Efficient trade and ownership on networks*

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Abstract

Consider the problem of placing a valuable resource on a network before demands are realized, anticipating costs of subsequent transportation. We say that a node *reach dominates* another if it has cumulatively more neighbors for any distance such that transportation could be ex post efficient. Assuming IID demands, we show that the expected social surplus maximizing placement is confined to reach dominant nodes and that this extends to the optimal placement by a planner that maximizes expected profits and, if the resources is indivisible, to the second-best. Stochastic reach dominance generalizes reach dominance to account for different distributions and different distances between nodes. Further, we obtain a universal impossibility result: for sufficiently high transportation costs, there is no initial placement that permits ex post efficient reallocation, assuming that placement confers ownership and that the reallocation mechanism must respect incentive compatibility, individual rationality, and no-deficit constraints.

Keywords: mechanism design, networks, reach dominance, partnership models, transaction costs, supply chains

JEL Classification: D44, D82, L41

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1 Introduction

Suppose some resources have to be placed on nodes on a network before demands are realized, anticipating that reallocation will involve costly transportation. Examples range from the placement of medical or military personnel and equipment before the onslaught of a pandemic or an attack by enemy forces to the placement of production and storage facilities to the allocation of resources within an organization or across different firms. Where should the resources be placed? This is the question we address in this paper.

We assume that nodes in the network represent agents whose private values are independent draws from commonly known distributions. Transportation occurs along the edges of the network and involves a constant marginal cost, which, like the network structure, is commonly known. To fix ideas, consider first the problem with identical distributions, assuming that ex post efficient reallocation is always possible, with ex post efficient reallocation dictating that, for any given initial resource placement, the resources be shipped to the agent with the highest value net of transportation costs, which involves of course the possibility that they are not shipped at all.

To see what governs the placement that maximizes expected social surplus, consider a H-network with six nodes. If the transportation cost is so large that the resources are only ever shipped the length of one edge, the number of immediate neighbors of a node entirely determines the value of placing the resources at that node. Because the nodes at the opposite ends of the horizontal line in this H-network each have three immediate neighbors while all the other ones only have one, it follows that the optimal placement can be confined to these nodes when the marginal cost of transportation is so large. As this cost decreases, eventually shipping the length of two nodes will sometimes become ex post efficient. Because all nodes have two neighbors at a distance of two, it follows that optimal placements are still confined to these two nodes. In fact, this extends to any smaller marginal cost of transportation since these two nodes have more and closer neighbors than any of the other nodes. Formally, call the vector that contains the fraction of other nodes at various distances a node's reach vector. In our example, the reach vectors are $(2/5, 3/5)$ for the nodes at the opposite ends of the horizontal line and $(1/5, 2/5, 2/5)$ for all the other nodes, using the convention of not displaying elements that are zero. The nodes at the opposite ends of the horizontal line *reach dominate* the other ones for any marginal cost of transportation because cumulatively they have a higher fraction of nodes at distances of no more $k \in \{1, 2, 3\}$ links away. As we show, if there are nodes that are reach dominant, optimal placements are confined to these, and optimal placements never involve nodes that are reach dominated. Notice that the reach vector of a completely connected node is one, which implies that with completely connected agents optimal placements are confined to these. This means that, for example, in a star or wheel network, it is optimal to place all resources at the hub.

In general, whether a node is reach dominant may depend on the marginal cost of transportation. To see this, consider a wide H-network in which there is an additional node in the middle of horizontal line, so that there are seven nodes. The reach vectors of the agents at the opposite ends of the horizontal line are $(1/2, 1/6, 1/3)$, while the reach vector of the agent in its middle is $(1/3, 2/3)$ and the reach vectors of all other nodes are $(1/6, 1/3, 1/6, 1/3)$. When the marginal cost of transportation is so large that only shipments of length one are ex post efficient, the nodes at the opposite ends of the horizontal line are reach dominant. As shipments of length two or more become ex post efficient, these nodes and the one in the middle can no longer be ranked by reach dominance. However, they always reach dominate the peripheral nodes, which means that the resources are never optimally placed at a peripheral node. One can show that for uniformly distributed values on the unit interval, the resources are optimally placed at the opposite ends of the horizontal line when the marginal cost of transportation is sufficiently large and at the node in the middle otherwise. This has the interesting and perhaps counterintuitive feature that a reduction in transportation (or transaction) costs leads to optimal placements at a node that is more "metroplitain" insofar as it has fewer close neighbors but more at an intermediate distance.

If placing resources at a node confers control or ownership rights over these to that node, then ex post efficient reallocation may not be possible, subject to incentive compatibility, individual rationality and no-deficit constraints. We show that ex post efficiency is never possible with extremal ownership and, possibly more surprisingly, for any ownership structure if the marginal cost of transportation is sufficiently large. With this in mind, we first derive the constrained efficient reallocation mechanism and then use this mechanism to determine the optimal ownership structure, which, loosely speaking, finds a balance between incentive and transportation costs. While in general this optimal ownership structure is not determined via reach dominance, we show that the reach dominance arguments extend to optimal ownership if the resource is indivisible, as in the case of a production plant, and the agents draw their values from identical distributions.

Because profit maximization is isomorphic to social surplus maximization, it follows directly that the reach dominance argument for optimal placements—that is, when placement does not confer control—extends to the case where the planner's objective is profit maximization rather than social surplus. Further, assuming ex post efficient reallocation, the idea of reach dominance can be extended to allow for both heterogeneous distributions and links between nodes that are of different lengths. The key is to replace the fraction of agents at a given distance from a node by the distribution of the highest draw from the agents not further away from a node (including the distribution of the agent at that node) than some given distance. The ranking is then based on stochastic dominance, which is why we refer to this generalization as *stochastic reach dominance*. We also show that the analysis extends to the case in which the cost of transportation is a fixed cost, which seems an appropriate description when the cost relates to difficulties of communication.

This paper relates to the literature on the (im)possibility of ex post efficient trade initiated by Vickrey (1961) and Myerson and Satterthwaite (1983) and debates surrounding the Coase Theorem (Coase, 1960). That extremal ownership prevents ex post efficient trade follows from an extension of the impossibility theorem of Myerson and Satterthwaite to costly transportation.[1] That ex post efficient trade is not possible for any ownership vector if the marginal cost of transportation is sufficiently large is, to our knowledge, a new impossibility theorem. The fact that, if the marginal cost of transportation is small, the designer trades off incentive costs against transportation costs builds on the insight from the partnership literature that, without costly transportation, ex post efficient trade is possible with appropriately structured ownership; see, for example, Cramton et al. (1987), Che (2006), or Figueroa and Skreta (2012). We show that this insight extends to costly transportation, provided the marginal cost of transportation is sufficiently small. To solve the designer's problem, the paper builds on the work related to optimal trading mechanisms for asset markets—problems in which each agent's trading positions (buy, sell, remain inactive) are determined endogenously—and partnership models by Lu and Robert (2001) and Loertscher and Wasser (2019).[2] There is a related literature on networks,[3] including Akbarpour and Jackson (2018), which examines how diffusion patterns depend on the network placement of heterogeneous agents, and Houde et al. (2023), which shows that incentives for tax avoidance led Amazon to distort its distribution network in a way that increased transportation costs. In contrast, we examine how trade patterns depend on the ownership (or placement) of resources, holding fixed the network locations of the agents.

The remainder of this paper is structured as follows. Section 2 contains the setup together with the definitions of the various problems of interest and basic results. Sections 3 analyzes the problem when resource placement does not confer control, and 4 analyzes the problem when it does confer control. Extensions are presented in Section 5, and Section 6 concludes the paper.

[1] Of course, it is also an extension of the bilateral trade setup to settings with multiple buyers and one seller, but that extension is already in the literature (see e.g. Gresik and Satterthwaite, 1989).

[2] The term "asset market" has been used by Loertscher and Marx (2020, 2023) and Delacrétaz et al. (2022). Analyses of asset market problems (that do not use that label) are also provided by Lu and Robert (2001) and Li and Dworczak (2021).

[3] Condorelli et al. (2017) also examine trade on a network, but their focus is on dynamic bilateral bargaining with binary types and no transportation costs.

2 Setup

We assume n agents indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$ and a resource whose total supply is 1. Each agent $i \in \mathcal{N}$ is located at a node in an undirected graph that connects all agents, where $d_{ij} \in \{0, 1, \ldots, n-1\}$ is the length of the minimum path through the network between agents i and j (for all $i \in \mathcal{N}$, $d_{ii} = 0$). Before trade occurs, each agent $i \in \mathcal{N}$ holds a resource amount $r_i \in [0, 1]$ and $\sum_{i \in \mathcal{N}} r_i = 1$. Each agent i has constant marginal value v_i for the resource, which is independently drawn from the distribution F_i, which we assume has support that is bounded by 0 and 1 and that contains both 0 and 1. For some of our results, we assume that all agents draw their values from the same distribution F.

We assume that the cost of transporting x units of the resource from agent i to agent j is xcd_{ij}, where $c \geq 0$ is the commonly known marginal transportation cost per edge traveled. The $n \times n$ symmetric matrix $C = (C_{ij})_{i,j \in \mathcal{N}}$ is called a transportation cost matrix, with component C_{ij} representing the transportation cost between agents i and j, if for all $i, j \in \mathcal{N}$, $C_{ii} = 0$ and $C_{ij} = C_{ji} = cd_{ij}$.

Two agents i and j are directly connected if $d_{ij} = 1$. We say that a network is *complete* if every agent is directly connected to every other agent. Agent i is said to be *completely connected* if agent i is directly connected to every other agent, and we say that agent i is *maximally connected* if no other agent is directly connected to a larger number of other agents than is agent i.

For example, the *star* network with $n \geq 3$ agents is defined as having agent 1 as the hub and a transportation cost matrix such that for $i > 1$, $C_{1i} = c$, and for $1 < i < j$, $C_{ij} = 2c$. We define a *wheel* network with $n \geq 5$ agents to also have agent 1 as the hub, but with a transportation cost between two agents i and j with $i < j$ of $C_{ij} = c$ if either (i) $i = 1$ or (ii) $i > 1$ and $j - i \in \{1, n-2\}$, and otherwise a transportation cost of $C_{ij} = 2c$.

A trading mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ consists of allocation rule $\mathbf{Q} = (Q_i)_{i \in \mathcal{N}}$, where $Q_i(\mathbf{v})$ specifies agent i's quantity following trade, and payment rule $\mathbf{M}$, where $M_i(\mathbf{v})$ specifies the payment made by agent i to the mechanism. For the allocation rule, $Q_i : [0, 1]^n \to [0, 1]$ such that $\sum_{i \in \mathcal{N}} Q_i(\mathbf{v}) = 1$. For the payment rule, $M_i : [0, 1]^n \to \mathcal{R}$.

2.1 Ex post efficient trade

Given realized types $\mathbf{v}$, define the $n \times n$ binary matrix $V^e(\mathbf{v})$, each of whose rows sums to 1 by:[4]

$$V_{ij}^e(\mathbf{v}) = \begin{cases} 1 & \text{if } v_j - C_{ij} \geq \max_\ell v_\ell - C_{i\ell} \text{ and } v_j - C_{ij} > \max_{\ell<j} v_\ell - C_{i\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

This says that $V_{ij}^e(\mathbf{v}) = 1$ only if moving agent i's resources to agent j maximizes value net of transportation costs. If there are ties, then we (arbitrarily) identify the lowest indexed agent j with whom the maximum net value is achieved.

The ex post efficient allocation rule assigns to agent i the resources of agents j with $V_{ji}^e(\mathbf{v}) = 1$, that is $Q_{i,\mathbf{r}}^e(\mathbf{v}) \equiv \sum_{j \in \mathcal{N}} V_{ji}^e(\mathbf{v}) r_j$. Maximized social surplus is therefore

$$SS_{\mathbf{r}}^e(\mathbf{v}) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} (v_i - C_{ji})\, V_{ji}^e(\mathbf{v}) r_j, \tag{1}$$

with expected value

$$ss^e(\mathbf{r}) \equiv \mathbb{E}_{\mathbf{v}}[SS_{\mathbf{r}}^e(\mathbf{v})] = \sum_{j \in \mathcal{N}} r_j \mathbb{E}_{\mathbf{v}} \Big[\sum_{i \in \mathcal{N}} (v_i - C_{ji})\, V_{ji}^e(\mathbf{v}) \Big]. \tag{2}$$

Total transportation costs under the ex post efficient allocation rule are $T_{\mathbf{r}}^e(\mathbf{v}) \equiv \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} C_{ji} V_{ji}^e(\mathbf{v}) r_j$, and their expected value is

$$t_{\mathbf{r}}^e \equiv \mathbb{E}_{\mathbf{v}}[T_{\mathbf{r}}^e(\mathbf{v})] = \sum_{j \in \mathcal{N}} r_j \mathbb{E}_{\mathbf{v}} \Big[\sum_{i \in \mathcal{N}} C_{ji} V_{ji}^e(\mathbf{v}) \Big]. \tag{3}$$

2.2 Placement problems

In a *placement problem*, a central authority or social planner places resources with agents before types are realized. In this setting, we refer to $\mathbf{r}$ as the placement vector. The planner retains control over the resources, meaning that after agents' types are realized, the planner can direct the reallocation of resources subject to incentive compatibility and individual rationality constraints, where each agent's outside option is zero.

