Helicopter Drops and Liquidity Traps*

Manuel Amador[†] Javier Bianchi[‡]

April 2023

Abstract

We show that if the central bank operates without commitment and faces constraints on its balance sheet, helicopter drops can be a useful stabilization tool during a liquidity trap. In our model, even with balance sheet constraints, helicopter drops are at best irrelevant under commitment.

Keywords: Zero lower bound, helicopter drops, liquidity traps, central bank independence, monetary policy, time consistency problem

JEL: E31, E52, E58, E61, E63

*We thank V.V.Chari and Anmol Bhandari for conversations that led us to write this paper. We also thank Mark Aguiar and Marco Bassetto for helpful discussions. Sang Min Lee provided excellent research assistance. The views expressed here are entirely those of the authors. They do not necessarily represent the views of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

[†]University of Minnesota, Federal Reserve Bank of Minneapolis, and NBER

[‡]Federal Reserve Bank of Minneapolis

1 Introduction

During the 2008 global recession, nominal interest rates in the developed world dropped to zero. Lacking the ability to further reduce nominal rates, central banks utilized new policy instruments to manage the ongoing liquidity trap. Among these, quantitative easing and forward guidance eventually became a part of the monetary policy toolkit.

When nominal rates again reached zero during the recession generated by the COVID pandemic, central banks continued to rely on these new instruments. Yet, the extreme downturn and questions about the adequacy of central banks' policy toolkit generated a quest for further policy options.[1] One that has received substantial attention is "helicopter drops"—that is, a policy by which a central bank prints money and gives it to the public.[2] The unorthodox feature is that unlike conventional monetary policy operations, the use of helicopter drops involves an increase in the central bank's monetary liabilities without a corresponding increase in its assets.

According to standard economic theory, however, helicopter drops are irrelevant in a liquidity trap. At zero nominal interest rates, money and bonds are perfect substitutes from the point of view of the private sector. Ricardian equivalence thus implies that when the central bank prints money and provides transfers, households keep higher money balances without inducing any changes in real or nominal variables. The result of the irrelevance of helicopter drops closely mirrors the classic irrelevance of open market operations at the zero lower bound (Wallace, 1981; Eggertsson and Woodford, 2003). Hence, a helicopter drop, as radical as it sounds, is just as ineffective as a conventional open market operation in a liquidity trap.

In this paper, we provide a model of how helicopter drops can be effective during a liquidity trap. At the heart of our model is an explicit separation between the fiscal authority's and monetary authority's budget constraints. We show that when the monetary authority faces balance sheet constraints and lacks commitment to future policies, there is a role for helicopter drops during a liquidity trap.

We consider a simple New Keynesian model, in which the central bank is required, as it is in practice, to remit excess earnings to the fiscal authority. In addition, there are no capital injections from the fiscal authority, and the central bank is limited in its ability to borrow using interest-bearing securities. We first consider the set of private sector equilibria for a given set

[1] Regarding forward guidance, the current Chair of the Federal Reserve, Jay Powell, noted in a speech in 2019 that: "Part of the problem is that when the time comes to deliver the inflationary stimulus, that policy is likely to be unpopular—what is known as the time consistency problem in economics."

[2] The term helicopter drops was coined by Friedman (1969) as a parable to describe monetary injections. For recent proposals advocating helicopter drops, see, for example, Bartsch, Boivin, Fischer, Hildebrand and Wang (2019) and Gali (2020). Earlier, in 2002, Ben Bernanke at the time a member of the Federal Reserve's Board of Governors, also suggested that this policy would be an option in the context of Japan's liquidity trap (Bernanke, 2002). For discussions in the financial media, see e.g., The Economist, 2016.

of policies. We show that in this environment, any private sector equilibrium that arises from a policy with a helicopter drop remains an equilibrium with an alternative policy without the helicopter drop, even in the presence of balance sheet constraints. Hence, when the central bank can commit to future policies, there is no scope for helicopter drops. Under commitment, the central bank can already credibly promise expansionary policies in the future to help mitigate the recession originating from the liquidity trap (Krugman, 1998; Eggertsson and Woodford, 2003; Jung, Teranishi and Watanabe, 2005; Werning, 2011). Resorting to helicopter drops would only inefficiently limit the central bank's ability to stabilize output and inflation in the future. Thus, helicopter drops should not be used under commitment.

We then examine the policy equilibria when the central bank cannot commit, which is our main focus. We study a Markov perfect equilibrium in which the central bank at each point in time chooses the best available policy, taking as given its inherited states (including its balance sheet) and how the private sector's expectations react to its policy choices. A Markov equilibrium constrains these private sector expectations to be consistent with the central bank's future behavior. Using both theoretical and numerical results, we argue that a lower net worth pushes the central bank towards a more expansionary policy (i.e., higher output and inflation). The reason is as follows. To reduce monetary liabilities, the monetary authority has to either sell assets or reduce the transfers it makes to the fiscal authority. Because the monetary authority is constrained to transfer at a minimum its excess earnings and has a limited ability to borrow, it faces a limit to how much it can reduce its monetary liabilities. A lower net worth implies that the monetary authority needs to keep higher nominal balances to satisfy the balance sheet constraints, and this results in lower interest rates and higher levels of output and inflation.

We argue that helicopter drops during a liquidity trap are a credible way for the central bank to commit to an expansionary monetary policy in the future. We highlight that open market operations remain irrelevant in a liquidity trap in our environment. However, when the central bank expands the money supply without a corresponding increase in assets, this generates expectations of lower interest rates going forward, which imply high output and inflation in the future that help mitigate the recession today. In a simulation, the optimal helicopter drop achieves a degree of stabilization that comes close to the optimal policy under commitment. The downside, however, is that helicopter drops induce a more protracted period of inflation.

Related literature. A seminal paper on monetary and fiscal policy interactions is Sargent and Wallace (1981). They show that fiscal policy imposes constraints on the central bank's ability to control inflation. In particular, fiscal deficits force the central bank to eventually collect higher seigniorage revenues to balance the consolidated government budget constraint.

A more recent literature has unbundled the budget constraints of the government and the central bank and studied the implications for monetary policy.[3] Sims (2004) is an early paper exploring the implications of different institutional configurations for price level determinacy. Hall and Reis (2015) take as given that the central bank is committed to an inflation target and evaluate when the central bank faces the risk of becoming insolvent, in the sense of facing an explosive path for liabilities. Del Negro and Sims (2015) study how the central bank balance sheet can determine whether price determinacy can be achieved and the extent to which fiscal support is needed. Bassetto and Messer (2013) examine the fiscal consequences of paying interest on reserves, arguing that although it increases the flexibility of the balance sheet, it can increase the risk of insolvency. Benigno and Nisticò (2020) explore the neutrality of changes in the central bank's balance sheet (given transfer policies for the central bank and treasury) and highlight how lack of fiscal support may constrain the central bank's choices when it can suffer losses. We instead focus on the case where the Central Bank lacks commitment, and ask whether the Central Bank will voluntarily weaken its net worth through a helicopter drop.

There is a large literature on Central Bank policy during a liquidity trap. A key insight from this literature (Krugman, 1998; Eggertsson and Woodford, 2003; Jung et al., 2005; Werning, 2011) is that the Central Bank should commit to delivering high inflation and output in the future (after the liquidity trap is over) to help mitigate the contraction in output when the zero lower bound constraint binds. Without commitment, however, this policy is not feasible, as the government would find it optimal to stabilize output and inflation when the liquidity trap is over.

Our paper is more directly related to those exploring the role of helicopter drops during liquidity traps. Auerbach and Obstfeld (2005) and Galí (2019) show that money-financed stimulus can be expansionary at the zero lower bound. However, it is assumed that money supply remains high beyond the period of the liquidity trap, a policy that is not necessarily time-consistent. Benigno and Nisticò (2022) consider a model where the fiscal and the monetary authorities are separated (as they are here) and where the fiscal theory of the price level is operative. They study how, under commitment, helicopter drops can reflate the economy. Also in a setup with commitment, Michau (2022) analyzes the effects of helicopter drops in a model with secular stagnation and where households have a utility preference for wealth.[4] In contrast to these contributions, our paper focuses on the time-consistency problem for the central bank in a liquidity trap and shows that helicopter drops of money, by constraining the central bank's future behavior, can be a credible way to raise expectations of higher inflation and output in the future.

[3]See Bassetto and Sargent (2020) for a review of the research on the interactions between monetary and fiscal policy.

[4]In the context of models with heterogeneous agents, studies such as Buera and Nicolini (2020) show that a transfer to constrained agents financed by printing money can have stimulative effects. However, the same effects can be achieved with issuing debt, a manifestation of the classic irrelevance of open market operations at the zero lower bound.

Finally, our paper is related to the literature exploring the connection between the maturity structure of the government debt and the time consistency of optimal fiscal and monetary policy. In the context of a real model, Lucas and Stokey (1983) argue that a carefully chosen maturity structure for government debt can make optimal fiscal and monetary policies time consistent when taxation is costly. Alvarez, Kehoe and Neumeyer (2004) study an economy with nominal debt and show that a portfolio can be chosen to deliver time-consistent policies if the Friedman rule is optimal. Calvo and Guidotti (1992) show that a lower level of nominal government debt can help reduce the temptation to inflate ex post. Closer to our paper is the work of Bhattarai, Eggertsson and Gafarov (2022), who study quantitative easing in a liquidity trap where the problem is that the lack of commitment generates deflationary bias. They show that by shortening the maturity of the consolidated government debt, quantitative easing provides incentives for the central bank to slow down the exit from the liquidity trap because it is averse to losses in its balance sheet. In our model, the central bank does not suffer losses on its balance sheet but intentionally engineers a reduction in net worth through a helicopter drop.[5]

The paper is organized as follows. Section 2 presents the model. Sections 3 and 4 present the analysis of optimal monetary policy with and without commitment. Section 5 concludes.

2 Model

We consider a deterministic closed economy New Keynesian model with a representative agent, a single final consumption good, a monetary authority, and a fiscal authority. Aside from the monetary authority's balance sheet constraints that we introduce, the model is entirely standard.

Monetary Authority. We start by considering the balance sheet of the monetary authority. The monetary authority issues monetary liabilities M_t, accumulates a nominal risk-free asset A_t, which pays a nominal interest rate ι_t, and makes nominal transfers τ_t to the fiscal authority.[6] The budget constraint of the monetary authority is given by

$$\frac{A_t}{1+\iota_t} + M_{t-1} + \tau_t = M_t + A_{t-1}. \tag{1}$$

[5]Also related is Jeanne and Svensson (2007), which shows how purchases of foreign currency assets can help avoid a liquidity trap when the central bank faces a lower bound on the level of capital. Amador, Bianchi, Bocola and Perri (2016) show how a large currency mismatch in a central bank can provoke an early abandonment of a floor on the exchange rate, because of the risk of facing losses.

[6]Alternatively, we could allow the monetary authority to make transfers directly to households. Because the model features Ricardian properties in terms of debt by the fiscal authority, this alternative is equivalent to the current one.

Note that we have written the risk-free assets as zero-coupon bonds. Correspondingly, let us define the nominal beginning-of-period net worth of the monetary authority as $N_t \equiv A_t - M_t$. Using this definition, we have that,

$$N_t = N_{t-1} + \frac{\iota_t}{1+\iota_t} A_t - \tau_t.$$

Let $i_t = \log(1 + \iota_t)$ be the corresponding instantaneous rate of interest, and we let

$$\tau^\star(A, i) \equiv A(1 - e^{-i}).$$

Then, $N_t < N_{t-1}$ if and only if $\tau_t > \tau^*(A_t, i_t)$. The value of τ^* denotes the nominal net gains from holding financial assets ($\iota \times A$) relative to the nominal cost of liabilities ($0 \times M$). When the monetary authority remits more than τ^* to the fiscal authority, the net worth of the monetary authority falls in nominal terms. Conversely, if remittances are lower than τ^*, the net worth increases in nominal terms.

