in a monetary context by Berentsen *et al.* (2007).

Buyers can access a record of all settlement and money market transactions before they lend.[6] A buyer who is a potential borrower in this market will be considered in good standing if he has always repaid his loans in the past. He will be considered a defaulter instead if he

[6]The central bank is assumed, however, not to be party to the borrowers' records. A bank's business depends on its reputation. Public default or bankruptcy is disastrous. We assume banks share information with each other, but not generally. The notion is that a defaulting bank would eventually make their lender whole, and that this technical default is not broadly disclosed.

has borrowed and failed to repay.[7] If having previously defaulted, agents are permanently excluded from the money market, so from any future borrowing and lending. We assume agents can, however, still use money to participate in the investment and settlement markets. The maximum amount a buyer will repay is equal to the difference in value between repaying loans and continuing in good standing, and not repaying but losing access to the money market. Writing B_{t+1} for the maximum repayment an agent can credibly promise to make in the settlement market of $t+1$, derived below, the borrowing constraint in the money market of period t is

$$B_{t+1}/(1+r_t^l) - l_t^\varepsilon \geq 0. \tag{1}$$

Notice that the outside option of depositing at the central bank puts a floor on the market rate: $r_t^l \geq r^E$. Market clearing requires $\int l_t^\varepsilon dF(\varepsilon) = 0$.

In the settlement market, the central bank pays interest and makes transfers, buyers produce and sell the settlement good in order to acquire money and repay debts, and sellers use money they acquired in the previous investment market to buy and consume the settlement good. Sellers will sell all of their money to purchase settlement goods, as they have no need for it in either the money market or the investment market. In any stationary equilibrium, the real value of money must be constant, so $\phi_t M_t = \phi_{t+1} M_{t+1}$ which, plugging in for money growth, gives $\phi_t/\phi_{t+1} = \gamma$.

2.3 Equilibrium

Consider sellers first. Let W_t^S denote their value of entering the investment market and $V_{t+1}^S(d_t^S)$ their value of entering the next settlement market with d_t^S units of deposits at the central bank.[8] Assuming that sellers will carry no money nor any other state variable into the investment market, which we will verify in equilibrium, then

$$W_t^S = \max_{q_t^S} -q_t^S + \beta V_{t+1}^S(p_t q_t^S) \tag{2}$$

Sellers work to produce q_t^S units of the investment good, sell them at price p_t and deposit the proceeds from such sales at the central bank. In the next settlement market, sellers' deposits earn interest from the central bank, and can be used to purchase $\phi_{t+1}(1+r^E)d_t^S$ units of the settlement good, yielding a value of

$$V_{t+1}^S(d_t^S) = x_t^S + W_{t+1}^S \quad \text{s.t.} \quad x_t^S \leq \phi_{t+1}(1+r^E)d_t^S. \tag{3}$$

Consolidating these, one can derive sellers' supply curve in the investment market as

[7] Given our assumption on unbounded support for $\tilde{\varepsilon}_t$ every buyer will want to obtain a loan with probability 1. Hence, the event that a buyer has never obtained a loan on the record, has probability zero. We can alternatively assume that a buyer is in good standing also if he has no history of borrowing in the past.

[8] Note, we assume $r^E \geq 0$, so deposits dominate money as a store of value between periods. No agent would ever carry fiat from one period to the next.

$$q_t^S(p_t) \in \begin{cases} 0 & \text{if } p_t \beta \phi_{t+1}(1+r^E) < 1, \\ [0,\infty] & \text{if } p_t \beta \phi_{t+1}(1+r^E) = 1, \\ \infty & \text{if } p_t \beta \phi_{t+1}(1+r^E) > 1. \end{cases} \tag{4}$$

From this, one can derive

Lemma 1 *For an equilibrium with positive output in the investment market, the following must hold:* $p_t = \left[\beta\phi_{t+1}(1+r^E)\right]^{-1}$.

Turning to buyers, we work backwards. A buyer's value entering the investment market with m_t units of fiat money, loans l_t^ε and preference shock ε is

$$W_t^B(m_t, l_t^\varepsilon|\varepsilon) = \max_{d_t^\varepsilon, q_t^\varepsilon} \left\{\varepsilon \log(q_t^\varepsilon) + \beta V_{t+1}^B(l_t^\varepsilon, d_t^\varepsilon) \quad \text{s.t.} \quad p_t q_t^\varepsilon + d_t^\varepsilon \leq m_t \right\}, \tag{5}$$

where $V_{t+1}^B(l_t^\varepsilon, d_t^\varepsilon)$ is the value of entering the next settlement market having deposited d_t^ε at the central bank, and owing l_t^ε in loans. Stepping backward, the value of entering the money market is

$$U_t^B(m_t|\varepsilon) = \max_{l_t^\varepsilon} \left\{ W_t^B(m_t + l_t^\varepsilon, l_t^\varepsilon|\varepsilon) \quad \text{s.t.} \quad \frac{B_{t+1}}{1+r_t^l} \geq l_t^\varepsilon \geq -m_t \right\}. \tag{6}$$

Finally, the value of entering the next settlement market with deposits d_t^ε and loans l_t^ε is

$$\begin{aligned} V_{t+1}^B(l_t^\varepsilon, d_t^\varepsilon) = & \max_{x_t, h_t, m_{t+1}} x_t - h_t + \mathbb{E}_\varepsilon \left[U_{t+1}^B(m_{t+1}|\varepsilon)\right] \\ & \text{s.t.} \quad h_t - x_t + \phi_{t+1}\left[(1+r^E)d_t^\varepsilon + \tau M_{t+1}\right] \geq \phi_{t+1}\left[(1+r_t^l)l_t^\varepsilon - m_{t+1}\right] \end{aligned} \tag{7}$$

where ϕ_{t+1} is the value of money in terms of the settlement good, m_{t+1} is the amount of fiat carried into the money market, $(1+r^E)d_t^\varepsilon$ is the payoff on deposits from the previous period, τM_{t+1} is the lump-sum transfer, and $(1+r_t^l)l_t^\varepsilon$ is payment on previous period's loans.

One can verify that the buyers' value function in the settlement market, (7), is linear in d_t^ε and l_t^ε with

$$\frac{\partial V_{t+1}^B}{\partial d_t^\varepsilon} = \phi_{t+1}(1+r^E) \quad \text{and} \quad \frac{\partial V_{t+1}^B}{\partial l_t^\varepsilon} = -\phi_{t+1}(1+r_t^l). \tag{8}$$

Hence, one can write $V_{t+1}^B(d_t^\varepsilon, l_t^\varepsilon) = \phi_{t+1}\left[(1+r^E)d_t^\varepsilon - (1+r^l)l_t^\varepsilon\right] + V_{t+1}^B$, where we define $V_{t+1}^B \equiv V_{t+1}^B(0,0)$. Plugging this into the investment and money market problems, (5) and (6), gives a combined money-investment problem for buyers:

$$\hat{W}_t^B(m_t|\varepsilon) = \max_{l_t^\varepsilon, d_t^\varepsilon, q_t^\varepsilon} \varepsilon \log(q_t^\varepsilon) + \beta\phi_{t+1}\left[(1+r^E)d_t^\varepsilon - (1+r_t^l)l_t^\varepsilon\right] + \beta V_{t+1}^B \tag{9}$$

$$\text{s.t.} \quad p_t q_t^\varepsilon \leq l_t^\varepsilon + m_t - d_t^\varepsilon \tag{10}$$

$$\frac{B_{t+1}}{1+r_t^l} \geq l_t^\varepsilon \geq -m_t \tag{11}$$

The derivation is provided in detail in the appendix, but, after plugging in the price from Lemma 1, one arrives at a solution characterized by two thresholds, ε_t^L and ε_t^B given by

$$\varepsilon_t^L = \beta\phi_{t+1}(1+r_t^l)m_t \quad \text{and} \quad \varepsilon_t^B = \beta\phi_{t+1}(1+r_t^l)\left(m_t + \frac{B_{t+1}}{1+r_t^l}\right).$$

Buyers with $\varepsilon < \varepsilon_t^L$ prefer to lend (if $r_t^l > r^E$) or are indifferent between lending and depositing (when $r_t^l = r^E$). Buyers with $\varepsilon_t^L < \varepsilon < \varepsilon_t^B$ borrow but are not borrowing constrained. And buyers with $\varepsilon > \varepsilon_t^B$ have such a high marginal utility of consumption that they choose to borrow up to the limit. The full solution in the case of $r_t^l > r^E$ is summarized in Table 1 where we let $\beta\phi_{t+1}\lambda_t^\varepsilon$ denote the multiplier on the budget constraint (10) so that $\partial\hat{W}_t^B(m_t|\varepsilon)/\partial m_t = \beta\phi_{t+1}\lambda_t^\varepsilon$.

$q_t^\varepsilon =$	$l_t^\varepsilon =$	$d_t^\varepsilon =$	$\lambda_t^\varepsilon =$	
$\frac{1+r^E}{1+r_t^l}\varepsilon$	$\frac{\varepsilon}{\beta\phi_{t+1}(1+r_t^l)} - m_t$	0	$1+r_t^l$	if $\varepsilon < \varepsilon_t^L$
$\frac{1+r^E}{1+r_t^l}\varepsilon$	$\frac{\varepsilon}{\beta\phi_{t+1}(1+r_t^l)} - m_t$	0	$1+r_t^l$	if $\varepsilon_t^L < \varepsilon < \varepsilon_t^B$
$\frac{1+r^E}{1+r_t^l}\varepsilon_t^B$	$\frac{B_{t+1}}{1+r_t^l}$	0	$(1+r_t^l)\frac{\varepsilon}{\varepsilon_t^B}$	if $\varepsilon_t^B < \varepsilon$.

Table 1: Solution to buyers' problem (9)

Stepping back to the settlement market, problem (7) can be rearranged as

$$V_t^B(l_{t-1}^\varepsilon, d_{t-1}^\varepsilon) = \max_{x_t,h_t,m_t} x_t - h_t + \mathbb{E}_\varepsilon\left[\hat{W}_t^B(m_t|\varepsilon)\right] \tag{12}$$
$$\text{s.t.} \quad h_t - x_t + \phi_t\left[(1+r^E)d_{t-1}^\varepsilon - (1+r_{lt-1})l_{t-1}^\varepsilon + \tau M_t\right] \geq \phi_t m_t$$

the buyer's first order condition for money is:

$$0 = -\phi_t + \mathbb{E}_\varepsilon\left[\partial\hat{W}_t^B(m_t|\varepsilon)/\partial m_t\right] = -\phi_t + \beta\phi_{t+1}\,\mathbb{E}_\varepsilon\left[\lambda_t^\varepsilon\right].$$

Noting that, in any steady state monetary equilibrium, $\gamma = \phi_t/\phi_{t+1}$, substituting λ_t^ε from Table 1, and rearranging gives the money choice equation for buyers:

$$\frac{\gamma}{\beta(1+r_t^l)} = \int_0^{\varepsilon_t^B} dF(\varepsilon) + \int_{\varepsilon_t^B}^\infty \frac{\varepsilon}{\varepsilon_t^B} dF(\varepsilon). \tag{13}$$

Turning to the limited commitment constraint, we assume an exclusion equilibrium as in Kehoe and Levine (1993). A buyer who defaults on its loan is forever excluded from borrowing and lending, but can still participate in the settlement and investment markets with cash. We refer to these excluded buyers "autarks" and denote their variables with the superscript A. Given the settlement market value for a autark, V_t^A, a buyer is willing to repay an amount B_t if the cost of repaying and continuing in good standing exceeds the value of continuing as an autark, which defines the endogenous borrowing limit B_t:

$$-\phi_t B_t + V^B_{t+1} \geq V^A_{t+1}. \tag{14}$$

Hence, we must derive this value V^A_t, so solve the problem of autarks.

Autarks' problem is similar buyers'. They work for money in the settlement market and use this money to buy goods in the investment market.[9] But exclusion from the money market comes with two costs. First, instead of being able to lend excess cash at the market rate r^l_t, autarks only have recourse to the central bank's deposit facility with rate r^E. Second, they can not borrow, so face a borrowing constraint with $B_{t+1} = 0$. Hence, autarks' problem in the investment market is

$$\hat{W}^A_t(m^A_t|\varepsilon) = \max_{d^{A\varepsilon}_t, q^{A\varepsilon}_t} \varepsilon \log(q^{A\varepsilon}_t) + \beta\phi_{t+1}\left[(1+r^E)d^{A\varepsilon}_t\right] + \beta V^A_{t+1} \tag{15}$$

$$\text{s.t.} \quad p_t q^{A\varepsilon}_t \leq m^A_t - d^{A\varepsilon}_t \tag{16}$$

Similarly to buyers', the solution to this problem is described by a single threshold,

$$\varepsilon^A_t = \beta\phi_{t+1}(1+r^E)m^A_t. \tag{17}$$

Autarks with $\varepsilon < \varepsilon^A_t$ are not liquidity constrained in the investment market, and deposit at the central bank all of their unspent money. Autarks with $\varepsilon > \varepsilon^A_t$ are constrained, spend all of their money, and make no deposits. The solution to autarks' problem in the investment market is characterized in the appendix and summarized in Table 2.

$q^{A,\varepsilon} =$	$l^{A,\varepsilon} =$	$d^{A,\varepsilon} =$	$\lambda^{A,\varepsilon} =$	
ε	0	$\frac{\varepsilon}{\beta\phi_{t+1}(1+r^E)} - m^A$	$1+r^E$	if $\varepsilon < \varepsilon^A_t$
ε^A_t	0	0	$(1+r^E)\frac{\varepsilon}{\varepsilon^A_t}$	if $\varepsilon^A_t < \varepsilon$.

Table 2: Solution to autarks' problem in the investment market

Also, similar to the buyers' problem, one derives the money choice equation for autarks as

$$\frac{\gamma}{\beta(1+r^E)} = \int_0^{\varepsilon^A_t} dF(\varepsilon) + \int_{\varepsilon^A_t}^{\infty} \frac{\varepsilon}{\varepsilon^A_t} dF(\varepsilon). \tag{18}$$

Given all of this, we can write the value of an autark in the settlement market as

$$V^A_t = -\phi_t m^A_t + \mathbb{E}_\varepsilon\left[\varepsilon \log(q^{A,\varepsilon}_t) + \beta\phi_{t+1}(1+r^E)d^{A,\varepsilon}_t\right] + \beta V^A_{t+1}. \tag{19}$$

And, for completeness, we can rewrite the value for buyers entering the settlement market in (12) as

$$V^B_t = -\phi_t m_t + \mathbb{E}_\varepsilon\left[\varepsilon \log(q^\varepsilon_t) + \beta\phi_{t+1}\left((1+r^E)d^\varepsilon_t - (1+r^l_t)l^\varepsilon_t\right)\right] + \beta V^B_{t+1}. \tag{20}$$

[9]Note, there are no autarks along the equilibrium path, so they do not affect equilibrium prices, $\{\phi_t, p_t\}$.

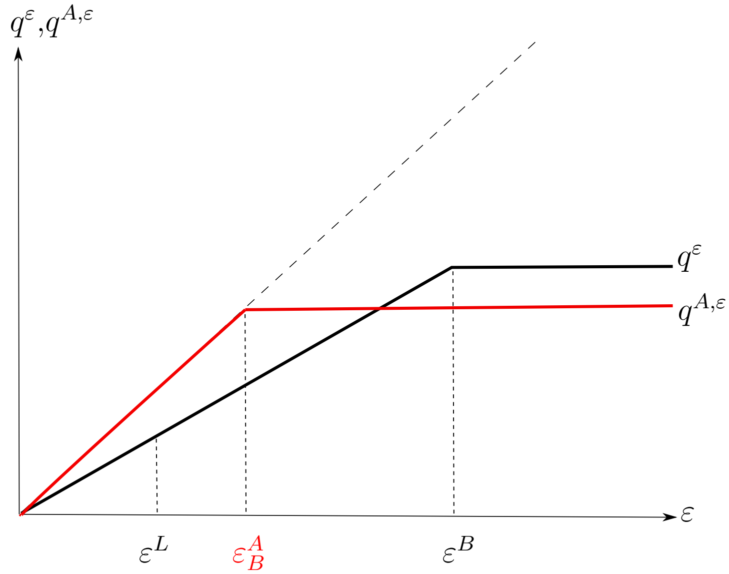

Figure 1: Consumption allocation for buyers and autarks.

The solution to buyers' and autarks decision problems in the investment market, summarized in Table 1 and Table 2, is also graphically represented in Figure 1. Buyers with a shock $\varepsilon \leq \varepsilon^L$ are unconstrained in their consumption of the investment good, and are lenders in the money market; buyers with a shock $\varepsilon \in (\varepsilon_L, \varepsilon_B]$ borrow on the money market less than their borrowing limit, and are thus unconstrained in their consumption of the investment good; buyers with a shock $\varepsilon > \varepsilon_B$ are borrowing constrained on the money market and in their consumption of the investment good. The first best sets $q^{\varepsilon} = \varepsilon$, and runs along the forty-five degree line. Unconstrained buyers consume less than this because they face a different marginal rate than sellers as they have the opportunity to lend. Autarks, however, consume along the forty-five degree line up to their cash constraint because their recourse is the same as sellers – depositing at the central bank and earning IOR.

Having described the economy and agents' decision problems, we are ready to define an equilibrium.

Definition 1 *An equilibrium is value functions for sellers, buyers and autarks,* V_t^S, V_t^B, V_t^A, $W_t^S, \hat{W}^B, \hat{W}^A$, *allocation in the settlement, money and investment market for sellers, buyers and autarks,* x_t^S, x_t, x_t^A, h_t^S, h_t, h_t^A, m_t, m_t^A, l_t^{ε}, $q_t^{\varepsilon}, q_t^{A,\varepsilon}$, $d_t^S, d_t^{\varepsilon}, d_t^{A\varepsilon}$, *prices of the settlement good, of the loan and of the investment good* ϕ_t, r_t^l, p_t *and borrowing limit* B_t, *such that i) given prices and the borrowing limit the allocation solves sellers', buyers' and autarks' problems (2), (3), (9), (15) and yield the value functions; ii) given the allocation and borrowing limit, prices clear the settlement, money and investment markets* $\int_0^1 m_t = M_t$, $\int_0^{\infty} l_t^{\varepsilon} dF(\varepsilon) = 0$, $\int_0^{\infty} q_t^{\varepsilon} dF(\varepsilon) = \int_0^1 q_t^S$ *and iii) given prices, allocation and value functions, the borrowing limit is defined by the endogenous repayment constraint (14).*

The settlement market clearing condition requires that the aggregate demand for money by buyers equals the supply of money by the central bank, which pins down the price of money in terms of settlement good. The money market clearing condition requires that loans are in zero net supply, as there is no net injection of funds and money is simply reallocated among buyers after they receive the $\tilde{\varepsilon}$ shock. The investment market clearing condition requires that aggregate demand of the investment good by buyers equals the aggregate supply by sellers.

2.4 Analysis

Our analysis begins with an elementary observation which applies to any limited commitment model of this type: there always exists a no-borrowing equilibrium.

Proposition 1 *There always exists an equilibrium with $B_t = 0$ and $r_l = r_E$.*

By way of proof, observe that, if $B_t = 0$, then market clearing requires $l^\epsilon = 0$ for all ϵ. Hence, all buyers with $\epsilon < \epsilon_t^L$ must be willing to deposit their excess funds at the central bank, so $r^l = r^E$. Since these were exactly the restrictions on the buyers' problem which yielded the autarks' problem, we must have $V_t^B = V_t^A$, confirming the equilibrium assertion that $B_t = 0$.

Credit in this type of economy must be self-confirming. If potential lenders believe other agents will not repay, no one will lend. If no one lends, continued access to the market has no value, so no one would repay any loan. In the same way that monetary economies have a non-monetary equilibrium where nobody values money, this type of credit economy always has a no-credit equilibrium where nobody values the money market. Further, when there is no borrowing or lending, market clearing requires rates be at the floor: the market rate, r_l, equals the deposit rate, r_E.

The rest of our analysis focuses on stationary equilibria, where the real allocation is constant over time. To ease exposition of the results in this section we adopt the following notation: current period and next period's variables are denoted x and x' respectively, real money holdings of buyers and autarks, in terms of the current settlement market, are denoted respectively by $z = \phi m$ and $z^A = \phi m^A$; the real value of the borrowing limit, in terms of the current settlement market is denoted $L = \phi B$ (so, in the current period, the borrowing limit is $l \leq \rho_l B'$), and $\zeta = \beta/\gamma$. Also, let $\rho_l = \frac{1}{1+r_l}$, $\rho_E = \frac{1}{1+r_E}$ and let the difference in utility from consumption between buyers and autarks be simply denoted by

$$\Delta\,\mathbb{E}[u] = \int \varepsilon \left(u(q_\varepsilon) - u(q_\varepsilon^A)\right) dF.$$

Given this, from buyers' and autarks' maximization problems we obtain:

$$\varepsilon_B = \frac{\beta\phi'\left(m + \rho_l B'\right)}{\rho_l} = \frac{\beta\phi'\left(z + \rho_l \gamma \phi' B'\right)}{\phi\rho_l} = \frac{\zeta\left(z + \rho_l \gamma L'\right)}{\rho_l} \qquad \text{and} \qquad \varepsilon_E^A = \frac{\zeta}{\rho_E} z^A.$$

Proposition 1 begs the question of whether floor systems can feature positive lending. In this subsection, we further characterize equilibria for economies running a floor system and derive a necessary condition for a floor system to obtain: Either $L = 0$ or $\gamma\rho_E = 1$.

Proposition 2 *If $\gamma\rho_E \neq 1$ then $\rho_l = \rho_E$ and $L = 0$. If $\gamma\rho_E = 1$ there exists a continuum of Pareto ranked equilibria with $L \geq 0$.*

Proof. See appendix A. ■

The intuition behind the results in proposition 2 stems from the substitutability between money and borrowing in agents' portfolios, and the assumption $\gamma\rho_E \neq 1$ plays an important role in this respect. An agent can hold money or borrow to pay for consumption goods. But how much of the good is a unit of borrowed money going to buy?

First, notice that the borrower has to compensate the lender for inflation because the lender acquired money in the current period and will be repaid in nominal terms (i.e. with money) in the following period. This economic mechanism is summarized by the term γ in the assumption of proposition 2.

Second, a borrower has to pay interest on the loan, which is important for the borrowing limit as the borrowing limit sets an upper bound on the loan *cum interest*. This economic mechanism is summarized by the term $\rho_E = \rho_l$ in the assumption of proposition 2.

The actual loan that an agent can obtain is net of both terms: inflation and interest. Therefore, a unit of "borrowing ability" (i.e. L') buys only $\gamma\rho_l$ units of consumption at the relevant price. A unit of money, on the other hand, buys 1 unit of consumption at the relevant price. When $\gamma\rho_l = \gamma\rho_E < 1$ then the ability to buy goods with loans is strictly smaller than the ability to buy goods with money. As a consequence, $L' = 0$. If instead $\gamma\rho_l = \gamma\rho_E > 1$ the opposite result holds, but this yields to a non stationary path for the borrowing limit. In fact, conditional on being able to borrow L' in the future (i.e. being a borrower in good standing) then an agent wants to borrow more than L' today.[10] As a consequence, no stationary equilibrium exists in this case.

Finally, the last result in proposition 2 shows that when money deposited at the central bank is remunerated at a rate just equal to the inflation rate, borrowers are indifferent between holding money or loans in their portfolios, and lenders are indifferent between lending or depositing at the central bank. As a consequence many equilibria exists, one with $L = 0$ and many with $L > 0$. The latter Pareto dominate the former because they support the same consumption allocation with less money, which is costly in aggregate as it needs to be purchased before consumption and is subject to inflation.[11] In other words, the money market performs liquidity insurance: this reallocation of liquidity allows agents to purchase the same amount of consumption goods as they would purchase using money only, but without having to hold as much money in the first place. Therefore, the money market performs a reallocation of liquidity ex post (i.e. after the shock ε) which reduces the inflation tax on the economy as a whole.