Reallocation in the placement problem is a one-sided allocation problem in which the agents are buyers with their outside options and worst-off types equal to zero. Because the planner can, for example, run a second-price auction, where each agent's bid is adjusted

[4] This specification of V^e uses a particular tie-breaking rule, but given our assumptions ties are zero probability events and so the particular tie-breaking rule does not affect our results.

for the required transportation cost, the reallocation phase always permits an incentive compatible and individually rational solution that does not run a deficit.[5]

We consider objectives for the planner of either social surplus maximization or profit maximization. In either case, the planner first places resources on the network and then implements an incentive compatible, individually rational mechanism that reallocates the resources and collects payment from the agents, with the planner paying for the associated transportation costs. A social-surplus-maximizing planner uses the incentive compatible, individually rational mechanism that maximizes expected social surplus net of transportation costs. A profit-maximizing planner uses the incentive compatible, individually rational mechanism that maximizes the planner's revenue net of transportation costs.

2.3 Ownership problems

In an *ownership problem*, a market designer determines resource ownership by the agents, where ownership gives an agent property rights or control over the resources. In this setting, we refer to $\mathbf{r}$ as the ownership vector. Then, following type realizations, the designer implements an incentive compatible, individually rational reallocation mechanism that does not run a deficit, i.e., the expected revenue to the designer is sufficient to cover expected transportation costs. The individual rationality constraints affecting the mechanism vary with the resource ownership because each agent's outside option is to consume its owned resources. Thus, an ownership problem is more constrained than a placement problem, where the agents' outside options are zero.

In the an ownership problem, if there is extremal resource ownership, i.e., $r_i = 1$ for some agent i, then the reallocation phase is a two-sided allocation problem with one seller (agent i) and $n-1$ buyers. In contrast, if resource ownership is dispersed among multiple agents, then the reallocation phase becomes what is sometimes called an "asset market" because the trading positions of the agents—buy, sell, or do not trade—depend, in general, on their own realized types and the realized types of all other agents.

For some, but not all, ownership vectors, there exists an incentive compatible, individually rational, deficit-free trade mechanism that implements ex post efficient trade. For ownership vectors for which ex post efficient trade is not possible, a social-surplus maximizing designer specifies the constrained-efficient mechanism, i.e., the mechanism that maximizes expected social surplus subject to incentive compatibility, individual rationality, and revenues that at least cover the transportation costs. A profit-maximizing designer specifies the incentive

[5]By standard arguments, it can always be made to balance the budget; see e.g., Börgers and Norman (2009).

compatible, individually rational mechanism that maximizes the designer's expected revenue from the agents net of the transportation costs.

For the analysis involving constrained-efficient and profit-maximizing reallocation mechanisms (defined below), we assume continuous distributions with densities $f_i > 0$ on $[0, 1]$ and that each agent i's virtual type functions,

$$\Psi_i^B(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)} \quad \text{and} \quad \Psi_i^S(v) \equiv v + \frac{F_i(v)}{f_i(v)},$$

are increasing. Despite this monotonicity of the virtual types, which corresponds to what Myerson (1981) calls the "regular" case, the mechanism design problem in the reallocation phase will not be regular away from extremal ownership. In the case of identical distributions, we simply write $\Psi^B(v)$ and $\Psi^S(v)$.

With the designer's trade mechanism in hand, whether it be ex post efficient, constrained efficient, or profit maximizing, we can then work backwards to determine the ownership vector that maximizes the designer's objective.

3 Optimal placement

As mentioned above, the first-best involves optimal resource placement followed by ex post efficient trade. In this section, we first characterize the optimal placement for a social-surplus-maximizing planner, and then we consider the case of a profit-maximizing planner.

3.1 Extremal placement is always optimal

The linearity in $\mathbf{r}$ of social surplus under ex post efficient trade, $SS^e_{\mathbf{r}}(\mathbf{v})$, which is defined in (1), implies that its expectation, $ss^e(\mathbf{r})$ defined in (2), is also linear in $\mathbf{r}$. This in turn implies that an extremal placement, i.e., $r_i = 1$ for some $i \in \mathcal{N}$, is always optimal.[6] We state this in the following proposition:

Proposition 1. *The placement problem has a solution involving extremal placement followed by ex post efficient trade.*

Proof. See Appendix A.

While extremal placement is not necessarily uniquely optimal, and while not any extremal placement will be optimal, one extremal placement always will be. Specifically, noting that

[6] Linearity of $SS^e_{\mathbf{r}}(\mathbf{v})$ in $\mathbf{r}$ means the for any $\mathbf{r}, \mathbf{r}'$ and any $a \in [0, 1]$, we have $SS^e_{a\mathbf{r}+(1-a)\mathbf{r}'}(\mathbf{v}) = aSS^e_{\mathbf{r}}(\mathbf{v}) + (1 - a)SS^e_{\mathbf{r}'}(\mathbf{v})$.

$r_n = 1 - \sum_{i=1}^{n} r_i$ and writing expected social surplus as a function only of $\mathbf{r}_{-n}$, then, given agent $j \in \{1, \ldots, n-1\}$ with $\frac{\partial ss^e(\mathbf{r}_{-n})}{\partial r_j} = \max_{\ell \in \{1,\ldots,n-1\}} \frac{\partial ss^e(\mathbf{r}_{-n})}{\partial r_\ell} \geq 0$, the extremal placement that has $r_j = 1$ is optimal; and if $\frac{\partial ss^e(\mathbf{r}_{-n})}{\partial r_j} < 0$ for all $j \in \{1, \ldots, n-1\}$, then the extremal placement that has $r_n = 1$ is uniquely optimal. If $\frac{\partial ss^e(\mathbf{r}_{-n})}{\partial r_j} = 0$ for all $j \in \{1, \ldots, n-1\}$, as is the case, for example, if $c = 0$, then any $\mathbf{r}$ is optimal.

3.2 Reach dominance and the first-best

In this subsection, we assume that $F_i = F$ for all $i \in \mathcal{N}$. This is mainly done for illustrative purposes since, as we show in Section 5.2, the key insights and mechanics hold more generally.

To develop an understanding of what determines optimal placement in the planner's problem, assume first that $c \in (1/2, 1)$. In this case, only the immediate neighbors of an agent are candidate trading partners for that agent. We denote by $n_i(1)$ the *degree centrality* of agent i, which is defined as the number of immediate neighbors of agent i divided by $n-1$, where $n-1$ is the maximum possible number of immediate neighbors (Jackson, 2008, p. 38). It then follows that $r_i = 1$ is optimal if and only if i has the maximum number of immediate neighbors, that is,

$$n_i(1) = \max_{j \in \mathcal{N}} n_j(1).$$

If there are multiple agents with the maximum number of immediate neighbors, then optimal placement can be dispersed across these agents.

For c less than $1/2$, we also have to take into account agents other than the immediate neighbors of agent i when determining the value of having $r_i = 1$. For any given $c \in (0, 1)$, the maximum reach that needs to be considered is $\min\{\lceil 1/c \rceil, n-1\}$, where $\lceil x \rceil$ denotes the largest integer no larger than x.

For any agent i, we let $\mathbf{n}_i = (n_i(1), \ldots, n_i(n-1))$ be the $n-1$-dimensional vector where for $\ell \in \{1, \ldots, n-1\}$,

$$n_i(\ell) \equiv \frac{1}{n-1} \left| \{ j \in \mathcal{N} \backslash \{i\} \mid d_{ij} = \ell \} \right|.$$

In words, $n_i(\ell)$ is the number of agents at distance ℓ from agent i, normalized by $n-1$. We refer to $\mathbf{n}_i$ as the *reach vector* of agent i. Then for any $i \in \mathcal{N}$, we have

$$\sum_{j=1}^{n-1} n_i(j) = 1.$$

Let $s(\mathbf{n})$ denote the expected social surplus associated with placement of the resource to an agent with reach vector $\mathbf{n}$, anticipating ex post efficient trade. Because closer neighbors are more valuable as potential trading partners, it follows that $s(\cdot)$ increases in the number of close neighbors. Formally, given $\mathbf{n}$ such that $n(\ell+1) > 0$ for some $\ell \in \{1, \ldots, n-2\}$, and given $\hat{\mathbf{n}}$ that is derived from $\mathbf{n}$ by moving a neighbor at distance $\ell+1$ to be one unit closer, that is, $\hat{\mathbf{n}}$ satisfies $\hat{n}(j) = n(j)$ for all $j \notin \{\ell, \ell+1\}$ and $\hat{n}(\ell) = n(\ell) + \frac{1}{1-n}$ and $\hat{n}(\ell+1) = n(\ell+1) - \frac{1}{1-n}$, we have

$$s(\hat{\mathbf{n}}) \geq s(\mathbf{n}),$$

where the inequality is strict if and only if $\ell < \lceil 1/c \rceil$.

Drawing on the concept of reach centrality from the graph theory literature, given $k \in \{1, \ldots, n-1\}$, agent i's *k-step reach centrality* is:[7]

$$\sigma_i(k) \equiv \sum_{j=1}^{k} n_i(j).$$

We can then employ k-step reach centrality to define reach dominance: agent i *reach dominates* (RDs) agent h given c if for all $k \in \{1, \ldots, \lceil 1/c \rceil\}$,

$$\sigma_i(k) \geq \sigma_h(k), \tag{4}$$

with a strict inequality for at least one k. Reach dominance induces an incomplete order and is equivalent to first-order stochastic dominance, with better outcomes having cumulatively higher probability, but, because there is is nothing stochastic here, we use the alternative term.

Given c, let $\mathcal{D}(c)$ denote the set of of agents who are reach dominated by some other agent. That is,

$$\mathcal{D}(c) = \{h \in \mathcal{N} \mid \exists\, i \in \mathcal{N} \text{ s.t. } i \text{ RDs } h \text{ given } c\}.$$

If follows that if $\mathbf{r}$ is part of the solution of the planner's problem, then $r_j = 0$ for all $j \in \mathcal{D}(c)$.

We define the (possibly empty) set of *reach dominant* agents given c, denoted $\mathcal{T}(c)$, to be the set of agents such that any agent not in $\mathcal{T}(c)$ is reach dominated given c by every agent in $\mathcal{T}(c)$ and for all $i, j \in \mathcal{T}(c)$ and all $\ell \in \{1, \ldots, \lceil 1/c \rceil\}$, (4) holds with equality. For example, if $c \in (1/2, 1)$, $\mathcal{T}(c)$ is the set of agents with the maximum number of immediate neighbors. It follows that:

Proposition 2. *Assume $F_i = F$ for all $i \in \mathcal{N}$. Under the first-best, resources are never placed with agents in $\mathcal{D}(c)$. If $\mathcal{T}(c)$ is nonempty, then under the first-best, resources are only placed with agents in $\mathcal{T}(c)$.*

A first corollary to Proposition 2, is, as foreshadowed above, the following:

[7] See, e.g., Borgatti et al. (2018, Chapter 10.3.5), Sosnowska and Skibski (2018).

Corollary 1. *Assume $F_i = F$ for all $i \in \mathcal{N}$. If $c \in (1/2, 1)$, then $\mathcal{T}(c)$ is nonempty and consists of the agents with the maximum number of immediate neighbors. Further, if the set of completely connected agents $\mathcal{J}$ is nonempty, then $\mathcal{J} = \mathcal{T}(c)$ for any $c \in (0, 1)$.*

The reason is simple: Completely connected agents are reach dominant for any $c \in (0, 1)$. Consequently, we have for the star and wheel networks:

Corollary 2. *Assume $F_i = F$ for all $i \in \mathcal{N}$. For star and wheel networks with $c \in (0, 1)$, all resources are optimally placed at the hub.*

As an illustration and application, begin by considering a line network with five nodes. We label the three inner nodes from left to right as 1, 2 and 3, and we label the peripheral nodes as 0 and 4 as shown in Figure 1(a). For this network, the reach vectors are:

$$\mathbf{n}_1 = \mathbf{n}_3 = (2/4, 1/4, 1/4)\,, \quad \mathbf{n}_2 = (2/4, 2/4) \quad \text{and} \quad \mathbf{n}_0 = \mathbf{n}_4 = (1/4, 1/4, 1/4, 1/4)\,.$$

It follows that agents 0 and 4 are reach dominated for any c by agents 1, 2, and 3. For $c \in (1/2, 1)$, $\mathcal{T}(c) = \{1, 2, 3\}$, and for $c < 1/2$, $\mathcal{T}(c) = \{2\}$. Thus, this is a network for which the set of first-best placement is determined entirely by reach dominance. While $\mathcal{T}(c)$ varies with c, it does so monotonically in the sense of set inclusion.

