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Existence of an equilibrium in arrowian markets for consumption externalities ☆

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Abstract

We study the existence of quasi-equilibria and equilibria for pure exchange economies with consumption externalities and Arrowian markets with personalized Lindahl prices. We provide examples showing first that quasi-equilibria and equilibria of the externality economy fail to exist under assumptions guaranteeing existence for economies without externalities. We show that the externality economy has identical equilibrium allocations of an appropriately defined constant returns to scale production economy without externalities. We exploit this equivalence to map sufficient conditions for the existence of quasi-equilibria and equilibria of the production economy into sufficient conditions of the pure exchange economy with externalities, thereby unveiling suitable irreducibility conditions and survival conditions.

JEL classification: C62; D50; D62

Keywords: Consumption externalities; Markets for externalities; Existence of an equilibrium; Irreducibility and survival

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1. Introduction

In the presence of consumption externalities, restoring the two Fundamental Theorems of Welfare Economics requires setting up markets where individuals face personalized Lindahl prices and choose allocations, that is, profiles of consumption bundles one for each of the individuals in the economy. This was observed a long time ago by Samuelson (1954), Arrow (1969), and Laffont (1976).

Surprisingly, the literature on the existence of competitive equilibria in economies with markets for externalities (hereafter Arrow-Lindahl equilibria) is very limited. Foley (1970) proves the existence of Arrow-Lindahl equilibria in economies with private and public goods, without consumption externalities and with monotonic preferences. Bergstrom (1976b) proves the existence of an equilibrium for so-called communal commodities and shows that it encompasses the case of markets for external effects through a technical construction.[1] All existence results in the literature are based on specific fixed-points like arguments.

We study economies with consumption externalities in preferences, and find general conditions for the existence of quasi-equilibria and equilibria. This is not a trivial task. Even when preferences are locally non-satiated and individual endowments are strictly positive, quasi-equilibria and equilibria may fail to exist. We provide two examples that make this point. The existence failures of markets for externalities were hidden in the literature by technical assumptions playing the role of the survival and irreducibility assumptions.[2]

We do not develop a direct argument, but rather follow the intuition of Arrow that in externality economies, "individual i's consumption is regarded as the production of joint outputs, one for each individual whose utility is affected by individual i's consumption" (Arrow, 1969, see pages 9-10). We show that a pure exchange economy with consumption externalities where trade takes place in Arrowian markets for externalities yields the same equilibrium allocations of an appropriately defined production economy with constant returns to scale and no externalities. Once this equivalence is established, existence of equilibrium, quasi-equilibrium, irreducibility and survival for economies with consumption externalities follow from well known and understood results for standard production economies.

The heavy lifting in our argument is done by the construction of the production economy and in particular of its technology. The technology transforms inputs of physical commodities into a collection of as many identical allocations as individuals in the economy. The total amount of physical commodities in an allocation cannot exceed the total amount of inputs used in the production process. The technology displays constant returns to scale, thereby generating zero equilibrium profits and making it irrelevant to specify individual property rights over production activities. Further, the chosen form of the technology restricts equilibrium prices to satisfy the classical Lindahl compatibility conditions. The latter establishes equivalence in equilibrium prices, and thus equilibrium allocations of the two economies.

[1] There are other more recent papers on the subject. Crès (1996) focuses on the symmetry breaking properties of equilibria with consumption externalities. Existence is generic in the space of preferences under strong conditions of smoothness and irreducibility. Conley and Smith (2005) proves existence of quasi-equilibria and equilibria treating externality markets directly instead of indirectly through Arrowian commodities. Thus, there is no relation between the prices of the ordinary commodities and the prices of the externality rights. del Mercato and Florenzano (2009), and del Mercato (2010) prove the existence of Arrow-Lindalh quasi-equilibria in economies with consumption externalities.

[2] See, for example, Assumptions (S), (E), and (F) in Bergstrom (1976b).

We exploit the equivalence of feasible allocations to map classical irreducibility and survival conditions of the production economy into equivalent such conditions of the economy with externalities. In doing this, we understand the reasons for the failure of existence of quasi-equilibria and equilibria of the externality economy.

Without free disposal in the externality economy, the equivalent production economy does not satisfy the standard weak survival assumption, explaining the lack of quasi-equilibria in our first example. In the second example there is free disposal, and thus quasi-equilibria exist, but equilibria do not because the equivalent production economy is not irreducible. Indeed, irreducibility for externality economies is more demanding. In economies without externalities, irreducibility means that at any feasible allocation, any group of individuals can be made better off (in the Pareto sense) by adding to their consumption bundles something of the privately owned resources of the complementary group. For economies with externalities, any group must choose an allocation that specifies not only their consumption bundles, but also those of the individuals in the complementary group. This makes it harder to find an allocation (Pareto) dominating the target feasible allocation for all the members of the chosen group.

The paper is organized as follows. Section 2 describes the fundamentals, the basic assumptions and the structure of the markets. Section 3 provides examples of nonexistence of quasi-equilibrium and equilibrium in economies satisfying the assumptions of Section 2. Section 4 is the key part of the paper describing the equivalent production economy without externalities. Section 5 shows arguments for the existence of a quasi-equilibrium and equilibrium and it defines notions of irreducibility and survival for the economies with externalities.

2. The basic model and its assumptions

We consider a pure exchange economy with a finite number of physical commodities labeled by the superscript $\ell \in \mathcal{L} = \{1, \dots, L\}$, and a finite set of individuals labeled by the subscript $i \in \mathcal{I} = \{1, \dots, I\}$. Each individual i is characterized by a consumption set $X_i \subset \mathbb{R}^L$, an endowment of commodities $e_i \in \mathbb{R}^L$, and preferences represented by a utility function allowing for possible consumption external effects:

$$u_i : x \in X = \prod_{i \in \mathcal{I}} X_i \mapsto u_i\,(x) \in \mathbb{R},$$

where $x = (x_i)_{i \in \mathcal{I}} \in X$ is an allocation. An economy with consumption externalities is $\mathcal{E} = (X_i, u_i, e_i)_{i \in \mathcal{I}}$.

Individuals' utility functions, consumption sets and endowments satisfy classical assumptions listed below.

Assumption A.

(1) *X_i is closed, convex, bounded from below, and $0 \in X_i$.*
(2) *u_i is continuous, quasi-concave, and locally non-satiated on X.*[3]
(3) $e_i \in \mathbb{R}^L_+$.[4]

[3] That is, for every $x \in X$ the set $\{x' \in X : u_i\,(x') \geq u_i\,(x)\}$ is convex, and for every open neighborhood N of x there exists $x' \in N \cap X$ such that $u_i\,(x') > u_i\,(x)$.

[4] $0 \in X_i$ and $e_i \geq 0$ are needed to guarantee that the budget set defined in Subsection 2.1 is nonempty for every conceivable price.

We also assume the existence of a free disposal technology of physical commodities, i.e., an allocation $x = (x_i)_{i\in\mathcal{I}} \in X$ is a feasible allocation of the economy $\mathcal{E}$ if $\sum_{i\in\mathcal{I}} x_i \leq \sum_{i\in\mathcal{I}} e_i$.

While in economies without externalities Assumption A is standard, Starrett (1972) pointed out that negative external effects may create nonconvexities in preferences. As is well known, without convexity competitive equilibria may fail to exist. Since this paper is concerned with existence of equilibrium in a finite economy, we must assume that utilities are quasi-concave.[5]

2.1. Arrowian markets for consumption externalities

Market structure and equilibrium follow the classical definitions in Arrow (1969) and Laffont (1976).