[10] Recall that in a model of endogenous limited commitment a borrower can obtain a loan today only if he will need to borrow in the future. From the perspective of a current lender, a borrower with no need or incentive to borrow in the future is a borrower who has all the incentives to default on his current loan. Therefore no loans would be granted to such a borrower.

[11] In other words, this is model of nominal loans, such as the money market, not a model of trade credit.

Proposition 3 *If the central bank deposit rate equals or exceeds the rate of inflation, $\gamma\rho_E \leq 1$, then in any equilibrium the private money market rate equals the deposit rate, $\rho_l = \rho_E$.*

Proof. See appendix A. ∎

The main idea for this result stems from two observations: first, consider the result in proposition 2 that at the floor ($\rho_l = \rho_E$) there is no activity in money markets ($L' = 0$) when $\gamma\rho_E < 1$. Hence, to revive borrowing and lending it is necessary that the economy be away from the floor: $\rho_l < \rho_E$. Therefore, loans must be preferred to holding money for a borrower, and making loans must be preferred to depositing money at the central bank for lenders. However, when $\gamma\rho_E < 1$ the central bank remunerates reserves at a higher rate than inflation, which prevents the substitution away from money and into loans in buyers' portfolios that is necessary for the money market to clear. In other words there is too much money in the economy relative to what would be necessary for the money market rate to rise above the interest on reserves. The case $\gamma\rho_E = 1$ follows from a similar argument but relying on the agents' strict indifference between borrowing (lending) and holding money (depositing at the central bank) as money held across periods is taxed at the same rate as the rate at which it is remunerated: the interest on reserves just equals the inflation rate.

3 Voluntary Reserve Targets

This section considers an alternate central bank deposit facility termed "Voluntary Reserve Targets" which asks agents to report a targeted deposit level at the beginning of the period and pays a high rate on balances up to the target, a low rate on balances over the target, and charges a fee for shortages relative to the target. This differentially affects buyers who can borrow and those who can not because the former have the ability to adjust their balances through borrowing and lending, so hit their targets more precisely.

The central bank offers a deposit facility to all agents. At the beginning of the settlement market all agents report a target to the central bank. Given a target, T, the deposit facility offers remuneration according to a three part schedule. Balances up to the target earn a rate r_T, balances in excess of the target earn a rate r_E, and shortages are charged a penalty r_P. We assume that deposits are voluntary and can be costlessly withdrawn as cash. Hence, if the central bank payed negative rates, the deposit facility would not be used. Hence, we assume $r_i \geq 0$ for $i \in \{E, T, P\}$. Moreover, assume $r_T \geq r_E$ and $r_T \geq r_P$. One can write this as

$$\tilde{R}(d|T) = \begin{cases} (1+r_T)d - r_p(T-d) & \text{if } T > d \\ (1+r_T)T + (1+r_E)(d-T) & \text{if } d \geq T \end{cases}$$

If one writes $\rho_P = (1 + r_T + r_P)^{-1}$, $\rho_T = (1 + r_T)^{-1}$, and $\rho_E = (1 + r_E)^{-1}$, then this schedule can be written in a form that will prove more convenient:[12] $\tilde{R}(d|T) = T/\rho_T + R(d - T)$

[12]Notice that $\rho_E < 1$ is a feasible policy. This relates to our discussion of negative interest rates with an

where

$$R(d-T) = \begin{cases} (d-T)/\rho_P & \text{if } d-T<0 \\ (d-T)/\rho_E & \text{if } d-T \geq 0. \end{cases}$$

Turning to households' decisions, the value of entering this market for agents $i \in \{B, S, A\}$ is

$$V^i(\hat{m}, l, d|T) = \max_{(X,H,m^+,T^+)} X - H + W^i(m^+|T^+)$$

$$\text{s.t.} \quad X + \phi m^+ = \phi \hat{m} + H - \phi l/\rho_l + \phi(T/\rho_T + R(d-T)) - \phi \tau M$$

where $\hat{m}$ denotes, money holdings brought into the settlement market from the previous goods' market. Substituting for $X - H$ gives

$$V^i(\hat{m}, l, d|T) = \phi\left(\hat{m} + T/\rho_T + R(d-T) - l/\rho_l - \tau M\right) + \max_{(m^+,T^+)} \left\{-\phi m^+ + W^i(m^+|T^+)\right\}.$$

The first order conditions with respect to m^+ and T^+, respectively, give

$$W^i_{m^+}(m^+|T^+) \leq \phi \quad (\;=\; \text{if } m^+ > 0) \quad \text{and} \quad W^i_{T^+}(m^+|T^+) \leq 0 \quad (\;=\; \text{if } T^+ > 0). \tag{21}$$

Linear disutility of labor makes $-\phi$ the marginal cost of balances which must equal the marginal value of carrying those balances into the next market. Agents are free to choose any positive target.

Envelope conditions give

$$V^i_{\hat{m}} = \phi; \quad V^i_l = -\phi/\rho_l; \quad V^i_d = \begin{cases} \phi/\rho_P & \text{if } d<T, \\ \phi/\rho_E & \text{if } d>T; \end{cases} \text{ and } V^i_T = \phi/\rho_T - \begin{cases} \phi/\rho_P & \text{if } d<T, \\ \phi/\rho_E & \text{if } d>T. \end{cases} \tag{22}$$

Let us consolidate the money and goods market, and characterize the sellers' problem first. As sellers do not participate in the money market, they carry no loans, $l = 0$, so we drop this argument from V^S.[13]

exemption threshold that can be analyzed within this framework (thus connecting the framework with some of the actual frameworks currently adopted by central banks.

[13]That sellers do not participate in the money market is a result, which follows from the comparison of their first order condition for m:

$$-\phi + \beta \frac{\phi^+}{\rho_l} \leq 0$$

with that of a buyer:

$$-\phi + \beta \phi^+ \mathbb{E}_\varepsilon W^B_m(m|T, \varepsilon) = 0$$

where $W^B_m(m|T,\varepsilon)$ denotes the partial derivative of $W^B(m|T,\varepsilon)$ with respect to m. Intuitively, this says that buyers have a higher expected marginal utility of money, as they might be borrowing constrained in the money market, and, because of that, the price of money is higher in the settlement market than what sellers would be willing to pay just to lend out the funds in the money market.

$$W^S(m|T) = \max_h -h + \beta V^S(d|T) \quad \text{s.t.} \quad d = ph.$$

Given the envelope on V above, we get the following supply curve:

$$h \in \begin{cases} \infty & \text{if } p > \rho_E/\beta\phi^+, \\ [T/p, \infty] & \text{if } p = \rho_E/\beta\phi^+, \\ T/p & \text{if } \rho_P/\beta\phi^+ < p < \rho_E/\beta\phi^+, \\ [0, T/p] & \text{if } p = \rho_P/\beta\phi^+, \text{ and} \\ 0 & \text{if } p < \rho_P/\beta\phi^+. \end{cases} \tag{23}$$

The maximized value is

$$\begin{cases} \infty & \text{if } p > \rho_E/\beta\phi^+, \\ T\left[\beta\phi^+/\rho_T - 1/p\right] & \text{if } \rho_P/\beta\phi^+ < p \leq \rho_E/\beta\phi^+, \\ -T\beta\phi^+/\rho_P & \text{if } p \leq \rho_P/\beta\phi^+. \end{cases}$$

This gives our first result:

Lemma 2 *For a positive and finite target choice by sellers, in equilibrium:*

$$p\beta\phi^+/\rho_T = 1, \text{ and } T^s = pD(p)$$

where $D(p)$ is the demand for goods from buyers and T^s is the choice of sellers' target. If sellers are not allowed to choose targets, in equilibrium $p\beta\phi^+/\rho_E = 1$.

The second result in lemma 2 shows that interpreting sellers strictly as non-financial firms would introduce a slight change to the above characterization. Without being account holders at the central bank, sellers would make no target setting decision and would earn ρ_E on any money balances carried over from one period to the next.[14] As a result, equilibrium prices would adjust accordingly.[15]

Turning to the characterization of buyers' decision problem, let $W^B(m|T, \varepsilon)$ denote the value of a buyer entering the money and goods market conditional on receiving shock ε, so that $W^B(m|T) = \mathbb{E}_\varepsilon\left[W^B(m|T, \varepsilon)\right]$.

$$W^B(m|T, \varepsilon) = \max_{(q^\varepsilon, l^\varepsilon, d^\varepsilon)} \varepsilon u(q^\varepsilon) + \beta V^B(\overline{m}', l^\varepsilon, d^\varepsilon|T)$$

[14] In practice, non-financial firms could deposit idle balances overnight at a financial institution and earn at least the interest in excess of the target set by that institution. The equilibrium pass-through rate from the financial institution to the non-financial firm would depend on the characteristics of the deposit contracts and the market structure where such contracts are traded, all of which are outside of our model.

[15] If sellers were not allowed to set targets then their decision problem would be equivalent to that of sellers in the benchmark economy of the previous section, where all balances are remunerated at a flat rate.

$$\text{s.t.}\quad m + l^{\varepsilon} - pq^{\varepsilon} - d^{\varepsilon} - \overline{m}' \geq 0, \quad \text{and} \quad \rho_l B - l^{\varepsilon} \geq 0.$$

where $\overline{m}'$ denotes money holdings carried over to the next settlement market rather than spent on the goods' market or deposited at the central bank. Write $\beta\phi^{+}\lambda_{\varepsilon}$ and $\beta\phi^{+}\lambda_{B}$ as the constraints on the budget and repayment constraints, respectively. The first order conditions with respect to $\overline{m}'$, q^{ε}, d^{ε}, and l^{ε} are, respectively,[16]

$$-\beta\phi\lambda_{\varepsilon} + \beta V^{B}_{\overline{m}'}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) \leq 0 \tag{24}$$

$$\varepsilon u'(q^{\varepsilon}) - p\beta\phi^{+}\lambda_{\varepsilon} = 0, \tag{25}$$

$$\beta V^{B}_{d_{\varepsilon}}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) - \beta\phi'\lambda_{\varepsilon} \leq 0 \tag{26}$$

with "=" if $d_{\varepsilon} > 0$.

$$\beta V^{B}_{l_{\varepsilon}}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) + \beta\phi'\lambda_{\varepsilon} - \beta\phi'\lambda_{B} = 0 \tag{27}$$

with "=" if $l_{\varepsilon} > 0$. Notice that $l_{\varepsilon} < 0$ denotes lending.

Using $V^{B}_{d_{\varepsilon}}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) = \phi' R_d'$ and $V^{B}_{l_{\varepsilon}}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) = -\frac{\phi}{\rho_l}$, the first order condition for d_{ε} can be rearranged as:

$$0 \geq -\lambda_{\varepsilon} + \begin{cases} 1/\rho_P & \text{if } 0 < d < T \\ 1/\rho_E & \text{if } d > T \end{cases} \tag{28}$$

and the first order condition for l_{ε} can be rearranged as:

$$-1/\rho_l + \lambda_{\varepsilon} - \lambda_B = 0 \tag{29}$$

The first order condition for $\overline{m}'$ can be simplified to

$$\beta\phi'(1 - \lambda_{\varepsilon}) \leq 0$$

with "=" if $\overline{m}' > 0$. Thus $\lambda_{\varepsilon} > 1$ is a necessary and sufficient condition for $\overline{m}' = 0$.

Lemma 3 *Suppose $\rho_E > \rho_l > \rho_P$. There exist numbers $\varepsilon_l < \varepsilon_B < \varepsilon_T < \varepsilon_M$ such that the following hold: Buyers with $\varepsilon < \varepsilon_l$ lend. Buyers with $\varepsilon_l < \varepsilon < \varepsilon_B$ borrow and fully fund their targets. Buyers with $\varepsilon_B < \varepsilon < \varepsilon_T$ are borrowing constrained and fully fund their target.*

[16] These follow from the Lagrangian:

$$L(z) = \varepsilon u(q_{\varepsilon}) + \beta V^{B}(\overline{m}', l_{\varepsilon}, d_{\varepsilon}|T) + \beta\phi'\lambda_{\varepsilon}(m + l_{\varepsilon} - d_{\varepsilon} - pq_{\varepsilon} - \overline{m}') + \beta\phi'\lambda_{B}(\rho_l B - l_{\varepsilon}) + \beta\phi'\lambda_d d_{\varepsilon} + \beta\phi'\lambda_q q_{\varepsilon}$$

where $z \in \mathbb{R}^4$, $z = (\overline{m}', q_{\varepsilon}, l_{\varepsilon}, d_{\varepsilon})$ with $\overline{m}'$ denoting money holdings not spent or deposited but carried over to the next settlement market, so that the budget constraint in the goods market is $pq_{\varepsilon} + d_{\varepsilon} + \overline{m}' \leq m + l_{\varepsilon}$. Also, $\beta\phi'\lambda_B$ and $\beta\phi'\lambda_{\varepsilon}$ denotes the Lagrange multiplier on the borrowing constraint and the budget constraint in the goods' market, and $\beta\phi'\lambda_d$, $\beta\phi'\lambda_q$ denote the Lagrange multipliers on non negativity constraints on d_{ε} and q_{ε} respectively.

Buyers with $\varepsilon_T < \varepsilon < \varepsilon_M$ are borrowing constrained and partially fund their target. Buyers with $\varepsilon_M < \varepsilon$ are borrowing constrained, and make no deposits. These values solve

$$\varepsilon_l = \beta\phi^+(m-T)/\rho_l, \quad \varepsilon_B = \beta\phi^+(m-T)/\rho_l + \beta\phi^+ B,$$
$$\varepsilon_T = \varepsilon_B\rho_l/\rho_P = \beta\phi^+(m-T)/\rho_P + \beta\phi^+ B\rho_l/\rho_P, \quad \textit{and} \quad \varepsilon_M = \beta\phi^+ m/\rho_P + \beta\phi^+ B\rho_l/\rho_P.$$

Proof. See Appendix B. ■

The result derives from the assumption on the ordering of prices: $\rho_P < \rho_l < \rho_E$. For low levels of ε, a buyer fills their target and borrows or lends. Because of the gap between ρ_l and ρ_P, a region of buyers consume a constant amount until a critical threshold is reached such that consumption is more valuable than funding the target. This is followed by a region where targets are partially funded and then a final region where the deposit facility is not used, and a borrowing constrained buyer spends all its funds on consumption.

Buyers in autarky effectively face $\rho_l = 0$ and $B = 0$. This alters behavior in the money and goods market. As above, write $W^A(m|T) = \mathbb{E}_\varepsilon\left[W^A(m|T,\varepsilon)\right]$.

$$W^A(m|T,\varepsilon) = \max_{q_\varepsilon^A, d_\varepsilon^A} \varepsilon u(q_\varepsilon^A) + \beta V^A(d_\varepsilon|T)$$

$$\text{s.t.} \quad m - pq_\varepsilon^A - d_\varepsilon^A \geq 0.$$

Writing $\beta\phi^+\lambda_A$ as the constraint on the budget, the first order conditions are as follows:

$$\varepsilon u'(q^\varepsilon) - p\beta\phi^+\lambda_\varepsilon = 0,$$

$$0 \geq -\lambda_\varepsilon + \begin{cases} 1/\rho_P & \text{if } 0 < d < T \\ 1/\rho_E & \text{if } d > T. \end{cases}$$

Lemma 4 *There exist $\varepsilon_E^A, \varepsilon_P^A, \varepsilon_M^A$ such that autarkic borrowers with $\varepsilon < \varepsilon_E^A$ deposit more than their target, those with $\varepsilon_E^A < \varepsilon < \varepsilon_P^A$ just meet their target, those with $\varepsilon_P^A < \varepsilon < \varepsilon_M^A$ partially meet their targets, and those with $\varepsilon > \varepsilon_M^A$ make no deposits. These thresholds satisfy*

$$\varepsilon_E^A = \frac{\beta\phi^+(m-T)}{\rho_E}, \quad \varepsilon_P^A = \frac{\beta\phi^+(m-T)}{\rho_P}, \quad \varepsilon_M^A = \frac{\beta\phi^+ m}{\rho_P}$$

Proof. See Appendix B. ■

Figure 2 summarizes the results in lemmas 3 and 4.

Turning to target setting decisions, targets will be interior only if $\rho_P < \rho_T < \rho_E$. If $\rho_T = \rho_E$, then there is no cost of exceeding one's target, so agents will set $T = 0$. Alternately, if $\rho_P = \rho_T$, there is no cost of falling below one's target, and agents set $T = \infty$. While we will return to these observations, suppose for now that the inequalities hold strictly. We proceed to solve for each type's targets in turn.

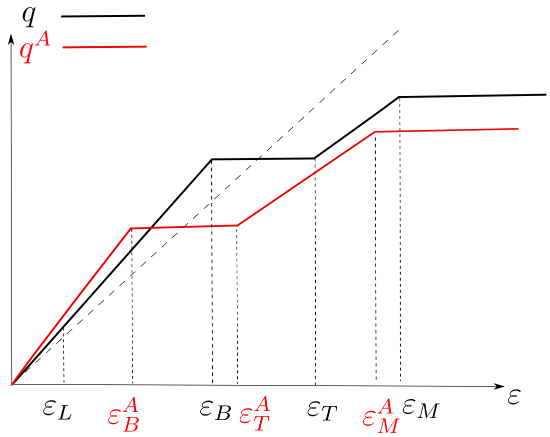

Figure 2: Consumption allocation for buyers (black) and autarks (red)

Sellers face no uncertainty over their end of period deposits. For fixed deposits, the remuneration schedule is maximized by setting the target equal to deposits. This is the intuition behind Lemma 2 which ensures that $T^S = pD(p)$.

Turning to buyers, from their decision problem in the settlement market, voluntary targets must satisfy $W_T^i = 0$. Because R is not differentiable in T, a simple envelope theorem will not hold. Alternatively, one can substitute the maximized values of q^ε, l^ε, and d^ε and differentiate this. The following lemma formalizes this result for both buyers and autarks

Lemma 5 *A voluntary target for buyers, T^B, must satisfy*

$$\frac{1}{\rho_T} = \frac{1}{\rho_l}\left(\int_0^{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_B}^{\varepsilon_T}\left(\frac{\varepsilon}{\varepsilon_B}\right) dF(\varepsilon)\right) + \frac{1}{\rho_P}\int_{\varepsilon_T}^{\infty} dF(\varepsilon) \tag{30}$$

which implies $\rho_l \in [\rho_P, \rho_T]$.

A voluntary target for autarkic buyers, T^A, must satisfy

$$\frac{1}{\rho_T} = \frac{1}{\rho_E}\left(\int_0^{\varepsilon_E^A} dF(\varepsilon) + \int_{\varepsilon_E^A}^{\varepsilon_P^A}\frac{\varepsilon}{\varepsilon_E^A} dF(\varepsilon)\right) + \frac{1}{\rho_P}\int_{\varepsilon_P^A}^{\infty} dF(\varepsilon) \tag{31}$$

Proof. See Appendix B. ■

Lemma 5 shows that the optimal choice of target for a buyer trades off the remuneration rate on any unit of money deposited below the target, on the left hand side of equation (30), with its expected cost, on the right hand side of equation (30). The first term on the right hand side of equation (30) captures the marginal cost of increasing targets for buyers who would be lenders or unconstrained borrowers. Lenders would earn the money market rate on any unit of money not deposited towards the target, and unconstrained borrowers need to pay the market rate in order to borrow a unit of money and deposit it towards meeting the target. The second term captures the marginal cost of increasing targets for buyers who would be giving up consumption to meet the targets, whose opportunity cost is the marginal utility of consumption (i.e. $\frac{\varepsilon}{\varepsilon_B}$). The third term captures the marginal cost of increasing targets for buyers who would be consuming their entire money holdings, thus missing the target and paying the penalty rate on any shortage of balances with respect to the target. Equation (31) captures a similar trade-off for autarks.

Our analysis of a voluntary reserve target framework is focused on symmetric stationary equilibria, which requires all agents maximize, all markets clear and that all real quantities are constant over time and across agents of the same type. Policy variables are $\tau, \gamma, \rho_T, \rho_E$ and ρ_P. Endogenous variables are the target choices T^B, T^S, and T^A; prices ρ_l and p; preference cutoffs $\varepsilon_l, \varepsilon_B, \varepsilon_T, \varepsilon_M, \varepsilon_E^A, \varepsilon_P^A$, and ε_T^A; along with the borrowing limit B.

Consider a buyer who repays his money market loan from the previous period. Because his repayment is publicly recorded then his continuation value at the end of the settlement market, before choosing target and money holdings, is $V^B(0,0,0|0)$, as defined in (65) in the appendix. To save on notation, let $V^B = V^B(0,0,0|0)$. Consider now a buyer who plans to default on his money market loan from the previous period. Analogously to a buyer, his continuation value in the settlement market, before choosing target and money holdings, is $V^A(0,0|0) = \max_{\{m^{A+},T^{A+}\}} -\phi m^{A+} + W^A(m^{A+}|T^{A+})$. To save on notation, let $V^A = V^A(0,0|0)$. The payoff to a buyer who repays in the settlement market is $-\phi\frac{l_\varepsilon}{\rho_\varepsilon} + V^B$, since the loan was in terms of money, while the payoff to a buyer who defaults is V^A. Therefore, the repayment constraint can be rewritten as

$$l_\varepsilon \leq \frac{\rho_l}{\phi}(V^B - V^A) \tag{32}$$

which, as shown in Appendix B.6 can be rearranged as:

$$\begin{aligned}\phi B &= \phi(m^{A+} - m^+) + \mathbb{E}_\varepsilon[\varepsilon u(q_\varepsilon) - \varepsilon u(q_\varepsilon^A)] \\ &\quad + \mathbb{E}_\varepsilon \beta\phi' \Big\{ (\frac{T^+ - T^{A+}}{\rho_T}) - \frac{l_\varepsilon}{\rho_l} + R(d_\varepsilon - T^+) - R(d_\varepsilon^A - T^{A+}) \Big\} + \beta\phi' B'\end{aligned} \tag{33}$$

Notice that none of the autarky terms depend on B, while ρ_l, T^+ and the allocation of consumption and deposits do.

Definition 2 *Given policy variables $\tau, \rho_P, \rho_T, \rho_E$, an equilibrium is an allocation for sellers h, d and target choice T^S, an allocation for buyers $m^+, \hat{m}', q_\varepsilon, d_\varepsilon, l_\varepsilon$ and target choice T^+, an allocation for autarkic buyers $m^{A+}, \hat{m}^{A'}, q_\varepsilon^A, d_\varepsilon^A$ and target choice T^{A+}, prices p, ρ_l, money growth rate $\gamma = \frac{M^+}{M}$, value functions for buyers and autarkic buyers V^B, W^B and V^A, W^A, endogenous borrowing constraint $\phi B = V^B - V^A$ such that: i) taking prices as given the allocations solve the respective agents' decision problem and targets solve (30) for buyers and (31) for autarkic buyers, ii) markets clear and iii) B satisfies (33).*

Market clearing conditions are characterized below for each market.

Equilibrium in the goods market is given by a price p where

$$q^S = \int q_\varepsilon dF(\varepsilon).$$

Lemma 2 implies $p = \rho_T/\beta\phi^+$, or $p = \rho_E/\beta\phi^+$ if sellers are not allowed to set targets, and that sellers' optimal targets satisfy $T^S = p\int q_\varepsilon dF(\varepsilon)$.

When entering the money market, only buyers with sufficiently low preference shock lend. All other buyers borrow, with buyers experiencing a sufficiently high shock being borrowing constrained. The following lemma characterizes the money market clearing condition and the money market rate: ρ_l.