We assume that the monetary authority needs to remit at least $\tau^\star$ every period:

$$\tau_t \geq \tau^\star(A_t, i_t), \text{ for all } t, \tag{2}$$

which implies that net worth cannot increase. In effect, this also means that the monetary authority cannot count on equity injections by the fiscal authority. We also assume the only liability of the monetary authority is the monetary base, M_t. That is,

$$A_t \geq 0, \text{ for all } t. \tag{3}$$

These two constraints, (2) and (3), imply that there is a lower bound on monetary liabilities given by

$$M_t \geq -N_{t-1}. \tag{4}$$

According to (4), lower net worth forces the monetary authority to keep a higher level of monetary liabilities. The reason is as follows. If the monetary authority would like to reduce M_t, it has to either sell assets or reduce the transfers it makes to the fiscal authority. Because the monetary authority is constrained to pay the minimum transfer and has to hold a minimum amount of assets, it faces a limit to how much it can reduce its monetary liabilities.[7] In particular, selling all

[7]Notice that we could have allowed for a borrowing limit different from zero in eq. (3) (i.e., we could have written it as $A_t/P_t \geq \bar{a}$ for given $\bar{a}$). In that case, the constraint (4) would be $M_t \geq -N_{t-1} + \bar{a}P_t$ and the results would be similar, although the reduction in net worth necessary for (4) to bind would be larger.

its assets and paying the minimum transfer τ^*, it can reduce the monetary liabilities up to $-N_{t-1}$.[8]

We refer to constraints (2) and (3) as the "balance sheet constraints", and they constitute the key departure from a standard model. Absent either of these two constraints on the monetary authority, the model would have in effect a consolidated budget constraint for the government. However, there is extensive evidence that balance sheet constraints on central banks can actually have implications for monetary policy, an issue raised in early work by Stella (1997, 2005). As they note, countries generally have different accounting rules that govern the required level of capital for the central bank, the rules for dividends, and remittances, and these in turn, determine the ability to operate with a low level of capital or to deal with operating losses.[9] Our modeling assumptions attempt to capture these frictions between the fiscal and monetary authorities. Some recent literature (e.g., Hall and Reis, 2015) has focused on the effects of central bank losses. We note that in our analysis, the monetary authority always makes non-negative profits. This is due to the fact that we do not have valuation effects emerging from long-term assets or interest rate payments on the monetary liabilities.[10] Following the language of Benigno and Nisticò (2020), constraint (2) implies that the monetary authority is *financially independent*, and balance sheet policies are not necessarily neutral.

Fiscal Authority. There is a fiscal authority that collects the revenue received from the monetary authority. In addition, it issues bonds and makes lump-sum transfers/taxes to households. Let B_t denote the level of debt of the fiscal authority. The fiscal authority budget constraint is

$$\frac{B_t}{1+\iota_t} + \tau_t = T_t + B_{t-1}, \tag{5}$$

where T_t represents the lump-sum transfers to households.

[8] Given that we abstract from reserves held by commercial banks, we do not consider a policy of interest on reserves. Allowing for payments on interest on reserves can help shift the composition of monetary liabilities away from currency in circulation, thus achieving higher interest rates for a given net worth. Even though such an instrument grants more flexibility to the monetary authority facing balance sheet constraints, there are also potential costs and limits to its use. For example, given constraints on banks' leverage, raising interest on reserves can lead to a contraction in bank credit (see Bianchi and Bigio, 2022). We leave these issues for future research.

[9] In the US, for example, the Federal Reserve Act requires the Reserve Banks to remit every year excess earnings after expenses to the U.S. Treasury. When these excess earnings turn negative, a so-called "deferred asset" is accumulated. After that, no remittance is made until earnings, through time, have been sufficient to cover that loss. However, a large accumulation of the deferred asset would eventually call for fiscal support from the Treasury and potentially compromise the independence of the Federal Reserve.

[10] Our analysis hinges on the requirement to transfer operating profits to the fiscal authority and a lower bound on asset holdings (i.e, constraints (2) and (3)). Introducing additional elements to the model is left for future work.

Households. There is a representative household that has the following preferences over consumption and real money balances:

$$\sum_{t=0}^{\infty} \beta(t) \left[u(C_t) + v\left(\frac{M_t}{P_t}\right) \right]$$

with $\beta(t) = \Pi_{s=0}^{t} e^{-(\rho+\xi_{s-1})} < 1$ the discount factor from period 0 to t. The value of ξ_t is a time-varying exogenous disturbance to the discount rate (and, as in Werning, 2011, this is the only non-stationary parameter and the reason for a liquidity trap). The function u is assumed to be strictly increasing, strictly concave, and differentiable in $\mathbb{R}^{+}$. The function v is differentiable in $\mathbb{R}^{+}$; strictly increasing, and strictly concave in $[0, \overline{m}]$; reaching a satiation point at $\overline{m}$: $v(m) = v(\overline{m})$ for $m > \overline{m}$.

The household faces the following sequential budget constraint:

$$P_t C_t + M_t + \frac{W_t}{1+\iota_t} \leq M_{t-1} + W_{t-1} + T_t + P_t Y_t, \tag{6}$$

where P_t is the price level, C_t is the households' real consumption, Y_t is the households' real income (composed of real labor income and profits), and W_t is the financial wealth net of money holdings.

The household is subject to the following No-Ponzi condition:[11]

$$\lim_{t\to\infty} \frac{W_t + M_t}{\prod_{s=0}^{t}(1+\iota_s)} \geq 0. \tag{7}$$

The household's problem is to maximize utility subject to the budget constraint, (6), and the No-Ponzi condition, (7), taking prices and transfers as given. Letting $\pi_t \equiv \log(P_t/P_{t-1})$, we have the following sufficiency result.[12]

Lemma 1 (Sufficiency)**.** *Take a sequence of prices and transfers $\{P_t, \iota_t, T_t\}$ and initial conditions P_{-1}, W_{-1}, ι_{-1}. Let $\{C_t, M_t, W_t\}$ be such that (i) $C_t > 0$ for all $t \geq 0$, (ii) the household budget constraint (6) holds, (iii) the No-Ponzi condition (7) holds, (iv) the following first-order conditions (an Euler equation,*

[11] Buiter and Sibert (2007) propose to use an alternative No-Ponzi condition that does not include money holdings. Part of their argument is a potential difficulty with obtaining sufficiency results for the household problem. We show in Lemma 1 that this is not an issue.

[12] We will impose the sufficient conditions of this Lemma as requirements for household optimality. For the necessity of the transversality condition, we refer the reader to Kamihigashi (2002).

and a money demand condition)

$$u'(C_t) = e^{i_t - \rho - \xi_t - \pi_{t+1}} u'(C_{t+1}),$$
$$v'\left(\frac{M_t}{P_t}\right) = u'(C_t)\frac{\iota_t}{1+\iota_t}$$

hold, and (v) the following transversality condition

$$\lim_{t\to\infty} \frac{W_t + M_t}{\prod_{s=0}^{t}(1+\iota_s)} \le 0,$$

holds. Then the sequence $\{C_t, M_t, W_t\}$ *solves the household's problem.*

Proof. In the Appendix. □

Note that we can collapse the No-Ponzi condition and the transversality condition into just one condition at an optimal solution:

$$\lim_{t\to\infty} \frac{W_t + M_t}{\prod_{s=0}^{t}(1+\iota_s)} = 0, \tag{8}$$

Firms. The firms' side of the model is in the standard New Keynesian tradition. The model features monopolistic competition and Calvo-style sticky prices. Because these features are standard, we work directly with the log-linearized version of the price setting equation (Phillips curve):

$$\pi_t = \beta\pi_{t+1} + \kappa(y_t - \bar{y}), \tag{9}$$

where $\beta \equiv e^{-\rho}$ and for some parameters $\bar{y}$ and $\kappa \ge 0$. For simplicity, we set $\bar{y} = 0$ in the rest of the analysis.

Market Clearing. In an equilibrium, the goods market clears $Y_t = C_t$. In what follows, we assume that $u(C) = \frac{C^{1-1/\sigma}}{1-1/\sigma}$. Letting $y_t \equiv \log(Y_t)$, we have that the household Euler equation can be written as:

$$y_t = y_{t+1} - \sigma(i_t - \pi_{t+1} - \rho - \xi_t), \tag{10}$$

which is the usual log-linear Euler equation (or the dynamic IS curve).

The first order condition for money holdings requires that $\iota_t \ge 0$. Let h denote the inverse of

v', and using i_t instead of ι_t, we let

$$L(c,i) \equiv h\left(u'(e^c)(1-e^{-i})\right),$$

which is defined for $i \geq 0$, increasing in c, decreasing in i, and with $L(c,0) = \overline{m}$.

Thus, the money market clearing condition, together with $C_t = Y_t$, is equivalent to

$$\frac{M_t}{P_t} \geq L(y_t, i_t); \text{ with equality if } i_t > 0. \tag{11}$$

2.1 Private Sector Equilibrium.

Let us define a private sector equilibrium. The economy starts at $t = 0$ with the monetary authority's balance sheet given by A_{-1} and M_{-1}, an initial fiscal authority debt level B_{-1}, and an initial price level P_{-1}.[13]

Definition 1. A *private sector equilibrium* is a sequence $\{y_t, \pi_t, i_t, P_t, M_t, A_t, B_t, \tau_t, T_t\}$ such that for all $t \geq 0$ and

(i) households optimize; that is, (6), (8), (10), (11), hold for $W_t = B_t - A_t$ for all $t \geq -1$.

(ii) firms optimize; that is, (9) holds.

(iii) the zero lower bound constraint holds, $i_t \geq 0$.

(iv) the budget constraint of the monetary authority, (1), holds.

(v) the budget constraint of the fiscal authority, (5), holds.

From the definition of equilibrium, it follows that the monetary authority's balance sheet does not restrict the private equilibrium set for $\{y_t, \pi_t, i_t\}$. To see this, take a sequence $\{y_t, \pi_t, i_t\}$ that satisfies (10), (9), and the zero lower bound. Then, we can use the initial price level to solve for the corresponding $\{P_t\}$ and use the condition (11) with equality to recover a sequence for $\{M_t\}$ that satisfies money market clearing. With this, we can then obtain $\{A_t, B_t\}$ by setting any combination such that $(A_t - B_t) = M_t$ for all t large enough, which guarantees that the transversality condition, (8), holds. Finally, the implied transfers and taxes make sure that all the budget constraints hold. This is a standard result, and justifies the common focus on just two equations, the Euler equation, (10), and the price setting equations, (9), together with the zero lower bound constraint when studying optimal policy.

[13] In what follows, we use the notation $\{a_t, b_t, ...\}$ to refer to the sequences $\{a_t\}_{t=0}^{\infty}$, $\{b_t\}_{t=0}^{\infty}$, etc.

Helicopter Drops and Balance Sheet Constraints. We can rewrite the budget constraint of the monetary authority as follows:

$$(M_t - M_{t-1}) = (A_t - A_{t-1}) + (\tau_t - \tau^\star(A_t, i_t)).$$

Thus, a change in the monetary base can be decomposed in a change in A, and a deviation of τ from $\tau^\star$. We will call the first component an "open market operation" and the second, a "helicopter drop." This means that if the monetary authority wants to increase M_t, without a helicopter drop, it needs to engage in an open market operation and purchase an amount of nominal assets equal to the increase in the money supply. Alternatively, the monetary authority could increase M without engaging in asset purchases by simply increasing τ above $\tau^\star$. This leads to our next definition:

Definition 2 (Helicopter Drops)**.** A private sector equilibrium features *a helicopter drop* at time t if $\tau_t > \tau^\star(A_t, i_t)$.

