Outside options, reputations, and the partial success of the Coase conjecture

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Abstract

A buyer and seller bargain over a good's price in continuous time. The buyer has a private value and positive outside option. Additionally, bargainers can be either rational or committed to some fixed price. If the sets of buyer values and commitment types are rich and the probability of commitment vanishes, outcomes are partially consistent with the Coase conjecture: the seller chooses a price below the maximum of the lowest outside option and half the lowest value; the buyer immediately accepts or exits, taking her outside option. There is minimal delay, but outcomes are inefficient when the buyer exits.

Keywords: Bargaining, reputation, coase conjecture

1 Introduction

What effect do outside options have on bargaining with incomplete information? The existing literature suggests a surprisingly dramatic impact. Most notably, consider an infinite horizon game where in every period the seller proposes a price for a good to a buyer with private information about her value $v \in [\underline{v}, \overline{v}]$. If there are no outside options, then the Coase conjecture holds: when players are sufficiently patient (offers are frequent) the seller proposes a price of $\underline{v}$ almost immediately if there is "gap", $\underline{v} > 0$, or buyer strategies are stationary (Fudenberg and Tirole (1985), Gul et al. (1986)). The reason is that if today's offer $p > \underline{v}$ is rejected, the seller will update her beliefs and cut her price tomorrow, but in which case even a high value buyer would not accept today unless the price is already low $p \approx \underline{v}$. However, if the buyer can get a strictly

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positive outside option $w \in [\underline{w}, \overline{w}]$ by exiting the market at the end of each period, then Board and Pycia (2014) (henceforth BP) show the seller acts as if she had commitment: she can choose any price in the first period, and the buyer either accepts it or exits the market.[1] The logic is that if bargaining continues into period 2, the seller will never charge a price below the lowest possible net value of any buyer type that remains, giving that type a continuation payoff below her outside option, and so she would prefer to exit in period 1 instead to avoid discounting (net values are $v - w$).

BP's result has a paradoxical implication. If the buyer knew her type, and could take her outside option before the start of bargaining (period 0), the market would completely unravel. No buyer would ever negotiate, as the lowest net value type who did would get a discounted continuation payoff below her outside option.[2,3] We might avoid that paradoxical implication if the buyer could sometimes make counteroffers, allowing her lowest net value type, to receive a continuation payoff larger than her outside option.[4] However, offers by the informed party introduces signalling. Off-path buyer offers can then be "punished with beliefs", interpreted as coming from the highest net value buyer, $\overline{v} - \underline{w}$, and so support a wide variety of on path play.[5,6]

One way to mitigate the power of belief punishments is to introduce a small probability of commitment into the model, types which always propose some fixed price and never back down. Abreu and Gul (2000) showed that such commitment types imply an essentially unique equilibrium despite two sided incomplete information in a dollar division bargaining problem, independent of the fine details of the bargaining protocol, when offers are frequent.

In this paper, I introduce commitment types into a model where a seller and buyer sequentially announce prices, before playing a continuous time concession game. Players are equally patient. A rational buyer has private information about her value and positive outside option. For each possible buyer value v there is some probability of the

[1] Chang (2021) shows that if outside options are dispersed and positively related to values, then the seller's optimal dynamic mechanism may feature declining prices over time.

[2] Similar unravelling can arise in auctions when there must be at least two bids and bids incur a sunk participation cost; Lauermanny and Wolinsky (2021) investigate some of these issues.

[3] BP partially address this critique, by suggesting buyers only learn their values after bargaining starts.

[4] Certainly, if the buyer made all offers instead, she could get the good for free. An alternative assumption to stop unravelling is that buyers don't know their value for good until bargaining starts.

[5] For similar reasons there are multiple equilibria in BP if the buyer sometimes has no outside option. In some the seller chooses any ultimatum and the buyer is believed to have a high outside option if she remains in period 2. Others have a Coasean structure, with low and declining prices.

[6] Chatterjee et al. (2022) get clear predictions in stationary equilibria of a coalitional bargaining game where a veto player has a privately known outside option and can sometimes make offers. The outside option is sufficiently large that the veto player either accepts the first offer she receives, or exits. Again, if she could exit before the start of bargaining, she always would.

lowest outside option, $\underline{w} > 0$.

Equilibria have a similar structure to existing reputational models. Rational players always imitate commitment types' prices, however, if the seller's price is unacceptable to a rational buyer, $v - p_s < w$, she only imitates the lowest such price $\underline{p} > 0$. After demand announcements, concession and exit behavior ensure the rational seller and the buyer with the highest remaining value are indifferent between conceding at one instant or the next (the skimming property holds: lower value buyers concede later if at all). Eventually both players reach a probability 1 reputation for commitment at the same time $T^* < \infty$.

My main result shows that if the sets of buyer values and commitment types are rich and the probability of commitment vanishes, then bargaining outcomes are partially Coasean: they are approximately those which would arise if the seller could issue an ultimatum at a price bounded above by $p^* = \{\underline{v}/2, \underline{w}\}$. The buyer either immediately accepts such a price, or exits. Loosely, the set of buyer values is rich if for any $d \in [\underline{v}, \overline{v}]$ there is a possible buyer value close to d, while the set of commitment types is rich if for any $d' \in [0, \overline{v} - \underline{w}]$ there is some type which makes a demand close to d'.

The result suggests the original Coase conjecture predictions of low prices and high efficiency are robust to the presence of outside options so long as $\underline{v}$ and $\underline{w}$ are small, in contrast to BP. Bargaining delays are also minimal, consistent with both the original Coase conjecture and BP.

However, the result also diverges from some features of the original Coase conjecture in a similar direction to BP. Notably, prices can be high (if $\underline{v}$ or $\underline{w}$ are[7]). For instance, if $v = 1$ is known, but w is approximately uniformly distributed on $[0, 1]$, then net values $v - w$ are also uniform on $[0, 1]$ and the seller will charge $p^* = 1/2$ for a profit of $1/4$.

Of course, p^* is only an upper bound, and the seller may choose a lower price. A particular case of interest when she does so is when she faces a single rational buyer type (v, w). If $w < v/2$ then the seller charges $p_s \approx p^* = v/2$, but if $w > v/2$ then she charges $p_s \approx v - w < p^* = w$. This price offers the buyer approximately his outside option when he accepts, $v - p_s \approx w$ as he would not accept higher prices. This prediction matches Binmore et al. (1989)'s with outside options in an alternating offers game under complete information (without commitment types).

What is the reasoning behind the main result? In particular, what's special about p^*?

[7] For instance, a seller may know her product (e.g. bespoke software) provides high buyer value $\underline{v} >> 0$ (e.g. in cost savings) without knowing the outside option created by competitors offers. Or she might know a competitor's product (e.g. off-the-shelf software with a nationally set price) offers a high outside option $\underline{w} >> 0$, without knowing how the buyer values her own product.

I say that the seller's offer is more generous than the buyer's if the buyer gets greater utility from conceding, $v - p_s > p_b$. With a rich set of buyer values and commitment types, a seller's offer is more generous than any counteroffer of the lowest value buyer who eventually concedes (a buyer with value $v^{1,p_s} = \min\{v > p_s + \underline{w}\}$), if and only if $p_s \leq p^*$. Recall that in any equilibrium, a player (she) concedes at a rate which makes her opponent (he) indifferent between conceding at one instant or the next. That rate is, therefore, proportional to the generosity of her offer (which determines his cost of delaying his concession) and so determines how fast her reputation for commitment grows over time. When the prior probability of commitment is arbitrarily small, so are updated reputations when all high value buyers have conceded ($v > v^{1,p_s}$), after which reputations must still grow a lot to reach probability 1. Hence, if the seller's offer is more generous than the remaining v^{1,p_s} buyer (e.g. if $p_s \leq p^*$), she always builds reputation faster than the buyer, and in order for both players to reach a probability 1 reputation at the same time, the buyer must either concede or exit with probability close to 1 at time 0. On the other hand, if the seller's offer is less generous than the remaining v^{1,p_s} buyer (e.g. $p_s > p^* \approx p_b$), she eventually builds reputation slower than the buyer, and must therefore concede with probability close to 1 at time 0.

The limit outcome's dependence only on the generosity of the seller compared to a value v^{1,p_s} buyer highlights the model's Coasean force. Eventually the seller realizes she faces such a buyer and then what matters is which player has the greatest incentive to give in (the less generous player). A greater willingness to eventually concede translates into immediate concession.

In Section 5, I present additional implications and extensions of the model. In particular: I show the same results hold in a discrete-time alternating offers game; I extend the results to unequal buyer and seller patience, in particular showing that outcomes are equivalent to BP if the seller is much more patient than the buyer; I show the seller's profits can increase in the buyer's outside option or in a sunk cost the buyer must pay to initiate bargaining (the market need not unravel, unlike in BP); and show the seller may successfully charge higher prices ($p_s >> p^*$) if buyer values are not rich.

The rest of this section highlights additional literature, then Section 2 outlines the model, Section 3 characterizes equilibria, Section 4 presents the main result, and Section 5 considers extensions. Proofs are in the Appendix, unless stated otherwise.

1.1 Additional Literature

Hwang and Li (2017) show that the Coase conjecture may hold if a buyer's outside option arrives stochastically. The seller makes all offers, and the buyer's outside option arrives publicly at the end of each period with some probability (after the seller's offer). With frequent offers, the seller almost immediately offers a price that makes the lowest value buyer indifferent to waiting for the outside option. The logic driving the result is that buyers must immediately take an outside option when it arrives. Otherwise the lowest buyer type which did not, would receive a continuation payoff equal to that outside option in subsequent periods (as in BP) and so she would prefer to avoid delay. If the stochastic arrival is not publicly observable multiple equilibria exist, some of which are Coasean and some not.

Nava and Schiraldi (2019) highlight what they call a robust Coase conjecture, when the seller can offer differentiated goods. The seller makes all offers and after purchasing one variety, the consumer receives no value from buying a second variety. When offers are frequent, the market clears instantaneously, with the buyer purchasing one of the varieties offered, however, the seller retains some market power and the outcome is not efficient. The seller offers a low price (possibly 0) for one variety, and a high price for the other and allows consumers to select between them. The low price for the first variety, effectively creates a consumer outside option, which (by BP) allows her to charge a monopolistic price for the second variety. The authors suggest BP's result is similarly consistent with a properly understood Coase conjecture. However, seemingly many prices can clear the market with outside options, and the low prices identified in my analysis when $\underline{v} \approx \underline{w} \approx 0$ seem "more Coasean" than BP's.

In addition to outside options, the existing literature has identified many reasons the Coase conjecture may fail: a monopolist may rent rather than sell, or under-invest in capacity (Bulow (1982)), or use best-price provisions (Butz (1990)), or buyers may use non-stationary strategies if there is no "gap" between their values and the seller's (Ausubel and Deneckere (1989)). However, other factors that might be thought to interfere with the Coasean logic, merely see it confirmed in different guises. For instance, if a second buyer may arrive to compete with the first, the seller's profit is driven down to what she would get from waiting for that buyer's arrival (Fuchs and Skrzypacz (2010)).

Abreu et al. (2015) introduced commitment types into a bargaining game where one of the rational players had private information about her discount rate (high or low) and there are no outside options. They showed that if the set of demands made by commitment types is sufficiently rich, then outcomes must be Coasean as the probability of commitment vanishes: there is immediate agreement on the same terms as would have been agreed if the informed party were known to be her most patient type.[8] Inderst

[8]They also show that if commitment types sometimes delay making their fixed demand, non-Coasean

(2005) and Kim (2009) identify Coasean results when a seller has a vanishing probability of being a commitment type, and the buyer has private information about her value but no outside option and is never a commitment type (the good is sold for $\underline{v}$). Peski (2021) also identifies a Coasean result when dividing two pies, one bargainer has private information about her relative value of the pies, and there is a vanishingly small probability that each bargainer is committed to some menu of divisions.

Compte and Jehiel (2002) suggests commitment types have little effect on rational players' behavior when they have outside options. In an alternating offer protocol with a fixed surplus, and commitment types that aggressively offer an opponent less than her outside option, rational players never imitate commitment types. My model allows for more generous commitment types, and I show rational agents do choose to imitate them.

In a contemporary paper, Pei and Vairo (2022) investigate a related model, where effectively a seller and a buyer with a private value (but no outside option) simultaneously announce prices before becoming committed to them with positive probability (Kambe (1999)'s protocol). There is always a Coasean equilibrium (all players announce the same price $p_s = p_b$ which approaches $\underline{v}/2$ as commitment vanishes). But if buyer values are sparse (not rich), there can also be non-Coasean equilibria (the seller charges $p_s > \underline{v}$, as do buyers with $v > p_s$, but buyers with $v \leq p_s$ counterdemand $p_b = 0$ and never concede). I highlight a similar (unique) equilibrium limit with sparse buyer values and high seller prices ($p_s >> p^*$) in Section 5.

Atakan and Ekmekci (2013) show that reputational bargaining with outside options endogenously determined by a search market can lead to inefficiency. Endogenous outside options are also central to Özyurt (2015), who shows that even vanishingly small reputational concerns allow a wide range of prices in Bertrand-competition like setting with two sellers and a single buyer.

2 The model

In this section I outline a simple baseline model; I extend it in several directions in Section 5.

A buyer and seller bargain in a continuous time concession/exit game, where the seller has a single indivisible good. Time 0 is subdivided into 4 times, $0^1 < 0^2 < 0^3 < 0^4$,

limit outcomes are possible as patient players try to signal their type. Analyzing such types in the current setting, or even types that vary their demands over time in history contingent ways (see Abreu and Pearce (2007) and Fanning (2016)) has the potential to be very challenging and is left for future work.

to allow for sequential decisions to be made with no discounting of payoffs between them. At time 0^1 the seller proposes a price $p_s \in P$ where $P \subset (0, \infty)$ is some finite set. At time 0^2, the buyer can observe the seller's price p_s and can: immediately concede (accept her opponent's price; action c), counterdemand $p_b \in (0, p_s) \cap P$, or exit the market (action e). If the game continues to time 0^3: the seller observes p_b and chooses a stopping time $t^s \in \{0^3, 0^4\} \cup (0, \infty]$ to concede, while the buyer chooses a stopping time $t^b \in \{0^3, 0^4\} \cup (0, \infty]$ and an action $a \in \{c, e\}$, where (t^b, c) denotes a decision to concede at time t^b, and (t^b, e) denotes a decision to exit. If players choose the same stopping time ($t^s = t^b$), each player's chosen action occurs with probability 1/2 (concession or possibly exit).

Both buyer and seller can either be rational or a commitment type. A rational seller has no value for the good and no outside option.[9] A rational buyer of type (v, w) has an outside option $w > 0$ and a value $v > w$.[10] If the good is traded at price p at time $t \geq 0$ then a rational seller gets a payoff $e^{-rt}p$, and a rational buyer gets a payoff $e^{-rt}(v - p)$, where r is a common discount rate. If instead, the rational buyer exits the market at time t she gets a payoff $e^{-rt}w$, and a rational seller gets a payoff of 0.

The distribution of rational buyer types has finite support Θ with probability mass function g, so that $\sum_{(v,w)\in\Theta} g(v, w) = 1$. Let $V = \{v : (v, w) \in \Theta\}$ and $W = \{w : (v, w) \in \Theta\}$ so $\underline{v} = \min V$, $\overline{v} = \max V$ and $\underline{w} = \min W$, and $\overline{w} = \max W$. I assume $g(v, \underline{w}) > 0$ for all $v \in V$, so there is always a chance of the minimum outside option; this is implicitly an assumption about the richness of types (without it, the main results require a more complicated definition of a rich type set). I further assume that $v - w \neq p$ and $(v - p)/(p - p') \neq w'/(v' - p' - w')$ and $v - p \neq p'$ for all $(v, w), (v', w') \in \Theta$ and $p, p' \in P$ with $p > p'$; given the finiteness of Θ and P, equalities are non-generic.

The probability of player i being a commitment type is $z_i \in (0, 1)$. There is a finite set $P_i \subset P$ of commitment types for player i. Conditional of being committed, she is of type $p_i \in P_i$ with probability $\pi_i(p_i) \in (0, 1)$. Type p_i demands the price p_i in the bargaining game, concedes only if offered a better price (i.e. if $p_s < p_b$ for the buyer) and never exits the market. To simplify the exposition, I assume $\max P_b \leq \max P_s$, so the highest seller commitment price is higher than the buyer's. I also assume the lowest price is a buyer commitment demand, $\underline{p} = \min P_b = \min P$; this helps with several arguments, but is not needed or assumed in the alternating offer game of Section 5 (which admits a continuum of prices).

[9]Results extend if the seller has a known outside option: as commitment vanishes, outcomes are equivalent to the seller making any ultimatum offer below the maximum of p^* and her outside option.

[10]An buyer with $v < w$ will never agree to any price for the good and would immediately exit at 0^2; I explore endogenous participation in bargaining more generally in section 5.

Let $\mu_s(p_s)$ be the probability that a rational seller proposes a price $p_s \in P$ at 0^1, and given p_s let $\mu_b^{p_s,v,w}(a)$ be the probability that a rational seller of type (v,w) chooses action $a \in P \cup \{e,c\}$ at 0^2. Hence, immediately after a seller's demand $p_s \in P_s$ and buyer's counterdemand $p_b \in P_b$, the bargainers' reputations for commitment are:

$$\bar{z}_s^{p_s} = \frac{z_s\pi_s(p_s)}{z_s\pi_s(p_s) + (1-z_s)\mu_s(p_s)}, \quad \bar{z}_b^{p_s,p_b} = \frac{z_b\pi_b(p_b)}{z_b\pi_b(p_b) + (1-z_b)\sum_{(v,w)\in\Theta}\mu_b^{p_s,v,w}(p_b)}$$

and $\bar{z}_s(p_s) = 0$ if $p_s \notin P_s$ and $\bar{z}_b^{p_s}(p_b) = 0$ if $p_b \notin P_b$.[11] If $\mu_b^{p_s,v',w'}(p_b) > 0$ for some (v',w') then the probability that the buyer is of type (v,w) conditional on rationality is:

$$\bar{g}^{p_s,p_b}(v,w) = \frac{g(v,w)\mu_b^{p_s,v,w}(p_b)}{\sum_{(v',w')\in\Theta'} g(v',w')\mu_b^{p_s,v',w'}(p_b)}.$$

Conditional on reaching a continuation game at 0^3 with demands p_s, p_b, let the probability that player i concedes by time $t \in \{0^3, 0^4\} \cup (0,\infty]$ be $F_i^{p_s,p_b}(t)$, and let the probability that buyer exits by time t in that continuation game be $E_b^{p_s,p_b}(t)$. We can later back out the behavior of rational players from these objects. At time t, the seller's updated reputation for commitment is then $\bar{z}_s^{p_s,p_b}(t) = \bar{z}^{p_s}/(1-F_s^{p_s,p_b}(t))$ while buyer's updated reputation is $\bar{z}_b^{p_s,p_b}(t) = \bar{z}^{p_s,p_b}/(1-E_b^{p_s,p_b}(t) - F_b^{p_s,p_b}(t))$. A rational seller's utility in the continuation game at 0^3 when she concedes at time t is:

$$\begin{aligned} U_s^{p_s,p_b}(t) = &\int^{\tau<t} p_s e^{-r\tau} dF_b^{p_s,p_b}(\tau) + (1 - F_b^{p_s,p_b}(t) - E_b^{p_s,p_b}(t))p_b e^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_b^{p_s,p_b}(t) - F_b^{p_s,p_b}(t_-))(p_s+p_b) + (E_b^{p_s,p_b}(t) - E_b^{p_s,p_b}(t_-))p_b\right) \end{aligned}$$

where $G(t_-) = \sup_{\tau<t} G(\tau)$ with $G(0^3_-) = 0$ for $G : \{0^3, 0^4\} \cup (0,\infty] \to [0,1]$. The utility of a rational buyer with value v that concedes at time t is:

$$\begin{aligned} U_b^{p_s,p_b,v,c}(t) = &\int^{\tau<t} (v-p_b)e^{-r\tau} dF_s^{p_s,p_b}(\tau) + (1-F_s^{p_s,p_b}(t))(v-p_s)e^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_s^{p_s,p_b}(t) - F_s^{p_s,p_b}(t_-))(2v-p_s-p_b)\right) \end{aligned}$$

The utility of a rational buyer with type (v,w) that exits at time t is:

$$\begin{aligned} U_b^{p_s,p_b,v,w,e}(t) = &\int^{\tau<t} (v-p_b)e^{-r\tau} dF_s^{p_s,p_b}(\tau) + (1-F_s^{p_s,p_b}(t))we^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_s^{p_s,p_b}(t) - F_s^{p_s,p_b}(t_-))(w+v-p_b)\right). \end{aligned}$$

[11] This is without loss of generality even if $\mu_s(p_s) = 0$ or $\mu_s^{p_s}(p_b) = 0$; commitment types can't deviate.

I analyze weak perfect Bayesian equilibria of this game, where at each information set (0^1, 0^2 and 0^3) players' strategies must be optimal given their beliefs, beliefs are consistent with Bayes' rule when possible (even off the equilibrium path), and a player's actions do not affect her belief about her opponent. However, my main result, providing tight bounds on equilibrium outcomes also holds for any Nash equilibrium.