Lemma 6 *Supply of funds is given by*

$$S(\rho_l) = \int_0^{\varepsilon_L} \frac{\rho_l}{\rho_T} p(\varepsilon - \varepsilon_L) dF(\varepsilon). \tag{34}$$

Demand for funds is given by

$$D(\rho_l) = \int_{\varepsilon_l}^{\varepsilon_B} \frac{\rho_l}{\rho_T} p(\varepsilon - \varepsilon_L) dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \rho_l B dF(\varepsilon) \tag{35}$$

The money market interest rate satisfies:

$$0 = \frac{\rho_l \gamma}{\beta} \int_0^{\varepsilon_B} \varepsilon dF(\varepsilon) - [\rho_l \phi B + \phi m^+ - \phi T^+] F(\varepsilon_B) + \rho_l \phi B$$

Proof. See Appendix B. ■

We are now ready to characterize the buyers' choice for settlement period balances.[17] From the settlement period problem, the first order condition on money balances in (21) implies

$$\frac{\gamma}{\beta} = \int_0^{\varepsilon_B} \frac{1}{\rho_l} + \int_{\varepsilon_B}^{\varepsilon_T} \frac{1}{\rho_l} \frac{\varepsilon}{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_T}^{\varepsilon_M} \frac{1}{\rho_P} dF(\varepsilon) + \int_{\varepsilon_M}^{\infty} \frac{1}{\rho_P} \frac{\varepsilon}{\varepsilon_M} dF(\varepsilon)$$

The optimal choice of money holdings for a buyer trades off its cost, on the left hand side, with its present expected value, on the right hand side. Acquiring money is costly due to inflation, γ. The expected benefit of holding money stems from (i) the ability to lend it out or not having to borrow at the money market rate, $\frac{1}{\rho_l}$, (ii) increased consumption when borrowing constrained and meeting the target, (iii) reduced penalties on shortages with respect to the target when giving up consumption to partially fund the target, (iv) increased consumption when failing to meet the target entirely.

Similarly, the optimal choice of money holdings for autarks satisfies

$$\frac{\gamma}{\beta} \quad = \quad F(\varepsilon_E^A) \frac{1}{\rho_E} + \int_{\varepsilon_E^A}^{\varepsilon_P^A} \frac{\varepsilon}{\rho_E \varepsilon_E^A} dF(\varepsilon) + \int_{\varepsilon_P^A}^{\varepsilon_M^A} \frac{1}{\rho_P} + \int_{\varepsilon_M^A}^{\infty} \frac{\varepsilon}{\varepsilon_E^A \rho_P} \tag{36}$$

[17] See Appendix B.5.

4 Three types economy

To derive analytical results about the comparison between VRT and the flat rate we simplify the model as follows. Let buyers' preference shock be a discrete random variable with support $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and with π_i denoting the probability that a type is ε_i, $i = 1, 2, 3$. We focus on stationary equilibria and construct equilibrium strategies so that buyers experiencing taste shock ε_1 are lenders in the money market under both frameworks; buyers experiencing taste shock ε_2 are lenders in the flat rate framework and unconstrained borrowers in the VRT framework; buyers experiencing taste shock ε_3 are constrained borrowers in both frameworks, but spending part of their income towards meeting their targets in a VRT framework. Specifically, let $\tilde{\varepsilon}_B$ and $\tilde{\varepsilon}_B^A$ denote the endogenous thresholds in the economy where monetary policy is implemented via a flat rate remuneration framework, and ε_B, ε_T, ε_M, ε_B^A, ε_T^A, ε_M^A denote the endogenous thresholds in the economy where monetary policy is implemented via a VRT. Then, in the VRT economy we construct an equilibrium where preference shocks are drawn so that $\varepsilon_1 < \min\left(\varepsilon_B^A, \varepsilon_B\right)$, $\varepsilon_B^A < \varepsilon_2 < \varepsilon_B$, $\max\left(\varepsilon_T, \varepsilon_T^A\right) < \varepsilon_3 < \min\left(\varepsilon_M, \varepsilon_M^A\right)$. In the flat rate economy we construct an equilibrium where preference shocks are drawn so that $\varepsilon_1 < \min\left(\tilde{\varepsilon}_B^A, \tilde{\varepsilon}_B\right), \varepsilon_2 < \min\left(\tilde{\varepsilon}_B^A, \tilde{\varepsilon}_B\right), \varepsilon_3 > \max\left(\tilde{\varepsilon}_B, \tilde{\varepsilon}_B^A\right)$.

Tables 3 and 4 summarize these equilibrium conjectures.

Type	Flat Rate		VRT	
ε_1	$< \tilde{\varepsilon}_B$	lend	$< \varepsilon_B$	lend
ε_2	$< \tilde{\varepsilon}_B$	lend	$< \varepsilon_B$	borrow to meet target
ε_3	$> \tilde{\varepsilon}_B$	borrow to consume	$\in (\varepsilon_T, \varepsilon_M)$	borrow and partially fund target

Table 3: Constructed equilibrium strategies for buyers

Type	Flat Rate		VRT	
ε_1	$< \tilde{\varepsilon}_B^A$	deposit	$< \varepsilon_B^A$	deposit
ε_2	$< \tilde{\varepsilon}_B^A$	deposit	$> \varepsilon_B^A$	give up consumption to meet target
ε_3	$> \tilde{\varepsilon}_B^A$	consume all money	$\in (\varepsilon_T^A, \varepsilon_M^A)$	partially fund target

Table 4: Constructed equilibrium strategies for autarks

4.1 VRT

Consider an economy where monetary policy is implemented with a VRT framework. Without loss of generality, let sellers deposit unused money balances overnight at the central bank without choosing targets. As discussed in section 3 this assumption has no impact on our results while it simplifies the exposition of the analysis, as lemma 2 shows that equilibrium prices satisfy $p\beta\phi^+ = \rho_E$. In this economy, we can prove the following results.

Proposition 4 *An equilibrium is characterized by the following equations*

$$\begin{aligned}
\varepsilon_B &= \frac{\pi_1\varepsilon_1 + \pi_2\varepsilon_2}{(1-\pi_3)} + \frac{\beta\phi^+}{(1-\pi_3)}B^+ \\
\varepsilon_B^A &= \frac{\pi_2\varepsilon_2}{\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)} \\
\rho_l &= \frac{(\pi_1+\pi_2)}{\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P}\right)} \qquad (37) \\
\phi &= \frac{\beta\phi^+}{\rho_T} \qquad (38)
\end{aligned}$$

and in a stationary equilibrium:

$$\begin{aligned}
L\left(1 - \frac{\beta\rho_l}{\rho_T}\right) &= (\pi_1\varepsilon_1 + \pi_2\varepsilon_2)\log\frac{\rho_l}{\rho_E} + \\
&\quad + \pi_2\varepsilon_2\left[\log\frac{\rho_E}{\pi_2}\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)\right] \qquad (39)
\end{aligned}$$

where $B^+ = \frac{V^{B+} - V^{A+}}{\phi^+} = \frac{L^+}{\phi^+}$ *and* $L = L^+$.

Proof. See appendix C. ■

Notice that the equilibrium condition (38) ties the remuneration rate of targeted reserves to the inflation rate: $\frac{\gamma}{\beta} = \frac{1}{\rho_T}$. Intuitively, this means that every unit of targeted reserve is remunerated at what would be the Friedman rule if it was applied to all reserves/unspent money. However, this economy is still away from the Friedman rule as buyers miss their targets with positive probability in state ε_3.

The equilibrium condition (37) links the money market rate to the penalty rate: in particular, the money market rate, $\frac{1}{\rho_l}$, decreases in the penalty rate, $\frac{1}{\rho_P}$. The economic mechanism works thorough target setting behavior, as a higher penalty rate gives buyers incentives to reduce their targets. In turn, lower targets imply less need to borrow for buyers, which results in lower money market rate as long as the supply of loans doesn't fall too much. Real money holdings, in fact, do not fall as much as targets do in response to an increase in the penalty rate because money is necessary to fund consumption as well as to meet the target.

The equilibrium condition (39) characterizes the real borrowing limit in the constructed equilibrium: the term on the right hand side is the utility differential between buyers and autarks from consuming in the goods market, whereas the left hand side includes a term $(\frac{\rho_l}{\rho_T})$ measuring a buyer's savings over an autark in having to acquire money in the settlement market.

Turning to characterizing the conditions for the existence of the constructed equilibrium, let the functions $\Delta\mathbb{E}u(\alpha)$ and $\alpha(\rho_P)$ denote, respectively, the sum of the utility differential between buyers and autarks from consuming in the goods market, and a mapping from the penalty rate to the money market rate:

$$\begin{aligned} \Delta\mathbb{E}u(\alpha) &= (\pi_1\varepsilon_1 + \pi_2\varepsilon_2)\log\frac{\rho_l}{\rho_E} + \pi_2\varepsilon_2\left[\log\frac{\rho_E}{\pi_2}\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)\right] \\ \alpha(\rho_P) &= \frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} \end{aligned}$$

Then, in a stationary equilibrium,

$$L\left(1 - \frac{\beta\rho_l}{\rho_T}\right) = \Delta\mathbb{E}u(\alpha)$$

Lemma 7 *Assume* $\left(\alpha - \frac{\pi_1}{\rho_E}\right) > 0$*. If* $\underline{\rho}_{Pu} > \rho_P > \overline{\rho}_{Pu}$ *then* $\Delta\mathbb{E}u(\alpha) > 0$ *and* $\frac{\partial\Delta\mathbb{E}u(\alpha)}{\partial\alpha} < 0$*, where* $\overline{\rho}_{Pu}$ *is defined by (79) holding at equality and* $\underline{\rho}_{Pu}$ *by (78) holding at equality.*

Proof. See appendix C. ■

Lemma 7 characterizes the effect of the penalty rate, $r_P = 1/\rho_P$, on the utility differential. The economic mechanism works through the money market rate and through autarks' target choice.

First, a higher penalty rate causes the market rate to decrease as described above. As a consequence of the lower interest rate (i) lenders are more willing to consume than to lend, both on the intensive and the extensive margin, and (ii) the borrowing constraint of all buyers is relaxed.[18]

Second, the penalty rate affects the consumption of constrained autarks: a larger r_P causes an increase in autarks' consumption in state ε_2, when they are constrained by their money holdings and consume ε_B^A in order to meet the target.[19] In fact, higher penalty rates induce autarks to lower their target choices, resulting in effectively more money that they can allocate to consumption when constrained. The condition $\rho_P > \overline{\rho}_{Pu}$ guarantees that the marginal contribution of the second effect is smaller than that of the first effect, target choices depend on the marginal remuneration and penalty rates. The condition $\rho_P < \underline{\rho}_{Pu}$ guarantees that the average contribution of the second effect is smaller than the first, or, in other words, that the effect working through autarks' target choices is second order.

Then we can state a result characterizing sufficient conditions for the equilibrium borrowing limit to be positive.

[18] In the three types example the extensive margin may not matter as the threshold ε_L may not fall below the lowest buyers' types ε_1.

[19] The penalty rate also affects consumption of both buyers and autarks in state ε_3 as the equilibrium is constructed with both buyers and autarks failing to meet the target in state ε_3. However, because both buyers and autarks face the same opportunity cost of consumption (i.e. the penalty rate) then the net effect of the penalty rate on the utility differential is zero in state ε_3.

Proposition 5 *If $\underline{\rho}_{Pu} > \rho_P > \overline{\rho}_P$, with $\overline{\rho}_P$ defined in (83), then $L > 0$. Conditional on autarky as off equilibrium punishment, this stationary equilibrium is unique.*

Proof. See appendix C. ■

The intuition for the result in proposition 5 is similar to the economic mechanism described for the results in lemma 7, and works together with it. The penalty rate, $r_P = 1/\rho_P$, affects autarks' as well as buyers' consumption via their target choices. A higher penalty rate reduces autarks' target choices, which results in larger autarks' consumption even when they are constrained by their money holdings. Hence, a possibly tighter upper bound on the penalty rate than that in lemma 7 is necessary to ensure that the endogenous thresholds $\varepsilon_B, \varepsilon_B^A$ are consistent with the constructed strategies.

To conclude our analysis of the VRT economy, the following proposition provides a characterization of the relationship between the borrowing limit and the penalty rate.

Proposition 6 *Maintain the assumptions in proposition 5. Then $\frac{\partial L(\rho_P)}{\partial \rho_P} < 0$.*

Proof. See appendix C. ■

The intuition for the result in proposition 6 lays in the indirect effect of the penalty rate on the market rate. An increase in the penalty rate causes a reduction in the interest rate (i.e. an increase in ρ_l), which, in turn, causes an increase in the amount that a buyer can borrow per unit of loan (i.e. for a fixed L), as implied by the borrowing constraint $\phi l_\varepsilon \leq \rho_l L$. Because an increase in the borrowing amount per unit of loan reduces the money needed (i) to purchase consumption in the goods' market and (ii) to meet the target, then it increases the value of a buyer over that of an autark.

4.2 Flat Rate Remuneration

Consider an economy where monetary policy is implemented with a flat rate remuneration framework. We can prove the following results.

Lemma 8 *An equilibrium is characterized by the following conditions:*

$$\tilde{\varepsilon}_L^A = \frac{\pi_3 \tilde{\varepsilon}_3}{\frac{\phi}{\beta \phi^+} \rho_E - (\pi_1 + \pi_2)}$$

$$\tilde{\varepsilon}_B = \tilde{\varepsilon}_L = \frac{\gamma L^+ + \pi_1 \varepsilon_1 + \pi_2 \varepsilon_2}{(1 - \pi_3)}$$

$$\rho_l = \frac{(\gamma L^+ + \overline{\varepsilon})(1 - \pi_3)}{\gamma L^+ + \pi_1 \varepsilon_1 + \pi_2 \varepsilon_2} \frac{\beta}{\gamma} = \frac{(\gamma L^+ + \overline{\varepsilon})}{\tilde{\varepsilon}_B} \frac{\beta}{\gamma} \tag{40}$$

$$
\begin{aligned}
L(1-\beta+\gamma) &= \frac{(\gamma L^{+}+a+b)(1-\pi_3)}{\gamma+\frac{a}{L^{+}}}\beta+ \\
&\quad (a+b)\log\left(\gamma L^{+}+a+b\right)-a\log\left(\gamma L^{+}+a\right)+K \qquad (41)
\end{aligned}
$$

with $a=\bar{\varepsilon}_1+\bar{\varepsilon}_2$, $b=\bar{\varepsilon}_3$, *for* $\bar{\varepsilon}_i=\pi_i\varepsilon_i$ $i=1,2,3$, $K=-\bar{\varepsilon}+K_1+K_2$, *and*

$$
\begin{aligned}
K_1 &= \left(\phi-\frac{\beta\phi^{+}}{\rho_E}(1-\pi_3)\right)\frac{\rho_E\tilde{\varepsilon}_B^A}{\beta\phi^{+}}+\frac{\beta\phi^{+}}{\rho_E}\left[\pi_1\frac{\rho_E\varepsilon_1}{\beta\phi^{+}}+\pi_2\frac{\rho_E\varepsilon_2}{\beta\phi^{+}}\right] \\
K_2 &= (\bar{\varepsilon}_1+\bar{\varepsilon}_2)\log(1-\pi_3)-\bar{\varepsilon}\log\rho_E+\bar{\varepsilon}\log\frac{\beta}{\gamma}-\pi_3\varepsilon_3\log\tilde{\varepsilon}_B^A
\end{aligned}
$$

Proof. See appendix D. ■

The equilibrium condition (40) characterizes the money market rate ($\frac{1}{\rho_l}$), which increases with the size of the loan by constrained buyers and with the present value of inflation, captured by the term $\frac{\gamma}{\beta}$. With respect to the equilibrium condition defining the borrowing limit in VRT, no endogenous adjustment of targets to monetary conditions appears in (41).

Turning to characterizing the conditions for the existence of the constructed equilibrium, let $FL(L)$ and $FR(L)$ denote the left and right hand sides of (41) respectively:

$$
\begin{aligned}
FL(L) &= L(1-\beta+\gamma) \qquad (42)\\
FR(L) &= \gamma\rho_l L+\Delta\mathbb{E}u+K \qquad (43)\\
&= \frac{(\gamma L+a+b)(1-\pi_3)}{\gamma+\frac{a}{L}}\beta+ \\
&\quad (a+b)\log(\gamma L+a+b)-a\log(\gamma L+a)+K
\end{aligned}
$$

Notice that $(1-\beta+\gamma)>1$. We now turn to characterizing the function $FR(L)$.

Lemma 9 *The function $\rho_l L$ is increasing and concave in L, $\rho_l L\,(L=0)=0$ and $\lim_{L\to\infty}\frac{\partial\rho_l L}{\partial L}=\frac{\beta}{\gamma}(1-\pi_3)<1$. The function $\Delta\mathbb{E}u$ is increasing in L, concave for large enough L ,$\lim_{L\to\infty}\frac{\partial\Delta\mathbb{E}u}{\partial L}=0$ and $\Delta\mathbb{E}u\ (L=0)-K_2=\log\frac{(a+b)^{a+b}}{a^a}>0$.*

Proof. See appendix D. ■

We are now ready to state and prove our result about the existence of the constructed equilibrium when monetary policy is implemented with a flat rate remuneration scheme.

Proposition 7 *If $2-\frac{1}{\beta}<\pi_3$ and $\frac{b}{a}\leq\frac{\left(\frac{1}{\beta}-1\right)}{(1-\pi_3)}-1$, and if $a>\underline{a}$ and $b\in\left(\underline{b},\bar{b}\right)$, there exists a unique $L>0$ solving (41). Moreover, the function $\Phi(L)=FL(L)-FR(L)$ is monotonically increasing.*

Proof. See appendix D. ■ The assumptions in proposition 7 guarantee that the slope of the left hand side of equation (41) is smaller than that of the right hand side, and, thus, that a solution to (41) exists and is unique.

4.3 VRT vs Flat Rate remuneration

Let L^{VRT} and L^{FR} denote the equilibrium borrowing limit in the VRT and the flat rate economy respectively. Proposition 8 characterizes the sufficient conditions such that the former exceeds the latter. Proposition 9 characterizes the sufficient conditions such that the result in proposition 8 translates into a result on welfare across the two economies.

Proposition 8 *Maintain all assumptions of propositions 4-7. If* $\underline{\rho}_P > \rho_P > \overline{\rho}_P$ *with* $\underline{\rho}_P$ *defined in (90), then* $L^{VRT} > L^{FR}$.

Proof. See appendix E. ■

Intuitively, the effects of the penalty rate on the welfare gain from VRT work through the effect of the penalty rate on the market rate. This comes into play in various channels. First, the consumption of types ε_1, ε_2 is less distorted with respect to the efficient allocation when the money market rate is relatively low, which happens when the penalty rate is relatively high due to the endogenous adjustment of targets described above. Hence, lenders are more willing to consume than to lend, and unconstrained borrowers have access to cheaper borrowing. For both types consumption increases. Second, cheaper borrowing affects saving on net-of-target money holdings, as each unit of money borrowed is cheaper at relatively low interest rates. Third, autarks' welfare loss from being banned from the market is higher in a VRT than in a flat rate remuneration framework because the VRT introduces an additional productive opportunity where money is valuable. Besides being used to finance consumption in the goods market, money is valuable to meet the target. Hence, the ability to access a market where money is reshuffled from buyers with lower to buyers with higher marginal value, is relatively more valuable in a VRT framework.

Finally, drawing conclusions about welfare across the two frameworks requires mapping the borrowing limit to the value of buyers. The following proposition characterizes sufficient conditions on policy parameters such that buyers' welfare is higher with a VRT framework.

Let $\gamma^{FR}\rho_E^{FR}$ denote the inverse of the remuneration rate on reserves, adjusted by inflation, in the economy implementing monetary policy with a flat rate remuneration framework, and $\gamma^{VRT}\rho_E^{VRT}$ the inverse of the remuneration rate on reserves above the target, adjusted by inflation, in the economy implementing monetary policy with a VRT framework. Furthermore, let $V^{B,VRT}$, $V^{B,FR}$ denote, respectively, the value of buyers at the beginning of every period in the economy with a flat rate remuneration framework and with a VRT framework.

Proposition 9 *If* $\gamma^{VRT}\rho_E^{VRT} = \gamma^{FR}\rho_E^{FR}$ *then* $V^{B,VRT} > V^{B,FR}$.

Proof. See appendix E. ■

Notice that the adoption of policies such that $\gamma^{VRT}\rho_E^{VRT} = \gamma^{FR}\rho_E^{FR}$ is feasible in both frameworks as lump sum taxes are assumed to be available to the consolidated government-central bank.

The assumptions in proposition 9 are sufficient for the value of autarky in the economy with a VRT framework is greater than or equal to the value of autarky in the economy with a flat rate remuneration framework. Intuitively, autarks are always at least as well off with a VRT framework as with a flat rate framework if any allocation feasible in the latter framework is also feasible in the former framework by choosing targets to be zero. When the real value of the remuneration rate on reserves in excess of the target in a VRT framework is the same as the real value of the remuneration rate on all reserves in a flat rate framework, then choosing zero targets renders the VRT equivalent to the flat rate framework.

5 Conclusion

In response to the 2007-2009 financial crisis several central banks adopted a variety of unprecedented measures to support liquidity in various markets. Faced with dramatically larger reserve balances across financial institutions, the Federal Reserve moved away from an implementation framework for monetary policy operating through changes in the supply of reserves to the banking system - i.e. open market operations paired with reserve requirements.

In particular, the Federal Reserve started operating a so-called floor system, whereby the money market rate collapses to the central bank's administer rate, resulting in a substantial reduction in trading activity in the federal funds market. While such reduction in trading activity is consistent with banks holding abundant excess reserves and rarely needing to borrow, we show that it is also consistent with banks being unable to borrow. When money markets are unsecured, borrowers repay their loans only if it is in their best interest to do so. A trustworthy borrower is then one who needs frequent access to credit, as such borrower would suffer from being excluded from money markets were he to default on a loan. Hence, banks' rare need to borrow causes a collapse in the endogenous borrowing limit that banks face in unsecured money markets. Despite they would want to borrow, banks are unable to do so because their borrowing limit evaporates.

We characterize the set of parameters which lead to a floor system in the long run and show that an "ample" reserves regime is the locally worst policy. On the one hand, discretely reducing reserves would introduce more frequent need to borrow and, as a consequence, result in increased trading in interbank markets. On the other hand, increasing reserves, by raising their remuneration rate, would partially close the gap with the Friedman rule resulting in a lower inflation tax on idle money holdings.

Furthermore, we investigate the welfare properties of an alternative framework for monetary policy implementation, called a voluntary reserve target framework, designed to preserve interbank trade regardless of the level of aggregate reserves in the economy. We show that, by endogenously forcing banks to trade on a regular basis to meet their voluntary targets, such framework relaxes borrowing constraints and improves welfare.

Finally, a voluntary reserve target framework would constitute an alternative to truly ample levels of reserves for relieving liquidity stress in short term funding markets, thus weighing in on the controversial debate about whether the Federal Reserve should aim, in the long run, for a small or a large balance sheet.

Appendices

A Flat Rate

A.1 Proof of proposition 2.

The proof is developed in steps as described by the following lemmas.

Lemma 10 *The two money-choice equations can be equivalently expressed as*

$$(z + \rho_l \gamma L') = \mathbb{E}\varepsilon + \int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon,$$

and

$$z^A = \mathbb{E}_\varepsilon \varepsilon + \int_0^{\varepsilon_E^A} F(\varepsilon)\, d\varepsilon.$$

Proof. First, note that $\mathbb{E}[\varepsilon] = \int_0^{\varepsilon_M} \varepsilon dF + \int_{\varepsilon_M}^{\infty} \varepsilon dF$. Also, after integration by parts,

$$\int_0^{\varepsilon_M} \varepsilon dF = [\varepsilon F(\varepsilon)|_0^{\varepsilon_M} - \int_0^{\varepsilon_M} F(\varepsilon) d\varepsilon. \tag{44}$$

Hence, we can write

$$\int_{\varepsilon_M}^{\infty} \varepsilon dF = \mathbb{E}[\varepsilon] - \int_0^{\varepsilon_M} \varepsilon dF = \mathbb{E}[\varepsilon] + \int_0^{\varepsilon_M} F(\varepsilon) d\varepsilon - \varepsilon_M F(\varepsilon_M).$$

For buyers:

$$\begin{aligned}
\frac{\gamma}{\beta} &= \int_0^{\varepsilon_B} \frac{1}{\rho_l} dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\beta \phi'(m + \rho_l B')} dF(\varepsilon) \\
\frac{\gamma \rho_l}{\beta} &= \int_0^{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF(\varepsilon) \\
\frac{\rho_l}{\zeta} \varepsilon_B &= \varepsilon_B F(\varepsilon_B) + \int_{\varepsilon_B}^{\infty} \varepsilon dF(\varepsilon) \qquad (45) \\
(z + \rho_l \gamma L') &= \mathbb{E}\varepsilon + \int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon \qquad (46)
\end{aligned}$$

where the last equation follows by integrating by parts.