To see what happens beyond networks with this property, we amend the above network by adding a link to node 1 and a link to node 3, which we denote by $0'$ and $4'$, respectively, as shown in Figure 1(b). For this "H network," the reach vectors are:

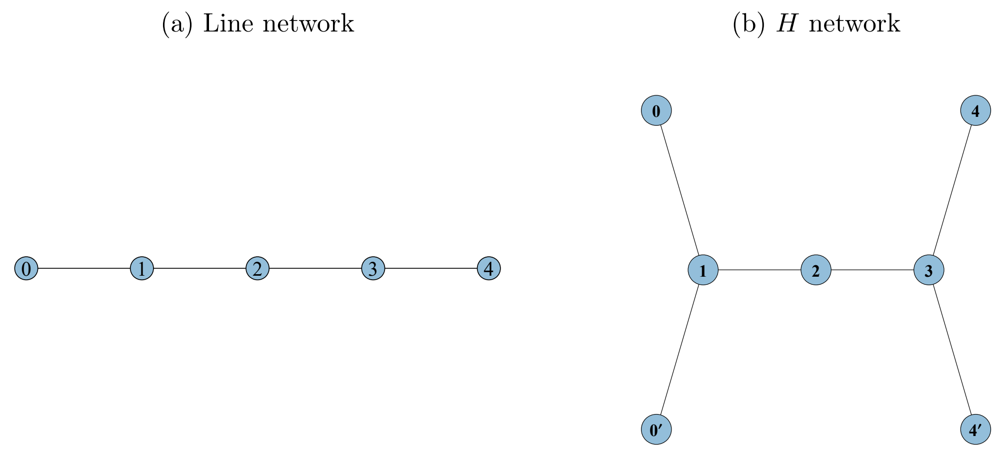

Figure 1: Panel (a): A network in which first-best placement varies monotonically with c. Panel (b): A network in which first-best placement varies nonmonotonically with c.

$$\mathbf{n}_1 = \mathbf{n}_3 = (3/6, 1/6, 2/6)\,, \quad \mathbf{n}_2 = (2/6, 4/6)\,, \ \text{and} \ \mathbf{n}_i = (1/6, 2/6, 1/6, 2/6)\,,$$

for $i \in \{0, 0', 4, 4'\}$.

The set of reach dominated agents now consists of the nodes $\{0, 0', 4, 4'\}$. For $c \in (1/2, 1)$, $\mathcal{T}(c) = \{1, 3\}$, and otherwise $\mathcal{T}(c) = \emptyset$. Thus, for $c < 1/2$, determining the optimal placement depends on c and requires computation. It will, of course, be confined to agents that are not reach dominated. For example, for F uniform, $r_1 = 1$ and $r_3 = 1$ are optimal for $c > 0.09$, and $r_2 = 1$ is optimal otherwise. Thus, the set of nodes that are most central—to whom resources are optimally placed—varies with c in a nonmonotonic way. For c large, resources are placed with agents 1 or 3, which have a more immediate neighbors. In contrast, when c is small, agent 2, which has few immediate neighbors but has all agents within a distance of two, is optimally the sole initial holder of the resources.

3.3 Optimal placement under profit maximization

While it is sensible to think of a planner as maximizing social surplus, it is also conceivable that a planner, i.e., an entity that retains control of the resources after placement, maximizes its expected profit, subject to the agents' incentive compatibility and individual rationality constraints. With that in mind, we now examine the profit-maximizing mechanism and placement for the planner. Its profit is defined as payments from the agents minus transportation costs. We begin by taking the placement as given and deriving the profit-maximizing mechanism, and then we optimize over the placement. Throughout this subsection, we assume that for all $i \in \mathcal{N}$, F_i is a continuous distribution with support $[0, 1]$ and density $f_i > 0$ that exhibits an increasing virtual value function $\Psi_i^B(v)$.

Because the planner retains control over the resources, it acts as a seller with all agents acting as buyers, including the agent with whom the resources are initially placed. Thus, the planner's optimal mechanism reallocates units to agents in order according to their virtual values net of transportation costs if and only if the net virtual value is positive. Specifically, the planner's profit-maximizing allocation rule is given by

$$Q_{i,\mathbf{r}}^P(\mathbf{v}) \equiv \sum_{j \in \mathcal{N}} V_{ji}^P(\mathbf{v}) r_j,$$

where

$$V_{ij}^P(\mathbf{v}) \equiv \begin{cases} 1 & \text{if } \Psi_j^B(v_j) - C_{ij} = \max_\ell \Psi_\ell^B(v_\ell) - C_{i\ell} \geq 0 \\ & \quad \text{and } \Psi_j^B(v_j) - C_{ij} > \max_{\ell < j} \Psi_\ell^B(v_\ell) - C_{i\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

To define the planner's expected profit, it will be useful to have the following lemma, which follows from standard mechanism design arguments:

Lemma 1. *For the planner's problem, given an incentive compatible mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, agent i's expected payment to the mechanism is* $\mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}\left[\Psi_i^B(v_i)Q_i(\mathbf{v})\right]$.

Using Lemma 1, the expected profit to the planner not including transportation costs is

$$\Pi_{\mathbf{r}}^P \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i\in\mathcal{N}} \Psi_i^B(v_i)Q_{i,\mathbf{r}}^P(\mathbf{v})\Big] = \mathbb{E}_{\mathbf{v}}\Big[\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} \Psi_i^B(v_i)V_{ji}^P(\mathbf{v})r_j\Big],$$

and expected transportation costs are

$$t_{\mathbf{r}}^P \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} C_{ji}V_{ji}^P(\mathbf{v})r_j\Big],$$

giving the planner maximized expected profit conditional on $\mathbf{r}$ of $\Pi_{\mathbf{r}}^P - t_{\mathbf{r}}^P$, which, notably, is linear in $\mathbf{r}$.

Thus, just as in the case of a planner that maximizes expected social surplus, extremal ownership is optimal for a planner that maximizes its expected profit. In particular, it is optimal to assign all resources to the agent j with the highest value of

$$\sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\mathbf{v}}[(\Psi_i^B(v_i) - C_{ji})V_{ji}^P(\mathbf{v})]\Big).$$

The key observation regarding optimal placement under profit maximization is a simple isomorphism between the maximizing expected social surplus and profit. To see this, let $\tilde{F}_i(\psi) = F_i(\Psi_i^{B^{-1}}(\psi))$ for $\psi \in (0,1]$, and $\tilde{F}_i(\psi) = F_i(\Psi_i^{B^{-1}}(0))$ otherwise, be the distribution of i's virtual value, conditional on its being positive. Note that, just like the values, the nonnegative virtual values are independent random variables, whose distributions are $\tilde{F}_i$ rather than F_i, and their support is $[0,1]$. Letting $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$, we then have $V_{ji}^P(\mathbf{v}) = V_{ji}^e(\boldsymbol{\psi})$ and thus

$$\sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\mathbf{v}}[(\Psi_i^B(v_i) - C_{ji})V_{ji}^P(\mathbf{v})]\Big) = \sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\boldsymbol{\psi}}[(\psi_i - C_{ji})V_{ji}^e(\boldsymbol{\psi})]\Big).$$

This basic observation yields the corollaries that follow.[8] The first one is an implication of Proposition 1:

Corollary 3. *Extremal ownership is optimal for a profit-maximizing planner.*

[8] The observation that profit and social surplus maximization are isomorphic in this sense was made and exploited by Loertscher et al. (2022). Whether that was the first explicit formalization of that fact we do not know. Clearly, it is implicit in the analysis of regular mechanism design problems, such as optimal auctions or bilateral trade problems à la Myerson (1981) and Myerson and Satterthwaite (1983).

In the case of identical distributions, that is, $F_i = F$ for all $i \in \mathcal{N}$, we have of course $\tilde{F}_i(\psi) = \tilde{F}(\psi)$ for all $i \in \mathcal{N}$ and all $\psi \in [0,1]$, where $\tilde{F}(\psi) = F(\Psi^{B^{-1}}(\psi))$ for $\psi \in (0,1]$, and $\tilde{F}(\psi) = F(\Psi^{B-1}(0))$ otherwise. Proposition 2 then yields the next corollary:

Corollary 4. *Assuming $F_i = F$ for all $i \in \mathcal{N}$ and a profit-maximizing planner, resources are never placed with agents in $\mathcal{D}(c)$, and if $\mathcal{T}(c)$ is nonempty, then resources are only placed with agents in $\mathcal{T}(c)$.*

Of course, our earlier Corollaries 1 and 2 also extend to the setting with profit maximization. For example, in the H network with $n = 7$ shown in Figure 1 and uniformly distributed types, we show above that for a social-surplus-maximizing planner, $r_1 = 1$ and $r_3 = 1$ are both optimal for $c \in (0.09, 1)$, but only $r_2 = 1$ is optimal for $c \in (0, 0.09)$. In the case of a profit-maximizing planner, the range where $r_2 = 1$ is uniquely optimal extends to all $c \in (0, 0.175)$. Thus, the profit-maximizing planner places resources with the agent with fewer immediate neighbors (but with all agents within a distance of 2) for a larger range of costs. As intuition, notice that the planner's expected profit in the profit-maximizing mechanism is the same as in the ex post efficient mechanism, but with types drawn from different distributions, i.e., with agent i's type drawn from the distribution of $\Psi_i^B(v_i)$. Because $\Psi_i^B(v_i) < v_i$ for $v_i \in [0,1)$, it is as if the profit-maximizing planner faces agents with a worse distribution. When facing a worse distribution, the planner values having the extra "draws" within close range (specifically within a distance of 2), that come with having the resources placed with agent 2 rather than with agents 1 or 3. As the distribution becomes worse, having access to additional type realizations becomes more valuable.[9]

4 Optimal ownership

We now consider ownership problems, which as mentioned above, arise when agents are endowed with ownership of resources at their nodes. We first derive conditions for ex post efficiency to be (im)possible. Then we derive the constrained-efficient mechanism when ex post efficient trade is not possible and characterize conditions under which only constrained-efficient trade is possible. We can then derive the optimal ownership for a social-surplus-

[9] For example, consider ex post efficient trade and types drawn from $F_i = F$, where F is the uniform distribution on $[\ell, 1]$, where $\ell < 1$. Then for $\ell = 0$, the problem is one of a social-surplus-maximizing planner facing types drawn from the uniform distribution on $[0,1]$. For $\ell = -1$, the problem is one of a profit-maximizing planner facing types drawn from the uniform distribution on $[0,1]$ because in that case the agents' virtual values are uniformly distributed on $[-1,1]$. For the H network with $n = 7$, $r_2 = 1$ is uniquely optimal for $c \in (0, \bar{c}(\ell))$, and $r_1 = 1$ and $r_3 = 1$ are optimal for $c \in (\bar{c}(\ell), 1)$, where $\bar{c}(\ell)$ is decreasing in ℓ, implying that the range of costs for which $r_2 = 1$ is optimal decreases with ℓ.

maximizing and a profit-maximizing designer. Throughout this section, assume that for all $i \in \mathcal{N}$, F_i is a continuous distribution with support $[0,1]$ and density $f_i > 0$.

4.1 Impossibility of ex post efficient trade

We begin by establishing two sets of impossibility results. The first, in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983), is based on the observation that ex post efficient trade is impossible under extremal ownership. This implies that the first-best is not possible when the first-best dictates extremal ownership, which is, for example, the case for the star and wheel networks with identical distributions. Second, we show that ex post efficiency is impossible for any ownership vector when $c \geq 1/2$.

Impossibility with extremal ownership

Consider an extremal ownership vector in which $r_1 = 1$, so that agent 1 is the seller whenever there is trade. Trade between agent 1 and agent $i \in \{2, \ldots, n\}$ is ex post efficient if and only if $v_i - C_{1i} = \max_{j \in \{2,\ldots,n\}} v_j - C_{1j}$ and $v_i - C_{1i} > v_1$. We denote by i_{eff} the index of such an agent i. Consider then the market-clearing (Walrasian) prices that establish ex post efficient trade given types $\mathbf{v}$. Without loss of generality, we let the seller bear the transportation cost.

If $(p_1^W, \ldots, p_n^W)$ is a Walrasian price vector, then it has to satisfy $v_1 + C_{1i_{\text{eff}}} \leq p_{i_{\text{eff}}}^W \leq v_{i_{\text{eff}}}$ so that agent 1 is willing to pay the transportation cost $C_{1i_{\text{eff}}}$ to sell to agent i_{eff} at price $p_{i_{\text{eff}}}^W$ and so that agent i_{eff} is willing to buy at this price. In addition, for $j \in \{2, \ldots, n\} \backslash i_{\text{eff}}$, we require that $v_j \leq p_j^W$ so that agent j does not want to buy at price p_j^W and $p_j^W - C_{1j} \leq p_{i_{\text{eff}}}^W - C_{1i_{\text{eff}}}$ so that agent 1 does not want to sell to agent j instead of agent i_{eff}. Putting these together, the largest range of trading prices involves $p_j^W = v_j$ for all $j \in \{2, \ldots, n\} \backslash i_{\text{eff}}$ and $p_{i_{\text{eff}}}^W$ such that

$$\underline{p}^W \equiv \max_{j \in \{2,\ldots,n\} \backslash i_{\text{eff}}} \{v_1, v_j - C_{1j}\} + C_{1i_{\text{eff}}} \leq p_{i_{\text{eff}}}^W \leq v_{i_{\text{eff}}} \equiv \overline{p}^W,$$

where $\underline{p}^W$ and $\overline{p}^W$ denote the lowest and highest Walrasian prices, respectively, at which trade occurs.