Thus, an extended consumption vector of individual i is

$$\widetilde{x}_i = (x_{ih})_{h\in\mathcal{I}} \in X,$$

where x_{ii} is the effective consumption of individual i, while x_{ih} is the external effect of individual h's consumption as perceived by individual i. Equivalently, x_{ih} is the demand of i for the consumption of h. A price system is

$$\widetilde{p} = \left(p, \left((p_{ih})_{h\neq i}\right)_{i\in\mathcal{I}}\right) \in \mathbb{R}^L \times \mathbb{R}^{L(I-1)I},$$

where $p \in \mathbb{R}^L$ is the price of the L physical commodities, while $p_{ih} \in \mathbb{R}^L$ is the price paid by individual i to h for the consumption externality created by individual h on i. Individual i faces a *personalized* price for her own consumption, i.e., the price $p_{ii} = p - \sum_{h\neq i} p_{hi}$. By definition of p_{ii}, $\widetilde{p}$ satisfies the classical compatibility condition on prices à la Lindahl, that is:

$$p = \sum_{h\in\mathcal{I}} p_{hi}, \forall i \in \mathcal{I}.$$

With this notation, individual i's budget constraint is defined below.[6]

$$B_i\left(\widetilde{p}\right) = \left\{\widetilde{x}_i \in X : \left(p - \sum_{h\neq i} p_{hi}\right) \cdot x_{ii} + \sum_{h\neq i} p_{ih} \cdot x_{ih} \leq p \cdot e_i\right\}.$$

At equilibrium, the extended consumption vectors are coherent, that is

$$x_{ih} = x_{hh} = x_h \in \mathbb{R}^L_+, \forall (i, h).$$

Hence, $p_{ih} \cdot x_{ih}$ $(= p_{ih} \cdot x_{hh})$ is the value transfer (positive or negative) from individual i to h.

Definition 1 *(Equilibrium with markets for externalities).* An equilibrium of $\mathcal{E}$ is a pair of an allocation $x^* = \left(x_i^*\right)_{i\in\mathcal{I}} \in X$ and a price system $\tilde{p}^* \neq 0$ with $p^* \geq 0$, satisfying the two following properties.[7]

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5 All papers dealing with the existence of Arrow-Lindhal equilibria in finite economies with externalities adopt assumption A.(2), see Foley (1970), Bergstrom (1976b), Crès (1996), and Conley and Smith (2005).

6 The unique budget constraint reflects the completeness of the market structure. In principle one could generalize the analysis of this paper by explicitly introducing time, uncertainty and an incomplete set of assets.

7 As is known, the existence of a free disposal technology means that the aggregate production set of physical commodities of the economy is $\mathbb{R}^L_-$. Then profit maximization on $\mathbb{R}^L_-$ implies that $p^* \geq 0$.

(1) For all i, $x^* \in \operatorname{argmax}\{u_i(x) : x \in B_i(\tilde{p}^*)\}$, and
(2) x^* is feasible, i.e., $\sum_{i\in\mathcal{I}} x_i^* \leq \sum_{i\in\mathcal{I}} e_i$.

By local non-satiation, at equilibrium, all budget constraints are satisfied with equality, and therefore $p^* \cdot \sum_{i\in\mathcal{I}} \left(x_i^* - e_i\right) = 0$.

For a quasi-equilibrium, the cost of x^* is equal to the value of e_i, and any consumption bundle that is weakly preferred to x^* cannot cost less. That is, (1) in the definition above is replaced by

$$\left(p^* - \sum_{h\neq i} p_{hi}^*\right) \cdot x_i^* + \sum_{h\neq i} p_{ih}^* \cdot x_h^* = p^* \cdot e_i,$$

and all $x \in X$, $u_i(x) \geq u_i(x^*)$ implies that

$$p^* \cdot e_i \leq \left(p^* - \sum_{h\neq i} p_i^*\right) \cdot x_i + \sum_{h\neq i} p_{ih}^* \cdot x_h.$$

Under local non-satiation, an equilibrium is a quasi-equilibrium, but not vice versa.

3. Examples of non-existence of (quasi-)equilibrium

This section consists of two examples. The first shows that quasi-equilibria (hence equilibria) may fail to exist in the absence of a free disposal technology. The second example suggests that the notion of irreducibility is far more demanding for economies with externalities than for standard economies. Indeed, strictly positive endowments and local non-satiation are not sufficient for irreducibility.

3.1. Non-existence of a quasi-equilibrium without free disposal

For standard pure exchange economies (without external effects) satisfying Assumption A and $e_i \gg 0$, all $i \in \mathcal{I}$, quasi-equilibria exist even when the presence of free disposal is dispensed with, that is, when feasibility reads $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i$.[8] The example shows that this is not the case for economies with consumption externalities. We come back on this issue in Subsection 5.1.

Consider an economy with one physical commodity, three individuals, $X_i = \mathbb{R}_+$, $e_i \in \mathbb{R}_{++}$, for $i \in \mathcal{I} = \{1, 2, 3\}$, and utility functions:

$$u_1(x_1, x_2, x_3) = x_1, \text{ and } u_i(x_1, x_2, x_3) = x_i - 2\sum_{h\neq i} x_h, \ i = 2, 3.$$

We show that this economy does not have quasi-equilibria.

We break the argument in three simple steps. Suppose by contradiction that a quasi-equilibrium $(x^*, \widetilde{p}^*)$ exists. Through out the argument we use the fact that by definition $p_{ii}^* = p^* - \sum_{h\neq i} p_{hi}^*$, for all i.

Step 1: *It is* $\left(p_{11}^*, p_{12}^*, p_{13}^*\right) \geq 0$ *and* $p^* \geq 0$.

[8] See Hart and Kuhn (1975).

Proof. Since individual 1's utility function is nondecreasing in (x_1, x_2, x_3), the (first order) optimality conditions of the cost minimization problem imply that $\left(p^*_{11}, p^*_{12}, p^*_{13}\right) \geq 0$. Therefore, since at a quasi-equilibrium $\sum_h p^*_{1h} x^*_h = p^* e_1$, and since $e_1 > 0$, it is both $p^* e_1 \geq 0$ and $p^* \geq 0$. □

The market clearing (or feasibility) condition $\sum_{i \in \mathcal{I}} x^*_i = \sum_{i \in \mathcal{I}} e_i$ implies that aggregate consumption is strictly positive, i.e., $\sum_{i \in \mathcal{I}} x^*_i > 0$. Therefore, since

$$u_2\left(x^*\right) + u_3\left(x^*\right) = -\left(x^*_2 + x^*_3\right) - 4x^*_1 < 0,$$

either $u_2\left(x^*\right) < 0$ or $u_3\left(x^*\right) < 0$.

Assume without loss of generality that $u_2\left(x^*\right) < 0$. Recall that by Step 1, $p^* \geq 0$. Then we prove the following step.

Step 2: *It is* $\left(p^*_{21}, p^*_{22}, p^*_{23}\right) \geq 0$ *and* $p^* = 0$.

Proof. If $\left(p^*_{21}, p^*_{22}, p^*_{23}\right) \notin \mathbb{R}^3_+$, it is $\sum_h p^*_{2h} x_h < p^* e_2$ for some $x \in \mathbb{R}^3_{++}$, and similarly if $p^* > 0$, it is $\sum_h p^*_{2h} x'_h < p^* e_2$ for some $x' \in \mathbb{R}^3_{++}$. In both cases, the interior of the budget set $B_2\left(\widetilde{p}\right)$ is nonempty implying that x^* (solution to the cost minimization problem) is a solution to individual 2's utility maximization problem.[9] Further, since $p^* e_2 \geq 0$, in both cases 0 is in the budget set. But, then $u_2\left(x^*\right)$ must be greater than $u_2\left(0\right) = 0$ contradicting $u_2\left(x^*\right) < 0$. □

Step 1 and Step 2 have established that

$$\left(p^*_{11}, p^*_{12}, p^*_{13}\right) \geq 0,\ \left(p^*_{21}, p^*_{22}, p^*_{23}\right) \geq 0,\ \text{and } p^* = 0.$$

However, by definition $0 = p^* = \sum_{h=1}^3 p^*_{hi}$ for all i, thereby implying that

$$p^*_{3i} \leq 0, \forall i.$$

Step 3: *It is* $\widetilde{p}^* = 0$.