Recall that $N_t = A_t - M_t$; thus, a helicopter drop necessarily reduces the nominal net worth of the monetary authority. Note also that helicopter drops have no impact on the set of allocations $\{y_t, \pi_t, i_t\}$ consistent with private sector equilibria. As we discussed above, for any path of $\{y_t, \pi_t, i_t\}$ that satisfies (10), (9), and the zero lower bound, any combination of $\{A_t, B_t\}$ such that $(A_t - B_t) = M_t$ where $\{M_t\}$ satisfies (11) is consistent with a private sector equilibrium. It turns out that helicopter drops also have no effect on the set of private sector equilibria in the presence of balance sheet constraints. To see this, let us first define what we mean by an equilibrium consistent with balance sheet constraints:

Definition 3. A private sector equilibrium $\{y_t, \pi_t, i_t, P_t, M_t, A_t, B_t, \tau_t, T_t\}$ *is consistent with balance sheet constraints* if equations (2) and (3) hold for all t.

We now show that the presence of helicopter drops does not enlarge the set of allocations that are consistent with equilibrium *and* balance sheet constraints.

Lemma 2 (Irrelevance of Helicopter Drops)**.** *Consider a private sector equilibrium* $\{y_t, \pi_t, i_t, P_t, M_t, A_t, B_t, \tau_t, T_t\}$ *that is consistent with balance sheet constraints. Let* $\{\hat{\tau}_t, \hat{A}_t, \hat{B}_t\}$ *be such that* $\hat{\tau}_t = \tau^\star(\hat{A}_t, i_t)$, $\hat{A}_t = M_t - M_{-1} + A_{-1}$, *and* $\hat{B}_t = \hat{A}_t - A_t$. *Then* $\{y_t, \pi_t, i_t, P_t, M_t, \hat{A}_t, \hat{B}_t, \hat{\tau}_t, T_t\}$ *is also a private sector equilibrium consistent with balance sheet constraints.*

Proof. Given $\hat{\tau}_t$ and $\hat{A}_t$, the budget constraint of the monetary authority is

$$\hat{A}_t = M_t - M_{t-1} + \hat{A}_{t-1} = M_t - M_{-1} + A_{-1},$$

where the last follows from repeated substitution of the value of $\hat{A}$.

For the balance sheet constraints, letting $\hat{A}_{-1} = A_{-1}$, we have

$$\hat{A}_t - \hat{A}_{t-1} = M_t - M_{t-1} \geq M_t - M_{t-1} - (\tau_t - \tau^\star(A_t, i_t)) = A_t - A_{t-1},$$

where we use that the original sequence satisfies $\tau_t \geq \tau^\star(A_t, i_t)$. Thus $\hat{A}_t - \hat{A}_{t-1} \geq A_t - A_{t-1}$. Given that $\hat{A}_{-1} = A_{-1}$, it follows that $\hat{A}_t \geq A_t$ and thus condition (3) holds at the new sequence given that it holds at the old. For the new sequence, condition (2) holds with equality. Thus, the balance sheet constraints are satisfied.

The budget constraint of the fiscal authority holds, given the construction of $\hat{B}_t$. Given that $B_t - A_t = \hat{B}_t - \hat{A}_t$. the transversality condition (8) holds given that it was holding at the initial equilibrium.

The new sequence then satisfies all of the conditions for a private sector equilibrium. □

The above Lemma tells us that if there is a helicopter drop at some t (that is, $\tau_t > \tau^\star$), then it is possible to find an alternative policy with $\tau_t = \tau^\star$ that is also consistent with equilibrium conditions and where the balance sheet constraints are satisfied. For example, suppose that starting from an original equilibrium, the monetary authority has increased M_t at some time t while simultaneously setting $\tau_t > \tau^\star$. Such a policy reduces the monetary authority's net worth at time t (this is a "helicopter drop"). Consider instead an alternative in which the monetary authority conducts an open market operation at time t; that is, it increases M_t by the same amount but uses the proceeds to purchase assets, A_t, while leaving $\tau_t = \tau^\star$. The lemma above establishes that this new policy is also consistent with the same original equilibrium outcome for $\{y_t, \pi_t, i_t, P_t, M_t\}$ and consistent with balance sheet constraints.[14]

We highlight here that the reverse of Lemma 2 (that a helicopter drop can be introduced without requiring a change in the equilibrium allocation) does not generally hold. To the extent that a helicopter drop generates a reduction in net worth, this may lead to a violation of the balance sheet constraint (4), thus implying that the original allocation cannot longer be part of an equilibrium.

So helicopter drops do not expand the set of private sector equilibria, even in the presence of balance sheet constraints. Do balance sheet constraints matter at all for the set of private

[14]In the Lemma, the fiscal authority changes B_t to adjust the budget. Alternatively, we could have the fiscal authority to arbitrarily change T_t. This would also work as long as the transversality condition of the households remains satisfied after the change.

sector equilibria?[15] Take any allocation $\{y_t, \pi_t, i_t\}$ that satisfies the Euler equation, the price setting equation, and the zero lower bound constraint. Then, as long as the monetary authority is well capitalized, we can find a policy under which this allocation constitutes a private sector equilibrium.

Lemma 3 (Balance-sheet irrelevance with sufficient initial net worth)**.** *Suppose that $N_{-1} = A_{-1} - M_{-1} \geq 0$. Consider a sequence $\{y_t, \pi_t, i_t\}$ that satisfies the Euler equation,* (10)*, the price setting equation,* (9)*, and the zero lower bound constraint, $i_t \geq 0$. Let $\{P_t\}$ be the corresponding price level given $\{\pi_t\}$ and P_{-1}. Then, there exists a policy $\{M_t, A_t, B_t, \tau_t, T_t\}$ such that $\{y_t, \pi_t, i_t, P_t, M_t, A_t, B_t, \tau_t, T_t\}$ is a private sector equilibrium consistent with balance sheet constraints.*

Proof. The proof is constructive. For $t \geq 0$, let

$$\begin{aligned} M_t &= P_t L(e^{y_t}, i_t) \\ A_t &= M_t + A_{-1} - M_{-1} \\ \tau_t &= \tau^\star(A_t, i_t) \\ B_t &= A_t - M_t \\ T_t &= \frac{B_t}{1+\imath_t} + \tau_t - B_{t-1} \end{aligned}$$

Condition (i) of the private sector equilibrium definition holds, as the household budget constraint, (6), holds with $W_t = B_t - A_t$, the Euler equation (10) holds by assumption, by construction the money demand condition, (11), holds as well, and the private sector total net financial wealth (inclusive of money holdings) $M_t + B_t - A_t = 0$ for all $t \geq 0$, implying that the transversality condition (8) holds. Conditions (ii) and (iii) hold. And by construction, the budget constraints of the monetary and the fiscal authority hold as well. The sequence constitutes a private sector equilibrium. Given that $A_{-1} - M_{-1} \geq 0$, it follows that $A_t \geq 0$, as $M_t \geq 0$ and (3) holds. Finally, by construction condition (2) holds with equality. Thus, the balance sheet constraints are satisfied. □

Lemma 3 thus tells us that, as long as the monetary authority initial net worth is non-negative, any sequence that satisfies the Euler equation, the price setting equation, and the zero lower bound constitutes a private sector equilibrium for some policy. Lemma 2 in addition tells us (and the proof of Lemma 3 explicitly shows) that it suffices to consider policies without helicopter drops.

[15] Although related, our analysis is different from Benigno and Nisticò (2020) analysis of the neutrality of the monetary authority's balance sheet. Benigno and Nisticò (2020) considers an environment where the transfer policy is fixed and the monetary authority chooses an interest on reserves and an asset portfolio. In this environment, they analyze whether a sequence $\{\pi_t, i_t\}$ is a private sector equilibrium for a given transfer/balance sheet policy. In our case, we examine an endogenous transfer (while abstracting from interest on reserves and a portfolio decision), and we ask instead whether the sequence $\{y_t, \pi_t, i_t\}$ constitutes a private sector equilibrium for at least one such policy.

Note that for these irrelevance results to hold, we require changes to the policies followed by the monetary and fiscal authorities. But we have not yet discussed how these policies are decided and whether the monetary and fiscal authorities will actually follow them. Our next goal is to analyze how policies are chosen. For that, we need to specify the monetary and fiscal policy objectives.

Monetary Policy Objective. We assume that the monetary authority seeks to minimize departures from zero for both inflation and the output gap. Specifically, the monetary authority evaluates welfare according to the following objective function:

$$\sum_{t=0}^{\infty} e^{-\rho t} W(\pi_t, y_t), \tag{12}$$

where W encapsulates the objective of keeping inflation and output close to the target and is strictly maximized at $W(0,0)$.

Fiscal Policy Objective. The fiscal authority is assumed to select a path of $\{B_t, T_t\}$ such that its budget constraint holds at all times, and the consolidated debt of the government satisfies the following condition:

$$\lim_{t\to\infty} \frac{B_t + M_t - A_t}{\prod_{s=0}^{t}(1+\iota_s)} = 0, \tag{13}$$

This implies that the transversality condition of the households will be satisfied. And also implies that the sequence of taxes chosen by the fiscal authority does not affect the rest of the equilibrium. Under this assumption, *the fiscal theory of the price level* would not play a role in our analysis; as the consolidated government sector follows a "Ricardian" fiscal policy.[16]

However, in our analysis of the equilibrium without commitment, we will impose the restriction that the real *assets* of the fiscal authority do not explode:

Condition 1. *In a private sector equilibrium, the fiscal authority does not accumulate unbounded assets on the private sector:*

$$\lim_{t\to\infty} \frac{B_t}{\prod_{s=0}^{t}(1+\iota_s)} \geq 0.$$

As we will see below, in our environment, this restriction rules out the deflationary trap

[16] We follow here Benhabib, Schmitt-Grohé and Uribe (2001a) who refer to a fiscal policy that guarantees that the household transversality condition holds for all possible paths of prices as a "Ricardian policy."

With regards to the fiscal theory of the price level, see Woodford (2001), and Kocherlakota and Phelan (1999) for summaries. For a game-theoretical analysis of the equilibrium selection issues that arise, see Bassetto (2002). For an analysis of helicopter drops under commitment that relies on the fiscal theory of the price level, see Benigno and Nisticò (2022).

equilibria of Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b, 2002) that otherwise will be pervasive in the analysis without commitment.

To see the role that this constraint can play, let us iterate forward on the budget constraint of the monetary authority. Using Condition 1, together with the household's transversality condition, we get that in a private sector equilibrium, we must have that

$$N_{t-1} + \sum_{s=t}^{\infty} \frac{\frac{\iota_s}{1+\iota_s} M_s}{\prod_{\tau=t}^{s-1}(1+\iota_\tau)} \geq \sum_{s=t}^{\infty} \frac{\tau_s}{\prod_{\tau=t}^{s-1}(1+\iota_\tau)} \tag{14}$$

The left-hand side can be thought of as the "equity value" of the monetary authority: net worth plus the future discounted value of "profits" (i.e. seigniorage revenue). This value cannot be lower than the present discounted value of the "dividends", that is, the transfers that the monetary authority makes.

Without Condition 1, the monetary authority's net worth can explode to minus infinity in a private sector equilibrium, even in the presence of balance sheet constraints. In that case, the fiscal authority must accumulate assets to compensate for the ever increasing monetary authority's net liabilities. That is, exploding negative net worth can be part of a private sector equilibrium if the fiscal authority effectively *provides the demand for the monetary authority liabilities*, which is intermediated, in equilibrium, through the household sector. Condition 1 restricts this fiscal behavior, and as a result, (14) generates a lower bound on the monetary authority net worth given that $\tau_s \geq 0$ (and assuming that the present value of seigniorage revenue has an upper-bound).

