GENERAL EQUILIBRIUM THEORY FOR CLIMATE CHANGE

ROBERT M. ANDERSON, HAOSUI DUANMU, AND M. ALI KHAN

Abstract. We propose two general equilibrium (GE) models—quota equilibrium and tax equilibrium—that incorporate regulatory schemes to control total net pollution emissions into GE models of the generality and rigor of the Arrow-Debreu Model. In these models, the government first specifies quotas or taxes on emissions, then refrains from further action. Equilibrium is defined by the equilibration of supply and demand, in which consumers maximize preferences and firms maximize profit. We show the existence of a quota equilibrium. Assume that the only externality arises from the total net pollution emission, then the quota equilibrium consumption-production plan is Pareto Optimal among all feasible consumption-production plans with the same total net emissions. We show that every quota equilibrium can be realized as a tax equilibrium and vice versa. However, given specified tax rates, there may be no tax equilibrium consistent with those rates.

1. Introduction

The mitigation of climate change requires the reduction of greenhouse gas emissions. In the absence of governmental regulation, market mechanisms have proven insufficient in achieving the necessary reduction. Various regulatory schemes—notably carbon taxes and cap-and-trade—have been proposed. However, these regulatory schemes have yet to be incorporated into a general equilibrium model of the generality and rigor of the Arrow-Debreu Model [AD54]. In this paper, we do so.

Free-disposal equilibrium and non-free-disposal equilibrium are two classical equilibrium notions and they differ in the resource feasibility constraint. The demand is allowed to be less than or equal to the supply for free-disposal equilibrium while non-free-disposal equilibrium notion requires demand to be exactly equal to the supply for each commodity.[1] As pointed out in Hara [Har05], the free-disposal equilibrium

[1]While Arrow and Debreu [AD54] used free-disposal equilibrium as the equilibrium concept, McKenzie [McK81] used the non-free-disposal equilibrium as the equilibrium concept. See also Bergstrom [Ber76], Hart and Kuhn [HK75] and Polemarchakis and Siconolfi [PS93].

notion allows bads to be freely disposed and hence trivializes the problem of the efficient allocation of bads. Florenzano [Flo03] formulated the notion of a disposal cone, which allows the exogenous specification of certain commodities that can be freely disposed, and other goods that may not be disposed.

Florenzano's work is a substantial improvement on the pre-existing literature, but it has a significant limitation: while one can prohibit the emissions of a given pollutant, the cone formulation does not allow for setting a positive cap on those emissions. In Florenzano's formulation, a "complementary slackness" condition holds: the value of disposal at equilibrium of any good must be zero. Hence, it is not possible to charge a positive tax per unit of pollution. The complementary slackness condition is a consequence of budget balance in the Arrow-Debreu model. If a tax on pollution generates positive revenues, these revenues would evaporate from the model, leaving consumers without sufficient to buy the goods produced by the firms.

In this paper, we define a quota equilibrium in which a "government" sets global limits on pollution emissions, and charges firms for pollution emissions that aggregate up to those limits. The government also establishes a rebate scheme under which its revenues from the sale of pollution rights are distributed to consumers. This distribution ensures that the revenue stays in the model, so that the complementary slackness condition need no longer hold at equilibrium: firms pay a positive price to emit pollution. After having specified the pollution limits and the distribution scheme, the government refrains from further intervention. Equilibrium is determined, as in the Arrow-Debreu setting, by balancing supply and demand. The limits and the distribution scheme are exogenous to the market-clearing process.

We show the existence of a quota equilibrium at which the total net pollution emission is within the pre-specified quota, see Theorem 2.6. The existence is proven by constructing a closely-related fictitious economy that satisfies the assumptions of Florenzano's theorem. Specifically, we introduce a government-sponsored firm, equipped with a fictitious technology allowing it to costlessly dispose of pollution up to the set limits. In effect, the government-sponsored firm's technology allows it to sell the right to emit pollution. The government also assigns shares of the government-sponsored firm to the consumers, thus specifying how the revenues from the sale of pollution rights will be distributed. Florenzano's theorem implies the existence of an equilibrium for the fictitious economy, which is a quota equilibrium of our real economy.

The desire to regulate pollution emission is driven by an important externality, that a given amount of total emissions results in a corresponding increase in the average global temperature. If there are no other externalities in the economy, then a version of the First Welfare Theorem holds: taking the specified pollution limits as given, then for any distribution scheme, the resulting equilibrium consumption-production pair is Pareto optimal among the set of all feasible consumption-production pairs with the same total net emission, see Theorem 2.12, Corollary 2.14 and **??**. For a fixed distribution scheme, changing the quota alters the welfare of consumers, and it is possible that the equilibrium consumption-production pair for one quota may Pareto dominate the equilibrium consumption-production plan for a different quota, see Example 2.16. But once the government has established the quota, no further government intervention is required to achieve constrained Pareto Optimality through market forces. In particular, changing the distribution scheme alters the welfare of consumers, but it cannot result in a Pareto improvement.

We also define a tax equilibrium, in which the government sets tax rates on the net emissions of pollutants and a rebate scheme to distribute the tax revenues to consumers. After having specified the tax rates and the distribution scheme, the government refrains from further intervention. Equilibrium is determined, again, by balancing supply and demand. As in the case of quota equilibrium, the tax rates and the distribution scheme are exogenous to the market-clearing process.

We show that every quota equilibrium can be realized as a tax equilibrium with the same distribution scheme, and vice versa, see Theorem 3.3 and Theorem 3.4. However, if we start with a specified tax rate, there may be no tax equilibrium consistent with that tax rate, see Example 3.5. The specified tax rate ties down the relative price of pollution against other commodities, and there may not be enough degrees of freedom remaining to balance supply and demand in the commodities market.

We now conclude the introduction by laying out the plan of the paper for reader's convenience. In Section 2, we define a general equilibrium model for finite production economies with quota. The existence of a quota equilibrium is presented in Section 2.1, and we present a version of the First Welfare Theorem establishing the constrained Pareto optimality of the quota equilibrium consumption-production pair in Section 2.2. In Section 3, we define a general equilibrium model for finite production economies with tax. We show that every quota equilibrium can be realized as a tax equilibrium, and vice versa. However, we show with an example that there need not be a tax equilibrium consistent with a specified tax rate. Finally, in Appendix A.1, we present a technical result on the promotion of a quasi-equilibrium to an equilibrium.

2. Production Economy with Quota

In this section, we present a general equilibrium model that is compatible with the presence of bads and incorporates quota regulatory schemes on total net pollution emission, and introduce two equilibrium notions known as quota equilibrium and revenue-maximizing quota equilibrium. In particular, we generalize the usual feasibility constraints so that, at equilibrium, the total net pollution emission is under the prespecfied level. For revenue-maximizing quota equilibrium, we in addition require that the government maximizes its revenue from quota selling given the equilibrium price. We show that, given any quota on pollution, there exists a revenue-maximizing quota equilibrium. Moreover, we establish a welfare theorem for our equilibrium concept: Every quota equilibrium consumption-production pair is *constrained Pareto optimal*, that is, it is Pareto optimal among all feasible consumption-production pairs with the same total net pollution emission. Finally, as Example 2.16 indicates, while it is impossible to achieve full optimality for quota equilibrium by setting the quota, revenue-maximizing quota equilibrium can be full Pareto optimal if the government sets the right quota.

We start this section by introducing the following characterization of agents' preferences as presented in Hildenbrand [Hil74]:

Definition 2.1. The set $\mathcal{P}$ of preferences on an Euclidean space $\mathbb{R}^\ell$ consists of elements of the form $(X, \succ)$, where

- The *consumption set* $X \subset \mathbb{R}^\ell_{\geq 0}$ is closed and convex;
- $\succ$ is a continuous and irreflexive preference relation defined on X.

For every $(X, \succ) \in \mathcal{P}$ and $a, b \in X$, $a \succ b$ means a is strictly preferred to b. Note that we require neither completeness nor transitivity of $\succ$ in Definition 2.1. A preference $\succ$ on X is *continuous* if, for every $x, y \in X$ with $x \succ y$, there exist relatively open sets $U \ni x$ and $V \ni y$ such that $a \succ b$ for all $a \in U$ and $b \in V$. The set $\mathcal{P}$ is equipped with the topology of closed convergence, which makes it a compact metric space as indicated in Hildenbrand [Hil74]. For two elements $y_1, y_2 \in \mathbb{R}^\ell$, we have $(y_1, y_2) \in (X, \succ)$ if and only if $y_1, y_2 \in X$ and $y_1 \succ y_2$. A preference $P = (X, \succ)$ is *convex* if $\{y \in X : y \succ x\}$ is convex for every $x \in X$, and we use $\mathcal{P}_H$ to denote the set of convex preference from $\mathcal{P}$. Let $\Delta = \{p \in \mathbb{R}^\ell : \|p\| = \sum_{k=1}^{l} |p_k| = 1\}$ be the set of all prices. Note that we allow for negative prices which can be interpreted as fees for disposal of bads. We now give the formal definition of a finite production economy with quota.

Definition 2.2. A finite production economy with quota

$$\mathcal{E} \equiv \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$$

is a list such that

(i) Ω is a finite set of agents and J is a finite set of producers,

(ii) For every agent $\omega \in \Omega$, its consumption set $X(\omega)$ is a non-empty, closed and convex subset of $\mathbb{R}^\ell_{\geq 0}$. We sometimes write X_ω for $X(\omega)$,

(iii) $Y_j \subset \mathbb{R}^\ell$ is a non-empty subset denoting the production set of producer $j \in J$ with $Y = \prod_{j\in J} Y_j$,

(iv) the set of allocations is $\mathcal{A} = \prod_{\omega\in\Omega} X_\omega$ is equipped with the product topology,

(v) Let $M_\omega = \mathcal{A} \times Y \times \Delta \times X_\omega$ for every $\omega \in \Omega$. The *social preference relation* of agent ω is $\succ_\omega \subset M_\omega \times M_\omega$. For $m, m' \in M_\omega$, we write $m \succ_\omega m'$ to mean that the agent ω strictly prefers m over m'. The *preference map* of agent ω is a map $P_\omega : \mathcal{A} \times Y \times \Delta \to \mathscr{P}(X_\omega \times X_\omega)$ given by

$$P_\omega(x, y, p) = \{(a, b) \in X_\omega \times X_\omega | (x, y, p, a) \succ_\omega (x, y, p, b)\}.$$

For every $\omega \in \Omega$, P_ω satisfies:

- The range of P_ω is $\mathcal{P}$. By Definition 2.1, $P_\omega(x, y, p)$ can be written as $(X_\omega, \succ_{x,y,\omega,p})$;

- For $x, x' \in \mathcal{A}$ with $x(i) = x'(i)$ for all $i \neq \omega$, $P_\omega(x, y, p) = P_\omega(x', y, p)$ for all $(y, p) \in Y \times \Delta$;
- P_ω is continuous in the norm topology on $\mathcal{A} \times Y \times \Delta$,

(vi) $\theta(\omega)(j)$ is agent ω's share of firm j such that $\sum_{\omega\in\Omega} \theta(\omega)(j) = 1$ for all $\omega \in \Omega$. We sometimes write $\theta_{\omega j}$ for $\theta(\omega)(j)$,

(vii) $e \in (\mathbb{R}^\ell_{\geq 0})^\Omega$ is the initial endowment for each agent such that each coordinate of $\sum_{\omega\in\Omega} e(\omega)$ is positive,

(viii) let $k \leq \ell$ be some natural number and $m = (m_1, m_2, \ldots, m_k) \in \mathbb{R}^k_{\leq 0}$. The vector $-m \in \mathbb{R}^k_{\geq 0}$ denotes the vector of quotas for the first k commodities. The *disposal region* $\mathcal{Z}(m) = \prod_{n\leq \ell} \mathcal{Z}(m)_n$ is a convex subset of $\mathbb{R}^\ell_{\leq 0}$, where $\mathcal{Z}(m)_n = [m_n, 0]$ for all $n \leq k$ and $\mathcal{Z}(m)_n$ is either $\{0\}$ or $\mathbb{R}_{\leq 0}$ for $k < n \leq \ell$,

(ix) $\theta_0 \in \mathbb{R}^\Omega_{\geq 0}$ is the government's rebate share of agents with $\sum_{\omega\in\Omega} \theta_0(\omega) = 1$. The government rebates the revenue of quota selling to agents according to θ_0.

Remark 2.3. The two main features of our model are:

(1) Item (v) characterizes each agent's preference through the social preference relation $\succ_\omega$ and the preference map P_ω. The preference relation $\succ_\omega$ represents the agent's preference on all agents' consumption, production, prices and her own consumption. The agent, however, has no control over other agent's consumption, production and prices. Hence, given all other agent's choices, production and prices, the agent chooses her bundle according to the preference map P_ω. For the existence of equilibrium, one only need to work with the preference map P_ω. However, the preference relation $\succ_\omega$ is essential for studying welfare properties and potential Pareto improvement of consumption-production pairs.

(2) Our model incorporates a quota regulatory scheme on total net pollution emission by defining the disposal region $\mathcal{Z}(m)$, which reflects the society's choice on which commodities to dispose, and at what quantity. The society wishes to limit the total net emission of the first k commodities by setting quotas on them. The government distributes the revenue from selling the quotas to agents according to the rebate distribution scheme θ_0, which ensures that the revenue stays within the model.