Proof. Since u_3 is strictly increasing in x_3, the optimality conditions of individual 3 for cost minimization imply that $p^*_{33} \geq 0$. By Steps 1 and 2, $p^*_{13} \geq 0$, $p^*_{23} \geq 0$, and $p^* = 0$. Therefore, $0 = p^* = \sum_h p^*_{h3}$ implies that $0 = p^*_{13} = p^*_{23} = p^*_{33}$. Then, if it were either $p^*_{31} < 0$ or $p^*_{32} < 0$, since $p^*_{33} = 0$, there could not be solution to the cost minimization problem of individual 3. Therefore, $p^*_{31} = p^*_{32} = 0$. Then since all prices are non negative and $0 = p^* = \sum_h p^*_{hi}$, all i, it must be $\widetilde{p}^* = 0$. □

Step 3 shows that a quasi-equilibrium cannot exist as the latter requires $\widetilde{p}^* \neq 0$. Therefore, without free disposal, quasi-equilibria may fail to exist.

With free disposal, the allocation $x^* = 0$ and the price $\widetilde{p}^*$ defined by $p^* = p^*_{12} = p^*_{13} = p^*_{21} = p^*_{31} = 0$, and $p^*_{23} = p^*_{32} = -1$ is a quasi-equilibrium. Indeed, $\sum x^*_i < \sum e_i$, $\left(p^* - \sum_{h \neq i} p^*_{hi}\right) \cdot x^*_i + \sum_{h \neq i} p^*_{ih} \cdot x^*_h = p^* \cdot e_i$, and it is easily verified that at $\tilde{p}^*$ the allocation x^* solves the cost minimization problems of every individual.

[9] As is known, when consumption sets are convex, utilities continuous, and the interior of $B_i\left(\tilde{p}\right)$ is non-empty, solutions to i's cost minimization are solutions to i's utility maximization.

3.2. Non-existence of an equilibrium

The economy satisfies Assumption A with strictly positive endowments and individuals' utility functions strictly increasing in their own consumption. Under such conditions, all quasi-equilibria of economies without externalities are equilibria. However, the example shows that for economies with externalities equilibria may fail to exist.

There is one physical commodity and three individuals, $X_i = \mathbb{R}_+$, $e_i \in \mathbb{R}_{++}$ for $i \in \mathcal{I} = \{1, 2, 3\}$. Utility functions are:

$$u_1(x_1, x_2, x_3) = x_1, \; u_2(x_1, x_2, x_3) = x_2 - x_3, \; \text{and } u_3(x_1, x_2, x_3) = x_3 - x_2.$$

Let $(x^*, \widetilde{p}^*)$ be an equilibrium. Recall that, by definition, $p^*_{ii} = p^* - \sum_{h \neq i} p^*_{hi}$ for all i. The existence of a solution to the individual maximization problems has the following immediate implications.

(1) Since u_i is strictly increasing in x_i, $p^*_{ii} > 0$ for every i.
(2) If u_i is independent of x_h, then $p^*_{ih} \geq 0$.

Claim: *There cannot be a competitive equilibrium.*

Proof. By (2), $\left(p^*_{12}, p^*_{13}, p^*_{21}, p^*_{31}\right) \geq 0$. By (1), $p^*_{11} > 0$ and then $p^* = p^*_{11} + p^*_{21} + p^*_{31} > 0$. Consequently, as $e_i > 0$, it is $p^* e_i > 0$ for every i. Then, each individual can reach a strictly positive utility level in her budget constraint. Therefore, at equilibrium, $u_2\left(x^*_1, x^*_2, x^*_3\right) = x^*_2 - x^*_3 > 0$ as well as $u_3\left(x^*_1, x^*_2, x^*_3\right) = x^*_3 - x^*_2 > 0$, which is clearly impossible. □

We conclude with two further observations. This economy has a continuum of quasi-equilibria. They are described by $p^* = p^*_{12} = p^*_{13} = p^*_{21} = p^*_{31} = 0$, $p^*_{23} = p^*_{32} = -1$, and $x^*_2 = x^*_3 \leq \frac{1}{2}\left(r - x^*_1\right)$ with $x^*_1 \in [0, r]$ and $r = e_1 + e_2 + e_3$.

In the economy with individuals 2 and 3 only, equilibria exist with $p^* = 0$, $p^*_{23} = p^*_{32} = -1$, and $x^*_2 = x^*_3 \leq \frac{1}{2}(e_2 + e_3)$. Hence, the presence of individual 1, who is not affected by any external effects and whose consumption does not generate external effects on any other individual prevents the existence of an equilibrium. We further discuss this issue in Subsection 5.2.

4. The production economy behind the consumption externalities

In this section, we construct a standard production economy without externalities, $\mathcal{E}_Y$. We show that there is a simple one-to-one mapping between the equilibria of the two economies $\mathcal{E}_Y$ and $\mathcal{E}$. Thus existence of (quasi-)equilibria of $\mathcal{E}$ follows from the existence of equilibria of $\mathcal{E}_Y$. Similarly, survival and irreducibility conditions for $\mathcal{E}$ are derived from survival and irreducibility conditions for $\mathcal{E}_Y$. The main idea behind the construction of $\mathcal{E}_Y$ is in the definition of the production set Y that transforms inputs of L physical commodities into I identical allocations, one for each individual.

The production economy $\mathcal{E}_Y$ has $L + (LI)I$ commodities. A generic bundle of $\mathbb{R}^L \times \mathbb{R}^{(LI)I}$ is denoted as

$$(a, \widetilde{a}_1, \ldots, \widetilde{a}_i, \ldots, \widetilde{a}_I)$$

where $a \in \mathbb{R}^L$ and $\widetilde{a}_i = (a_{ih})_{h \in \mathcal{I}} \in \mathbb{R}^{LI}$ for every $i \in \mathcal{I}$.

The production technology is described by the production set

$$Y = \left\{ y = (-z, \widetilde{y}_1, \ldots, \widetilde{y}_i, \ldots, \widetilde{y}_I) \in \mathbb{R}^L_- \times \mathbb{R}^{(LI)I} : \widetilde{y}_i = \widetilde{y}_h, \forall i, h \in \mathcal{I} \text{ and } \sum_{i \in \mathcal{I}} y_{ii} \le z \right\}$$

It is verified that Y is a closed convex cone that satisfies the possibility of inaction and the impossibility of free production, i.e., $Y \cap \left(\mathbb{R}^L_+ \times \mathbb{R}^{(LI)I}_+\right) = \{0\}$. The condition $\sum_{i \in \mathcal{I}} y_{ii} \le z$ allows for free disposal of physical commodities.

The consumption set of individual i is now defined as:

$$\Xi_i = \left\{ \xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, \widetilde{x}_i, 0_{LI}, \ldots, 0_{LI}) \in \mathbb{R}^L \times \mathbb{R}^{(LI)I} : \widetilde{x}_i \in X \right\}. \tag{1}$$

The utility function of individual i is $\widetilde{u}_i : \Xi_i \mapsto \mathbb{R}$ defined by $\widetilde{u}_i(\xi_i) = u_i(\widetilde{x}_i)$. The endowment of individual i is $\eta_i = (e_i, 0_{LI}, \ldots, 0_{LI}, \ldots, 0_{LI}) \in \mathbb{R}^L_+ \times \mathbb{R}^{(LI)I}$.

Let $\Xi = \prod_{i \in \mathcal{I}} \Xi_i$. An allocation of the production economy is a pair $(\xi, y) \in \Xi \times Y$, which is feasible if $\sum_{i \in \mathcal{I}} \xi_i = \sum_{i \in \mathcal{I}} \eta_i + y$.

The price system of $\mathcal{E}_Y$ is

$$\pi = (p, \widetilde{p}_1, \ldots, \widetilde{p}_i, \ldots, \widetilde{p}_I) \in \mathbb{R}^L_+ \times \mathbb{R}^{(LI)I}.$$

The next lemma describes the relation between the feasible allocations of the economies $\mathcal{E}$ and $\mathcal{E}_Y$.