As is reasonably well known and easily established, a trading buyer's payment in the VCG mechanism is $\underline{p}^W$ and a trading seller's payment is $\overline{p}^W$ (see e.g. Delacrétaz et al., 2022).[10]

[10] The gains from trade with agent i_{eff} present, but excluding its value for the allocation, are $-v_1 - C_{1i_{\text{eff}}}$, whereas gains from trade with agent i_{eff} reporting a value of 0 are $\max\{0, \max_{j \in \{2,\ldots,n\} \backslash i_{\text{eff}}} v_j - C_{1j} - v_1\}$. Hence, the VCG transfer of agent i_{eff} is $\max\{0, \max_{j \in \{2,\ldots,n\} \backslash i_{\text{eff}}} v_j - C_{1j} - v_1\} - (-v_1 - C_{1i_{\text{eff}}}) = \max\{v_1, \max_{j \in \{2,\ldots,n\} \backslash i_{\text{eff}}} v_j - C_{1j}\} + C_{1i_{\text{eff}}} = \underline{p}^W$. Similarly, gains from trade with agent 1 present, but without its value for the allocation, are $v_{i_{\text{eff}}}$, whereas they are 0 with the seller reporting a cost of 1. Hence, the VCG payment that agent 1 receives is $v_{i_{\text{eff}}} = \overline{p}^W$.

Consequently, if trade occurs under ex post efficiency, then the revenue of the mechanism is

$$\underline{p}^W - \overline{p}^W \leq 0,$$

where the inequality is strict unless $\max_{j\in\{2,\dots,n\}\setminus i_{\text{eff}}} v_j - C_{1j} = v_{i_{\text{eff}}} - C_{1i_{\text{eff}}}$. Because ties have probability 0 with continuous distributions, it follows that the VCG mechanism almost always runs a deficit when trade is ex post efficient (and never a budget surplus). Consequently, in expectation, the VCG mechanism runs a deficit. Because the ex post and hence interim expected payoffs are zero for buyers of type 0 and for the seller of type 1, it follows that the VCG mechanism satisfies the interim individual rationality constraints with equality. By the payoff equivalence theorem, this implies that no other ex post efficient, (Bayesian or dominant strategy) incentive compatible, and interim individually rational mechanism runs a smaller deficit. Because the VCG mechanism runs a deficit, it follows that ex post efficiency is impossible for any network when ownership is extremal. We summarize this in the following result:

Proposition 3. *If $r_i = 1$ and $C_{ij} < 1$ for some $j \in \mathcal{N}\setminus\{i\}$, then ex post efficient trade is impossible.*

If $C_{1i} \geq 1$ for all $i \neq 1$, then trade is never ex post efficient, and ex post efficient trade is possible in the same trivial way that it would be possible in Myerson and Satterthwaite (1983) if the upper bound of the support of the buyer's value distribution were less than the lower bound of the support of the seller's cost distribution.

Corollary 2 and Proposition 3 imply immediately:

Corollary 5. *In an ownership problem, for star and wheel networks with $F_i = F$ for all $i \in \mathcal{N}$ and $c \in (0,1)$, the first-best cannot be achieved.*

Universal impossibility

Proposition 3 and Corollary 5 are, as foreshadowed, impossibility results in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983) insofar as they depend on extremal ownership. They imply that one will need to consider constrained-efficient trade if $r_i = 1$ for some $i \in \mathcal{N}$. But they leave open the question of whether there exists nonextremal ownership vectors that permit ex post efficient trade. Intuition based on Cramton et al. (1987) may suggest that the answer is affirmative.

With that in mind, our next result is probably unexpected because it states that for

$c \geq 1/2$, ex post efficient trade is impossible for *any* ownership vector.[11] As intuition for the result, note that under ex post efficiency, any agent of type $v \leq 1-c$ ever only trades as a seller, and any agent of type $v \geq c$ ever only trades as a buyer. Consequently, for $c \geq 1/2$, agents with types $v \in [1-c, c]$ never trade and have payoffs of 0. As a result, for $c \geq 1/2$, the trading problem is not only ex post two-sided but already *ad interim*—knowing only its type, every agent knows whether it will trade as a buyer (if $v > c$) or as a seller (if $v < 1-c$) if it trades and agents with types between $1-c$ and c know that they will never trade. As in the proof of Proposition 3, it therefore suffices to verify that transportation costs are not covered under VCG transfers, and that the VCG mechanism satisfies the agents' ex post individual rationality constraints with equality. This then also means that it satisfies interim individual rationality with equality.

Proposition 4. *In an ownership problem, for $c \geq 1/2$, ex post efficient reallocation is impossible for any network and any ownership vector.*

Proof. See Appendix A.

Proposition 4 provides a simple sufficient condition for ex post efficient reallocation to be impossible. The result is a form of "Non-Coase Theorem" because it provides a condition under which ex post efficient reallocation is impossible for any ownership vector. While Proposition 4 shows that incentive costs together with sufficiently large transportation costs imply insurmountable transactions costs, it has the positive implication that reducing transportation costs has the additional, seemingly overlooked, benefit of making markets work better.

4.2 Possibility for small costs and dispersed ownership

We now provide a necessary and sufficient condition for ex post efficient trade to be possible. To do so, it will be useful to begin with two lemmas, which characterize, for any incentive compatible mechanism, agents' worst-off types and expected payments to the mechanism.

Consider an incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and let $u_i(v) \equiv q_i(v)v - m_i(v) - r_i v$ denote agent i's interim expected gains from participation in the mechanism, net of its outside option. Incentive compatibility implies that q_i is nondecreasing, from which it follows that the first-order condition $u_i'(v) = q_i(v) - r_i = 0$ characterizes a global minimum for agent i interim expected payoff, provided that it is satisfied for some v. The following lemma, a version of which was first established by Cramton et al. (1987), characterizes the set of worst-off types for any allocation rule such that q_i is nondecreasing:

[11] This uses our assumption that the support of the agents' type distribution is $[0, 1]$. For a more general support of $[\underline{v}, \overline{v}]$, the required condition on transportation costs is that $c \geq (\overline{v} + \underline{v})/2$.

Lemma 2. *Given an incentive compatible, individually rational mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, if there is a* v_i *such that* $q_i(v_i) = r_i$*, then the set of worst-off types for agent* i *is* $\{v_i \mid q_i(v_i) = r_i\}$*. If* $q_i(v_i) \neq r_i$ *for all* $v_i \in [0, 1]$*, then the set of worst-off types for agent* i *is the singleton set* $\{v_i \mid q_i(v) < r_i \ \forall v < v_i \text{ and } q_i(v) > r_i \ \forall v > v_i\}$.

As observed by Cramton et al. (1987), intuitively, the worst-off type of an agent expects on average to be neither a net buyer nor a net seller, and therefore an agent with the worst-off type has no incentive to overstate or understate its valuation and so does not need to be compensated to induce truthful reporting, which is why it is the worst-off type.

Given an incentive compatible mechanism, we can use standard mechanism design techniques to write an agent's expected payment to the mechanism in terms of its worst-off type and its virtual type functions. Defining

$$\Psi_i(v; \omega) \equiv \begin{cases} \Psi_i^S(v) & \text{if } v \leq \omega, \\ \Psi_i^B(v) & \text{if } v > \omega, \end{cases}$$

we have:

Lemma 3. *For a placement problem with ownership* $\mathbf{r}$ *and incentive compatible trade mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, for any agent* i *and* $\omega_i \in [0, 1]$*, agent* i*'s expected payment to the mechanism can be written as*

$$\mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}\left[\Psi_i(v_i; \omega_i) Q_i(\mathbf{v})\right] - r_i \omega_i - u_i(\omega_i),$$

where $u_i(\omega_i) \equiv \mathbb{E}_{\mathbf{v}_{-i}}[Q_i(\omega_i, \mathbf{v}_{-i})\omega_i - M_i(\omega_i, \mathbf{v}_{-i})] - r_i \omega_i$.

Proof. See Appendix A.

Letting $\omega_{i,\mathbf{r}}^e$ denote agent i's worst-off type (or one of its worst-off types) under the ex post efficient allocation rule $Q_{i,\mathbf{r}}^e$, and using Lemma 3, we obtain an expression for the expected budget surplus (not including transportation costs) of an ex post efficient reallocation mechanism that satisfies the agents' individual rationality constraints with equality:

$$\Pi_{\mathbf{r}}^e \equiv \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi_i(v_i; \omega_{i,\mathbf{r}}^e) Q_{i,\mathbf{r}}^e(\mathbf{v})\right] - \sum_{i=1}^{n} \omega_{i,\mathbf{r}}^e r_i.$$

Thus, ex post efficient reallocation is possible without running a deficit if and only if $\Pi_{\mathbf{r}}^e \geq t_{\mathbf{r}}^e$. It follows that the necessary and sufficient condition for the possibility of ex post efficient reallocation is

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\Psi_i(v_i; \omega_{i,\mathbf{r}}^e) - C_{ji}) V_{ji}^e(\mathbf{v}) r_j\right] \geq \sum_{i=1}^{n} \omega_{i,\mathbf{r}}^e r_i. \tag{5}$$

Condition (5) implicitly defines the set of combinations of ownership vectors $\mathbf{r}$ and transportation cost matrices C such that ex post efficient trade is possible. We can use (5) to calculate, for a given $\mathbf{r}$, the maximum c such that ex post efficient trade is possible, denoted by $c_n^{max}(\mathbf{r})$ (and defined to be $-\infty$ if no such c exists). Further, for each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, the boundary of the set of ownership vectors such that ex post efficient trade is possible is defined by vectors $\mathbf{r}$ that satisfy (5) with equality. For example, if we consider a star or wheel network with $\mathbf{r} = (r, (1-r)/(n-1), \ldots, (1-r)/(n-1))$, then for each r and each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, we can calculate the maximum r, such that ex post efficient trade is possible, denoted by $\overline{r}_n(c)$. We illustrate this in Figure 2 for uniformly distributed types.

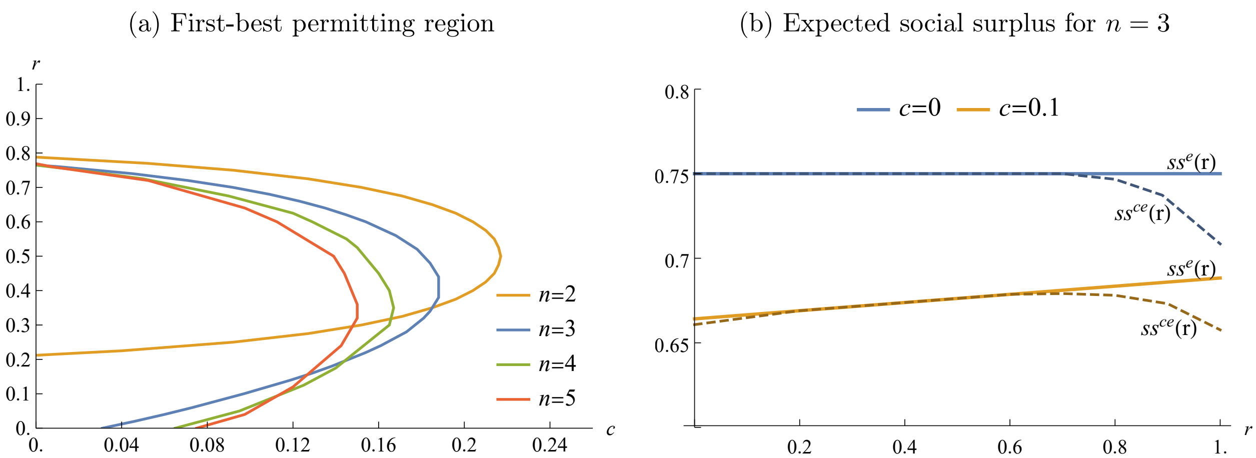

Figure 2: Panel (a): Values for (r, c) that permit ex post efficient reallocation and expected social surplus for star networks. As illustrated, $\overline{r}_2(0) = 0.7887$, $\overline{r}_3(0) = 0.7654$, $\overline{r}_4(0) = 0.7647$, and $\overline{r}_5(0) = 0.7689$. Further, $c_2^{max} = 0.217$, $c_3^{max} = 0.187$, $c_4^{max} = 0.1675$, and $c_5^{max} = 0.150$. Panel (b): Expected social surplus under ex post efficient reallocation, $ss^e(\mathbf{r})$, and under constrained-efficient reallocation, $ss^{ce}(\mathbf{r})$. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

As illustrated in Figure 2, for a star network with $n = 2$ and $c = 0$, the first-best is possible for all $r \in [0.21, 0.79]$, which corresponds to the values obtained by Cramton et al. (1987). For $n = 2$, if $c > c_2^{max}$, then $\mathcal{R}(c)$ is empty (for uniformly distributed types, this occurs for $c > 0.217$), in which case the constrained-efficient mechanism is used, implying that the ownership $\mathbf{r} = (1/2, 1/2)$ maximizes expected social surplus. While Figure 2 illustrates that $\overline{r}_n(0)$ need not be monotone in n, if one properly accounts for the expansion in the number of agents by calculating the resources accounted for by the first $x \in [0, 1]$ share of agents, giving us a distribution of resources G_n defined by

$$G_n(x) \equiv \begin{cases} nx\overline{r}_n(0) & \text{if } x \leq 1/n, \\ 1 - (1-x)n^{\frac{1-\overline{r}_n(0)}{n-1}} & \text{if } x > 1/n, \end{cases}$$

then one finds that, at least for uniformly distributed types, G_n first-order stochastically dominates $G_{n'}$ if $n < n'$ (see Appendix ??). Thus, with more agents, the boundary of $\mathcal{R}(0)$ shifts towards greater concentration at the hub.