The total net pollution emission depend on agents' consumption and the aggregated production. In particular, for any $(x, y) \in \mathcal{A} \times Y$ (not necessarily feasible), $C(x, y) = \pi_k\big(\sum_{\omega\in\Omega} e(\omega) + \sum_{j\in J} y(j) - \sum_{\omega\in\Omega} x(\omega)\big)$ is the total net emission of the first k commodities, where π_k is the projection map onto the first k coordinates. For $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the government's total revenue from selling the quota is $-(\pi_k(p)) \cdot C(x, y)$. For every $\omega \in \Omega$, $p \in \Delta$ and $(x, y) \in \mathcal{A} \times Y$, the *quota budget set* $B^m_\omega(x, y, p)$ is defined to be:

$$\left\{ z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} p \cdot y(j) - \theta_0(\omega)\pi_k(p) \cdot C(x, y) \right\}.$$

So an agent's budget consists of the value of her endowment, her dividend of firms and her rebate from the government's revenue of quota selling. Given $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the *quota demand set* $D^m_\omega(x, y, p)$ of agent ω consists of all elements in the quota budget set $B^m_\omega(x, y, p)$ that maximize the agent's preference given (x, y, p). In particular, $D^m_\omega(x, y, p)$ is given by:

$$\{z \in B^m_\omega(x, y, p) : w \succ_{x,y,\omega,p} z \implies p \cdot w > p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} p \cdot y(j) - \theta_0(\omega)\pi_k(p) \cdot C(x, y)\}.$$

For each $j \in J$, let $S_j(p) = \operatorname*{argmax}_{z\in Y_j} p \cdot z$ denote the (possibly empty) supply set at $p \in \Delta$. This implies that producers are profit maximizers and their profits depend only on prices and their own production.[2]

Free-disposal equilibrium and non-free-disposal equilibrium are two classical general equilibrium notions, and they differ only in the resource feasibility constraint. While free-disposal equilibrium only requires demand to be no more than the supply, non-free-disposal equilibrium requires demand to be exactly equal to the supply for each commodity. Free-disposal equilibrium allows for the bad to be freely disposed of and hence precludes any control on total net pollution emission. While non-free-disposal equilibrium is widely used in GE models with bads,[3] it does not allow one to limit net pollution emission to a positive amount. While it might be desirable to eliminate

[2]We assume producers are profit maximizers. Makarov [Mak81] established a general equilibrium existence theorem which allows for firm objectives other than profit maximization.

[3]McKenzie [McK59] used the non-free-disposal equilibrium with possible negative prices, which is followed by Bergstrom [Ber76], Hart and Kuhn [HK75], Polemarchakis and Siconolfi [PS93], and others.

CO_2 emissions at some future date, it is not practical to do so in the near term. Our equilibrium notion generalizes these two classical equilibrium notions by allowing for positive quotas on certain pollution emissions.

Definition 2.4. Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota. A $\mathcal{Z}(m)$-disposal quota equilibrium is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that the following conditions are satisfied:

(1) $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$;
(2) $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$. So every firm is profit maximizing given the price $\bar{p}$;
(3) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$.

A $\mathcal{Z}(m)$-disposal quota equilibrium $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-*disposal revenue-maximizing quota equilibrium* if, in addition, $-\pi_k(\bar{p}) \cdot C(\bar{x}, \bar{y}) \in \operatorname{argmax}_{z\in M} \pi_k(\bar{p}) \cdot z$, where $M = \prod_{n\leq k} \mathcal{Z}(m)_n$. That is, given the equilibrium price $\bar{p}$, the government maximizes its revenue from selling the quota.

Our feasibility constraint $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$ implies, at equilibrium, that total net emission of the first k commodities is within the pre-specified quota. Thus, the disposal region allows the society to choose which commodities are disposed, and in what quantities. Note that the government sets quota on the whole economy instead of individual firms, and the allocation of the quota among firms is determined endogeneously through market forces.

The astute reader may wonder about the motivation for introducing the notion of revenue-maximizing quota equilibrium. To address this question, we first observe that a quota equilibrium with respect to a smaller quota is also a quota equilibrium with respect to any larger quota. In particular, the following result follows easily from Definition 2.4:

Theorem 2.5. *Let $m \leq m' \in \mathbb{R}^k_{\leq 0}$. Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ and $\mathcal{E}' = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m'), \theta_0\}$. Then every $\mathcal{Z}(m')$-disposal quota equilibrium is a $\mathcal{Z}(m)$-disposal quota equilibrium.*

Proof. Let $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m')$-disposal quota equilibrium. Note that $B^m_\omega(\bar{x}, \bar{y}, \bar{p}) = B^{m'}_\omega(\bar{x}, \bar{y}, \bar{p})$. Hence, we have $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. Note that $\bar{y}(j) \in S_j(\bar{p})$

for all $j \in J$. Finally, we have

$$\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m') \subset \mathcal{Z}(m). \tag{2.1}$$

Hence, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium. □

Theorem 2.5 implies that each quota is, in general, associated with multiple equilibria with different total net pollution emission. Many of these equilibria are highly undesirable in the short run (e.g., quota equilibria with 0 quota), but formal definition of quota equilibrium does not rule them out. On the other hand, for revenue-maximizing quota equilibria, the government revenue-maximizing condition $-\pi_k(\bar{p}) \cdot C(\bar{x}, \bar{y}) \in \operatorname*{argmax}_{z\in M} \pi_k(\bar{p}) \cdot z$ is equivalent to the condition:

- For $n \leq k$, $\big(C(\bar{x}, \bar{y})\big)_n = 0$ if $\bar{p}_n > 0$, $\big(C(\bar{x}, \bar{y})\big)_n = -m_n$ if $\bar{p}_n < 0$, and $\big(C(\bar{x}, \bar{y})\big)_n \in [0, -m_n]$ if $\bar{p}_n = 0$,

which implies that the quota is binding for pollutants with negative equilibrium price. Hence, revenue-maximizing quota equilibrium reduces multiplicity of equilibria by getting rid of many equilibria that are either inplausible or undesirable.[4]

2.1. **Existence of Revenue-Maximizing Quota Equilibrium.** In this section, for any given quota, we establish the existence of a revenue-maximizing quota equilibrium for production economies with quota as defined in Definition 2.2, hence also establishing the existence of a quota equilibrium. Our result generalizes Proposition 3.2.3 in Florenzano [Flo03], in which the disposal region is a cone. As we have discussed in the introduction, while disposal cone allows the society to prohibit the emissions of a given pollutant, it does not allow for setting a positive cap on net pollution emissions. However, Proposition 3.2.3 in Florenzano [Flo03] plays a crucial role in establishing our main existence theorem, and a version of Proposition 3.2.3 tailored for our setting is proved in the Appendix A.3 (see Theorem A.3). The proof of the existence of revenue-maximizing quota equilibrium consists of the following major steps:

[4] As we will see in Example 2.16, revenue-maximizing quota equilibrium, in general, has better social welfare property than quota equilibrium.

(1) We construct a derived production economy with quota $\mathcal{E}'$ by introducing a fictitious firm which has the production technology of sequestering pollution up to the level specified by the given quota. The disposal region of $\mathcal{E}'$ is a cone;
(2) We apply Theorem A.3 to show that $\mathcal{E}'$ has a quota equilibrium;
(3) Finally, we show every quota equilibrium of $\mathcal{E}'$ is a revenue-maximizing quota equilibrium of $\mathcal{E}$.

We now state and prove our main existence result.

Theorem 2.6. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota as in Definition 2.2. Suppose $\mathcal{E}$ satisfies the following conditions:*

(i) for all $\omega \in \Omega$, we have $0 \in X_\omega$, P_ω takes value in $\mathcal{P}_H$ and $e_\omega \in \mathrm{int}(X_\omega - \sum_{j\in J} \theta_{\omega j} Y_j)$;
(ii) for all $\omega \in \Omega$, for each $(x, y) \in \mathcal{O}$ with $x_\omega \in X_\omega$, there is $u \in X_\omega$ such that $(u, x_\omega) \in \bigcap_{p\in\Delta\cap(\mathcal{Z}')^0} P_\omega(x, y, p)$, where $\mathcal{Z}' = \{z \in \mathcal{Z}(m) : (\forall n \le k)(z_n = 0)\}$ and $(\mathcal{Z}')^0 = \{p \in \Delta : (\forall z \in \mathcal{Z}')(p \cdot z \le 0)\}$ is the polar cone of $\mathcal{Z}'$;[5]
(iii) $\bar{Y}$ is closed, convex, and $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}^\ell_{\ge 0} = \{0\}$.

Then, there exists a $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium.

Proof. Note that Theorem A.3 is not directly applicable since $\mathcal{Z}(m)$ is not a cone. To overcome this difficulty, we introduce a fictitious firm 0 with the production set Y_0':

$$Y_0' = [m_1, 0] \times [m_2, 0] \times \ldots \times [m_k, 0] \times \{0\} \times \ldots \times \{0\} \subset \mathbb{R}^\ell_{\le 0}. \qquad (2.2)$$

Let $\mathcal{E}' = \{(X, R'_\omega, P'_\omega, e_\omega, \theta'_\omega)_{\omega\in\Omega}, (Y'_j)_{j\in J'}, \mathcal{Z}', \theta_0\}$ be a finite production economy with quota where:

(1) $J' = \{0, 1, 2, \ldots, J\}$ is the set of firms, which is the set of firms J of $\mathcal{E}$ plus the fictitious firm 0;
(2) $Y'_j = Y_j$ for all $j \in J$ and the production set Y'_0 for the fictitious firm 0 is defined as above. Let $Y' = \prod_{j\in J'} Y'_j$;

[5]This condition is a generalization of the classical non-satiation property on preferences. In fact, this condition is equivalent to the non-satiation property on preferences if the preferences have no externality. In the presence of externality, the choice of u depends on $\omega \in \Omega$ and $(x, y) \in \mathcal{O}$, but is independent of prices in $\Delta \cap (\mathcal{Z}')^0$.

(3) The agent's preference R'_ω and the induced preference map P'_ω are independent of the fictitious firm's production plan, hence are the same as R_ω and P_ω[6];
(4) $\theta'(j) = \theta(j)$ for all $j \in J$ and agents share $\theta'(0) \in \mathbb{R}^{\Omega}_{\geq 0}$ of the fictitious firm 0 is the government's rebate share θ_0;
(5) Let $\mathcal{Z}' = \{z \in \mathcal{Z}(m) : (\forall n \leq k)(z_n = 0)\}$.

To show that the derived economy $\mathcal{E}'$ has a $\mathcal{Z}'$-disposal equilibrium with quota, we must verify that $\mathcal{E}'$ satisfies the assumptions of Theorem A.3. It is easy to see that:

(1) As P'_ω is independent of the production plan of the fictitious firm 0 for all $\omega \in \Omega$, P'_ω takes value in $\mathcal{P}_H$ for all $\omega \in \Omega$. As $0 \in Y'_0$, we have $e(\omega) \in \operatorname{int}(X_\omega - \sum_{j \in J'} \theta'_{\omega j} Y'_j)$ for all $\omega \in \Omega$;
(2) It is clear that $\mathcal{Z}'_n = \{0\}$ for all $n \leq k$.

Claim 2.7. *For all $\omega \in \Omega$ and all feasible consumption-production pairs (x, y) of $\mathcal{E}'$ with $x(\omega) \in X_\omega$, there exists $u \in X_\omega$ so that $(u, x(\omega)) \in \bigcap_{p \in \Delta \cap (\mathcal{Z}')^0} P'_\omega(x, y, p)$.*

Proof. Fix $\omega \in \Omega$. Let (x, y) be a feasible consumption-production pair of $\mathcal{E}'$ with $x(\omega) \in X_\omega$. Let $y_{\mathcal{E}} = (y_1, y_2, \dots, y_J)$ be the production vector for firms in $\{1, 2, \dots, J\}$. Then $(x, y_{\mathcal{E}})$ is a feasible consumption-production pair of $\mathcal{E}$ with $x(\omega) \in X_\omega$. So there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta \cap (\mathcal{Z}')^0} P_\omega(x, y_{\mathcal{E}}, p)$. As $P'_\omega(x, y, p) = P_\omega(x, y_{\mathcal{E}}, p)$ for all $p \in \Delta$, we have $(u, x(\omega)) \in \bigcap_{p \in \Delta \cap (\mathcal{Z}')^0} P'_\omega(x, y, p)$. □

Claim 2.8. *Let $\bar{Y}' = \{\sum_{j \in J'} y(j) : y \in Y'\}$. Then, $\bar{Y}'$ is closed and convex, $\bar{Y}' \cap \mathbb{R}^{\ell}_{\geq 0} = \{0\}$ and the set $\hat{Y}'_j$ of feasible production plans is relatively compact for all $j \in J$.*

Proof. Note that $\bar{Y}' = Y'_0 + \bar{Y}$. As Y'_0 is compact and convex, $\bar{Y}'$ is closed and convex. As $0 \in Y'_0$, we have $0 \in \bar{Y}' \cap \mathbb{R}^{\ell}_{\geq 0}$. On the other hand, as $a \leq 0$ for all $a \in Y'_0$, we conclude that $\bar{Y}' \cap \mathbb{R}^{\ell}_{\geq 0} = \{0\}$. As $\bar{Y} \cap (-\bar{Y}) = \{0\}$, every feasible production set of $\mathcal{E}$ is relatively compact. As $\hat{Y}_j = \hat{Y}'_j$ for all $j \geq 1$, $\hat{Y}'_j$ is relatively compact for all $j \geq 1$. As Y'_0 is compact, $\hat{Y}'_0$ is relatively compact. □

By Theorem A.3 and Footnote 16, there is a $\mathcal{Z}'$-disposal quota equilibrium $(\bar{x}, \bar{y}, \bar{p})$ for $\mathcal{E}'$. We first show that $(\bar{x}, \bar{y}_{\mathcal{E}}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium for $\mathcal{E}$. It is easy to see that:

[6]As we will see, the fictitious firm's production plan at equilibrium is completely determined by agents' consumption and other firms' production plan.