Lemma 2.

(1) *Let $(\xi, y) \in \Xi \times Y$ be a feasible allocation of $\mathcal{E}_Y$. Let $(\widetilde{x}_i)_{i \in \mathcal{I}}$, $\widetilde{x}_i \in X$, be such that $\xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, \widetilde{x}_i, 0_{LI}, \ldots, 0_{LI})$ for all $i \in \mathcal{I}$. Then, $x = \widetilde{x}_1$ is a feasible allocation of $\mathcal{E}$.*

(2) *Conversely, let $x \in X$ be a feasible allocation of $\mathcal{E}$. Let*

$$\xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, x, 0_{LI}, \ldots, 0_{LI})$$

for all $i \in \mathcal{I}$ and $y = \left(-\sum_{i \in \mathcal{I}} e_i, x, \ldots, x, \ldots, x\right)$, then $(\xi, y) \in \Xi \times Y$ is a feasible allocation of $\mathcal{E}_Y$.

Proof. (1) By definition of Y, $\widetilde{x}_1 = \widetilde{x}_i = \widetilde{y}_i$ for all $i \in \mathcal{I}$. Then setting $x = \widetilde{x}_1$ and recalling that $\widetilde{y}_i = (y_{ih})_{h \in \mathcal{I}}$, for every $i \in \mathcal{I}$, the definition of Y implies that $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} y_{ii} \le \sum_{i \in \mathcal{I}} e_i$. Therefore $x = \widetilde{x}_1$ is a feasible allocation of $\mathcal{E}$.

(2) Since $x \in X$ is a feasible allocation of $\mathcal{E}$, $\sum_{i \in \mathcal{I}} x_i \le \sum_{i \in \mathcal{I}} e_i$ and, by definition of Y, $y = \left(-\sum_{i \in \mathcal{I}} e_i, x, \ldots, x, \ldots, x\right)$ belongs to Y. Further, $\sum_{i \in \mathcal{I}} \xi_i = (0_L, x, \ldots, x, \ldots, x)$ and

$$\sum_{i \in \mathcal{I}} \eta_i + y = \left(\sum_{i \in \mathcal{I}} e_i - \sum_{i \in \mathcal{I}} e_i, x, \ldots, x, \ldots, x \right) = (0_L, x, \ldots, x, \ldots, x)$$

implying that $(\xi, y) \in \Xi \times Y$ is a feasible allocation of $\mathcal{E}_Y$. □

Since the production set Y exhibits constant returns to scale, at equilibrium optimal profit must be zero. Thus profit maximization restricts the search of an equilibrium price of $\mathcal{E}_Y$ to the price domain

$$\Pi^* = \left\{ \pi \in \mathbb{R}^L_+ \times \mathbb{R}^{(LI)I} : \pi \cdot y \le 0, \forall y \in Y \right\}.$$

A price π is in Π^* if and only if it satisfies restrictions identical to those generated by the Lindahl conjecture in the price domain of $\mathcal{E}$, the economy with externalities. This is spelled out in the next key lemma.

Lemma 3. *The price system π is in Π^* if and only if $0 \leq \sum_{h\in\mathcal{I}} p_{hi} \leq p$ and $\sum_{h\in\mathcal{I}} p_{hi}$ does not depend on i.*

Proof. By definition, $y \in Y$ implies that $z = \sum_{i\in\mathcal{I}} y_{ii} + b$ for some $b \in \mathbb{R}^L_+$. Thus, profits at (π, y) are

$$\pi \cdot y = \sum_{i\in\mathcal{I}} \left(\sum_{h\in\mathcal{I}} p_{hi} \right) \cdot y_{ii} - p \cdot z = \sum_{i\in\mathcal{I}} \left(\sum_{h\in\mathcal{I}} p_{hi} - p \right) \cdot y_{ii} - p \cdot b$$

Therefore, $\pi \in \Pi^*$ iff the system of linear inequalities

$$\begin{aligned} \sum_{i\in\mathcal{I}} \left(\sum_{h\in\mathcal{I}} p_{hi} - p \right) \cdot y_{ii} - p \cdot b > 0, \\ \sum_{i\in\mathcal{I}} y_{ii} + b \geq 0, \\ b \geq 0 \end{aligned}$$

does not have a solution $\left((y_{ii})_{i\in\mathcal{I}}, -b\right)$. Let v be the vector of dimension $1 \times (LI + L)$ defined as $v = \left(\sum_h p_{h1} - p, \ldots, \sum_h p_{hI} - p, p\right)$. Then

$$v \cdot \left((y_{ii})_{i\in I}, -b\right) = \sum_i \left(\sum_h p_{hi} - p \right) \cdot y_{ii} - p \cdot b$$

Let A be a matrix of dimension $2L \times (LI + L)$ defined as

$$A = \begin{bmatrix} I_L & \cdot & \cdot & \cdot & I_L & -I_L \\ 0 & \cdot & \cdot & \cdot & 0 & -I_L \end{bmatrix}$$

Then,

$$A\left((y_{ii})_{i\in I}, -b\right) = \left(\sum_i y_{ii} + b, b \right).$$

Thus $\pi \in \Pi^*$ iff the system of linear inequalities $v \cdot c > 0, Ac \geq 0$ does not have a solution $c \in \mathbb{R}^{LI+L}$. By the Theorem of the Alternative, this is true iff there exists $\theta = (\theta_1, \theta_2) \in \mathbb{R}^{2L}_+ \setminus \{0\}$ such that $-v = A^T\theta$. That is, $\left(p - \sum_h p_{hi}\right) = \theta_1 \geq 0$, for all $i \in \mathcal{I}$, and $p = \theta_1 + \theta_2 \geq 0$. □

Remark 4. Let $y = \left(-z, (\tilde{y}_i)_{i\in\mathcal{I}}\right) \in Y$ be a profit maximization bundle at $\pi \in \Pi^*$. Lemma 3 implies that $\sum_{h\in\mathcal{I}} p^\ell_{hi} = p^\ell$ whenever $z^\ell > 0$. Then, since feasibility in $\mathcal{E}_Y$ entails $z^\ell = \sum_{i\in\mathcal{I}} e^\ell_i$, at an equilibrium of the production economy, the Lindahl compatibility condition $\sum_{h\in\mathcal{I}} p^\ell_{hi} = p^\ell$ holds true for all commodities ℓ.

As $\mathcal{E}_Y$ is a standard production economy (without externalities), we omit the definitions of equilibrium and quasi-equilibrium.

The next proposition exploits Lemmas 2 and 3 in order to show the existence of a simple one-to-one mapping between the equilibria of the economies $\mathcal{E}$ and $\mathcal{E}_Y$. Importantly such a mapping yields equivalent allocation-price pairs in the sense specified by the statement of the next proposition.

Proposition 5.