Note that if transfers τ_t were equal to or larger than the seigniorage revenue at all times, then (14) would imply that net worth could not be negative. However, the minimum remittance constraint, (2) operates on a nominal mark-to-market rule, as in practice.[17] This leaves space for $\tau_s < \iota_s M_s/(1+\iota_s)$, and thus a monetary authority that operates with negative net worth can, in principle, be part of a private sector equilibrium.

It is interesting to note that the irrelevance of the helicopter drops, stated in Lemma 2, remains to hold even in the presence of Condition 1. Inspection of the proof of the lemma shows that the alternative allocation without helicopter drops features a fiscal debt that is weakly higher than the original allocation ($\hat{B}_t \geq B_t$ for all t), and thus if Condition 1 held at the original, it remains to hold as well in the alternative. The reverse of the Lemma (that is, that a helicopter drop can be done without affecting the allocation) is not generally true with balance sheet constraints even in the absence of Condition 1, as we discussed previously.

Finally, we highlight that our subsequent analysis in the case without commitment can be done without imposing Condition 1, by focusing away from the deflationary traps that emerge.

[17] See Hall and Reis (2015) for more details on this.

3 Commitment Solution

We start by analyzing the optimal monetary policy under commitment. The monetary authority's problem is as follows: the monetary authority maximizes its objective function by choosing a sequence $\{y_t, \pi_t, i_t, M_t, A_t, \tau_t\}$ while the fiscal authority selects a sequence of $\{B_t, T_t\}$ such that $\{y_t, \pi_t, i_t, M_t, A_t, \tau_t, B_t, T_t\}$ constitutes a private sector equilibrium consistent with balance sheet constraints. In this case under commitment, we are assuming that the monetary authority chooses the equilibrium outcome as long as it satisfies the required equilibrium conditions. We are left to guarantee that the path of the fiscal authority's real debt remains bounded, as specified in Condition 1. We will discuss this last point at the end of this section.

Lemma 3 implies that if $A_{-1} \geq M_{-1}$, we can ignore balance sheets *at all times* and focus on $\{y_t, \pi_t, i_t\}$ when solving for the optimal policy under commitment. In the rest of the paper, we will narrow attention to this case of $A_{-1} \geq M_{-1}$, as it starkly showcases the difference between the commitment and the no-commitment solutions.

3.1 Liquidity Trap under Commitment

We illustrate the commitment solution in the event of a liquidity trap. We let the monetary authority's objective be[18]

$$W(\pi, y) = -\Big[(1-\varphi)\pi^2 + \varphi y^2\Big].$$

for $\varphi \in [0, 1]$. And let ξ_t be

$$\xi_t = \begin{cases} \tilde{\xi}; & \text{if } t = 0, \\ 0; & \text{otherwise.} \end{cases} \tag{15}$$

To solve the model under commitment, we assume that the initial net worth is positive (so that the balance sheet constraints do not bind). In this case, the problem under commitment is in effect, a discrete-time version of Werning (2011), with the liquidity trap lasting for one period (See also Eggertsson and Woodford (2003) and Jung et al. (2005) for related setups).

Numerical Results. Each period represents a quarter. We set values for the discount rate to $\rho = 0.01$ and the intertemporal elasticity of substitution to $\sigma = 0.5$. The value for the slope of the Phillips curve is set to $\kappa = 0.35$, which is in the middle range of those typically used in calibrations of New Keynesian models. The coefficient on the output gap in the loss function is

[18] This objective can be derived from a second-order approximation to welfare around an efficient equilibrium with zero inflation and following Woodford (2011)'s cashless limit. Efficiency requires a subsidy on production to undo the monopolistic distortions (see Woodford, 2011, chapter 6, for details).

set to $\varphi = 0.05$.[19] In addition, we set the shock to the natural rate to $\tilde{\xi} = -0.12$. In the solution without helicopter drops, this shock will generate a fall in output of 6%. Finally, we let money demand be approximated by $L(y, i) = \theta e^{y - \eta i}$ (although, as we discussed above in Lemma 3, this does not affect the commitment solution for y_t, π_t, i_t). We set the interest rate elasticity of money demand to $\eta = 0.5$, a standard value (see, e.g., Benati, Lucas, Nicolini and Weber, 2021), and the intercept θ to match a ratio of currency to GDP of 10%.

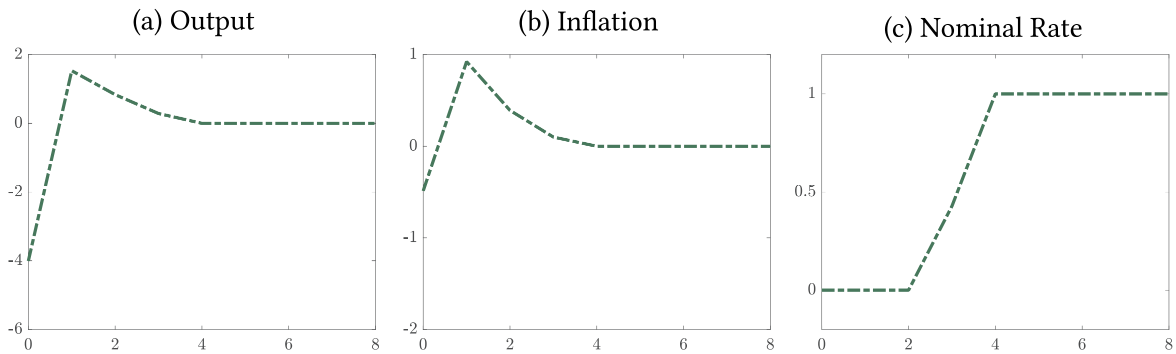

Figure 1: Simulation under Commitment

Note: Output gap, inflation, and the nominal interest rate are expressed as percentages. The x-axes represent time.

Figure 1 presents the numerical results. The figure shows the path for output, inflation, and the nominal interest rate. In line with existing results in the literature, the economy experiences deflation and a negative output gap at $t = 0$; to optimally manage the liquidity trap, the monetary authority maintains nominal interest rates low beyond the duration of the liquidity trap. In the example presented, lift-off occurs after 4 periods, while ξ goes back to zero for $t \geq 1$.

To complete the analysis, we can specify the balance sheets and transfers. Ricardian equivalence means that these are not pinned down, but it suffices to use the sequence described in the proof of Lemma 3, which guarantees that the fiscal authority debt position satisfies Condition 1.

4 Policy without Commitment

We now study the equilibrium in which the monetary authority cannot commit. We focus on Markov equilibria. This is the equilibrium concept used in several previous papers, such as Albanesi,

[19]This value may seem small, but it is standard in the literature. Indeed, a second-order approximation of the welfare function in the canonical New Keynesian model delivers a relative weight of κ divided by the elasticity of substitution between differentiated inputs. Although we do not model this, our choice of φ would be consistent with the value we set for κ and an elasticity of substitution of 7.

Chari and Christiano (2003); Eggertsson (2003); Adam and Billi (2007); Armenter (2018); Nakata and Schmidt (2019). We will narrow attention to the monetary authority's problem. We assume that the fiscal authority selects a path of debt and taxes consistent with its budget constraint and the consolidated government's transversality condition. We rule out as an equilibrium outcome any allocations where the fiscal path violates Condition 1.

4.1 Markov Equilibrium: The Stationary Case

First, let us consider the stationary case, in which $\xi_t = 0$ at all times. The state variables at time t are the inherited balance sheet positions, M_{-}, A_{-}, and the previous period price level, P_{-}. As usual in monetary models, it suffices to carry $M_{-}/P_{-} = m_{-}$ and $A_{-}/P_{-} = a_{-}$ as state variables. But, as we show below, we can further simplify the state space.

In a Markov equilibrium, the monetary authority takes as given the private sector expectation functions of next-period's output and inflation. At the beginning of the period, the monetary authority inherits its balance sheet, m_{-} and a_{-}, and chooses y, π, i, m, a, and $\tilde{\tau} = e^{\pi}\tau/P_{-}$ to maximize its objective function, subject to the Euler equation, the price setting equation, the zero lower bound constraint, the money market clearing condition; its budget constraint, and the two balance sheet constraints. The budget and balance sheet constraints (1)-(3) can be written as

$$a \geq 0; \quad e^{\pi - i}a + m_{-} + \tilde{\tau} = e^{\pi}m + a_{-}; \text{ and } \tilde{\tau} \geq (1 - e^{-i})a.$$

Defining $n \equiv a_{-} - m_{-}$, and $n' \equiv a - m$, these constraints are equivalent to

$$m \geq -n'; \quad n \geq e^{\pi}n'.$$

The next step is to notice that the inherited values of m_{-} and a_{-} do not affect any other constraint, and thus we can simply use real net worth, n, as the only state variable.

We let $V(n)$ denote the value function of the monetary authority as a function of its net worth and Ω its domain set. The functions $\mathcal{Y}(n)$ and $\Pi(n)$ denote the private sector expectations of future output and inflation. The monetary authority's problem can be written as[20]

$$V(n) = \max_{(y,\pi,i,n'\in\Omega)} W(\pi,y) + \beta V(n') \tag{16}$$

subject to:

$$y = \mathcal{Y}(n') - \sigma(i - \Pi(n') - \rho) \tag{17}$$

$$\pi = \beta\Pi(n') + \kappa y \tag{18}$$

$$i \geq 0, \tag{19}$$

$$L(e^y, i) \geq -n' \text{ if } i > 0 \tag{20}$$

$$n \geq e^\pi n'. \tag{21}$$

The first two constraints, (17) and (18), are the Euler and price setting equations. The next constraint, (19), is the zero lower bound on the nominal interest rate. Constraint (20) is the balance sheet constraint ($a \geq 0$) with the money market equilibrium. The final constraint, (21), is the bound on net worth implied by the lack of fiscal support ($\tau \geq \tau^\star$).

The last two constraints of Problem (16) reflect the key innovation of the paper. Notice that constraint (20) is relaxed with a choice of a higher n'. Intuitively, a higher end-of-period net worth implies that higher monetary liabilities must be carried to satisfy the lower bound constraint on asset holdings. On the other hand, a higher end-of-period net worth tightens constraint (21) as it would require higher inflation to satisfy the minimum real remittances that the monetary authority must transfer. Note that together these two constraints imply $e^\pi L(e^y, i) \geq -n$. The monetary authority cannot increase i (and reduce the money supply) unless it has enough net worth.

Further inspection of the monetary authority's problem, (16), shows that the inherited net worth, n, only appears in the last constraint. It follows then that increases in n relax the monetary authority's problem, which guarantees that the value function $V(n)$ must be (weakly) increasing in net worth. The result that the value function is monotonic in n raises the question of why would the monetary authority ever conduct a helicopter drop that would deplete its net worth and thus lower its continuation value. The answer has to do with how, during a liquidity trap, a low net worth induces higher inflation and output in the future, affecting private sector expectations (the $\mathcal{Y}$ and Π functions), something we will discuss below.

Let $\mathcal{N}(n)$ denote the corresponding policy functions for the net worth evolution. With this, we are ready to define a Markov equilibrium for this case:

[20] In writing this, we use that if $i = 0$ then, it is always possible to find an m' large enough such that the money market clearing condition holds, $m' \geq L(e^y, i)$, and $n' \geq -m$; and thus these constraints on m' can be ignored. If $i > 0$, however, then we must have that $m' = L(e^y, i) \geq -n'$.

Definition 4. A *Markov equilibrium* (in the stationary case) is given by a set Ω and functions $V, \mathcal{Y}, \Pi, \mathcal{N}$ such that V solves the monetary authority's problem given $\mathcal{Y}$ and Π for all $n \in \Omega$; and $\mathcal{Y}$, Π, $\mathcal{N}$ are optimal policy functions for output, inflation, and real net worth.