(1) As each firm is profit maximizing in the economy $\mathcal{E}'$, each firm is profit maximizing in $\mathcal{E}$;

(2) As $\sum_{\omega\in\Omega}\bar{x}(\omega)-\sum_{\omega\in\Omega}e(\omega)-\sum_{j\in J'}\bar{y}(j)\in\mathcal{Z}'$, we have

$$\sum_{\omega\in\Omega}\bar{x}(\omega)-\sum_{\omega\in\Omega}e(\omega)-\sum_{j\in J}\bar{y}_{\mathcal{E}}(j) \tag{2.3}$$

$$=\sum_{\omega\in\Omega}\bar{x}(\omega)-\sum_{\omega\in\Omega}e(\omega)-\sum_{j\in J'}\bar{y}(j)+\bar{y}(0)\in\mathcal{Z}(m). \tag{2.4}$$

Claim 2.9. *$\bar{x}(\omega)\in D^m_\omega(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ for all $\omega\in\Omega$.*

Proof. Note that $\pi_k(\bar{y}(0))=\pi_k(\sum_{\omega\in\Omega}\bar{x}(\omega)-\sum_{\omega\in\Omega}e(\omega)-\sum_{j\in J}\bar{y}_{\mathcal{E}}(j))$. Hence, we have $\bar{p}\cdot\bar{y}(0)=-(\pi_k(\bar{p}))\cdot C(\bar{x},\bar{y}_{\mathcal{E}})$. Thus, for all $\omega\in\Omega$, the budget set under quota $B'_\omega(\bar{x},\bar{y},\bar{p})$ for agent ω of the economy $\mathcal{E}'$ can be written as:

$$\left\{z\in X_\omega:\bar{p}\cdot z\leq\bar{p}\cdot e(\omega)+\sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}_{\mathcal{E}}(j)-\theta_0(\omega)\pi_k(\bar{p})\cdot C(\bar{x},\bar{y}_{\mathcal{E}})\right\}, \tag{2.5}$$

which is the same as the budget set under quota $B^m_\omega(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ of the economy $\mathcal{E}$. As $P_\omega=P'_\omega$, for all $\omega\in\Omega$, the demand set under quota $D'_\omega(\bar{x},\bar{y},\bar{p})$ for agent ω of the economy $\mathcal{E}'$ is the same as the demand set under quota $D^m_\omega(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ of the economy $\mathcal{E}$. Thus, we conclude that $\bar{x}(\omega)\in D^m_\omega(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ for all $\omega\in\Omega$. □

By Claim 2.9, $(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ is a $\mathcal{Z}$-disposal quota equilibrium. Note that the fictitious firm is profit maximizing. Thus, for $n\leq k$, the fictitious firm's production for commodity n is m_n if $\bar{p}_n<0$, is 0 if $\bar{p}_n>0$, and can be anything in $[m_n,0]$ if $\bar{p}_n=0$, which implies that $-\pi_k(\bar{p})\cdot C(\bar{x},\bar{y})\in\operatorname*{argmax}_{z\in M}\pi_k(\bar{p})\cdot z$, where $M=\prod_{n\leq k}\mathcal{Z}(m)_n$. Hence, $(\bar{x},\bar{y}_{\mathcal{E}},\bar{p})$ is a $\mathcal{Z}$-disposal revenue-maximizing quota equilibrium. □

Theorem 2.6 shows that the production economy with quota model, defined in Definition 2.2, has a revenue-maximizing quota equilibrium under moderate assumptions. At any such equilibrium, the total net emission of the first k commodities is under the pre-specified quota, and the government is maximizing its revenue from quota selling given the equilibrium price, which implies that all quotas are binding for commodities with a negative equilibrium price. As we will see in the next sub-section, binding quotas at equilibrium yields more desirable welfare properties of the equilibrium.

2.2. **Welfare Theorem for Production Economies with Quota.** To compare a house in Miami with a house in Minneapolis, the net CO_2 emission is an important factor to consider since net CO_2 affects the temperature. In this section, we focus on this specific type of externality and investigate the welfare property of quota equilibrium and revenue-maximizing quota equilibrium. In particular, we assume that the only externality arises from the total net emission of the first k commodities and establish a version of the first welfare theorem, which shows that every quota equilibrium consumption-production pair is Pareto optimal among all feasible consumption-production pairs with the same total net emission of the first k commodities. As it is possible for a quota equilibrium consumption-production pair to Pareto dominates another quota equilibrium consumption-production pair with a different total net emission of the first k commodities, a quota equilibrium consumption-production pair is, in general, not full Pareto optimal among all feasible consumption-production pairs. Moreover, as indicated by Theorem 2.5, a pre-specified quota is usually associated with multiple quota equilibria with possibly different total net emission of the first k commodities, so it is impossible to guarantee full Pareto optimality among quota equilibria by setting a quota. On the other hand, as revenue-maximizing quota equilibrium implies that the quota is binding if the equilibrium price of a commodity is negative, one can achieve full Pareto optimality among revenue-maximizing quota equilibria by if the government sets the right quota, provided that the equilibrium price is negative for the first k commodities. The welfare property comparison between quota equilibria and revenue-maximizing quota equilibria is illustrated in Example 2.16.

Agents' preference exhibit externality, which needs to be taken into account in defining Pareto domination. Recall that $C(f, y)$ denote the total net emission of the first k commodities of the consumption-production pair $(f, y) \in \mathcal{A} \times Y$. As the only externality arises from the total net pollution emission, an agent ω's social preference relation $\succ_\omega$ is a preference defined on $X \times E$, where $E = \pi_k\big(\sum_{\omega\in\Omega} e(\omega) + \bar{Y} - \bar{X}\big)$[7] denote the set of all possible total net emission of the first k commodities. We define *Pareto domination* and *full Pareto optimality* as:

[7] $\bar{X} = \sum_{\omega\in\Omega} X_\omega$ is the aggregated consumption set.

Definition 2.10. For two feasible consumption-production pairs $(f, y), (f', y') \in \mathcal{A} \times Y$, we say (f, y) *Pareto dominates* (f', y') if:

- for all $\omega \in \Omega$, $(f'(\omega), C(f', y')) \not\succ_\omega (f(\omega), C(f, y))$;
- there exists some $\omega_0 \in \Omega$ such that $(f(\omega_0), C(f, y)) \succ_{\omega_0} (f'(\omega_0), C(f', y'))$.

A consumption-production pair (g, h) *strongly Pareto dominates* another consumption-production pair (g', h') if $(g(\omega), C(g, h)) \succ_\omega (g'(\omega), C(g', h'))$ for all $\omega \in \Omega$. A consumption-production pair (f, y) is *(weakly) Pareto optimal* among $F \subset \mathcal{A} \times Y$ if no consumption-production pair in F (strongly) Pareto dominates (f, y). A consumption-production (f, y) is *(weakly) full Pareto optimal* if it is (weakly) Pareto optimal among all feasible consumption-production pairs.

It is too much to hope that quota equilibria are full Pareto optimal for any given quota. After all, the quota equilibrium with 0 quota yields an immediate return to a pre-industrial society. Therefore, we need to introduce a weakened notion of Pareto optimality to better characterize the social welfare property of quota equilibria. For two consumption-production pairs $(f, y), (f', y') \in \mathcal{A} \times Y$, we say (f, y) is *equivalent in total net emission* to (f', y') and write $(f, y) \sim_{\text{total}} (f', y')$ if $C(f, y) = C(f', y')$. It is easy to verify that $\sim_{\text{total}}$ is an equivalent relation on $\mathcal{A} \times Y$. For $v \in E$, let $[v]_{\text{total}}$ be the set of $(f, y) \in \mathcal{A} \times Y$ with $C(f, y) = v$.[8] As we assume that the only externality arises from the total net emission of the first k commodities, for all $v \in E$, all $(f, y) \in [v]_{\text{total}}$ and all $\omega \in \Omega$, we can write $P_\omega(v)$ to denote $P_\omega(f, y, p)$ for all $p \in \Delta$. Hence, for two consumption-production pairs $(f, y), (f', y') \in [v]_{\text{total}}$, the consumption-production pair (f, y) Pareto dominates (f', y') if:

- for all $\omega \in \Omega$, $\big(f'(\omega), f(\omega)\big) \notin P_\omega(v)$;
- there exists some $\omega_0 \in \Omega$ such that $\big(f(\omega_0), f'(\omega_0)\big) \in P_{\omega_0}(v)$.

Similarly, (f, y) strongly Pareto dominates (f', y') if $\big(f(\omega), f'(\omega)\big) \in P_\omega(v)$ for all $\omega \in \Omega$. We now formally present the weakened definition of Pareto optimality.

Definition 2.11. A feasible consumption-production pair $(f, y) \in \mathcal{A} \times Y$ is *(weakly) constrained Pareto optimal* if there is no feasible $(g, h) \in [C(f, y)]_{\text{total}}$ (strongly) Pareto dominates (f, y).

[8] In other words, $[v]_{\text{total}}$ is the set of consumption-production pairs such that the total net emission of the first k commodities is v.

Our next result shows that any quota equilibrium consumption-production pair is constrained Pareto optimal among.

Theorem 2.12. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota, and the only externality arises from the total net emission of the first k commodities. Let $(\bar{f}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-disposal quota equilibrium. Then:*

1. *$(\bar{f}, \bar{y})$ is weakly Pareto optimal among $[C(\bar{f}, \bar{y})]_{\mathrm{total}}$, i.e., $(\bar{f}, \bar{y})$ is weakly Pareto optimal among all feasible production-consumption pairs with the same total net emission of the first k commodities;*
2. *Suppose $P_\omega(v)$ is negatively transitive and locally non-satiated for all $\omega \in \Omega$ and all $v \in \pi_k(\sum_{\omega\in\Omega} e(\omega) + \bar{Y} - \bar{X})$. Then $(\bar{f}, \bar{y})$ is Pareto optimal among $[C(\bar{f}, \bar{y})]_{\mathrm{total}}$, i.e., $(\bar{f}, \bar{y})$ is Pareto optimal among all feasible production-consumption pairs with the same total net emission of the first k commodities.*

Proof. Suppose there exists some feasible $(\hat{f}, \hat{y}) \in [C(\bar{f}, \bar{y})]_{\mathrm{total}}$ that strongly Pareto dominates $(\bar{f}, \bar{y})$. Then, we have $\big(\hat{f}(\omega), \bar{f}(\omega)\big) \in P_\omega\big(C(\bar{f}, \bar{y})\big)$ for all $\omega \in \Omega$. As $(\bar{f}, \bar{y}, \bar{p})$ is a $\mathcal{Z}$-disposal quota equilibrium, we have

$$\begin{aligned}\bar{p} \cdot \hat{f}(\omega) &> \bar{p} \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j) - \theta_0(\omega)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y}) \\ &\geq \bar{p} \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} \bar{p} \cdot \hat{y}(j) - \theta_0(\omega)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y})\end{aligned}$$

for all $\omega \in \Omega$. Thus, we have

$$\bar{p}\big(\sum_{\omega\in\Omega} \hat{f}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j)\big) > -\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y}).$$

As $(\hat{f}, \hat{y})$ is feasible, we know that $\sum_{\omega\in\Omega} \hat{f}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j) \in \mathcal{Z}(m)$. Note that $\bar{p}_n \geq 0$ for all $n > k$ with $\mathcal{Z}_n = \mathbb{R}_{\leq 0}$. As $(\hat{f}, \hat{y}) \in [C(\bar{f}, \bar{y})]_{\mathrm{total}}$, we have

$$\bar{p}\big(\sum_{\omega\in\Omega} \hat{f}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j)\big) \leq -\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y}),$$

which leads to a contradiction. Hence $(\bar{f}, \bar{y})$ is weakly Pareto optimal among $[C(\bar{f}, \bar{y})]_{\mathrm{total}}$.

We now show that $(\bar{f}, \bar{y})$ is Pareto optimal among $[C(\bar{f}, \bar{y})]_{\mathrm{total}}$ if $P_\omega(v)$ is negatively transitive and locally non-satiated for all $\omega \in \Omega$ and all $v \in \pi_k(\sum_{\omega\in\Omega} e(\omega) + \bar{Y} - \bar{X})$. Suppose there exists some feasible $(\hat{f}, \hat{y}) \in [C(\bar{f}, \bar{y})]_{\mathrm{total}}$ that Pareto dominates $(\bar{f}, \bar{y})$.

Then there exists some $\omega_0 \in \Omega$ such that $\big(\hat{f}(\omega_0), \bar{f}(\omega_0)\big) \in P_{\omega_0}\big(C(\bar{f}, \bar{y})\big)$. As $(\bar{f}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium, we have:

$$\bar{p} \cdot \hat{f}(\omega_0) > \bar{p} \cdot e(\omega_0) + \sum_{j \in J} \theta_{\omega_0 j} \bar{p} \cdot \hat{y}(j) - \theta_0(\omega_0)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y}).$$

To complete the proof, we need the following key result:

Claim 2.13. $\bar{p} \cdot \hat{f}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j) - \theta_0(\omega)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y})$ *for all* $\omega \in \Omega$.

Proof. Suppose there exists some $\omega_1 \in \Omega$ such that

$$\bar{p} \cdot \hat{f}(\omega_1) < \bar{p} \cdot e(\omega_1) + \sum_{j \in J} \theta_{\omega_1 j} \bar{p} \cdot \bar{y}(j) - \theta_0(\omega_1)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y}).$$

As $P_{\omega_1}\big(C(\bar{f}, \bar{y})\big)$ is locally non-satiated, then there exists some $u \in X_{\omega_1}$ such that $\big(u, \hat{f}(\omega_1)\big) \in P_{\omega_1}\big(C(\bar{f}, \bar{y})\big)$ and $\bar{p} \cdot u < \bar{p} \cdot e(\omega_1) + \sum_{j \in J} \theta_{\omega_1 j} \bar{p} \cdot \bar{y}(j) - \theta_0(\omega_1)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y})$. As $P_{\omega_1}\big(C(\bar{f}, \bar{y})\big)$ is negatively transitive, we have $\big(u, \bar{f}(\omega_1)\big) \in P_{\omega_1}\big(C(\bar{f}, \bar{y})\big)$. This leads to a contradiction since $(\bar{f}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium. $\square$

By Claim 2.13, we have $\bar{p} \cdot \hat{f}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \hat{y}(j) - \theta_0(\omega)\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y})$ for all $\omega \in \Omega$. So we have $\bar{p}\big(\sum_{\omega \in \Omega} \hat{f}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \hat{y}(j)\big) > -\pi_k(\bar{p}) \cdot C(\bar{f}, \bar{y})$. By the same argument as in the first paragraph, we conclude that $(\bar{f}, \bar{y})$ is Pareto optimal among $[C(\bar{f}, \bar{y})]_{\text{total}}$. $\square$

Theorem 2.12 shows that, after the quota is set, constrained Pareto optimality of the equilibrium consumption-production pair is achieved without further intervention from the government. As a revenue-maximizing quota equilibrium is a quota equilibrium, Theorem 2.12 is valid for revenue-maximizing quota equilibria. If in addition, we assume that agents can not consume any of the first k commodities,[9] then, since endowments are fixed, the total net emissions of the first k commodities depend only on the production. The following result is similar to Theorem 2.12 except that the total net emissions of the first k commodities are replaced by the total net production of the first k commodities.