(1) *Let $(x^*, \widetilde{p}^*) \in X \times \mathbb{R}^L_+ \times \mathbb{R}^{(L-I)I}$ be an equilibrium (resp. a quasi-equilibrium) of the economy $\mathcal{E}$. Consider the pair (ξ, y) where for all $i \in \mathcal{I}$, $\xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, x^*, 0_{LI}, \ldots, 0_{LI})$, and $y = \left(-\sum_{i\in\mathcal{I}} e_i, x^*, \ldots, x^*, \ldots, x^*\right)$. Further define the price system $\pi = (p, \widetilde{p}_1, \ldots, \widetilde{p}_i, \ldots, \widetilde{p}_I)$ as $p = p^*$ and $p_{ih} = p^*_{ih}$ for $h \neq i$ and $p_{ii} = p^* - \sum_{h\neq i} p^*_{hi}$, for every $i \in \mathcal{I}$. Then, $(\xi, y, \pi) \in \Xi \times Y \times \mathbb{R}^L_+ \times \mathbb{R}^{(LI)I}$ is an equilibrium (resp. a quasi-equilibrium) of the production economy $\mathcal{E}_Y$.*

(2) *Conversely, let $(\xi, y, \pi) \in \Xi \times Y \times \mathbb{R}^L_+ \times \mathbb{R}^{(LI)I}$ be an equilibrium (resp. a quasi-equilibrium) of the production economy $\mathcal{E}_Y$. Let $x^* \in X$ such that $\xi_i = (0_L, 0_{LI}, \ldots, x^*, \ldots, 0_{LI})$ and $\widetilde{p}^* = \left(\sum_{h\in\mathcal{I}} p_{h1}, \left((p_{ih})_{h\neq i}\right)_{i\in\mathcal{I}}\right)$. Then, $(x^*, \widetilde{p}^*)$ is an equilibrium (resp. a quasi-equilibrium) of the economy $\mathcal{E}$.*

Proof. First, by Lemma 2, the mappings between $(\xi, y) \in \Xi \times Y$ and $x^* \in X$ preserve feasibility. Next we pick π, y as defined in (1) and we show that $\pi \in \Pi^*$ and that y maximizes profits at π. Since by construction $\sum_{h\in\mathcal{I}} p_{hi} = p^* \geq 0$, for all $i \in \mathcal{I}$, Lemma 3 implies $\pi \in \Pi^*$. Then,

$$
\begin{aligned}
\pi \cdot y &= \sum_{i\in\mathcal{I}} \left(\sum_{h\in\mathcal{I}} p_{hi}\right) \cdot y_{ii} - p \cdot z \\
&= \sum_{i\in\mathcal{I}} \left(\sum_{h\in\mathcal{I}} p_{hi}\right) \cdot x^*_i - p \cdot \sum_{i\in\mathcal{I}} e_i \\
&= \sum_{i\in\mathcal{I}} \left(p_{ii} + \sum_{h\neq i} p_{hi}\right) \cdot x^*_i - p \cdot \sum_{i\in\mathcal{I}} e_i \\
&= p^* \cdot \sum_{i\in\mathcal{I}} \left(x^*_i - e_i\right) = 0
\end{aligned}
$$

implies that y maximizes profits at π.

Therefore, in order to conclude the argument it suffices to verify the optimality of individuals' choices in (1) and (2) at respectively π and $\widetilde{p}^*$. This is implied by the equalities $\pi \cdot \eta_i = p^* \cdot e_i$ and $\pi \cdot \xi'_i = \widetilde{p}^*_i \cdot \widetilde{x}'_i$, for $\xi'_i = \left(0, \ldots, 0, \widetilde{x}'_i, 0, \ldots, 0\right)$, all $\widetilde{x}'_i \in X$, and $i \in \mathcal{I}$. It is verified that the later holds true since, by Lemma 3, $\sum_{h\in\mathcal{I}} p_{h1} = \sum_{h\in\mathcal{I}} p_{hi}$, for all $i \in \mathcal{I}$. □

Example (a). We describe the production economy $\mathcal{E}_Y$ and some of its properties by means of a simple example. Consider a 2×2 economy with the following characteristics. Endowments are $e_1 \gg 0$ and $e_2 \gg 0$, the consumption sets are $X_1 = X_2 = \mathbb{R}_+ \times \mathbb{R}_{++}$, and the quasi-linear utilities are:

$$u_1(x_1, x_2) = x^1_1 + \ln x^2_1 - \alpha x^1_2 \text{ and } u_2(x_1, x_2) = x^1_2 + \ln x^2_2,$$

where $\alpha \neq 0$ is a parameter measuring the intensity of the external effect of the second consumer on the first one, which could be either negative, $\alpha > 0$, or positive, $\alpha < 0$. We assume that $e_1^1 > 1$ and $e_2^1 > 1 + \alpha > 0$.[10] The production set of $\mathcal{E}_Y$ is:

$$Y = \left\{ y = (-z, (y_{11}, y_{22}), (y_{11}, y_{22})) \in \mathbb{R}^2_- \times \mathbb{R}^8 \mid y_{11} + y_{22} \leq z \right\}.$$

The consumption sets of $\mathcal{E}_Y$ are:

$$\Xi_1 = \{\xi_1 = (0_2, x_{11}, x_{12}, 0_2, 0_2) \mid (x_{11}, x_{12}) \in X_1 \times X_2\},$$
$$\Xi_2 = \{\xi_2 = (0_2, 0_2, 0_2, x_{21}, x_{22}) \mid (x_{21}, x_{22}) \in X_1 \times X_2\}.$$

The utility functions are $\tilde{u}_1(\xi_1) = u_1(x_{11}, x_{12})$ and $\tilde{u}_2(\xi_2) = u_2(x_{21}, x_{22})$, and the initial endowments $\eta_i = (e_i, 0_4, 0_4)$, all $i = 1, 2$. The price vector is:

$$\pi = (p, p_{11}, p_{12}, p_{21}, p_{22}) \in \mathbb{R}^2_+ \times \mathbb{R}^8.$$

Equilibrium prices satisfy:

(1) $p_{ii}^* \gg 0$, all $i = 1, 2$, because $\widetilde{u}_i$ is strongly monotone in x_{ii}.
(2) $p_{21}^* \geq 0$, since $\widetilde{u}_2$ does not depend on x_{21}.
(3) By Lemma 3, $p_{11}^* + p_{21}^* = p_{12}^* + p_{22}^* = q^*$ and $0 \leq q^* \leq p^*$.

The profit maximization problem is then

$$\begin{array}{ll} \max & q^* \cdot (y_{11} + y_{22}) - p^* \cdot z \\ \text{subject to} & \begin{cases} z \geq 0 \\ y_{11} + y_{22} \leq z \end{cases} \end{array}$$

The three properties above imply that $q^* \gg 0$ and therefore that the technological constraint is binding, i.e., $y_{11} + y_{22} = z$. Further, by Lemma 2-(2), at equilibrium, $x_{12} = x_{22} = x_2$, $x_{21} = x_{11} = x_1$, $y_{ii} = x_{ii} = x_i$, all $i = 1, 2$, and $z = e_1 + e_2$. Then, at equilibrium, the market clearing conditions must hold true in the market of physical commodities, i.e.,

$$x_1 + x_2 = e_1 + e_2.$$

Further, since $(e_1 + e_2) \gg 0$, by Remark 4, it must be that $q^* = p^*$.

We determine equilibrium allocations of the economy $\mathcal{E}$. We choose the price normalization $p^{*2} = 1$ and limit attention to interior equilibrium allocations $(x_1^*, x_2^*) \gg 0$. Equilibrium price and allocations of $\mathcal{E}$ are:

$$\tilde{p}^* = \left(\left(\tfrac{r^2}{2+\alpha}, 1 \right), \left(-\tfrac{\alpha r^2}{2+\alpha}, 0 \right), (0, 0) \right),$$
$$x_1^* = \left(e_1^1 - 1 + \tfrac{\alpha\left(e_2^1 - 1 - \alpha\right)}{1+\alpha} + \tfrac{(2+\alpha)\left[(1+\alpha)e_1^2 + \alpha e_2^2\right]}{(1+\alpha)r^2}, \tfrac{r^2}{2+\alpha} \right),$$
$$x_2^* = \left(\tfrac{e_2^1 - 1 - \alpha}{1+\alpha} + \tfrac{(2+\alpha)e_2^2}{(1+\alpha)r^2}, \tfrac{(1+\alpha)r^2}{2+\alpha} \right),$$

where $r^2 = e_1^2 + e_2^2$. By Proposition 5, the corresponding competitive equilibrium of the production economy $\mathcal{E}_Y$ is $\xi_1^* = (0_2, x_1^*, x_2^*, 0_2, 0_2)$ and $\xi_2^* = (0_2, 0_2, 0_2, x_1^*, x_2^*)$, with the pro-

[10] As is well known, with quasi-linear preferences, some lower bounds on the endowments are required to get an equilibrium allocation $(x_1^*, x_2^*) \gg 0$.

duction $y^* = \left(-(e_1+e_2), x_1^*, x_2^*, x_1^*, x_2^*\right)$, and the extended price $\pi^* = \left(\left(\frac{r^2}{2+\alpha}, 1\right), \left(\frac{r^2}{2+\alpha}, 1\right), \left(-\frac{\alpha r^2}{2+\alpha}, 0\right), (0,0), \left(\frac{(1+\alpha)r^2}{2+\alpha}, 1\right)\right)$.