For autarks:

$$\begin{aligned}
\phi &= \int_0^{\varepsilon_E^A} \frac{\beta\phi'}{\rho_E} dF(\varepsilon) + \int_{\varepsilon_E^A}^{\infty} \varepsilon u'\left(\frac{m^A}{p}\right)\frac{1}{p} dF(\varepsilon) \\
\frac{\gamma}{\beta} &= \frac{1}{\rho_E}\left[F\left(\varepsilon_E^A\right) + \int_{\varepsilon_E^A}^{\infty} \frac{\varepsilon}{\varepsilon_E^A} dF(\varepsilon)\right] \\
\frac{\rho_E}{\zeta} &= F\left(\varepsilon_E^A\right) + \int_{\varepsilon_E^A}^{\infty} \frac{\varepsilon}{\varepsilon_E^A} dF(\varepsilon) \\
\frac{\rho_E}{\zeta}\varepsilon_E^A &= \varepsilon_E^A F\left(\varepsilon_E^A\right) + \int_{\varepsilon_E^A}^{\infty} \varepsilon dF(\varepsilon) \qquad (47) \\
&= \mathbb{E}_\varepsilon \varepsilon + \int_0^{\varepsilon_E^A} F(\varepsilon) d\varepsilon \qquad (48)
\end{aligned}$$

and finally

$$z^A = \mathbb{E}_\varepsilon \varepsilon + \int_0^{\varepsilon_E^A} F(\varepsilon) d\varepsilon$$

■

Lemma 11 *Deposits of autarks can be written as*

$$\phi\mathbb{E}\left[d_\varepsilon^A\right] = \frac{\rho_E}{\zeta}\left[z^A - \mathbb{E}(\varepsilon)\right].$$

Proof. Deposits for autarks consist of all unspent funds. That is

$$\begin{aligned}
\mathbb{E}d_\varepsilon^A &= \int_0^{\varepsilon_E^A}\left(m^A - \frac{\varepsilon\rho_E}{\beta\phi'}\right) dF(\varepsilon) \\
\beta\phi'\mathbb{E}\frac{\left[d_\varepsilon^A\right]}{\rho_E} &= \frac{\beta\phi'}{\rho_E}\int_0^{\varepsilon_E^A}\left(m^A - \frac{\varepsilon\rho_E}{\beta\phi'}\right) dF(\varepsilon) \\
&= \frac{\zeta}{\rho_E}\int_0^{\varepsilon_E^A}\left(\phi m^A - \frac{\varepsilon\rho_E\phi}{\beta\phi'}\right) dF(\varepsilon) \\
&= \frac{\zeta}{\rho_E}\int_0^{\varepsilon_E^A}\left(z^A - \frac{\varepsilon\rho_E}{\zeta}\right) dF(\varepsilon) \\
&= \frac{\zeta}{\rho_E}\left(z^A F\left(\varepsilon_E^A\right) - \frac{\rho_E}{\zeta}\int_0^{\varepsilon_E^A} \varepsilon dF(\varepsilon)\right) \\
&= \frac{\zeta}{\rho_E}\left(\frac{\rho_E}{\zeta}\varepsilon_E^A F\left(\varepsilon_E^A\right) - \frac{\rho_E}{\zeta}\int_0^{\varepsilon_E^A} \varepsilon dF(\varepsilon)\right).
\end{aligned}$$

where the last equation follows from the definition of ε_E^A. Then we can integrate by parts to get

$$\begin{aligned}
\beta\phi'\mathbb{E}\frac{\left[d_\varepsilon^A\right]}{\rho_E} &= \varepsilon_E^A F\left(\varepsilon_E^A\right) - \int_0^{\varepsilon_E^A} \varepsilon dF(\varepsilon) = \int_0^{\varepsilon_E^A} F(\varepsilon) d\varepsilon \\
&= z^A - \mathbb{E}_\varepsilon \varepsilon
\end{aligned}$$

where the second equation follows from the first order condition for money holdings for autarks $z^A = \frac{\rho_E}{\zeta}\varepsilon_E^A = \varepsilon_E^A F\left(\varepsilon_E^A\right) + \int_{\varepsilon_E^A}^{\infty} \varepsilon dF(\varepsilon)$. Substituting $\gamma = \frac{\phi}{\phi'}$ and $\zeta = \frac{\beta}{\gamma}$ yields the result.

■

Lemma 12 *The money market clearing condition can be written as $\beta L' = \int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon$.*

Proof. The money market clearing condition is $\mathbb{E}_\varepsilon[-l_\varepsilon] = 0$, that is

$$
\begin{aligned}
0 &= \int_0^{\varepsilon_B} (pq_\varepsilon - m)\, dF(\varepsilon) + \rho_l B' \int_{\varepsilon_B}^{\infty} dF(\varepsilon) \\
&= \int_0^{\varepsilon_B} \left(\frac{\rho_E \phi}{\beta \phi'} \varepsilon \frac{\rho_l}{\rho_E} - z\right) dF(\varepsilon) + \frac{\phi}{\phi'} \rho_l \phi' B' \int_{\varepsilon_B}^{\infty} dF(\varepsilon) \\
&= \int_0^{\varepsilon_B} \left(\frac{\rho_l}{\zeta}\varepsilon - z\right) dF(\varepsilon) + \gamma \rho_l L' \int_{\varepsilon_B}^{\infty} dF(\varepsilon) \\
&= \int_0^{\varepsilon_B} \left(\frac{\rho_l}{\zeta}\varepsilon - (z + \gamma\rho_l L')\right) dF(\varepsilon) + \gamma \rho_l L' \\
&= \int_0^{\varepsilon_B} \left(\frac{\rho_l}{\zeta}\varepsilon - \frac{\varepsilon_B \rho_l}{\zeta}\right) dF(\varepsilon) + \gamma \rho_l L' \\
&= \frac{\rho_l}{\zeta} \int_0^{\varepsilon_B} (\varepsilon - \varepsilon_B)\, dF(\varepsilon) + \gamma \rho_l L' \\
&= \frac{\rho_l}{\zeta}\left[\int_0^{\varepsilon_B} \varepsilon dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \varepsilon dF(\varepsilon) - \varepsilon_B \left(\int_0^{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF(\varepsilon)\right)\right] + \gamma \rho_l L' \\
&= \frac{\rho_l}{\zeta}\left[\mathbb{E}_\varepsilon \varepsilon - \varepsilon_B \left(\int_0^{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF(\varepsilon)\right)\right] + \gamma \rho_l L' \\
&= \frac{\rho_l}{\zeta}\left[\mathbb{E}_\varepsilon \varepsilon - \varepsilon_B \frac{\gamma \rho_l}{\beta}\right] + \gamma \rho_l L'
\end{aligned}
$$

where the last equation follows from the first order condition for money holdings for buyers, rewritten as $\frac{\gamma\rho_l}{\beta} = \int_0^{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF(\varepsilon)$. Then we have

$$
\begin{aligned}
\varepsilon_B \frac{\rho_l}{\zeta} - \mathbb{E}_\varepsilon \varepsilon &= \beta L' \\
\int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon &= \beta L'
\end{aligned}
$$

■

Lemma 13 *When $\rho_E > \rho_l$, so buyers make no deposits, the equation defining the endogenous borrowing limit in a stationary equilibrium can be written as*

$$(1-\beta) L = \mathbb{E}_\varepsilon \varepsilon - z + \Delta \mathbb{E}_\varepsilon u(q_\varepsilon)$$

Proof. Assuming $\rho_E > \rho_l$ allows one to ignore the potential for buyers' deposits. The equation defining L is then

$$L = \phi B = V^B - V^A \;\; = \;\; z^A - z + \Delta \mathbb{E}_\varepsilon u + \beta \phi' \mathbb{E}_\varepsilon \left[-\frac{l_\varepsilon}{\rho_l} + R\left(d_\varepsilon\right) + \frac{l_\varepsilon^A}{\rho_l} - R\left(d_\varepsilon^A\right) \right] + \beta L'$$

Under the assumption $\rho_l < \rho_E$ then $R\left(d_\varepsilon\right) = 0$, because autarks are shut out of money markets then $l_\varepsilon^A = 0$, and money market clearing requires $\mathbb{E}_\varepsilon \left[-\frac{l_\varepsilon}{\rho_l} \right] = 0$. Then we are left with

$$\begin{aligned} L &= z^A - z + \Delta \mathbb{E}_\varepsilon u - \beta \phi' \mathbb{E}_\varepsilon \left[\frac{d_\varepsilon^A}{\rho_E} \right] + \beta L' \\ &= z^A - z + \Delta \mathbb{E}_\varepsilon u - \left(z^A - \mathbb{E}_\varepsilon \varepsilon \right) + \beta L' \end{aligned}$$

which, in a stationary equilibrium, is simply $(1-\beta) L = \mathbb{E}_\varepsilon \varepsilon - z + \Delta \mathbb{E}_\varepsilon u\left(q_\varepsilon\right)$. ■

Lemma 14 *The difference in expected utility from consumption, $\Delta\,\mathbb{E}[u]$, decreases in ρ_l and is zero at when $\rho_l = \rho_E$.*

Proof. That $\Delta\,\mathbb{E}[u] = 0$ when $\rho_l = \rho_E$ follows from two observations. First, consider the two money holding equations. If $\rho_l = \rho_E$ the equations become the same, so $\varepsilon_B = \varepsilon_M$. Second, if $\rho_l = \rho_E$ and $\varepsilon_B = \varepsilon_M$, then $q_\varepsilon = q_\varepsilon^A$. Identical consumption, of course, implying equal utility from such, given common preferences.

For the next part, we need the derivative of ε_B with respect to ρ_l. To calculate this, consider the money holding equation (45) from lemma 10:

$$\frac{\rho_l}{\zeta} \varepsilon_B = \mathbb{E}[\varepsilon] + \int_0^{\varepsilon_B} F(\varepsilon) d\varepsilon$$

Moving the ε_B / ζ over and differentiating with respect to ε_B gives

$$\frac{\partial}{\partial \varepsilon_B} \left[\zeta \frac{\mathbb{E}[\varepsilon] + \int_0^{\varepsilon_B} F(\varepsilon) d\varepsilon}{\varepsilon_B} \right] = \zeta \frac{F(\varepsilon_B)\varepsilon_B - \left(\mathbb{E}[\varepsilon] + \int_0^{\varepsilon_B} F(\varepsilon) d\varepsilon \right)}{\varepsilon_B^2} = -\frac{\zeta}{\varepsilon_B^2} \int_{\varepsilon_B}^{\infty} \varepsilon dF < 0$$

where the last line follows from integration by parts, as in equation (44). This, then, implies that

$$\frac{\partial \varepsilon_B}{\partial \rho_l} = -\frac{\varepsilon_B^2}{\zeta \int_{\varepsilon_B}^{\infty} \varepsilon dF}.$$

To show that $\Delta\,\mathbb{E}[u]$ is decreasing, consider its derivative. Because q_ε^A is independent of ρ_l

then

$$
\begin{aligned}
\frac{\partial \Delta \mathbb{E}_{\varepsilon} u}{\partial \rho_l} &= \frac{\partial \left[\int \varepsilon \left(u\left(q_{\varepsilon}\right) - u\left(q_{\varepsilon}^{A}\right)\right) dF\left(\varepsilon\right)\right]}{\partial \rho_l} \\
&= \frac{\partial}{\partial \rho_l}\left[\int_0^{\varepsilon_B} \varepsilon u\left(\frac{\rho_l \varepsilon}{\rho_E}\right) + \int_{\varepsilon_B}^{\infty} \varepsilon u\left(\frac{\rho_l \varepsilon_B}{\rho_E}\right) dF\left(\varepsilon\right)\right] \\
&= \int_0^{\varepsilon_B} \frac{\varepsilon}{\rho_l} dF\left(\varepsilon\right) + \int_{\varepsilon_B}^{\infty} \varepsilon \frac{\rho_E}{\rho_l \varepsilon_B} \frac{1}{\rho_E}\left(\varepsilon_B + \rho_l \frac{\partial \varepsilon_B}{\partial \rho_l}\right) dF\left(\varepsilon\right) \\
&= \int_0^{\varepsilon_B} \frac{\varepsilon}{\rho_l} dF\left(\varepsilon\right) + \int_{\varepsilon_B}^{\infty}\left(\frac{\varepsilon}{\rho_l} + \frac{\varepsilon}{\varepsilon_B}\frac{\partial \varepsilon_B}{\partial \rho_l}\right) dF\left(\varepsilon\right) \\
&= \int_0^{\infty} \frac{\varepsilon}{\rho_l} dF\left(\varepsilon\right) + \int_{\varepsilon_B}^{\infty}\left(\frac{\varepsilon}{\varepsilon_B}\frac{\partial \varepsilon_B}{\partial \rho_l}\right) dF\left(\varepsilon\right) \\
&= \int_0^{\infty} \frac{\varepsilon}{\rho_l} dF\left(\varepsilon\right) + \int_{\varepsilon_B}^{\infty}\left(\frac{\varepsilon}{\varepsilon_B}\left(-\frac{\varepsilon_B^2}{\zeta \int_{\varepsilon_B}^{\infty} \varepsilon dF\left(\varepsilon\right)}\right)\right) dF\left(\varepsilon\right) \\
&= \frac{1}{\rho_l}\int_0^{\infty} \varepsilon dF\left(\varepsilon\right) + \frac{1}{\varepsilon_B}\left(-\frac{\varepsilon_B^2}{\zeta \int_{\varepsilon_B}^{\infty} \varepsilon dF\left(\varepsilon\right)}\right)\int_{\varepsilon_B}^{\infty} \varepsilon dF\left(\varepsilon\right) \\
&= \frac{1}{\rho_l}\int_0^{\infty} \varepsilon dF\left(\varepsilon\right) - \frac{\varepsilon_B}{\zeta} \\
&= \frac{1}{\rho_l}\int_0^{\infty} \varepsilon dF\left(\varepsilon\right) - \frac{\varepsilon_B}{\zeta}
\end{aligned}
$$

then using $\frac{\rho_l}{\zeta}\varepsilon_B = \mathbb{E}\varepsilon + \int_0^{\varepsilon_B} F\left(\varepsilon\right) d\varepsilon$ from equation (45) yields

$$
\begin{aligned}
\frac{\partial \Delta \mathbb{E}_{\varepsilon} u}{\partial \rho_l} &= \frac{1}{\rho_l}\int_0^{\infty} \varepsilon dF\left(\varepsilon\right) - \frac{1}{\rho_l}\left(\mathbb{E}\varepsilon + \int_0^{\varepsilon_B} F\left(\varepsilon\right) d\varepsilon\right) \\
&= -\frac{1}{\rho_l}\int_0^{\varepsilon_B} F\left(\varepsilon\right) d\varepsilon < 0 \qquad (49)
\end{aligned}
$$

■

Lemma 15 *There always exists a stationary equilibrium where $\rho_l = \rho_E$ and $L = 0$. If $\gamma\rho_E \neq 1$ then $\rho_l = \rho_E \Rightarrow L = 0$.*

Proof. Consider the first order conditions for money holdings for buyers and autarks, (45) and (47), and evaluate them at $\rho_l = \rho_E$. The first equation pins down $\varepsilon_B \in (0, \infty)$ uniquely: in fact, if $\varepsilon_B \to \infty$ then (45) is violated as $\frac{\gamma\rho_E}{\beta} > 1$ because the economy is away from the Friedman rule by assumption. The left hand side of (45) is larger than the right hand side:

$$
\frac{\gamma\rho_E}{\beta} > 1 = \lim_{\varepsilon_B \to \infty}\left\{F\left(\varepsilon_B\right) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF\left(\varepsilon\right)\right\}.
$$

Similarly if $\varepsilon_B \to 0$ then (45) is violated as the left hand side is smaller than the right hand side:

$$
\frac{\gamma\rho_E}{\beta} < \infty = \lim_{\varepsilon_B \to 0}\left\{F\left(\varepsilon_B\right) + \int_{\varepsilon_B}^{\infty} \frac{\varepsilon}{\varepsilon_B} dF\left(\varepsilon\right)\right\}.
$$

Let $g(\varepsilon_B)$ denote the right hand side of (45), and notice that $\frac{\partial g(\varepsilon_B)}{\partial \varepsilon_B} = f(\varepsilon_B) + \int_{\varepsilon_B}^{\infty} -\frac{\varepsilon f(\varepsilon)}{\varepsilon_B^2} d\varepsilon - f(\varepsilon_B) < 0$, where the last inequality is strict is we assume that f has no mass point. Since the left hand side of (45) is finite and given, and the right hand side of (45) is strictly decreasing in ε_B, then there exist a unique $\varepsilon_B^* \in (0, \infty)$ solving (45). Following the same argument equation (47) pins down $\varepsilon_E^A \in (0, \infty)$ uniquely. Because $\rho_l = \rho_E$ then $\varepsilon_B = \varepsilon_E^A \in (0, \infty)$. using this result and rearranging the first order conditions (45) and (47) as (46) and (48) yields $z + \gamma \rho_E L' = z^A$. This means that buyers and autarks have the same portfolio of liquid assets (i.e. assets that allow agents to purchase consumption goods in the goods' market). Because of this then buyers and autarks consume the same allocation in the goods market and, thus, enjoy the same utility from consumption: $\Delta \mathbb{E}_{\varepsilon} u(q_{\varepsilon}) = 0$. In fact, with $\rho_l = \rho_E$ the consumption allocations characterized in section 2.3 become:

$q^{\varepsilon} =$	$q^{A\varepsilon} =$	
$\frac{\varepsilon \rho_l}{\rho_E} = \varepsilon$	ε	if $\varepsilon \in (0, \varepsilon_B = \varepsilon_E^A]$
$\frac{\rho_l \varepsilon_B}{\rho_E} = \varepsilon_E^A$	ε_E^A	if $\varepsilon > \varepsilon_B = \varepsilon_E^A$

Consider now the definition of endogenous borrowing limit derived in lemma 13 and evaluated it at a stationary equilibrium, so that $L = L'$. Substituting $\Delta \mathbb{E}_{\varepsilon} u(q_{\varepsilon}) = 0$ and using (46) yields

$$(1-\beta) L = \mathbb{E}_{\varepsilon} \varepsilon - z \tag{50}$$

$$(1-\beta) L = \rho_l \gamma L' - \int_0^{\varepsilon_B} F(\varepsilon) d\varepsilon. \tag{51}$$

Lemma 12 then implies

$$(1-\gamma\rho_l) L = (1-\gamma\rho_E) L = 0.$$

Hence a solution to this equation is $L = 0$. If $\gamma \rho_E \neq 1$ then the only solution to this equation is $L = 0$.

■

A.2 Proof of proposition 3.

Proof. Note that $\rho_l > \rho_E$ is not possible: no one would lend money at a worse rate than can be had from the central bank. We will require the derivatives of z and L with respect to ρ_l.

For the derivative of ε_B with respect to ρ_l

$$\frac{\partial \rho_l}{\partial \varepsilon_B} = -\frac{\zeta}{\varepsilon_B^2} \int_{\varepsilon_B}^{\infty} \varepsilon dF(\varepsilon) \qquad \text{implies} \qquad \frac{\partial \varepsilon_B}{\partial \rho_l} = -\frac{\varepsilon_B^2}{\zeta \int_{\varepsilon_B}^{\infty} \varepsilon dF(\varepsilon)}$$

Then, Lemma 12 implies

$$\frac{\partial L}{\partial \rho_l} = \frac{F(\varepsilon_B)}{\beta} \frac{\partial \varepsilon_B}{\partial \rho_l} \tag{52}$$

Then rewriting z in terms of L'

$$z = \frac{\rho_l}{\zeta}\varepsilon_B - \rho_l \gamma L'$$

Then

$$\begin{aligned}
\frac{\partial z}{\partial \rho_l} &= \frac{\varepsilon_B}{\zeta} + \frac{\rho_l}{\zeta}\frac{\partial \varepsilon_B}{\partial \rho_l} - \gamma L' - \rho_l \gamma \frac{\partial L'}{\partial \rho_l} \\
&= \frac{\varepsilon_B}{\zeta} + \frac{\rho_l}{\zeta}\frac{\partial \varepsilon_B}{\partial \rho_l} - \gamma L' - \rho_l \gamma \frac{F(\varepsilon_B)}{\beta}\frac{\partial \varepsilon_B}{\partial \rho_l} \\
&= \frac{\varepsilon_B}{\zeta} + \left(\frac{1}{\zeta} - \gamma\frac{F(\varepsilon_B)}{\beta}\right)\rho_l \frac{\partial \varepsilon_B}{\partial \rho_l} - \gamma L' \qquad (53)
\end{aligned}$$

then from Lemma 13

$$0 = \mathbb{E}_\varepsilon \varepsilon - z + \Delta \mathbb{E}_\varepsilon u(q_\varepsilon) - (1-\beta) L$$

Letting $\Gamma(\rho_l)$ denote the right hand side of the above equation, and differentiating it with respect to ρ_l, and substituting out from (53),(52), and (49), yields

$$\begin{aligned}
\Gamma'(\rho_l) &= -\frac{\partial z}{\partial \rho_l} + \frac{\partial \Delta \mathbb{E}_\varepsilon u}{\partial \rho_l} - (1-\beta)\frac{\partial L}{\partial \rho_l} \\
&= -\left[\frac{\varepsilon_B}{\zeta} + \left(\frac{1}{\zeta} - \gamma\frac{F(\varepsilon_B)}{\beta}\right)\rho_l \frac{\partial \varepsilon_B}{\partial \rho_l} - \gamma L'\right] + \int_0^\infty \frac{\varepsilon}{\rho_l} dF(\varepsilon) + \\
&\quad + \int_{\varepsilon_B}^\infty \left(\frac{\varepsilon}{\varepsilon_B}\frac{\partial \varepsilon_B}{\partial \rho_l}\right) dF(\varepsilon) - (1-\beta)\frac{F(\varepsilon_B)}{\beta}\frac{\partial \varepsilon_B}{\partial \rho_l}
\end{aligned}$$