[9] In the classical general equilibrium model developed Arrow and Debreu [AD54], equilibrium assigns ownership which conveys the right to consume the commodity, but does not entail the obligation to dispose of it, which we believe is not the right interpretation for the consumption of bads. As consumers may not have means to dispose certain type of pollutants such as CO_2, it seems reasonable to assume agents can not consume these commodities at all. (!) Bob will modify this footnote.

Corollary 2.14. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota, and the only externality arises from the total net emission of the first k commodities. Suppose $\pi_k(X_\omega) = \{0\}$ for all $\omega \in \Omega$. Let $(\bar{f}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium. Then:*

(1) *$(\bar{f}, \bar{y})$ is weakly Pareto optimal among all feasible (f, y) with $\pi_k\big(\sum_{j\in J} y(j)\big) = \pi_k\big(\sum_{j\in J} \bar{y}(j)\big)$, i.e., $(\bar{f}, \bar{y})$ is weakly Pareto optimal among all feasible production-consumption pairs with the same total production of the first k commodities;*

(2) *Suppose $P_\omega(v)$ is negatively transitive and locally non-satiated for all $\omega \in \Omega$ and all $v \in \pi_k\big(\sum_{\omega\in\Omega} e(\omega) + \bar{Y}\big)$. Then $(\bar{f}, \bar{y})$ is Pareto optimal among all feasible (f, y) with $\pi_k\big(\sum_{j\in J} y(j)\big) = \pi_k\big(\sum_{j\in J} \bar{y}(j)\big)$, i.e., $(\bar{f}, \bar{y})$ is Pareto optimal among all feasible production-consumption pairs with the same total production of the first k commodities.*

Corollary 2.14 is also valid for revenue-maximizing quota equilibrium. Since the total net emission of the first k commodities is likely to affect agents' preferences, there may be a Pareto ranking among (revenue-maximizing) quota equilibrium with different total net emission of the first k commodities.

2.2.1. *Full Pareto Optimality.* When the only externality arises from the total net emission of the first k commodities, an agent ω's social preference relation $\succ_\omega$ is a preference defined on $X \times E$, where $E = \pi_k\big(\sum_{\omega\in\Omega} e(\omega) + \bar{Y} - \bar{X}\big)$. Given $(f, y), (f', y') \in \mathcal{A} \times Y$, we say (f, y) Pareto dominates (f', y') if:

- for all $\omega \in \Omega$, $(f'(\omega), C(f', y')) \not\succ_\omega (f(\omega), C(f, y))$;
- there exists some $\omega_0 \in \Omega$ such that $(f(\omega_0), C(f, y)) \succ_{\omega_0} (f'(\omega_0), C(f', y'))$.

A consumption-production pair (g, h) strongly Pareto dominates another consumption-production pair (g', h') if $(g(\omega), C(g, h)) \succ_\omega (g'(\omega), C(g', h'))$ for all $\omega \in \Omega$. A consumption-production pair (f, y) is (weakly) Pareto optimal among $F \subset \mathcal{A} \times Y$ if no consumption-production pair in F (strongly) Pareto dominates (f, y). A consumption-production (f, y) is (weakly) full Pareto optimal if it is (weakly) Pareto optimal among all feasible consumption-production pairs.

Theorem 2.15. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota, and the only externality arises from the total net emission* *of the first k commodities. Suppose $\succ_\omega$ is continuous and negatively transitive for all $\omega \in \Omega$. Suppose there exists a non-empty $Q \subset \mathbb{R}^k_{\leq 0}$ such that*

(1) *for any $m \in Q$, if p is the equilibrium price for a $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium, then every coordinate of $\pi_k(p)$ is negative;*
(2) *The set $\mathcal{Q} = \{(g, z) \in \mathcal{A} \times Y : (g, z)$ is feasible and $C(g, z) \in -Q\}$ is compact;*
(3) *If $(f, y) \in \mathcal{Q}$ is Pareto optimal among $\mathcal{Q}$, then (f, y) is full Pareto optimal.*

We can then conclude that:

(1) *There exists a $(\bar{f}, \bar{y}) \in \mathcal{Q}$ that is Pareto optimal among $\mathcal{Q}$;*
(2) *For all $m_0 \in Q$ where there exists a $(f, y) \in \mathcal{Q}$ such that (f, y) is Pareto optimal among $\mathcal{Q}$ and $C(f, y) = -m_0$, every $\mathcal{Z}(m_0)$-disposal revenue-maximizing quota equilibrium is full weakly Pareto optimal. If we assume, in addition, there is a single consumer, then every $\mathcal{Z}(m_0)$-disposal revenue-maximizing quota equilibrium is full Pareto optimal.*

Proof. As $\mathcal{Q}$ is compact, by Lemma A.5, there exists a $(\bar{f}, \bar{y}) \in \mathcal{Q}$ that is Pareto optimal among $\mathcal{Q}$, hence is full Pareto optimal. Let $m_0 = -C(\bar{f}, \bar{y})$ and (g, s) be a $\mathcal{Z}(m_0)$-disposal revenue-maximizing quota equilibrium consumption-production pair. As each of the first k coordinates of the equilibrium price associated with (g, s) is negative, we have $C(g, s) = -m_0$. By Theorem 2.12, (g, s) is Pareto optimal among all feasible consumption-production pairs with total net pollution emission being $-m_0$. Hence, (g, s) is weakly full Pareto optimal. If Ω is a singleton, then the notion of full Pareto optimality is the same as weakly full Pareto optimality, which implies that (g, s) is full Pareto optimal. □

Theorem 2.15 shows that full weak Pareto optimality can be achieved through market forces if the government sets the right quota. The motivations of these conditions are:

(1) The set Q represents the collection of *acceptable quotas* for the government. For all quotas in Q, the equilibrium prices of pollution determined by the market are negative. Thus, the economic interpretation of the first condition is: if the government sets a sufficiently low quota, the equilibrium prices of pollution emissions are negative. Although this condition is endogeneous, it

is satisfied, and easy to check, in many economic scenarios, as we will see in Example 2.16, Example 3.5 and Example 3.9;

(2) The second condition is similar to assuming compactness of the set of feasible allocations, as in Debreu [Deb59]. If each firm's production set is closed, for production economies satisfying the assumptions of Theorem 2.6, this condition is satisfied;

(3) The purpose for pollution control is to improve the overall welfare of the society. The third condition asserts that, if a consumption-production pair is Pareto optimal among all consumption-production pairs under acceptable quotas (quotas which are elements of Q), then it is full Pareto optimal. This condition reflects that pollution control at least does not make the society's welfare worse off.

We conclude this section with the following example which demonstrates the applicability of Theorem 2.15. The example also illustrates the difference of welfare properties between the quota equilibrium and revenue-maximizing quota equilibrium. In particular, we show that revenue-maximizing quota equilibrium is full Pareto optimal for a carefully chosen quota. However, one can not achieve full Pareto optimality for quota equilibrium by setting the quota alone.

Example 2.16. Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota:

(1) The economy $\mathcal{E}$ has three commodities CO_2, coal and electricity, which we denote by c_1, c_2 and c_3;

(2) There is a single agent with consumption set $X = \{0\} \times \mathbb{R}^2_{\geq 0}$ and endowment $e = (0, 1, 0)$. Given the total net emission v of CO_2, the utility function $u_v(c_1, c_2, c_3) = c_3 - v^2$;

(3) There are two producers with production sets $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$ and $Y_2 = \{(-2r, 0, -r) : r \in \mathbb{R}_{\geq 0}\}$. So the first producer has the production technology to burn r units of coal to generate r units of electricity and r units of CO_2 as byproduct. The second producer has the production technology to use r units of electricity to sequester $2r$ units of CO_2;

(4) The disposal region $\mathcal{Z}(m) = [m, 0] \times \mathbb{R}^2_{\leq 0}$ for $m \in [-1, 0]$;

(5) Since there is only one agent, we have $\theta_\omega = \theta_0(\omega) = 1$.

It is clear that $\bar{Y} = Y_1 + Y_2$ is a convex and closed set. For every $y \in \bar{Y}$, we have $y_2 \leq 0$, and if $y_2 = 0$ then $y_1, y_3 \leq 0$. Hence, we conclude that $\bar{Y} \cap \mathbb{R}^3_{\geq 0} = \{0\} = \bar{Y} \cap (-\bar{Y})$. It can be verified that $e \in \text{int}(X - \bar{Y})$.[10] So $\mathcal{E}$ satisfies the conditions of Theorem 2.6 for all $m \in [-1, 0]$, which implies that $\mathcal{E}$ has a $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium $(\tilde{x}_m, \tilde{y}_m, \tilde{p}_m)$ for all $m \in [-1, 0]$. The total net emission of CO_2 is bounded by $-m$ for all $\mathcal{Z}(m)$-disposal quota equilibrium. We now verify all the assumptions of Theorem 2.15 for this example.

Claim 2.17. *If $\bar{p}$ is an equilibrium price for a $\mathcal{Z}(m)$-disposal quota equilibrium with $m > -1$, then $\bar{p}_1 < 0$.*

Proof. Suppose $\bar{p}_1 \geq 0$. By the form of the agent's utility function, we know that $\bar{p}_3 > 0$, which implies that the equilibrium production plan for the second firm is $(0, 0, 0)$. Note that $\bar{p}_2 \geq \bar{p}_3$ since the first firm's profit will be unbounded otherwise. As the agent would sell all its coal for electricity, the equilibrium production plan for the first firm is $(1, -1, 1)$, which implies that the total net CO_2 emission is 1. □

Let $Q = [-0.9, 0]$. By Claim 2.17, the first assumption of Theorem 2.15 is satisfied. By Proposition 2.2.4 of Florenzano [Flo03] and the fact that Q is compact, the set $\mathcal{Q} = \{(g, z) \in X \times Y : (g, z) \text{ is feasible and } C(g, z) \in [0, 0.9]\}$ is compact.[11] So the second assumption of Theorem 2.15 is satisfied. It remains to validate the third assumption of Theorem 2.15. To do so, we first compute a (revenue-maximizing) quota equilibrium for any total net emission of CO_2 that is no greater than 1.

Claim 2.18. *Let $\hat{p} = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. For every total net CO_2 emission level $0 \leq v \leq 1$, let $\hat{x}_v = (0, 0, \frac{1+v}{2})$ and $\hat{y}_v = \big((1, -1, 1), (v - 1, 0, \frac{v-1}{2})\big)$. Then $(\hat{x}_v, \hat{y}_v, \hat{p})$ is a $\mathcal{Z}(-v)$-disposal revenue-maximizing quota equilibrium.*

[10] In fact, we have $X - \bar{Y} = \{(a, b, c) | |a| \leq b\}$ which implies that $e \in \text{int}(X - \bar{Y})$.

[11] Proposition 2.2.4 of Florenzano [Flo03] states that every feasible production set for a single firm is relatively compact. Since the production sets in this example are compact, the feasible production set for both firms are compact, which further implies that the total feasible production set is compact. Proposition 2.2.4 of Florenzano [Flo03] also states that the feasible consumption set is compact. The set $\mathcal{Q}$ is a closed subset of the product of feasible consumption set and the total feasible production set, hence is compact.

Proof. Pick $0 \leq v \leq 1$. Note that $\hat{x}_v - e - \sum_{j\in J} \hat{y}_v(j) = (-v, 0, 0) \in \mathcal{Z}(-v)$. Both firms are profit maximizing. The quota budget set for the agent is $\{z \in X : \hat{p} \cdot z \leq \frac{1}{4}(1+v)\}$. Hence, $\hat{x}_v$ is an element of the quota demand set $D^{-v}(\hat{x}_v, \hat{y}_v, \hat{p})$, which implies that $(\hat{x}_v, \hat{y}_v, \hat{p})$ is a $\mathcal{Z}(-v)$-disposal quota equilibrium. As $\hat{p}_1 = -\frac{1}{4} < 0$, $(\hat{x}_v, \hat{y}_v, \hat{p})$ is a $\mathcal{Z}(-v)$-disposal revenue-maximizing quota equilibrium. □

The total net CO_2 emission of $(\hat{x}_v, \hat{y}_v, \hat{p})$ is v. Moreover, by Theorem 2.12, $(\hat{x}_v, \hat{y}_v)$ is constrained Pareto optimal, i.e., $(\hat{x}_v, \hat{y}_v)$ is Pareto optimal among all feasible production-consumption pairs such that the total net CO_2 emission is v.

Let $\mathcal{V} = \{(\hat{x}_v, \hat{y}_v) : v \in [0, 0.9]\}$. The agent's utility among $\mathcal{V}$, as a function of the total net emission v of CO_2, is $\frac{1+v}{2} - v^2$. By taking the derivative, the total net CO_2 emission that maximizes the agent's utility is at $\hat{v} = \frac{1}{4}$, with the agent's utility being $\frac{9}{16}$. Note that $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}}, \hat{p})$ is a $\mathcal{Z}(-\frac{1}{4})$-disposal revenue-maximizing quota equilibrium, and the equilibrium consumption-production pair $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$ Pareto dominates $(\hat{x}_v, \hat{y}_v)$ for all $v \in [0, 1]$ such that $v \neq \frac{1}{4}$. Let $(\bar{f}, \bar{y}) \in X \times Y$ be feasible and Pareto optimal among $\mathcal{Q}$. Then the agent's utility at $(\bar{f}, \bar{y})$ must be no less than $\frac{9}{16}$ since $\mathcal{V} \subset \mathcal{Q}$. On the other hand, if $(f, y) \notin \mathcal{Q}$ is feasible, the agent's utility at (f, y) is bounded up by $1 - 0.9^2 = 0.19$, which is less than $\frac{9}{16}$. Hence, $(\bar{f}, \bar{y})$ is Pareto optimal among all feasible consumption-production pairs. Hence, by Claim 2.17, the fact that there is only one consumer and Theorem 2.15, there exists $m_0 \in Q$ such that every $\mathcal{Z}(m_0)$-disposal revenue-maximizing quota equilibrium is full Pareto optimal.