Since utilities are separable, competitive equilibrium allocations of the economy without externalities ($\alpha = 0$) are competitive equilibrium allocations of the economy with externalities ($\alpha \neq 0$) without markets for externalities. Such equilibrium allocations are never Pareto optimal when $\alpha \neq 0$. The introduction of markets for externalities restores the Pareto optimality in the economy $\mathcal{E}$. Indeed, the economy $\mathcal{E}_Y$ is a standard production economy and as such its equilibrium allocations satisfy the First Fundamental Theorem of Welfare Economics. Then, by Proposition 5, the same properties hold true for the equilibrium allocations of economy with externalities $\mathcal{E}$.

It is verified that the introduction of markets for externalities reduces the negative external effect on the first individual when $\alpha > 0$, that is, $x_2^{*1}(\alpha) < x_2^{*1}(0)$. The converse holds true when $\alpha < 0$, that is, $x_2^{*1}(\alpha) > x_2^{*1}(0)$.

5. Existence of an equilibrium with markets for externalities

We prove that quasi-equilibria exist and we provide sufficient conditions under which quasi-equilibria are equilibria, namely an irreducibility condition combined with a survival condition. Contrary to what happens in exchange economies without externalities, the argument requires an irreducibility condition even with strictly positive endowments and locally non-satiated preferences.

5.1. Existence of a quasi-equilibrium with markets for externalities

The following theorem is a consequence of Lemma 3, Proposition 5, and a general result, Proposition 2.2.2 in Florenzano (2003), for the existence of a quasi-equilibrium in production economies without externalities.

Theorem 6 *(Existence of a quasi-equilibrium of the economy $\mathcal{E}$). Under Assumption A, there exists a quasi-equilibrium $(x^*, \widetilde{p}^*)$ of the economy $\mathcal{E}$ with $p^* \geq 0$.*

Proof. Consider the production economy $\mathcal{E}_Y$ defined in Section 4. By Assumption A, consumption sets are closed, convex and bounded from below, the sets $P_i(\xi_i) = \left\{\xi_i' \in \Xi_i : \tilde{u}_i\left(\xi_i'\right) > \tilde{u}_i(\xi_i)\right\}$ are nonempty and convex, while the sets $P_i^{-1}\left(\xi_i'\right) = \left\{\xi_i \in \Xi_i : \xi_i' \in P_i(\xi_i)\right\}$ are open in Ξ_i. Further, by construction, the production set Y is closed, convex, and satisfies the possibility of inaction and the impossibility of free production. Therefore, the attainable consumption sets are nonempty and compact. Further, the weak survival assumption $\eta_i \in \Xi_i - Y$ for all $i \in \mathcal{I}$ is satisfied since $0 \in \Xi_i$ and $-\eta_i \in Y$. Thus the assumptions of Proposition 2.2.2 in Florenzano (2003) are satisfied: $\mathcal{E}_Y$ has a quasi-equilibrium with $\pi \neq 0$ that by Lemma 3 and Proposition 5 implies the existence of a quasi-equilibrium of $\mathcal{E}$ with $p^* \geq 0$. □

Example in Subsection 3.1 shows that without free disposal the economy with externalities $\mathcal{E}$ may fail to have a quasi-equilibrium. This is because, in the absence of the free disposal condition $\sum_{i\in\mathcal{I}} y_{ii} \leq z$, the production economy $\mathcal{E}_Y$ does not satisfy the weak survival condition $\eta_i \in \Xi_i - Y$ of Florenzano (2003).

5.2. Irreducibility conditions

We derive equivalent irreducibility conditions of the economy with externalities from known conditions of irreducibility for production economies without externalities. We adapt the classical irreducibility conditions of McKenzie (1959, 1961, 1981) and Bergstrom (1976a) to the production economy $\mathcal{E}_Y$.

Assumption I. *The economy $\mathcal{E}_Y$ is McKenzie-Bergstrom irreducible if for every feasible allocation $\left((\xi_i)_{i\in\mathcal{I}}, y\right) \in \Xi \times Y$ and for every pair of non-empty sets I_1 and I_2 with $I_1 \cup I_2 = \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$, there exist $\theta_h > 0$, $h \in I_2$, $\left(\xi_i'\right)_{i\in\mathcal{I}} \in \Xi$ and $y' \in Y$ satisfying the two following conditions:*

(1) $\sum_{k\in I_1}\left(\xi_k' - \eta_k\right) + \sum_{h\in I_2}\theta_h\left(\xi_h' - \eta_h\right) = y'$,
(2) $\tilde{u}_k\left(\xi_k'\right) \geq \tilde{u}_k\left(\xi_k\right)$ *for all* $k \in I_1$ *and* $\tilde{u}_{k_0}\left(\xi_{k_0}'\right) > \tilde{u}_{k_0}\left(\xi_{k_0}\right)$ *for some* $k_0 \in I_1$.

It is verified that:

(1) if $Y = \{0\}$ and I_2 is a singleton, Assumption I coincides with the irreducibility condition of Bergstrom (1976a) for a classical exchange economy without free disposal, and
(2) if the economy satisfies the irreducibility conditions of McKenzie (1959, 1961, 1981), then it satisfies Assumption I.

We now translate Assumption I into an equivalent irreducibility condition for the economy with consumption externalities $\mathcal{E}$. An externality economy is irreducible when any subset of individuals can pick a Pareto improving allocation whose total resources are less or equal than the sum of the endowments of the first group plus a weighted sum of endowments of the residual.

Assumption E *(Irreducibility in $\mathcal{E}$). The economy $\mathcal{E}$ is irreducible if for every feasible allocation $x \in X$ and for every pair of non-empty sets I_1 and I_2 with $I_1 \cup I_2 = \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$, there exist $\theta_h > 0$, $h \in I_2$, and an allocation $\bar{x} = (\bar{x}_i)_{i\in\mathcal{I}} \in X$ satisfying the two following conditions:*

(1) $\sum_{i\in\mathcal{I}}\bar{x}_i \leq \sum_{k\in I_1} e_k + \sum_{h\in I_2}\theta_h e_h$,
(2) $u_k(\bar{x}) \geq u_k(x)$ *for all* $k \in I_1$ *and* $u_{k_0}(\bar{x}) > u_{k_0}(x)$ *for some* $k_0 \in I_1$.

The next proposition exploits the definition of the production technology Y to show that Assumptions I and E are equivalent.

Proposition 7. *The economy $\mathcal{E}_Y$ is McKenzie-Bergstrom irreducible if and only if the economy $\mathcal{E}$ satisfies Assumption E.*[11]

Proof. Let us assume that Assumption I is satisfied. Let x be a feasible allocation of $\mathcal{E}$ and I_1 and I_2 as in Assumption E. Then, by definition of Y and feasibility, $y = \left(-\sum_{i\in\mathcal{I}} e_i, x, \ldots, x, \ldots, x\right)$

[11] Definition 4 in Geistdorfer-Florenzano (1982) is more general than Assumption I as it allows $\theta_k > 0$, $k \in I_1$. However, by the definition of our technology Y, the allocation $\bar{x}$ in (2) of Assumption E must be invariant in I_1 implying that the weights θ_k must be invariant in I_1.

and $\xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, x, 0_{LI}, \ldots, 0_{LI})$ for all $i \in \mathcal{I}$ is a feasible allocation of $\mathcal{E}_Y$. Let $y' \in Y$ and $\left(\xi'_i\right)_{i\in\mathcal{I}}$ as given by Assumption I. Then $y' = \left(-z', \left(\widetilde{y}'_i\right)_{i\in\mathcal{I}}\right)$ with $z' \in \mathbb{R}^L_+$ and $\widetilde{y}'_i = \bar{x}$ for all $i \in \mathcal{I}$ for some $\bar{x} = (\bar{x}_i)_{i\in\mathcal{I}} \in \mathbb{R}^{LI}$ such that $\sum_{i\in\mathcal{I}} \bar{x}_i \leq z'$.