We can now show that given any initial $n_{-1} \in \Omega$ and some initial B_{-1}, the resulting sequence of aggregates constitutes a private sector equilibrium. To see this, iterate the net worth policy function, $\mathcal{N}$, to obtain a sequence of $\{n_t\}$ starting from n_{-1}. Let $\{y_t, \pi_t\}$ denote the associated sequence of output and inflation given the policies $\mathcal{Y}, \Pi$. We let $\{i_t\}$ be uniquely defined by $\mathcal{Y}(n_t) = \mathcal{Y}(n_{t+1}) - \sigma(i_t - \Pi(n_{t+1}) - \rho)$. For $i_t > 0$, we let $M_t = P_t L(e^{y_t}, i_t)$. For $i_t = 0$, we let $M_t = P_t \max\{L(e^{y_t}, i_t), -n_{t+1}\}$. We let $A_t = P_t n_t - M_t$. Finally $\tau_t = M_t - P_{t-1} n_{t-1} - \frac{1}{1+\imath_t} A_t$. We can then set $B_t = A_t - M_t$, and $T_t = \frac{B_t}{1+\imath_t} + \tau_t - B_{t-1}$, with B_{-1} given. It is straightforward to check that the sequence $\{y_t, \pi_t, i_t, P_t, M_t, A_t, B_t, \tau_t, T_t\}$ we have just constructed *constitutes a private sector equilibrium consistent with balance sheet constraints.*

As we discussed previously, there are many sequences of $\{B_t, T_t\}$ that also constitute an equilibrium, as long as (13) holds. But do any of them satisfy Condition 1? The answer relies on the long-run behavior of the sequence $\{n_t\}$. To see this, suppose that the sequence $\{n_t \prod_{s=0}^{t} e^{\pi_s - i_s}\}$ does not converge to a non-negative number. Then, we have that

$$0 > \lim_{t\to\infty} -n_t \prod_{s=0}^{t} e^{\pi_s - i_s} = \lim_{t\to\infty} \frac{M_t - A_t}{P_t} \prod_{s=0}^{t} \frac{P_s/P_{s-1}}{1+\imath_s} = \lim_{t\to\infty} \frac{M_t - A_t}{P_{-1}\prod_{s=0}^{t}(1+\imath_s)}.$$

For the household transversality condition to hold, it would be necessary that

$$\lim_{t\to\infty} \frac{B_t}{P_{-1}\prod_{s=0}^{t}(1+\imath_s)} < 0,$$

a contradiction of Condition 1. If however, the sequence $\{n_t \prod_{s=0}^{t} e^{\pi_s - i_s}\}$ converges to a non-negative number, then it is possible to construct a fiscal policy that generates a private sector equilibrium where Condition 1 holds. We will restrict attention to Markov equilibria consistent with this:

Definition 5. A Markov equilibrium $(\Omega, V, \mathcal{Y}, \Pi, \mathcal{N})$ is *fiscally consistent* if for all $n_0 \in \Omega$, the resulting equilibrium path of $\{n_t, i_t, \pi_t\}$ is such that

$$\lim_{t\to\infty} n_t \prod_{s=0}^{t} e^{\pi_s - i_s} \geq 0. \tag{22}$$

Ruling Out The Deflationary Trap. The environment we are analyzing has the potential for multiple Markov equilibria, a characteristic that has been highlighted by Albanesi et al. (2003);

Armenter (2018); Nakata and Schmidt (2019).

Armenter (2018) and Nakata and Schmidt (2019) show that there exists a "deflationary trap": in our case $\pi = -\rho, y = -\rho(1-\beta)/\kappa$. If the private sector expects these levels of inflation and output tomorrow, then setting $i = 0$ delivers them as outcomes today per equations (17) and (18). And with these expectations, there is not much the monetary authority can do, as raising the nominal interest rate above zero further lowers inflation and reduces output (and thus welfare).[21] The next question is whether this deflationary trap is consistent with the fiscal condition 1.

Using condition (21), we can see that in this expectational trap (as $i = 0$),

$$n_t \leq e^{-\pi} n_{t-1} = e^{i-\pi} n_{t-1} \leq \ldots \leq e^{(i-\pi)(t+1)} n_{-1}$$
$$e^{(\pi-i)(t+1)} n_t \leq n_{-1} \Rightarrow \lim_{t\to\infty} e^{(\pi-i)(t+1)} n_t \leq n_{-1}$$

And thus, if $n_{-1} < 0$, the deflationary trap is not fiscally consistent, as it necessarily violates (22). Note that we can use the above to also rule out transitional paths that converge to the deflationary trap.[22] This argument does not work if $n_{-1} \geq 0$, but in this case, given that we know that the value function is monotonic, reducing n' suffices to also eliminate the trap as an equilibrium outcome, as we discuss below.

The Best Possible Equilibrium. Once we have ruled out the trap outcome, the next question is whether the best allocation ($\pi_t = 0$ and $y_t = 0$ at all t) constitutes an equilibrium. Note that in this case, $i_t = \rho > 0$, and thus the zero lower bound constraint is automatically satisfied. Let us define $n^\star \equiv -L(1,\rho)$. We assume that $n^\star \in \text{interior}(\Omega)$. Note that $n^\star < 0$.

For $n = n^\star < 0$, the deflationary trap is not an equilibrium, as we discussed above. In addition, the $\{\pi_t = 0, y_t = 0\}$ allocation can be implemented as an outcome of a Markov equilibrium in which the monetary authority maintains a constant level of net worth, $n_t = n^\star$, and chooses $\pi = 0, y = 0$ every period. Given that this allocation achieves the maximum welfare possible, there are no incentives for the monetary authority to deviate from it at any time if future monetary authorities follow this strategy. We assume that this is indeed the equilibrium at $n = n^\star$. For values of $n > n^\star$, the deflationary trap cannot be an equilibrium even if $n > 0$. This follows from

[21] The existence of this deflationary trap is anticipated by the self-fulfilling liquidity traps of Benhabib et al. (2001b; 2002) in their analysis of a monetary authority that adheres to a (non-optimizing) Taylor rule but faces a zero lower bound constraint on nominal rates.

[22] Benigno (2020) shows that a policy commitment by the monetary authority to remit its profits, as well as Condition 1 holding with equality, can lead to price level determinacy in a flexible price model where the monetary policy follows a Taylor rule. In his analysis, it is an exploding positive net worth by the monetary authority that is ruled out by the policy, while in ours, is an exploding negative one. In our model, the monetary authority is operating without commitment and it is not following a Taylor rule, but the basic insight that eliminates the deflationary trap is the same as in his model.

the monotonicity of the value function. Alternatively, one can see that setting $n' = n^\star$, $\pi = 0$, and $y = 0$ is a feasible choice for $n > n^\star$ that guarantees the best equilibrium outcome as the continuation.

However, if $n < n^\star$, then the $\pi = 0, y = 0$ allocation is no longer a feasible choice: the monetary authority lacks sufficient net worth to implement it. Let's explore the following alternative feasible policy. At $t = 0$ (given that $n < 0$), the monetary authority chooses a sufficiently high level of inflation to guarantee that $n' = n^\star$ is a feasible choice. In this way, the monetary authority can guarantee that the first best allocation is achieved from period $t = 1$ onward at the cost of a positive inflation rate in $t = 0$. The monetary authority may choose a less drastic policy of setting inflation above zero at $t = 0$ and increasing n' above n, but not enough to get to $n^\star$. Doing so may result in an equilibrium where inflation and output are above their first best values for longer than one period.

To provide some additional intuition, we will assume below that the Markov equilibrium delivers inflation and output policy functions that are monotonically decreasing in n below $n^\star$ (a result we will confirm in the simulations below). Together with a continuity condition, these imply that *helicopter drops are not used away from the zero lower bound*:

Lemma 4 (No Helicopter Drops Away from ZLB)**.** *Suppose that* $\Pi(n)$ *and* $\mathcal{Y}(n)$ *are weakly decreasing and continuous functions of* n *and that* $\Pi(n)$ *is strictly decreasing for* $n < n^\star$*. If for* $n < n^\star$*, the solution to the monetary authority's problem features* $i > 0$*, then, it must also feature that constraint* (21) *holds with equality.*

Proof. In the Appendix. □

If constraint (21) holds with equality, nominal net worth is unchanged and $\tau = \tau^\star$, that is, there is no helicopter drop. A key intermediate step for the result of the Lemma is that given that $n < n^\star$, the monetary authority necessarily chooses positive inflation. If the monetary authority were to engineer a helicopter drop, it could achieve the same level of output with a lower helicopter drop (increasing its continuation value) and a lower interest rate. This satisfies the balance sheet constraints, generates less inflation, and is feasible as long as $i > 0$. Thus the alternative policy constitutes an improvement relative to the solution with a helicopter drop.

The result connects the potential use of helicopter drops to a binding zero lower bound constraint on nominal rates. So, if helicopter drops are to be used in a Markov equilibrium (that satisfies the assumptions of the lemma), it is because nominal rates have already reached zero. Whether or not helicopter drops are used in equilibrium is something we will discuss in further detail in the liquidity trap section below.

The next lemma shows that the nominal interest rate is increasing in n if both balance sheet constraints are binding, and, when conducting a helicopter drop, the monetary authority induces *lower* levels for the nominal interest rate going forward.

Lemma 5. *Suppose that $\Pi(n)$ and $\mathcal{Y}(n)$ are weakly decreasing, continuous and differentiable functions of n. If the solution to the monetary authority's problem features $i > 0$ and constraints (20) and (21) hold with equality, then i is increasing in n, and π and y are decreasing in n.*

Proof. In the Appendix. □

Let us provide some intuition for this result. Suppose that indeed $\mathcal{Y}$ and Π are decreasing functions of n, achieving the best possible values for $n \geq n^\star$. Accordingly, the monetary authority would like to choose a high n', to reduce deviations from the best allocation tomorrow and a nominal interest rate consistent with low inflation and output deviations today. But a low value of net worth today, $n < n^*$, limits its ability to do this. To see this, let us put together (20) and (21) and obtain the necessary condition we mentioned previously: $e^{\pi} L(e^{y}, i) \geq -n$. This implies that low real net worth today effectively imposes a lower bound on money balances. As a result, the nominal interest rate cannot be too high (as L is decreasing in i). For a fixed n', from (17) and (18), it follows that a lower nominal interest rate leads to higher inflation and output today. A lower choice of n' reinforces this effect, operating through the expectation functions $\mathcal{Y}$ and Π. The overall effect is therefore monotonic: the lower the net worth is, the lower the nominal interest rate, resulting in higher inflation and output.

The above lemmas necessitated restrictions on endogenous objects in order to provide intuitions for the forces at play. We are not able to further characterize analytically the equilibrium outcome. But we now proceed to study the equilibrium numerically.

Numerical Results. To solve the model, we impose that the equilibrium delivers the best allocation for $n \geq n^\star$. We then extend the equilibrium to the rest of the state space.[23] Here, we continue to use the same parameter values as in the previous section.

Figure 2 presents the results that arise from a Markov perfect equilibrium in the stationary environment. The vertical line represents the value of $n^\star$. Panels (a), (b), and (c) show output, inflation, and the nominal interest rate as a function of current net worth. Notice that there is a kink at the level of net worth at which the balance sheet constraints become binding. For relatively high levels of net worth (i.e., $n \geq n^\star$), the monetary authority can choose the first-best level of inflation and output by setting $i = \rho$, as reflected by the flat region in the figures. For relatively low levels of net worth, the nominal interest rate is increasing in current net worth, while inflation and output are decreasing in current net worth, as established in Lemma 5.

[23] Even though we ruled out the deflationary trap outcome (and paths that converge to it), we do not have a proof of uniqueness of Markov Equilibria.