4.3 Constrained-efficient reallocation mechanism

An implication of Proposition 3 is that we must have either a nonextremal ownership or trade that is not ex post efficient, or both. And as illustrated above, for some transportation costs, ex post efficient trade is not an option for any ownership vector. Thus, we next characterize constrained-efficient reallocation mechanisms. This analysis assumes that for each $i \in \mathcal{N}$, the virtual types functions $\Psi_i^B(v)$ and $\Psi_i^S(v)$ are increasing.

The constrained-efficient reallocation mechanism maximizes the sum of the agents' expected surpluses subject to incentive compatibility, individual rationality, and no deficit, which requires that the expected budget surplus of the mechanism must be sufficient to cover the expected transportation costs. To define the mechanism, it is useful to introduce the notion of weighted virtual types and their ironed counterparts. For $a \in [0,1]$, we denote by $\Psi_{i,a}(v;\hat{v})$ the weighted virtual type of agent i with type v and threshold type $\hat{v}$,

$$\Psi_{i,a}(v;\hat{v}) \equiv \begin{cases} \Psi_{i,a}^S(v) & \text{if } v \le \hat{v}, \\ \Psi_{i,a}^B(v) & \text{if } v > \hat{v}, \end{cases}$$

where $\Psi_{i,a}^S(v) \equiv v + (1-a)\frac{F_i(v)}{f_i(v)}$ and $\Psi_{i,a}^B(v) \equiv v - (1-a)\frac{1-F_i(v)}{f_i(v)}$ are agent i's weighted virtual cost and virtual value functions. (With identical distributions, we write $\Psi_a^S(v) \equiv v + (1-a)\frac{F(v)}{f(v)}$, $\Psi_a^B(v) \equiv v - (1-a)\frac{1-F(v)}{f(v)}$ and $\Psi_a(v;\hat{v})$ in lieu of $\Psi_{i,a}(v;\hat{v})$.) Although, as noted above, we assume that $\Psi_i^S(v)$ and $\Psi_i^B(v)$ are increasing, which implies that $\Psi_{i,a}^S(v)$ and $\Psi_{i,a}^B(v)$ are increasing for all $a \in [0,1]$, nevertheless, $\Psi_{i,a}(v;\hat{v})$ is not monotone, and so, as we shall see, will require ironing. We let $\overline{\Psi}_{i,a}(v;\hat{v})$ denote the ironed weighted virtual type of an agent with type v and threshold type $\hat{v}$, defined by

$$\overline{\Psi}_{i,a}(v;\hat{v}) \equiv \begin{cases} \Psi_{i,a}^S(v) & \text{if } \Psi_{i,a}^S(v) < z, \\ z & \text{if } \Psi_{i,a}^B(v) \le z \le \Psi_{i,a}^S(v), \\ \Psi_{i,a}^B(v) & \text{if } z < \Psi_{i,a}^B(v), \end{cases}$$

where the ironing parameter z satisfies

$$\int_0^{\hat{v}} \max\{0, \Psi_{i,a}^S(v) - z\} dF_i(v) = \int_{\hat{v}}^1 \max\{0, z - \Psi_{i,a}^B(v)\} dF_i(v). \tag{6}$$

The constrained-efficient mechanism, as shown by Loertscher and Wasser (2019), is the solution to a saddle point problem that simultaneously chooses the allocation rule to maximize expected social surplus given agents' worst-off types, subject to constraints, and chooses the agents' worst-off types to minimize their expected payoffs given the allocation rule.

Focusing on the maximization problem for the moment, let ω_i denote agent i's worst-off type. Then letting ρ be the Lagrange multiplier on the no-deficit constraint and μ_i be the Lagrange multiplier on agent i's individual rationality constraint, and using Lemma 3, we have the Lagrangian

$$\begin{aligned}\mathcal{L} \;\equiv\; & \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}(Q_i(\mathbf{v})v_i - Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) + r_i\omega_i + u_i(\omega_i))\right.\\ & \left.+\rho\left(\sum_{i=1}^{n}(Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) - r_i\omega_i - u_i(\omega_i)) - T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))\right)\right] + \sum_{i=1}^{n}\mu_i u_i(\omega_i),\end{aligned}$$

where $T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))$ is the total transportation cost under allocation rule $\mathbf{Q}$ and type vector $\mathbf{v}$ when the ownership vector is $\mathbf{r}$. Rearranging this, we have

$$\mathcal{L} = \rho\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}Q_i(\mathbf{v})\Psi_{i,\frac{1}{\rho}}(v_i;\omega_i) - T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))\right] + (1-\rho)\sum_{i=1}^{n}r_i\omega_i + \sum_{i=1}^{n}(1-\rho+\mu_i)u_i(\omega_i).$$

Given $\boldsymbol{\omega}$ and ρ, we can then solve for $\mathbf{Q}$ pointwise, subject to the constraint that $\mathbf{Q}$ is nondecreasing (thus, requiring ironing). Specifically, given Lagrange multiplier ρ and worst-off types $\boldsymbol{\omega}$, the constrained-efficient reallocation rule for agent i is given by

$$Q_{i,\mathbf{r}}^{ce}(\mathbf{v};\rho,\boldsymbol{\omega}) \equiv \sum_{j\in\mathcal{N}} V_{ji}^{ce}(\mathbf{v};\rho,\boldsymbol{\omega})r_j,$$

where V^{ce} is defined analogously to V^e, but with actual types replaced by ironed virtual types:

$$V_{ij}^{ce}(\mathbf{v};\rho,\boldsymbol{\omega}) \equiv \begin{cases} 1 & \text{if } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - C_{ij} \geq \max_{\ell}\overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - C_{i\ell} \\ & \quad \text{and } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - C_{ij} > \max_{\ell<j}\overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - C_{i\ell}, \\ 0 & \text{otherwise.}\end{cases}$$

Using Lemma 3, the expected budget surplus under binding individual rationality is

$$\Pi_{\mathbf{r}}^{ce}(\rho,\boldsymbol{\omega}) \;\equiv\; \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}\Psi_i(v_i;\omega_i)Q_{i,\mathbf{r}}^{ce}(\mathbf{v};\rho,\boldsymbol{\omega})\right] - \sum_{i=1}^{n}\omega_i r_i.$$

Expected transportation costs under the constrained-efficient allocation rule are:

$$t_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}[\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ji} V_{ji}^{ce}(\mathbf{v}; \rho, \boldsymbol{\omega}) r_j].$$

Given this, we can state the following result:

Proposition 5. *The constrained-efficient reallocation rule is the same as the ex post efficient reallocation rule if $\Pi_{\mathbf{r}}^e \geq t_{\mathbf{r}}^e$, and otherwise it is defined by $\mathbf{Q}_{\mathbf{r}}^{ce}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*)$, where $\boldsymbol{\omega}^*$ and ρ^* are such that for all $i \in \mathcal{N}$, $\mathbb{E}_{\mathbf{v}_{-i}}[Q_{i,\mathbf{r}}^{ce}(\omega_i^*, \mathbf{v}_{-i}; \rho^*, \boldsymbol{\omega}^*)] = r_i$ and $\rho^* = \arg\min_{\rho}\{\rho \geq 1 \mid \Pi_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}^*) \geq t_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}^*)\}$.*

The constrained-efficient reallocation mechanism has the allocation rule specified by Proposition 5 along with the payment rule given by Lemma 3, with $\boldsymbol{\omega}$ equal to $\boldsymbol{\omega}^*$. Figure 3 illustrates the contrast between the ex post efficient and constrained-efficient reallocation rules for the case of two agents. In setups with identical distributions and no transportation costs, the constrained-efficient reallocation rule coincides with the ex post efficient reallocation when both agents have small values and when both agents have large values if ironing occurs in the interior (see Loertscher and Wasser, 2019; Loertscher and Marx, 2022). To see this, assume $v_1 > v_2$ and observe that under the optimal mechanism with $c = 0$ trade occurs if and only if $\Psi_a^S(v_1) > \Psi_a^S(v_2)$ when both types are small, respectively $\Psi_a^B(v_1) > \Psi_a^B(v_2)$ when both types are large. With identical distributions this is equivalent to $v_1 > v_2$.

Interestingly, this feature does not extend to a settings with positive transportation costs, in which case it is easy to obtain, locally, more trade than under ex post efficiency. To see this, assume ironing occurs in the interior and consider v_1 and v_2 with $v_1 > v_2$, both of which are sufficiently small so that trade of r_2 occurs if and only if $\Psi_a^S(v_1) > \Psi_a^S(v_2) + c$, which is equivalent to

$$v_1 > v_2 + (1-a)\left[\frac{F(v_2)}{f(v_2)} - \frac{F(v_1)}{f(v_1)}\right] + c. \tag{7}$$

Under the constrained-efficient mechanism, $a = 1/\rho^* < 1$, and so the right side of (7) is smaller than $v_2 + c$—which is the condition for trade under ex post efficiency—if F/f is increasing. (And when both types are large, trade of r_2 occurs if and only if $v_1 > v_2 + (1 - a)\left[\frac{1-F(v_1)}{f(v_1)} - \frac{1-F(v_2)}{f(v_2)}\right] + c$, whose right-hand side is less than $v_2 + c$ if $(1-F)/f$ is decreasing.) These hazard rate properties are satisfied, for example, by the uniform distribution. This possibility of locally excessive trade is illustrated in Figure 3(c), which is plotted for the uniform distribution. At the boundaries, the contours of the constrained-efficient reallocation lie "inside" the contours for the ex post efficient allocation. Away from the boundaries, the constrained-efficient reallocation is shifted towards giving a greater quantity to the agent with greater initial resources (agent 1 in Figure 3), relative to the ex post efficient allocation.[12]

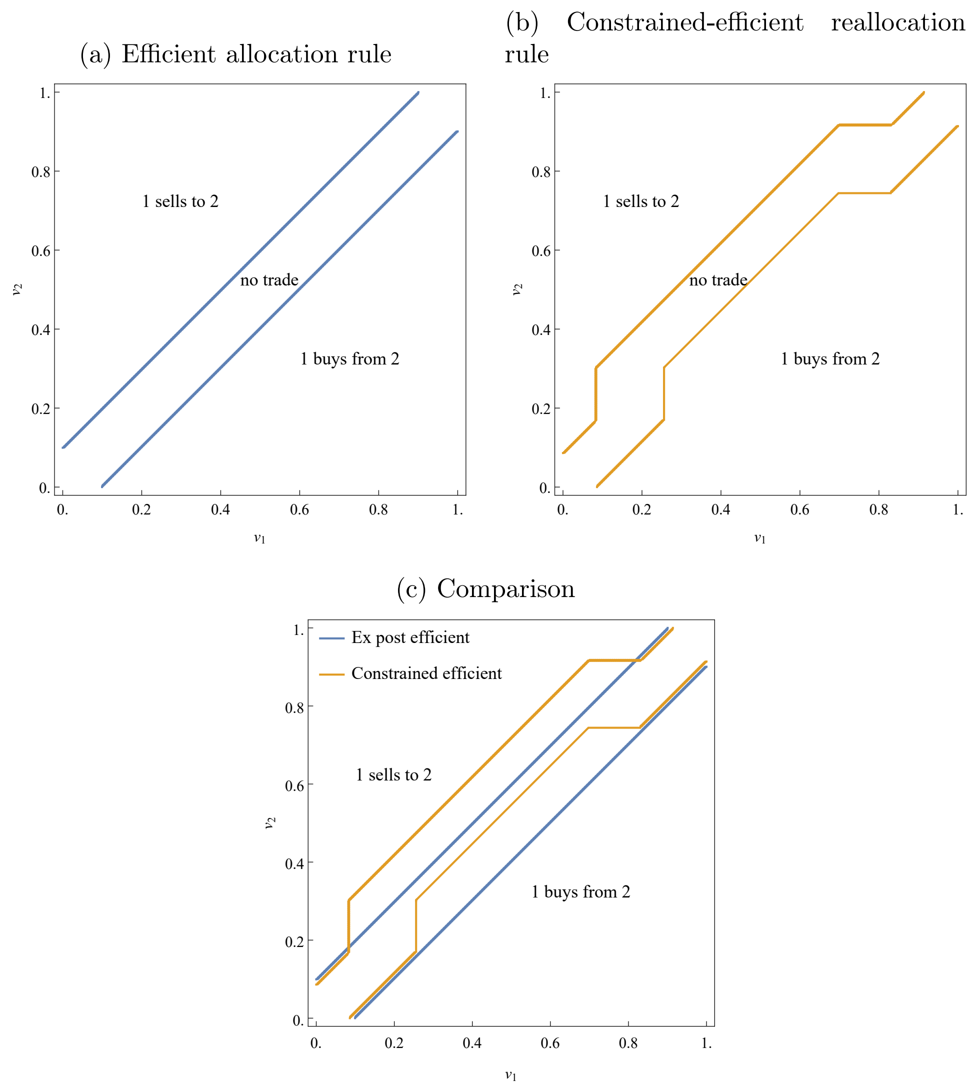

Figure 3: Efficient and constrained-efficient reallocation rules. Assumes $n = 2$, $\mathbf{r} = (0.9, 0.1)$, $c = 0.1$, and uniformly distributed types. The constrained-efficient results are based on numerical calculations of $\rho^* = 1.18$ and ironing parameters $z_1^* = 0.8047$ and $z_2^* = 0.1953$.