Grouping the terms in $\frac{\partial \varepsilon_B}{\partial \rho_l}$ yields

$$\begin{aligned}
\Gamma'(\rho_l) &= -\frac{\partial \varepsilon_B}{\partial \rho_l}\left[\left(\frac{1}{\zeta} - \gamma\frac{F(\varepsilon_B)}{\beta}\right)\rho_l + (1-\beta)\frac{F(\varepsilon_B)}{\beta} - \int_{\varepsilon_B}^\infty \left(\frac{\varepsilon}{\varepsilon_B}\right) dF(\varepsilon)\right] - \frac{\varepsilon_B}{\zeta} + \int_0^\infty \frac{\varepsilon}{\rho_l} dF(\varepsilon) + \gamma L' \\
&= -\frac{\partial \varepsilon_B}{\partial \rho_l}\left[(1 - F(\varepsilon_B))\frac{\gamma \rho_l}{\beta} + (1-\beta)\frac{F(\varepsilon_B)}{\beta} - \int_{\varepsilon_B}^\infty \left(\frac{\varepsilon}{\varepsilon_B}\right) dF(\varepsilon)\right] \\
&\quad -\frac{1}{\rho_l}\left[\mathbb{E}\varepsilon + \int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon\right] + \int_0^\infty \frac{\varepsilon}{\rho_l} dF(\varepsilon) + \gamma L' \\
&= -\frac{\partial \varepsilon_B}{\partial \rho_l}\left[(1 - F(\varepsilon_B))\frac{\gamma \rho_l}{\beta} + \frac{(1-\beta)}{\beta}F(\varepsilon_B) - \left(\frac{\gamma \rho_l}{\beta} - \int_0^\infty dF(\varepsilon)\right)\right] - \frac{\int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon}{\rho_l} + \gamma L' \\
&= -\frac{\partial \varepsilon_B}{\partial \rho_l}\left[\left(\frac{(1-\beta)}{\beta} - \frac{\gamma \rho_l}{\beta}\right)F(\varepsilon_B) + \int_0^{\varepsilon_B} dF(\varepsilon)\right] - \frac{\int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon}{\rho_l} + \gamma \frac{\int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon}{\beta}
\end{aligned}$$

where the last equation follows from Lemma 12. Then we have

$$\Gamma'(\rho_l) = -\frac{\partial \varepsilon_B}{\partial \rho_l}\left[(1 - \beta - \gamma \rho_l)\frac{1}{\beta}F(\varepsilon_B) + \int_0^{\varepsilon_B} dF(\varepsilon)\right] + \left(\frac{\gamma}{\beta} - \frac{1}{\rho_l}\right)\int_0^{\varepsilon_B} F(\varepsilon)\, d\varepsilon \qquad (54)$$

The definition of ε_B in (45) then implies that the last term is

$$\begin{aligned}\left(\frac{\gamma}{\beta}-\frac{1}{\rho_l}\right)\int_0^{\varepsilon_B}F(\varepsilon)\,d\varepsilon &= \frac{1}{\rho_l}\left(\frac{\rho_l}{\zeta}-1\right)\int_0^{\varepsilon_B}F(\varepsilon)\,d\varepsilon \\ &= \frac{\int_0^{\varepsilon_B}F(\varepsilon)\,d\varepsilon}{\rho_l}\left[F(\varepsilon_B)+\int_{\varepsilon_B}^{\infty}\frac{\varepsilon}{\varepsilon_B}dF(\varepsilon)-1\right] \\ &> 0\end{aligned}$$

where the last inequality follows from

$$\int_{\varepsilon_B}^{\infty}\frac{\varepsilon}{\varepsilon_B}dF(\varepsilon) > \int_{\varepsilon_B}^{\infty}dF(\varepsilon)=1-F(\varepsilon_B)$$

If $\left[(1-\beta-\gamma\rho_l)\frac{1}{\beta}F(\varepsilon_B)+\int_0^{\varepsilon_B}dF(\varepsilon)\right]\geq 0$ then (54) implies $\Gamma'(\rho_l)>0$. Notice that

$$\begin{aligned}(1-\beta-\gamma\rho_l)\frac{1}{\beta}F(\varepsilon_B)+\int_0^{\varepsilon_B}dF(\varepsilon) &= (1-\beta-\gamma\rho_l)\frac{1}{\beta}F(\varepsilon_B)+\frac{1}{\beta}\beta F(\varepsilon_B) \\ &= (1-\gamma\rho_l)\frac{1}{\beta}F(\varepsilon_B)\end{aligned}$$

Since $\rho_E>\rho_l$ the a sufficient condition for $\left[(1-\beta-\gamma\rho_l)\frac{1}{\beta}F(\varepsilon_B)+\int_0^{\varepsilon_B}dF(\varepsilon)\right]\geq 0$ is $1\geq\gamma\rho_E$, in turn implying $\Gamma'(\rho_l)>0$. Because $\Gamma(\rho_l)$ is strictly increasing then there exists at most one value $\rho_l\in(0,1)$ such that $\Gamma(\rho_l)=0$. We know that $\rho_l=\rho_E$ is one such point. Hence, it must be the unique solution. ■

B VRT

B.1 Proof of Lemma 3

Proof. Consider the money market decision problem of a buyer with preference shock ε.

B.1.1 Region $\varepsilon\leq\varepsilon_L$

In this region, maintaining the assumption $\rho_l\in(\rho_P,\rho_E)$, buyers are lending and fully fund their target: $l_\varepsilon<0$, $\lambda_B=0$, $d_\varepsilon=T$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l}+\lambda_\varepsilon=<0$$

$$1/\rho_l=\lambda_\varepsilon\geq R_d'\geq\frac{1}{\rho_E}$$

$$\frac{\varepsilon}{q_\varepsilon}=p\beta\phi'\lambda_\varepsilon$$

Thus: $\frac{1}{\rho_E} < \frac{1}{\rho_l}$, and $\lambda_\varepsilon = \frac{1}{\rho_l}$, $l_\varepsilon = T - m + pq_\varepsilon$ and $pq_\varepsilon = \frac{\varepsilon \rho_l}{\beta\phi'}$. The largest value of ε consistent with this solution is $\varepsilon_L = \{\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}] : l_\varepsilon = 0\}$ (i.e. the marginal buyer stops lending in the money market). This implies $pq_\varepsilon = m - T$ and we can define $\varepsilon_L = (m-T)\beta\phi'\lambda_\varepsilon = (m-T)\frac{\beta\phi'}{\rho_E}$. In this case, the first order condition for l_ε evaluated at l_{ε_L} implies $-\frac{1}{\rho_l} + \lambda_\varepsilon = 0$. Thus we can rewrite

$$\varepsilon_L = (m - T)\beta\phi'\lambda_\varepsilon = (m - T)\frac{\beta\phi'}{\rho_l}. \tag{55}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned} d_\varepsilon &= T \\ \lambda_\varepsilon &= \frac{1}{\rho_l} \\ l_\varepsilon &= pq_\varepsilon + T - m < 0 \\ pq_\varepsilon &= \frac{\varepsilon\rho_l}{\beta\phi'} \end{aligned}$$

where we can substitute $p\beta\phi' = \rho_T$ to get $q_\varepsilon = \frac{\varepsilon\rho_l}{\rho_T}$.

B.1.2 Region $\varepsilon_L < \varepsilon \leq \varepsilon_B$

In this region a buyer borrows in the money market, but is not borrowing constrained, and fully funds his target. Therefore $l_\varepsilon > 0$, $\lambda_B = 0$, $d_\varepsilon = T$. Optimality conditions in this case imply:

$$\lambda_\varepsilon = \frac{1}{\rho_l}$$

$$\lambda_\varepsilon = \frac{1}{\rho_l} \geq R'_d \geq \frac{1}{\rho_E}$$

So $\lambda_\varepsilon = \frac{1}{\rho_l}$. Furthermore:

$$\frac{\varepsilon}{q_\varepsilon} = p\beta\phi'\lambda_\varepsilon \quad l_\varepsilon = pq_\varepsilon + T - m$$

where $pq_\varepsilon = \frac{\varepsilon}{\beta\phi'\lambda_\varepsilon} = \frac{\varepsilon\rho_l}{\beta\phi'}$ The largest value of ε consistent with this solution is $\varepsilon_B = \{\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}] : l_\varepsilon = \rho_l B\}$ (i.e. the marginal buyer starts being borrowing constrained in the money market). Thus, combining the first order condition for q_ε with the budget constraint we have

$$\varepsilon_B = \beta\phi'\lambda_\varepsilon(\rho_l B + m - T) = \frac{\beta\phi'}{\rho_l}(m - T) + \beta\phi' B. \tag{56}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned} d_\varepsilon &= T \\ \lambda_\varepsilon &= \frac{1}{\rho_l} \\ l_\varepsilon &= T - m + pq_\varepsilon > 0 \\ pq_\varepsilon &= \frac{\varepsilon\rho_l}{\beta\phi'} \end{aligned}$$

where we can substitute $p\beta\phi' = \rho_T$ to get $q_\varepsilon = \frac{\varepsilon\rho_l}{\rho_T}$. Notice that using (55), which can be rearranged as $\varepsilon_L\rho_l = (m - T)\beta\phi'$, we can rewrite $l_\varepsilon = \frac{\rho_l}{\beta\phi'}(\varepsilon - \varepsilon_L)$.

B.1.3 Region $\varepsilon_B < \varepsilon \leq \varepsilon_T$

In this region a buyer borrows in the money market and is borrowing constrained, but fully funds his target because the marginal return on a deposit before meeting the target (i.e. $\frac{1}{\rho_P}$) exceeds the marginal utility of consumption. Therefore $l_\varepsilon > 0$, $\lambda_B > 0$, $d_\varepsilon = T$. Optimality conditions in this case imply:[20]

$$-\frac{1}{\rho_l} + \lambda_\varepsilon = \lambda_B$$

And from the binding borrowing constraint we have $l_\varepsilon = \rho_l B$. Furthermore:

$$\frac{\varepsilon}{q_\varepsilon} = p\beta\phi'\lambda_\varepsilon$$

where q_ε is pinned down by the budget constraint in the goods' market: $pq_\varepsilon = m + \rho_l B - T$. The largest value of ε consistent with this solution is $\varepsilon_T = \{\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}] : d_\varepsilon < T\}$ (i.e. the marginal buyer starts funding his target only partially). Thus, the first order condition for d_ε evaluated at ε_T yields $R'_d = \frac{1}{\rho_P}$ so that $\lambda_{\varepsilon_T} = \frac{1}{\rho_P}$ and, combining the first order condition for q_ε with the budget constraint, we have $\lambda_B = \frac{1}{\rho_P} - \frac{1}{\rho_l}$ and

$$\varepsilon_T = (m + \rho_l B - T)\beta\phi'(\frac{1}{\rho_P}) \tag{57}$$

To summarize, the allocation in this region satisfies:

$$\begin{aligned} d_\varepsilon &= T \\ l_\varepsilon &= \rho_l B \\ pq_\varepsilon &= m + \rho_l B - T \end{aligned}$$

Using then (56) and substituting out $p\beta\phi' = \rho_T$, we can rewrite the last equation as

$$\frac{\rho_T}{\beta\phi'}q_\varepsilon = \frac{\rho_l\varepsilon_B}{\beta\phi'}$$

which simplifies to

$$q_\varepsilon = \frac{\rho_l\varepsilon_B}{\rho_T}.$$

Notice that consumption in this region is constant with respect to ε: this is due to the kink in the remuneration of reserves at the central bank, which makes the value function not differentiable at the value of d_ε where the kink occurs. Indeed in this region the first order condition for q_ε holds with inequality because the marginal value of a unit of deposit when $d_\varepsilon = T$ is simply $\frac{1}{\rho_E}$ which is smaller than the marginal utility evaluated at q_ε.

$$\frac{\varepsilon}{q_\varepsilon} > p\beta\phi'\frac{1}{\rho_E}$$

[20] Recall R'_d is not differentiable at $d = T$.

However the same first order condition for q_ε holds with the inequality reversed if the marginal value of a unit of deposit when $d_\varepsilon < T$ is $\frac{1}{\rho_P}$:

$$\frac{\varepsilon}{q_\varepsilon} < p\beta\phi' \frac{1}{\rho_P}$$

In order to pin down λ_ε, we combine the first order condition for q_ε with the budget constraint, to obtain:

$$\frac{\varepsilon}{\frac{\rho_l \varepsilon_B}{\rho_T}} = p\beta\phi'\lambda_\varepsilon = \rho_T \lambda_\varepsilon$$

where the last equation follows from lemma 2, which implies $p\beta\phi' = \rho_T$. Thus

$$\lambda_\varepsilon = \frac{\varepsilon}{\varepsilon_B \rho_l}$$

B.1.4 Region $\varepsilon_T < \varepsilon \leq \varepsilon_M$

In this region a buyer is borrowing constrained and partially funds his target: $l_\varepsilon = \rho_l B$, $\lambda_B > 0$, $d_\varepsilon < T$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l} + \lambda_\varepsilon = \lambda_B$$

$$\lambda_\varepsilon = R'_d = \frac{1}{\rho_P}$$

So that $\frac{1}{\rho_P} - \frac{1}{\rho_l} = \lambda_B > 0$. Furthermore:

$$\frac{\varepsilon}{q_\varepsilon} = p\beta\phi'\lambda_\varepsilon = p\beta\phi' \frac{1}{\rho_P}$$

$$pq_\varepsilon + d_\varepsilon = \rho_l B + m$$

The largest value of ε consistent with this solution is $\varepsilon_M = \{\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}] : d_\varepsilon = 0\}$ (i.e. the marginal buyer can no longer fund his target at all). Thus, from the budget constraint we have $pq_\varepsilon = \rho_l B + m$ and

$$\varepsilon_M = \frac{\beta\phi'}{\rho_P}(\rho_l B + m) \tag{58}$$

so that $d_\varepsilon = \rho_l B + m - \frac{\varepsilon \rho_P}{\beta\phi'}$ To summarize, the allocation in this region satisfies:

$$d_\varepsilon = \frac{p\rho_P}{\rho_T}(\varepsilon_M - \varepsilon) < T \tag{59}$$

$$l_\varepsilon = \rho_l B \tag{60}$$

$$q_\varepsilon = \frac{\varepsilon \rho_P}{\rho_T} \tag{61}$$

which we obtain by rearranging the first order condition for q_ε as

$$\varepsilon \rho_P = p\beta\phi' q_\varepsilon \Rightarrow q_\varepsilon = \frac{\varepsilon \rho_P}{\rho_T}$$

and by rearranging the first order condition for d_ε using the definition of ε_M, (58), as follows:

$$d_\varepsilon = (m + \rho_l B) - pq_\varepsilon = \frac{\rho_P \varepsilon_M}{\beta\phi'} - \frac{\rho_P \varepsilon}{\beta\phi'}$$

$$d_\varepsilon = \frac{p\rho_P}{\rho_T}(\varepsilon_M - \varepsilon)$$

B.1.5 Region $\varepsilon_M < \varepsilon$

In this region a buyer is borrowing constrained and makes no deposit at all, entirely missing his target: $l_\varepsilon = \rho_l B$, $\lambda_B > 0$, $d_\varepsilon = 0$. Optimality conditions in this case imply:

$$-\frac{1}{\rho_l} + \lambda_\varepsilon = \lambda_B$$

$$\lambda_\varepsilon > R_d' = \frac{1}{\rho_P}$$

and from the budget constraint $pq_\varepsilon = m + \rho_l B$. Using the definition of ε_M, (58), we can rearrange this as $pq_\varepsilon = \frac{\varepsilon_M \rho_P}{\beta\phi'}$, which, substituting the equilibrium value of p, implies $q_\varepsilon = \frac{\varepsilon_M \rho_P}{\rho_T}$. We then obtain λ_ε combining the first order condition for q_ε with the budget constraint:

$$\frac{\varepsilon}{\frac{\varepsilon_M \rho_P}{\rho_T}} = \rho_T \lambda_\varepsilon \quad \Rightarrow \lambda_\varepsilon = \frac{\varepsilon}{\varepsilon_M \rho_P}$$

where $p\beta\phi' = \rho_T$ has been substituted out.

Consumption, loans, deposits, and the envelope condition satisfy

$q^\varepsilon =$	$l^\varepsilon =$	$d^\varepsilon =$	$W_m^B(m\vert T,\varepsilon) =$	
$\varepsilon\rho_l/\rho_T$	$\frac{\rho_l}{\rho_T}p(\varepsilon-\varepsilon_l)$	T	$(1/p)(\rho_T/\rho_l)$	if $\varepsilon < \varepsilon_l$
$\varepsilon\rho_l/\rho_T$	$\frac{\rho_l}{\rho_T}p(\varepsilon-\varepsilon_l)$	T	$(1/p)(\rho_T/\rho_l)$	if $\varepsilon_l < \varepsilon < \varepsilon_B$
$\varepsilon_B\rho_l/\rho_T$	$\rho_l B$	T	$\frac{1}{p}\frac{\varepsilon}{\varepsilon_B}\frac{\rho_T}{\rho_l}$	if $\varepsilon_B < \varepsilon < \varepsilon_T$
$\varepsilon\rho_P/\rho_T$	$\rho_l B$	$p\frac{\rho_P}{\rho_T}[\varepsilon_M - \varepsilon]$	$(1/p)(\rho_T/\rho_P)$	if $\varepsilon_T < \varepsilon < \varepsilon_M$
$\varepsilon_M\frac{\rho_P}{\rho_T}$	$\rho_l B$	0	$\frac{1}{p}\frac{\varepsilon}{\varepsilon_M}\frac{\rho_T}{\rho_P}$	if $\varepsilon > \varepsilon_M$.

■

B.2 Proof of Lemma 4

Proof. Consider the money market decision problem of an autark with preference shock ε.

B.2.1 Region where $\varepsilon_E^A \geq \varepsilon$

In this region buyers in autarky deposit more than their target: $d_\varepsilon > T$. Optimality conditions imply

$$R_d' = \frac{1}{\rho_E}, \quad \lambda_\varepsilon = \frac{1}{\rho_E}, \quad \varepsilon = pq_\varepsilon\beta\phi'\lambda_\varepsilon = pq_\varepsilon\beta\phi'(\frac{1}{\rho_E})$$

Finally the budget constraint implies $d_\varepsilon = m - pq_\varepsilon = m - \frac{\varepsilon\rho_E}{\beta\phi'}$ The largest value of ε consistent with this solution is $\varepsilon_E^A = \{\max_{\{\varepsilon\in[\underline{\varepsilon},\bar{\varepsilon}]\}} \varepsilon : d_\varepsilon = T\}$ (i.e. the marginal buyer can just fund his target). This implies that $pq_\varepsilon = m - T$ and

$$\varepsilon_E^A = (m-T)\beta\phi'\lambda_\varepsilon = (m-T)\frac{\beta\phi'}{\rho_E}. \tag{62}$$

so that

$$q_\varepsilon = \frac{\varepsilon\rho_E}{p\beta\phi'} = \frac{\varepsilon\rho_E}{\rho_T}$$

And using the budget constraint and the definition of ε_E^A in (62), we have:

$$\begin{aligned} d_\varepsilon &= \frac{\rho_E\varepsilon_E^A}{\beta\phi'} + T - pq_\varepsilon \\ &= \frac{\rho_E\varepsilon_E^A}{\beta\phi'} + T\frac{\rho_E\varepsilon}{\beta\phi'} \\ &= \frac{\rho_E(\varepsilon_E^A - \varepsilon)}{\beta\phi'} + T \end{aligned}$$

B.2.2 Region where $\varepsilon_E^A < \varepsilon \leq \varepsilon_P^A$

In this region autarkic buyer deposit just to fund their target: $d_\varepsilon = T$ so that from the budget constraint we have $d_\varepsilon = T = m - pq_\varepsilon$, which, combined with the definition of ε_E^A in (62) and multiplying both sides of the budget constraint by $\beta\phi'$ yields $\beta\phi' pq_\varepsilon = \beta\phi'(m-T)$, which can be rearranged as:

$$\rho_T q_\varepsilon = \varepsilon_E^A\rho_E$$

which is constant in ε in this region. Notice in fact that, similarly to the case of non autarkic buyers, the first order condition for q_ε satisfies:

$$\frac{p\beta\phi'}{\rho_P} > \frac{\varepsilon}{q_\varepsilon} > \frac{p\beta\phi'}{\rho_E}$$

as the marginal utility of consumption is larger than the marginal return on an additional unit of deposits at the central bank when deposits are above the target, but it is smaller than the marginal return on an additional unit of deposits at the central bank when deposits are below the target.

The largest value of ε consistent with this solution is $\varepsilon_P^A = \{\max_{\varepsilon\in[\underline{\varepsilon},\bar{\varepsilon}]} \varepsilon : d_\varepsilon = T\}$ (i.e. the marginal buyer who chooses to fully fund his target). This implies that $\lambda_\varepsilon = \frac{1}{\rho_P}$ and, as $d_\varepsilon = T$, that $pq_\varepsilon = m - T$ and

$$\varepsilon_P^A = (m-T)\beta\phi'\lambda_\varepsilon = (m-T)\frac{\beta\phi'}{\rho_P}. \tag{63}$$

To summarize, in this region $d_\varepsilon = T$, $q_\varepsilon = \frac{\varepsilon_E^A\rho_E}{\rho_T}$. The first order condition for q_ε then implies:

$$\frac{\varepsilon}{\frac{\varepsilon_E^A\rho_E}{\rho_T}} = \rho_T\lambda_\varepsilon$$

where $p\beta\phi' = \rho_T$ has been substituted out, to yield $\lambda_\varepsilon = \frac{\varepsilon}{\rho_E\varepsilon_E^A} = \frac{\varepsilon}{\rho_P\varepsilon_P^A}$.

B.2.3 Region where $\varepsilon_P^A < \varepsilon \leq \varepsilon_M^A$

In this region autarkic buyer only partially fund their target: $0 < d_\varepsilon < T$ so that from the first order condition for d_ε we have $R'_d = \frac{1}{\rho_P} = \lambda_\varepsilon$. The first order condition for q_ε yields: $\varepsilon = pq_\varepsilon \beta\phi' \frac{1}{\rho_P}$. Thus, since lemma 2 implies that $p\beta\phi' = \rho_T$, we have $q_\varepsilon = \frac{\varepsilon\rho_P}{\rho_T}$. From the budget constraint we have $d_\varepsilon = m - pq_\varepsilon$, which, combined with the first order condition for q_ε, yields $d_\varepsilon = m - \frac{\varepsilon\rho_P}{\beta\phi'}$. The largest value of ε consistent with this solution is $\varepsilon_M^A = \{\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}] : d_\varepsilon = 0\}$ (i.e. the marginal buyer can no longer fund his target at all). This implies that $pq_\varepsilon = m$ and

$$\varepsilon_M^A = m\frac{\beta\phi'}{\rho_P}. \tag{64}$$

Using (64) we can then rewrite deposits as $d_\varepsilon = \frac{\beta\phi' m - \varepsilon\rho_P}{\beta\phi'} = \frac{\rho_P(\varepsilon_M^A - \varepsilon)}{\beta\phi'}$. To summarize, in this region we have $q_\varepsilon = \frac{\varepsilon\rho_P}{\rho_T}$, $d_\varepsilon = \frac{\rho_P(\varepsilon_M^A - \varepsilon)}{\beta\phi'}$ and $\lambda_\varepsilon = \frac{1}{\rho_P}$.

B.2.4 Region where $\varepsilon > \varepsilon_M^A$

In this region autarkic buyers do not deposit at all: $d_\varepsilon = 0$ so that from the first order condition for d_ε implies $\lambda_\varepsilon > R'_d = \frac{1}{\rho_P}$, and the budget constraint implies that $pq_\varepsilon = m$. We can then use the definition of ε_M^A in (64) to get

$$\beta\phi' pq_\varepsilon = \rho_T q_\varepsilon \quad = \quad \rho_P \varepsilon_M^A = \beta\phi' m$$

yielding $q_\varepsilon = \frac{\rho_P \varepsilon_M^A}{\rho_T}$. The first order condition for q_ε is

$$pq_\varepsilon \beta\phi' \frac{1}{\rho_P} < pq_\varepsilon \beta\phi' \lambda_\varepsilon = \varepsilon$$

as buyers in this region are constrained by the non negativity of deposits and, as a consequence, $\lambda_\varepsilon > \frac{1}{\rho_P} = R'_d$.[21] Thus, combining the first order condition for q_ε with the budget constraint we get $\lambda_\varepsilon = \frac{\varepsilon}{\rho_T q_\varepsilon} = \frac{\varepsilon}{\rho_P \varepsilon_M^A}$.