We now explicitly compute the quota m_0 on CO_2 that would lead to full Pareto optimality of $\mathcal{Z}(m_0)$-disposal revenue-maximizing quota equilibrium. By Theorem 2.12, $(\hat{x}_v, \hat{y}_v)$ is constrained Pareto optimal for all $v \in [0, 0.9]$. Thus, $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$ is weakly Pareto optimal among $\mathcal{Q}$. Since the agent space is a singleton, $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$ is Pareto optimal among $\mathcal{Q}$, hence is full Pareto optimal. Thus, every $\mathcal{Z}(-\frac{1}{4})$-disposal revenue-maximizing quota equilibrium consumption-production pair is Pareto optimal among all feasible consumption-production pairs. In addition, as $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$ is the only Pareto optimal consumption-production pair in $\mathcal{V}$, $\frac{1}{4}$ is the only quota such that the associated revenue-maximizing quota equilibrium is full Pareto optimal. As a result, after the government sets the quota to be $\frac{1}{4}$, full Pareto optimality of the equilibrium consumption-production pair is achieved without further intervention from the government.

On the other hand, it is impossible to guarantee full Pareto optimality for quota equilibrium. By Theorem 2.5, any quota m is associated with with multiple $\mathcal{Z}(m)$-disposal quota equilibrium with different total net CO_2 emission, but at most one of these quota equilibrium is Pareto optimal in this example. In particular, if the government sets the quota $m > -\frac{1}{4}$, the resulting $\mathcal{Z}(m)$-disposal quota equilibrium consumption-production pair is Pareto dominated by $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$. If the government sets the quota $m \leq -\frac{1}{4}$, any $\mathcal{Z}(m)$-disposal quota equilibrium consumption-production pair (f, y) such that $C(f, y) \neq \frac{1}{4}$ is Pareto dominated by $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$. Thus, one can not achieve full Pareto optimality for quota equilibria by setting the quota alone.

3. Production Economy with Tax

An alternative to setting quotas is to set tax rates on pollution emissions. In this section, we present a general equilibrium model that incorporates a tax regulatory scheme on net pollution emissions, and focus on its connection with quota equilibrium. While quota equilibrium always exists and is effective in limiting the total net pollution emission under the quota, we demonstrate in Theorem 3.3, Theorem 3.4 and Example 3.5 that:

(1) Every quota equilibrium is a tax equilibrium and vice versa;
(2) Given specific tax rates, there may be no tax equilibrium consistent with those tax rates;
(3) Even if an emission tax equilibrium exists for a given tax rate, there may be multiple equilibria and there is no guarantee that emissions will lie under a pre-specified level of total net pollution emissions for every equilibrium.

On the other hand, while quota equilibrium is, in general, only constrained Pareto optimal, Example 3.9 shows that it might be possible to achieve full Pareto optimality through an emission tax.

In Section 3.1, we consider an alternative formulation of a general equilibrium model with tax, which imposes add-on taxes on commodities that generate pollution as a byproduct in the production process. We demonstrate in Example 3.11 that add-on tax equilibrium is Pareto dominated by the Pareto optimal emission tax equilibrium, indicating that an emission tax may be more effective in achieving Pareto optimality.

We start by giving a rigorous description of a general equilibrium model that incorporates tax on emission of pollution:

Definition 3.1. A finite production economy with emission tax

$$\mathcal{F} \equiv \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \theta_0\}$$

is a list such that

(i) $(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}$, $(Y_j)_{j\in J}$ and θ_0 are the same as in Definition 2.2;

(ii) The disposal region $\mathcal{V}$ takes the form of $\prod_{n\leq\ell}\mathcal{V}_n$ where $\mathcal{V}_n = \mathbb{R}_{\leq 0}$ for all $n \leq k$ and $\mathcal{V}_n$ is either $\{0\}$ or $\mathbb{R}_{\leq 0}$ for all $n > k$;

(iii) $t \in \pi_k(\Delta)$ is an emission tax rate on the net emission of the first k commodities.

Recall that $C(x, y) = \pi_k\big(\sum_{\omega\in\Omega} e(\omega) + \sum_{j\in J} y(j) - \sum_{\omega\in\Omega} x(\omega)\big)$ is the total net emission of the first k commodities. For every $\omega \in \Omega$, $p \in \Delta$ and $(x, y) \in \mathcal{A} \times Y$, the *emission tax budget set* $B^t_\omega(x, y, p)$ is defined to be:

$$\{z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j\in J}\theta_{\omega j} p \cdot y(j) + \theta_0(\omega) t \cdot C(x, y)\}.$$

So an agent's budget consists of the value of her endowment, her dividend of firms and her rebate from the government's emission tax revenue. For each $\omega \in \Omega$ and $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the *emission tax demand set* $D^t_\omega(x, y, p)$ consists of all elements in the emission tax budget set $B^t_\omega(x, y, p)$ that maximize the agent's preference given (x, y, p). In particular, $D^t_\omega(x, y, p)$ is defined as:

$$\{z \in B^t_\omega(y, p) : w \succ_{x,y,\omega,p} z \implies p \cdot w > p \cdot e(\omega) + \sum_{j\in J}\theta_{\omega j} p \cdot y(j) + \theta_0(\omega) t \cdot C(x, y)\}.$$

The equilibrium notion for finite production economies with emission tax is:

Definition 3.2. Let $\mathcal{F} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \theta_0\}$ be a finite production economy with emission tax. A $\mathcal{V}$-disposal emission tax equilibrium is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that the following conditions are satisfied:

(1) $\pi_k(\bar{p}) = -t$;
(2) $\bar{x}(\omega) \in D^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$;
(3) $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$;
(4) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{V}$.

As $\mathcal{V}_n = \mathbb{R}_{\leq 0}$ for all $n \leq k$, we allow for arbitrary net emission for the first k commodities. On the other hand, the government charges an emission tax t on the first k commodities. Our next two results shed light on the connection between quota equilibrium and emission tax equilibrium:

Theorem 3.3. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota as in Definition 2.2 and $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-disposal quota equilibrium. Let $\mathcal{F} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \theta_0\}$ be a finite production economy with emission tax such that*

(1) $\mathcal{V}_n = \mathbb{R}_{\leq 0}$ for all $n < k$ and $\mathcal{V}_n = \mathcal{Z}(m)_n$ for all $n \geq k$;
(2) $t = -\pi_k(\bar{p})$.

Then $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium for $\mathcal{F}$.

Proof. As profit maximization does not depend on the disposal region, we have $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$. As $\mathcal{V} \supset \mathcal{Z}$, we have $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{V}$. Note that $B^m_\omega(\bar{x}, \bar{y}, \bar{p}) = B^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. As $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$, we have $\bar{x}(\omega) \in D^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. Hence, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium for the economy $\mathcal{F}$. □

On the other hand, every emission tax equilibrium is a quota equilibrium with the quota being minus the total net emission of the first k commodities in the emission tax equilibrium:

Theorem 3.4. *Let $\mathcal{F} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \theta_0\}$ be a finite production economy with emission tax and $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{V}$-disposal emission tax equilibrium. Define a finite production economy with quota $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ where $m = -C(\bar{x}, \bar{y})$ and $\mathcal{Z}(m)_n = \mathcal{V}_n$ for all $n \geq k$. Then $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium for $\mathcal{E}$.*

Proof. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium, we have $m = -C(\bar{x}, \bar{y}) \leq 0$. Hence, we have $\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}(m)$. As profit maximization does not depend on the disposal region, we have $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$. As $t = -\pi_k(\bar{p})$, we have $B^m_\omega(\bar{x}, \bar{y}, \bar{p}) = B^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. Hence, we have $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. So $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium. □

Theorem 3.3 and Theorem 3.4 show that a quota equilibrium can be realized as an emission tax equilibrium and vice versa. In the next example, we study the existence of emission tax equilibrium and whether an emission tax rate can be chosen to ensure that emissions will lie under a pre-specified level of total net pollution emissions for every emission tax equilibrium consistent with the chosen emission tax rate.

Example 3.5. Let $\mathcal{F}$ be a finite production economy with emission tax which is defined exactly as the finite production economy with quota $\mathcal{E}$ in Example 2.16 except that the disposal region $\mathcal{V} = \mathbb{R}^3_{\leq 0}$ (i.e., we eliminate the quota on CO_2 and replace it by imposing an emission tax on CO_2).

Existence of Emission Tax Equilibrium: We show that $\mathcal{F}$ has a $\mathcal{V}$-disposal emission tax equilibrium if and only if the tax rate $t \leq \frac{1}{4}$. Hence, emission tax equilibrium might not exist for specific emission tax rates.

Claim 3.6. *There is a $\mathcal{V}$-disposal emission tax equilibrium for emission tax rate $t \leq \frac{1}{4}$.*

Proof. Pick $t_0 \leq \frac{1}{4}$ to be the tax rate on CO_2. Let $\bar{x} = (0, 0, 1), \bar{y} = \big((1, -1, 1), (0, 0, 0)\big)$ and $\bar{p} = (-t_0, \frac{1}{2} - t_0, \frac{1}{2})$. We claim that $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-disposal tax equilibrium with tax rate t_0 on CO_2. At the equilibrium price $\bar{p}$, both firms are profit maximizing, which are both 0. The budget set for the agent is:

$$\{z \in X : \bar{p} \cdot z \leq \frac{1}{2} - t_0 + t_0 = \frac{1}{2}\}. \tag{3.1}$$

Hence, $\bar{x}$ is an element of the demand set $D^t(\bar{x}, \bar{y}, \bar{p})$. Note that $\bar{x} - e - \sum_{j \in J} \bar{y}(j) = (-1, 0, 0) \in \mathcal{V}$. Hence, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-disposal tax equilibrium with tax rate t_0, which proves the desired result. □

We now show that there is no $\mathcal{V}$-disposal emission tax equilibrium if the emission tax rate is greater than $\frac{1}{4}$. Suppose $t > 0$ is an emission tax rate on CO_2 under which there is a $\mathcal{V}$-disposal emission tax equilibrium $(\hat{x}, \hat{y}, \hat{p})$. By definition, we know that

$\hat{p}_1 = -t$. The equilibrium price $\hat{p}_3$ must be no less than $2t$ since otherwise the second firm's profit is unbounded. For the same reason, we know that $\hat{p}_2 \geq \hat{p}_3 - t > 0$. As the endowment $e = (0, 1, 0)$, the agent's budget at equilibrium is positive. The equilibrium production for the first firm must not be $(0, 0, 0)$ since the agent has a positive budget which she will spend entirely on electricity. Hence, we conclude that $\hat{p}_3 = t + \hat{p}_2$. As $\bar{p} \in \Delta$, we have $2t + 2\hat{p}_2 = 1$, which implies that $\hat{p}_3 = \frac{1}{2}$. As $\hat{p}_3 \geq 2t$, we know that $t \leq \frac{1}{4}$. By Claim 3.6, we conclude that $\mathcal{F}$ has a $\mathcal{V}$-disposal emission tax equilibrium if and only if the emission tax rate $t \leq \frac{1}{4}$.

Limiting CO_2 Emission via Emission Tax: We now investigate whether emission tax rate can ensure the total net CO_2 emission is under a pre-specified level.

Claim 3.7. *The total net emission of* CO_2 *at any* $\mathcal{V}$*-disposal emission tax equilibrium is* 1 *if the emission tax rate* $t < \frac{1}{4}$.

Proof. Pick $t_0 < \frac{1}{4}$ to be the emission tax rate of CO_2, under which there is a $\mathcal{V}$-disposal emission tax equilibrium $(\hat{x}, \hat{y}, \hat{p})$. By the same argument as in the previous paragraph, we conclude that $\hat{p}_3 = \frac{1}{2}$ and $\hat{p}_2 = \frac{1}{2} - t_0$. As $\hat{p}_3 > 2t_0$, the equilibrium production for the second firm is $(0, 0, 0)$. Suppose the equilibrium production for the first firm is $(r, -r, r)$ for some $r < 1$. Then the emission tax budget set is:

$$\{z \in X : \hat{p} \cdot z \leq \frac{1}{2} - t_0 + rt_0\}. \tag{3.2}$$

As $t_0 < \frac{1}{4}$ and $r < 1$, we have $\frac{1}{2}r < \frac{1}{2} - (1 - r)t_0$, which implies that the consumption $(0, 0, r)$ is in the agent's budget set. However, the agent has extra budget to consume more electricity but the total production of the electricity is r units. So $(0, 0, r)$ is not in the demand set $D^t(\hat{x}, \hat{y}, \hat{p})$. Hence, the equilibrium production for the first firm is $(1, -1, 1)$. So the total net emission of CO_2 is 1. □

By Claim 3.7, if the government sets the emission tax rate to be less than $\frac{1}{4}$, then the total net CO_2 emission is 1 unit. We now consider the case where the emission tax rate is $\frac{1}{4}$. Note that, when the emission tax rate is $\frac{1}{4}$, the equilibrium price must be $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

Claim 3.8. *Let the emission tax rate be* $\frac{1}{4}$ *and* $0 \leq v \leq 1$. *Then there exists a unique* $\mathcal{V}$*-disposal emission tax equilibrium such that the total net emission of* CO_2 *is* v.

Proof. Pick some $v_0 \in [0,1]$. To achieve this pre-specified total net emission of CO_2, all possible equilibrium production plans take the following form:

- The first firm produces at $(r_1, -r_1, r_1)$ where $0 \leq r_1 \leq 1$. The second firm produces at $(-2r_2, 0, -r_2)$ where $0 \leq r_2 \leq \frac{r_1}{2}$. We also require $r_1 - 2r_2 = v_0$.