3 Equilibrium

This section characterizes equilibria of the game for arbitrary parameters. I follow a heuristic approach leaving many details for the appendix. I first characterize equilibria in the continuation game at 0^3, and then consider players' initial demand choices.

In any equilibrium, a rational seller must imitate a commitment demand $p_s \in P_s$. This is a standard result in the reputational bargaining literature: if not, she would reveal her rationality, and so need to immediately concede to any possibly committed buyer, due to the reputational Coase conjecture (Abreu and Gul (2000)).[12] Henceforth, therefore, assume $p_s \in P_s$. It is also true that rational buyers must imitate commitment demands, but to show that I need to characterize continuation equilibria in the continuation game at 0^3 that might arise if they did not. I now proceed to characterize such equilibria.

3.1 Equilibria in the continuation game

I first describe equilibria in the continuation game at 0^3 assuming commitment demands $p_i \in P_i$ before highlighting what happens when the buyer makes a non-commitment demand ($p_b \notin P_b$). Since p_s and p_b are fixed, I drop them in superscripts on variables.

If there is just one rational buyer type who prefers to concede rather than exit, $v - p_s > w$, there is a unique equilibrium in the continuation game, which resembles that in Abreu and Gul (2000); since the buyer will never choose her outside option it becomes irrelevant. The equilibrium is characterized by three properties: (i) at most one player concedes at time 0; (ii) both players reach a probability 1 reputation at the same time $T^* < \infty$; and (iii) players are indifferent between conceding at any time on $(0, T^*]$.

Property (iii) implies that the seller and buyer must concede at the constant rates λ_s^v and

[12] If such a seller (she) doesn't immediately concede to such a buyer (he), she must expect he will soon concede to her, and if he doesn't, will eventually become convinced of his commitment and so concede. If $T > 0$ is the last time she concedes, however, even a rational buyer won't concede just before T, and so the seller won't wait that long, a contradiction. Given that, the seller can always do weakly better demanding $\max P_s \geq \max P_b$ than $p_s \notin P_s$.

λ_b respectively on $(0, T^*)$, where:

$$\lambda_s^v := \frac{r(v - p_s)}{p_s - p_b}, \quad \lambda_b := \frac{r p_b}{p_s - p_b}.$$

The numerator of player i's concession rate is her opponent's instantaneous cost of delaying his concession, while denominator is the capital gain he receives when she concedes instead of him (so the rate equalizes his costs and benefits of waiting). Let $T_i = -ln(\bar{z}_i)/\lambda_i$ be the time it would take that player i to reach a probability 1 reputation given that she concedes at rate λ_i on $(0, T^*)$ but not at time 0. Then we must have $T^* = \min\{T_s, T_b\}$ and time 0 concession satisfies $F_i(0^4) = 1 - \min\{\bar{z}_i \bar{z}_j^{-\lambda_i/\lambda_j}, 1\}$.

Given indifference to concession on $(0, T^*)$, the seller's continuation game payoff must be $U_s = F_b(0^4) p_s + (1 - F_b(0^4)) p_b$ and the buyer's payoff must be $U_b = F_s(0^4)(v - p_b) + (1 - F_s(0^4))(v - p_s)$; these only exceed what an opponent offers if that opponent concedes at time 0 with positive probability. A player who is more generous to her opponent, offering him more utility when he concedes, concedes faster and so builds reputation quicker, so he must concede more often at time 0 in order for both players to reach probability 1 reputation at the same time. More precisely, the probability the buyer (seller) concedes at time 0 increases (decreases) in the relative generosity of the seller's offer compared to the buyer's, $(v - p_s)/p_b = \lambda_s^v/\lambda_b$.

Adding more buyer types who find the seller's price acceptable, $v - p_s > w$, doesn't greatly change the equilibrium structure highlighted above. The only difference is that property (iii) must be modified to account for the skimming property: on $(0, T^*)$ the seller and highest remaining buyer type are indifferent between conceding at one instant and the next. The skimming property says high value buyers concede before low value buyers, since they face greater (instantaneous) costs of delaying their concession, $r(v - p_s)$. Enumerate these buyers' values $v^1 < v^2 < ... < v^K$ and let t^k be the first time that all buyers with value v^k have conceded, and $t^{K+1} = 0$. The buyer and seller, therefore, concede at rates λ_b and $\lambda_s^{v^k}$ on the interval (t^{k+1}, t^k), where $t^{k+1} \leq t^k$ and $T^* = t^1$ by the skimming property. This equilibrium is still unique.

There is a Coasean force already at work in this equilibrium. Namely, high value buyers benefit from the presence of low value buyers. This is because the seller (he) concedes at a slower rate to low value buyers, which means he concedes with greater probability at time 0 to ensure both players reach a probability 1 reputation at the same time T^* than if all buyers had high value. More precisely, $F_s(0^4)$ is increasing in $\bar{g}(v^1, w)$.

To make the above description more precise, let $\Theta^c = \{(v, w) \in \Theta : v - w > p_s\}$ be the set of rational buyer types for whom the seller's price is acceptable (such types eventually concede= c). Let $\Theta^e = \{(v,w) \in \Theta : v - w < p_s\} = \Theta \setminus \Theta^c$ be the set of types for whom the seller's price is not acceptable (such types eventually exit= e). Let $v^1 = \min\{v \in V : v > \underline{w} + p_s\}$ and $v^{k+1} = \min\{v \in V : v > v^k\}$ until $v^K = \overline{v}$ for some $K < \infty$. and

$$t^k = \min\Big\{t \geq 0^4 : F_b(t) \geq (1 - \bar{z}_b) \sum_{(v,w)\in\Theta^c : v \geq v^k} \overline{g}(v,w)\Big\}.$$

Adding some Θ^e buyer types who find the seller's price unacceptable, $w > v - p_s$, further modifies the equilibrium structure. Let $v^1 < v^2 < ... < v^K$ and $t^1 \geq t^2 \geq ... \geq t^{K+1}$ be defined as above with respect to Θ^c types that eventually concede. Additionally, define $\underline{\lambda}^{v,w}$ as the concession rate that would make a buyer of type (v,w) indifferent between immediately exiting and waiting an instant to do so:

$$\underline{\lambda}^{v,w} := \frac{rw}{v - p_b - w}.$$

In equilibrium: buyer and seller must still concede at rates λ_b and $\lambda_s^{v^k}$ on the interval (t^{k+1}, t^k), to keep the seller and highest remaining Θ^c value buyer indifferent between conceding at one instant or the next. Given this, buyer type $(v,w) \in \Theta^e$ will choose to exit at: time $0 = t^{K+1}$ if $\underline{\lambda}^{v,w} > \lambda_s^{\overline{v}}$; at time t^k if $\underline{\lambda}^{v,w} \in (\lambda_s^{v^{k-1}}, \lambda_s^{v^k})$; and at $T^* = t^1$ if $\underline{\lambda}^{v,w} > \lambda_s^{v^1}$. If the buyer exits at $t^k < T^*$ with positive probability, she must also concede with positive probability; if she didn't, the seller (he) would prefer to concede just before t^k rather than just after. On the other hand, if $t^k > 0$ she can't concede too often or he would prefer to concede just after t^k than before. More precisely, suppose the buyer exits with (conditional) probability α at time t^k. If $t^k < T^*$ then the buyer must also concede with probability greater than $\alpha p_b/(p_s - p_b)$. If $t^k > 0$, she must not concede with probability greater than $\alpha p_b/(p_s - p_b)$. If $t^k \in (0, T^*)$, therefore, she must concede with probability $\alpha p_b/(p_s - p_b)$ exactly.

The presence of Θ^e buyers that exit means that both players may concede with positive probability at time 0, contrary to property (i) with only Θ^c buyers. This is because the seller may concede at 0^3 when the buyer both concedes and exits at 0^4 with positive probability. Without loss of generality, however, the buyer never concedes at 0^3 and the seller never concedes at 0^4. The equilibrium may no longer be unique. This is because the value of any buyers who concede at $t^k \in (0, T^*)$ is not determined (which can shift t^k and T^*), and total concession at T^* is not determined (which can shift T^*).

How should the above equilibrium characterization be modified if the buyer makes a

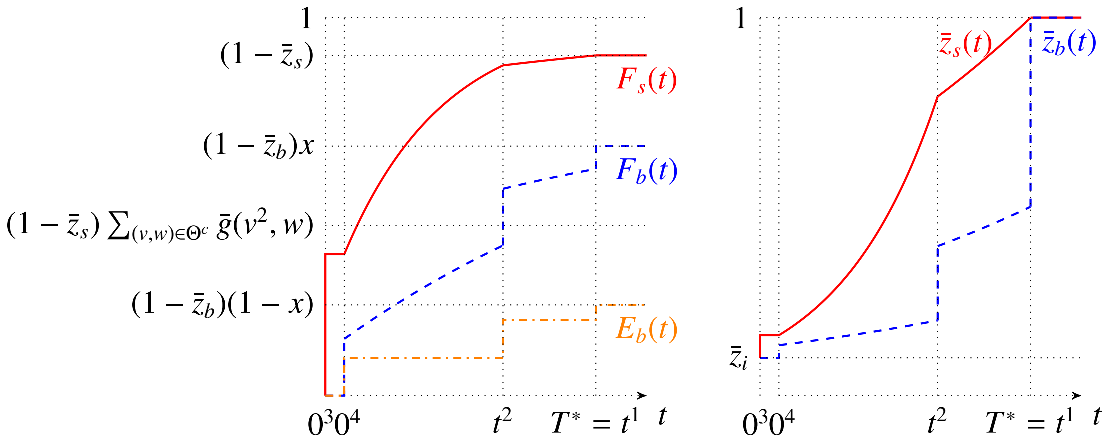

Figure 1. An equilibrium. Left: concession/exit probabilities. Right: reputations.

non-commitment demand ($p_b \notin P_b$)?[13] To accommodate this situation, I generalize the definition of T^*. Let $T_b = \min\{t \geq 0^4 : F_b(t) = (1-\bar{z}_b)x\}$, $T_s = \min\{t \geq 0^4 : F_s(t) = 1-\bar{z}_b\}$ and $T^* = \min\{T_s, T_b\}$ where $x = \sum_{(v,w)\in\Theta^c}\bar{g}(v,w)$ is the probability that a buyer is a Θ^c type, conditional on being rational. A rational buyer must have conceded or exited with probability 1 by time $T^* < \infty$ (as when $p_b \in P_b$). However, a rational seller may, or may not, have conceded with probability 1 by T^*. Prior to T^*, the other properties of equilibria (described above for $p_b \in P_b$) must still hold. Those properties ultimately allow me to show that the buyer only makes commitment demands ($p_b \in P_b$); this is discussed in the next subsection.

Figure 1 displays an equilibrium of this sort. I summarize the above equilibrium characterization into the following lemma.

Lemma 1. *Consider a continuation game at 0^3 after demands $p_s \in P_s$ and $p_b \in P$. Without loss of generality, the buyer never concedes or exits at 0^3 and the seller never concedes at 0^4. The skimming property holds before $T^* < \infty$ and:*

(1) the seller concedes at rate $\lambda_s^{v^k}$ and the buyer at rate λ_b on $(t^{k+1}, t^k) \cap (0, T^*)$.

(2) buyer $(v,w) \in \Theta^e$ exits at time $t^k \in \{t^1, ..., t^{K+1}\} \cap [0, T^*]$ if $\underline{\lambda}^{v,w} \in (\lambda_s^{v^{k-1}}, \lambda_s^{v^k})$ where $\lambda_s^{v^0} = 0$ and $\lambda_s^{v^{K+1}} = \infty$.

(3) if the buyer exits at time $t^k \in \{t^1, ..., t^{K+1}\} \cap [0, T^*]$ with probability α then she must concede with probability greater than $\alpha p_b/(p_s - p_b)$ whenever $t^k < T^*$ and with probability less than $\alpha p_b/(p_s - p_b)$ if $t^k \in (0, T^*]$.

(4) $F_b(T^*) + E_b(T^*) = 1 - \bar{z}_b$ and if $p_b \in P_b$ then $F_s(T^*) = 1 - \bar{z}_s$.

[13] While maintaining that the seller makes a commitment demand ($p_s \in P_s$).

3.2 Equilibrium demand choices

I next consider the buyer's demand choice at 0^2 after the seller has announced a commitment demand. I show that Θ^e buyer types (who eventually exit) always imitate the lowest commitment demand $\underline{p}$ (or exit immediately). Whereas, Θ^c buyer types (who eventually concede) imitate some commitment type, and weakly (possibly strictly) prefer lower price demands. This is an important result for establishing the paper's main findings because it means Θ^e buyers demands are always extremely ungenerous to the seller when there is a rich set of commitment types, $\underline{p} \approx 0$, who consequently has little incentive to concede to them. The lemma below provides a formal statement:

Lemma 2. *Consider any equilibrium in the continuation game at 0^2 after a seller demand $p_s \in P_s$. Without loss of generality: Θ^c buyers never announce a non-commitment price $p_b \notin P_b$, and receive a weakly higher continuation payoff from $p_b \in P_b$ than from $p_b' > p_b$. Θ^e buyers never announce a non-minimal price $p_b > \underline{p} \in P_b$.*

First consider the behavior of the Θ^c buyers who eventually concede; this dictates the behavior of the Θ^e buyers who exit. Suppose some Θ^c buyers announce price $p_b' \in P$ with positive probability (allowing for $p_b' \notin P_b$). Then some Θ^c buyers must also imitate smaller commitment prices, $p_b < p_b'$ where $p_b \in P_b$. If this wasn't true, a rational seller would believe buyers who demand p_b are either committed or will eventually exit, and so would immediately concede. But in that case, demanding $p_b' > p_b$ cannot be optimal for any buyer (who would prefer the lower price).

Since a value $\overline{v} = v^K$ buyer always concedes first (by the skimming property), she must be indifferent between demanding p_b' or p_b and then conceding at any time $t \in [0^4, t^{K,p_b}]$. Suppose then that a value v^k buyer is indifferent between the options: (i) demand p_b and concede at t^{k,p_b} and (ii) demand p_b' and concede at $t^{k,p_b'}$. The proof of the lemma establishes that a value v^{k-1} buyer is either indifferent between (i) and (ii) or strictly prefers the lower price (i). Moreover, if she is indifferent between these options then so are all buyers, and there is a greater time discounted probability that the seller has not conceded after the lower price p_b, that is: $e^{-rt^{k,p_b'}}(1 - F_s^{p_b'}(t^{k,p_b'})) \leq e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b}))$, strictly if $F_s^{p_b'}(t^{k,p_b'}) > 0$.

The reason a value v^{k-1} buyer must (weakly) prefer the lower price, option (i) over (ii), is that there is "more delay" after the lower price since the seller concedes more slowly ($\lambda_s^{v^k,p_b} < \lambda_s^{v^k,p_b'}$), and the value v^{k-1} buyer has a lower cost of delay than v^k who is indifferent between the options. For a more precise argument, consider the special case of $k = K$, and suppose (by way of contradiction) a value v^{K-1} buyer strictly preferred option (i) over (ii). All buyers with value $v < v^{K-1}$ must then similarly prefer (ii), because the expected utility of (i) and (ii) are linear in the buyer's value. Since no buyer with value $v \leq v^{K-1}$ adopts (i) we would then have $T^{*,p_b} = t^{K,p_b}$ and since the seller concedes slower after the lower price ($\lambda_s^{v^k,p_b} < \lambda_s^{v^k,p_b'}$) we would have $t^{K,p_b'} < t^{K,p_b} = T^{*,p_b}$. Consider then a third option (iii), demand p_b and then concede at $t^{K,p_b'} \in (0, t^{K,p_b})$. The difference in utility for value v^K between option (iii) and (ii) is 0. However, since the seller concedes slower after the lower price ($\lambda_s^{v^k,p_b} < \lambda_s^{v^k,p_b'}$), there is more delay in (iii) than (i), and because delay is less costly for value v^{K-1} than for v^K, the utility difference between (iii) and (ii) must be strictly positive for v^{K-1}. The value v^{K-1} buyer would then strictly prefer (i) over (iii) over (ii). To make a value v^{K-1} buyer indifferent between (i) and (ii) we need $t^{K,p_b} < t^{K,p_b'}$ (and so $t^{K,p_b} < T^{*,p_b}$) unless $t^{K,p_b} = t^{K,p_b'} = 0^4$.

To see why $e^{-rt^{k,p_b'}}(1 - F_s^{p_b'}(t^{k,p_b'})) \leq e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b}))$ when a value v^{k-1} buyer is indifferent between (i) and (ii), consider the special case of $t^{k,p_b'} = t^{k,p_b} = 0^4$.[14] For the value v^{k-1} buyer to be indifferent between (i) and (ii), the seller must concede more often after the higher price p_b'. If the seller conceded with the same strictly positive probability after each demand then the v^{k-1} buyer would strictly prefer the lower price option (i). More precisely, we need $F_s^{p_b'}(0^3) = F_s^{p_b}(0^3)(p_s - p_b)/(p_s - p_b') \geq F_s^{p_b}(0^3)$.

I now turn to the claim that Θ^e buyers (who eventually exit) only ever make the minimal commitment demand $\underline{p} \in P_b$ or immediately exit. Suppose a value v^{k-1} buyer is indifferent between option (i) demand p_b before conceding at t^{k,p_b} and (ii) demand $p_b' > p_b$ before conceding at $t^{k,p_b'}$, then type $(v', w') \in \Theta^e$ is also indifferent. Consider then options (i') demand p_b before exiting at t^{k,p_b} and (ii') demand p_b' before exiting at $t^{k,p_b'}$. Buyer type $(v', w') \in \Theta^e$ receives a payoff gain switching from (i) to (i') that is proportional to the discounted probability that the seller has not conceded, $e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b}))(w' - (v' - p_s))$. That gain is strictly larger than the gain switching from (ii) to (ii') if $e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b})) > e^{-rt^{k,p_b'}}(1 - F_s^{p_b'}(t^{k,p_b'}))$, and so (i') is preferred to (ii'). If $F_s^{p_b}(t^{k,p_b}) = F_s^{p_b'}(t^{k,p_b'}) = 0$ and option (ii') is ever an optimal strategy for type (v', w') it is without loss of generality to assume she exits at time 0^2 instead. A similar argument can be made when a value v^{k-1} buyer strictly prefers option (i) over (ii).

Given that a Θ^e buyer only demands $\underline{p} \in P_b$, a Θ^c buyer cannot make a non-commitment demand, $p_b \notin P_b$. If she did, standard arguments imply she must immediately concede to a possibly committed seller (the reputational Coase conjecture).

Finally, we can move back to the start of the game and the seller's demand choice at 0^1. I do not attempt a precise characterization here, but merely establish that an equilibrium

[14]This can occur if the buyer exits and concedes at 0^4.

exists, Proposition 1. The proof is in the Online Appendix. It first identifies a particular continuation equilibrium structure in the continuation game at 0^3 that is continuous in players' beliefs, $\bar{z}_i$ and $\bar{g}$. Given that, existence incorporating demand choice follows by a standard Kakutani fixed point argument.

Proposition 1. *An equilibrium exists.*

4 Vanishing commitment

This section presents the paper's main result: when the set of buyer values and commitment types are rich and commitment vanishes, bargaining outcomes are approximately equivalent to those when the seller can propose an ultimatum at a price below $p^* = \max\{\underline{w}, \underline{v}/2\}$. To get there, I again first focus on the continuation game at 0^3 (as players' initial reputations vanish), before considering demand choices.

4.1 Vanishing commitment in the continuation game

First consider the simple case in which there is only a single rational buyer type who eventually concedes $v - p_s > w$. Recall, this setting is equivalent to Abreu and Gul (2000); the outside option is irrelevant. If players' initial reputations in the continuation game vanish at same rate ($\bar{z}_i^n \to 0$, $\bar{z}_i^n/\bar{z}_j^n \in [1/L, L]$ for some $L \geq 1$) then the player who is less generous than her opponent must concede with probability approaching 1 at time 0; the seller immediately concedes if $p_b > v - p_s$, and the buyer does if $p_b < v - p_s$. This occurs because the generosity of player i's offer is proportional to the cost of delay of her opponent (him) and thus proportional to her concession rate, λ_i. Those concession rates determine the exponential growth rate of a player's reputation during the continuation game, $(d\bar{z}_i(t)/dt)/\bar{z}_i(t) = \lambda_i$. When initial reputations are vanishingly small, reputations must grow a lot to reach probability 1 (it takes infinitely long in the limit). Absent time 0 concession, therefore, the faster growth rate of the more generous player's reputation means she would reach a probability 1 reputation much faster than her opponent. To ensure both players reach a probability 1 reputation at the same time, therefore, the less generous player must concede immediately (with probability approaching 1).

By similar logic, in a continuation game with many buyer types and $p_b > \underline{p}$, where the seller is *always* more generous than the buyer ($v^1 - p_s > p_b$), the buyer must concede at time 0 with probability approaching 1 if players' initial reputations vanish at the same rate. Since $p_b > \underline{p}$, the buyer never exits (by Lemma 2). The more generous seller concedes faster and so builds reputation faster. To ensure both players reach a probability 1 reputation at the same time, the buyer concedes immediately in the limit.