Panel (d) shows the next-period net worth, n', as a function of current net worth, n. Again, when current net worth is high, the balance sheet constraints are not binding, and the monetary authority can choose to keep real (and nominal) net worth constant. On the other hand, when $n < n^{\star}$, we have that nominal net worth remains constant, implying that real net worth grows at the rate of inflation. This is because as shown in Lemma 4, the monetary authority chooses not to do helicopter drops in the stationary case.

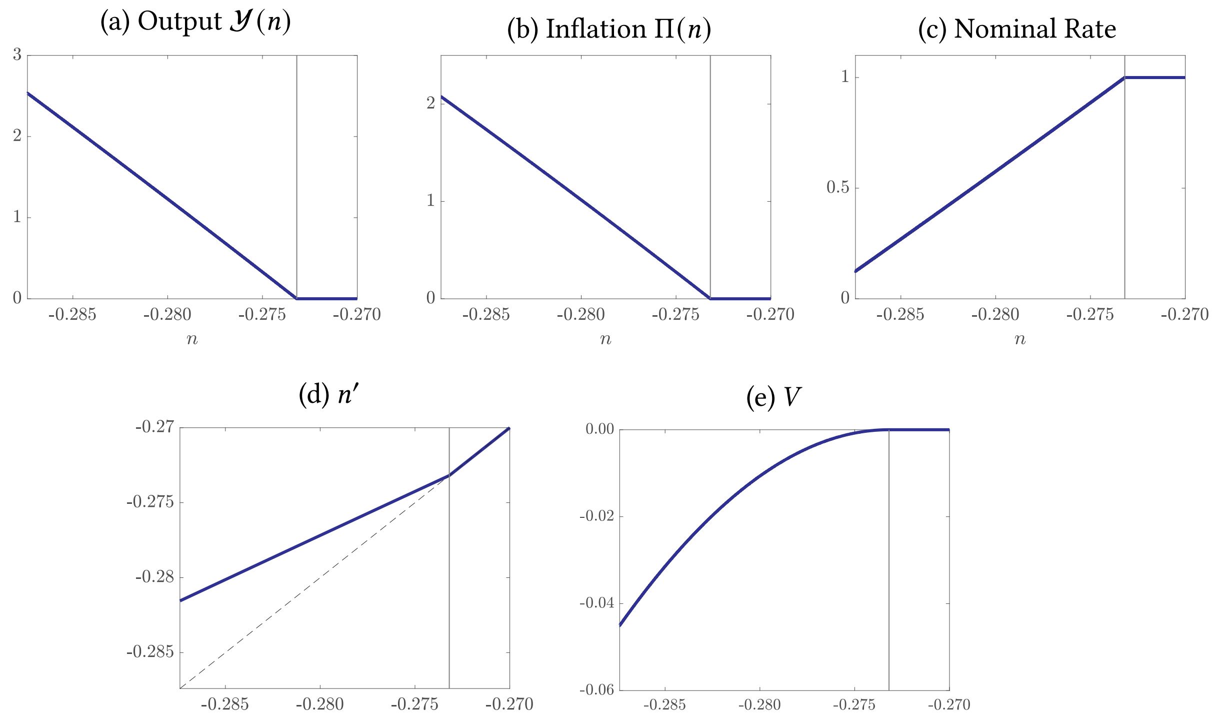

Figure 2: Stationary Environment and Markov Equilibrium

Note: This figure shows the equilibrium policies y, π, i, n' and value V as a function of net worth, n. The vertical line indicates the value of net worth at which the balance sheet constraints cease to bind. All variables in the y-axes, with the exception of net worth, are expressed as percentages.

Panel (e) of Figure 2 shows the value function. The value function is constant and achieves a zero loss for $n > n^{\star}$, as discussed. The value function falls as net worth falls below $n^{\star}$ given the negative output gap and positive inflation that arise in equilibrium.

Finally, let us highlight that helicopter drops are never used in this stationary environment: constraint (21) always holds with equality, as stated in Lemma 4. There is no need to use helicopter drops in this stationary environment as long as $i > 0$. We will see below how this result changes once we move to the liquidity trap environment.

4.2 Markov Equilibrium: A Liquidity Trap

We now return to the liquidity trap environment of Section 3.1 and let the path of ξ be as in (15): temporarily low at $t = 0$ and zero afterwards. Note that from period $t = 1$ onward, the equilibrium is as described in the stationary environment section, and thus we can use the previously computed $\mathcal{Y}(n)$, $\Pi(n)$, and $V(n)$ as the continuation policies and value function.

We are interested in the Markov equilibrium starting from period $t = 0$, the period of the liquidity trap. In this period, the problem of the monetary authority is

$$V_0(n) = \max_{(y,\pi,i,n' \in \Omega)} W(\pi, y) + \beta V(n')$$

subject to:

$$\begin{aligned}
&y = \mathcal{Y}(n') - \sigma(i - \Pi(n') - \rho - \tilde{\xi}) \\
&\pi = \beta\Pi(n') + \kappa y \\
&i \geq 0, \\
&L(e^y, i) \geq -n' \text{ if } i > 0 \\
&n \geq e^\pi n'
\end{aligned}$$

Note that the only difference with the stationary case is the presence of the discount factor shock $\tilde{\xi}$ in the Euler equation.

We will assume, as in the commitment analysis, that the initial net worth is sufficiently high so that the balance sheet constraints do not bind at time $t = 0$. For simplicity, we set it to zero. Using the same numerical parameters of previous sections, we proceed to compute the equilibrium policy.

Figure 3 shows the equilibrium outcomes that would result in period $t = 0$ as functions of the level of net worth chosen in period 0 (i.e., the starting value of 0 minus the size of the helicopter drop). The vertical line represents $n^\star$, and the solid dots represent the optimal policies chosen by the monetary authority.

The top panels of Figure 3 show the effects on output and inflation of a helicopter drop in period $t = 0$. A larger helicopter drop generates higher inflation and output in period 0, relative to the case without helicopter drops (which corresponds to the levels above $n^\star$, at the flat region). As shown before in Figure 2, a lower net worth starting in period 1 implies higher inflation and output in the future, and this increases current output and inflation.

The bottom panels show the trade-off faced by the monetary authority. In the bottom right panel, one can see the helicopter drop that maximizes the objective function during the liquidity trap. The bottom left panel shows that it is possible to further increase the current value W_0 with

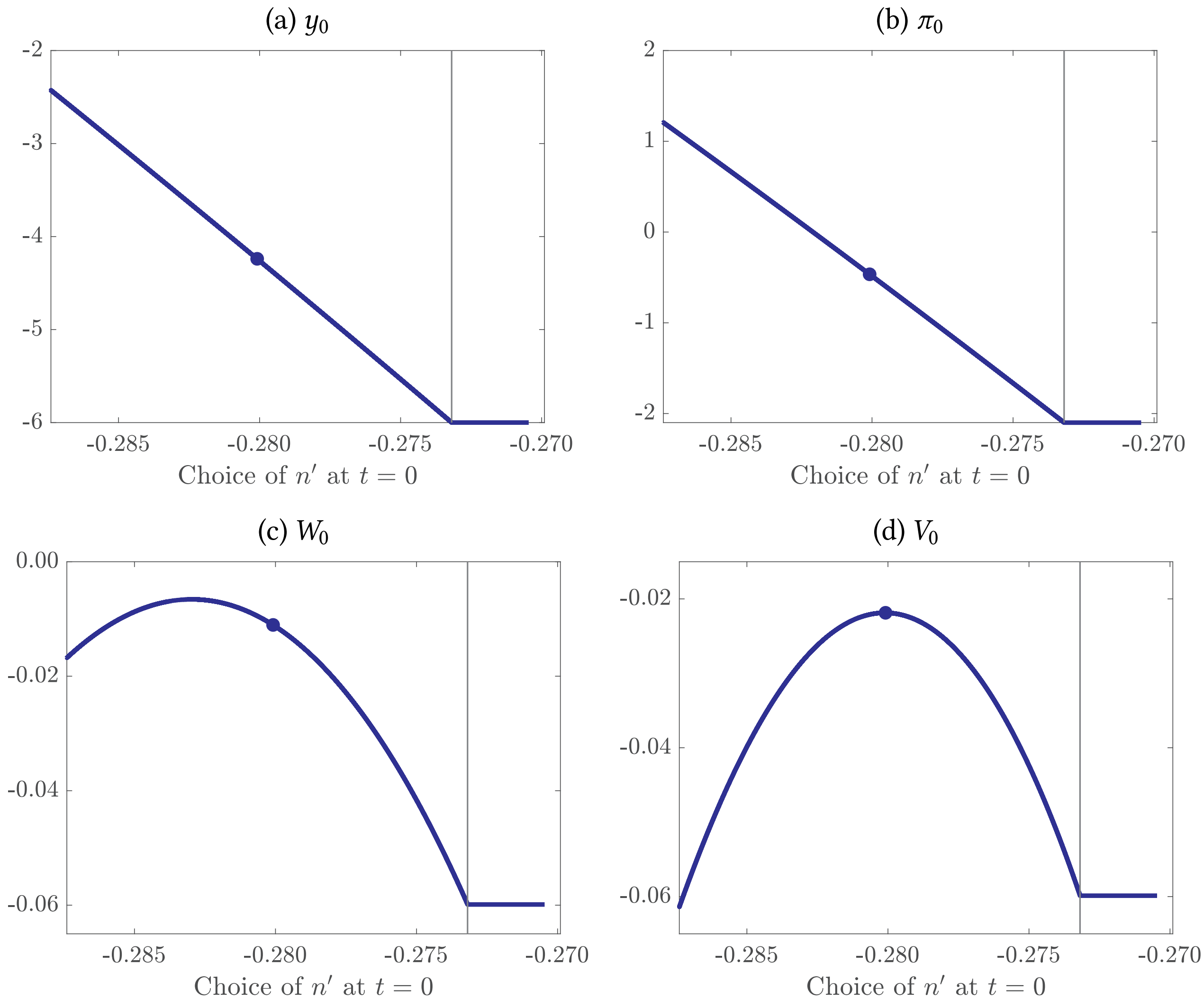

Figure 3: Liquidity Trap and the Markov Equilibrium

Note: This figure shows how different levels of net worth chosen at $t = 0$ affect allocations and welfare at $t = 0$. All variables in the y-axes are expressed as percentages.

an even larger helicopter drop, but this would come at the expense of raising future losses. The optimal helicopter drop turns out to be roughly 2.5 percentage points of annual GDP.

Comparison with Commitment Solution. We now compare the simulation outcomes *over time* of the economy under commitment with the simulations for the Markov perfect equilibrium with and without helicopter drops. In all cases, we initialize the economy at $t = 0$ with a high enough net worth so that balance sheet constraints do not initially bind, and we feed the economy with the same shock $\tilde{\xi}$ used above.

Figure 4 presents the results. The red dashed line represents the Markov perfect equilibrium *without helicopter drops*. This economy features the standard outcome of $\pi_t = 0, y_t = 0$ from $t = 1$ onward. One can see that this economy experiences a much larger recession and deflation compared to the commitment solution. This is because when the government can commit, it promises inflation and an output boom from $t = 1$ onward, which mitigates the recession at $t = 0$.