[12]This possibility of locally excessive trade depends simultaneously on transportation costs and on the ironing ranges being interior. For example, if $c > 0$ is the fixed cost of producing a public good, in the optimal mechanism production occurs if and only if $\sum_i \Psi_a^B(v_i) > c$, which for any $a < 1$ is more restrictive than the condition for production under ex post efficiency. Likewise, if c is a transportation cost but ironing ranges are at the bounds, for example because $r_2 = 1$, trade occurs if and only if $\Psi_a^B(v_1) > \Psi_a^S(v_2) + c$, where for any any $a < 1$, the left-hand side is less than v_1 and the right-hand is larger than $v_2 + c$.

4.4 Optimal ownership under social-surplus maximization

Above we show that in a placement problem, extremal placement is optimal. This raises the question whether extremal ownership is optimal in an ownership problem. We next show that it is not, which implies that the first-best is not possible in an ownership problem whenever extremal placement is uniquely optimal in the placement problem. In an ownership problem, extremal ownership creates a tension with incentives, and this tension is at the center of the analysis that follows.

The results of this section establish that given a network with at least one completely connected agent: for $c > 0$ sufficiently small, an ownership problem with a social-surplus-maximizing designer is solved by ownership that differs from the optimal placement in the corresponding placement problem; for $c < 1$ sufficiently large, the optimal ownership matches the optimal placement, albeit with a different reallocation mechanism; and for intermediate c, the ownership problem is solved by both different ownership and a different reallocation mechanism than for the placement problem.

It will be useful to define four sets of ownership vectors, parameterized by the transportation cost by c. First, let $\mathcal{R}(c)$ be the set of ownership vectors, possibly empty, satisfying (5), where ex post efficient trade is possible. Second, let $\mathcal{R}^B(c)$ denote the set of ownership vectors on the "boundary" of $\mathcal{R}(c)$ that minimize expected transportation costs subject to allowing ex post efficient trade, if such a ownership vectors exists. Thus, for c such that $\mathcal{R}(c) \neq \emptyset$, we define $\mathcal{R}^B(c) \equiv \{\mathbf{r} \in \mathcal{R}(c) \mid \mathbf{r} \in \arg\min_{\mathbf{r}} t^e_{\mathbf{r}}\}$. Third, we let $\mathcal{R}^P(c)$ denote the planner's set of optimal placement vectors, and, fourth, we let $\mathcal{R}^D(c)$ denote the designer's set of optimal ownership vectors.

We consider different levels of transportation cost in turn, beginning with a result for the case of zero transportation costs. In that case, any ownership vector that allows ex post efficient trade is optimal for the designer in the ownership problem:

Proposition 6. *For $c = 0$, the ownership problem with a social-surplus-maximizing designer is solved by any $\mathbf{r} \in \mathcal{R}(0)$, i.e., $\mathcal{R}^D(0) = \mathcal{R}^B(0) = \mathcal{R}(0)$.*

Further, for a network with one completely connected agent, for $c \in (0, 1)$, $\mathcal{R}^P(c)$ contains only extremal ownership, and using Proposition 3, $\mathcal{R}(0)$ does not contain any extremal ownership. Thus, by continuity, in such a network, for $c > 0$ sufficiently close to zero, the set of solutions to the ownership problem has an empty intersection with the set of solutions to the placement problem:

Proposition 7. *For a network with one completely connected agent, there exists $\hat{c} \in (0, 1)$ such that for all $c \in (0, \hat{c})$, $\mathcal{R}^D(c) \cap \mathcal{R}^P(c) = \emptyset$.*

Turning to the case of sufficiently high transportation costs, we begin by noting that this case is simplified by each agent having the same worst-off type.

Lemma 4. *If $c \geq 1/2$, then $1/2$ is a worst-off type for every agent.*

Proof. See Appendix A.

Using Proposition 5, the maximized objective under the constrained-efficient reallocation rule can be written as:

$$\begin{aligned}\mathcal{L}^*(\mathbf{r}) \quad \equiv \quad & \rho^* \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\Psi_{i,\frac{1}{\rho^*}}(v_i; \omega_i^*) - C_{ji} \right) V_{ji}^{ce}(\mathbf{v};\rho^*, \boldsymbol{\omega}^*) r_j \right] \\ & + (1-\rho^*) \sum_{j=1}^{n} \omega_j^* r_j + \sum_{j=1}^{n} (1 - \rho^* + \mu_j^*) u_j(\omega_j^*).\end{aligned} \tag{8}$$

For $c \geq 1/2$, we have $\omega_1^* = \cdots = \omega_n^* = 1/2$, so in that case $\sum_{j=1}^{n} \omega_j^* r_j = 1/2$, and the only direct effects of $\mathbf{r}$ occur in the expression in (8) in square brackets. Further, as the following lemma shows, we can rewrite the expectation of the term in square brackets in (8) in terms of the ironed rather than unironed weighted virtual types:

Lemma 5. *For $c \geq 1/2$ and $F_i = F$ for all $i \in \mathcal{N}$, the maximized objective $\mathcal{L}^*(\mathbf{r})$ can be written as:*

$$\mathcal{L}^*(\mathbf{r}) = \rho^* \mathbb{E}_{\mathbf{v}} \Big[\sum_{j=1}^{n} \sum_{i=1}^{n} \left(\overline{\Psi}_{i,\frac{1}{\rho^*}}(v_i; 1/2) - C_{ji} \right) V_{ji}^{ce}(\mathbf{v};\rho^*, \mathbf{1/2}) r_j \Big] + \frac{1-\rho^*}{2} + \sum_{j=1}^{n} (1 - \rho^* + \mu_j^*) u_j(1/2).$$

Proof. See Appendix A.

Using Lemma 5, we see that the expression in square brackets in (8) is the same as the objective for the unconstrained problem of maximizing $\mathbb{E}_{\mathbf{x}} \Big[\sum_{j=1}^{n} \sum_{i=1}^{n} (x_i - C_{ji}) V_{ji}^{e}(\mathbf{x}) r_j \Big]$, where x_i is drawn from the distribution of $\overline{\Psi}_{i,\frac{1}{\rho^*}}(v_i; 1/2)$. And, for a network in which the set of completely connected agents is nonempty, this objective is maximized by giving ownership of all the resources at one or more of the completely connected agents. Thus, using Lemma 5, we have the following result:

Proposition 8. *Assuming that $F_i = F$ for all $i \in \mathcal{N}$, if the set of completely connected agents $\mathcal{I}$ is nonempty, then for all $c \geq 1/2$ and $\mathbf{r} \in \mathcal{R}^D(c)$, we have $r_i = 0$ for all $i \in \mathcal{N} \backslash \mathcal{I}$.*

Proposition 8 implies that for symmetric distributions and star or wheel networks, the hub optimally has ownership of all resources for sufficiently large transportation costs.

In cases in which ex post efficient trade is possible, we have $\rho^* = 1$, and so

$$\mathcal{L}^*(r) = \mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}(v_i - C_{ji})\,V^e_{ji}(\mathbf{v})r_j\Big] + \sum_{j=1}^{n}\mu^*_j u_j(\omega^*_j),$$

where again the term in square brackets is the objective for the unconstrained problem and so solved with extremal ownership. Thus, for a star or wheel network with solution to the ownership problem of $\mathbf{r}^* = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and $F_i = F$ for all $i \in \mathcal{N}$, if $\mathbf{r}^* \in \mathcal{R}(c)$, then $r^*_1 = \max\{r_1 \mid (r_1, \mathbf{r}_{-1}) \in \mathcal{R}(c)\}$. This says that for a star or wheel network, when the solution to the ownership problem involves ex post efficient trade, then the optimal ownership vector is on the boundary of the region permitting ex post efficient trade.

Combining these results, we see that for a star or wheel network, for a range of intermediate values for c, the solution to the ownership problem has $\mathbf{r}^*$ that does not permit ex post efficient trade, and is not extremal, and so the solution is intermediate between the solution to the ownership problem with $c = 0$ and with $c \geq 1/2$. We illustrate this in Figure 4.

(a) $\mathcal{R}(c)$ and $r^D(c)$ for a star network with $n = 3$

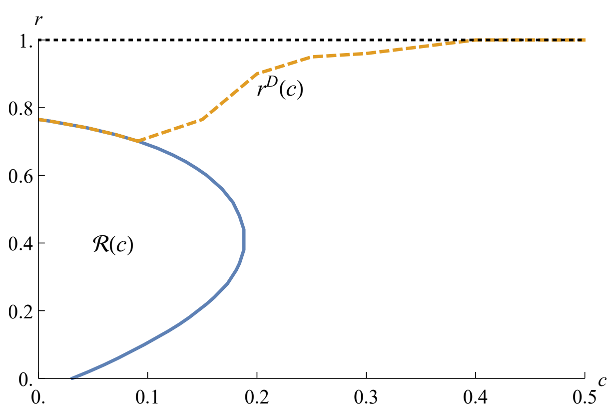

Figure 4: Values for (r, c) that permit ex post efficient trade and $r^D(c)$ for a star network. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

Wheel networks

For wheel networks, the solution to placement problem is the same as for a star network and involves placing all resources with the hub. But the solution to the ownership problem will differ between the two network structures because in a wheel network, peripheral agents can trade with their immediate neighbors if they have positive endowments. As a result, for wheel networks and small c, ex post efficient trade is possible with a more extreme (closer to the planner's optimum) ownership, i.e., increased resources at the hub. We illustrate the contrast in Figure 5, which shows that for c close to zero and $n = 5$, the region permitting ex post efficient trade for the wheel network includes larger values of r than does the corresponding region the star network.

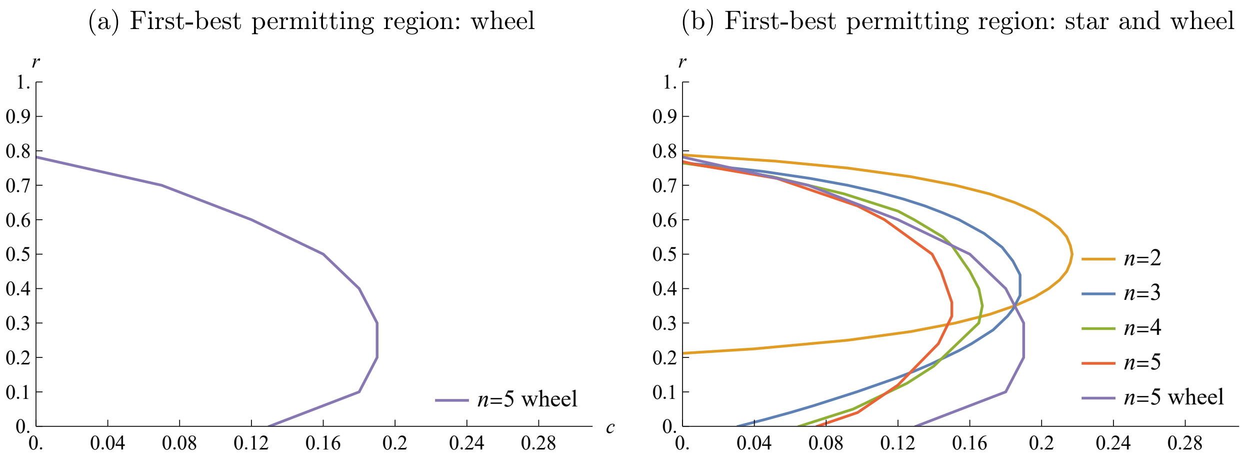

Figure 5: Values for (r, c) that permit ex post efficient trade for a wheel network and also contrasted with star networks. For the $n = 5$ wheel, $c^w_{max} = 0.19$ and $\overline{r}^w_5(0) = 0.782$. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

Results for a wheel network emphasize that having more trades does not mean that a market achieves greater efficiency. For example, for the wheel network with $n = 5$ and $c = 0.05$, the optimal placement has all the resources at the hub. In this case the probability of there being no trade is approximately 25%, and the expected number of trades is approximately 0.75. In contrast, for ownership with $r = 0.7280$, which is on the boundary of the region permitting ex post efficient trade, the probability of no trade is 0.01% and the expected number of trades is approximately 0.77. While the planner delivers the more efficient outcome, there are more trades under the designer.

Table 1: Wheel network outcomes for the planner with $\mathbf{r} = (1, 0, 0, 0, 0)$ and the designer with $\mathbf{r} = (r, (1 - r)/(n-1), \ldots, (1-r)/(n-1))$ with $r = 0.7280$, followed by ex post efficient trade. Assumes $n = 5$, $c = 0.05$, and uniformly distributed types. Based on a simulation using 10,000 draws of $\mathbf{v} \in [0, 1]^5$.

	no trade	only 1 agent is a buyer	2 agents are buyers	expected trades
planner	25.12%	74.88%	0	0.7488
designer	0.01%	87.95%	12.04%	0.7662

In the placement problem, the optimal ownership has resources at the hub, so the possibilities are that we have no trade at all or one trading cluster whereby the hub sells to the spoke with the highest value. In contrast, for the ownership problem with $r \in (0, 1)$, we can have zero, one, or two trading clusters. These correspond to whether there are zero, one, or two agents that are allocated more than their initial holdings, which we refer to here as "buyers." In an $n = 5$ wheel, it is not possible to have more than two buyers. As illustrated in Table 1, in the ownership problem, of the 88% of cases in which only one agent was a buyer, those can be divided into 20% of cases in which the trading cluster involved the hub and 68% in which trading was along the ring road only. The 12% of cases in which there were two trading clusters, can be divided into 4% that had one cluster involving the hub and one along the ring road and 8% where there were two trading clusters, neither of which involved the hub, implying that there were two separate trading clusters both along the ring road.