More interestingly, in a continuation game where the lowest value buyer who concedes is more generous than the seller ($v^1 - p_s < p_b$) the seller must concede immediately with probability approaching 1 if players' initial reputations vanish at the same rate and the lowest value buyer's probability doesn't ($\lim_n \sum_{(v^1,w)\in\Theta^c} \bar{g}^n(v^1, w) > 0$). This prediction highlights the key Coasean force which drives the main results: any possibility of a generous low value buyer makes the seller immediately back down.

The reason for the result is: The buyer's positive concession rate means that all high value buyers (who may be less generous than the seller, e.g. $\overline{v} - p_s > p_b$) must have conceded in some bounded length of time. At that point, players' updated reputations must still be arbitrarily small, and so we are effectively back in the case of a single rational buyer type: the additional time it takes for players to reach a probability 1 reputation is unbounded, and during that time the (more generous) low value buyer concedes quicker than the seller and so her reputation grows quicker. To ensure both players reach a probability 1 reputation at the same time, therefore, the seller must concede immediately in the limit.

Figure 2 illustrates this logic in an example with two buyer values $v^2 = 6$, $v^1 = 4$ and outside option $\underline{w} < 1$. The announced prices $p_s = 3$ and $p_b = 2$ imply that the seller is more generous than the high value buyer, and so concedes at a faster rate $\lambda_s^{v^2} = 3 > \lambda_b = 2$ on the interval $(0, t^2)$. However, the seller is less generous than the low value buyer, and so concedes slower thereafter $\lambda_s^{v^1} = 1 < \lambda_b = 2$. Initial reputations are small, $\bar{z}_i = 1/100$, and so remain small at t^2 after all high value buyers have conceded. Even though the buyer is much more likely to have a high rather than low value, $\bar{g}(v^2, \underline{w})/\bar{g}(v^1, \underline{w}) = 3$, it takes much less time for all high value types to concede than all low value types, $t_2/T^* < 1/3$ because of the concave shape of buyer's concession probability, $F_b(t) = 1 - e^{-\lambda_b t}$ ($t_2/T^* \to 0$ as $\bar{z}_b^n \to 0$). This means most reputation building occurs after t^2 when the buyer has a reputation building advantage (since $\lambda_s^{v^1} < \lambda_b$). To ensure that both players reach a probability 1 reputation at the same time therefore the seller must concede with high probability at time 0, $F_s(0^3) = 0.61$.

What happens in the continuation game as initial reputations vanish, when some types exit (so $p_b = \underline{p}$)? If there is some type which waits until T^* to exit ($\underline{\lambda}^{v,w} < \lambda_s^{v^1}$ for some $(v, w) \in \Theta^e$) then the seller must again concede with probability approaching 1 at time 0. The buyer's positive concession rate means that the time at which only this exiting type (or a commitment type) remains is bounded. For the seller's reputation to reach

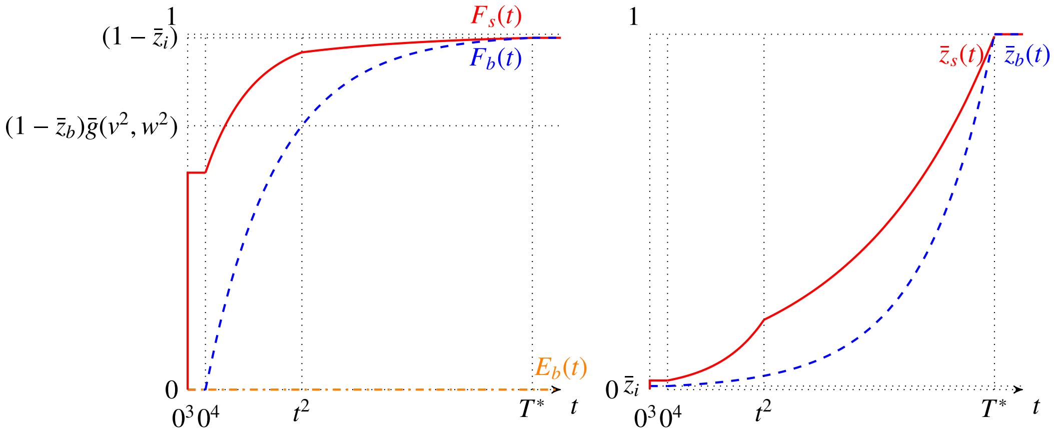

Figure 2. Continuation equilibrium with parameters: $v^2 = 6$, $v^1 = 4$, $\underline{w} < 1$, $p_s = 3$, $p_b = 2$, $\bar{g}(v^1, w) = 1/4$, $\bar{z}_i = 1/100$. Left: Concession. Right: Updated reputations.

probability 1 by that time, therefore, she must concede immediately in the limit.

On the other hand, if $p_b = \underline{p}$ is sufficiently small and no buyers exit at T^* ($\underline{\lambda}^{v,w} > \lambda_s^{v^1}$ for all $(v, w) \in \Theta^e$), then the buyer must immediately concede or exit with probability approaching 1. When $p_b = \underline{p}$ is very small (see Lemma 3 part (f) for a precise cutoff) not only is the seller more generous than any buyer who concedes $p_b = \underline{p} \leq \underline{w} < v^1 - p_s$ but the buyer is so ungenerous that the seller would wait to receive concession from $(v^1, w) \in \Theta^c$ buyers at time t^2 even if all other buyer types exit at t^2. This means that players' reputations must be vanishingly small at t^2 and since the seller concedes faster at that point, for both players to reach a probability 1 reputation at the same time, the buyer must concede and exit immediately in the limit.

Finally, unsurprisingly, if the seller's reputation vanishes but the buyer's doesn't ($\lim_n \bar{z}_b^n > 0$) then the seller must immediately concede with probability approaching 1 or the buyer would reach a probability 1 reputation before the seller. Similarly, if the buyer's reputation vanishes but the seller's doesn't ($\lim_n \bar{z}_s^n > 0$), and no buyers exit ($p_b > \underline{p}$) then the buyer immediately concedes with probability approaching 1. Lemma 3 formally establishes all these results.

Lemma 3. *Consider some fixed demands $p_s \in P_s$ and $p_b \in P_b$ and sequence of continuation games at 0^3 with updated equilibrium beliefs $(\bar{z}_i^n, \bar{g}^n)$.*
Suppose $\bar{z}_s^n \to 0$ and $\bar{z}_b^n \geq L\bar{z}_s^n$ for some constant $L > 0$.

(a) If $\lim_n \bar{z}_b^n > 0$, then $\lim_n F_s^n(0) = 1$.

(b) If $\lim_n \bar{g}^n(v, w) > 0$ for some $(v, w) \in \Theta^c$ with $v - p_s < p_b$, then $\lim_n F_s^n(0) = 1$.

(c) If $p_b = \underline{p}$ and $\underline{\lambda}^{v,w} < \lambda_s^{v^1}$ for some $(v, w) \in \Theta^e$, then $\lim_n F_s^n(0) = 1$.

Suppose instead $\bar{z}_b^n \to 0$ *and* $\bar{z}_s^n \geq L\bar{z}_b^n$ *for some constant* $L > 0$.

(d) If $p_b > \underline{p}$ *and* $\lim_n \bar{z}_b^n > 0$*, then* $\lim_n F_b^n(0^4) = 1$.

(e) If $p_b > \underline{p}$ *and* $v^1 - p_s > p_b$*, then* $\lim_n F_b^n(0^4) = 1$.

(f) If $p_b = \underline{p} < v^1 - p_s$ *and* $p_b(1 - \lim_n x^n) < (p_s - p_b) \lim_n \sum_{(v^1,w)\in\Theta^c} \bar{g}^n(v^1, w)$ *and* $\underline{\lambda}^{v,w} > \lambda_s^{\overline{v^1}}$ *for all* $(v, w) \in \Theta^e$*, then* $\lim_n F_b^n(0^4) = 1 - \lim_n E_b^n(0^4) = \lim_n x^n$.

4.2 The main results

This subsection presents the paper's main results. Proposition 2 identifies tight bounds on the seller's payoff as commitment vanishes when the set of buyer's values and commitment types are rich: it is approximately the same as what she could get by making an ultimatum offer to the buyer at a price restricted to be below $p^* = \max\{\underline{v}/2, \underline{w}\}$. It also shows that $\max\{v - p^*, w\}$ is an approximate lower bound on the buyer's payoff. Proposition 3 provides additional regularity conditions that identify the ultimatum price the seller would charge, and so precisely predict limit equilibrium outcomes.

In order to formally present these results, I must first define what makes the sets of buyer values and commitment types rich. I say that the set of buyer values is $\varepsilon > 0$ rich if for any $d \in [\underline{v}, \bar{v}]$, there exists some $v \in V$ such that $|v - d| < \varepsilon$. This means that the difference between two consecutive buyer values must be less than 2ε. Given a rational buyer's type distribution, I say that the sets of players' commitment types are $\varepsilon' > 0$ rich if for any $d' \in [0, \bar{v} - \underline{w}]$, there exists some $p_i \in P_i$ such that $|p_i - d'| < \varepsilon'$ for $i = 1, 2$. When I say that set of buyer values and commitment types are rich, I informally mean that they are respectively $\varepsilon > 0$ and $\varepsilon' > 0$ rich where $\varepsilon \approx \varepsilon' \approx 0$. However, in fact, my results first fix the ε richness of the buyer's values, and then choose the ε' richness of commitment types. I also define $H(p) = \sum_{(v,w):v-w<p} g(v, w)$ for any $p \in [0, \infty)$, as the probability a rational buyer's net value $v - w$ is less than p; for instance, if net values are approximately uniformly distributed on $[0, 1]$ then $H(p) \approx p$.

Proposition 2 presents precise upper and lower bounds on the seller's equilibrium payoff, V_s and a lower bound on the buyer's equilibrium payoff $V_b^{v,w}$. For any $\delta > 0$, it fixes an arbitrary $\varepsilon > 0$ rich rational buyer type distribution. It then identifies a sufficiently small $\varepsilon' > 0$ such that if the distribution of commitment types is ε' rich, and players' prior probabilities of commitment vanish at the same rate ($z_i^n \to 0$, $z_i^n/z_j^n \in [1/L, L]$ for some $L \geq 1$): The seller's limit payoff is at most δ more than her payoff from making an ultimatum with prices restricted to be below $p^* + 2\varepsilon$ (the upper bound). Her limit payoff can also not be less than 2ε below her payoff from making an ultimatum with prices restricted to be below p^* (the lower bound). This tightly pins down the seller's payoff for small ε. The buyer can likewise guarantee a payoff of at least $\{v - p^* - 2\varepsilon, w\}$.

Proposition 2. *For any $\delta > 0$, and any $\varepsilon > 0$ rich distribution of rational buyer types (g, Θ), there exists some $\varepsilon' > 0$ such that for any sequence of bargaining games $(z_i^n, \pi_i, P_i, g, \Theta, P)_{i \in s,b}$ with a ε' rich distribution of commitment types, $z_i^n \to 0$ and $z_s^n / z_b^n \in [1/L, L]$ for some $L \geq 1$, the seller's payoffs satisfy:*

$$\max_{p \in [0, p^*]} (1 - H(p))p - 2\varepsilon \leq \liminf_n V_s^n \leq \limsup_n V_s^n \leq \max_{p \in [0, p^* + 2\varepsilon]} (1 - H(p))p + \delta.$$

and the buyer's payoffs satisfy: $\liminf_n V_b^{v,w,n} \geq \max\{w, v - p^* - 2\varepsilon\}$

To explain the logic for this result, I first highlight something special about $p^* = \max\{\underline{v}/2, \underline{w}\}$. It is the highest price p^* such that the seller can always guarantee her offer is more generous than the counterdemand of any buyer who eventually concedes.

To see that a seller proposing $p_s \leq p^*$ is always more generous than the counterdemand of any buyer who eventually concedes, notice that if $p_s \leq \underline{v}/2$ then $\underline{v}/2 \leq \underline{v} - p_s$ and so $p_b < p_s \leq \underline{v}/2 \leq \underline{v} - p_s$. Similarly, $p_s \leq \underline{w}$ implies $p_b < p_s \leq \underline{w} < v^{1,p_s} - p_s$.

This feature of p^* helps establish the lower bound on the seller's payoff. Suppose price $\hat{p}_s$ maximizes the seller's payoff when she can issue an ultimatum at a price below p^*. In a reputational bargaining game with a rich set of buyer values and commitment types, we can find a slightly small commitment demand $p_s \in P_s$ (so $p_s \geq \hat{p}_s - 2\varepsilon$ when buyer values are ε rich, and commitment types are $\varepsilon' \approx 0$ rich), which ensures the buyer either immediately concedes or exit with probability approaching 1 as commitment vanishes, and so provide a limit profit of approximately $\hat{p}_s(1 - H(\hat{p}_s))$. Any counterdemand made with positive probability in the limit would make the buyer's updated reputation in the continuation game vanish at a weakly faster rate than the seller's. If that counterdemand is not minimal, $p_b > \underline{p}$, it is only made by Θ^{c,p_s} buyers who eventually concede, and because they are less generous than the seller (since $p_s \leq p^*$), they must then immediately concede with probability approaching 1 (Lemma 3, part (d)). If that counterdemand is minimal, however, then it is very ungenerous, $p_b = \underline{p} \approx 0$, since the set of commitment types is rich. The seller's limit demand p_s is chosen to ensure there is no Θ^{e,p_s} buyer type that would wait until T^* to exit.[15] And so, in this continuation game the buyer immediately concedes or exits with probability approaching 1 (Lemma 3, part (f)).

[15]If $\hat{p}_s \leq \underline{v} - \underline{w}$ then no Θ^{e,p_s} will wait until T^* for any $p_s < \hat{p}_s$. If $\hat{p}_s > \underline{v} - \underline{w}$ then there is some $p_s \approx v^{1,p_s} - \underline{w}$ which ensures no such waiting. Since $\underline{\lambda}^{v,\underline{w},p_s,\underline{p}}$ is decreasing in v and $\underline{\lambda}^{v^{1,p_s},\underline{w},p_s,\underline{p}} \approx \lambda_s^{v^{1,p_s},p_s,\underline{p}}$ given $p_s \approx v^{1,p_s} - \underline{w}$ we have $\underline{\lambda}^{v,\underline{w},p_s,\underline{p}} > \lambda_s^{v,p_s,\underline{p}}$ for $v \leq v^{1,p_s}$. Furthermore $\underline{\lambda}^{v,w,p_s,\underline{p}} \geq \underline{\lambda}^{v,\underline{w},p_s,\underline{p}}$ for $w \geq \underline{w}$.

On the other hand, when the seller proposes $p_s > p^*$ and the set of buyer values and commitment types is rich, there is a counterdemand $p_b \approx p^*$ for the lowest value buyer who concedes, v^{1,p_s} that is more generous than the seller's offer (more precisely if $p_s > p^* + 2\varepsilon$ then there is a more generous counterdemand $p_b < p^* + 2\varepsilon$ for the buyer). To see this, it is useful to distinguish between: case (i) $p_s < \underline{v} - \underline{w}$, when some $\underline{v}$ buyers eventually concede; and case (ii) $p_s \in (\underline{v}-\underline{w}, \overline{v}-\underline{w})$ when all $\underline{v}$ buyers eventually exit. In case (i) we must have $p_s > p^* = \underline{v}/2$ and so $p_s > \underline{v} - p_s > \underline{v}/2$ and so a counterdemand $p_b \approx p^* = \underline{v}/2$ by a value $\underline{v} = v^{1,p_s}$ buyer, is more generous than the seller's offer; in fact, this is also true for counterdemands as low as $p_b \approx (\underline{v} - p_s) < \underline{v}/2$. In case (ii) if there is a rich set of buyer values, then the lowest value buyer that ever concedes v^{1,p_s} must be close to indifferent to taking the lowest outside option $v^{1,p_s} - p_s \approx \underline{w}$. Any counterdemand $p_b > v^{1,p_s} - p_s$ for a value v^{1,p_s} buyer is more generous than the seller's offer. With a rich set of commitment types, there is such a (more generous) counterdemand $p_b \approx v^{1,p_s} - p_s \approx \underline{w} \leq p^* < p_s$.

This second feature of p^* establishes the upper bound on the seller's payoff, and the lower bound on the buyer's payoff. Suppose the seller charged $p_s > p^*$ with positive limit probability as commitment vanished, then her updated reputation vanishes (at a weakly faster rate than the buyer's). As highlighted above, given a rich set of buyer values and commitment types there are commitment counterdemands $p_b \in P_b$ which would make the lowest value buyer who concedes more generous than seller, $p_b > v^{1,p_s} - p_s$, with either $p_b < p^* = \underline{v}/2$ (in case (i)) or $p_b \approx \underline{w} \leq p^*$ (in case (ii)). The lowest value buyer, v^{1,p_s}, *will* make the lowest counterdemand p_b that is more generous than the seller's offer with positive limit probability in equilibrium: if she didn't some higher value buyer must instead, and the fast seller concession rate to that buyer would cause v^{1,p_s} to deviate to p_b. Following that counterdemand, therefore, the seller immediately concedes with probability approaching 1 (Lemma 3 part b). Other buyer types can also imitate that counterdemand to guarantee a limit payoff of at least $\max\{v - p_b, w\}$.

From the seller's perspective then, the game with a rich set of values and commitment types is approximately equivalent to an ultimatum game with an upper bound on prices of p^*, however, which ultimatum price she chooses has a large effect on the buyer's payoff. To get a tighter prediction Proposition 3 imposes conditions on the seller's ultimatum game payoff function. It effectively says that if there is a unique price $\hat{p}_s$ that maximizes the seller's payoff in the ultimatum game when she can't charge more than p^*, then the seller charges approximately $\hat{p}_s$ in equilibrium and the buyer either accepts it or exits; in fact, the condition is slightly more involved to account for the richness of the sets of agents' types.

Proposition 3. *Fix any $\varepsilon > 0$ rich distribution of rational buyer types (g, Θ), and define $\check{p}(p) = \min\{p, \max\{v - \underline{w} \leq p : v \in V\}\}$ where $\max \emptyset = \infty$. If $\check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) > p(1 - H(p))$ for $\hat{p}_s \leq p^*$ and all $p \in [0, \check{p}(\hat{p}_s)) \cup (\hat{p}_s, p^* + 2\varepsilon]$, then $\varepsilon' > 0$ exists such that for any sequence of bargaining games $(z_i^n, \pi_i, P_i, g, \Theta, P)_{i \in s,b}$ with a ε' rich distribution of commitment types, $z_i^n \to 0$ and $z_s^n / z_b^n \in [1/L, L]$ for some $L \geq 1$: $\lim_n \sum_{p_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s)} \mu_s^n(p_s) = 1$, and without loss of generality $\lim_n \mu_b^{p_s, v, w, n}(\{c, e\}) = 1$ when $\lim_n \mu_s^n(p_s) > 0$. So, $\max\{w, v - \hat{p}_s\} \leq \liminf_n V_b^{v,w,n} \leq \limsup_n V_b^{v,w,n} \leq \max\{w, v - \hat{p}_s - 2\varepsilon\}$.*

5 Extensions and discussion

This section discusses additional implications of the model, and extensions of it.

5.1 Alternating offers

Below, I outline an alternating offers bargaining protocol where outcomes must converge to those of the continuous time game as offers become frequent. The protocol is a minimal modification of BP.

In period 1 the seller can propose any price $p_s^1 \in [0, \infty)$. The buyer observes this and can then accept, reject, or exit (taking her outside option). If still bargaining in period $n \geq 2$, the buyer can propose any price $p_b^n \in [0, \infty)$. The seller observes this and can accept, or make any counterdemand $p_s^n \in [0, \infty)$. If the seller makes a counterdemand the buyer observes this and can accept, reject, or exit. If one of the players accepts, or exits, the game ends. If the price p is agreed in period n, a rational seller gets $\delta^{n-1} p$ and a rational buyer $\delta^{n-1}(v - p)$ where $\delta = e^{-r\Delta}$ for some period length $\Delta > 0$. If the buyer exits in period n, rational payoffs are 0 and $\delta^{n-1} w$ respectively. The description of types is unchanged except now assume $\underline{p} = \min P_b < \min_{(v,w) \in \Theta} v - w$.

In this model, the buyer never reveals rationality in equilibrium, before the game ends. This is an immediate consequence of Lemma 1 from BP. If a buyer did reveal rationality, a rational seller will never propose or accept a price strictly below that of the lowest net value type she considers feasible. Given that, the lowest net value buyer's continuation payoff will be (weakly) less than her outside option w. Of course, if this buyer faces a committed seller, her continuation payoff is also below $\max\{v - p_s, w\}$. However, the buyer could have obtained the payoff $\max\{v - p_s, w\}$ in the previous period, avoiding discounting, and so would never wait.