For all i, $\xi'_i = \left(0_L, 0_{LI}, \ldots, 0_{LI}, \widetilde{x}'_i, 0_{LI}, \ldots, 0_{LI}\right)$ for some $\widetilde{x}'_i \in X$. Since $\eta_i = (e_i, 0_{LI}, \ldots, 0_{LI}, \ldots, 0_{LI})$ for all $i \in \mathcal{I}$, Condition 1 in Assumption I reads:

$$\begin{cases} \forall k \in I_1,\ \widetilde{x}'_k = \widetilde{y}'_k = \bar{x} \\ \forall h \in I_2,\ \theta_h \widetilde{x}'_h = \widetilde{y}'_h = \bar{x} \\ z' = \sum_{k\in I_1} e_k + \sum_{h\in I_2} \theta_h e_h \end{cases}$$

Then $\bar{x} \in X$ because $\bar{x} = \widetilde{x}'_k \in X$ for all $k \in I_1$. Therefore, the economy $\mathcal{E}$ satisfies Assumption E since for all $k \in I_1$, $u_k(\bar{x}) = \widetilde{u}_k\left(\xi'_k\right)$ and $u_k(x) = \widetilde{u}_k(\xi_k)$.

Conversely, assume now that the economy $\mathcal{E}$ satisfies Assumption E. Let $\left((\xi_i)_{i\in\mathcal{I}}, y\right)$ be a feasible allocation of $\mathcal{E}_Y$ and I_1 and I_2 as in Assumption I. Then, x such that $\xi_1 = (0, x, 0, \ldots, 0)$ is a feasible allocation of $\mathcal{E}$. Let $\bar{x}$ as given by Assumption E.

Set $\gamma_h = \max\{1, \theta_h\} \geq 1$. Then, since by Assumption A, $e_h \geq 0$:

$$\sum_{i\in\mathcal{I}} \bar{x}_i \leq \sum_{k\in I_1} e_k + \sum_{h\in I_2} \theta_h e_h \leq \sum_{k\in I_1} e_k + \sum_{h\in I_2} \gamma_h e_h.$$

For $k \in I_1$ take $\widetilde{x}'_k = \bar{x}$, and for $h \in I_2$ set $\widetilde{x}'_h = (1/\gamma_h)\,\bar{x}$. Since $0 \in X$, X is convex and $\gamma_h \geq 1$, it must be that $\widetilde{x}'_h \in X$ for all $h \in I_2$. Then the economy $\mathcal{E}_Y$ is McKenzie-Bergstrom irreducible with $y = \left(-\left(\sum_{k\in I_1} e_k + \sum_{h\in I_2} \gamma_h e_h\right), \bar{x}, \ldots, \bar{x}\right)$, $\xi'_k = (0, \ldots, 0, \bar{x}, 0, \ldots, 0)$ for $k \in I_1$, and $\xi'_h = (0, \ldots, 0, (1/\gamma_h)\,\bar{x}, 0, \ldots, 0)$ for $h \in I_2$, with the coefficients γ_h for $h \in I_2$. □

Assumption E is satisfied in an economy without externalities if $e_i \gg 0$, all $i \in \mathcal{I}$, and the utility functions are locally non-satiated. It is also satisfied with externalities if there are only two consumers, $e_i > 0$, all $i = 1, 2$, and preferences are strongly monotonic in own consumption. In such a situation, without loss of generality, set $I_1 = \{1\}$ and $I_2 = \{2\}$ and choose $\bar{x}_2 = x_2$ and $\bar{x}_1 \leq x_1 + e_2$ with $u_1(\bar{x}_1, \bar{x}_2) > u_1(x_1, \bar{x}_2)$. However, in the economy of the example in Subsection 3.2 with three individuals, although $e_i > 0$ and preferences are strongly monotonic in own consumption, an equilibrium does not exist. Hence such an economy does not satisfy Assumption E with $I_1 = \{2, 3\}$, $I_2 = \{1\}$ at the feasible allocation $(0, r/2, r/2)$ with $r = e_1 + e_2 + e_3$.

In general, whether or not an economy is irreducible depends on the sign and strength of the externalities.[12] If some commodities do not display negative external effects, under standard monotonicity in own consumption, the economy is irreducible. The next definition and proposition make this idea precise.

Definition 8. Good ℓ^* is socially desirable if for every pair of allocations x and x' with $x'^{\ell}_i = x^{\ell}_i$, $\ell \neq \ell^*$, and $x'^{\ell^*}_i \geq x^{\ell^*}_i$, all i, with at least one strict inequality, it holds true that

$$u_i\left(x'\right) \geq u_i(x), \text{ for all } i, \text{ with } u_i\left(x'\right) > u_i(x) \text{ if } x'^{\ell^*}_i > x^{\ell^*}_i.$$

[12] It is verified that Assumption N in Crès (1996) implies Assumption E.

If there is at least one socially desirable commodity and individuals are positively endowed with it, the economy is irreducible. This is stated in the next proposition whose simple proof is left to the reader.

Proposition 9. *Under Assumption A, suppose that there exists at least one socially desirable commodity $\ell^* \in \mathcal{L}$, and that $e_i^{\ell^*} > 0$ for all $i \in \mathcal{I}$. Then, Assumption E is satisfied.*

Thus, if the economy is not irreducible, all commodities necessarily display negative externalities. The example below suggests that the converse is not always true. In one commodity economies with more than two individuals, irreducibility fails if there are at least two individuals whose consumption has negative and sizable effects on the other. Therefore, even when all commodities display negative external effects, irreducibility holds whenever these effects are not too strong.

Example (b). Consider a slightly different version of the economy given in Subsection 3.2, where $u_1(x_1, x_2, x_3) = x_1$ and the utility functions of individuals 2 and 3 are now

$$u_2(x_1, x_2, x_3) = x_2 - \alpha x_3 \text{ and } u_3(x_1, x_2, x_3) = x_3 - \delta x_2,$$

with parameters $\alpha \geq 0$ and $\delta \geq 0$ controlling the size of the negative externality. Irreducibility is satisfied if and only if $\alpha\delta < 1$. Hence, irreducibility fails if both $\alpha > 0$ and $\delta > 0$ and the combined effect (measure by $\alpha\delta$) is large. To show this, notice that since $e_i > 0$, all i, and $\theta_h > 0$, $h \in \mathcal{I}_2$, Assumption E holds true if and only if for each feasible allocation x there is an allocation $\bar{x}$ satisfying condition (2) in Assumption E. If $\mathcal{I}_1 = \{i\}$, the allocation $\bar{x}_i > x_i$ and $\bar{x}_h = x_h$, for all $h \in \mathcal{I}_2$, satisfies condition (2). Similarly, if $\mathcal{I}_1 = \{1, i\}$ with $i \neq 1$, the allocation $\bar{x}_1 > x_1$, $\bar{x}_i = x_i$ and $\bar{x}_h = x_h$ satisfies condition (2). Thus, the economy is irreducible if and only if condition (2) is satisfied for $\mathcal{I}_1 = \{2, 3\}$. Condition (2) holds if and only if $(\bar{x}_2 - x_2) - \alpha(\bar{x}_3 - x_3) \geq 0$ and $(\bar{x}_3 - x_3) - \delta(\bar{x}_2 - x_2) \geq 0$, with at least one strict inequality. By Farkas's Lemma those inequalities are simultaneously satisfied if and only if $\alpha\delta < 1$.