■

Consumption, deposits, and the envelope condition satisfy

$q^\varepsilon =$	$d^\varepsilon =$	$W_m^A(m\|T,\varepsilon) =$	
$\varepsilon\rho_E/\rho_T$	$T + \frac{\rho_E}{\rho_T}p(\varepsilon_E^A - \varepsilon)$	$(1/p)(\rho_T/\rho_E)$	if $\varepsilon < \varepsilon_E^A$
$\varepsilon_E^A\rho_E/\rho_T$	T	$\frac{1}{p}\frac{\varepsilon}{\varepsilon_E^A}\frac{\rho_T}{\rho_E}$	if $\varepsilon_E^A < \varepsilon < \varepsilon_P^A$
$\varepsilon\rho_P/\rho_T$	$\frac{\rho_P}{\rho_T}p(\varepsilon_M^A - \varepsilon)$	$(1/p)(\rho_T/\rho_P)$	if $\varepsilon_P^A < \varepsilon < \varepsilon_M^A$
$\varepsilon_M^A\frac{\rho_P}{\rho_T}$	0	$\frac{1}{p}\frac{\varepsilon}{\varepsilon_M^A}$	if $\varepsilon_M^A < \varepsilon$.

[21]Referring to the Lagrangian in section 3 the first order condition for d_ε is $\beta V_d^B - \beta\phi'\lambda_\varepsilon + \beta\phi'\lambda_d = 0$, with $\beta\phi'\lambda_d$ denoting the multiplier on the non negativity constraint on d_ε. With $\beta\phi'\lambda_d > 0$ the first first order condition for d_ε becomes $\frac{1}{\rho_P} + \lambda_d = \lambda_\varepsilon$.

B.3 Proof of Lemma 5

Proof. In order to solve for the marginal value on T, strip beginning of period money balances from the settlement function and bring them back to the previous period's money and goods markets. That is, because we can write

$$V^B(l, d|T) = \phi\left(T/\rho_T + R(d-T) - l/\rho_l\right) + V^B(0,0|0),$$

we can also write a buyer's value in the money and goods market as

$$W^B(m|T) = \varepsilon u(q^\varepsilon) + \beta\phi^+\left(T/\rho_T + R(d^\varepsilon - T) - l^\varepsilon/\rho_l\right) + \beta V^B(0,0|0)$$

That is, using the definition of a buyer's value in the settlement market $V^B(\hat{m}, l_\varepsilon, d_\varepsilon|T)$, we can define

$$V^B(\hat{m},0,0|0) = \max_{\{m^+,T^+\}}\{\phi m^+ + W^B(m^+|T^+)\} \tag{65}$$

so that

$$V^B(\hat{m}, l_\varepsilon, d_\varepsilon|T) = \phi\left(\frac{T}{\rho_T} + R(d_\varepsilon - T) - \frac{l_\varepsilon}{\rho_l}\right) + V^B(\hat{m},0,0|0),$$

and the buyer's value in the money and goods' market as

$$W^B(m|T) = \varepsilon u(q^\varepsilon) + \beta\phi^+\left(\frac{T}{\rho_T} + R(d^\varepsilon - T) - \frac{l_\varepsilon}{\rho_l} + (\hat{m}' - \tau M)\right) + \beta V^B(\hat{m}',0,0|0).$$

Substituting for q^ε, l^ε, and d^ε we get $W^B(m|T,\varepsilon) - \beta V^B(\hat{m},0,0|0) =$

Substituting for q^ε, l^ε, and d^ε we get $W^B(m|T,\varepsilon) - \beta V^B(0,0|0) =$

$$\begin{cases} \varepsilon\log(\varepsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \frac{\rho_l}{\rho_T}p\left(\varepsilon - \varepsilon_l\right)\rho_l\right) & \text{if } \varepsilon < \varepsilon_B, \\ \varepsilon\log(\varepsilon_B\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \rho_l B/\rho_l\right) & \text{if } \varepsilon_B < \varepsilon < \varepsilon_T, \\ \varepsilon\log(\varepsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\left(\varepsilon_M - \varepsilon\frac{\rho_P}{\rho_l}\right) - T\right) - \rho_l B/\rho_l\right) & \text{if } \varepsilon_T < \varepsilon < \varepsilon_M, \\ \varepsilon\log(\varepsilon_M) + \beta\phi^+\left(T/\rho_T + R(-T) - \rho_l B/\rho_l\right) & \text{if } \varepsilon > \varepsilon_M. \end{cases}$$

Substituting in for ε_i and simplifying gives $W^B(m|T,\varepsilon) - \beta V^B(0,0|0) =$

$$\begin{cases} \varepsilon\log(\varepsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T - \frac{p}{\rho_T}\varepsilon + (m-T)/\rho_l\right) & \text{if } \varepsilon < \varepsilon_B, \\ \varepsilon\log\left(\frac{m-T+B\rho_l}{p}\right) + \beta\phi^+\left(T/\rho_T - B\right) & \text{if } \varepsilon_B < \varepsilon < \varepsilon_T, \\ \varepsilon\log\left(\varepsilon\rho_P/\rho_T\right) + \beta\phi^+\left(T/\rho_T - B + (m+\rho_l B - T)/\rho_P\right) - \varepsilon & \text{if } \varepsilon_T < \varepsilon < \varepsilon_M, \\ \varepsilon\log\left(\frac{m+B\rho_l}{p}\right) + \beta\phi^+(T/\rho_T - B - T/\rho_P) & \text{if } \varepsilon_M \le \varepsilon. \end{cases} \tag{66}$$

I get a few different values in the above tables, which I rewrote as follows:

$$\begin{cases} \varepsilon\log(\varepsilon\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - \frac{p}{\rho_T}\left(\varepsilon - \varepsilon_l\right)\right) & \text{if } \varepsilon < \varepsilon_B, \\ \varepsilon\log(\varepsilon_B\rho_l/\rho_T) + \beta\phi^+\left(T/\rho_T + R(T-T) - B\right) & \text{if } \varepsilon_B < \varepsilon < \varepsilon_T, \\ \varepsilon\log(\varepsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\frac{\rho_P}{\rho_T}\left(\varepsilon_M - \varepsilon\right) - T\right) - B\right) & \text{if } \varepsilon_T < \varepsilon < \varepsilon_M, \\ \varepsilon\log(\varepsilon_M\frac{\rho_P}{\rho_T}) + \beta\phi^+\left(T/\rho_T + R(-T) - \rho_l B/\rho_l\right) & \text{if } \varepsilon > \varepsilon_M. \end{cases}$$

Substituting in for ε_i and simplifying gives
$W^B(m|T,\varepsilon) - \beta V^B(0,0|0) =$

$$\begin{cases} \varepsilon \log(\varepsilon\rho_l/\rho_T) + \beta\phi^+ \left(T/\rho_T - \frac{p}{\rho_T}\varepsilon + (m-T)/\rho_l\right) & \text{if } \varepsilon < \varepsilon_B, \\ \varepsilon \log\left(\frac{m-T+B\rho_l}{p}\right) + \beta\phi^+ \left(T/\rho_T - B\right) & \text{if } \varepsilon_B < \varepsilon < \varepsilon_T, \\ \varepsilon \log\left(\varepsilon\rho_P/\rho_T\right) + \beta\phi^+ \left(T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) + \frac{(m+B\rho_l)}{\rho_P} - \frac{\varepsilon}{\beta\phi'} - B\right) & \text{if } \varepsilon_T < \varepsilon < \varepsilon_M, \\ \varepsilon \log\left(\frac{m+B\rho_l}{p}\right) + \beta\phi^+(T/\rho_T - B - T/\rho_P) & \text{if } \varepsilon_M \leq \varepsilon. \end{cases} \tag{67}$$

To see this consider each case in turn:

1. $\varepsilon < \varepsilon_B$

$$W^B(m|T) = \varepsilon u(q_\varepsilon) + \beta\phi'[\frac{T}{\rho_T} - \frac{p}{\rho_T}(\varepsilon - \frac{\beta\phi'(m-T)}{\rho_l})]$$

where $\varepsilon_l = \frac{\beta\phi'(m-T)}{\rho_l}$.

2. $\varepsilon \in (\varepsilon_B, \varepsilon_T)$

$$W^B(m|T) = \varepsilon u(\frac{\rho_l}{\rho_T}[\beta\phi'\frac{(m-T)}{\rho_l} + \beta\phi' B]) + \beta\phi'[\frac{T}{\rho_T} - B]$$

that we can rearrange as

$$W^B(m|T) = \varepsilon u(\frac{(m-T+\rho_l B)}{p}) + \beta\phi'[\frac{T}{\rho_T} - B]$$

where we can substitute $\varepsilon_B = \beta\phi'[\frac{(m-T+\rho_l B)}{\rho_l})]$ and $p\beta\phi' = \rho_T$ to get

$$W^B(m|T) = \varepsilon u(\frac{\rho_l}{\rho_T}\varepsilon) + \beta\phi'[\frac{T}{\rho_T} - B]$$

3. $\varepsilon \in (\varepsilon_T, \varepsilon_M)$

$$W^B(m|T) = \varepsilon u(\frac{\varepsilon\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} - \frac{1}{\rho_P}(\frac{p\rho_P}{\rho_T}(\varepsilon_M\varepsilon) - T) - B)]$$

which we can rearrange as

$$W^B(m|T) = \varepsilon u(\frac{\varepsilon\rho_P}{\rho_T}) + \beta\phi'[T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) + \frac{(m+B\rho_l)}{\rho_P} - \frac{\varepsilon}{\beta\phi'} - B)]$$

and, using $\varepsilon_M = \frac{\beta\phi'}{\rho_P}(m+B\rho_l)$, as

$$W^B(m|T) = \varepsilon\log(\varepsilon\rho_P/\rho_T) + \beta\phi^+\left(T/\rho_T + R\left(p\frac{\rho_P}{\rho_T}\left(\varepsilon_M - \varepsilon\right) - T\right) - B\right)$$

4. $\varepsilon > \varepsilon_M$

$$W^B(m|T) = \varepsilon \log(\varepsilon_M \frac{\rho_P}{\rho_T}) + \beta\phi^+ (T/\rho_T + R(-T) - \rho_l B/\rho_l)$$

That can be rearranged as

$$W^B(m|T) = \varepsilon \log(\varepsilon_M \frac{\rho_P}{\rho_T}) + \beta\phi^+ \left(T(\frac{1}{\rho_T} - \frac{1}{\rho_P}) - B\right)$$

This, finally, gives us our expression for the marginal value of a target

$$W_T^B(m|T) = \beta\phi^+ \left[\frac{1}{\rho_T} - \frac{1}{\rho_l}\left(F(\varepsilon_B) + \int_{\varepsilon_B}^{\varepsilon_T}\left(\frac{\varepsilon}{\varepsilon_B}\right)dF(\varepsilon)\right) - \frac{1}{\rho_P}(1 - F(\varepsilon_T))\right].$$

Setting $W_T^B(m|T) = 0$ gives the equation (30). That $\rho_l \in [\rho_P, \rho_T]$ follows from the fact that $\varepsilon/\varepsilon_B \in [\rho_P^{-1}, 1]$ as $\varepsilon \in [\varepsilon_B, \varepsilon_T]$. Indeed, so long as F has unbounded support, we must have $\rho_l > \rho_P$. Similarly, if 0 is in the support of F, then $\rho_l < \rho_T$.

Consider now autarks. Recall that

$$W^A(m|T) = \varepsilon u(q_\varepsilon) + \beta\phi'[\frac{T}{\rho_T} + R(d_\varepsilon - T)] + \beta V^A(\hat{m}', 0|0)$$

Then, case by case we derive the marginal value of a target for an autarkic buyer:

1. $\varepsilon < \varepsilon_E^A$.

   In this case $\lambda_\varepsilon = \frac{1}{\rho_E}$ so that $\beta\phi'\lambda_\varepsilon = \frac{\rho_T}{p\rho_E}$, $q_\varepsilon = \frac{\varepsilon\rho_E}{\rho_T}$ and $d_\varepsilon = T + \frac{\rho_E}{\beta\phi'}(\varepsilon_E^A - \varepsilon)$. Thus

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \varepsilon u(\frac{\varepsilon\rho_E}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + \frac{\varepsilon_E^A - \varepsilon}{\beta\phi'}] \\ &= \varepsilon u(\frac{\varepsilon\rho_E}{\rho_T}) + \frac{T}{p} + \varepsilon_E^A - \varepsilon \end{aligned}$$

   So that

$$\frac{\partial W^A}{\partial T} = \frac{1}{p} - \frac{\beta\phi'}{\rho_E} = \frac{1}{p}(1 - \frac{\rho_T}{\rho_E})$$

2. $\varepsilon \in (\varepsilon_E^A, \varepsilon_P^A)$.

   In this case $q_\varepsilon = \varepsilon_E^A \frac{\rho_E}{\rho_T}$, $d_\varepsilon = T$, $\lambda_\varepsilon = \frac{\varepsilon}{\rho_E \varepsilon_E^A}$, where the last equation is derived from the first order condition for q_ε, which, with log utility, is $\frac{\varepsilon}{q_\varepsilon} = p\beta\phi'\lambda_\varepsilon$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \varepsilon u(\varepsilon_E^A \frac{\rho_E}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + R(0)] \\ &= \varepsilon u(\frac{m-T}{p}) + \beta\phi'\frac{T}{\rho_T} \end{aligned}$$

   where the last equation uses the definition of $\varepsilon_E^A = \frac{(m-T)\beta\phi'}{\rho_E}$. So that

$$\begin{aligned} \frac{\partial W^A}{\partial T} &= \varepsilon(-\frac{1}{p})u'(\frac{m-T}{p}) + \frac{1}{p} \\ &= \frac{1}{p}(1 - \varepsilon u'(\frac{m-T}{p})) \\ &= \frac{1}{p}[1 - \frac{\rho_T \varepsilon}{\rho_E \varepsilon_E^A}] \end{aligned}$$

where, in the last equation we have rewritten $\varepsilon u'(\frac{m-T}{p})$ using the first order condition for q_ε as

$$\varepsilon u'(q_\varepsilon) = p\beta\phi'\lambda_\varepsilon = \rho_T \frac{\varepsilon}{\rho_E \varepsilon_E^A}$$

3. $\varepsilon \in (\varepsilon_P^A, \varepsilon_M^A)$.
   In this case $d_\varepsilon = m - \frac{\varepsilon\rho_P}{\beta\phi'} < T$, $q_\varepsilon = \varepsilon\frac{\rho_P}{\rho_T}$, $\lambda_\varepsilon = \frac{1}{\rho_P}$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \varepsilon u(\varepsilon\frac{\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + R(m - \frac{\varepsilon\rho_P}{\beta\phi'} - T)] \\ &= \varepsilon u(\varepsilon\frac{\rho_P}{\rho_T}) + \beta\phi'[\frac{T}{\rho_T} + \frac{1}{\rho_P}(m - \frac{\varepsilon\rho_P}{\beta\phi'} - T)] \\ &= \varepsilon u(\varepsilon\frac{\rho_P}{\rho_T}) + \beta\phi' T[\frac{1}{\rho_T} - \frac{1}{\rho_P}] + \varepsilon_E^A - \varepsilon \end{aligned}$$

   where the last equation follows from substituting

$$m - \frac{\varepsilon\rho_P}{\beta\phi'} = \frac{1}{\beta\phi'}(\rho_P\varepsilon_E^A - \varepsilon\rho_P)$$

   So that

$$\frac{\partial W^A}{\partial T} = \beta\phi'(\frac{1}{\rho_T} - \frac{1}{\rho_P}) = \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})$$

4. $\varepsilon > \varepsilon_M^A$.
   In this case $d_\varepsilon = 0$, $\lambda_\varepsilon = \frac{\varepsilon}{\rho_P\varepsilon_E^A} < \frac{1}{\rho_P}$ and $q_\varepsilon = \frac{\rho_P}{\rho_T}\varepsilon_T^A$. Then

$$\begin{aligned} W^A(m|T) - \beta V^A(\hat{m}', 0|0) &= \varepsilon u(\frac{\rho_P}{\rho_T}\varepsilon_T^A) + \beta\phi'[\frac{T}{\rho_T} + R(-T)] \\ &= \varepsilon u(\frac{\rho_P}{\rho_T}\varepsilon_T^A) + \beta\phi' T[\frac{1}{\rho_T} - \frac{1}{\rho_P}] \end{aligned}$$

   So that

$$\frac{\partial W^A}{\partial T} = \beta\phi'(\frac{1}{\rho_T} - \frac{1}{\rho_P}) = \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})$$

Combining $\frac{\partial W^A}{\partial T}$ from all cases, yields:

$$\begin{aligned} \frac{\partial W^A}{\partial T} &= F(\varepsilon_E^A)\frac{1}{p}(1 - \frac{\rho_T}{\rho_E}) + \int_{\varepsilon_E^A}^{\varepsilon_P^A} \frac{1}{p}[1 - \varepsilon u'(\frac{\varepsilon_E^A\rho_E}{\rho_T})]dF(\varepsilon) + \\ &+ \int_{\varepsilon_P^A}^{\varepsilon_M^A} \frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P})dF(\varepsilon) + [1 - F(\varepsilon_M^A)]\frac{\rho_T}{p}(\frac{1}{\rho_T} - \frac{1}{\rho_P}) \\ &= \frac{1}{p}\Big\{F(\varepsilon_E^A)(1 - \frac{\rho_T}{\rho_E}) + \int_{\varepsilon_E^A}^{\varepsilon_P^A}(1 - \frac{\rho_T\varepsilon}{\rho_E\varepsilon_E^A})dF(\varepsilon) + (1 - \frac{\rho_T}{\rho_P})(1 - F(\varepsilon_P^A))\Big\} \\ &= \frac{1}{p}\Big\{1 - F(\varepsilon_E^A)\frac{\rho_T}{\rho_E} - \int_{\varepsilon_E^A}^{\varepsilon_P^A}\frac{\rho_T\varepsilon}{\rho_E\varepsilon_E^A}dF(\varepsilon) - \frac{\rho_T}{\rho_P}(1 - F(\varepsilon_P^A))\Big\} \end{aligned}$$

so that $\frac{\partial W^A}{\partial T} = 0$ if and only if

$$\frac{1}{\rho_T} = \frac{F(\varepsilon_E^A)}{\rho_E} + \int_{\varepsilon_E^A}^{\varepsilon_P^A} \frac{\varepsilon}{\rho_E \varepsilon_E^A} dF(\varepsilon) + \frac{1-F(\varepsilon_P^A)}{\rho_P}$$

■

B.4 Proof of Lemma 6

Proof. Consider supply of funds first. Using the characterization in lemma 3 it follows that $l_\varepsilon < 0$ if and only if a buyer receives shock $\varepsilon \leq \varepsilon_L$. In this case $pq_\varepsilon = \frac{\varepsilon \rho_l}{\beta \phi'}$ and $l_\varepsilon = pq_\varepsilon + T - m$. Using also the definition of $\varepsilon_L = (m - T\frac{\beta\phi'}{\rho_l})$ yields

$$l_\varepsilon = \frac{\varepsilon \rho_l}{\beta \phi'} - \frac{\varepsilon_L \rho_l}{\beta \phi'} = (\varepsilon - \varepsilon_L)\frac{p\rho_l}{\rho_T}$$

where the last equation follows from lemma 2, which implies $p\beta\phi' = \rho_T$. This yields (34). Consider demand for funds. Using the characterization in lemma 3 it follows that buyers with a shock $\varepsilon \geq \varepsilon_B$ are borrowing constrained, so they borrow $l_\varepsilon = \rho_l B$. Buyers with $\varepsilon \in (\varepsilon_L, \varepsilon_B)$ borrow $l_\varepsilon = (\varepsilon - \varepsilon_L)\frac{p\rho_l}{\rho_T}$. This yields (35). Thus the money market clearing condition, pinning down ρ_l is

$$\begin{aligned} 0 &= \int_0^{\varepsilon_B} \frac{\rho_l}{\rho_T} p(\varepsilon - \varepsilon_L)\, dF(\varepsilon) + \int_{\varepsilon_B}^{\infty} \rho_l B\, dF(\varepsilon) \\ 0 &= \frac{\rho_l \gamma}{\beta} \int_0^{\varepsilon_B} \varepsilon dF(\varepsilon) - [\phi m^+ - \phi T^+]F(\varepsilon_B) + \rho_l \phi B(1 - F(\varepsilon_B)) \\ 0 &= \frac{\rho_l \gamma}{\beta} \int_0^{\varepsilon_B} \varepsilon dF(\varepsilon) - [\rho_l \phi B + \phi m^+ - \phi T^+]F(\varepsilon_B) + \rho_l \phi B \end{aligned} \tag{68}$$

■

B.5 Settlement market

From the settlement period problem, we have the first order condition on money balances which guarantees that $\phi = W_M^B(m|T)$, where $W_M^B(m|T)$ is just the expectation of $W_M^B(m|T,\varepsilon)$ which is just $\beta\phi^+\lambda_\varepsilon$. This final quantity can be derived from the FOC for q: $\varepsilon u'(q) - p\beta\phi^+\lambda_\varepsilon = 0$.

The envelope condition from the decision problem in the settlement market is:

$$V^i(\hat{m}, l, d|T) = \phi\left(\hat{m} + T/\rho_T + R(d - T) - l/\rho_l - \tau M\right) + \max_{(m^+, T^+)} \left\{-\phi m^+ + W^i(m^+|T^+)\right\}.$$

where the first order condition for m^+ yields $-\phi + \frac{\partial W^i}{\partial m^+} \leq 0$, with $W^B(m^+|T^+) = \mathbb{E}_\varepsilon[W^B(m^+, T^+|\varepsilon)]$. Consider then the decision problem in money and goods market:

$$W^B(m|T, \varepsilon) = \max_{(q^\varepsilon, l^\varepsilon, d^\varepsilon)} \varepsilon u(q^\varepsilon) + \beta V^B(\hat{m}', l^\varepsilon, d^\varepsilon|T)$$

$$\text{s.t.} \quad m + l^\varepsilon - pq^\varepsilon - d^\varepsilon - \hat{m}' \geq 0, \quad \text{and} \quad \rho_l B - l^\varepsilon \geq 0.$$

and, with $z = (\hat{m}', q_\varepsilon, l_\varepsilon, d_\varepsilon)$, the Lagrangian

$$L(z) = \varepsilon u(q_\varepsilon) + \beta V^B(\hat{m}', l_\varepsilon, d_\varepsilon|T) + \beta\phi'\lambda_\varepsilon(m + l_\varepsilon - d_\varepsilon - pq_\varepsilon - \hat{m}') + \beta\phi'\lambda_B(\rho_l B - l_\varepsilon) + \beta\phi'\lambda_d d_\varepsilon + \beta\phi'\lambda_q q_\varepsilon$$

we then have $\frac{\partial W^B(m|T,\varepsilon)}{\partial m^+} = \beta\phi'\lambda_\varepsilon$ and $W^B(m^+|T^+) = \mathbb{E}_\varepsilon\beta\phi'\lambda_\varepsilon$. Thus the first order condition for m^+ is simply $-\phi + \mathbb{E}_\varepsilon[\beta\phi'\lambda_\varepsilon] \leq 0$.