On the other hand, if the seller reveals rationality, she must almost immediately concede to a possibly committed buyer when offers are frequent, due to the reputational Coase conjecture (see Abreu and Gul (2000)'s Lemma 1). For any $\varepsilon > 0$, there exists $\overline{\Delta} > 0$ such that if $\Delta < \overline{\Delta}$ and the seller has revealed rationality but the buyer has not, then the buyer's continuation payoff is at least $\max\{v - p_b, w\} - \varepsilon$ and the seller's is at most $p_b + \varepsilon$. As offers become frequent $\Delta \to 0$, therefore, equilibria converge to those of the continuous time game (see Abreu and Gul (2000)'s Proposition 4 for a similar proof).

Equivalent results would also hold under BP's protocol with only seller offers if the buyer could send a cheap talk message at the start of bargaining, indicating which offers she would accept (assuming commitment types indicate truthfully). Both the above protocols would give BP's results if there were no commitment types (the seller can effectively make any ultimatum).

For a much larger set of discrete-time protocols, there will still be equilibria that converge to the continuous time equilibria (as offers become frequent). The seller believes her price must be acceptable to any buyer who reveals rationality, $v - p_s > w$, and such a buyer must then concede almost immediately due to the reputational Coase conjecture. However, it might also be possible to construct other equilibria.

In some settings, the buyer may seem to not even have cheap talk opportunities to indicate her willingness to pay. If so, but there is still a rich set of commitment types for both buyer and seller (who won't accept less than a target price), then a rational seller can effectively choose any ultimatum price as commitment vanishes, consistent with BP. The reason is that if the seller ever reveals rationality, she must almost immediately drop her price to be acceptable to the lowest buyer commitment type $\underline{p} \approx 0$ when offers are frequent (due to Abreu and Gul (2000)'s reputational Coase conjecture). Hence, effectively, rational buyers can only announce $\underline{p} \approx 0$, which guarantees the seller is more generous than the buyer. The buyer must, therefore, immediately concede or exit as commitment vanishes (see Lemma 3). This highlights the benefit to the buyer of being able to indicate generosity (showing she isn't committed to $\underline{p} \approx 0$).[16]

5.2 Unequal discount rates

All the paper's results generalize straightforwardly when seller and buyer have different discount rates, r_s and r_b. In particular, as commitment vanishes with a rich set

[16]This conclusion depends on strictly positive buyer outside options. With no outside options, the seller would propose a price below $\underline{v}$ as commitment vanished, to ensure no rational buyer waited forever. That prediction is broadly consistent with Inderst (2005), who assumes the buyer is always rational and cannot make offers (and so accepts $\underline{v}$), while the seller might be a commitment type.

of buyer values and commitment types, outcomes are approximately equivalent to the seller choosing an ultimatum below the price $p^{**} = \max\{r_b \underline{w}/r_s, r_b \underline{v}/(r_s + r_b)\}$.

If the seller is much more patient than the buyer, $r_b/r_s > (\overline{v} - \underline{w})/\underline{w}$, then $p^{**} > \overline{v} - \underline{w}$ so there is effectively no constraint on the seller's prices (she would never choose $p_s > \overline{v} - \underline{w}$); predictions are then equivalent to BP. On the other hand, if the buyer is much more patient than the buyer we have $p^{**} \approx 0 < \min\{v - w > 0 : (v, w) \in \Theta\}$, implying an efficient outcome.

The large effect of discount rates contrasts with their irrelevance in the original Coase conjecture setting with frequent seller offers and no commitment types (the good is sold almost immediately for $\underline{v} > 0$). Discount rates are also irrelevant in BP.[17]

The logic for the result is identical to Proposition 2. By charging $p_s \leq p^{**}$ the seller ensures that she always concedes at a faster rate than the buyer, $\lambda_s^v > \lambda_b$, so the buyer must concede or exit at time 0. However, if $p_s > p^{**}$ then after a counterdemand $p_b \approx p^*$, the lowest value buyer that concedes v^{1,p_s}, would concede faster than the seller, $\lambda_s^{v^{1,p_s}} < \lambda_b$, so the seller must concede at time 0. A player's concession rate is proportional to her opponent's cost of delay, and so proportional to his discount rate:

$$\lambda_b := \frac{r_s p_b}{p_s - p_b}, \quad \lambda_s^v := \frac{r_b(v - p_s)}{p_s - p_b}$$

5.3 Seller benefit from buyer outside options

The seller's payoff can increase in the buyer's outside option, because this allows her to charge higher prices. As an example, suppose $v \sim U[1, 5]$ approximately and w is known, so $v - w \sim [1 - w, 5 - w]$. For $w \in [0.5, 1.25]$, the seller will charge $p_s \approx p^* = w$ for a payoff of $w(5 - 2w)/4$, which is strictly increasing in w.[18] By contrast, seller payoffs always decrease in the buyer's outside options in BP (so long as they are strictly positive). This suggests sellers' may benefit from (differentiated) competition.

5.4 Endogenous entry and sunk initiation costs

What happens if players must wait before bargaining, or incur some other sunk cost? Buyer participation is then endogenous. Such costs can sometimes completely deter

[17] I focused on common discount rates in the main analysis in order to simplify an already complicated environment and because that is assumed by BP.

[18] For $w > 1$, some buyers have $w > v$ and so there are no gains from trade with the seller. I previously assumed $v > w$ for all $(v, w) \in \Theta$ to simplify the exposition, but this was unimportant for any results. Whether or not $w > 1$, all buyers with $v \leq 2w$ will choose to immediately exit as commitment vanishes.

efficient buyer participation due to the hold up problem. When buyers do participate, however, outcomes appear fully Coasean (in particular efficient). The seller's payoff can increase in the buyer's sunk cost by deterring low value buyer participation.

Suppose buyers can either immediately exit (take their outside option), or wait a length of time $T > 0$ before bargaining, so subsequent payoffs are discounted by $\delta = e^{-rT} < 1$. Continue to assume a rich set of commitment types that always wait.

For any strictly positive probability of seller commitment, $z_s > 0$, if $T \to 0$ (so $\delta \to 1$), then all positive net value rational buyer types wait (i.e. Θ). This is because all such types will receive bargaining payoffs that strictly exceed their outside option, since a committed seller type sometimes demands $\min P_s < \min\{v - w > 0 : (v, w) \in \Theta\}$. This contrasts with BP, where even negligible delay causes the market to unravel.

What happens if the delay before bargaining is non-negligible? In particular, fix $\delta < 1$ and let commitment vanish ($z_i^n \to 0$). The benefit to low net value buyer types outlined above (occasional low prices $\min P_s$ from committed sellers) vanishes, and so no longer justifies delay. If $\overline{v} < 2\underline{w}/\delta$, the probability that buyers wait must vanish (the market unravels).[19] If $\overline{v} > 2\underline{w}/\delta$ the seller charges $p_s \approx p^\dagger = \max\{\underline{v}/2, \underline{w}/\delta\}$, which is immediately accepted by all buyers who wait, where the lowest such value buyer is $\underline{v}^\dagger = \min\{v \geq 2\underline{w}/\delta : (v, w) \in \Theta\}$, and so $p^\dagger = \underline{v}^\dagger/2 = \max\{\underline{v}^\dagger/2, \underline{w}\}$.

This characterization is due to the hold-up problem. The lowest value buyer who waits effectively pays a sunk cost investment $\underline{w}$ to initiate bargaining and create value $\delta v^\dagger$ (only buyers who eventually purchase will wait). Bargaining allows the seller to appropriate $\delta v^\dagger/2$ of this value (by charging $p^\dagger = v^\dagger/2$), so the benefit from investment is $\delta v^\dagger/2 - \underline{w}$, which is positive only if $v^\dagger > 2\underline{w}/\delta$.

With respect to buyers who do wait, outcomes then always appear Coasean: there is immediate agreement at a price of half the buyer's lowest value. This is same price as would be agreed if the buyer was known to have type $(\underline{v}^\dagger, \underline{w})$. With respect to those buyers who wait, it may also appear that buyers' "small" outside options are irrelevant to bargaining since $v - w > v/2 \geq p^\dagger$ (as in Binmore et al. (1989) under complete information). However, both appearances are misleading. First, there can be substantial inefficiency, as many types with $\delta v > w$ will choose not to wait. Second, outside options do determine prices by affecting who turns up to bargain, $\underline{p}^\dagger = \underline{w}/\delta$ if $\underline{w}/\delta \geq \underline{v}/2$.

Interestingly, outcomes can be discontinuous between the models without and with delay as $\delta \to 1$ (first letting commitment vanish, $z_i^n \to 0$). The upper bound on the price

[19] If $\delta\overline{v} < \underline{w}$, then rational buyers never wait. If $\delta\overline{v} \in (\underline{w}, 2\underline{w})$ then a rational buyer type $(\overline{v}, \underline{w})$ waits with vanishing probability, and receives a continuation payoff of exactly $\underline{w}$ from doing so.

the seller may charge is continuous in this limit, $p^\dagger \to p^* = \max\{\underline{v}/2, \underline{w}\}$, however, in the model without delay the seller may have actually chosen a lower price. For example suppose $v \sim U[1,2]$ and $w = 1$ so $v - w \sim U[0,1]$. If there is no delay the seller will choose $p_s = 1/2$ for a payoff of $1/4$. However, whenever there is delay with $\delta < 1$ we have $\overline{v} = 2 < 2w/\delta$ and so buyers never wait in the limit and the seller's payoff is 0.

In the above example, the seller is hurt by delay. This is generally true when players have equal discount rates, even though it can allow for higher prices. If the buyer does wait, she subsequently accepts the price $p^\dagger = \max\{\underline{v}/2, \underline{w}/\delta\}$, giving the seller a discounted payoff $\delta p^\dagger = \max\{\delta\underline{v}/2, \underline{w}\}$, but all such buyers would have also have accepted the price $p_s = \max\{\delta\underline{v}/2, \underline{w}\} \leq p^*$ in the model without delay.

By contrast, if the seller is more patient than the buyer, then she can strictly benefit from delay. With different discount rates, let $\delta_s = e^{-r_s T}$ and $\delta_b = e^{-r_b T}$ for length of delay T. If $\overline{v} \leq \underline{w}(r_b + r_s)/(r_s\delta_b)$ then the probability a buyer waits vanishes. Otherwise the seller charges $p_s \approx p^\ddagger = \max\{r_b\underline{v}/(r_s + r_b), r_b\underline{w}/(\delta_b r_s)\} = \underline{v}^\ddagger r_b/(r_s + r_b)$, where $\underline{v}^\ddagger = \min\{v \geq \underline{w}(r_b + r_s)/(\delta_b r_s) : (v, w) \in \Theta\}$ is the lowest value buyer to wait. Clearly, the market unravels with no buyers choosing to wait if the seller is much more patient than the buyer, $r_s/r_b \approx 0$, because the hold-up problem is too severe. On the other hand, buyers wait efficiently (whenever $\delta_b v > w$) if they are much more patient, $r_b/r_s \approx 0$.

The seller can benefit from delay if she is slightly more patient than the buyer: assume $v \sim U[3,25]$ and $w = 1$ so $v - w \sim U[2,24]$, with $r_s = 1$ and $r_b = 2$. Without delay the seller charges $p^{**} = \max\{r_b\underline{w}/r_s, r_b\underline{v}/(r_s + r_b)\} = 2$ for a payoff of 2. With delay, the seller charges $p^\ddagger = r_b\underline{w}/(\delta_b r_s) = 2e^{2T}$ which is accepted by buyers with $v \geq v^\ddagger = \underline{w}(r_b + r_s)/(\delta_b r_s) = 3e^{2T}$ for a profit $\pi = \delta_s p^\ddagger(25 - v^\ddagger)/22 = (25e^T - 3e^{3T})/11$, which is maximized at $\pi \approx 2.52$ by $T = ln(25/9)/2 > 0$. The high cost of delay for low value buyers deters their participation, which allows the seller to charge much higher prices that more than compensate for her own smaller cost of delay.

Analogous predictions hold when the buyer (alone) must pay an additive sunk cost $c > 0$ to initiate bargaining (perhaps for travel costs or legal/advisor fees); in particular, the seller's payoff can increase in c. With no delay, the lowest value buyer who initiates bargaining is $\underline{v}^\dagger = \min\{v \geq 2(c + \underline{w}) : (v, w) \in \Theta\}$, and there is again immediate agreement at a price of $p^\dagger = \underline{v}^\dagger/2$. If $v \sim U[0,1]$ and $w \approx 0$ then $v - w \approx U[0,1]$, $p^\dagger \approx c$ and the seller's payoff is approximately $c(1 - 2c)$ which is increasing in $c \leq 1/4$.

The finding that the seller can benefit from buyer sunk costs to initiate bargaining (so long as any sunk cost the seller has to pay isn't similarly large) can help explain why some sellers appear to make their goods intentionally hard to purchase, beyond just restricting supply (e.g. Birkin bags[20]).

5.5 Rich type space requirements

What can happen if the sets of players' types are not rich, as commitment vanishes?

If the set of buyer values is not sufficiently rich, prices may be much higher than the Coase conjecture would lead us to expect. For instance, consider the case of binary values, $v \in \{\underline{v}, \overline{v}\}$, with $\min\{\overline{v}/2, 2\underline{w}\} > \underline{v}$. In this case, the seller can charge $p_s = \overline{v}/2$, which can be much larger than $p^* = \underline{w}$ (the buyer either immediately accepts this price or exits as commitment vanishes). This is consistent with Proposition 2, which said the seller couldn't charge more than $p^* + 2\varepsilon$ in a ε rich type space because a binary type space is only ε-rich if $2\varepsilon > \overline{v} - \underline{v}$. This price $p_s = \overline{v}/2$ is unacceptable to the low value buyer, but is more generous than any counterdemand of the high value seller, $\overline{v} - p_s > p_b$, so the seller concedes faster after time 0, and the buyer must immediately concede or exit (in the limit).[21,22]

A rich set of buyer values is only needed because outside options are positive, $\underline{w} > 0$. If $\underline{w} = 0$ my results easily extend to show the buyer would choose her best ultimatum price $p_s \leq p^* = \underline{v}/2$ regardless of the richness of buyer values. In that case, if the seller made a demand $p_s > \underline{v}$ with positive limit probability, then buyers with type $(\underline{v}, 0)$ would demand $\underline{p} > 0$ and wait until at least T^* to exit, which would mean the seller must immediately concede with probability approaching 1 (by the same logic as Lemma 3, part (c)). If $p_s \in (\underline{v}/2, \underline{v})$, however, then type $(\underline{v}, 0)$ will make a more generous counterdemand $p_b \approx \underline{v} - p_s < \underline{v}/2$, to which the seller must immediately concede.

The assumption of a positive probability of the lowest outside option $\underline{w}$ for any buyer value, $g(v, \underline{w}) > 0$, is also implicitly an assumption about the richness of buyer types. Without a similar assumption, the seller may charge higher prices. For instance, suppose that instead $w = h(v) \geq v/2$ where $h(v)/v$ is strictly decreasing (e.g. $h(v) = 1 + v/2$ with $\underline{v} > 2$), then outcomes are approximately equal to those where the seller can choose *any* ultimatum price. In this case, any seller price p_s is more generous than the counterdemand of any buyer who eventually concedes, and so buyers immediately concede or

[20] For example see *https://baghunter.com/blogs/insights/how-to-get-birkin-bag-from-hermes* on the obstacles to acquiring such bags.

[21] Moreover, after the ungenerous counterdemand $\underline{p} \approx 0$, a low value buyer never waits as the seller concedes at rate $\lambda_s^{\overline{v},\underline{p},p_s} = r(\overline{v} - p_s)/(p_s - \underline{p}) \approx r$ since $\underline{\lambda}^{\underline{v},\underline{w}} = r\underline{w}/(\underline{v} - \underline{p} - \underline{w}) < r$.

[22] Introducing a third buyer value v' that is slightly higher than $\overline{v}/2 + \underline{w}$ rules out the price $p_s = \overline{v}/2$, because this buyer could counterdemand slightly more than $\underline{w}$ and be more generous than the seller, $v' - p_s < p_b$. Hence, the seller would immediately concede in the limit.

exit as commitment vanishes: $v - p_s > h(v)$ implies $p_s < v - h(v) \leq h(v)$ and so for any counterdemand $p_b < p_s < h(v) < v - p_s$, and thus $\lambda_s^v > \lambda_b$.[23]

The main result also depends on a rich set of commitment types, in particular buyer commitment types that make ungenerous offers $\underline{p} \approx 0$. Examples show that all types of rational buyer could benefit if they were constrained to make more generous offers, $\underline{p} >> 0$. The reason is that the exit of buyers who demand $\underline{p} >> 0$ is much more painful for the seller, and so substantial buyer concession (simultaneous with that exit) can help the buyer quickly reach a probability 1 reputation.[24]

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[23]The seller's price can be chosen to ensure $\lambda_s^{v^{1,p_s},p_s,\underline{p}} < \underline{\lambda}^{v,g(v),p_s,\underline{p}}$ for all buyers with $v - p_s < h(v)$ given that $h(v)/v$ is decreasing in v.

[24]An example: The buyer has values 5 and 6 with probability 0.24, value 13 with probability 0.48, values $\{7, 8, ..., 12\}$ with probability 1/150, and outside option $w = 3 = p^*$. With a rich set of commitment types the seller chooses $p_s \approx p^* = 3$, for a limit payoff 2.28. But if the lowest commitment price is $\underline{p} = 1.5$, there are multiple equilibrium limits. In one, the seller always proposes $p_s \approx 2$, which is always accepted. When the seller charges higher prices (with vanishing probability), the buyer mixes between immediately conceding and demanding $\underline{p} = 1.5$. After counterdemand $\underline{p} = 1.5$, players concede at rates $\lambda_s^{p_s,\underline{p},\bar{v}}$ and $\lambda_b^{p_s,\underline{p}}$ until time T^*, when the buyer exits and concedes with positive probability.

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A Appendix: Proofs of results

I first prove the supporting lemma below, which helps establish Proposition 1.

Lemma 4. *In any equilibrium in the continuation game at* 0^3 *after demands* p_s *and* p_b

(a) It is without loss of generality to assume $\bar{g}(v,w) = 0$ *if* $v - w < p_b$ *(henceforth, this is assumed throughout).*

(b) *If* $x = 0$*, as when* $\Theta^c = \emptyset$*, then without loss of generality* $F_s(0^3) = 1 - \bar{z}_s$*,* $E_b(0^3) = 0$*,* $E_b(0^4) = 1 - \bar{z}_b$*.*

(c) *If* $p_s \in P_s$ *then* $T^* < \infty$*.*

(d) *If* $p_s \notin P_s$*, but* $p_b \in P_b$*, then* $F_s(0^4) = 1$ *(and so clearly* $T^* < \infty$*).*

(e) *If* $p_s \in P_s$*,* $p_b \notin P_b$ *and* $x = 1$ *then* $F_b(0^4) = 1$*.*

(f) *If* $p_s \in P_s$*, then* $F_b(T^*) = (1 - \bar{z}_b)x$ *and* $E_b(T^*) = (1 - \bar{z}_b)(1 - x)$*. Similarly, if* $p_b \in P_b$ *then* $F_s(T^*) = (1 - \bar{z}_s)$*.*

(g) *If* F_s *jumps at* $t \geq 0^3$ *then* F_b *and* E_b *are constant on* $[t - \varepsilon, t]$ *for some* $\varepsilon > 0$*.*

(h) F_s *is continuous at* $t > 0^4$*.*

(i) *If* F_s *is continuous at* t *then so is* $U_b^{c,v,w}$ *and* $U_b^{e,v,w}$*. Likewise if* F_b *and* E_b *are continuous at* t *then so is* U_s*.*

(j) F_s *and* F_b *are strictly increasing on* $(0, T^*]$*.*

(k) *The skimming property holds: if a buyer with value* v *concedes at* t *then a buyer with value* $v' > v$ *will not concede after* $\max\{t, 0^4\}$*.*

Proof. For (a), suppose that $\mu_b^{p_s,v,w}(p_b) > 0$ for some $v - w < p_b$, then such a player would certainly always exit before 0^4 as her best payoff in the continuation game is less than w, and if the buyer ever conceded to her with positive probability she would have a strictly profitably deviation of conceding at 0^2 instead.

For (b), notice that since buyer can never concede in equilibrium, without loss of generality, $F_b(t) = 0$. Suppose that $F_s(t) < 1 - \bar{z}_s$ for $t > 0$ then deviating to concede at 0^3 is always a profitable deviation. This deviation would also be profitable for the seller if she conceded at 0^4 and $E_b(0^4) > 0$, while if $E_b(0^4) = 0$ then it is still weakly better for the seller to concede at 0^3 than 0^4, hence in all cases we may assume $F_s(0^3) = 1 - \bar{z}_s$. If $F_s(0^3) > 0$, then clearly $E_b(0^3) = 0$ (as exit at 0^4 would be a profitable deviation for the buyer given (a)). If $\bar{z}_s < 1$, then we must, however, have $E_b(0^4) = 1 - \bar{z}_b$ given $w > 0$.

The argument for (c) is standard (e.g. see the proof of Lemma 1 in Abreu and Gul (2000) for more detail). If $p_s \in P_s$, then if a rational buyer does not concede or exit, she must believe the seller will concede shortly afterwards, and so her belief that the seller is committed increases if there is no concession. Repeating this argument, the buyer must eventually become convinced of the seller's commitment by some time $T^* < \infty$ and will then concede or exit.

