Contributions to the discussion of K.J. Arrow's paper on
"The Possibility of a Universal Social Welfare Function"

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The purpose of the present document is to contribute to the discussion of Arrow's paper in two ways: a) by reformulating some of the basic axioms and some of the "consequences" in non-technical language (Part I). This task has been carried out in consultation with Arrow and making use of various suggestions advanced during the first discussion of the paper; b) by formulating certain criticisms for further discussion (Part II).

Part I. Non-technical reformulation of certain basic points.

A. Reformulation of the three basic axioms to be satisfied by the social welfare function (S.W.F.) (Arrow's paper, pages 9 and 10).

Condition I. The S.W.F. must give us a consistent ranking (weak ordering) of the alternatives confronting the community for every possible concrete form of the preference relations $R_1 \ldots \ldots R_n$ of the n individuals composing the community.

Condition 2. Consider two alternative sets of individual tastes: the first set of tastes is expressed by the individual rankings $R_1 \ldots \ldots R_n$; the second set is expressed by the rankings $R_1' \ldots \ldots R_n'$. Suppose the second set R_i' has the property that (a) the ranking of all alternatives, excluding a certain alternative x, is the same as under R_i and (b) the position of the alternative x, relative to every other alternative, is not lower than under R_i. Then, if the S.W.F. is such that under the set of tastes R_i it prescribes that x shall be preferred to y, we require that under the set of tastes R_i', it shall also prescribe that x is preferred to y.

Condition 3. Consider again two possible sets of individual tastes expressed by the systems of rankings R_i and R_i'. Suppose that for a sub-set of alternatives S, the individual rankings implied in R_i coincide with the rankings implied in R_i', while for alternatives not included in S the rankings may be the same or not. Then as long as we are concerned only with the subset of alternatives S we require that the S.W.F. be such as to prescribe the choice of the very same alternative (or alternatives) under the set of tastes R_i as under the set of tastes R_i'.

Condition 4. For the purpose of the discussion it is suggested that the original condition 4 be replaced by the condition 4' as given in the text.

B. Reformulation of certain definitions and consequences.

Definition 10 (p. 11). A group V of individuals is said to be decisive for alternative x against y if the S.W.F. prescribes that x shall be preferred to y whenever all the individual members of V prefer x to y.

Consequence 1 (p. 11). For a set V of individuals to be decisive for x against y it is sufficient that the S.W.F. prescribe that x is preferred to y whenever every member of V prefers x to y even if every individual not in V prefers y to x.

Consequence 2. A social welfare function satisfying the postulates has the property that for every x and y there must be a decisive set for x against y.

Proof: If the S.W.F. is not conventional there must be at least one possible set of tastes $R_1 \ldots R_n$, such that the S.W.F. will prescribe the choice of x against y. Let V be the group of individuals that under this set of tastes prefer x to y or are indifferent between x and y. Consider now another set of tastes R_i' such that every member of V now actually prefers x to y. This implies that for this group the rank of x relative to y has either remained unchanged or has increased (since some that were indifferent now

prefer x to y). Then even if every individual not in V prefers again y to x

under the second set of tastes R_i', the social welfare function must prescribe

the choice of x against y by Condition 2. Therefore the group V is decisive

for x against y, by consequence 1 (and by condition 3).

Consequence 4 (p. 12). This is the consequence which I have tried to

disprove during the first discussion by presenting a counter example. I have

become convinced that the counter example is not valid but I believe it will

be useful to see just why it fails. For this purpose it is sufficient to con-

sider only the first part of consequence 4, namely:

Consequence 4.I. If V' is decisive for x against y the V' is decisive

for x against z.

The hypothesis states: 1) xPy if 2) xP_1y even if 3) yP_ix.

The consequence states: 4) xPz if 5) xP_1z even if 6) zP_ix.

Nothing is stated so far as to the relation between y and z either in the S.W.F.

or for individual 1 or for every other individual. If we assume with Arrow

that IA) yP_1z and IIA) yP_iz, then it follows by consequence 3 that IIIA) yPz

and from 1) above and condition 1 it follows that IVA) xPz even if 6) holds.

This is Arrow's proof.

In my counter example I assumed instead IIM) zP_iy and further assumed

that the S.W.F. then specifies IIIM) zPy; I further accepted 6) and assumed

that the S.W.F. then specifies IVM) zPx. Under these conditions it would

follow that individual 1 is not decisive for x against z, since IVM holds

though 5) holds.

The failure of my counter example is as follows. IVM and 5) imply that

the set V* of all individuals but individual 1 is decisive for z against x.

But a.S.W.F. prescribing this will violate one of the conditions. In fact

suppose that the set of tastes of the group V* now changes so that IIA holds instead of IIM. Then by consequence 3 we must have (since everybody prefers y to z) IIIA) yPz; at the same time if V* is decisive for z against x, we must have on account of 6), zPx; but the set of rankings 1) xPy, IIA) yPz, IVM) zPx, contradicts condition 1 (consistent ranking). Thus the assumption that V* and not V' is decisive for z against x is inconsistent with the axioms.

Part II. Criticism.

The main criticism I want to suggest for further discussion refers to the conditions 2 and 3 stated on p. 10 of Arrow's paper and reformulated on p. 1 of this paper.

Admittedly each one of these axioms separately appears very reasonable, yet when taken together they have some very unsatisfactory immediate implications. This may be best illustrated with an example.

Consider a first set of tastes R_i, with the following property: the first m individuals prefer intensely x to y, individual m/1 is indifferent and the remaining individuals prefer y to x but very slightly. Under these conditions, and especially if m is a substantial proportion of the population close to 1/2, it is very reasonable to demand that the S.W.F. should prescribe to choose x over y. Assume that this is in fact prescribed by the S.W.F. Consider now another set of tastes R_i', with the following property: the first m individuals prefer x to y but very slightly, individual m/1 prefers x to y but very slightly, the remaining individuals prefer y to x intensely, let us say that they actually consider x unbearable (I use this expression purposely as a reference to Marschak's remarks about revolutions).

Then if the S.W.F. is to satisfy conditions 2 and 3, under the set of tastes R_i' it must prescribe the choice x over y.

I submit that this is a very unreasonable requirement to impose on the S.W.F. It will be readily appreciated that this example relies heavily on the notions of "intense" like and dislike as contrasted with the simple notion of relative position in a ranking order. I feel that some methods must be found to introduce a weighted ranking. Though Arrow has discussed two examples of such weighted rankings methods on p. 15 and shown that they are unsatisfactory, I strongly feel that other alternatives of this type should be investigated, possibly making use of the notion of "indemnification" of "classical" welfare economics.