4.5 Optimal ownership under profit maximization

We conclude this section with an analysis of a designer that seeks to maximize its expected profit. This analysis continues to assume that for each $i \in \mathcal{N}$ the virtual types functions $\Psi_i^B(v)$ and $\Psi_i^S(v)$ are increasing.

Given worst-off types $\boldsymbol{\omega}$, the designer's profit-maximizing allocation rule $\mathbf{Q}^D$ is defined analogously to $\mathbf{Q}^{ce}$, but with the ironed weighted virtual types replaced by the ironed unweighted (i.e., weight equal to zero) virtual types:

$$Q_{i,\mathbf{r}}^D(\mathbf{v};\boldsymbol{\omega}) \equiv \sum_{j\in\mathcal{N}} V_{ji}^D(\mathbf{v};\boldsymbol{\omega}) r_j,$$

where V^D is defined by

$$V_{ij}^D(\mathbf{v};\boldsymbol{\omega}) \equiv \begin{cases} 1 & \text{if } \overline{\Psi}_{j,0}(v_j;\omega_j) - C_{ij} \geq \max_\ell \overline{\Psi}_{\ell,0}(v_\ell;\omega_\ell) - C_{i\ell} \\ & \quad \text{and } \overline{\Psi}_{j,0}(v_j;\omega_j) - C_{ij} > \max_{\ell<j} \overline{\Psi}_{\ell,0}(v_\ell;\omega_\ell) - C_{i\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following result:

Proposition 9. *The designer's profit-maximizing allocation rule is* $\mathbf{Q}_{\mathbf{r}}^D(\mathbf{v};\boldsymbol{\omega}^*)$, *where* $\boldsymbol{\omega}^*$ *is such that for all* $i \in \mathcal{N}$, $\mathbb{E}_{\mathbf{v}_{-i}}[Q_{i,\mathbf{r}}^D(\omega_i^*, \mathbf{v}_{-i};\boldsymbol{\omega}^*)] = r_i$.

Using Lemma 3, given $\boldsymbol{\omega}$, the expected profit to the designer not including transportation costs is

$$\Pi_{\mathbf{r}}^D(\boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\left[\sum_{i\in\mathcal{N}} \Psi_i(v_i;\omega_i) Q_{i,\mathbf{r}}^D(\mathbf{v};\boldsymbol{\omega})\right] - \sum_{i\in\mathcal{N}} \omega_i r_i,$$

and expected transportation costs are:

$$t_{\mathbf{r}}^{D}(\boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} C_{ji}V_{ji}^{D}(\mathbf{v};\boldsymbol{\omega})r_j\Big].$$

Thus, the designer's maximized expected profit is

$$\Pi_{\mathbf{r}}^{D}(\boldsymbol{\omega}^*) - t_{\mathbf{r}}^{D}(\boldsymbol{\omega}^*),$$

where $\boldsymbol{\omega}^*$ is as defined in Proposition 9.

Turning to the optimal ownership vector for a profit-maximizing designer, in the case of the designer, ownership affects not only transportation costs, but also the individual rationality constraint, which affects the profit-maximizing allocation rule. The tradeoffs differ somewhat from the case of a social-surplus-maximizing designer because it is as if the profit-maximizing designer faces agents with worse distributions, i.e., sellers with higher types and buyers with lower types.

5 Extensions

In this section, we provide extensions that consider the designer's problem with an indivisible resource, define stochastic reach dominance for problems with heterogeneous distributions, and allow for fixed costs of transportation per edge traveled.

5.1 Reach dominance and optimal ownership

Characterizing the optimal ownership is, in general, plagued by the problem that optimal ownership may be shared, which implies that the constrained-efficient mechanism varies non-trivially with $\mathbf{r}$. The "asset market" nature of this mechanism renders characterizing optimal ownership difficult in general. However, the problem simplifies dramatically if the resource is indivisible, as is the case, for example, for a network with a single production plant. In this case, $r_i = 1$ for some $i \in \mathcal{N}$. In this case, the constrained-efficient reallocation mechanism is simply the constrained-efficient mechanism for a two-sided allocation problem in which agent i is the seller and all other agents are buyers. This constrained-efficient mechanism is an extension of the second-best mechanism derived by Myerson and Satterthwaite (1983) to a setting with multiple buyers and costly transportation. We now show that, with identical distributions and indivisible resource, the optimal ownership is governed by reach dominance in the same way as is the optimal placement.

Proposition 10. *If the resource is indivisible and the agents' draw their types from identical distributions, then the optimal ownership is confined to the set of reach dominant agents, provided this set is nonempty. Agents who are reach dominated are never given positive ownership.*

Proof. See Appendix A.

The proof shows that, with identical distributions, if agent i reach dominates agent j, then the expected social surplus under the constrained-efficient reallocation mechanism is larger when resources are owned by agent i. The argument relies on a revealed preference argument that shows that the mechanism at i could treat the agents the same way it does at j, thereby either directly generating more social surplus or positive revenue, which can then be used to increase social surplus by reoptimizing.[13]

5.2 Stochastic reach dominance

Our analysis of optimal placement based on reach dominance rested on the assumption that the agents' distributions were identical and, like the entire analysis up to this point, that links were of equal length, implying that the cost of transportation between any two neighbors is the same. We now show that both of these assumptions can be dropped simultaneously without qualitatively altering the conclusions or mechanics at work.

To this end, let us first allow for heterogeneous distributions while keeping the length of each link the same. For every agent i, let $A_i(k)$ be the set of agents that are $k \in \{0, 1, \dots, \lceil 1/c \rceil\}$ links away from i, and let $\mathcal{A}_i(k) \equiv \cup_{h=1}^{k} A_i(h)$ be the set of agents that are within k links of agent i. Denote by

$$L_k^i(v) \equiv \prod_{j \in \mathcal{A}_i(k)} F_j(v)$$

the distribution of the highest draw among all neighbors of agent i that are not farther away than k links. Accordingly agent (or node) i is said to *stochastically reach dominate (SRD)* agent j given c if for all $k \in \{0, 1, \dots, \lceil 1/c \rceil\}$ and all $v \in [0, 1]$

$$L_k^i(v) \leq L_k^j(v) \tag{9}$$

holds, with a strict inequality for some v and k. Stochastic reach dominance extends the insight that more and closer neighbors are better, which holds under identical distributions,

[13] The argument is similar to the proof in Loertscher and Marx (2019) that shows that a merger between two suppliers harms a powerful buyer.

to something like "stronger agents with stronger, more, and closer neighbors are better." With identical distributions, reach dominance and stochastic reach dominance are equivalent because having more draws and stochastic dominance are equivalent with identical distributions.

To see that the concept of stochastic reach dominance extends straightforwardly to settings in which links between agents are not necessarily of equal length, for any distance $x \in (0, 1/c)$ from i, $\mathcal{A}_i(x)$ is now the set of agents not farther away from i than x and, accordingly, $L_x^i(v) = \prod_{j \in \mathcal{A}_i(x)} F_j(v)$ is the distribution of the highest draw among i's neighbors that are not farther away than x, and i SRDs j if $L_x^i(v) \leq L_x^j(v)$ holds for all $x \in (0, 1/c)$ and all $v \in [0, 1]$, with strict inequality for some.

5.3 Fixed cost of communication

In some applications, it is more appropriate to think of transportation as involving a fixed cost per link that is independent of the amount being shipped. For example, the agent shipping and the agent receiving the good may need to communicate about the specifics of the shipment and what it requires. This communication may be costly due to lack of a common language or cultural differences, but once the cost is borne and a common understanding is established, the shipment is free and hence the cost does not vary with the quantity shipped. This kind of problem is pervasive for resource (re-)allocation within organizations, where different units and departments have their own culture and language.

If agents i and j are directly linked, then it is ex post efficient for agent i to ship r_i to agent j if and only if $c < r_i(v_j - v_i)$ or equivalently

$$v_i + \frac{c}{r_i} < v_j.$$

The larger is r_i, the more likely is agent i thus to ship r_i units to agent j. Moreover, the gains from trade $r_i(v_j - v_i) - c$ increase in r_i.

If $c \in [1/2, 1)$, then with nonextremal placement, at most one agent will be able to ship because $c/r_i \geq 1$ for any $r_i \leq 1/2$. If $c \in [0, 1/2)$, then the analysis above related to reach dominant agents applies. This gives us the following result:

Proposition 11. *Assume that $F_i = F$ for all $i \in \mathcal{N}$. If $c \in [1/2, 1)$, then optimal placement has $r_i = 1$ for some i; and if $c \in [0, 1/2)$, then optimal placement has $r_i = 1$ for an agent i that is reach dominant agent whenever such an agent exists.*

Proof. See Appendix A.

Proposition 11 provides conditions under which the placement problem with fixed costs is solved by extremal placement when agents have identical distributions. More generally, if one of the solutions to the placement problem under constant marginal cost of transportation involves extremal placement, then with fixed costs, extremal placement is uniquely optimal. In all of these cases in which the first-best requires extremal placement, we of course obtain the result that in the ownership problem, the first-best is impossible

6 Conclusions

We study trade on networks with linear transportation costs. We show that absent individual rational and no-deficit constraints an extremal placement of resources, that is, placing the entire resource with a specific single agent, is *always* optimal for the planner, irrespective of agents' type distributions, the transportation cost, and the network structure. For incomplete networks, such as the star with three or more agents, or the wheel with five or more agents, placing the entire resource at the hub is uniquely optimal with identical distributions, provided that transportation cost is positive but not prohibitively large. However, as we show, if resource holdings bestow an agent with property rights over the resource, then extremal ownership conflicts with individual rationality and no-deficit constraints. We then solve for optimal ownership, anticipating that a constrained-efficient rather than ex post efficient reallocation mechanism may be required for trade once agents' types are realized. Even though the optimal placement is the same for the star and the wheel networks, the optimal ownership strcutures that account for individual rationality and no-deficit constraints differ because the wheel offers additional opportunities of trade along the ring road.

References

Akbarpour, M. and M. O. Jackson (2018): "Diffusion in Networks and the Virtue of Burstiness," *Proceedings of the National Academy of Sciences*, 115, E6996–E7004.

Borgatti, S. P., M. G. Everett, and J. C. Johnson (2018): *Analyzing Social Networks*, Sage Publications, second ed.

Börgers, T. and P. Norman (2009): "A Note on Budget Balance Under Interim Participation Constraints: The Case of Independent Types," *Economic Theory*, 39, 477–489.

Che, Y.-K. (2006): "Beyond the Coasian Irrelevance: Asymmetric Information," Unpublished Lecture Notes, Columbia University.

Coase, R. H. (1960): "The Problem of Social Cost," *Journal of Law & Economics*, 3, 1–44.

Condorelli, D., A. Galeotti, and L. Renou (2017): "Bilateral Trading in Networks," *Review of Economic Studies*, 84, 82–105.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a Partnership Efficiently," *Econometrica*, 55, 615–632.

Delacrétaz, D., S. Loertscher, and C. Mezzetti (2022): "When Walras Meets Vickrey," *Theoretical Economics*, 17, 1803–1845.

Figueroa, N. and V. Skreta (2012): "Asymmetric Partnerships," *Economics Letters*, 115, 268–271.

Gresik, T. and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

Houde, J.-F., P. Newberry, and K. Seim (2023): "Nexus Tax Laws and Economies of Density in E-Commerce: A Study of Amazon's Fulfillment Center Network," *Econometrica*, 91, 147–190.

Jackson, M. O. (2008): *Social and Economic Networks*, Princeton University Press.

Li, J. and P. Dworczak (2021): "Are Simple Mechanisms Optimal when Agents are Unsophisticated?" Working Paper, Northwestern University.

Loertscher, S. and L. M. Marx (2019): "Merger Review for Markets with Buyer Power," *Journal of Political Economy*, 127, 2967–3017.

——— (2020): "A Dominant-Strategy Asset Market Mechanism," *Games and Economic Behavior*, 120, 1–15.

——— (2022): "Vertical Integration with Incomplete Information," Working Paper, University of Melbourne.

——— (2023): "Asymptotically Optimal Prior-Free Asset Market Mechanisms," *Games and Economic Behavior*, 137, 68–90.

Loertscher, S., E. V. Muir, and P. G. Taylor (2022): "Optimal market thickness," *Journal of Economic Theory*, 200.

Loertscher, S. and C. Wasser (2019): "Optimal Structure and Dissolution of Partnerships," *Theoretical Economics*, 14, 1063–1114.