The reasoning for (d) is similar (again see the proof of Lemma 1 in Abreu and Gul (2000) for more detail): given $p_b \in P_b$ if the seller does not immediately concede, she must eventually become convinced of the buyer's commitment by some $T^* < \infty$ and will then concede. Given that, however, no rational buyer will concede to her on $[T^* - \varepsilon, T^*]$ for sufficiently small $\varepsilon > 0$ (strictly preferring to wait for the seller's concession), implying that she must have conceded by $T^* - \varepsilon$, a contradiction. The argument for (e) is analogous.

For the first part of (f), notice that at time T^*, the buyer knows that the seller is committed to her demand and so will immediately either concede or exit. For the second part, the seller likewise knows that the buyer will never concede after T^*, and so will immediately concede herself.

For (g), we can assume that $v - p_b > w$ for all buyers by (a). Given $p_b < p_s$ and the seller's positive concession at t, the buyer would strictly prefer to concede, or respectively exit, an instant after time t than on $[t - \varepsilon, t]$ for $\varepsilon > 0$ small. Given (g), if F_s jumped at $t > 0^4$ then F_b is constant on $[t - \varepsilon, t]$, and hence the seller would prefer to concede strictly before t, a contradiction which implies (h). Part (i) is immediate from the definitions.

Suppose that (j) did not hold, so that $F_i(t) = F_i(t')$ for some $0 < t < t' \leq T^*$ and i. Let $t_i^* = \sup\{\tau : F_i(\tau) = F_i(t)\}$. Clearly, player j will not concede at $\tau \in (t, t_i^*)$ as conceding slightly beforehand would strictly improve her payoff, and hence $t_s^* = t_b^*$. Since F_s and $U_b^{c,v,w}$ are continuous at $t > 0^4$ by (h) and (i), conceding at or slightly after t_b^* delivers a strictly lower buyer payoff than conceding at $\tau \in (t, t_b^*)$. Hence, t_b^* cannot be the supremum, a contradiction.

For (k), given that conceding at t is optimal for type (v, w) we can assume $t \geq 0^4$ and that F_s is continuous at t (if the seller conceded with positive probability at 0^3 or 0^4 then the buyer wouldn't), and at t' by (g) and (h). So let $D(v) = U_b^{v,c}(t) - U_b^{v,c}(t')$ for $t' > t$:

$$D(v) = -\int^{\tau \in (t,t')} (v - p_b) e^{-r\tau} dF_s(\tau) + (v - p_s)\left((1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'}\right) \geq 0$$

Notice that

$$\begin{aligned} dD(v)/dv &= -\int^{\tau \in (t,t')} e^{-r\tau} dF_s(\tau) + (1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'} \\ &\geq \left(1 - \frac{(v - p_s)}{(v - p_b)}\right)\left((1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'}\right) > 0 \end{aligned}$$

where the first inequality uses $D(v) \geq 0$, and the second uses $(1 - F_s(t))e^{-rt} > (1 - F_s(t'))e^{-rt'}$ and $p_s > p_b$. Hence, $D(v') > 0$. □

Proof of Proposition 1. Part (f) of Lemma 4 establishes the part (4) of the Proposition. By part (a) of Lemma 4 we can assume $\bar{g}(v, w) = 0$ if $v - w < p_b$. Part (b) of Lemma 4 means we can focus on continuation games where $\Theta^c \neq \emptyset$, so v^1 is well defined.

For such games, I next establish parts (1) and (2) of the Proposition. By Lemma 4 part (j), F_i is strictly increasing on $(0, T^*)$. This implies that if $T^* > 0$, the set of times O_i^c at which it is optimal for some type of player i to concede, must be dense in $(0, T^*) \cap (t^{k+1}, t^k)$. By the skimming property only types $(v^k, w) \in \Theta^c$ concede on (t^{k+1}, t^k). By Lemma 4 parts (h) and (i), we also have that F_s and so $U_b^{c,v,w}$ are continuous at $t > 0$. Combined with the density of O_b^c in (t^{k+1}, t^k) therefore, we must have that $U_b^{c,v^k,w}$ is differentiable on that interval, with a derivative equal to zero, $dU_b^{c,v^k,w}(t)/dt = 0$. This immediately implies that the seller must concede at rate $\lambda_s^{v^k}$ on that interval.

A buyer of type $(v, w) \in \Theta^e$, with $\underline{\lambda}^{v,w} > \lambda_s^{v^k}$ prefers to exit earlier on (t^{k+1}, t^k) than later, as the inequality implies $dU_b^{e,v,w}(t)/dt < 0$ on that interval. Moreover, given the skimming property and the continuity of $U_b^{e,v,w}$ at $t > 0$ (parts (h) and (i) of Lemma 4), such a buyer would prefer to concede at some point in (t^{k+1}, t^k) rather than at a later time $t \geq t^k$ (as any seller concession after t^k is even slower). Likewise, if $\underline{\lambda}^{v,w} < \lambda_s^{v^k}$ for $(v, w) \in \Theta^e$, then such a buyer prefers to concede later on (t^{k+1}, t^k) than earlier as $dU_b^{e,v,w}(t)/dt > 0$. Given the skimming property, therefore, she will not concede before t^k (as any seller concession before t^{k+1} is even faster).

I next claim that F_b is continuous on (t^{k+1}, t^k). If F_b jumped at some $t \in (t^{k+1}, t^k)$, then F_s would necessarily be constant on $[t - \varepsilon, t]$, for some small $\varepsilon \in (0, t - t^{k+1}]$, because we have established that the buyer will not exit on (t^{k+1}, t^k), while the seller prefers that the buyer concedes to her, rather than that she concedes. This, however, would contradict the required seller concession rate of $\lambda_s^{v^k}$ on that interval.

Given the continuity of F_b and E_b on (t^{k+1}, t^k), U^s is also continuous, by Lemma 4 part (i). Combined with the fact that O_s^c is dense in (t^{k+1}, t^k), we must then have $dU_s(t)/dt = 0$ and so the buyer must concede at rate λ_b.

I next argue that (without loss) the buyer never concedes or exits at 0^3 and the seller never concedes at 0^4. Suppose instead that a seller conceded with positive probability at time 0^4, then certainly a rational buyer cannot concede or exit at 0^3 or 0^4 (or the buyer would strictly prefer to concede or exit an instant after 0^4). Hence, outcomes are not affected by switching such seller concessions to time 0^3. Likewise, if the buyer conceded or exited at 0^3, then certainly the seller cannot concede at 0^3 or 0^4, or the buyer's decision would not be optimal. Hence, outcomes are not affected by moving any buyer concession or exit to time 0^4.

Next consider part (3) or the proposition, and suppose the buyer concedes with probability strictly greater than $\alpha p_b/(p_s - p_b)$ at t^k. If $t^k \in (0, T^*]$, since F_b has at most finitely many jumps at times $t^K, ..., t^1$, there exists some $\varepsilon > 0$ such that the seller would prefer to concede an instant after t^k than on $(t^k - \varepsilon, t^k]$. However, this would contradict that F_s is strictly increasing on $(0, T^*)$, Lemma 4 part (j). Hence, such large concession requires $t^k = 0^4$ (recall, we can assume no buyer concession or exit at 0^3). Clearly, in this case seller will not concede at 0^3 (or 0^4).

Now suppose the buyer concedes with probability strictly less than $\alpha p_b/(p_s - p_b)$ at time t^k. If $t^k < T^*$, then the seller would prefer to concede at t^k compared to conceding on $(t^k, t^k + \varepsilon]$ for sufficiently small $\varepsilon > 0$. However, this would contradict that F_s is strictly increasing on $(0, T^*)$, Lemma 4 part (j). Hence, such small concession requires $t^k = T^*$. If $t^k = 0^4$, a rational seller would prefer to concede at 0^3 rather than at 0^4 or $(0, \varepsilon)$, and hence without loss, any rational buyer must have always conceded by $0^4 = T^*$.

□

Proof of Lemma 2. First notice that if seller never concedes at 0^3 or 0^4, then any buyer exit and concession by a rational buyer at 0^3 or 0^4 can instead be moved to 0^2 without affecting outcomes or incentives. Henceforth, therefore, assume that if $F_s^{p_b}(0^4) = 0$ for all buyer demands $p_b \in P$ made with positive probability, then for such demands $E_b^{p_b}(0^4) = F_b^{p_b}(0^4) = 0$.

Let $\tilde{v}^{p_b} = \max\{v : \sum_{(v,w)\in\Theta^c} \mu_s^{p_s}(p_b) > 0\}$. By the skimming property (Lemma 4) the payoff for $(\tilde{v}^{p_b}, w) \in \Theta^c$ from demanding $p_b \in P$ is $F_s^{p_b}(0^4)(\tilde{v}^{p_b} - p_b) + (1 - F_s^{p_b}(0^4))(\tilde{v}^{p_b} - p_s)$, while her payoff from demanding p'_b is at least $F_s^{p'_b}(0^4)(\tilde{v}^{p_b} - p_b) + (1 - F_s^{p'_b}(0^4))(\tilde{v}^{p_b} - p_s)$. If $F_s^{p'_b}(0^4)(p_s - p'_b) > F_s^{p_b}(0^4)(p_s - p_b)$ then type $\tilde{v}^{p_b}$ will not imitate type p_b (a contradiction). On the other hand, if $F_s^{p'_b}(0^4)(p_s - p'_b) < F_s^{p_b}(0^4)(p_s - p_b)$ then type $\tilde{v}^{p'_b}$ will not imitate type p'_b. Hence, if p_b and p'_b are both imitated with positive probability by some buyer in Θ^c then $F_s^{p'_b}(0^4)(p_s - p'_b) = F_s^{p_b}(0^4)(p_s - p_b)$; if $p_b < p'_b$ therefore $F_s^{p'_b}(0^4) \geq F_s^{p_b}(0^4)$. If $p'_b > p_b \in P_b$, and p'_b is demanded with positive probability by some rational buyer, then some Θ^c buyer must demand p_b with positive probability: if not, $(1-\bar{z}_b^{p_b})x^{p_b} = 0$, so a rational seller will immediately concede and $F_s^{p'_b}(0^4)(p_s - p'_b) < F_s^{p_b}(0^4)(p_s - p_b)$, a contradiction.

Let $\check{v}^{p_b}$ be the maximum value buyer such that some $(\check{v}^{p_b}, w) \in \Theta^c$ demands p_b with positive probability, but has not always conceded by time 0^4. Suppose that $p'_b > p_b \in P_b$ is demanded with positive probability by rational buyers but $\check{v}^{p'_b}$ is not well defined because all those buyers concede or exit by 0^4, $F_b^{p'_b}(0^4) + E_b^{p'_b}(0^4) = 1 - \bar{z}_b > 0$. A rational seller must therefore concede at 0^3 after p'_b with strictly positive probability (or we could move the buyer's concession and exit to 0^2), and so $F_s^{p'_b}(0^4)(p_s - p'_b) = F_s^{p_b}(0^4)(p_s - p_b) > 0$. The payoff of type $(v, w) \in \Theta^e$ who demands p'_b and exits at 0^4 is then $w + F_s^{p'_b}(0^4)(v - p'_b - w)$. Her payoff to demanding p_b and exiting at 0^4 is then strictly larger $w + F_s^{p_b}(0^4)(v - p_b - w) = w + F_s^{p'_b}(0^4)(v - p_b - w)(p_s - p'_b)/(p_s - p_b)$ since $(v - p_b - w)/(p_s - p_b)$ is decreasing in $p_b < p'_b$ when $v - p_s > w$. This implies $x^{p'_b} = 1$, and so since the buyer concedes at 0^4 after p'_b with positive probability ($F_b^{p'_b}(0^4) = 1 - \bar{z}_b$) the seller strictly prefers to concede an instant after 0^4 than at 0^3, a contradiction. This shows $\check{v}^{p'_b}$ is well defined. We know that $F_s^{p'_b}(0^4)(p_s - p'_b) = F_s^{p_b}(0^4)(p_s - p_b)$ and so must have $F_s^{p_b}(0^4) < 1 - \bar{z}_b = F_s^{p_b}(T^{*,p_b})$ for $p_b \in P_b$ with $p_b < p'_b$, and hence $\check{v}^{p_b}$ is also well defined. The argument above shows more generally that (without loss of generality) a buyer with value $(v, w) \in \Theta^e$ will never demand $p'_b > p_b \in P_b$ and then exit at time 0.

I next claim that $\check{v}^{p_b} = \check{v}^{p'_b}$. Suppose instead $\check{v}^{p_b} < \check{v}^{p'_b}$. The payoff to $(\check{v}^{p_b}, w) \in \Theta^c$ from demanding p_b is $F_s^{p_b}(0^4)(p_s - p_b) + (\check{v}^{p_b} - p_s)$, which is also her payoff from demanding p'_b and then conceding an instant after 0^4 (we established $F_s^{p'_b}(0^4)(p_s - p'_b) = F_s^{p_b}(0^4)(p_s - p_b)$ above). However, the payoff to type $(\check{v}^{p_b}, w)$ from demanding p'_b and waiting to concede after the positive interval on which she receives concession at rate $\lambda_s^{\check{v}^{p'_b}, p'_b} > \lambda_s^{\check{v}^{p_b}, p'_b}$, (by Lemma 1) is strictly larger, a contradiction. Similarly, if $\check{v}^{p_b} > \check{v}^{p'_b}$ then $\check{v}^{p'_b}$ will profitably deviate by demanding p_b. Hence, $\check{v}^{p_b'} = \check{v}^{p_b}$.

Recall that a player with value v^m with $m \geq 1$ is indifferent between conceding at any point in the interval $[t^{m+1,p_b}, t^{m,p_b}]$ after demanding p_b. Now assume: (i) for any v, a rational buyer with that value is indifferent between demanding p_b before conceding at t^{m+1,p_b} or demanding $p_b' > p_b$ before conceding at $t^{m+1,p_b'}$; and (ii) $t^{m+1,p_b'} \geq t^{m+1,p_b}$ and $F_s^{p_b'}(t^{m+1,p_b'}) \geq F_s^{p_b}(t^{m+1,p_b})$ (both strictly if $t^{m+1,p_b'} > 0$). Clearly (ii) implies $e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) \geq e^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'}))$. Let the difference in payoffs for a buyer with value v between demanding $p_b \in P_b$ before conceding at t^{m,p_b} or demanding $p_b' > p_b$ before conceding at time t^{m,p_b} be $D^m(v) = U^{c,p_b,v}(t^{m,p_b}) - U^{c,p_b',v}(t^{m,p_b'})$. Given (i) we have $D^m(v) = D^m(v) - D^{m+1}(v)$, and so:

$$D^m(v) = \int^{t^{m+1,p_b}<\tau<t^{m,p_b}} e^{-r\tau}(v - p_b)dF_s^{p_b}(\tau) - (e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) - e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b})))(v - p_s)$$
$$- \int^{t^{m+1,p_b'}<\tau<t^{m,p_b'}} e^{-r\tau}(v - p_b')dF_s^{p_b'}(\tau) + (e^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'})) - e^{-rt^{m,p_b'}}(1 - F_s^{p_b}(t^{m,p_b'})))(v - p_s)$$

which implies,

$$\begin{aligned}
\frac{dD^m(v)}{dv} &= \int^{t^{m+1,p_b}<\tau<t^{m,p_b}} e^{-r\tau}dF_s^{p_b}(\tau) - (e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) - e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b}))) \\
&\quad - \int^{t^{m+1,p_b'}<\tau<t^{m,p_b'}} e^{-r\tau}dF_s^{p_b'}(\tau) + (e^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'})) - e^{-rt^{m,p_b'}}(1 - F_s^{p_b}(t^{m,p_b'}))) \\
&= -\frac{p_s - p_b}{v^m - p_b} e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b}))\Big(1 - e^{-r(t^{m,p_b} - t^{m+1,p_b})}\frac{1 - F_s^{p_b}(t^{m,p_b})}{1 - F_s^{p_b}(t^{m+1,p_b})}\Big) \\
&\quad + \frac{p_s - p_b'}{v^m - p_b'} e^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'}))\Big(1 - e^{-r(t^{m,p_b'} - t^{m+1,p_b'})}\frac{1 - F_s^{p_b'}(t^{m,p_b'})}{1 - F_s^{p_b'}(t^{m+1,p_b'})}\Big)
\end{aligned}$$

where the second line imposes that type v^m is indifferent between conceding at t^{m+1,p_b} or t^{m,p_b} for any demand p_b (as required by Lemma 1), that is:

$$\int^{t^{m+1,p_b}<\tau<t^{m,p_b}} e^{-r\tau}dF_s^{p_b}(\tau) = (e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) - e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b}))\frac{v^m - p_s}{v^m - p_b}$$

and also $(v^m - p_s)/(v^m - p_b) - 1 = -(p_s - p_b)/(v^m - p_b)$.

Other things equal it is clear that $dD^m(v)/dv$ is strictly decreasing in t^{m,p_b} and strictly increasing in $t^{m,p_b'}$ and equals 0 when both $t^{m,p_b'} = t^{m+1,p_b'}$ and $t^{m,p_b} = t^{m+1,p_b}$.

Given some equilibrium $t^{m,p_b'}$, $t^{m+1,p_b'}$ and t^{m+1,p_b} we must have $T^{*,p_b} \geq t^{m+1,p_b} + t^{m,p_b'} - t^{m+1,p_b'}$. Suppose not, then let $t^{m,p_b'} - t^{m+1,p_b'} = q > T^{*,p_b} - t^{m+1,p_b}$. Since $\lambda_s^{v,p_b} \leq \lambda_s^{v^m,p_b} < \lambda_s^{v^m,p_b'}$ for all $v \leq v^m$ (since $p_b < p_b'$) we have $(1 - F_s^{p_b}(T^{*,p_b}))/(1 - F_s^{p_b}(t^{m+1,p_b})) > e^{-\lambda_s^{v^m,p_b}q} \geq e^{-\lambda_s^{v^m,p_b'}q} = (1 - F_s^{p_b'}(t^{m,p_b'}))/(1 - F_s^{p_b'}(t^{m+1,p_b'}))$, and so given $F_s^{p_b'}(t^{m+1,p_b'}) \geq F_s^{p_b}(t^{m+1,p_b})$ by (ii) we would then have $(1 - F_s^{p_b}(T^{*,p_b})) > 1 - F_s^{p_b}(t^{m,p_b'})) \geq \bar{z}_s$, a contradiction since $p_b \in P_b$. Suppose next that $q = t^{m,p_b'} - t^{m+1,p_b'} = t^{m,p_b} - t^{m+1,p_b}$ and in this case let $\hat{D}^v(q)$ be $dD^m(v)/dv$ defined as a function of q. We then have:

$$\frac{d\hat{D}^v(q)}{dq} = -re^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b}))e^{-(r+\lambda_s^{v^m,p_b})q} + re^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b}))e^{-(r+\lambda_s^{v^m,p'_b})q}$$

where I use the identity $r + \lambda_s^{v^m,p_b} = r(v^m - p_b)/(p_s - p_b)$. Given that $e^{(r+\lambda_s^{v^m,p'_b})q}d\hat{D}^v(q)/dq$ is strictly decreasing in q (since $\lambda_s^{v^m,p'_b} > \lambda_s^{v^m,p_b}$) and $e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) \geq e^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b}))$ (because of (ii)) we have $d\hat{D}^v(0)/dq \leq 0$, and $d\hat{D}^v(q)/dq < 0$ for all $q > 0$. Since $\hat{D}^v(0) = 0$ we must have $\hat{D}^v(q) < 0$ for all $q > 0$. Since $dD^m(v)/dv$ is strictly decreasing in t^{m,p_b}, if $t^{m,p_b} - t^{m+1,p_b} \geq t^{m,p'_b} - t^{m+1,p'_b}$ then $dD^m(v)/dv \leq 0$ with $dD^m(v)/dv < 0$ when $t^{m,p'_b} > t^{m+1,p'_b}$. On the flip side, if $dD^m(v)/dv \geq 0$ then we must certainly have $t^{m,p_b} - t^{m+1,p_b} \leq t^{m,p'_b} - t^{m+1,p'_b}$.

I next claim that we always have $dD^m(v)/dv \leq 0$. Suppose not, so that $dD^m(v)/dv > 0$. Since $D^m(v^m) = 0$ we must have $D(v) < 0$ for all $v < v^m$. Hence, all such buyers would strictly prefer to demand p'_b and concede at t^{m,p'_b} than demand p_b and concede at t^{m,p_b}. This would then imply that $T^{*,p_b} = t^{m,p_b}$. However, we observed above that $T^{*,p_b} \geq t^{m+1,p_b} + t^{m,p'_b} - t^{m+1,p'_b}$ and so $t^{m,p_b} - t^{m+1,p_b} \geq t^{m,p'_b} - t^{m+1,p'_b}$, but this implies $dD^m(v)/dv \leq 0$, a contradiction.