The Markov perfect equilibrium with helicopter drops is represented by the solid blue line. As the figure shows, helicopter drops are quite effective at alleviating the liquidity trap. Relative to the equilibrium without helicopter drops, the monetary authority is able to reduce the output gap

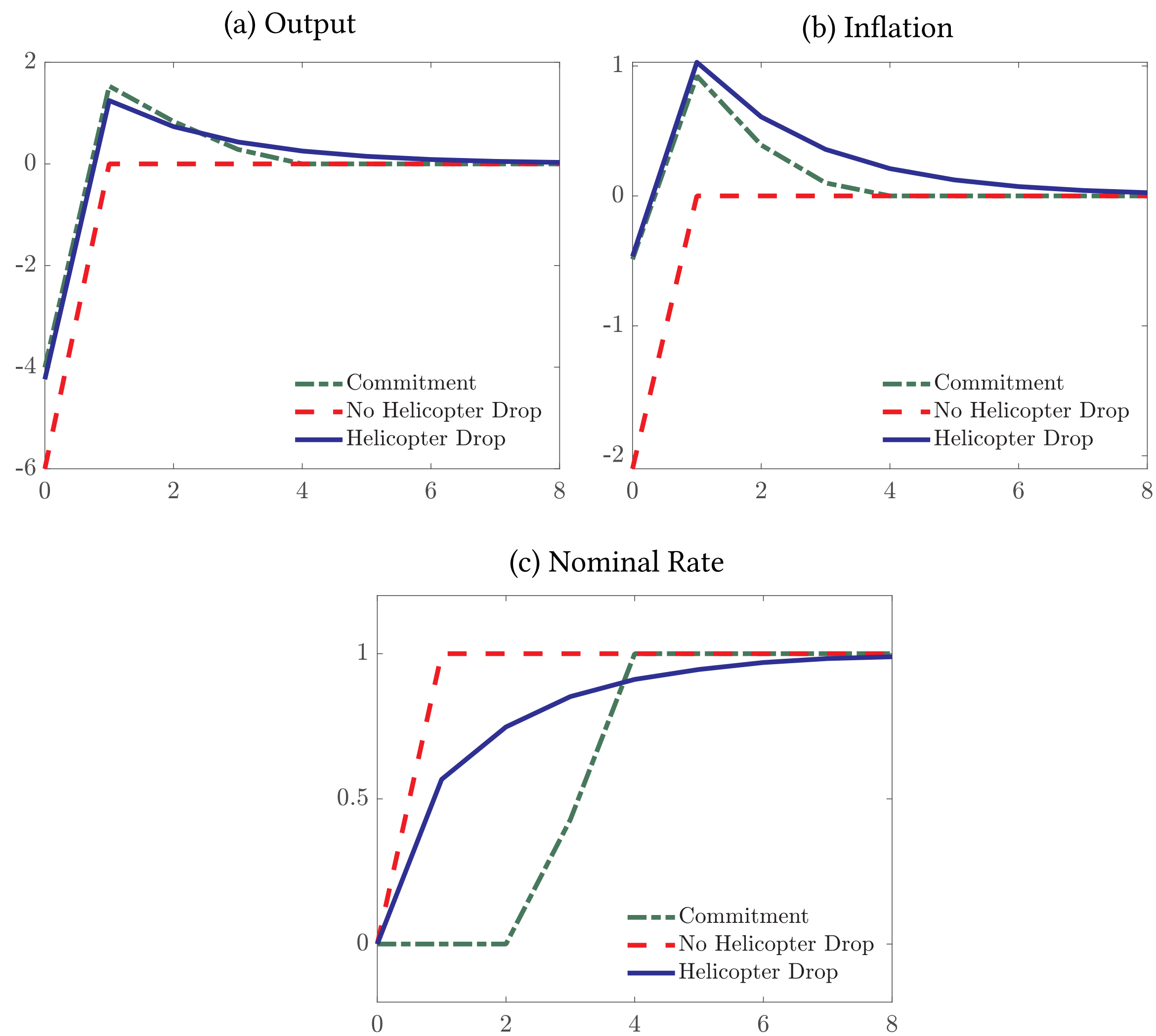

Figure 4: Simulation Comparison

Note: This figure shows the simulations for the three economies starting from a liquidity trap at $t = 0$. All variables in the y-axes are expressed as percentages. The x-axes represent time.

from -6% to -4% and deflation from -2.2% to -0.5%. As explained above, this is because the use of helicopter drops implies that the monetary authority will keep interest rates lower for longer after the liquidity trap is over. It is worth highlighting that the optimal use of helicopter drops achieves almost as much output stabilization as the commitment solution. The downside, however, lies in the evolution of inflation. While the amount of inflation is more short-lived under commitment, the economy with helicopter drops experiences a more protracted period of inflation.

5 Conclusion

We presented a simple theory of how helicopter drops of money can be effective during a liquidity trap, in contrast to standard irrelevance results. The key elements of the theory are the presence of balance sheet constraints and the lack of commitment of the monetary authority. When the monetary authority engages in helicopter drops, it reduces its net worth and is induced to keep the money supply higher in the future. As a result, this triggers expectations of higher inflation and output and mitigates the recessionary effects of a liquidity trap.

We finish with a few caveats and observations. First, central banks usually do not have the ability to provide direct transfers to households, but this restriction may change with changes in the political environment or innovations in financial markets. Second, as with any commitment tool, there is a trade-off between commitment and flexibility: a helicopter drop provides commitment but removes future flexibility in the conduct of monetary policy. Indeed, this is why a helicopter drop can be useful during a liquidity trap. But it could become costly if the central bank needs to unexpectedly reverse a previous monetary expansion. Finally, our theory introduced a balance sheet constraint that takes the form of a sharp limit on the evolution of the central bank's net worth. Future work can more clearly identify and measure the presence and nature of such constraints and their interactions with fiscal and monetary policy.

References

Adam, Klaus and Roberto M. Billi, "Discretionary monetary policy and the zero lower bound on nominal interest rates," *Journal of Monetary Economics*, Apr 2007, *54* (3), 728–752.

Albanesi, Stefania, V. V. Chari, and Lawrence J. Christiano, "Expectation Traps and Monetary Policy," *The Review of Economic Studies*, 2003, *70* (4), 715–741.

Alvarez, Fernando, Patrick J Kehoe, and Pablo Andrés Neumeyer, "The time consistency of optimal monetary and fiscal policies," *Econometrica*, 2004, *72* (2), 541–567.

Amador, Manuel, Javier Bianchi, Luigi Bocola, and Fabrizio Perri, "Reverse speculative attacks," *Journal of Economic Dynamics and Control*, 2016, *72*, 125–137.

Armenter, Roc, "The Perils of Nominal Targets," *The Review of Economic Studies*, 2018, *85* (1 (302)), 50–86.

Auerbach, Alan J and Maurice Obstfeld, "The case for open-market purchases in a liquidity trap," *American Economic Review*, 2005, *95* (1), 110–137.

Bartsch, Elga, Jean Boivin, Stanley Fischer, Philipp Hildebrand, and S Wang, "Dealing with the next downturn: from unconventional monetary policy to unprecedented policy coordination," *Macro and Market Perspectives*, 2019.

Bassetto, Marco, "A Game-Theoretic View of the Fiscal Theory of the Price Level," *Econometrica*, 2002, *70* (6), 2167–2195.

Bassetto, Marco and Thomas J. Sargent, "Shotgun Wedding: Fiscal and Monetary Policy," *Annual Review of Economics*, 2020, *12* (1), 659–690.

Bassetto, Marco and Todd Messer, "Fiscal consequences of paying interest on reserves," *Fiscal Studies*, 2013, *34* (4), 413–436.

Benati, Luca, Robert E Lucas, Juan Pablo Nicolini, and Warren Weber, "International evidence on long-run money demand," *Journal of monetary economics*, 2021, *117*, 43–63.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe, "Monetary Policy and Multiple Equilibria," *American Economic Review*, Mar 2001, *91* (1), 167–186.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe, "The Perils of Taylor Rules," *Journal of Economic Theory*, Jan 2001, *96* (1), 40–69.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martín Uribe, "Avoiding Liquidity Traps," *Journal of Political Economy*, Jun 2002, *110* (3), 535–563.

Benigno, Pierpaolo, "A Central Bank Theory of Price Level Determination," *American Economic Journal: Macroeconomics*, Jul 2020, *12* (3), 258–283.

Benigno, Pierpaolo and Salvatore Nisticò, "The economics of helicopter money," 2022. Centre for Economic Policy Research.

Benigno, Pierpaolo and Salvatore Nisticò, "Non-neutrality of Open-Market Operations," *American Economic Journal: Macroeconomics*, Jul 2020, *12* (3), 175–226.

Bernanke, Ben, "Deflation: Making Sure 'It' Doesn't Happen Here," 2002. Remarks on November 21 2002, https://www.federalreserve.gov/boarddocs/Speeches/2002/20021121/default.htm.

Bhattarai, Saroj, Gauti B Eggertsson, and Bulat Gafarov, "Time Consistency and Duration of Government Debt: A Model of Quantitative Easing," *The Review of Economic Studies*, 2022.

Bianchi, Javier and Saki Bigio, "Banks, liquidity management, and monetary policy," *Econometrica*, 2022, *90* (1), 391–454.

Buera, Francisco J and Juan Pablo Nicolini, "Liquidity traps and monetary policy: Managing a credit crunch," *American Economic Journal: Macroeconomics*, 2020, *12* (3), 110–38.

Buiter, Willem H. and Anne C. Sibert, "DEFLATIONARY BUBBLES," *Macroeconomic Dynamics*, Sep 2007, *11* (4), 431–454.

Calvo, Guillermo A and Pablo E Guidotti, "Optimal maturity of nominal government debt: an infinite-horizon model," *International Economic Review*, 1992, pp. 895–919.

Del Negro, M. and C. Sims, "When Does A Central Bank's Balance Sheet Require Fiscal Support?," *Journal of Monetary Economics*, 2015, *73*, 1–19.

Eggertsson, Gauti and Michael Woodford, "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, 2003, *34* (1), 139–235.

Eggertsson, Gauti B., "How to Fight Deflation in a Liquidity Trap: Committing to Being Irresponsible," *IMF Working paper*, 2003. Working Paper No. 2003/064.

Friedman, Milton, *The optimum quantity of money*, Routledge, 1969.

Galí, Jordi, "The effects of a money-financed fiscal stimulus," *Journal of Monetary Economics*, 2019.

Gali, Jordi, "Helicopter money: The time is now," *VOXEU column*, 2020. March 17 2020, https://cepr.org/voxeu/columns/helicopter-money-time-now.

Hall, R. and R. Reis, "Maintaining Central-Bank Financial Stability under New-Style Central Banking," NBER Working Paper 21173 2015.

Jeanne, Olivier and Lars EO Svensson, "Credible commitment to optimal escape from a liquidity trap: The role of the balance sheet of an independent central bank," *American Economic Review*, 2007, *97* (1), 474–490.

Jung, Taehun, Yuki Teranishi, and Tsutomu Watanabe, "Optimal monetary policy at the zero-interest-rate bound," *Journal of Money, credit, and Banking*, 2005, *37* (5), 813–835.

Kamihigashi, Takashi, "A simple proof of the necessity of the transversality condition," *Economic Theory*, Sep 2002, *20* (2), 427–433.

Kocherlakota, Narayana and Christopher Phelan, "Explaining the fiscal theory of the price level," *Federal Reserve Bank of Minneapolis Quarterly Review*, 1999, *23* (4), 14–23.

Krugman, Paul R, "It's baaack: Japan's slump and the return of the liquidity trap," *Brookings Papers on Economic Activity*, 1998, *1998* (2), 137–205.

Lucas, Robert E and Nancy L Stokey, "Optimal fiscal and monetary policy in an economy without capital," *Journal of monetary Economics*, 1983, *12* (1), 55–93.

Michau, Jean-Baptiste, "Helicopter Drops of Money under Secular Stagnation," 2022. Mimeo, Paris School of Economics.

Nakata, Taisuke and Sebastian Schmidt, "Conservatism and liquidity traps," *Journal of Monetary Economics*, Jun 2019, *104*, 37–47.

Sargent, Thomas J and Neil Wallace, "Some unpleasant monetarist arithmetic," *Federal Reserve Bank of Minneapolis, Quarterly Review*, 1981, *5* (3), 1–17.