Putting this all together gives

$$W_M^B = \frac{\rho_T}{p}\left(\int_0^{\varepsilon_B} \frac{1}{\rho_l} + \int_{\varepsilon_B}^{\varepsilon_T} \frac{1}{\rho_l}\frac{\varepsilon}{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_T}^{\varepsilon_M} \frac{1}{\rho_P} dF(\varepsilon) + \int_{\varepsilon_M}^{\infty} \frac{1}{\rho_P}\frac{\varepsilon}{\varepsilon_M} dF(\varepsilon)\right).$$

Setting this equal to ϕ and given that $\rho_t/p = \beta\phi^+$ with $\phi/\phi^+ = \gamma$ we get

$$\frac{\gamma}{\beta} = \int_0^{\varepsilon_B} \frac{1}{\rho_l} + \int_{\varepsilon_B}^{\varepsilon_T} \frac{1}{\rho_l}\frac{\varepsilon}{\varepsilon_B} dF(\varepsilon) + \int_{\varepsilon_T}^{\varepsilon_M} \frac{1}{\rho_P} dF(\varepsilon) + \int_{\varepsilon_M}^{\infty} \frac{1}{\rho_P}\frac{\varepsilon}{\varepsilon_M} dF(\varepsilon)$$

Similarly, the choice of money holdings for autarkic buyers satisfies $-\phi + \beta\phi'\mathbb{E}_\varepsilon\lambda_\varepsilon \leq 0$, that, for $m^{A+} > 0$, is

$$\frac{\gamma}{\beta} = F(\varepsilon_E^A)\frac{1}{\rho_E} + \int_{\varepsilon_E^A}^{\varepsilon_P^A} \frac{\varepsilon}{\rho_E\varepsilon_E^A} dF(\varepsilon) + \int_{\varepsilon_P^A}^{\varepsilon_M^A} \frac{1}{\rho_P} + \int_{\varepsilon_M^A}^{\infty} \frac{\varepsilon}{\varepsilon_E^A\rho_P} \tag{69}$$

$$\tag{70}$$

B.6 Endogenous borrowing limit

Consider a buyer who repays his money market loan from the previous period. Because his repayment is publicly recorded then his continuation value at the end of the settlement market, before choosing target and money holdings, is $V^B(0,0,0|0)$, as defined in (65) in the appendix. To save on notation, let $V^B = V^B(0,0,0|0)$. Consider now a buyer who plans to default on his money market loan from the previous period. Analogously to a buyer, his continuation value in the settlement market, before choosing target and money holdings, is $V^A(0,0|0) = \max_{\{m^{A+},T^{A+}\}} -\phi m^{A+} + W^A(m^{A+}|T^{A+})$. To save on notation, let $V^A = V^A(0,0|0)$. The payoff to a buyer who repays in the settlement market is $-\phi\frac{l_\varepsilon}{\rho_\varepsilon} + V^B$, since the loan was in terms of money, while the payoff to a buyer who defaults is V^A. Therefore, the repayment constraint can be rewritten as

$$l_\varepsilon \leq \frac{\rho_l}{\phi}(V^B - V^A) \tag{71}$$

More specifically:

$$V^B = \max_{\{m^+,T^+\}} -\phi m^+ + W^B(m^+|T^+) + \beta V^{B'} \tag{72}$$

$$= -\phi m^+ + \mathbb{E}_\varepsilon\Big\{\varepsilon\ u(q_\varepsilon) + \beta V^B(\hat{m}', l_\varepsilon, d_\varepsilon|T_\varepsilon)\} + \beta V^{B'} \tag{73}$$

$$= -\phi m^+ + \mathbb{E}_\varepsilon\{\varepsilon\ u(q_\varepsilon) + \beta\phi^+[\hat{m}' + \frac{T^+}{\rho_T} + R(d_\varepsilon - T^+) - \frac{l_\varepsilon}{\rho_l} - \tau M]\Big\} + \beta V^{B'} \tag{74}$$

Since in equilibrium $\hat{m} = \hat{m}' = 0$ this boils down, for buyers and autarkic buyers respectively, to:

$$V^B = -\phi m^+ + \mathbb{E}_\varepsilon\Big\{\varepsilon\ u(q_\varepsilon) + \beta\phi^+[\hat{m}' + \frac{T^+}{\rho_T} + R(d_\varepsilon - T^+) - \frac{l_\varepsilon}{\rho_l} - \tau M]\Big\} + \beta V^{B'} \quad (75)$$

$$V^A = -\phi m^{A+} + \mathbb{E}_\varepsilon\Big\{\varepsilon\ u(q_\varepsilon^A) + \beta\phi^+[\hat{m}^{A'} + \frac{T^{A+}}{\rho_T} + R(d_\varepsilon^A - T^{A+}) - \tau M]\Big\} + \beta V^{A'} \quad (76)$$

so that $\phi B = V^B - V^A$ and

$$\begin{aligned}\phi B &= \phi(m^{A+} - m^+) + \mathbb{E}_\varepsilon[\varepsilon u(q_\varepsilon) - \varepsilon u(q_\varepsilon^A)] \\ &\quad + \mathbb{E}_\varepsilon \beta\phi'\Big\{(\frac{T^+ - T^{A+}}{\rho_T}) - \frac{l_\varepsilon}{\rho_l} + R(d_\varepsilon - T^+) - R(d_\varepsilon^A - T^{A+})\Big\} + \beta\phi' B' \end{aligned} \quad (77)$$

where we used the result that $\hat{m}' = 0 = \hat{m}^{A'}$. Notice that none of the autarky terms depend on B, while ρ_l and T^+ as well as the consumption allocation do.

C Three types economy: VRT

C.1 Proof of Proposition 4

Proof. Under the conjectured strategies the analysis in lemmas 3 and 4 implies that the money market clearing condition is

$$\left(\frac{\pi_1\varepsilon_1 + \pi_2\varepsilon_2}{\beta\phi_3}\right) - (1-\pi_3)\frac{\varepsilon_{B_2}}{\beta\phi_3} + B^+ = 0$$

Notice that $B^+ = \frac{V^{B+} - V^{A+}}{\phi^+} = \frac{L^+}{\phi^+}$. This equation pins down ε_B:

$$\varepsilon_B = \frac{\pi_1\varepsilon_1 + \pi_2\varepsilon_2}{(1-\pi_3)} + \frac{\beta\phi_3}{(1-\pi_3)}B^+$$

Then the FOC for choice of targets for buyers pins down ρ_l

$$\rho_l = \frac{(\pi_1 + \pi_2)}{\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P}\right)}$$

The FOC for the choice of targets for autarks pins down ε_B^A

$$\varepsilon_B^A = \frac{\pi_2\varepsilon_2}{\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)}$$

Combining the FOC for money holdings for buyers with (37) yields

$$\phi = \frac{\beta\phi^+}{\rho_T}$$

We are now interested in $V^B - V^A$:

$$\begin{aligned}
V^B - V^A &= \phi\left(m^A - m\right) + \\
&+\pi_1\varepsilon_1\left[u\left(\varepsilon_1\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_1\right)\right] + \\
&+\pi_2\varepsilon_2\left[u\left(\varepsilon_2\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_B^A\right)\right] + \\
&+\beta\phi^+\left[\frac{T}{\rho_T} + \pi_3\frac{m + \rho_l B^+ - \frac{\varepsilon_3\rho_P}{\beta\phi_3} - T}{\rho_P}\right] \\
&-\beta\phi^+\left[\frac{T^A}{\rho_T} + \pi_1\frac{m_2^A - \frac{\rho_E\varepsilon_1}{\beta\phi_3} - T^A}{\rho_E} + \pi_3\frac{m_2^A - \frac{\varepsilon_3\rho_P}{\beta\phi_3} - T^A}{\rho_P}\right] \\
&+\beta\left(V^{B+} - V^{A+}\right)
\end{aligned}$$

where consumption in the state ε_3 is the same for buyers and autarks (i.e. $q_\varepsilon^A = q_\varepsilon = \varepsilon_3\frac{\rho_P}{\rho_E}$). Also notice that autarks in state ε_1 deposit at the central bank, so they earn remuneration on those deposits. Using the above equilibrium conditions, we can rewrite this as

$$\begin{aligned}
V^B - V^A &= \left(\frac{\phi}{\beta\phi^+} - \frac{\pi_3}{\rho_P}\right)\left(\varepsilon_B^A\rho_E - \varepsilon_B\rho_l\right) + \phi\rho_l B^+ + \\
&+\pi_1\varepsilon_1\left[u\left(\varepsilon_1\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_1\right)\right] + \\
&+\pi_2\varepsilon_2\left[u\left(\varepsilon_2\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_B^A\right)\right] + \\
&-\pi_1\left(\varepsilon_B^A - \varepsilon_1\right) \\
&+\beta\left(V^{B+} - V^{A+}\right)
\end{aligned}$$

With log utility we can further rearrange the following terms are follows

$$\begin{aligned}
\left[u\left(\varepsilon_1\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_1\right)\right] &= \log\frac{\rho_l}{\rho_E} < 0 \\
\left[u\left(\varepsilon_2\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_B^A\right)\right] &= \log\varepsilon_2 + \log\frac{\rho_l}{\rho_E} - \log\varepsilon_B^A \\
&= \log\varepsilon_2 + \log\frac{\rho_l}{\rho_E} - \log\pi_2\varepsilon_2 + \log\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right) \\
&= \log\rho_l - \log\pi_2 + \log\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)
\end{aligned}$$

Then, using the expressions for the endogenous variables and $V^B - V^A = \phi B = \phi^+ B^+ = L$ in a stationary equilibrium, we have

$$\begin{aligned}
L\left(1 - \frac{\beta\rho_l}{\rho_T}\right) &= \left(\pi_1\varepsilon_1 + \pi_2\varepsilon_2\right)\log\frac{\rho_l}{\rho_E} + \\
&+\pi_2\varepsilon_2\left[\log\frac{\rho_E}{\pi_2}\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)\right]
\end{aligned}$$

where we rearranged

$$\phi\rho_l B^+ \;=\; \frac{\phi}{\beta\phi^+}\rho_l\beta\phi^+ B^+ = \frac{\rho_l}{\rho_T}\beta L^+$$

■

C.2 Proof of Lemma 7

Proof. Notice that $\Delta\mathbb{E}u(\alpha) > 0$ if and only if

$$\begin{aligned}(\pi_1\varepsilon_1+\pi_2\varepsilon_2)\left[\log(\pi_1+\pi_2)-\log\alpha\right]-\pi_1\varepsilon_1\log\rho_E + \\ \pi_2\varepsilon_2\left[\log\left(\alpha-\frac{\pi_1}{\rho_E}\right)-\log\pi_2\right] \;>\; 0\end{aligned} \tag{78}$$

Then

$$\frac{\partial\Delta\mathbb{E}u(\alpha)}{\partial\alpha} \;=\; \frac{\pi_2\varepsilon_2}{\left(\alpha-\frac{\pi_1}{\rho_E}\right)} - \frac{\pi_1\varepsilon_1+\pi_2\varepsilon_2}{\alpha}$$

Notice that $\left(\alpha-\frac{\pi_1}{\rho_E}\right) > 0$ by assumption, which guarantees that $\varepsilon^A_{B_2} > 0$. Then rearrange the above equation as

$$\frac{\partial\Delta\mathbb{E}u(\alpha)}{\partial\alpha} \;=\; \frac{\frac{\pi_1}{\rho_E}(\pi_1\varepsilon_1+\pi_2\varepsilon_2)-\alpha\pi_1\varepsilon_1}{\left(\alpha-\frac{\pi_1}{\rho_E}\right)\alpha}$$

which is negative if and only if the numerator is negative

$$\frac{(\pi_1\varepsilon_1+\pi_2\varepsilon_2)}{\rho_E\varepsilon_1} \;<\; \alpha \tag{79}$$

Since $\frac{\partial\alpha}{\partial\rho_P} > 0$, the utility differential is decreasing in ρ_P if and only if $\rho_P > \overline{\rho}_{Pu}$, where $\overline{\rho}_{Pu}$ is defined by (79) holding at equality. Furthermore, (78) is satisfied if and only if α is not too large, that is to say if the inverse of the penalty rate is not too large: $\rho_P < \underline{\rho}_{Pu}$, with $\underline{\rho}_{Pu}$ denoting the value of ρ_P such that (78) holds at equality. Hence, if $\rho_P < \underline{\rho}_{Pu}$ then $\Delta\mathbb{E}u(\alpha) > 0$. ■

C.3 Proof of Proposition 5

Proof. Sufficient conditions for $L > 0$ are (i) $1 - \frac{\beta\rho_{l2}}{\rho_T} > 0$, that is the coefficient on the left hand side of (39) is positive, and (ii) $\Delta\mathbb{E}u(\alpha) > 0$, that is the utility differential is positive. First, using the equilibrium value of ρ_l yields:

$$1 \;>\; \beta\frac{(\pi_1+\pi_2)}{\rho_T\alpha}$$

which sets an upper bound on r_P, say $\overline{r}_{PlL} = \frac{(1+r_T)(\pi_1+\pi_2)(1-\beta)}{\pi_3}$. Second, the previous lemma shows that a sufficient condition for $\Delta\mathbb{E}u(\alpha) > 0$ is $\overline{r}_{Pu} > r_P > \underline{r}_{Pu}$. Finally, we need to verify the equilibrium conjecture $0 < \varepsilon_B^A < \varepsilon_B$, where $\varepsilon_B^A > 0$ is necessary since we assume log utility.

First, for $\varepsilon_B^A > 0$:

$$\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E} = \alpha - \frac{\pi_1}{\rho_E} \quad > \quad 0 \tag{80}$$

Sufficient condition (80) can be stated as

$$r_P \quad < \overline{r}_{PeB} = \quad \frac{(1-\pi_3)(1+r_T)}{\pi_3} - \frac{\pi_1}{\pi_3\rho_E}$$

Second, for $\varepsilon_B^A < \varepsilon_B$:

$$\frac{\pi_2\varepsilon_2}{\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)} \quad < \quad \frac{\pi_1\varepsilon_1 + \pi_2\varepsilon_2}{(1-\pi_3)} + \frac{\beta\phi^+}{(1-\pi_3)}B^+ \tag{81}$$

which is equivalent to

$$\begin{aligned} &\left(\frac{1}{\beta} - \frac{(\pi_1+\pi_2)}{\rho_T\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P}\right)}\right)\left[\frac{\pi_2\varepsilon_2(1-\pi_3)}{\rho_E\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)} - (\pi_1\varepsilon_1 + \pi_2\varepsilon_2)\right] \quad < \\ &(\pi_1\varepsilon_1 + \pi_2\varepsilon_2)\log\frac{(\pi_1+\pi_2)}{\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P}\right)} + \pi_2\varepsilon_2\left[\log\frac{\rho_E}{\pi_2}\left(\frac{1}{\rho_T} - \frac{\pi_3}{\rho_P} - \frac{\pi_1}{\rho_E}\right)\right] \end{aligned} \tag{82}$$

where we know that the right hand side of (82) is $\Delta\mathbb{E}u(\alpha)$. Recall that $\Delta\mathbb{E}u(\alpha) > 0$ and $\frac{\partial\Delta\mathbb{E}u(\alpha)}{\partial\alpha} < 0$ if $\overline{r}_{Pu} > r_P > \underline{r}_{Pu}$, that is increasing in r_P since $r_P < \overline{r}_{Pu}$. For the left hand side of (82), the first term is strictly positive by our assumptions that $r_P < \overline{r}_{PlL}$. Then, with $\alpha(r_P) = [(1-\pi_3)(1+r_T) - \pi_3 r_P]$, rearrange the term in square brackets as $\frac{\pi_2\varepsilon_2(1-\pi_3)}{\rho_E\left(\alpha - \frac{\pi_1}{\rho_E}\right)} - (\pi_1\varepsilon_1 + \pi_2\varepsilon_2)$. Then, a sufficient condition for inequality (82) to be satisfied is that the left hand side is negative, because we already made assumptions for the utility differential to be positive, that is to say $\overline{r}_{Pu} > r_P > \underline{r}_{Pu}$, hence the right hand side is positive. Then, with $\min(\overline{r}_{PlL}, \overline{r}_{Pu}) > r_P > \underline{r}_{Pu}$, the left hand side of (82) is negative if and only if:

$$\frac{\pi_2\varepsilon_2(1-\pi_3)}{\rho_E\left(\alpha - \frac{\pi_1}{\rho_E}\right)} \quad < \quad (\pi_1\varepsilon_1 + \pi_2\varepsilon_2)$$

$$\begin{aligned} \pi_2\varepsilon_2(2\pi_1+\pi_2) \quad &< \quad (\rho_E\alpha - \pi_1)\pi_1\varepsilon_1 + \rho_E\alpha\pi_2\varepsilon_2 \\ r_P \quad &< \quad \overline{r}_{Pe} = \frac{(\pi_1+\pi_2)}{\pi_3\rho_T} - \frac{\pi_2\varepsilon_2(2\pi_1+\pi_2) + \pi_1^2\varepsilon_1}{\pi_3\rho_E(\pi_1\varepsilon_1 + \pi_2\varepsilon_2)} \end{aligned}$$

which we can restate as $r_P < \overline{r}_{Pe}$. Notice that $\overline{r}_{Pu} < \overline{r}_{Pe}$.[22] Let

$$\overline{\rho}_P = \frac{1}{\min\left(\overline{r}_{Pu}, \overline{r}_{PlL}, \overline{r}_{PeB}\right)} \tag{83}$$

■

C.4 Proof or 6

Proof. Rearrange the borrowing limit, defined by equation (39), as follows

$$L(\alpha) = \frac{\Delta \mathbb{E} u(\alpha)}{\left(1 - \beta \frac{(\pi_1 + \pi_2)}{\rho_T \alpha}\right)}$$

Then let $DL(\alpha) = \left(1 - \beta \frac{(\pi_1 + \pi_2)}{\rho_T \alpha}\right)$ and notice that $\frac{\partial DL(\alpha)}{\partial \alpha} > 0$. Then consider

$$\frac{\partial L(\alpha)}{\partial \alpha} = \frac{\frac{\partial \Delta \mathbb{E} u(\alpha)}{\partial \alpha} DL(\alpha) - \Delta \mathbb{E} u(\alpha) \frac{\partial DL(\alpha)}{\partial \alpha}}{\left[DL(\alpha)\right]^2}$$

The assumption $r_P < \overline{r}_{Pu}$ implies $\frac{\partial \Delta \mathbb{E} u(\alpha)}{\partial \alpha} < 0$, the assumption $r_P < \overline{r}_{PlL}$ implies $DL(\alpha) > 0$, and the assumption $r_P > \underline{r}_{Pu}$ implies $\Delta \mathbb{E} u(\alpha) > 0$. Then $\frac{\partial L(\alpha)}{\partial \alpha} < 0$, which implies $\frac{\partial L(r_P)}{\partial r_P} > 0$ since with $\alpha' < 0$. ■

[22] Consider the inequality pinning down $\overline{r}_{Pu}$:

$$\frac{\pi_2 \varepsilon_2}{\left(\alpha - \frac{\pi_1}{\rho_E}\right)} < \frac{(\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2)}{\alpha}$$

Then consider the inequality pinning down $\overline{r}_{Pe}$

$$\begin{aligned} \frac{\pi_2 \varepsilon_2 (1 - \pi_3)}{\rho_E \left(\alpha - \frac{\pi_1}{\rho_E}\right)} &< (\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2) \\ \frac{\pi_2 \varepsilon_2}{\left(\alpha - \frac{\pi_1}{\rho_E}\right)} &< \frac{\rho_E}{(1 - \pi_3)} (\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2) \\ &= \frac{\rho_E}{\rho_l \alpha} (\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2) \end{aligned}$$

So $r_P < \min\left(\overline{r}_{Pu}, \overline{r}_{Pe}\right)$:

$$\frac{\pi_2 \varepsilon_2}{\left(\alpha - \frac{\pi_1}{\rho_E}\right)} < \min\left(\frac{(\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2)}{\alpha}, \frac{\rho_E}{\rho_l \alpha} (\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2)\right)$$

where $\frac{\rho_E}{\rho_l} > 1$. Hence, if $r_P < \overline{r}_{Pu}$ then $r_P < \overline{r}_{Pe}$ as well.

D Flat Rate

D.1 Proof of Lemma 8

Proof. The FOC for money holdings for buyers

$$\phi = (\pi_1 + \pi_2) \frac{\beta\phi^+}{\rho_l} + \pi_3 \frac{\varepsilon_3}{(m_2 + \rho_l B^+)}$$

where $\frac{\tilde{\varepsilon}_B \rho_l}{\beta\phi^+} = (m + \rho_l B^+)$, so that

$$\phi = \left(\pi_1 + \pi_2 + \pi_3 \frac{\varepsilon_3}{\tilde{\varepsilon}_B}\right) \frac{\beta\phi^+}{\rho_l}$$

FOC for money holdings for autarks

$$\begin{aligned} \frac{\phi}{\beta\phi^+}\rho_E &= (\pi_1 + \pi_2) + \pi_3 \frac{\varepsilon_3}{\tilde{\varepsilon}_L^A} \\ &= (\pi_1 + \pi_2) + \pi_3 \frac{\rho_E \varepsilon_3}{\beta\phi^+ m^A} \end{aligned}$$

which pins down

$$\tilde{\varepsilon}_L^A = \frac{\pi_3 \varepsilon_3}{\frac{\phi}{\beta\phi^+}\rho_E - (\pi_1 + \pi_2)}$$

The money market clearing condition is

$$\begin{aligned} 0 &= (\pi_1\varepsilon_1 + \pi_2\varepsilon_2) \frac{\rho_l}{\beta\phi^+} - m_2 + \pi_3 \left(m + \rho_l B^+\right) \\ &= (\pi_1\varepsilon_1 + \pi_2\varepsilon_2) \rho_l - \frac{\beta}{\gamma}\phi m + \pi_3 \frac{\beta}{\gamma} \left(\phi m + \rho_l \phi B^+\right) \\ &= (\pi_1\varepsilon_1 + \pi_2\varepsilon_2) \rho_l - \frac{\beta}{\gamma}\phi m + \pi_3 \frac{\beta}{\gamma} \left(\phi m + \rho_l \gamma L^+\right) \end{aligned}$$

This can be further rearranged with m as a function of $\tilde{\varepsilon}_B$ and dividing both sides by ρ_l

$$\tilde{\varepsilon}_B = \frac{\gamma L^+ + \pi_1\varepsilon_1 + \pi_2\varepsilon_2}{(1 - \pi_3)}$$

Then, combining this with the money holding equation for buyers yields

$$\rho_l = \frac{(\gamma L^+ + \bar{\varepsilon})(1 - \pi_3)}{\gamma L^+ + \pi_1\varepsilon_1 + \pi_2\varepsilon_2} \frac{\beta}{\gamma} = \frac{(\gamma L^+ + \bar{\varepsilon})}{\tilde{\varepsilon}_B} \frac{\beta}{\gamma}$$

where $\bar{\varepsilon}$ denotes the average value of the taste shock and $\frac{\beta\phi^+}{\phi} = \frac{\beta}{\gamma}$.

We are now interested in $V^B - V^A$:

$$\begin{aligned}
V^B - V^A &= \phi\left(m^A - m\right) + \Delta\mathbb{E}u + \\
&\quad -\frac{\beta\phi^+}{\rho_E}\left[\pi_1\left(m^A - pq^A_{\varepsilon 1}\right) + \pi_2\left(m^A - pq^A_{\varepsilon 2}\right)\right] \\
&\quad +\beta\left(V^{B+} - V^{A+}\right) \\
&= \left(\phi - \frac{\beta\phi^+}{\rho_E}\left(1 - \pi_3\right)\right) m^A - \phi m + \Delta\mathbb{E}u + \\
&\quad +\frac{\beta\phi^+}{\rho_E}\left[\pi_1\frac{\rho_E\varepsilon_1}{\beta\phi^+} + \pi_2\frac{\rho_E\varepsilon_2}{\beta\phi^+}\right] \\
&\quad +\beta\left(V^{B+} - V^{A+}\right)
\end{aligned}$$

where $\Delta\mathbb{E}u$ denotes the differential in expected utility between buyers and autarks.