We have established that one of the following hold: (a) $dD^m(v)/dv < 0$ and no player with $v < v^m$ imitates p'_b only to concede, or (b) $dD^m(v)/dv = 0$. In case (b) we have $t^{m,p_b} - t^{m+1,p_b} \leq t^{m,p'_b} - t^{m+1,p'_b}$ and so given (ii), $t^{m,p'_b} \geq t^{m,p_b}$ and $F_s^{p'_b}(t^{m,p'_b}) \geq F_s^{p_b}(t^{m,p_b})$, strictly if $t^{m,p'_b} > 0$. Furthermore, in either case (a) or (b), given $D^{m+1}(v) = 0$ for all v by (i), there is some $\hat{t}^{m,p_b} \in [t^{m+1,p_b}, t^{m,p_b}]$ such that all buyer types are indifferent between demanding p'_b before conceding at t^{m,p'_b} or demanding p_b before conceding at $\hat{t}^{m,p_b}$ where $\hat{t}^{m,p_b} \leq t^{m,p'_b}$ and $F_s^{p'_b}(t^{m,p'_b}) \geq F_s^{p_b}(t^{m,p_b})$, both strictly if $t^{m,p'_b} > 0$ (where $\hat{t}^{m,p_b} = t^{m,p_b}$ if $dD^m(v)/dv = 0$). Since m is arbitrary, induction establishes that all $(v, w) \in \Theta^c$ weakly prefer $p_b \in P_b$ over $p'_b > p_b$.

Next, consider the incentives of a player $(v, w) \in \Theta^e$. I claim that such a buyer would never demand $p'_b > p_b \in P_b$. We already saw that (without loss of generality) such a buyer would never demand p'_b only to exit at 0^4. Suppose then that it was optimal for such a player to demand p'_b before exiting at $t^{m,p'_b} > 0$. We can assume $t^{m,p'_b} > t^{m+1,p'_b}$, as otherwise it is optimal to exit at t^{m+1,p'_b}, and hence $D^{m+1}(v) = 0$. Given $t^{m,p'_b} > t^{m+1,p'_b}$ we established that all rational buyers must be indifferent between demanding p_b before conceding at time $\hat{t}^{m,p_b}$ and demanding p'_b before conceding at time $t^{m,p'_b} > 0$. However, in that case our Θ^e a buyer must strictly prefer to demand p_b before exiting at $\hat{t}^{m,p_b}$ to demanding p'_b before exiting at t^{m,p'_b}. To see this, let $\hat{D}^m(v, w) = U^{e,v,w,p_b}(\hat{t}^{m,p_b}) - U^{e,v,w,p'_b}(t^{m,p'_b})$ be the increase in payoffs from this deviation:

$$\hat{D}^m(v, w) = (e^{-r\hat{t}^{m,p_b}}(1 - F_s^{p_b}(\hat{t}^{m,p_b})) - e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b})))(w - v + p_s) > 0$$

where the first equality follows from $U^{e,v,w,p_b}(t) = U^{c,v,p_b}(t) + e^{-rt}(1 - F_s^{p_b}(t))(w - v + p_s)$ and $U^{c,v,p_b}(\hat{t}^{m,p_b}) = U^{c,v,p'_b}(t^{m,p'_b})$, and the inequality from $e^{-r\hat{t}^{m,p_b}}(1 - F_s^{p_b}(\hat{t}^{m,p_b})) > e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b}))$ and $w > v - p_s$. Hence, demanding p'_b is never optimal for $(v, w) \in \Theta^e$.

Since Θ^e buyers only demand $\underline{p}$, any Θ^c buyer cannot demand $p_b \in P \setminus P_b$ because otherwise $x^{p_b} = 1$ so that the buyer (and not the seller) would immediately concede by Lemma 4 (part e). □

Proof of Lemma 3. I first establish parts (a), (b), and (c). Suppose by way of contradiction that $\lim_n F_s^n(0^4) < 1$. Since $\bar{z}_s^n = 1 - F_s^n(T^*) \geq (1 - F_s^n(0^4))e^{-\lambda_s^{\bar{v}} T^{*,n}}$ by Lemma 1, we must have $T^{*,n} \to \infty$.

For (a), define $t^* = -ln(\lim_n \bar{z}_b^n - \varepsilon)/\lambda_b < \infty$ for some $\varepsilon \in (0, \lim_n \bar{z}_b^n)$. For all large enough n we must have $T^{*,n} \leq t^*$ since $\lim_n \bar{z}_b^n - \varepsilon < \bar{z}_b^n = 1 - E_b^n(T^{*,n}) - F_b^n(T^{*,n}) \leq e^{-\lambda_b T^{*,n}}$ by Lemma 1, a contradiction. Given this, for claims (b), and (c) assume $\lim_n \bar{z}_s^n = 0$

For (b), notice that $1 - E_b^n(t) - F_b^n(t) \leq e^{-\lambda_b t}$ by Lemma 1. Hence by the skimming property (Lemma 4, part (k)), for any $\varepsilon \in (0, \lim_n \bar{g}^n(v,w))$ for large n, at time $t^* = -ln(\lim_n \bar{g}^n(v,w) - \varepsilon)/\lambda_b < \infty$ all remaining rational buyers with $(v',w') \in \Theta^c$ must have $v' \leq v < p_b + p_s$ and hence $\lambda_b > \lambda_s^v \geq \lambda_s^{v'}$. But since $\bar{z}_b^n = 1 - E_b^n(T^{*,n}) - F_b^n(T^{*,n}) \leq e^{-\lambda_b T^{*,n}}$ and $\bar{z}_s^n = 1 - F_s^n(T^{*,n}) \geq (1 - F_s^n(0^4))e^{-\lambda_s^{\bar{v}} t^* - \lambda_s^v (T^{*,n} - t^*)}$ (by Lemma 1) we have

$$1 - F_s^n(0^4) \leq \frac{\bar{z}_s^n}{\bar{z}_b^n} e^{(\lambda_s^v - \lambda_b)(T^{*,n} - t^*) + (\lambda_s^{\bar{v}} - \lambda_b)t^*} \leq L e^{(\lambda_s^v - \lambda_b)(T^{*,n} - t^*) + (\lambda_s^{\bar{v}} - \lambda_b)t^*}$$

where the right hand side converges to 0 as $T^{*,n} \to \infty$ since $\lambda_s^v - \lambda_b < 0$. This clearly contradicts $\lim_n F_s^n(0) \neq 1$.

For (c), notice that type $(v',w') \in \Theta^e$ only demands $\underline{p}$ (given Lemma 2 and $\bar{z}_s^n < 1$) so that $\lim_n \bar{g}^n(v',w') \geq g(v',w') > 0$, and will not exit until after any type $(v^1, w) \in \Theta^c$, by Lemma 1. For any $\varepsilon \in (0, \lim_n \bar{g}^n(v',w'))$ let $t^* = -ln(\lim_n \bar{g}^n(v,w) - \varepsilon)/\lambda_b < \infty$. Since $1 - E_b^n(t) - F_b^n(t) \leq e^{-\lambda_b t}$, for large n, by time t^* all $(v^1, w) \in \Theta^c$ must have conceded, and so $t^* \geq T^{*,n}$, which contradicts $T^{*,n} \to \infty$.

I now turn to the proof of parts (d) and (e) and (f). The logic for (d) and (e) is almost identical to that for (a) and (b). Given $p_b > \underline{p}$ we must have $x^n = 1$ (given Lemma 2) so that $\bar{z}_b^n = 1 - F_b^n(T^{*,n}) = (1 - F_b^n(0^4))e^{-\lambda_b T^{*,n}}$ by Lemma 1. Hence, if $\lim_n F_b^n(0^4) < 1$ then we must have $T^{*,n} \to \infty$.

For (d) notice that $\bar{z}_s^n = 1 - F_s^n(T^{*,n}) \leq e^{-\lambda_s^{v^1} T^{*,n}}$ by Lemma 1, which implies $\lim_n T^{*,n}$ is bounded above by $-ln(\lim_n \bar{z}_s)/\lambda_s^{v^1} < \infty$, a contradiction.

For (e), by assumption $v^1 - p_s > p_b$ and so $\lambda_s^{v^1} > \lambda_b$. For both players to reach a probability 1 reputation at $T^{*,n}$ we need $\bar{z}_s^n = 1 - F_s^n(T^{*,n}) \leq e^{-\lambda_s^{v^1} T^{*,n}}$ and $(1 - F_b^n(0^4)) \leq \bar{z}_b^n e^{\lambda_b T^{*,n}}$, and so

$$(1 - F_b^n(0^4)) \leq \frac{\bar{z}_b^n}{\bar{z}_s^n} e^{(\lambda_b - \lambda_s^{v^1})T^{*,n}} \leq L e^{(\lambda_b - \lambda_s^{v^1})T^{*,n}}$$

where the right-hand side clearly converges to 0 as $T^{*,n} \to \infty$, implying $\lim_n F_b^n(0^4) = 1$, a contradiction.

For (f), suppose $\lim_n F_b^n(0^4)+E_b^n(0^4) < 1$. By time $t^{2,n}$ only rational buyers with type $(v^1, w) \in \Theta^c$ remain; recall $t^{2,n} = \min\{t \geq 0^4 : F_b^n(t) \geq \sum_{(v,w)\in\Theta^c : v\geq v^2} \bar{g}^n(v,w)\}$. We clearly have $F_b^n(t_-^{2,n}) = \sup_{s<t^{2,n}} F_b^n(s) \leq \sum_{(v,w)\in\Theta^c : v>v^1} \bar{g}^n(v,w)$. First consider some subsequence for which $t^{2,n} > 0$ for all n. By Lemma 1 the probability of concession at $t^{2,n} > 0$ must satisfy $(F_b^n(t^{2,n}) - F_b^n(t_-^{2,n}))(p_s - p_b)/p_b \leq E_b^n(t^{2,n}) - E_b^n(t_-^{2,n})$ where the right hand side is certainly less than $1 - x^n$ and so for small enough $\varepsilon > 0$, for all sufficiently large n

$$F_b^n(t^{2,n}) \leq (1-x^n)p_b/(p_s - p_b) + \sum_{(v,w)\in\Theta^c : v>v^1} \bar{g}^n(v,w) < \lim_n \sum_{(v,w)\in\Theta^c} \bar{g}^n(v,w) - \varepsilon.$$

And so, we have $1 - F_b^n(t^{2,n}) - E_b^n(t^{2,n}) \geq \varepsilon$ for all sufficiently large n.

Similarly, suppose along some subsequence we always have $t^{2,n} = 0^4$, then $E_b^n(0^4) = (1-x^n)(1-\bar{z}^n)$. For this subsequence, since $\lim_n F_b^n(0^4) + E_b^n(0^4) < 1$, we must have $\lim_n F_b^n(0^4) < \lim_n x^n$ and so $1-F_b^n(t^{2,n})-E_b^n(t^{2,n}) \geq \varepsilon$ for some $\varepsilon > 0$. For any subsequence with $t^{2,n} = 0^4$ or $t^{2,n} > 0^4$, therefore, we must have $1 - F_b^n(t^{2,n}) - E_b^n(t^{2,n}) \geq \varepsilon$ for some $\varepsilon > 0$ for all large n. In that case, we must have $\bar{z}_b^n = 1 - F_b^n(T^{*,n}) - E_b^n(T^{*,n}) = (1 - F_b^n(t^{2,n}) - E_b^n(t^{2,n}))e^{-\lambda_b(T^{*,n}-t^{2,n})}$ and so clearly $(T^{*,n} - t^{2,n}) \to \infty$. Combined with $\bar{z}_s = 1 - F_s^n(T^{*,n}) \leq e^{-\lambda_s^{v^1} T^{*,n}}$ we get

$$(1 - F_b^n(t^{2,n}) - E_b^n(t^{2,n})) \leq \frac{\bar{z}_b}{\bar{z}_s} e^{(\lambda_b - \lambda_s^{v^1})(T^{*,n}-t^{2,n}) - \lambda_s^{v^1} t^{2,n}} \leq L e^{(\lambda_b - \lambda_s^{v^1})(T^{*,n}-t^{2,n})}$$

where the right hand side must converge to 0 given that $\lambda_b < \lambda_s^{v^1}$ and $(T^{*,n} - t^n) \to \infty$. This contradicts $(1 - F_b^n(t^{2,n}) - E_b^n(t^{2,n})) \geq \varepsilon > 0$ for large n. □

Proof of Proposition 2. In order to reduce notation, I will drop the superscript n from all variables (e.g. z_s^n becomes z_s) but will still take limits as $n \to \infty$ (e.g. $\lim_n z_s = 0$). I first present some preliminary observations.

Notice that by choosing $\varepsilon' > 0$ sufficiently small, a ε' rich commitment type space must have $\underline{p} \leq \varepsilon' < \min\{v - w : (v,w) \in \Theta\}$. Moreover, let $\tilde{p}_s = \max\{p_s \in P_s : p_s \leq \min\{p^*, v-w : (v,w) \in \Theta\}\}$, then for small enough $\varepsilon' > 0$, we have $\tilde{p}_s > \underline{p}$, and a rational seller will always demand $p_s \geq \tilde{p}_s$. To see this, notice that demanding $\tilde{p}_s$ guarantees that $x^{\tilde{p}_s} = 1$ and since $\tilde{p}_s \leq p^*$, any counterdemand $p_b \in P_b$ will imply $\lambda_s^{\underline{v},p_b,\tilde{p}_s} > \lambda_b^{p_b,\tilde{p}_s}$ given $p_b < \tilde{p}_s \leq \min\{p^*, \underline{v} - \underline{w}\}$. After any counterdemand $p_b \in P_b$ the buyer makes with positive limit probability (for some subsequence) we must have $\lim_n \bar{z}_b^{p_b,\tilde{p}_s}/\bar{z}_s^{\tilde{p}_s} \leq L'$ for some constant L'. Hence, by Lemma 3, the buyer must concede with probability approaching 1 in the limit. This would guarantee the seller a payoff of at least $\tilde{p}_s$ in the limit and so she certainly won't demand less. This means a rational seller will never demand $p_s > \bar{v} - \underline{w}$ for large n as she would then need to immediately concede against any counterdemand (Lemma 4, part (a)), giving her a limit payoff of $\underline{p}$.

Suppose the seller demands p_s with positive limit probability, then $\lim_n \bar{z}_s^{p_s} = 0$ and for any counterdemand $\lim_n \bar{z}_s^{p_s}/\bar{z}_b^{p_s,\underline{p}} \leq L'$ for some constant L'. If $p_s > \underline{v} - \underline{w}$ we can define $v^{0,p_s} = \max\{v \in V : v < v^{1,p_s}\}$. Suppose that $\underline{\lambda}^{v^{0,p_s},\underline{w},p_s,\underline{p}} < \lambda_s^{v^{1,p_s},p_s,\underline{p}}$, then Lemma 2, implies that $(v^{0,p_s}, \underline{w}) \in \Theta^{e,p_s}$ counterdemands $\underline{p}$. Lemma 3 then implies the seller must concede with probability approaching 1 in the limit, giving her with a limit payoff of $\underline{p}$ (as all buyers will then demand $\underline{p}$). This is a contradiction, as we already established the seller can guarantee a payoff of $\tilde{p}_s > \underline{p}$. Hence, a rational seller can't make such a demand with positive limit probability and we can effectively restrict attention to seller demands, $p_s \in P_s^* = \{p_s \in P_s : p_s < \underline{v} - \underline{w}$ or $\underline{\lambda}^{v^{0,p_s},\underline{w},p_s,\underline{p}} > \lambda_s^{v^{1,p_s},p_s,\underline{p}}\}$. Also notice that for any sufficiently small $\varepsilon' > 0$ we must also have that $(p_s - \underline{p})g(v, w) > \underline{p}(1 - g(v, w))$ for all $p_s \geq \tilde{p}_s$ and $(v, w) \in \Theta$; henceforth assume this.

I next establish the upper bound on the buyer's payoff in Proposition 2. Recall that if the seller demands $p_s \in P_s^*$, with $p_s \geq \tilde{p}_s$, then by Lemma 2 no buyer with $(v, w) \in \Theta^{e,p_s}$ will counterdemand $p_s > \underline{p}$, where $\underline{p} \leq \varepsilon' \leq \delta$ in a small $\varepsilon' > 0$ rich set of commitment types. Hence, the best case for the seller who demands price p_s is that all types $(v, w) \in \Theta^{c,p_s}$ accept her demand and all types $(v, w) \in \Theta^{e,p_s}$ demand $\underline{p}$, giving her a payoff of at most $(1 - H(p_s))p_s + \delta$. For the seller to obtain a limit payoff larger than $\max_{p \in [0, p^* + 2\varepsilon]}(1 - H(p))p + \delta$, therefore, she must demand $p_s > p^* + 2\varepsilon$ with positive limit probability; assume this.

Define $\hat{p}_b^{p_s} = \min\{p_b \in P_b : p_b > v^{1,p_s} - p_s\}$. I claim that $\hat{p}_b^{p_s}$ is well defined and $\hat{p}_b^{p_s} < p^* + 2\varepsilon$ given a small $\varepsilon' > 0$ rich set of commitment types. There are two cases to consider, (a) $p_s < \underline{v} - \underline{w}$ and (b) $p_s > \underline{v} - \underline{w}$. First consider case (a) where $v^{1,p_s} = \underline{v}$. Since $\underline{v} - \underline{w} > p_s > p^* + 2\varepsilon$ we must have $\underline{w} < \underline{v}/2 < p_s - 2\varepsilon$, and so $\underline{v} - p_s < p_s - 4\varepsilon$. When $\varepsilon' \leq \varepsilon/2$, there exists $p_b \in [\underline{v} - p_s, \underline{v} - p_s + \varepsilon] \cap P_b$ in any ε' rich commitment type space, and so $\hat{p}_b^{p_s}$ is not only well defined but $\hat{p}_b^{p_s} \leq \underline{v} - p_s + \varepsilon \leq \underline{v}/2 - \varepsilon < p^*$.

Next consider case (b). Let $\hat{\varepsilon} = \max_{d \in [\underline{v}, \bar{v}]} \min_{v \in V} |d - v|$. Given that a rational buyer's type space is ε rich, we must have that $\hat{\varepsilon} < \varepsilon$. Since $v^{1,p_s} \neq \underline{v}$ we have v^{0,p_s} well-defined. Moreover, since $v^{1,p_s} - 2\hat{\varepsilon} \leq v^{0,p_s} < p_s + \underline{w}$, we must have $v^{1,p_s} - p_s < \underline{w} + 2\hat{\varepsilon}$. Given $\varepsilon' \leq \varepsilon - \hat{\varepsilon}$, there must be some $p_b \in [\underline{w} + 2\hat{\varepsilon}, \underline{w} + 2\varepsilon) \cap P_b$ in a ε' rich commitment type space, and hence $\hat{p}_b^{p_s}$ is well-defined with $\hat{p}_b^{p_s} < \underline{w} + 2\varepsilon < p_s$. Also notice that $\hat{p}_b^{p_s} > v^{1,p_s} - p_s > \underline{w} > \underline{p}$.

Without loss of generality, I will assume that types $(v^{1,p_s}, w) \in \Theta^c$ never concede with positive probability at time 0^2. They will certainly never do so if the seller concedes at time 0^3 for some counterdemand, but if $F_s^{p_s,p_b}(0^4) = 0$ for all $p_b < p_s$ then it makes no difference having $(v^{1,p_s}, w) \in \Theta^c$ concede at 0^4 instead of at 0^2.

I next claim that $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = 1$. Suppose not, so that $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) < 1$. This clearly requires $\lim_n \bar{z}_b^{p_s,\hat{p}_b^{p_s}} = 0$ and $\lim_n \bar{g}^{p_s,\hat{p}_b^{p_s}}(v^{1,p_s}, w) = 0$ for all $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ by Lemma 3.

Our assumption $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) < 1$ implies the subclaim, $\lim_n t^{2,p_s,\hat{p}_b^{p_s}} = \infty$. To see this, notice that we need $\bar{z}_s^{p_s} e^{\lambda_s^{\bar{v},p_s,\hat{p}_b^{p_s}} t^{2,p_s,\hat{p}_b^{p_s}} + \lambda_s^{v^{1,p_s},p_s,\hat{p}_b^{p_s}}(T^{*,p_s,\hat{p}_b^{p_s}} - t^{2,p_s,\hat{p}_b^{p_s}})} \geq 1 - F_s^{p_s,\hat{p}_b^{p_s}}(0^4)$ for the seller to reach a probability 1 reputation at $T^{*,p_s,\hat{p}_b^{p_s}}$; this immediately implies $T^{*,p_s,\hat{p}_b^{p_s}} \to \infty$. For the buyer to do likewise we need $\bar{z}_s^{p_s,\hat{p}_b^{p_s}} e^{\lambda_b^{p_s,\hat{p}_b^{p_s}} T^{*,p_s,\hat{p}_b^{p_s}}} \le 1$ and so:

$$1 - F_s^{p_s,\hat{p}_b^{p_s}}(0^4) \le \frac{\bar{z}_s^{p_s}}{\bar{z}_b^{p_s,\hat{p}_b^{p_s}}} e^{(\lambda_s^{\bar{v},p_s,\hat{p}_b^{p_s}} - \lambda_b) t^{2,p_s,\hat{p}_b^{p_s}} + (\lambda_s^{v^{1,p_s},p_s,\hat{p}_b^{p_s}} - \lambda_b^{p_s,\hat{p}_b^{p_s}})(T^{*,p_s,\hat{p}_b^{p_s}} - t^{2,p_s,\hat{p}_b^{p_s}})}.$$

This inequality combined with $\lim_n T^{*,p_s,\hat{p}_b^{p_s}} = \infty$ and $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) < 1$ and $\lim_n \bar{z}_s^{p_s}/\bar{z}_b^{p_s,\hat{p}_b^{p_s}} \le L'$ for some constant L', implies the subclaim $\lim_n t^{2,p_s,\hat{p}_b^{p_s}} = \infty$.