Sims, Christopher A, "Limits to inflation targeting," in "The Inflation-Targeting Debate," University of Chicago Press, 2004, pp. 283–310.

Stella, Peter, "Do central banks need capital?," IMF Working Paper 97/83 1997.

Stella, Peter, "Central bank financial strength, transparency, and policy credibility," *IMF Staff Papers*, 2005, pp. 335–365.

The Economist, "Why central banks are talking about throwing money from helicopters," 2016. May 2 2016, https://www.economist.com/the-economist-explains/2016/05/02/why-central-banks-are-talking-about-throwing-money-from-helicopters.

Wallace, Neil, "A Modigliani-Miller theorem for open-market operations," *The American Economic Review*, 1981, *71* (3), 267–274.

Werning, Ivan, "Managing a liquidity trap: Monetary and fiscal policy," 2011. NBER Working Paper 17344.

Woodford, Michael, "Fiscal Requirements for Price Stability," *Journal of Money, Credit and Banking*, 2001, *33* (3), 669–728.

Woodford, Michael, *Interest and prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2011.

A Omitted Proofs

A.1 Proof of Lemma 1

Proof. The argument here follows Buiter and Sibert (2007), except for the use of a different No-Ponzi condition. Let $\{C_t^\star.M_t^\star, W_t^\star\}$ be a sequence that satisfies the conditions in the Lemma. Let $\{C_t, M_t, W_t\}$ be any alternative sequence.

Define D to be

$$
\begin{aligned}
D &\equiv \lim_{T\to\infty}\inf \sum_{t=0}^{T} \beta(t)\left[u(C_t) + v(M_t/P_t) - u(C_t^\star) - v(M_t^\star/P_t^\star)\right] \\
&\leq \lim_{T\to\infty}\inf \sum_{t=0}^{T} \beta(t)\left[u'(C_t^\star)(C_t - C_t^\star) + v'(M_t^\star/P_t^\star)\frac{M_t - M_t^\star}{P_t}\right] \\
&= \lim_{T\to\infty}\inf \sum_{t=0}^{T} \beta(t)\Bigg[u'(C_t^\star)\left(\frac{W_{t-1} + M_{t-1} - \frac{W_t}{1+\iota_t} - M_t}{P_t} - \frac{W_{t-1}^\star + M_{t-1}^\star - \frac{W_t^\star}{1+\iota_t} - M_t^\star}{P_t}\right) \\
&\qquad + v'(M_t^\star/P_t^\star)\frac{M_t - M_t^\star}{P_t}\Bigg] \\
&= \lim_{T\to\infty}\inf \Bigg\{\sum_{t=-1}^{T-1} \beta(t+1)u'(C_{t+1}^\star)\left(\frac{W_t + M_t}{P_{t+1}} - \frac{W_t^\star + M_t^\star}{P_{t+1}}\right) + \\
&\qquad + \sum_{t=0}^{T} \beta(t)\left[u'(C_t^\star)\left(\frac{\frac{W_t^\star}{1+\iota_t} + M_t^\star}{P_t} - \frac{\frac{W_t}{1+\iota_t} + M_t}{P_t}\right) + v'(M_t^\star/P_t^\star)\frac{M_t - M_t^\star}{P_t}\right]\Bigg\} \\
&= \lim_{T\to\infty}\inf \Bigg\{\sum_{t=0}^{T-1} \beta(t)u'(C_t^\star)\left(\frac{W_t + M_t - W_t^\star - M_t^\star}{(1+\iota_t)P_t}\right) + \\
&\qquad + \sum_{t=0}^{T} \beta(t)u'(C_t^\star)\left(\frac{W_t^\star + M_t^\star - W_t - M_t}{(1+\iota_t)P_t}\right) \\
&\qquad + \beta(0)u'(C_0^\star)\cancelto{0}{\left(\frac{W_{-1} + M_{-1}}{P_0} - \frac{W_{-1}^\star + M_{-1}^\star}{P_0}\right)}\Bigg\} \\
&= \lim_{T\to\infty}\inf \beta(T)u'(C_T^\star)\left(\frac{W_T^\star + M_T^\star - W_T - M_T}{(1+\iota_T)P_T}\right) \\
&= \lim_{T\to\infty}\inf \beta(0)u'(C_0^\star)\frac{P_T}{P_0\prod_{s=0}^{T-1}(1+\iota_s)}\left(\frac{W_T^\star + M_T^\star - W_T - M_T}{(1+\iota_T)P_T}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta(0)u'(C_0^\star)}{P_0} \lim_{T\to\infty}\inf \frac{W_T^\star + M_T^\star - W_T - M_T}{\prod_{s=0}^{T}(1+\iota_s)} \\
&\le \frac{\beta(0)u'(C_0^\star)}{P_0} \lim_{T\to\infty}\inf \frac{W_T^\star + M_T^\star}{\prod_{s=0}^{T}(1+\iota_s)},
\end{aligned}
$$

where the first inequality uses concavity, the first equality uses the household budget constraint, the second one rearranges terms, the third one uses the Euler equation and the first-order condition for money holdings, the fourth simplifies, the fifth uses the Euler equation once more, the sixth simplifies, and the last inequality uses the No-Ponzi condition for the alternative allocation $\{C_t, M_t, W_t\}$. The transversality condition stated in the Lemma implies that $D \le 0$, completing the proof. □

A.2 Proof of Lemma 4

Proof. Let $n < n^\star$ and let (y, π, i, n') be an optimal solution to the monetary authority problem with $i > 0$.

First, in an optimal solution, $\pi > 0$. To see this, note that if $\pi \le 0$, then $y \le 0$ as $\Pi(n') \ge 0$, and thus $y = (\pi - \beta\Pi(n'))/\kappa$. Now, if $\pi < 0$, then $y < 0$, and it is possible to reduce i and increase both, relaxing (20) and strictly increasing the objective. If $\pi = 0$, and $\Pi(n') = 0$, then $n' \ge n^\star$ (by the fact that Π is strictly decreasing for $n < n^\star$), and thus, $\mathcal{Y}(n') = 0$, and $y = 0$ and $i = \rho$. But this is inconsistent with $n < n^\star$, as the $(0, 0)$ is not feasible. So it must be that in any optimal policy, $\pi = 0$ and $y < 0$. In this case, reducing i increases both π and y, generating a first-order gain in the payoff function (and satisfying (20)). Thus, it must be that $\pi > 0$.

Towards a proof by contradiction, suppose that the statement lemma is false, and that constraint (21) is slack at an optimal solution. Consider an alternative policy choice. Let $\hat{n}' = n' + \epsilon_1$ for some $\epsilon_1 > 0$; and let $\hat{i}$ and $\hat{\pi}$ be

$$
\begin{aligned}
\hat{i} &\equiv \frac{1}{\sigma}\left[\mathcal{Y}(\hat{n}') - y + \sigma\Pi(\hat{n}') + \sigma\rho\right] \le \frac{1}{\sigma}\left[\mathcal{Y}(n') - y + \sigma\Pi(n') + \sigma\rho\right] = i \\
\hat{\pi} &\equiv \beta\Pi(\hat{n}') + \kappa y \le \beta\Pi(n') + \kappa y = \pi
\end{aligned}
$$

where the inequalities follow from the monotonicity of $\mathcal{Y}$ and Π. The values of $(y, \hat{\pi}, \hat{i})$ satisfy equations (17) and (18) by construction.

Continuity of $\mathcal{Y}$ and Π implies that we can find $\epsilon_1 > 0$ small enough such that $\hat{i} > 0$ and $\hat{\pi} \ge 0$. Given that (21) is slack at (n', π), continuity of the policies also implies that we can find ϵ_1 small enough so that (21) remains slack at $(\hat{n}', \hat{\pi})$. Now note that $L(e^y, \hat{i}) \ge L(e^y, i) \ge -n' > -\hat{n}'$, and thus $(y, \hat{\pi}, \hat{i}, \hat{n}')$ is such that constraint (20) is also slack.

The policy vector $(y, \hat{\pi}, \hat{i}, \hat{n}')$ so constructed satisfies all of the constraints of the monetary authority problem at n (for $\epsilon_1 > 0$ small enough). Note that as $\hat{\pi} \in [0, \pi]$, it follows that $W(\hat{\pi}, y) \geq W(\pi, y)$. In addition, $V(\hat{n}') \geq V(n')$ by the monotonicity of V. So the new policy vector is also an optimal policy. Now we would argue that we can use this alternative policy vector to generate an strict improvement over the original policy. We will consider two cases, depending on the value of n'.

If $n' < n^\star$. Then the fact that Π is strictly decreasing in that range implies that $\hat{\pi} < \pi$, then the policy $(y, \hat{\pi}, \hat{i}, \hat{n}')$ is an improvement over the original policy.

If $n' \geq n^\star$, then we have that $y = -\sigma(i - \rho)$ and $\pi = -\kappa\sigma(i - \rho)$. From (21), it follows that $-L(1, \rho) = n^\star > n > e^\pi n' \geq e^\pi n^\star = -e^\pi L(1, \rho)$, and thus $\pi > 0$, $i < \rho$ and $y > 0$. Note that $n' > n^\star$ also means that $\hat{i} = i$ and $\hat{\pi} = \pi$. Consider then the alternative policy:

$$\hat{i}_2 \equiv i + \epsilon_2; \quad \hat{y} \equiv y - \sigma\epsilon_2; \quad \hat{\pi}_2 \equiv \pi - \kappa\sigma\epsilon_2$$

The policy $(y, \hat{\pi}, \hat{i}, \hat{n}')$ satisfies (17) and (18) and $\hat{i}_2 > 0$ for $\epsilon_2 > 0$. Given that constraints (20), (21) are slack at $(y, \pi, i, \hat{n}')$, they remain slack at $(\hat{y}, \hat{\pi}_2, \hat{i}_2, \hat{n}')$ for small enough $\epsilon_2 > 0$. It follows that for small enough $\epsilon_2 > 0$, the allocation $(\hat{y}, \hat{\pi}_2, \hat{i}_2, \hat{n}')$ represents a strict improvement as output and inflation are strictly lower and closer to their optimal levels. And thus, the original policy could not have been optimal.

Thus, we have argued that if (21) does not hold with equality, we can construct a strict improvement under the assumptions of the lemma, generating a contradiction. □

A.3 Proof of Lemma 5

Proof. The equilibrium is given by the solution

$$y = \mathcal{Y}\left(\frac{n}{e^\pi}\right) - \sigma\left(i - \Pi\left(\frac{n}{e^\pi}\right) - \rho\right), \tag{A.1}$$

$$\pi = \beta\Pi\left(\frac{n}{e^\pi}\right) + \kappa y, \tag{A.2}$$

$$L(e^y, i)e^\pi = -n. \tag{A.3}$$

Totally differentiating the expressions above, we obtain

$$e^{\pi}dy + e^{\pi}\sigma di + n(\sigma\Pi'(n') + \mathcal{Y}'(n'))d\pi = (\sigma\Pi'(n') + \mathcal{Y}'(n'))dn \tag{A.4}$$

$$-\kappa e^{\pi}dy + (e^{\pi} + n\Pi'(n'))d\pi = \beta\Pi'(n')dn \tag{A.5}$$

$$\frac{\partial L}{\partial Y}e^{y}dy + \frac{\partial L}{\partial i}di + L(e^{y}, i)d\pi = -\frac{dn}{e^{\pi}} \tag{A.6}$$

The system (A.4)- (A.6) is a linear system in three equations and three unknowns $\{dy, di, d\pi\}$ for arbitrary dn. Solving for these variables, and using that $n < 0, \mathcal{Y}'(n) \leq 0, \Pi'(n) \leq 0$, we obtain $\frac{di}{dn} > 0, \frac{d\pi}{dn} < 0, \frac{dy}{dn} < 0$.

□