Lu, H. and J. Robert (2001): "Optimal Trading Mechanisms with Ex Ante Unidentified Traders," *Journal of Economic Theory*, 97, 50–80.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Myerson, R. B. and M. A. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Sosnowska, J. and O. Skibski (2018): "Path Evaluation and Centralities in Weighted Graphs – An Axiomatic Approach," *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence*, 3856–3862.

Vickrey, W. (1961): "Counterspeculation, Auction, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

A Proofs

Proof of Proposition 1. Letting $v^e_{ji}(C) = \mathbb{E}_{\mathbf{v}}[V^e_{ji}(\mathbf{v})]$, we have $t^e(\mathbf{r}) = \sum_{i=1}^n \sum_{j=1}^n C_{ji} v^e_{ji}(C) r_j$ and $ss^e(\mathbf{r}) = \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n \sum_{j=1}^n v_i V^e_{ji}(\mathbf{v}) r_j\right] - t^e(\mathbf{r})$. Using $r_n = 1 - \sum_{\ell=1}^{n-1} r_\ell$, we have

$$\begin{aligned} ss^e(\mathbf{r}) &= \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n \sum_{j=1}^{n-1} v_i V^e_{ji}(\mathbf{v}) r_j + \sum_{i=1}^n v_i V^e_{ni}(\mathbf{v}) \left(1 - \sum_{\ell=1}^{n-1} r_\ell\right)\right] \\ &\quad - \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ji} v^e_{ji}(C) r_j - \sum_{i=1}^n C_{ni} v^e_{ni}(C) \left(1 - \sum_{\ell=1}^{n-1} r_\ell\right), \end{aligned}$$

so for $j \in \{1, \ldots, n-1\}$, we have

$$\frac{\partial ss^e(\mathbf{r})}{\partial r_j} = \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n v_i \left(V^e_{ji}(\mathbf{v}) - V^e_{ni}(\mathbf{v})\right)\right] - \sum_{i=1}^n \left(C_{ji} v^e_{ji}(C) - C_{ni} v^e_{ni}(C)\right),$$

which is independent of r_j (and any other r_i). This implies that an extremal ownership vector is *always* optimal, independently of network structure and distributions. ■

Proof of Proposition 4. Suppose $c \geq 1/2$, in which case agents only ever trade with their immediate neighbors. Let $\mathcal{N}_i \subset \mathcal{N}$ be the set of agent i's immediate neighbors. Consider the mechanism in which agent i can buy from agent $j \in \mathcal{N}_i$ at (per-unit) price $\max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$ and agent i can sell to agent $j \in \mathcal{N}_i$ at (per-unit) price $v_j - c$. This mechanism induces agent i to demand r_j units from agent j if $v_i > \max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$, and zero units otherwise, and it induces agent i to offer r_i units to agent j if $v_i < v_j - c$ and $v_j = \max_{\ell \in \mathcal{N}_i} v_\ell$, and zero units otherwise. Thus, this mechanism induces the ex post efficient trade with trading buyers paying the lowest Walrasian price and trading sellers receiving the highest Walrasian price. Agents with type 1/2 do not trade and have zero payments. These types are worst-off, implying that worst-off types satisfy ex post and interim individual rationality constraints with equality. Turning to the budget surplus of this mechanism, if $v_i > \max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$, then agent i purchases r_j units from agent j and makes a payment of $r_j \max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$, while agent j is paid $r_j(v_i - c)$. Thus, budget surplus associated with trades involving agent i is

$$\sum_{\ell \in \mathcal{N}_i \text{ s.t. } v_i > \max\{v_\ell + c, \max_{h \in \mathcal{N}_\ell} v_h\}} \left(r_\ell \max\{v_\ell + c, \max_{h \in \mathcal{N}_\ell} v_h\} - r_\ell (v_i - c) \right)$$

$$= \sum_{\ell \in \mathcal{N}_i \text{ s.t. } v_i > \max\{v_\ell + c, \max_{h \in \mathcal{N}_\ell} v_h\}} r_\ell \left(\max\{v_\ell - v_i + 2c, \max_{h \in \mathcal{N}_\ell} v_h - v_i + c\} \right),$$

where $v_\ell - v_i + 2c = (v_\ell + c) - v_i + c < v_i - v_i + c = c$ and $\max_{h \in \mathcal{N}_\ell} v_h - v_i + c < v_i - v_i + c = c$, which says that the transportation costs are not covered (on a trade-by-trade basis). This completes the proof of the impossibility of ex post efficient trade. ■

Proof of Lemma 3. Define

$$u_i(v) \equiv q_i(v)v - m_i(v) - r_i v,$$

implying that the individual rationality condition can be stated as for all $v \in [\underline{v}, \overline{v}]$, $u_i(v) \geq 0$. By incentive compatibility, $u_i(v) = \max_{\hat{v}} q_i(\hat{v})v - m_i(\hat{v}) - r_i v$, which implies that u_i is differentiable almost everywhere and by the envelope theorem, whenever it is differentiable, we have

$$u_i'(v) = q_i(v) - r_i.$$

Thus, for all $\omega \in [\underline{v}, \overline{v}]$,

$$u_i(v) = \int_\omega^v (q(x) - r)dx + u_i(\omega).$$

From this, it follows that

$$m_i(v) = q_i(v)v - r_i v - \int_\omega^v (q(x) - r)dx - u_i(\omega)$$

and so

$$\begin{aligned}
\mathbb{E}_{v_i}[m_i(v_i)] &= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, x dF_i(x) - \int_{\underline{v}}^{\overline{v}} \int_\omega^y (q(x) - r) f_i(y) dx dy - u_i(\omega) \\
&= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, \Psi_i(x;\omega) dF_i(x) - u_i(\omega) \\
&= \int_{\underline{v}}^{\overline{v}} q_i(x)\Psi_i(x;\omega) dF_i(x) - r_i\omega - u_i(\omega) \\
&= \mathbb{E}_{v_i}[q_i(v_i)\Psi_i(v_i;\omega)] - r_i\omega - u_i(\omega),
\end{aligned}$$

which completes the proof. ■

Proof of Lemma 4. Let $q_i(v)$ be agent i's interim expected allocation when of type v and let r_i be its initial resources. For $v \geq 1-c$, we have $q_i(v) \geq r_i$ because i cannot never act as a seller. For $v \leq c$, we have $q_i(v) \leq r_i$ because i cannot never act as a buyer. This implies that for $c > 1/2$, in which case we have $1-c \leq c$, the agent's interim expected allocation satisfies $q_i(v) = r_i$ for all $v \in [1-c, c]$. Hence, for $c \geq 1/2$, all types $v \in [1-c, c]$ will be worst-off. ■

Proof of Lemma 5. Assume that $F_i = F$ for all $i \in \mathcal{N}$. Define the function $\hat{z}(\hat{v}, a)$ to be the implicit solution for the ironing parameter z that solves (6) (this is the same for all i given the assumption that $F_i = F$ for all $i \in \mathcal{N}$). When $c \geq 1/2$, ex post efficient trade is not possible by Proposition 4, so we have $\rho^* > 1$. Thus, $\hat{z}(1/2, 1/\rho^*) \in (0,1)$, and we can let $\hat{c} \in [1/2, 1)$ be such that for all $c \geq \hat{c}$, we have

$$1 - c < \hat{z}(1/2, 1/\rho^*) < c. \tag{A.1}$$

Focusing on the expression in (8) in square brackets, if $1 - c < z(1/\rho^*, \omega^*) \equiv z^* < c$, then in order to have $V_{ij}^{e} = 1$ for $i \neq j$, we require that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) < z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*)$, and $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega_j^*) > z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega_j^*) = \Psi_{\frac{1}{\rho^*}}(v_j; \omega_j^*)$. So the term in square brackets can be written as

$$\sum_{i=1}^{n} \sum_{j \in \mathcal{N} \setminus \{i\}} \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) - C_{ji}\right) V_{ji}^{ce}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*) r_j + \sum_{i=1}^{n} \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) V_{ii}^{ce}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*) r_i.$$

But notice that, dropping the arguments on V^{ce},

$$\begin{aligned}
& \mathbb{E}_{\mathbf{v}}\left[\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*)\right) V_{ii}^{ce}\right] \\
= \ & \mathbb{E}_{\mathbf{v}}\left[\left(z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*)\right) V_{ii}^{ce} \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*\right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*\right) \\
= \ & \mathbb{E}_{\mathbf{v}}\left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*, \, V_{ii}^{ce} = 1\right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*, \, V_{ii}^{ce} = 1\right) \\
= \ & \mathbb{E}_{\mathbf{v}}\left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*\right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*\right) \\
= \ & 0,
\end{aligned}$$

where the first equality uses the fact that the ironed and unironed virtual types are identical outside of the ironing range, the second equality uses the binary nature of V^{ce}, the third equality uses the result that if v_i is in the ironing range and $c > \hat{c}$, then it is not possible for agent i to trade and so $\hat{V}_{ii}^{ce} = 1$, and the final equality uses the definition of the ironing parameter given in (6). Thus, the expectation of the expression in (8) in square brackets is

$$\mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i;\omega_i^*) - C_{ji}\Big)\, V_{ji}^{ce}(\mathbf{v};\rho^*,\boldsymbol{\omega}^*)r_j\Big],$$

which completes the proof. ■

Proof of Proposition 10. We need to show that the expected social surplus of giving ownership to agent i is larger than giving it to agent j if i reach dominates j. Consider a vector of neighbors $\mathbf{n} = (n_1, \ldots, n_{\lceil 1/c\rceil})$ and a vector $\hat{\mathbf{n}}$ that reach dominates $\mathbf{n}$. The expected social surplus in the constrained-efficient mechanism generated when ownership is given to an agent with neighbors $\hat{\mathbf{n}}$ is strictly larger than for an agent with neighbors $\mathbf{n}$ because at the node with $\hat{\mathbf{n}}$, the agents could be treated the same as at the node with $\mathbf{n}$ in the following sense: closer agents in $\hat{\mathbf{n}}$ are moved "outward" to obtain $\tilde{\mathbf{n}} = (n_1, \ldots, n_{\lceil 1/c\rceil-1}, n_{\lceil 1/c\rceil} + h)$, where $h \geq 0$ is the number of additional agents in $\hat{\mathbf{n}}$. Agents that are moved outward in $\hat{\mathbf{n}}$ are then treated the same way as those in $\mathbf{n}$, i.e., they are allocated the good in the same instances as they would have been if their true location were where they are now in $\tilde{\mathbf{n}}$, and the additional h at distance $\lceil 1/c\rceil$ away are allocated the good only in the instance in which the the agent with the highest value among the other $n_{\lceil 1/c\rceil}$ agents at distance $\lceil 1/c\rceil$ is allocated the good when the h additional agents are not there and when the highest value of these h agents is larger than the highest value among the other $n_{\lceil 1/c\rceil}$ agents. This mechanism generates strictly more revenue than the constrained-efficient mechanism at $\mathbf{n}$ if $\hat{n}_k > n_k$ for some $k < \lceil 1/c\rceil$ because of transportation cost savings, which means that by reoptimzing over the allocation rule, one can generate strictly more social surplus, and it generates more social surplus if $\hat{n}_k = n_k$ for all $k < \lceil 1/c\rceil$, because the good is allocated to an agent with a higher value some of the time. (It also generates positive revenue because it can use a second-price auction among the h agents at distance $\lceil 1/c\rceil$ with a reserve equal to the highest value among the $n_{\lceil 1/c\rceil}$ agents when one of the h agents obtains the good.) ■

Proof of Proposition 11. First consider the case with $c \in [1/2, 1)$. If $r_j \leq c$ for all j, then there is no trade and social surplus is simply $\mathbb{E}[v]$. If $r_i > c$ for some agent i, then social surplus is $\mathbb{E}[v] + r_i GFT_i(r_i)$, where $GFT_i(r_i)$ is the expected gain in social surplus associated with trades involving agent i, necessarily as a seller, given r_i. Because $GFT_i(r_i)$ is positive and increasing in r_i (because c/r_i is less than one and decreases with r_i), social surplus is maximized for $r_i \in (c, 1]$ at $r_i = 1$. It then remains to choose the agent i to maximize $GFT_i(1)$.

Now consider the case with $c \in [0, 1/2)$. Suppose that agent i reach dominates agent j

given cost c, and consider the gains from trade associated with trades involving agents i and j. To allow for the possibility that there are gains from trade, we assume that the sum of ownership shares of agents i and j satisfies $r_i + r_j > c$, which makes sure that for sufficiently extremal ownership, there are positive gains from trade. Then we have

$$\begin{aligned} r_i GFT_i(r_i) + r_j GFT_j(r_j) &\leq r_i GFT_i(r_i) + r_j GFT_i(r_j) \\ &< r_i GFT_i(r_i + r_j) + r_j GFT_i(r_i + r_j) \\ &= (r_i + r_j) GFT_i(r_i + r_j), \end{aligned}$$

where the first inequality uses that agent i reach dominates agent j, and the second inequality is strict because $GFT_i(\cdot)$ is strictly increasing in its argument when it is positive. Thus, social surplus is increased by shifting ownership towards the reach dominating agent. It then follows that if the set of reach dominant agents is nonempty, then the optimal ownership places resources only with reach dominant agents. Further, if there are two risk dominant agents, then it is optimal to place all resources with a single one of them because of the increasing returns to scale in the gains from trade. ■