By demanding $\hat{p}_b^{p_s}$ type $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ secures a payoff of at least

$$\begin{aligned} &(v^{1,p_s} - \hat{p}_b^{p_s})\Big(F_s^{p_s,\hat{p}_b^{p_s}}(0^4) + \int^{0^4 < t < t^{2,p_s,\hat{p}_b^{p_s}}} e^{-rt} dF_s^{p_s,\hat{p}_b^{p_s}}(t)\Big) \qquad (1) \\ \ge &(v^{1,p_s} - \hat{p}_b^{p_s})\Big(F_s^{p_s,\hat{p}_b^{p_s}}(0^4) + \frac{v^{2,p_s} - p_s}{v^{2,p_s} - \hat{p}_b^{p_s}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(0^4) - e^{-rt^{2,p_s,\hat{p}_b^{p_s}}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(t^{2,p_s,\hat{p}_b^{p_s}})))\Big) \end{aligned}$$

where the second inequality follows from the fact that v^{2,p_s} would find it optimal to concede at $t^{2,p_s,\hat{p}_b^{p_s}}$ conditional on demanding $\hat{p}_b^{p_s}$, that is:

$$\int^{0^4 < t < t^{2,p_s,\hat{p}_b^{p_s}}} e^{-rt} dF_s^{p_s,\hat{p}_b^{p_s}}(t) \ge \frac{v^{2,p_s} - p_s}{v^{2,p_s} - \hat{p}_b^{p_s}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(0^4) - e^{-rt^{2,p_s,\hat{p}_b^{p_s}}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(t^{2,p_s,\hat{p}_b^{p_s}}))).$$

Since $t^{2,p_s,\hat{p}_b^{p_s}} \to \infty$, the right hand side of (1) converges to

$$(v^{1,p_s} - \hat{p}_b^{p_s})\Big(\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) + \frac{v^{2,p_s} - p_s}{v^{2,p_s} - \hat{p}_b^{p_s}}(1 - \lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4))\Big). \qquad (2)$$

Now suppose some type $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ optimally demands $p_b > \hat{p}_b^{p_s}$ with positive limit probability. Since $v^{1,p_s} - p_s < p_b$, Lemma 3 implies the seller must immediately in the limit, $\lim_n F_s^{p_s,p_b}(0^4) = 1$, giving the value v^{1,p_s} buyer a payoff of $v^{1,p_s} - p_b$. A type $(\bar{v}, \underline{w})$ buyer must be indifferent between p_b and $\hat{p}_b^{p_s}$ before conceding at 0^4 (see precise argument in the proof of Lemma 2) and so $F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = F_s^{p_s,p_b}(0^4)(p_s - p_b')/(p_s - \hat{p}_b^{p_s})$. This, however, means the lower bound limit payoff (2) equals $(v^{1,p_s} - \hat{p}_b^{p_s})(v^{2,p_s} - p_b')/(v^{2,p_s} - \hat{p}_b^{p_s})$. But that strictly exceeds $v^{1,p_s} - p_b'$ given that $(v^{2,p_s} - p_b')/(v^{2,p_s} - \hat{p}_b^{p_s})$ is strictly increasing in v^{2,p_s}, a contradiction.

Next suppose some type $(v^{1,p_s}, \underline{w}) \in \Theta^{c,p_s}$ optimally demands $p_b \in (\underline{p}, \hat{p}_b^{p_s}) \cap P_b$ with positive limit probability, then since $p_b < v^{1,p_s} - p_s$ the buyer must concede with probability approaching 1 (by Lemma 3), to give the value v^{1,p_s} buyer a limit payoff of $v^{1,p_s} - p_s$; and so $\lim_n F_s^{p_s,p_b}(0^4) = 0$. Clearly, this implies $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = 0$, but even then, the lower bound limit payoff (2) is $(v^{1,p_s} - \hat{p}_b^{p_s})(v^{2,p_s} - p_s)/(v^{2,p_s} - \hat{p}_b^{p_s})$, which strictly exceeds $v^{1,p_s} - p_s$ since $(v^{2,p_s} - p_s)/(v^{2,p_s} - \hat{p}_b^{p_s})$ is strictly increasing in v^{2,p_s}, a contradiction.

The final possibility is that all $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ only demand $\underline{p}$ in the limit, $\lim_n \mu_b^{p_s,v^{1,p_s},w}(\underline{p}) = 1$.

Given $p_s \in P_s^*$ we have $\underline{\lambda}^{v',w',p_s,\underline{p}} < \lambda_s^{v^{1,p_s},p_s,\underline{p}}$ for all $(v', w') \in \Theta^{c,p_s}$. Hence, since $(p_s - \underline{p})g(v^{1,p_s}, \underline{w}) > \underline{p}(1 - g(v^{1,p_s}, \underline{w}))$, the buyer must either concede or exit immediately with probability approaching 1 in the limit by Lemma 3, giving the value v^{1,p_s} buyer a limit payoff of $v^{1,p_s} - p_s$, which is again strictly less than the lower bound limit payoff of (2) she could have obtained by demanding $\hat{p}_b^{p_s}$ (even when $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = 0$), a contradiction.

This establishes the claim $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = 1$. Given that, no rational buyer will propose a price $p_b > \hat{p}_b^{p_s}$. This implies the seller's payoff is at most $(1-H(p_s))\hat{p}_b^{p_s} + \delta \leq \max_{p\in[0,p^*+2\varepsilon]}(1 - H(p))p + \delta$ since $\hat{p}_b^{p_s} < p^* + 2\varepsilon < p_s$. This also shows (whether or not the seller demands $p_s > p^* + 2\varepsilon$ with positive limit probability), that the buyer enjoys a limit payoff of at least $\max\{v - (p^* + 2\varepsilon), w\}$.

I now turn to the lower bound on seller payoffs. Let $\hat{p}_s \in \arg\max_{p\in[0,p^*]}(1 - H(p))$ and recall that $\check{p}(p) = \min\{p, \max\{v - \underline{w} \leq p : v \in V\}\}$ and $\hat{\varepsilon} = \max_{d\in[\underline{v},\overline{v}]} \min_{v\in V} |d - v| < \varepsilon$, so that $\check{p}(p) \in [p-2\hat{\varepsilon}, p]$. Let $\overline{p}_s = \max\{p_s \in P_s : p_s < \check{p}(\hat{p}_s)\}$, where $\overline{p}_s \geq \tilde{p}_s < \min\{v-w : (v, w) \in \Theta\}$ and $\overline{p}_s \in [\hat{p}_s - 2\varepsilon, \check{p}(\hat{p}_s)) \cap P_s$ in any $\varepsilon' \leq \varepsilon - \hat{\varepsilon}$ rich commitment type space. I next claim $\overline{p}_s \in P_s^*$ for a small $\varepsilon' > 0$ rich set of commitment types.

If $\overline{v} = \underline{v}$ then since $\hat{p}_s \leq \overline{v} - \underline{w}$ we have $\check{p}(\hat{p}_s) = p \leq \underline{v} - \underline{w}$, otherwise let $\check{\varepsilon} = \min\{v - v' : v \neq v' \in V\} \in (0, \overline{v})$ and assume that $2\varepsilon' \leq \underline{w}\check{\varepsilon}/(\overline{v} - \check{\varepsilon})$.

Suppose that $\check{p}(\hat{p}_s) \leq \underline{v} - \underline{w}$ (as when $\underline{v} = \overline{v}$). This implies $v^{1,\overline{p}_s} = \underline{v}$. For $v - w < p_s$, we have $\lambda_s^{\underline{v},\overline{p}_s,\underline{p}} \leq \lambda_s^{v,\overline{p}_s,\underline{p}} < \underline{\lambda}^{v,w,p_s,\underline{p}}$, where the first inequality holds because $v \geq \underline{v}$ and the second because $\underline{\lambda}^{v,w,p_s,\underline{p}}$ is increasing in w. This implies $\overline{p}_s \in P_s^*$.

Next suppose that $\check{p}(\hat{p}_s) > \underline{v} - \underline{w}$ and so $v^{1,\overline{p}_s} - \underline{w} = \check{p}(\hat{p}_s) < \overline{p}_s + 2\varepsilon'$ and $v^{0,p_s} = \max\{v \in V : v < v^{1,p_s}\} \leq v^{1,\overline{p}_s} - \check{\varepsilon}$. In this case we have,

$$\begin{aligned}
(v^{1,\overline{p}_s} - \overline{p}_s)(v^{0,\overline{p}_s} - \underline{w}) - \underline{w}\overline{p}_s &< (\underline{w} + 2\varepsilon')(v^{0,\overline{p}_s} - \underline{w}) - \underline{w}(v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon') \\
\leq (\underline{w} + 2\varepsilon')(v^{1,\overline{p}_s} - \check{\varepsilon} - \underline{w}) - \underline{w}(v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon') &\leq (\underline{w} + 2\varepsilon')(\overline{v} - \check{\varepsilon} - \underline{w}) - \underline{w}(\overline{v} - \underline{w} - 2\varepsilon') \leq 0
\end{aligned}$$

where the first inequality follows from $\overline{p}_s > v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon'$, the second from $v^{0,\overline{p}_s} \leq v^{1,\overline{p}_s} - \check{\varepsilon}$, the third from $v^{1,\overline{p}_s} \leq \overline{v}$ and the fourth from $2\varepsilon' \leq \underline{w}\check{\varepsilon}/(\overline{v} - \check{\varepsilon})$. Furthermore, notice that

$$(v^{1,\overline{p}_s} - \overline{p}_s)(v^{0,\overline{p}_s} - p_b - \underline{w}) - \underline{w}(\overline{p}_s - p_b) \tag{3}$$

is decreasing in p_b given $v^{1,\overline{p}_s} - \overline{p}_s > \underline{w}$ and so must be negative for any $p_b \in (0, \overline{p}_s)$. I claim this implies $\lambda_s^{\overline{p}_s,\underline{p},v^{1,\overline{p}_s}} < \underline{\lambda}^{\overline{p}_s,\underline{p},v,w}$ for all $(v, w) \in \Theta^{e,\overline{p}_s}$, so that $\overline{p}_s \in P_s^*$. Clearly, the inequality holds for $(v^{0,\overline{p}_s}, \underline{w})$ by the negativity of (3). Since (3) is increasing in $v^{0,\overline{p}_s}$, it must likewise hold for any $(v, \underline{w})$ with $v < v^{0,\overline{p}_s}$. Since (3) is decreasing in $\underline{w}$, the claim must also hold for all $(v, w) \in \Theta$ with $v \leq v^{0,\overline{p}_s}$ and $w \geq \underline{w}$. If $v \geq v^{1,\overline{p}_s}$ and $v - w < p_s$, we again have $\lambda_s^{v^{1,\overline{p}_s},\overline{p}_s,\underline{p}} \leq \lambda_s^{v,\overline{p}_s,\underline{p}} < \underline{\lambda}^{v,w,p_s,\underline{p}}$ (since the latter is increasing in w). And so, $\overline{p}_s \in P_s^*$.

Hence, suppose the seller demands $\overline{p}_s$, then since $\overline{p}_s \leq p^*$ for any counterdemand $p_b < \overline{p}_s$ we must have $v - \overline{p}_s > p_b$ for all $(v, w) \in \Theta^{c,\overline{p}_s}$. To see this, notice that if $\underline{v}/2 \geq \overline{p}_s$ then $v - \overline{p}_s \geq \underline{v}/2 \geq \overline{p}_s > p_b$ whereas if $\underline{w} \geq \overline{p}_s$ then $v - \overline{p}_s > w \geq \underline{w} \geq \overline{p}_s > p_b$ for $(v, w) \in \Theta^{c,\overline{p}_s}$. Hence, if a buyer with $(v, w) \in \Theta^{c,\overline{p}_s}$ demands $p_b > \underline{p}$ with positive probability in the limit, she must subsequently immediately concede with probability 1 in the limit by Lemma 3 to give her payoffs of $(v - \overline{p}_s)$ for all large n (since $\Theta^{e,\overline{p}_s}$ buyers only demand $\underline{p}$).

As argued previously, it is without loss of generality to assume that type $(v^{1,\overline{p}_s}, \underline{w})$ always makes some counterdemand $p_b \in P_b$, and so without loss she counterdemands $\underline{p}$ with probability approaching 1 in this limit (or she will get exactly $(v^{1,p_s} - \overline{p}_s)$). However, by Lemma 3, therefore, the buyer concedes with probability approaching $\lim_n x^{\overline{p}_s,\underline{p}}$ and exits with probability approaching $1 - \lim_n x^{\overline{p}_s,\underline{p}}$ at time 0^4 since $(\overline{p}_s - \underline{p})g(v^{1,\overline{p}_s}, \underline{w}) > \underline{p}(1 - g(v^{1,\overline{p}_s}, \underline{w}))$. And so, the demand $\overline{p}_s$ secures the seller a limit payoff of at least $(1 - H(\overline{p}_s))\overline{p}_s \geq \max_{p \in [0,p^*]}(1 - H(p))p - 2\varepsilon$ where the inequality follows from $\overline{p}_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s]$. □

Proof of Proposition 3. By assumption, for some $\hat{p}_s \leq p^*$ we have $\check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) > p(1 - H(p))$ for all $p \in [0, \check{p}(\hat{p}_s)) \cup (\hat{p}_s, p^* + 2\varepsilon]$. As in the proof of Proposition 2, let $\overline{p}_s = \max\{p_s \in P_s : p_s < \check{p}(\hat{p}_s)\}$ where $\overline{p}_s \in (\check{p}(\hat{p}_s) - 2\varepsilon', p^*]$ with a $\varepsilon' > 0$ rich set of commitment types. As argued in the proof of Proposition 2, for $\varepsilon' > 0$ small enough $\overline{p}_s \in P_s^*$. Moreover, for any $p_s \in P_s^*$ with $p_s \in [\tilde{p}_s, p^*]$, all buyers immediately concede or exit with probability approaching 1 in the limit so the seller's limit payoff is exactly $(1 - H(p_s))p_s$ (Lemma 3). Hence, the seller's payoff from demanding $\overline{p}_s$ is at least $(\check{p}(\hat{p}_s))(1 - H(\check{p}(\hat{p}_s))) - 2\varepsilon'$.

If $\overline{p}_s > \min\{v - w : (v, w) \in \Theta\}$ then let $p^\dagger = \max\{p < \check{p}(\hat{p}_s) : H(p) < H(\check{p}(\hat{p}_s))\}$; notice that H is constant on the non-degenerate interval $(p^\dagger, \check{p}(\hat{p}_s))$ and $(1 - H(p_s))p_s$ is increasing in p_s on this interval so that $(1 - H(p_s))p_s < (1 - H(\overline{p}_s))\overline{p}_s$ for $p_s \in (p^\dagger, \overline{p}_s)$. Let $2\varepsilon' < \check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) - \max_{p \leq p^\dagger} p(1 - H(p))$, where the right hand side is strictly positive by assumption. Given this, the seller's limit payoff from demanding $p_s < \overline{p}_s$ is less than from demanding $\overline{p}_s$ and so she won't make such a demand for large n.

On the other hand, suppose that the seller demands $p_s > p^* + 2\varepsilon$, then as shown in the proof of Proposition 2, for small $\varepsilon' > 0$ the buyer will counterdemand $p_b \leq p^* + 2\varepsilon$, which the seller will immediately concede to with strictly positive probability, giving her a payoff less than $(p^* + 2\varepsilon)(1 - H(p^* + 2\varepsilon)) + \varepsilon'$. And so, the seller's best possible limit payoff from demanding $p_s > \hat{p}_s$, is always less than $\max_{p \in (\hat{p}_s, p^*+2\varepsilon]} p(1 - H(p)) + \varepsilon'$. This payoff is strictly then less than her payoff from proposing $\overline{p}_s$ whenever $3\varepsilon' < \check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) - \max_{p \in (\hat{p}_s, p^*+2\varepsilon]} p(1 - H(p))$, where the right hand side is strictly positive by assumption. Hence, the seller will never demand $p_s > \hat{p}_s$ with positive limit probability. Hence, the seller only demands $p_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s] \cap P_s^*$ with positive limit probability, and since $\hat{p}_s \leq p^*$, buyers' either immediately concede or exit in the limit. The bound on the buyer's payoff is then immediate. □

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Proof of Proposition 1

In order to prove this result I first define what I call a "straightforward" equilibrium in the continuation game at 0^3 given $p_s \in P_s$ and $p_b \in P_b$ and beliefs $(\bar{z}_s, \bar{z}_b, \bar{g})$ where $\bar{z}_i > 0$.

I first define some preliminary objects that will help to describe such an equilibrium. For $y \in [0, (1-\bar{z}_b)x]$ let

$$\bar{k}(y) = \max\{k \leq K+1 : \sum_{(v^m,w)\in\Theta^c : m\geq k} \bar{g}(v^m, w)(1-\bar{z}_b) \geq y\},$$

Clearly, $\bar{k}(0) = K+1$, and $\bar{k}((1-\bar{z})x) = 1$ if $\bar{g}(v^1, w) > 0$ for $(v^1, w) \in \Theta^c$. This is decreasing and upper semi continuous in y. Loosely, if fraction y of buyers have conceded by time t then $t \in (t^{\bar{k}(y)+1}, t^{\bar{k}(y)}]$. Also define $\underline{k}(y) = \bar{k}(y)$ if $y < (1-\bar{z})x$ and $\underline{k}((1-\bar{z})x) = 0$.

For $k \in \{1, ..., K\}$ let

$$\bar{G}^e(k) = \sum_{(v,w)\in\Theta^e : \underline{\lambda}^{v,w} > \lambda_s^{v^k}} \bar{g}(v, w)$$

while $\bar{G}^e(K+1) = 0$ and $\bar{G}^e(0) = 1-x$. Notice that $\bar{G}^e(\bar{k}(y))$ is increasing and lower semi continuous in y.

Next define

$$\pi(y, \hat{y}) = (p_s - p_b)(\hat{y} - y) - p_b(1-\bar{z}_b)(G^e(\underline{k}(\hat{y})) - G^e(\bar{k}(y))).$$

Loosely, this the difference between the present value payoff of p_b a seller gets by conceding an instant before time t, and the payoff she would receive conceding an instant after t, if at time t a fraction $(\hat{y} - y)$ of buyers concede and $(1-\bar{z}_b)(G^e(\underline{k}(\hat{y})) - G^e(\bar{k}(y)))$ exit. And then let:

$$\tilde{y}(\hat{y}) = \min\{y \geq 0 : \pi(y, \hat{y}) \leq 0\}.$$

Loosely, $\hat{y} - \tilde{y}(\hat{y})$ is the maximum probability of concession at time t (consistent with optimal exit at t) such that the buyer prefers to concede an instant before t than an instant after t, when fraction $\hat{y}$ of buyers have conceded by t.

It is useful to outline equilibrium strategies starting at time $T^* = t^1$, which I relabel as "time" $\tau^1 = 0$, and more generally will define equilibrium objects in terms of $\tau = T^* - t \in [0, \infty)$. Define $\hat{F}_s^1(\tau^1) = (1-\bar{z}_s)$, $\hat{F}_b^1(\tau^1) = (1-\bar{z}_b)x$, $\hat{E}_b^1(\tau^1) = (1-\bar{z}_b)G^e(1)$, and then by induction for $k \in \{1, ..., K\}$ and $\tau \geq \tau^k$, let $1-\hat{F}_s^k(\tau) = (1-\hat{F}_s^k(\tau^k))e^{\lambda_s^{v^k}(\tau-\tau^1)}$, $\hat{E}_b^k(\tau) = \hat{E}_b^k(\tau^k)$, $1-\hat{E}_b^k(\tau^k)-\hat{F}_b^k(\tau) = (1-\hat{E}_b^k(\tau^k)-\hat{F}_s^k(\tau^k))e^{\lambda_b(\tau-\tau^k)}$. Effectively, $\hat{F}_s^k$ (respectively $\hat{F}_b^k$) correspond to the concession probability of the seller (buyer) assuming she concedes at rate $\lambda_s^{v^k}$ (λ_b) on $(t, t^k) = (T^*-\tau, T^*-\tau^k)$

if $F_i(t^k_-) = \hat{F}^k_i(\tau^k)$ and $E_b(t^k_-) = \hat{E}^k_b(\tau^k)$. Then for $k \leq K$ (where recall that $v^K = \overline{v}$) define

$$\tau^{k+1} = \min\{\tau \geq \tau^k : \tilde{y}(\hat{F}^k_b(\tau)) < \hat{F}^k_b(\tau) \text{ or } \overline{k}(\hat{F}^k_b(\tau)) > k\}$$

with $\hat{F}^{k+1}_s(\tau^{k+1}) = \hat{F}^k_s(\tau^{k+1})$, $\hat{F}^{k+1}_b(\tau^{k+1}) = \tilde{y}(\hat{F}^k_b(\tau))$ and $\hat{E}^{k+1}_b(\tau^{k+1}) = (1-\bar{z}_b)G^e(k+1)$. Notice that we can have $\tau^{k+1} = \tau^k$. In fact, define $\ell^k = \max\{\ell : \tau^\ell \leq \tau^k\} \geq k$ so that $\tau^{\ell^k} = \tau^k$.