5.3. Survival conditions

The survival assumption is a sufficient condition guaranteeing that at a quasi-equilibrium at least one individual's consumption is not a minimal expenditure bundle. When combined with the irreducibility conditions, it implies that at a quasi-equilibrium no individual's consumption is a minimal expenditure bundle, that is, that individuals consumptions are utility maximizers. Hence a quasi-equilibrium is an equilibrium. We posit the following assumption.

Assumption S *(Survival in $\mathcal{E}$). For all $i \in \mathcal{I}$,* $\text{Int}\, X_i \neq \emptyset$ *and* $\sum_{i\in\mathcal{I}} e_i \in \mathbb{R}^L_{++} \cap \left(\text{Int} \sum_{i\in\mathcal{I}} X_i\right)$.

Assumption S is clearly satisfied under standard survival assumptions for exchange economies without externalities, i.e., $X_i = \mathbb{R}^L_+$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} e_i \gg 0$.

The following proposition shows that if the economy $\mathcal{E}$ satisfies Assumption S, then the production economy $\mathcal{E}_Y$ satisfies the survival assumption of Florenzano (2003), under which at a quasi-equilibrium at least one individual's consumption is a utility maximizer.

Proposition 10. *If the economy $\mathcal{E}$ satisfies Assumptions A and S, then*

$$\sum_{i\in\mathcal{I}} \eta_i \in \operatorname{Int}\left(\sum_{i\in\mathcal{I}} \Xi_i - Y\right).$$

Proof. By Assumption A, consumption sets X_i are convex. By Assumption S, $\operatorname{Int} X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Thus, $\operatorname{Int}\sum_{i\in\mathcal{I}} X_i = \sum_{i\in\mathcal{I}} \operatorname{Int} X_i$. Then, by Assumption S, there exists $t > 0$ small enough and $\underline{x} = (\underline{x}_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \operatorname{Int} X_i$ such that $\sum_{i\in\mathcal{I}} e_i - t\mathbf{1}_L \in \mathbb{R}^L_{++}$ and $\sum_{i\in\mathcal{I}} e_i - t\mathbf{1}_L = \sum_{i\in\mathcal{I}} \underline{x}_i$, where $\mathbf{1}_L$ is the vector of $\mathbb{R}^L$ with all its entries equal to 1. Consider now the following open set:

$$A = \left\{ r \in \mathbb{R}^L_{++} \mid r \gg \sum_{i\in\mathcal{I}} e_i - t\mathbf{1}_L \right\} \times \left(\prod_{j\in\mathcal{I}} \operatorname{Int} X_j \right)^I .$$

For all $i \in \mathcal{I}$, pick $\widetilde{x}_i \in \prod_{j\in\mathcal{I}} \operatorname{Int} X_j$ and r such that $\left(r, (\widetilde{x}_i)_{i\in\mathcal{I}}\right) \in A$. For all $i \in \mathcal{I}$, set $\xi_i = (0_L, 0_{LI}, \ldots, 0_{LI}, \widetilde{x}_i, 0_{LI}, \ldots, 0_{LI})$, and $y = \left(-r, \underline{x}, \ldots, \underline{x}, \ldots, \underline{x}\right)$. Thus $\sum_{i\in\mathcal{I}} e_i - t\mathbf{1}_L = \sum_{i\in\mathcal{I}} \underline{x}_i \ll r$ implies that $y \in Y$, and then

$$\left(r, \widetilde{x}_1 - \underline{x}, \ldots, \widetilde{x}_i - \underline{x}, \ldots, \widetilde{x}_I - \underline{x}\right) = \sum_{i\in\mathcal{I}} \xi_i - y \in \sum_{i\in\mathcal{I}} \Xi_i - Y.$$

Let Φ be the affine mapping from $\mathbb{R}^L \times \mathbb{R}^{(LI)I}$ to itself defined by

$$\Phi\left(r, \widetilde{x}_1, \ldots, \widetilde{x}_i, \ldots, \widetilde{x}_I\right) = \left(r, \widetilde{x}_1 - \underline{x}, \ldots, \widetilde{x}_i - \underline{x}, \ldots, \widetilde{x}_I - \underline{x}\right).$$

The mapping Φ is clearly one-to-one and onto and Φ^{-1} is continuous. Thus, Φ is an open mapping, that is, the image of an open set is an open set. Therefore $\Phi(A)$ is contained in $\sum_{i\in\mathcal{I}} \Xi_i - Y$, so in its interior. Furthermore,

$$\sum_{i\in\mathcal{I}} \eta_i = \left(\sum_{i\in\mathcal{I}} e_i, 0_{LI}, \ldots, 0_{LI}, \ldots, 0_{LI}\right) = \Phi\left(\sum_{i\in\mathcal{I}} e_i, \underline{x}, \ldots, \underline{x}, \ldots, \underline{x}\right) \in \Phi(A).$$

Hence $\sum_{i\in\mathcal{I}} \eta_i$ belongs to the interior of $\sum_{i\in\mathcal{I}} \Xi_i - Y$. □

5.4. *Statement of the existence theorem*

The existence of an equilibrium with markets for externalities of $\mathcal{E}$ is the consequence of Proposition 5, Theorem 6, Propositions 7 and 10.

Theorem 11 *(Existence of an equilibrium with markets for externalities). Under Assumptions A, E and S, the economy $\mathcal{E}$ has an equilibrium with $p^* \geq 0$.*

Data availability

No data was used for the research described in the article.

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References

Arrow, K.J., 1969. The organization of economic activity: issues pertinent to the choice of market versus non-market allocation. In: The Analysis and Evaluation of Public Expenditures: The PPB System. Joint Economic Committee, Congress of the United States. Washington, D.C., pp. 47–64.

Bergstrom, T.C., 1976a. How to discard 'free disposability' - at no cost. J. Math. Econ. 3, 131–134.

Bergstrom, T.C., 1976b. Collective choice and the lindahl allocation method. In: Lin, S.Y. (Ed.), Theory and Measurement of Economic Externalities. Academic Press, New York, pp. 111–131.

Conley, J.P., Smith, S.C., 2005. Coasian equilibrium. J. Math. Econ. 41, 687–704.

Crès, H., 1996. Symmetric smooth consumption externalities. J. Econ. Theory 69, 334–366.

del Mercato, E.L., Florenzano, M., 2009. Externalities: Equilibrium and Optimality. In: 9th SAET Conference on Current Trends in Economics 2009. Ischia, Italy, pp. 1–23. Mimeo.

del Mercato, E.L., 2010. Markets for Externalities à la Arrow-Laffont: Equilibrium and Optimality. In: Summer Workshop in Economic Theory SWET 2010, University Paris 1 Panthéon-Sorbonne, and 13th SAET Conference on Current Trends in Economics 2013. MINES Paris Tech. France, pp. 1–28. Mimeo.

Foley, D.K., 1970. Lindahl's solution and the core of an economy with public goods. Econometrica 38, 66–72.

Florenzano, M., 2003. General Equilibrium Analysis: Existence and Optimality Properties of Equilibria. Kluwer Academic Publishers, Boston.

Geistdorfer-Florenzano, M., 1982. The Gale-Nikaido-Debreu lemma and the existence of transitive equilibrium with or without the free-disposal assumption. J. Math. Econ. 9, 113–134.

Hart, O., Kuhn, H.W., 1975. A proof of the existence of equilibrium without the free disposal assumption. J. Math. Econ. 2, 335–343.

Laffont, J.-J., 1976. Decentralization with externalities. Eur. Econ. Rev. 7, 359–375.

McKenzie, L.W., 1959. On the existence of general equilibrium for a competitive market. Econometrica 27, 54–71.

McKenzie, L.W., 1961. On the existence of general equilibrium: some corrections. Econometrica 29, 247–248.

McKenzie, L.W., 1981. The classical theorem on existence of competitive equilibrium. Econometrica 49, 819–841.

Samuelson, P.A., 1954. The pure theory of public expenditure. Rev. Econ. Stat. 36, 387–389.

Starrett, D.A., 1972. Fundamental nonconvexities in the theory of externalities. J. Econ. Theory 4, 180–199.