Thus, the agent's utility function is $(r_1 - r_2) - v_0^2$. As $r_2 = \frac{r_1 - v_0}{2}$, so the agent's utility maximizes by taking $r_1 = 1$. By Theorem 3.4 and Theorem 2.12, the only possible equilibrium production plan for the first firm is $(1,-1,1)$, which implies that the only possible equilibrium consumption-production pair is $\hat{x}_{v_0} = (0, 0, \frac{1+v_0}{2})$ and $\hat{y}_{v_0} = \big((1,-1,1), (v_0 - 1, 0, \frac{v_0-1}{2})\big)$. It remains to show that $(\hat{x}_{v_0}, \hat{y}_{v_0}, \hat{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium. Note that $\hat{x}_{v_0} - e - \sum_{j \in J} \hat{y}_{v_0}(j) = (-v_0, 0, 0) \in \mathcal{V}$. Both firms are profit maximizing. The emission tax budget set for the agent is $\{z \in X : \hat{p} \cdot z \leq \frac{1}{4}(1 + v_0)\}$. Hence, $\hat{x}_{v_0}$ is an element of the emission tax demand set $D^t(\hat{x}_{v_0}, \hat{y}_{v_0}, \hat{p})$, which implies that $(\hat{x}_{v_0}, \hat{y}_{v_0}, \hat{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium for the emission tax rate $\frac{1}{4}$. $\square$

By Claim 3.8, there are multiple $\mathcal{V}$-disposal emission tax equilibrium associated with the emission tax rate $\frac{1}{4}$. In fact, for any pre-specified total net emission $v < 1$ of CO_2, there is a $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate $\frac{1}{4}$ whose total net emission of CO_2 equals to v. Combining with Claim 3.7, we conclude that, in this example, it is impossible to get under the pre-specified total net emission of CO_2 by setting an emission tax CO_2.

The Welfare Property of Emission Tax Equilibrium and Comparison with Revenue-Maximizing Quota Equilibrium: We now consider the welfare property of $\mathcal{V}$-disposal emission tax equilibria. As in Example 2.16, there is a Pareto ranking among $\mathcal{V}$-disposal emission tax equilibria arising from the externality. The total net emission of CO_2 that maximizes the agent's utility is $\hat{v} = \frac{1}{4}$. As a result, the $\mathcal{V}$-disposal emission tax equilibrium consumption-production pair $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$[12] with emission tax rate $\frac{1}{4}$ Pareto dominates all other $\mathcal{V}$-disposal emission tax equilibrium consumption-production pairs. Thus, if the government sets the emission tax rate to be less than $\frac{1}{4}$, then the resulting $\mathcal{V}$-disposal emission tax equilibrium consumption-production pair is Pareto dominated. On the other hand, there are multiple $\mathcal{V}$-disposal

[12]In particular, we have $\hat{x}_{\frac{1}{4}} = (0, 0, \frac{5}{8})$, $\hat{y}_{\frac{1}{4}} = \big((1,-1,1), (-\frac{3}{4}, 0, -\frac{3}{8})\big)$ and $\hat{p} = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

emission tax equilibrium with the emission tax rate $\frac{1}{4}$ but only one of these equilibrium consumption-production pairs is Pareto optimal. So it is impossible to guarantee full Pareto optimality by setting an emission tax rate on CO_2.

To compare emission tax equilibrium with revenue-maximizing quota equilibrium, we consider the finite production economy with quota $\mathcal{E}$ in Example 2.16. Recall that $\mathcal{E}$ has a $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium for all $m \in [-1, 0]$. By the definition of revenue-maximizing quota equilibrium, it is straightforward to conclude that the total net emission of CO_2 is bounded by $-m$ for all $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium. Thus, unlike emission tax equilibrium, revenue-maximizing quota equilibrium always exists, and quota can be chosen to ensure that the total net CO_2 emission of every quota equilibrium will be under a pre-specified level. Moreover, by Theorem 2.12, every revenue-maximizing quota equilibrium consumption-production pair is constrained Pareto optimal. That is, every quota equilibrium consumption-production pair is Pareto optimal among all feasible consumption-production pairs with the same total net CO_2 emission. Finally, as indicated in Example 2.16, $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}}, \hat{p})$ is the only $\mathcal{Z}(-\frac{1}{4})$-disposal revenue-maximizing quota equilibrium of $\mathcal{E}$, and $(\hat{x}_{\frac{1}{4}}, \hat{y}_{\frac{1}{4}})$ is full Pareto optimal. Thus, once the government sets the quota on CO_2 emission to be $\frac{1}{4}$, full Pareto optimality of the equilibrium consumption-production pair is achieved without further intervention from the government.

Example 3.5 shows that emission tax equilibrium may not exist for certain emission tax rate and, when there does not exist a one-to-one correspondence between emission tax rate and total net CO_2 emission, setting an emission tax rate does not ensure that the total net CO_2 emission will be under a pre-specified level. However, as the next example illustrates, if there exists a one-to-one correspondence between emission tax rate and total net CO_2 emission, one can not only limit the total net CO_2 emission under a pre-specified level but also guarantee full Pareto optimality via emission tax rate, provided that the only externality arises from total net CO_2 emission.

Example 3.9. Let $\mathcal{F}$ be a finite production economy with tax as in Example 3.5 except that the second firm's production set is now given by

$$Y_2 = \{(-a, 0, -r^2) : (r \in \mathbb{R}_{\geq 0}) \wedge (0 \leq a \leq 2r)\}. \tag{3.3}$$

The second firm has the production technology to sequester CO_2 using electricity, and the marginal cost of electricity to sequester an additional unit of CO_2 increases as the amount of CO_2 that has been sequestered increases.

Existence and Properties of Emission Tax Equilibrium: We show that $\mathcal{F}$ has a unique $\mathcal{V}$-disposal emission tax equilibrium if and only if the emission tax rate $t \leq \frac{1}{4}$.

Claim 3.10. *There is a unique $\mathcal{V}$-disposal emission tax equilibrium for all emission tax rate $t \leq \frac{1}{4}$*

Proof. Pick $t_0 \leq \frac{1}{4}$. Let $\bar{x}_{t_0} = (0, 0, 1 - 4t_0^2)$, $\bar{y}_{t_0} = \big((1, -1, 1), (-4t_0, 0, -4t_0^2)\big)$ and $\bar{p}_{t_0} = (-t_0, \frac{1}{2} - t_0, \frac{1}{2})$. We claim that $(\bar{x}_{t_0}, \bar{y}_{t_0}, \bar{p}_{t_0})$ is a $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate t_0 on CO_2. It is clear that the first firm is profit maximizing and $\bar{x}_{t_0} - e - \sum_{j \in J} \bar{y}_{t_0}(j) = (4t_0 - 1, 0, 0) \in \mathcal{V}$. The second firm's profit at $\bar{p}_{t_0}$, as a function of production, is given by $2rt_0 - \frac{1}{2}r^2$. Thus, the second firm maximize its profit at $(-4t_0, 0, -4t_0^2)$, and its profit is $2t_0^2$. The agent's emission tax budget set is:

$$\{z \in X : \bar{p} \cdot z \leq \frac{1}{2} - t_0 + 2t_0^2 + (1 - 4t_0)t_0 = \frac{1}{2} - 2t_0^2)\}.$$

Hence, $\bar{x}_{t_0}$ is an element of the emission tax demand set $D^{t_0}(\bar{x}_{t_0}, \bar{y}_{t_0}, \bar{p}_{t_0})$, which implies that $(\bar{x}_{t_0}, \bar{y}_{t_0}, \bar{p}_{t_0})$ is a $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate t_0.

We now show that $(\bar{x}_{t_0}, \bar{y}_{t_0}, \bar{p}_{t_0})$ is the only $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate t_0. Suppose $(\hat{x}, \hat{y}, \hat{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate t_0. By the form of the agent's utility function, we know that $\hat{p}_3 > 0$. Note that the second firm's profit at $\hat{p}$, as a function of production, is given by $2rt_0 - r^2\hat{p}_3$. So the second firm maximize its profit by producing at $(-\frac{2t_0}{\hat{p}_3}, 0, -(\frac{t_0}{\hat{p}_3})^2)$. As $(\hat{x}, \hat{y}, \hat{p})$ is a $\mathcal{V}$-disposal emission tax equilibrium and the total emission of CO_2 can not exceed 1 unit, so we have $\frac{2t_0}{\hat{p}_3} \leq 1$, which implies that $\hat{p}_3 \geq 2t_0$. Note that we must have $t_0 + \hat{p}_2 \geq \hat{p}_3$ since otherwise the first firm's profit is unbounded. Thus, we conclude that $\hat{p}_2 \geq t_0$ so the agent's endowment is positive, which further implies that the agent's budget at equilibrium is positive. By the form of the agent's utility function, the agent spends all its budget to consume electricity hence the equilibrium production for the first firm must not be $(0,0,0)$. Thus, we must have $\hat{p}_3 = \hat{p}_2 + t_0$. Since $\hat{p} \in \Delta$, we have $\hat{p} = (-t_0, \frac{1}{2} - t_0, \frac{1}{2})$. So the second firm's equilibrium production is $(-4t_0, 0, -4t_0^2)$. Suppose the first firm's equilibrium production is $(r, -r, r)$. Then the agent's emission tax budget set is:

$$\{z \in X : \hat{p} \cdot z \leq \frac{1}{2} - t_0 + 2t_0^2 + (r - 4t_0)t_0 = \frac{1}{2} - (1-r)t_0 - 2t_0^2\}. \tag{3.4}$$

So the emission tax demand set $D^{t_0}(\hat{x}, \hat{y}, \hat{p})$ is $\{(0, 0, 1 - 2(1-r)t_0 - 4t_0^2)\}$. As $\hat{y}_1 + \hat{y}_2 = (r - 4t_0, -r, r - 4t_0^2)$, so there are $r - 4t_0^2$ unit of electricity in the economy available to the agent. If $r < 1$, then we have $r - 4t_0^2 < 1 - 2(1-r)t_0 - 4t_0^2$. So the agent has enough budget to consume more electricity than what is available to her. As a result, the first firm's equilibrium production must be $(1, -1, 1)$, which implies that $\hat{x} = (0, 0, 1 - 4t_0^2)$. Hence, $(\bar{x}_{t_0}, \bar{y}_{t_0}, \bar{p}_{t_0})$ is the unique $\mathcal{V}$-disposal tax equilibrium with the emission tax rate t_0. $\square$

We now show that there is no $\mathcal{V}$-disposal emission tax equilibrium if the emission tax rate is greater than $\frac{1}{4}$. Suppose $t > 0$ is an emission tax rate on CO_2 under which there is a $\mathcal{V}$-disposal emission tax equilibrium $(\bar{x}_t, \bar{y}_t, \bar{p}_t)$. By the same argument as in Claim 3.10, the second firm's equilibrium production is $(-4t_0, 0, -4t_0^2)$. As the total emission of CO_2 can not exceed 1 unit, this implies that $t_0 \leq \frac{1}{4}$. So there is no $\mathcal{V}$-disposal emission tax equilibrium if the tax rate is greater than $\frac{1}{4}$.

Although emission tax equilibrium does not exist for emission tax rate $t > \frac{1}{4}$, one can achieve any pre-specified total net CO_2 emission via emission tax rate alone, since there is a one-to-one correspondence between the emission tax rate and the total net CO_2 emission. In particular, by Claim 3.10, given a tax rate $t \leq \frac{1}{4}$, the total net CO_2 emission is $1 - 4t$. Hence, the government can limit the total net CO_2 emission under any pre-specifed level $v \leq 1$ by setting the emission tax rate to be no less than $\frac{1-v}{4}$.

The Welfare Property of Emission Tax Equilibrium and Comparison with Revenue-Maximizing Quota Equilibrium: We now consider the welfare property of $\mathcal{V}$-disposal emission tax equilibrium. As in Example 3.5, there is a Pareto ranking among $\mathcal{V}$-disposal emission tax equilibrium consumption-production pairs arising from the externality. By Claim 3.10, the agent's utility at a $\mathcal{V}$-disposal emission tax equilibrium, as a function of the tax rate, is given by $(1 - 4t^2) - (1 - 4t)^2$. By taking the derivative, the agent's utility maximizes uniquely at $t = \frac{1}{5}$ and the utility is $\frac{4}{5}$. Thus, the $\mathcal{V}$-disposal emission tax equilibrium consumption-production pair with emission tax rate $\frac{1}{5}$ Pareto dominates all other $\mathcal{V}$-disposal emission tax equilibrium consumption-production pairs. By Claim 3.10, there exists a unique $\mathcal{V}$-disposal emission tax equilibrium with emission tax rate $\frac{1}{5}$. Hence, full Pareto optimality can be achieved by setting the emission tax rate to be $\frac{1}{5}$.

We now consider the associated finite production economy with quota. In particular, let $\mathcal{F}'$ be the finite production economy with quota which is defined exactly the same as $\mathcal{F}$ except that the disposal region of $\mathcal{F}'$ is given by $\mathcal{Z}(m) = [m, 0] \times \mathbb{R}^2_{\leq 0}$ (i.e., we eliminate the emission tax on CO_2 and replace it by setting a quota on CO_2 emission). Note that Y_2 is closed and convex. Hence, we know that $\bar{Y} = Y_1 + Y_2$ is convex and closed. For every $y \in \bar{Y}$, we have $y_2 \leq 0$, and if $y_2 = 0$ then $y_1, y_3 \leq 0$. Hence, we conclude that $\bar{Y} \cap \mathbb{R}^3_{\geq 0} = \{0\} = \bar{Y} \cap (-\bar{Y})$. For the same reason as in Example 3.5, we know that $e \in \mathrm{int}(X - \bar{Y})$. So $\mathcal{F}'$ satisfies the conditions of Theorem 2.6, which implies that $\mathcal{F}'$ has a $\mathcal{Z}(m)$-disposal revenue-maximizing quota equilibrium for all $m \in [-1, 0]$. Moreover, as there exists a one-to-one correspondence between tax rate and total net CO_2 emission, it follows from Theorem 3.3 that $(\bar{x}_{\frac{1}{5}}, \bar{y}_{\frac{1}{5}}, \bar{p}_{\frac{1}{5}})$ is the only $\mathcal{Z}(-\frac{1}{5})$-disposal revenue-maximizing quota equilibrium. Thus, once the government sets the quota on CO_2 to be $\frac{1}{5}$, full Pareto optimality of the equilibrium consumption-production pair is achieved without further intervention from the government.

Example 3.5 and Example 3.9 jointly provide a comprehensive comparison between quota equilibrium and emission tax equilibrium, which lead us to the following conclusions:

(1) Quota equilibrium always exists and the total net pollution emission at any quota equilibrium is under the quota. Moreover, by Theorem 2.12, every quota equilibrium is constrained Pareto optimal. On the other hand, by Theorem 2.5, every quota equilibrium with respect to a smaller quota is a quota equilibrium with respect to a larger quota, which implies that a quota is usually associated with multiple quota equilibrium with different total net pollution emission.