Next define $\hat{F}_s(0) = (1-\bar{z}_s)$, $\hat{F}_b(0) = (1-\bar{z}_b)x$, $\hat{E}_b(0) = (1-\bar{z}_b)(1-x)$, and if $\tau \in (\tau^k, \tau^{k+1}]$ then $\hat{F}_s(\tau) = \hat{F}^k_s(\tau)$, $\hat{F}_b(\tau) = \hat{F}^k_b(\tau)$, $\hat{E}_b(\tau) = \hat{E}^k_b(\tau) = (1-\bar{z}_b)G^e(k)$. Let $\hat{F}_s(\tau) = \hat{F}^K_s(\tau)$ for $\tau \geq \tau^{K+1}$ and then define $\tau_s = \min\{\tau : \hat{F}_s(\tau) \geq 0\}$, $\tau_b = \tau^{K+1}$ and $\tau^* = T^* = \min\{\tau_b, \tau_s\}$. Finally, let $F_s(0^3) = \hat{F}_s(\tau^*)$, $E_b(0^3) = F_b(0^3) = 0$, then for $t \in [0^4, T^*]$ let $F_s(t) = \hat{F}_s(\tau^* - t)$, $F_b(t) = \hat{F}_b(\tau^* - t)$ and $E_b(t) = \hat{E}_b(\tau^* - t)$.

By construction, for $k \in \{1, ..., K\}$, we have $t^k = \tau^* - \tau^k$ if $\tau^k < \tau^*$ and $t^k = 0^4$ otherwise. Rational player concession and exit strategies can clearly be backed out from these functions by skimming property and Lemma 1; all such equilibria are payoff equivalent. Up to that equivalence, the equilibrium is unique by construction. Also by construction, no player has a profitable deviation (so such strategies form an equilibrium). In particular, concession on (t^{k+1}, t^k) is at rates λ_b and $\lambda^{v^k}_s$ respectively to make a rational seller or buyer $(v^k, w) \in \Theta^c$ indifferent between conceding on that interval. If $\tau^* = 0$ then $F_s(0^3) = (1-\bar{z}_s)$. Otherwise, buyer concession at $t^k \geq 0^4$ is calibrated to always leave a rational seller indifferent between conceding an instant before or after t^k (given the probability of exit at t^k). As the next lemma shows, such an equilibrium is continuous in players' beliefs.

Lemma 5. *Consider the continuation game at* 0^3 *after demands* $p_s \in P_s$ *and* $p_b \in P_b$ *with fixed* Θ. *A unique straightforward continuation equilibrium exists, for which players' continuation payoffs are continuous at the beliefs* $(\bar{z}_s, \bar{z}_b, \bar{g})$ *where* $\bar{z}_i \geq z_i\pi_i(p_i) > 0$.

Proof. To prove the result it is first necessary to establish the following inductive *Claim:* Consider an arbitrary sequence of distributions $(\bar{z}^n_s, \bar{z}^n_b, \bar{g}^n) \to (\bar{z}^n_s, \bar{z}^n_b, \bar{g}^n)$. If $\lim_n \tau^{k,n} = \tau^k$ as well as $\lim_n \hat{F}^{k,n}_b(\tau^{k,n}) = F^k_b(\tau^k)$ and $\lim_n \hat{E}^{k,n}_b(\tau^{k,n}) = E^k_b(\tau^k)$, then $\lim_n \tau^{\ell,n} = \tau^\ell$ for all $\ell \in \{k+1, ..., \ell^{k+1}\}$ and $\lim_n \hat{F}^{\ell^{k+1},n}_b(\tau^{\ell^{k+1},n}) = F^{\ell^{k+1}}_b(\tau^{\ell^{k+1}})$ and $\lim_n \hat{E}^{\ell^{k+1},n}_b(\tau^{\ell^{k+1},n}) = E^{\ell^{k+1}}_b(\tau^{\ell^{k+1}})$.

Subclaim 1. *For any* $\tau > \tau^k$ *we must have* $\tau > \tau^{k,n}$ *for large n,* $\lim_n \hat{F}^{k,n}_b(\tau) = F^k_b(\tau)$ *and* $\lim_n \overline{k}^n(\hat{F}^{k,n}_b(\tau)) \leq \overline{k}(\hat{F}^k_b(\tau))$ *taking subsequences if necessary (so limits are defined).* To see this, notice that $1 - \hat{F}^{k,n}_b(\tau) = (1 - \hat{F}^{k,n}_b(\tau^{k,n}))e^{\lambda^{v^k}_s(\tau - \tau^{k,n})} \to 1 - \hat{F}^k_b(\tau)$ then $\lim_n \overline{k}^n(\hat{F}^{k,n}_b(\tau)) \leq \overline{k}(\hat{F}^k_b(\tau))$ follows from the upper semi continuity of $\overline{k}$. More precisely, if $\sum_{(v^m,w)\in\Theta^c:m\geq k'} \bar{g}^n(v,w)(1-\bar{z}^n_b) \geq \hat{F}^{k,n}_b(\tau)$ for all n, then the inequality also holds in the limit.

Subclaim 2. *If* $\overline{k}(\hat{F}^k_b(\tau)) = k' > k$ *for* $\tau \geq \tau^k$ *then* $\lim_n \overline{k}^n(\hat{F}^{k,n}_b(\tau + \varepsilon)) \geq k'$ *for any* $\varepsilon > 0$, *and so if* $\overline{k}(\hat{F}^k_b(\tau^{k+1})) > k$ *then* $\lim_n \tau^{k+1,n} \leq \tau^{k+1}$. This follows from $\sum_{(v^m,w)\in\Theta^c:m\geq k'} \bar{g}^n(v,w)(1-\bar{z}^n_b) > \hat{F}^k_b(\tau + \varepsilon/2) \geq \hat{F}^{k,n}_b(\tau + \varepsilon)$ for all large n.

Subclaim 3. *We must have* $\lim_n \tau^{k+1,n} \geq \tau^{k+1}$. Suppose not, so that $\lim_n \tau^{k+1,n} < \tau^{k+1}$. Since

$\lim_n \overline{k}^n(\hat{F}_b^{k,n}(\tau)) \leq \overline{k}(\hat{F}_b^k(\tau)) = k$ for $\tau < \tau^{k+1}$, we have $y^n = \tilde{y}^n(\hat{F}_b^{k,n}(\tau^{k+1,n})) < \hat{F}_b^{k,n}(\tau^{k+1,n})$ and

$$\pi^n(y^n, \hat{F}_b^{k,n}(\tau^{k+1,n})) = (p_s - p_b)(\hat{F}_b^{k,n}(\tau^{k+1,n}) - y^n) - p_b(1-\bar{z})(G^{e,n}(k) - G^{e,n}(\overline{k}^n(y^n))) \leq 0$$

where the inequalities are preserved in the limit, $\pi(\lim_n y^n, \hat{F}_b^k(\lim_n \tau^{k+1,n})) \leq 0$. We must have $\lim_n y^n \leq \lim_n \hat{F}_b^{k,n}(\tau^{k+1}) = \hat{F}_b^k(\tau^{k+1})$ otherwise $y^n > \hat{F}_b^k(\tau)$ for some $\tau < \tau^{k+1}$ and all large n so that $\bar{k}^n(y^n) \leq \bar{k}^n(\hat{F}_b^k(\tau)) = k$, so $\pi^n(y^n, \hat{F}_b^{k,n}(\tau^{k+1,n})) = (p_s - p_b)(\hat{F}_b^{k,n}(\tau^{k+1,n}) - y^n) > 0$, a contradiction. This in turn implies $\lim_n y^n < \hat{F}_b^k(\lim_n \tau^{k+1,n})$ so that $\pi(\lim_n y^n, \hat{F}_b^k(\lim_n \tau^{k+1,n})) \leq 0$ contradicts the definition of $\tau^{k+1} > \lim_n \tau^{k+1,n}$, establishing the subclaim.

Subclaim 4. *We must have* $\lim_n \tau^{k+1,n} = \tau^{k+1}$. Suppose not so that $\lim_n \tau^{k+1,n} > \tau^{k+1} + \varepsilon$ for some $\varepsilon > 0$ and $\overline{k}^n(\hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon) = k$ for large n. For small enough $\varepsilon' > 0$, we must have $\pi(y, \hat{F}_b^k(\tau^{k+1}))$ is continuous and strictly decreasing in y on some interval $y \in [-\varepsilon', 0] + \tilde{y}(\hat{F}_b^k(\tau^{k+1}))$ and $\overline{k}(y)$ is constant. Then define $y^\delta = \min\{\tilde{y}(\hat{F}_b^k(\tau^{k+1})) - \delta, 0\}$, for small enough $\delta > 0$, we have

$$\begin{aligned}
&\lim_n (p_s - p_b)(\hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon) - y^\delta) - p_b(1-\bar{z})(G^{e,n}(k) - G^{e,n}(\overline{k}^n(y^\delta))) \\
=&(p_s - p_b)(\hat{F}_b^k(\tau^{k+1} + \varepsilon) - y^\delta) - p_b(1-\bar{z})(G^e(k) - G^e(\overline{k}(y^\delta))) < 0.
\end{aligned}$$

And so, for all sufficiently large n we must have $\pi^n(y^\delta, \hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon)) < 0$, which contradicts $\lim_n \tau^{k+1,n} > \tau^{k+1} + \varepsilon$, establishing the subclaim.

Subclaim 5. *We must have* $\tau^{\ell^{k+1}} = \tau^{k+1} = \lim_n \tau^{k+1,n} = \lim_n \tau^{\ell^{k+1},n}$ *and* $F_b^{\ell^{k+1},n}(\tau^{\ell^{k+1},n}) \leq F_b^{\ell^{k+1}}(\tau^{\ell^{k+1}})$. This adapts the arguments for subclaim 4. If the first part of subclaim 5 didn't hold, then $\lim_n \tau^{\ell^{k+1},n} > \tau^{k+1}$, and so $\lim_n \tau^{l,n} = \tau^{k+1}$ for $l \in \{k+1, ..., k'\}$ but $\lim_n \tau^{k'+1,n} > \tau^{k+1} + \varepsilon$ for some $\varepsilon > 0$ and $k' \in \{k+1, ..., \ell^k - 1\}$. Define $\check{y}^{l,n} = \hat{F}_b^{n,l}(\tau^{l,n})$, $\hat{y}^{l,n} = \hat{F}_b^{n,l}(\tau^{l+1,n})$, $\alpha^n(1) = k$ and $\alpha^n(j+1) = \overline{k}^n(\tilde{y}^n(\hat{y}^{j,n}))$. Again taking a subsequence if necessary, $\alpha^n(j)$ is constant in n for large n, and then let $k' = \alpha^n(j')$. Given $\pi^n(\check{y}^{\alpha^n(j+1),n}, \hat{y}^{\alpha^n(j),n}) = 0$, we have

$$\sum_{j=1}^{j'-1} \pi^n(\check{y}^{\alpha^n(j+1),n}, \hat{y}^{\alpha^n(j),n}) = (p_s - p_b)(\hat{y}^{k,n} - \check{y}^{k',n} + \sum_{j=2}^{j'-1}(\hat{y}^{\alpha^n(j),n} - \check{y}^{\alpha^n(j),n}) - p_b(1-\bar{z}_b)(G^{e,n}(k) - G^{e,n}(k')) = 0$$

Similarly, letting, $\check{y}^l = \hat{F}_b^{n,l}(\tau^{l,n})$, $\hat{y}^l = \hat{F}_b^n(\tau^{l+1})$ we know $\pi^n(\check{y}^{\ell^{k+1}}, \hat{y}^k) = 0$. Let $y^\delta = \min\{\check{y}^{\ell^{k+1}} - \delta, 0\}$ be defined as before, then for small $\delta > 0$ we get

$$\begin{aligned}
&\pi^n(y^\delta, \hat{F}_b^{n,k'}(\tau^{k+1} + \varepsilon)) = \pi^n(y^\delta, \hat{F}_b^n(\tau^{k+1} + \varepsilon)) + \sum_{j=1}^{j'-1} \pi^n(\check{y}^{\alpha^n(j+1),n}, \hat{y}^{\alpha^n(j),n}) - \pi^n(\check{y}^{\ell^{k+1}}, \hat{y}^k) \\
=&(p_s - p_b)((\check{y}^{\ell^{k+1}} - y^\delta) + (\hat{y}^{k,n} - \hat{y}^k) + (\hat{F}_b^{n,k'}(\tau^{k+1} + \varepsilon) - \check{y}^{k',n}) \\
&+ \sum_{j=2}^{j'-1}(\hat{y}^{\alpha^n(j),n} - \hat{y}^{\alpha^n(j),n})) - p_b(1-\bar{z}_b)((G^{e,n}(k) - G^e(k)) + (G^e(\ell^{k+1}) - G^{e,n}(\ell^{k+1}))) \\
\to&(p_s - p_b)((\check{y}^{\ell^{k+1}} - y^\delta) + (\lim_n \hat{F}_b^{n,k'}(\tau^{k+1} + \varepsilon) - \check{y}^{k',n}) < 0
\end{aligned}$$

where the limit follows from $\lim_n \tau^{l,n} = \tau^{k+1}$ for $l \in \{k+1, ..., k'\}$ and the inequality from $\lim_n \hat{F}_b^{n,k'}(\tau^{k+1}+\varepsilon) - \check{y}^{k',n} < 0$ and with $\delta > 0$ chosen sufficiently small. However, of course, this implies a contradiction to $\lim_n \tau^{k'+1,n} > \tau^{k+1} + \varepsilon$.

Finally, suppose that $\lim_n \check{y}^{\ell^{k+1},n} > \check{y}^{\ell^{k+1}}$ then for $\delta = (\check{y}^{\ell^{k+1}} - \lim_n \check{y}^{\ell^{k+1},n})/2 < 0$, we have $\overline{k}^n(y^\delta) = \ell^{k+1}$ for large n and so

$$\begin{aligned}
&\pi^n(y^\delta, \hat{y}^{k,n}) = \pi^n(y^\delta, \hat{y}^{k,n}) - \pi^n(\check{y}^{\ell^{k+1}}, \hat{y}^k) \\
=&(p_s - p_b)((\check{y}^{\ell^{k+1}} - y^\delta) + (\hat{y}^{k,n} - \hat{y}^k) - p_b(1-\bar{z}_b)((G^{e,n}(k) - G^e(k)) + (G^e(\ell^{k+1}) - G^{e,n}(\ell^{k+1})))
\end{aligned}$$

which converges to $(p_s - p_b)\delta < 0$, contradicting the definition of $\check{y}^{\ell^{k+1},n} = \tilde{y}(\hat{y}^{k,n}) > y^\delta$ for large n.

Subclaim 6. *We have* $\lim_n F_b^{\ell^{k+1},n}(\tau^{\ell^{k+1},n}) = F_b^{\ell^{k+1}}(\tau^{\ell^{k+1}})$.

Let $\alpha^n(j') = \ell^{k+1}$ then

$$\begin{aligned}
0 &= \sum_{j=1}^{j'-1} \pi^n(\check{y}^{\alpha^n(j+1),n}, \hat{y}^{\alpha^n(j),n}) \\
&= (p_s - p_b)(\hat{y}^{k,n} - \check{y}^{\ell^{k+1},n} + \sum_{j=2}^{j'-1}(\hat{y}^{\alpha^n(j),n} - \check{y}^{\alpha^n(j),n}) - p_b(1-\bar{z}_b)(G^{e,n}(k) - G^{e,n}(\ell^{k+1})) \\
&\to (p_s - p_b)(\hat{y}^k - \lim_n \check{y}^{\ell^{k+1},n}) - p_b(1-\bar{z}_b)(G^e(k) - G^e(\ell^{k+1})) = \pi(\lim_n \check{y}^{\ell^{k+1},n}, \hat{y}^k)
\end{aligned}$$

where the limit follows from $\tau^{k+1} = \lim_n \tau^{k+1,n} = \lim_n \tau^{\ell^{k+1},n}$. Hence, $\lim_n \check{y}^{\ell^{k+1},n} \geq \check{y}^{\ell^{k+1}}$, by the definition of $\tilde{y}$, establishing the subclaim, and completing the proof of the Claim.

Given the Claim, it is clear that $\tau^{k,n} \to \tau^k$, $\tau_b^n \to \tau_b$, $\tau_s^n \to \tau_s$, as well as $F_s^n(0^3) \to F_s(0^3)$. The payoff of a rational buyer with value v who concedes at t^k is

$$U_b^{v,c}(t^k) = (v - p_b)F_s(0^3) + (v - p_b)\int^{t\in(0,t^k)} e^{-rt}dF_s(t) + (v - p_s)e^{-rt^k}(1 - F_s(t^k))$$

Given that $F_s^n \xrightarrow{w} F_s$ where F_s is continuous at t^k, it is clear that $U_b^{v,n}(t^{k,n}) \to U_b^v(t^k)$. Similarly, the payoff of a rational buyer who exits at time t^k is $U_b^{v,w,e}(t^k) = U_b^{v,c}(t^k) + (w - v + p_s)e^{-rt^k}(1 - F_s(t^k))$ so that $U_b^{v,w,e,n}(t^{k,n}) \to U_b^{v,w,e}(t^k)$.

We now turn to the rational seller, who's payoff can be expressed as $V_s = \max\{p_b, U_s(T_+^*)\}$ where $U_s(T_+^*) = \int^{s\leq T^*} p_s e^{-rs}dF_b(s) + e^{-rT^*}(1-\bar{z}_s)p_b$ is the payoff from conceding an instant after T^*. Given that $\lim_n F_b^n(T^{*,n}) = F_b(T^*) = 1 - \bar{z}_b$, $\lim_n T^{*,n} = T^*$ and $F_i^n \xrightarrow{w} F_i$ it is immediate that $U_s^n(T_+^{*,n}) \to U_s(T_+^*)$ and so $V_s^n \to V_s$. This completes the proof.

□

We are now ready to complete the proof of Proposition 1. Given the parameters of a bargaining game $(z_i, \pi_i, g, \Theta)_{i=s,b}$, let $\Delta_s = \Delta(P_s)$ be the set of seller demand choice distributions at 0^1. Let $\Delta_s^{p_s} \subset \Delta(P_b \cup \{e\})$ be the set of rational buyer demand choice distributions at 0^2 after seller demand p_s such that $\mu_b^{v,w,p_s}(e) = \mathbb{1}_{v-w>\underline{p}}$ and $\mu_b^{v,w,p_s}(p_b) = 0$ for $p_b \geq p_s$. Then $\Delta_b = \prod_{p_s \in P_s} \Delta_b^{p_s}$. Let $U_b^{v,w,p_b,p_s}(\mu_s, \mu_b)$ be the expected payoff of rational buyer (v, w) at 0^3 given demands $p_i \in P_i$, the demand choice distributions, $\mu_s \in \Delta_s(P_s)$ and $\mu_b \in \Delta_b$ combined with straightforward equilibrium continuation play. Also let $U_s^{p_s}(\mu_s, \mu_b)$ be the expected payoff of the seller at 0^2 given the demand $p_s \in P_s$, the demand choice distributions $\mu_s \in \Delta_s(P_s)$ and $\mu_b^{p_s} \in \Delta_b$ with straightforward equilibrium continuation play at 0^3. We then define:

$$\begin{aligned} B(\mu_s, \mu_b) = \{(\hat{\mu}_s, \hat{\mu}_b) \in \Delta_s \times \Delta_b : \hat{\mu}_s(p_s) > 0 \Rightarrow U_s^{p_s}(\mu_s, \mu_b) \geq U_s^{p'_s}(\mu_s, \mu_b), \forall p'_s \in P_s, \\ \hat{\mu}_b^{p_s}(p_b) > 0 \Rightarrow U_b^{v,w,p_b,p_s}(\mu_b, \mu_s) \geq U_b^{v,w,p'_b}(\mu_s, \mu_b), \forall p'_b \in P_b\}. \end{aligned}$$

It is clear that this self-correspondence is non-empty and convex-valued and has a closed graph given that $U_b^{v,w,p_b,p_s}(\mu_s, \mu_b)$ and $U_s^{p_s}(\mu_s, \mu_b)$ are continuous in (μ_b, μ_s) by Lemma 5. Hence, by Kakutani, it admits a (non-empty) fixed-point. This fixed point describes equilibrium demand choices and beliefs after $p_i \in P_i$. After the demand $p_b \notin P_b$, the seller always believes the rational buyer has a type $(\overline{v}, \underline{w})$. The buyer then immediately concedes if $p_s \leq \overline{v} - \underline{w}$ and a rational seller immediately concedes otherwise. $\square$