Rearranging and substituting out for m^A and m as functions of $\tilde{\varepsilon}^A_B$ and $\tilde{\varepsilon}_B$, from $\tilde{\varepsilon}^A_B = \frac{\beta\phi^+}{\rho_E} m^A$ and $\tilde{\varepsilon}_B = \frac{\beta\phi^+}{\rho_l}\left(m + \rho_l B^+\right) = \frac{\beta}{\gamma\rho_l}\left(\phi m + \gamma\rho_l L^+\right)$:

$$\begin{aligned}
L &= \left(\phi - \frac{\beta\phi^+}{\rho_E}\left(1 - \pi_3\right)\right)\frac{\rho_E\tilde{\varepsilon}^A_B}{\beta\phi^+} - \left(\frac{\gamma\tilde{\varepsilon}_B\rho_l}{\beta} - \gamma\rho_l L^+\right) + \Delta\mathbb{E}u + \\
&\quad +\frac{\beta\phi^+}{\rho_E}\left[\pi_1\frac{\rho_E\varepsilon_1}{\beta\phi^+} + \pi_2\frac{\rho_E\varepsilon_2}{\beta\phi^+}\right] \\
&\quad +\beta L^+
\end{aligned}$$

where

$$\begin{aligned}
\Delta\mathbb{E}u &= \pi_1\varepsilon_1\left[u\left(\varepsilon_1\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_1\right)\right] + \\
&\quad +\pi_2\varepsilon_2\left[u\left(\varepsilon_2\frac{\rho_l}{\rho_E}\right) - u\left(\varepsilon_2\right)\right] + \\
&\quad +\pi_3\varepsilon_3\left[u\left(\tilde{\varepsilon}_B\frac{\rho_{l_2}}{\rho_E}\right) - u\left(\tilde{\varepsilon}^A_B\right)\right] +
\end{aligned}$$

$$\rho_l = \frac{\left(\gamma L^+ + \overline{\varepsilon}\right)\left(1 - \pi_3\right)}{\gamma L^+ + \pi_1\varepsilon_1 + \pi_2\varepsilon_2}\frac{\beta}{\gamma} = \frac{\left(\gamma L^+ + \overline{\varepsilon}\right)}{\tilde{\varepsilon}_B}\frac{\beta}{\gamma}$$

and

$$\begin{aligned}
\tilde{\varepsilon}^A_B &= \frac{\pi_3\varepsilon_3}{\frac{\phi}{\beta\phi^+}\rho_E - \left(\pi_1 + \pi_2\right)} = \frac{\beta}{\gamma}\frac{\pi_3\varepsilon_3}{\left[\rho_E - \frac{\beta}{\gamma}\left(\pi_1 + \pi_2\right)\right]} \\
\tilde{\varepsilon}_B &= \frac{\gamma L^+ + \pi_1\varepsilon_1 + \pi_2\varepsilon_2}{\left(1 - \pi_3\right)} \\
\rho_l\tilde{\varepsilon}_B &= \left(\gamma L^+ + \overline{\varepsilon}\right)\frac{\beta}{\gamma} \\
\rho_l L^+ &= \rho_l\frac{\left[\tilde{\varepsilon}_B\left(1 - \pi_3\right) - \pi_1\varepsilon_1 - \pi_2\varepsilon_2\right]}{\gamma}
\end{aligned}$$

and with log utility

$$\begin{aligned}\Delta\mathbb{E}u &= \overline{\varepsilon}\log\frac{\rho_{l_2}}{\rho_E}+\pi_3\varepsilon_3\left[\log\tilde{\varepsilon}_B-\log\tilde{\varepsilon}_B^A\right]\\
&= \overline{\varepsilon}\log\left(\gamma L^+ +\overline{\varepsilon}\right)-\left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\varepsilon_B-\overline{\varepsilon}\log\rho_E+\overline{\varepsilon}\log\frac{\beta}{\gamma}+\\
&\quad -\pi_3\varepsilon_3\log\tilde{\varepsilon}_B^A\\
\Delta\mathbb{E}u &= \overline{\varepsilon}\log\left(\gamma L^+ +\overline{\varepsilon}\right)-\left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\frac{\gamma L^+ +\pi_1\varepsilon_1+\pi_2\varepsilon_2}{\left(1-\pi_3\right)}-\overline{\varepsilon}\log\rho_E+\overline{\varepsilon}\log\frac{\beta}{\gamma}+\\
&\quad -\pi_3\varepsilon_3\log\tilde{\varepsilon}_B^A\end{aligned}$$

as ε_B^A is independent of L. Then

$$\begin{aligned}\Delta\mathbb{E}u &= \overline{\varepsilon}\log\left(\gamma L^+ +\overline{\varepsilon}\right)-\left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\left(\gamma L^+ +\pi_1\varepsilon_1+\pi_2\varepsilon_2\right)+ && (84)\\
&\quad \left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\left(1-\pi_3\right)-\overline{\varepsilon}\log\rho_E+\overline{\varepsilon}\log\frac{\beta}{\gamma}+ && (85)\\
&\quad -\pi_3\varepsilon_3\log\tilde{\varepsilon}_B^A && (86)\end{aligned}$$

We can rearrange the equation for the borrowing limit as terms that are functions of L and terms that are independent of L

$$\begin{aligned}L\left(1-\beta\right) &= -\frac{\gamma\tilde{\varepsilon}_B\rho_l}{\beta}+\gamma\rho_l L^+ +\Delta\mathbb{E}u+\\
&\quad \left(\phi-\frac{\beta\phi_3}{\rho_E}\left(1-\pi_3\right)\right)\frac{\rho_E\tilde{\varepsilon}_B^A}{\beta\phi^+}+\frac{\beta\phi^+}{\rho_E}\left[\pi_1\frac{\rho_E\varepsilon_1}{\beta\phi^+}+\pi_2\frac{\rho_E\varepsilon_2}{\beta\phi^+}\right]\end{aligned}$$

and, letting $K_1=\left(\phi-\frac{\beta\phi^+}{\rho_E}\left(1-\pi_3\right)\right)\frac{\rho_E\tilde{\varepsilon}_B^A}{\beta\phi^+}+\frac{\beta\phi^+}{\rho_E}\left[\pi_1\frac{\rho_E\varepsilon_1}{\beta\phi^+}+\pi_2\frac{\rho_E\varepsilon_2}{\beta\phi^+}\right]$, we can futher rearrange it as

$$L\left(1-\beta\right) = -\gamma L^+ -\overline{\varepsilon}+\frac{\left(\gamma L^+ +\overline{\varepsilon}\right)\left(1-\pi_3\right)}{\gamma+\frac{\pi_1\varepsilon_1+\pi_2\varepsilon_2}{L^+}}\beta+\Delta\mathbb{E}u+K_1$$

Substituting out for $\Delta\mathbb{E}u$ and letting K_2 denote the constant terms, so that

$$K_2 = \left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\left(1-\pi_3\right)-\overline{\varepsilon}\log\rho_E+\overline{\varepsilon}\log\frac{\beta}{\gamma}-\pi_3\varepsilon_3\log\tilde{\varepsilon}_B^A$$

we can futher rearrange the equation for the borrowing limit as

$$\begin{aligned}L\left(1-\beta\right) &= -\gamma L^+ -\overline{\varepsilon}+\frac{\left(\gamma L^+ +\overline{\varepsilon}\right)\left(1-\pi_3\right)}{\gamma+\frac{\pi_1\varepsilon_1+\pi_2\varepsilon_2}{L^+}}\beta+\\
&\quad \overline{\varepsilon}\log\left(\gamma L^+ +\overline{\varepsilon}\right)-\left(\overline{\varepsilon}_1+\overline{\varepsilon}_2\right)\log\left(\gamma L^+ +\pi_1\varepsilon_1+\pi_2\varepsilon_2\right)+K_1+K_2\end{aligned}$$

Finally let $K=-\overline{\varepsilon}+K_1+K_2$, $a=\overline{\varepsilon}_1+\overline{\varepsilon}_2$ and $b=\overline{\varepsilon}_3$ and we obtain (41). ■

D.2 Proof of Lemma 9

Proof. Recall $\rho_l L = \frac{(\gamma L+a+b)}{\gamma L+a} L \frac{\beta(1-\pi_3)}{\gamma}$. Then it is easy to see that it equals zero when $L = 0$. Also, it is easy to see that

$$\begin{aligned}
\frac{\partial \rho_l L}{\partial L} &= \frac{\beta}{\gamma}(1-\pi_3)\left\{\frac{[\gamma(\gamma L+a) - \gamma(\gamma L+a+b)]}{(\gamma L+a)^2}L + \frac{(\gamma L+a+b)}{\gamma L+a}\right\} \\
&= \frac{\beta}{\gamma}(1-\pi_3)\left\{\frac{-\gamma b}{(\gamma L+a)^2}L + \frac{(\gamma L+a+b)}{\gamma L+a}\right\} \\
&= \frac{\beta}{\gamma}\frac{(1-\pi_3)}{\gamma L+a}\left\{\frac{-\gamma bL + (\gamma L+a+b)(\gamma L+a)}{(\gamma L+a)}\right\} \\
&= \frac{\beta}{\gamma}\frac{(1-\pi_3)}{\gamma L+a}\left\{\frac{(\gamma L+a)^2 + ab}{(\gamma L+a)}\right\} > 0
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{L\to\infty}\frac{\partial \rho_l L}{\partial L} &= \lim_{L\to\infty}\frac{\beta}{\gamma}(1-\pi_3)\left\{1 + \frac{ab}{(\gamma L+a)^2}\right\} \\
&= \frac{\beta}{\gamma}(1-\pi_3) < 1
\end{aligned}$$

Finally

$$\begin{aligned}
\frac{\partial^2 \rho_l L}{\partial^2 L} &= \frac{\beta}{\gamma}(1-\pi_3)\left[\frac{2\gamma(\gamma L+a)^3 - \left((\gamma L+a)^2 + ab\right)2\gamma(\gamma L+a)}{(\gamma L+a)^4}\right] \\
&= \frac{\beta}{\gamma}(1-\pi_3)\left[\frac{-ab2\gamma(\gamma L+a)}{(\gamma L+a)^4}\right] < 0
\end{aligned}$$

Turning now to the utility differential, defined in (86), we have

$$\Delta\mathbb{E}u = K_2 + (a+b)\log(\gamma L+a+b) - a\log(\gamma L+a)$$

It is easy to see that $\Delta\mathbb{E}u\,(L=0) = K_2 + \log\frac{(a+b)^{a+b}}{a^a}$ and that

$$\begin{aligned}
\frac{\partial \Delta\mathbb{E}u}{\partial L} &= \gamma\left[\frac{(a+b)}{(\gamma L+a+b)} - \frac{a}{(\gamma L+a)}\right] \\
&= \gamma\left[\frac{(a+b)(\gamma L+a) - a(\gamma L+a+b)}{(\gamma L+a+b)(\gamma L+a)}\right] \\
&= \gamma^2\frac{bL}{(\gamma L+a+b)(\gamma L+a)} > 0
\end{aligned}$$

Furthermore:

$$\begin{aligned}
\lim_{L\to\infty}\frac{\partial \Delta\mathbb{E}u}{\partial L} &= \lim_{L\to\infty}\frac{\gamma^2 bL}{(\gamma L+a+b)(\gamma L+a)} \\
&= \lim_{L\to\infty}\frac{\gamma^2 b}{2\gamma(\gamma L+a)+\gamma b} = 0
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2 \Delta \mathbb{E} u}{\partial^2 L} &= b\gamma^2 \frac{(\gamma L + a + b)(\gamma L + a) - L[\gamma(\gamma L + a) + (\gamma L + a + b)\gamma]}{[(\gamma L + a + b)(\gamma L + a)]^2} \\
&= b\gamma^2 \frac{(\gamma L + a + b)(\gamma L + a) - L2\gamma(\gamma L + a) - b\gamma L}{[(\gamma L + a + b)(\gamma L + a)]^2} \\
&= b\gamma^2 \frac{(-\gamma L + a + b)(\gamma L + a) - b\gamma L}{[(\gamma L + a + b)(\gamma L + a)]^2} \\
&= b\gamma^2 \frac{a^2 - (\gamma L)^2 + ab}{[(\gamma L + a + b)(\gamma L + a)]^2}
\end{aligned}$$

clearly for L sufficiently close to zero $\frac{\partial^2 \Delta \mathbb{E} u}{\partial^2 L} > 0$ and for L sufficiently large $\frac{\partial^2 \Delta \mathbb{E} u}{\partial^2 L} < 0$. ■

D.3 Proof of Proposition 7

Proof. The equilibrium borrowing limit solves $\Phi(L) = 0$:

$$\begin{aligned}
L(1 - \beta + \gamma) &= \frac{(\gamma L + a + b)(1 - \pi_3)}{\gamma + \frac{a}{L}}\beta + \\
&\quad (a + b)\log(\gamma L + a + b) - a\log(\gamma L + a) + K \qquad (87)
\end{aligned}$$

The left hand side of (87) is linear in linear in L with $(1 - \beta + \gamma) > 1$. We now turn to the right hand side of (87).

Step 1.

Consider first $K = -\bar{\varepsilon} + K_1 + K_2$. Substituting out for $K_1$and K_2 yields

$$\begin{aligned}
K &= -(a + b) + \frac{\gamma \rho_E}{\beta}\tilde{\varepsilon}_B^A - \tilde{\varepsilon}_B^A(1 - \pi_3) + a + \\
&\quad a\log(1 - \pi_3) - (a + b)\log\rho_E + (a + b)\log\frac{\beta}{\gamma} - b\log\tilde{\varepsilon}_B^A
\end{aligned}$$

that can be rearranged as

$$\begin{aligned}
K &= -b\left(1 + \log\tilde{\varepsilon}_B^A\right) + \left(\frac{\gamma\rho_E}{\beta} - (1 - \pi_3)\right)\tilde{\varepsilon}_B^A + \\
&\quad a\log(1 - \pi_3) - (a + b)\log\rho_E + (a + b)\log\frac{\beta}{\gamma}
\end{aligned}$$

where $\tilde{\varepsilon}_B^A = \frac{b}{\frac{\gamma\rho_E}{\beta} - (1 - \pi_3)}$, so we can further rearrange this as

$$\begin{aligned}
K &= -b\log\left(\frac{b}{\frac{\gamma\rho_E}{\beta} - (1 - \pi_3)}\right) + \\
&\quad a\log(1 - \pi_3) - (a + b)\log\rho_E + (a + b)\log\frac{\beta}{\gamma} \\
&= -b\log b + b\log\left(\frac{\gamma\rho_E}{\beta} - (1 - \pi_3)\right) - b\log\frac{\gamma\rho_E}{\beta} + \\
&\quad a\log(1 - \pi_3) - a\log\rho_E + a\log\frac{\beta}{\gamma}
\end{aligned}$$

Step 2.

If $a \geq \underline{a}$ and $b \in \left(\underline{b}, \bar{b}\right)$then $FR\left(0\right) > 0$.

Notice that

$$\begin{aligned} FR(0) &= (a+b)\log a + b - a\log a + K \\ &= b\log a + b - b\log b + \\ & \quad b\log\left(\frac{\gamma\rho_E}{\beta} - (1-\pi_3)\right) - b\log\frac{\gamma\rho_E}{\beta} + \\ & \quad a\log\left(1-\pi_3\right) - a\log\rho_E + a\log\frac{\beta}{\gamma} \end{aligned}$$

which can be rearranged as

$$\begin{aligned} FR(0) &= b\left(1+\log\frac{a}{b}\right) + \\ & \quad b\log\left(1 - \frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right) + \\ & \quad a\log\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right) \end{aligned}$$

Notice that the second and third term are smaller than zero because $\frac{\gamma\rho_E}{\beta} - (1-\pi_3) > 0$ by assumption. Then $FR\left(0\right) > 0$ if and only if

$$b\left(1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a\right) - b\log b \quad > \quad -a\log\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right) \tag{88}$$

The right hand side of (88) is larger than zero. Consider the term in brackets on the left hand side of (88) and define $\underline{a}$ as the lowest value of a such that this term is greater than zero: $\underline{a} = \left\{a \in \mathbb{R} : 1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a > 0\right\}$. Hence $\left(1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a\right) > 0$ for all $a > \left[e\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)\right]^{-1}$.

Consider now the second term on the left hand side of (88):

$$\begin{aligned} \lim_{b\to 0} b\log b &= 0 \\ \lim_{b\to\infty} b\log b &= \infty \\ \frac{\partial\left(b\log b\right)}{\partial b} &= \log b + 1 \\ \lim_{b\to\infty}\log b + 1 &> \left(1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a\right) \\ &= \lim_{b\to\infty}\frac{\partial\left(b\left(1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a\right)\right)}{\partial b} \end{aligned}$$

and $(b\log b) < 0$ for all $b < \frac{1}{e}$ and $(b\log b) > 0$ for all $b > \frac{1}{e}$. Then, with $a > \underline{a}$ there is a unique value of b such that

$$b\left(1+\log\left(1-\frac{\beta}{\gamma\rho_E}\left(1-\pi_3\right)\right)+\log a\right) \quad = \quad b\log b$$

Let $\bar{b}$ denote such value. Then for $b < \bar{b}$ the left hand side of the inequality (88) is strictly larger than zero. Then (88) is satisfied for $b > \underline{b}$, with $\underline{b}$ defined by (88) holding at equality. Hence, if $a \geq \underline{a}$ and $b \in (\underline{b}, \bar{b})$then $FR(0) > 0$.

Step 3.

If $2 - \frac{1}{\beta} < \pi_3$ and $\frac{b}{a} \leq \frac{\left(\frac{1}{\beta}-1\right)}{(1-\pi_3)} - 1$ then $\frac{\partial FR}{\partial L} < \frac{\partial FL}{\partial L}$ for all L.

Consider FR as defined in (43). Differentiating yields

$$\begin{aligned}\frac{\partial FR}{\partial L} &= \frac{\partial}{\partial L}\gamma\rho_l L + \frac{\partial}{\partial L}\mathbb{E}u \\ &= \beta(1-\pi_3)\left[1 + \frac{ab}{(\gamma L + a)^2}\right] + \gamma^2 \frac{bL}{(\gamma L + a + b)(\gamma L + a)}\end{aligned}$$

Then $\frac{\partial FR}{\partial L} < \frac{\partial FL}{\partial L}$ if and only if

$$\beta(1-\pi_3)\left[1 + \frac{ab}{(\gamma L + a)^2}\right] + \gamma \frac{b\gamma L}{(\gamma L + a + b)(\gamma L + a)} < 1 - \beta + \gamma$$

that can be rearranged as

$$\begin{aligned}(\gamma L + a)\left[(\gamma L + a)^2(\beta(2-\pi_3) - 1 - \gamma) + ab(\beta(1-\pi_3) - \gamma)\right] \\ b\left[(\gamma L + a)^2(\beta(2-\pi_3) - 1) + \beta ab(1-\pi_3)\right] &< 0 \qquad (89)\end{aligned}$$

The first term in (89) is always negative. Consider the second term. Rearranging the term in square brackets, if

$$\left((\gamma L)^2 + a^2 + 2a\gamma L\right)(\beta(1-\pi_3) + \beta - 1) + \beta ab(1-\pi_3) \leq 0$$

then $\frac{\partial FR}{\partial L} < \frac{\partial FL}{\partial L}$. If $2 - \frac{1}{\beta} < \pi_3$ then $(\beta(1-\pi_3) + \beta - 1) < 0$ and an upper bound for the left hand side of the above inequality is

$$a^2(\beta(1-\pi_3) + \beta - 1) + \beta ab(1-\pi_3) \leq 0$$

which we can rearrange as

$$\frac{b}{a} \leq \frac{\left(\frac{1}{\beta} - 1\right)}{(1-\pi_3)} - 1.$$

Step 4.

Lemma 9 shows that $\rho_l L$ and $\mathbb{E}u$ are both strictly increasing, and that

$$\lim_{L\to\infty}\frac{\partial FR}{\partial L} = \lim_{L\to\infty}\frac{\partial \rho_l L}{\partial L} + \frac{\partial \mathbb{E}u}{\partial L} = \frac{\beta}{\gamma}(1-\pi_3) < 1 < \lim_{L\to\infty}\frac{\partial FL}{\partial L} = (1 - \beta + \gamma)$$

Moreover, $FR(0) > 0 = FL(0)$ if $a > \underline{a}, b \in (\underline{b}, \bar{b})$. Hence, there exists a unique L^* solving (41). Finally, with $\frac{\partial FR}{\partial L} < \frac{\partial FL}{\partial L}$ it follows that $\Phi(L) = FL(L) - FR(L)$ is monotonically increasing. ∎

E VRT vs Flat Rate

E.1 Proof of Proposition 8

Proof. Since Φ is monotonically increasing in L, then $L^{VRT} > L^{FR}$ if and only if $\Phi\left(L^{VRT}\right) > 0$ that is to say

$$
\begin{aligned}
L^{VRT}(1-\beta+\gamma) \quad > \quad & \frac{\left(\gamma L^{VRT}+a+b\right)(1-\pi_3)}{\gamma+\frac{a}{L^{VRT}}}\beta + \\
& (a+b)\log\left(\gamma L^{VRT}+a+b\right) - a\log\left(\gamma L^{VRT}+a\right) + K
\end{aligned}
$$

Because Φ is monotonically increasing in L, and L^{VRT} is monotonically increasing in r_P, then $\Phi\left(L^{VRT}\right) > 0$ if and only if $r_P > \underline{r}_{Pc}$ where $\underline{r}_{Pc} : \left\{r_P > 0 : \Phi\left(L^{VRT}(r_P)\right) = 0\right\}$. Let

$$
\underline{r}_P \quad = \quad \max\left(\underline{r}_{Pc}, \underline{r}_{Pu}\right) = \frac{1}{\underline{\rho}_P} \tag{90}
$$

Then the constructed VRT equilibrium features a higher borrowing limit than the constructed FR equilibrium. ■

E.2 Proof of Proposition 9

Proof. Consider the decision problem of an autark in VRT as defined in section 3.

$$
W^A = \max_{(m^A, T^A, q_\epsilon^A, d_\epsilon^A)} \left\{ -\phi m^A + \mathbb{E}_\epsilon \epsilon u(q_\epsilon^A) + \beta\phi^+ \left(\frac{T^A}{\rho_T} + R(d - T^A) - \tau M\right) + V^A \right\}
$$

$$
\text{s.t.} \quad m^A - p q_\epsilon^A - d_\epsilon^A \geq 0.
$$

Using $\gamma = \frac{\phi}{\phi^+}$, and $p = \frac{\rho_E}{\beta\phi^+}$ this can be rewritten as

$$
W^A = \max_{(m^A, T^A, q_\epsilon^A, d_\epsilon^A)} \left\{ -\phi m^A + \mathbb{E}_\epsilon \epsilon u(q_\epsilon^A) + \frac{\beta}{\gamma} \left(\frac{\phi T^A}{\rho_T} + R(\phi d - \phi T^A) - \tau\phi M\right) + V^A \right\}
$$

$$
\text{s.t.} \quad \phi m^A - \frac{\rho_E \gamma}{\beta} q_\epsilon^A - \phi d_\epsilon^A \geq 0.
$$

where

$$
R(\phi d - \phi T^A) = \begin{cases} \frac{(\phi d - \phi T^A)}{\rho_P} & \text{if } d - T^A < 0 \\ \frac{(\phi d - \phi T^A)}{\rho_E} & \text{if } d - T^A \geq 0. \end{cases}
$$

Since autarks are allowed to choose targets, a feasible choice is $T^A = 0$, although that might not be optimal. With $T^A = 0$ the above decision problem is simply

$$
W^A = \max_{(m^A, q_\epsilon^A, d_\epsilon^A)} \left\{ -\phi m^A + \mathbb{E}_\epsilon \epsilon u(q_\epsilon^A) + \beta\left(\frac{\phi d}{\gamma\rho_E} - \tau\phi M\right) + V^A \right\}
$$

$$\text{s.t.} \quad \phi m^A - \frac{\rho_E \gamma}{\beta} q_\epsilon^A - \phi d_\epsilon^A \geq 0.$$

With $\gamma^{VRT}\rho_E^{VRT} = \gamma^{FR}\rho_E^{FR}$ this decision problem is equivalent to that of an autark in the economy where monetary policy is implemented with a flat rate remuneration framework, as defined in section 2.3. Hence, with $T^A = 0$ being a feasible choice for autarks in a VRT framework, it must be that $W^{A,VRT} \geq W^{A,FR}$ and $V^{A,VRT} \geq V^{A,FR}$. Combining this with $L^{VRT} = V^{B,VRT} - V^{A,VRT} > L^{FR} = V^{B,FR} - V^{A,FR}$ yields $V^{B,VRT} - V^{B,FR} > V^{A,VRT} - V^{A,FR} \geq 0$. ■

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