This further implies that a quota equilibrium is, in general, not Pareto optimal among all quota equilibria;

(2) Emission tax equilibrium need not exist for certain emission tax rates. When there does not exist a one-to-one correspondence between emission tax rate and total net pollution emission, setting an emission tax rate can not ensure the total net pollution emission of an emission tax equilibrium will be under a pre-specified level, as illustrated in Example 3.5;

(3) When there is a one-to-one correspondence between emission tax rate and total net pollution emission, the government can limit the total net pollution emission at any pre-specified level by setting an emission tax. Moreover, if the only externality arises from total net pollution emission, one can achieve full Pareto optimality through an emission tax.

3.1. **Emission Tax Versus Add-on Tax.**

Example 3.11. In this example, we consider an alternative formation of the finite production economies with tax. Let $\mathcal{F}'$ be the same finite production economy with tax as in Example 3.9, except that the tax is imposed on the input of coal rather than on the emission of CO_2. We shall show that the analogue tax equilibrium arises from $\mathcal{F}'$ is less efficient than the emission tax equilibrium as in Example 3.9.

Let $t \geq 0$ be a tax rate on the input of the coal. Through normalization, for a price vector p, we require that $t + \sum_{k=1}^{\ell} |p_k| = 1$, that is, $(t,p) \in \Delta$. Given a production plan $y \in Y$, let $T_{\text{coal}}(y)$ denote the total input of coal in the production. For every $(t,p) \in \Delta$ and $y \in Y$, the *carbon tax budget set* $B^c_\omega(y,t,p)$ is defined to be:

$$\{z \in X : p \cdot z \leq p \cdot e + \sum_{j \in J} p \cdot y(j) + t \cdot T_{\text{coal}}(y)\}. \tag{3.5}$$

The *carbon tax demand set* is defined to be the collection of elements in $B^c_\omega(y,t,p)$ that maximizes the agent's utility function. A $\mathcal{V}$-disposal carbon tax equilibrium under the tax rate t is $(\bar{x},\bar{y},\bar{p})$ such that:

(1) $(t,\bar{p}) \in \Delta$;

(2) $\bar{x}$ is in the carbon tax demand set and both firms are profit maximizing at $(t,\bar{p})$;

(3) $\bar{x} - \sum_{j \in J} \bar{y}(j) - e \in \mathcal{V}$.

As shown in Claim 3.10, every tax rate $t \leq \frac{1}{4}$ is associated with the unique $\mathcal{V}$-disposal tax equilibrium $(\bar{x}_t, \bar{y}_t, \bar{p}_t)$, where $\bar{x}_t = (0, 0, 1 - 4t^2), \bar{y}_t = \big((1, -1, 1), (-4t, 0, -4t^2)\big)$ and $\bar{p}_t = (-t, \frac{1}{2} - t, \frac{1}{2})$. Recall that the agent's utility, as a function of the tax rate, is given by $(1 - 4t^2) - (1 - 4t)^2 + 1$. Note that the agent's utility is no less than 1, and the equality holds if and only if $t = 0$. We show that there exists a $\mathcal{V}$-disposal carbon tax equilibrium with the same tax rate, and it is Pareto dominated by the $\mathcal{V}$-disposal tax equilibrium. Let $\hat{p}_t = (0, \frac{1}{2} - t, \frac{1}{2})$, $\hat{y}_t = \big((1, -1, 1), (0, 0, 0)\big)$ and $\hat{x}_t = (0, 0, 1)$. It is easy to verify that $(\hat{x}_t, \hat{y}_t, \hat{p}_t)$ is a $\mathcal{V}$-disposal carbon tax equilibrium with tax rate t. The agent's utility at $(\hat{x}_t, \hat{y}_t, \hat{p}_t)$ is 1, which is less than the agent's utility at $(\bar{x}_t, \bar{y}_t, \bar{p}_t)$ unless $t = 0$. In fact, when $t = 0$, the $\mathcal{V}$-disposal tax equilibrium is the same as the $\mathcal{V}$-disposal carbon tax equilibrium. Thus, for tax rate $t \leq \frac{1}{4}$, the $\mathcal{V}$-disposal carbon tax equilibrium $(\hat{x}_t, \hat{y}_t, \hat{p}_t)$ has a higher electricity consumption at the cost of a higher total net CO_2 emission, which together results in a lower utility for the agent.

We now analysis the generic $\mathcal{V}$-disposal carbon tax equilibrium, and show that it is Pareto dominated by the Pareto optimal $\mathcal{V}$-disposal tax equilibrium in Example 3.9. Let $(\hat{x}, \hat{y}, \hat{p})$ be a $\mathcal{V}$-disposal carbon tax equilibrium under the tax rate t. By the form of the agent's utility function, we know that $\hat{p}_3 > 0$. We now break our analysis into the following two cases:

- **First Case:** We first assume that $\hat{p}_1 \geq 0$. As the second firm is profit maximizing, the second firm's production is $(0, 0, 0)$. If the first firm's equilibrium production plan is $(r, -r, r)$ for $r > 0$, its profit must be 0 since otherwise its profit is unbounded. Thus, the agent can at most consume r units of CO_2 so the agent's utility is bounded by $r - r^2 + 1$. By taking the derivative, the agent's utility is bounded by $\frac{5}{4}$. Recall from Example 3.9 that the tax rate and utility for the Pareto optimal $\mathcal{V}$-disposal tax equilibrium are $\frac{1}{5}$ and $\frac{9}{5}$,respectively. Hence, $(\hat{x}, \hat{y}, \hat{p})$ is Pareto dominated by the Pareto optimal $\mathcal{V}$-disposal tax equilibrium;
- **Second Case:** We now assume that $\hat{p}_1 < 0$. We shall show that there exists a $\mathcal{V}$-disposal tax equilibrium at which the agent consumes at least as much electricity under the same total net CO_2 emission, with strict inequality except

at $t_0 = \frac{1}{4}$. Note that the second firm's profit at $(t, \hat{p})$, as a function of production, is given by $-2r\hat{p}_1 - r^2\hat{p}_3$, which implies that the second firm maximize it profit by producing at $(2\frac{\hat{p}_1}{\hat{p}_3}, 0, -(\frac{\hat{p}_1}{\hat{p}_3})^2)$, and its profit is $\frac{\hat{p}_1^2}{\hat{p}_3}$. As $(\hat{x}, \hat{y}, \hat{p})$ is a $\mathcal{V}$-disposal carbon tax equilibrium and the total CO_2 emission can not exceed 1 unit, we conclude that $\hat{p}_3 \geq -2\hat{p}_1$. Since the second firm's equilibrium production plan is not $(0,0,0)$, the first firm's equilibrium production plan must not be $(0,0,0)$. Hence, we must have $\hat{p}_2 + t - \hat{p}_1 = \hat{p}_3$ since otherwise the first firm's profit is unbounded. Suppose the first firm's equilibrium production plan is $(r, -r, r)$ for some $r > 0$. Then the agent's carbon tax budget set is $\{z \in X : \hat{p} \cdot z \leq \hat{p}_2 + rt + \frac{\hat{p}_1^2}{\hat{p}_3}\}$. By the form of the agent's utility function, the agent's carbon tax demand set is either empty or contains the single point $(0, 0, \frac{\hat{p}_2+rt}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2)$. Note that $\frac{\hat{p}_2+rt}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2 = \frac{\hat{p}_3+\hat{p}_1-t+rt}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2 \leq \frac{\hat{p}_3+\hat{p}_1}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2$, with equality if and only if $r = 1$.

Claim 3.12. *Let the tax rate $t_0 = \frac{\hat{p}_1}{-2\hat{p}_3}$. Then $(0, 0, \frac{\hat{p}_2+rt}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2)$ is no greater than the single point in agent's demand set under tax t_0, with the equality if and only if $r = 1 = 4t_0$.*

Proof. Since $\hat{p}_3 \geq -2\hat{p}_1$ and $\mathcal{V}$-disposal tax equilibrium in Example 3.9 exists if and only if the tax rate is no greater than $\frac{1}{4}$, so there exists a $\mathcal{V}$-disposal tax equilibrium with tax rate t_0. Recall from Eq. (3.4) that the agent's budget in Example 3.9 is given by $\frac{1}{2} - t_0 + 2t_0^2 + (r - 4t_0)t_0$. Recall from Example 3.9 that the equilibrium price for electricity is $\frac{1}{2}$, so the agent can consume up to $\frac{\frac{1}{2}-t_0+2t_0^2+(r-4t_0)t_0}{\frac{1}{2}}$. As $-2t_0 = \frac{\hat{p}_1}{\hat{p}_3}$, we conclude that $\frac{\hat{p}_3+\hat{p}_1}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2 \leq \frac{\frac{1}{2}-t_0+2t_0^2+(r-4t_0)t_0}{\frac{1}{2}}$, and the equality holds if and only if $r = 4t_0 = 2\frac{\hat{p}_1}{\hat{p}_3}$. From the calculation before Claim 3.12, we know that $\frac{\hat{p}_2+rt}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2 = \frac{\hat{p}_3+\hat{p}_1}{\hat{p}_3} + (\frac{\hat{p}_1}{\hat{p}_3})^2$ if and only if $r = 1$. So we have the desired result. □

By Claim 3.12, If $\frac{\hat{p}_1}{\hat{p}_3} > -\frac{1}{2}$, by Claim 3.12, the agent can consume more electricity with the same total net CO_2 emission under the emission tax $t_0 = \frac{\hat{p}_1}{-2\hat{p}_3}$. If $\frac{\hat{p}_1}{\hat{p}_3} = -\frac{1}{2}$, by Claim 3.12, the agent has the same utility as under the emission tax $t_0 = \frac{\hat{p}_1}{-2\hat{p}_3} = \frac{1}{4}$. Note that, in Example 3.9, the Pareto optimal tax rate is $\frac{1}{5}$ with the utility being $\frac{9}{5}$. So we conclude that $(\hat{x}, \hat{y}, \hat{p})$ is Pareto dominated by the Pareto optimal $\mathcal{V}$-disposal tax equilibrium with tax rate $\frac{1}{5}$.

In both Example 3.9 and this example, the ratio of the equilibrium prices of CO_2 and electricity determines the level of sequestration, that is, the second firm's equilibrium production plan. As the equilibrium price for electricity is always $\frac{1}{2}$ in Example 3.9, the emission tax on CO_2 determines the second firm's equilibrium production plan hence the equilibrium. Given a $\mathcal{V}$-disposal tax equilibrium, an associated $\mathcal{V}$-disposal carbon tax equilibrium is a one such that $\frac{\hat{p}_1}{\hat{p}_3} = -2t_0$. Note that a $\mathcal{V}$-disposal tax equilibrium may be associated with multiple $\mathcal{V}$-disposal carbon tax equilibrium. By Claim 3.12, we know that the agent consumes at least as much electricity with the same total net CO_2 emission under the emission tax, with strict inequality except at $t_0 = \frac{\hat{p}_1}{-2\hat{p}_3} = \frac{1}{4}$.

A. Appendix

The goal of this appendix is to provide a complete proof of a special case of Theorem 2.6 where the disposal region is a cone, which further completes our proof of Theorem 2.6. To do so, we need to introduce the concept of quasi-equilibrium. Let

$$\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, \mathcal{Z}(m), \theta_0\} \tag{A.1}$$

be a finite production economy with quota as in Definition 2.2. Given $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the *quota quasi-demand set* $\bar{D}^m_\omega(x, y, p)$ is defined to be:

$$\{z \in B^m_\omega(x, y, p) : w \succ_{x,y,\omega,p} z \implies p \cdot w \geq p \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} p \cdot y(j) - \theta_0(\omega)\pi_k(p) \cdot C(x, y)\}.$$

A $\mathcal{Z}(m)$*-disposal quota quasi-equilibrium* is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that:

1. $\bar{x}(\omega) \in \bar{D}^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$;
2. $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$. So every firm is profit maximizing given the price $\bar{p}$;
3. $\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}(m)$.

Note that quasi-equilibrium is not stable since agents could, in principle, do better within their budget sets. Thus, the interest of the quasi-equilibrium concept is purely mathematical. The rest of this appendix is broken into the following two parts:

(1) To show that every quota quasi-equilibrium is a quota equilibrium under suitable regularity conditions;
(2) Prove the special case of Theorem 2.6 by first establishing the existence of a quota quasi-equilibrium, then applying the result mentioned in the previous item to show that the quota quasi-equilibrium is a quota equilibrium.

A.1. **Quasi-Equilibrium Versus Equilibrium.** In this section, we show that, under the classical survival assumption, every $\mathcal{Z}(m)$-disposal quota quasi-equilibrium is a $\mathcal{Z}(m)$-disposal quota equilibrium.

Lemma A.1. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy under quota and $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium. Suppose $e_\omega \in \operatorname{int}(X_\omega - \sum_{j\in J}\theta_{\omega j}Y_j)$ for all $\omega \in \Omega$ and $-(\pi_k(\bar{p}))\cdot C(\bar{x}, \bar{y}) \geq 0$. Then $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium.*

Proof. Let $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium. For each consumer ω, define a correspondence $\delta_\omega : \Delta \twoheadrightarrow X_\omega$ as

$$\delta_\omega(p) = \{x_\omega \in X_\omega : p\cdot x_\omega < p\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\sup\{p\cdot y : y \in Y_j\} - \theta_0(\omega)\pi_k(\bar{p})\cdot C(\bar{x}, \bar{y})\}.$$

We first show that $\delta_\omega(\bar{p}) \neq \emptyset$ for all $\omega \in \Omega$. Note that $\bar{p} \neq 0$. By the hypothesis of the lemma, for each agent $\omega \in \Omega$, pick $u \in \mathbb{R}^\ell$ such that $\bar{p}\cdot u < 0$ and that $(e(\omega)+z) \in (X_\omega - \sum_{j\in J}\theta_{\omega j}Y_j)$. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium and $-(\pi_k(\bar{p}))\cdot C(\bar{x}, \bar{y}) \geq 0$, we have $\bar{p}\cdot\tilde{x}_\omega < \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j) - \theta_0(\omega)\pi_k(\bar{p})\cdot C(\bar{x}, \bar{y})$ for some $\tilde{x}_\omega \in X_\omega$. So we have $\delta_\omega(\bar{p}) \neq \emptyset$.

For each $\omega \in \Omega$, pick $z_\omega \in \delta_\omega(\bar{p})$ and $\hat{x}_\omega \in X_\omega$ such that $(\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium, we have $\bar{p}\cdot\hat{x}_\omega \geq \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j) - \theta_0(\omega)\pi_k(\bar{p})\cdot C(\bar{x}, \bar{y})$. As $P_\omega(\bar{x}, \bar{y}, \bar{p})$ is continuous, there exists $\lambda \in (0, 1)$ such that

$$(\lambda z_\omega + (1-\lambda)\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p}).$$

Assume that $\bar{p}\cdot\hat{x}_\omega = \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j) - \theta_0(\omega)\pi_k(\bar{p})\cdot C(\bar{x}, \bar{y})$. Then we have $(\lambda z_\omega + (1-\lambda)\hat{x}_\omega, \bar{x}(\omega)) \in P_\omega(\bar{x}, \bar{y}, \bar{p})$ and $\lambda z_\omega + (1-\lambda)\hat{x}_\omega \in \delta_\omega(\bar{p})$. This furnishes us a contradiction since $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium. Therefore, we have $\bar{p} \cdot \hat{x}_\omega > \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \bar{p} \cdot \bar{y}(j) - \theta_0(\omega) \pi_k(\bar{p}) \cdot C(\bar{x}, \bar{y})$. Hence, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}$-disposal quota equilibrium. $\square$

The survival assumption $e_\omega \in \operatorname{int}(X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$ is an autarky assumption: it implies that an agent can survive without participating in any exchanges using her initial endowment and shares in production. From an empirical point of view, such an assumption is obviously questionable. There, however, exist many works on relaxing the survival assumption[13]. It is possible to obtain the same results in our setting under these more general survival assumptions.

A.2. **Existence of Quota Equilibrium with Disposal Cone.** In this section, we prove a special case of Theorem 2.6 when the disposal region is a cone. We start with the definition of feasible consumption-production pairs:

Definition A.2. The set of feasible consumption-production pair of $\mathcal{E}$ is

$$\mathcal{O} = \left\{ (x, y) \in \mathcal{A} \times Y : \sum_{\omega \in \Omega} x(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} y(j) \in \mathcal{Z}(m) \right\}. \tag{A.2}$$

The set $\hat{Y}_j$ of feasible production plans for the j-th producer is

$$\left\{ y_j \in Y_j : \exists (x, y') \in \mathcal{A} \times \prod_{i \neq j} Y_i, \sum_{\omega \in \Omega} x(\omega) - \sum_{\omega \in \Omega} e(\omega) - y_j - \sum_{i \neq j} y'(i) \in \mathcal{Z}(m) \right\}. \tag{A.3}$$

The set $\hat{X}_i$ of feasible consumption for the i-th agent is

$$\left\{ x_i \in X_i : \exists (x', y) \in \prod_{\omega \neq i} X_\omega \times \prod_{j \in J} Y_j, x_i + \sum_{\omega \neq i} x'(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} y(i) \in \mathcal{Z}(m) \right\}. \tag{A.4}$$

We now state and prove the main result of this section, which is similar to the Proposition 3.2.3 in Florenzano [Flo03].

Theorem A.3. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega \in \Omega}, (Y_j)_{j \in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with tax as in Definition 2.2. Let the polar cone of $\mathcal{Z}(m)$ be*

[13] See McKenzie [McK81], Debreu [Deb62], Arrow and Hahn [AH71], Bergstrom [Ber76] and Florig [Flo99]. Also see Florenzano [Flo03] for a detailed discussion.

$\mathcal{Z}^0 = \{p \in \Delta : (\forall z \in \mathcal{Z}(m))(p \cdot z \leq 0)\}$. *Suppose* $\mathcal{E}$, *in addition, satisfies the following conditions:*

(i) *for all* $\omega \in \Omega$, *we have* $0 \in X_\omega$, P_ω *takes value in* $\mathcal{P}_H$ [14] *and* $e_\omega \in \operatorname{int}(X_\omega - \sum_{j \in J} \theta_{\omega j} Y_j)$;

(ii) *for all* $\omega \in \Omega$, *for each* $(x, y) \in \mathcal{O}$ *with* $x_\omega \in X_\omega$, *there exists* $u \in X_\omega$ *such that* $(u, x_\omega) \in \bigcap_{p \in \Delta \cap \mathcal{Z}^0} P_\omega(x, y, p)$ [15];

(iii) $\bar{Y}$ *is closed, convex, and* $\bar{Y} \cap (-\bar{Y}) = \bar{Y} \cap \mathbb{R}^\ell_{\geq 0} = \{0\}$ [16], *where* $\bar{Y} = \left\{\sum_{j \in J} y(j) : y \in Y\right\}$ *is the aggregated production set;*

(iv) *the vector* m *of quotas is* 0, *so for all* $n \leq k$, *we have* $\mathcal{Z}_n(m) = \{0\}$.

Then, there exists $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ *such that:*

(1) $(\bar{x}, \bar{y}, \bar{p})$ *is a* $\mathcal{Z}(m)$*-disposal quota equilibrium;*

(2) *we have* $\bar{p} \in \mathcal{Z}^0$ *and* $\bar{p} \cdot \big(\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j)\big) = 0$.

Proof. As $\mathcal{Z}(m)_n = \{0\}$ for all $n \leq k$, we have $C(x, y) = 0$ for all feasible consumption-production pair (x, y). So there is no government's rebate to any agent under any feasible consumption-production pair. Theorem A.3 is similar to Proposition 3.2.3 in Florenzano [Flo03]. For $\omega \in \Omega$, define the correspondence $P'_\omega : \mathcal{A} \times Y \times \Delta \twoheadrightarrow X_\omega$ by

$$P'_\omega(x, y, p) = \{a \in X_\omega | (a, x_\omega) \in P_\omega(x, y, p)\}. \tag{A.5}$$

Note that P'_ω is lower hemicontinuous since the preference map P_ω is continuous. As P_ω takes value in $\mathcal{P}_H$, $x_\omega \notin \operatorname{conv}(P'_\omega(x, y, p))$ for all $(x, y, p) \in \mathcal{A} \times Y \times \Delta$ and all $\omega \in \Omega$. By Item (ii), we have $\bigcap_{p \in \Delta \cap \mathcal{Z}^0} P'_\omega(x, y, p) \neq \emptyset$ for all $(x, y) \in \mathcal{O}$ with $x_\omega \in X_\omega$.

Claim A.4. $\hat{X}_\omega$ *is compact for every* $\omega \in \Omega$, $\hat{Y}_j$ *is relatively compact for every* $j \in J$ *and* $\bar{Y} + \mathcal{Z}$ *is closed.*

Proof. For any set $B \subset \mathbb{R}^\ell$, let $\mathbb{C}(B)$ denote the recession cone of B. Note that $\bar{X} = \sum_{\omega \in \Omega} X_\omega$ is a subset of $\mathbb{R}^\ell_{\geq 0}$, hence $\mathbb{C}(\bar{X}) \subset \mathbb{R}^\ell_{\geq 0}$. Thus, we have $\mathbb{C}(\bar{X}) \cap$

[14] As noted in Florenzano [Flo03], this condition can be weakened to the following condition: for each $(x, y, p) \in \mathcal{O} \times (\Delta \cap \mathcal{Z}^0)$ and all $\omega \in \Omega$, $(x(\omega), x(\omega)) \notin \operatorname{conv}(P_\omega(x, y, p))$, where $\operatorname{conv}(P_\omega(x, y, p))$ denotes the convex hull of $P_\omega(x, y, p)$.

[15] If the preferences are price independent, then this condition is equivalent to assuming non-satiation on the set of attainable allocations.

[16] As noted in Florenzano [Flo03], the condition $\bar{Y} \cap (-\bar{Y}) = \{0\}$ can be weakened to requiring every attainable production set being relatively compact.

$(-\mathbb{C}(\bar{X})) = \{0\}$. As $(\bar{Y} + \mathcal{Z}(m)) \cap \mathbb{R}^{\ell}_{\geq 0} = \{0\}$, we have $\mathbb{C}(\bar{X}) \cap \mathbb{C}(\bar{Y} + \mathcal{Z}(m)) = \{0\}$. By Proposition 2.2.4 in Florenzano [Flo03], $\hat{X}_\omega$ is compact for every $\omega \in \Omega$.

Note that $\bar{Y} \cap (-\bar{Y}) = \{0\}$ implies that $\mathbb{C}(\bar{Y}) \cap (-\mathbb{C}(\bar{Y})) = \{0\}$. By Proposition 2.2.4 in Florenzano [Flo03] again, $\hat{Y}_j$ is relatively compact for every $j \in J$. Finally, as $\mathcal{Z}(m) \subset \mathbb{R}^{\ell}_{\leq 0}$ and $\bar{Y} \cap \mathbb{R}^{\ell}_{\geq 0} = \{0\}$, we have $\mathbb{C}(\bar{Y}) \cap (-\mathcal{Z}(m)) = \{0\}$, which, by Proposition 2.2.4 in Florenzano [Flo03], implies that $\bar{Y} + \mathcal{Z}(m)$ is closed. □

As $\mathcal{Z}(m)$ is a cone, by Proposition 3.2.3 in Florenzano [Flo03], we conclude that $\mathcal{E}$ has a $\mathcal{Z}(m)$-disposal quota quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$. Moreover, we have $\bar{p} \in \mathcal{Z}^0$ and $\bar{p} \cdot \big(\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \big) = 0$. As $C(\bar{x}, \bar{y}) = 0$, by Lemma A.1, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-disposal quota equilibrium. □

A.3. **Existence of Pareto Optimal Consumption-Production Pair.** In this section, we present a technical result on the existence of Pareto optimal consumption-production pair among a compact set of consumption-production pairs. This technical result plays a crucial role in establishing Theorem 2.15.

Lemma A.5. *Let $\mathcal{E} = \{(X, R_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, \mathcal{Z}(m), \theta_0\}$ be a finite production economy with quota, and the only externality arises from the total net emission of the first k commodities. Suppose $\succ_\omega$ is continuous and negatively transitive for all $\omega \in \Omega$. Then, for every compact $F \subset \mathcal{A} \times Y$, there exists a $(f, y) \in F$ such that it is Pareto optimal among F.*

Proof. Let $\succ_P$ be the partial order on $\mathcal{A} \times Y$ such that $(f, y) \succ_P (f', y')$ if (f, y) Pareto dominates (f', y'). Suppose there does not exist any consumption-production pair in F that is Pareto optimal among F.

Claim A.6. *$\succ_P$ is irreflexive and transitive.*

Proof. As $\succ_\omega$ is irreflexive for every $\omega \in \Omega$, $\succ_P$ is irreflexive. Let $(f, y), (g, z), (h, s) \in \mathcal{A} \times Y$ be consumption-production pairs such that $(f, y) \succ_P (g, z)$ and $(g, z) \succ_P (h, s)$. For every $\omega \in \Omega$, we have $\big(g(\omega), C(g, z)\big) \not\succ_\omega \big(f(\omega), C(f, y)\big)$ and $\big(h(\omega), C(h, s)\big) \not\succ_\omega \big(g(\omega), C(g, z)\big)$. As $\succ_\omega$ is negatively transitive, we have $\big(h(\omega), C(h, s)\big) \not\succ_\omega \big(f(\omega), C(f, y)\big)$. There also exists some $t \in \Omega$ such that $\big(g(t), C(g, z)\big) \succ_t \big(h(t), C(h, s)\big)$. Suppose $\big(f(t), C(f, y)\big) \not\succ_t \big(h(t), C(h, s)\big)$, as $\big(g(t), C(g, z)\big) \not\succ_t \big(f(t), C(f, y)\big)$, then

$\big(g(t), C(g,z)\big) \not\succ_t \big(h(t), C(h,s)\big)$, which is a contradiction. Hence $\big(f(t), C(f,y)\big) \succ_t \big(h(t), C(h,s)\big)$, which implies that $\succ_P$ is transitive. □

We now apply transfinite recursion to construct a net of consumption-production pairs so that larger elements in the net Pareto dominates smaller elements in the net.

Claim A.7. *For every ordinal α, there exists a net $\big((f_\beta, y_\beta)\big)_{\beta<\alpha}$ of elements in F such that $(f_i, y_i) \succ_P (f_j, y_j)$ for all $0 \leq j < i < \alpha$.*

Proof. For an ordinal α, we shall construct such a sequence by transfinite recursion. We start with an arbitrary element $(f_0, y_0) \in F$. For all ordinal $\beta < \alpha$, pick $(f_{\beta+1}, y_{\beta+1}) \in F$ such that $(f_{\beta+1}, y_{\beta+1}) \succ_P (f_\beta, y_\beta)$. Let $\lambda < \alpha$ be a limit ordinal. As F is compact, the net $\big((f_\beta, y_\beta)\big)_{\beta<\lambda}$ has a convergent sub-net $\big((f_{\beta_i}, y_{\beta_i})\big)_{i\in I}$ with limit $(\bar{f}, \bar{y}) \in F$. Let $(f_\lambda, y_\lambda) = (\bar{f}, \bar{y})$. We shall show that the net $\big((f_\beta, y_\beta)\big)_{\beta<\alpha}$ is the desired sequence.

For every ordinal $\beta < \alpha$, by construction, we have $(f_{\beta+1}, y_{\beta+1}) \succ_P (f_\beta, y_\beta)$. For a limiting ordinal $\lambda < \alpha$, there exists a sub-net $\big((f_{\beta_i}, y_{\beta_i})\big)_{i\in I}$ of the net $\big((f_\beta, y_\beta)\big)_{\beta<\lambda}$ that converges to (f_λ, y_λ). We now show that $(f_\lambda, y_\lambda) \succ_P (f_{\beta_i}, y_{\beta_i})$ for all $i \in I$. Pick $i_0 \in I$. Suppose there exists some $t \in \Omega$ such that $\big(f_{\beta_{i_0}}(t), C(f_{\beta_{i_0}}, y_{\beta_{i_0}})\big) \succ_t \big(f_\lambda(t), C(f_\lambda, y_\lambda)\big)$. By continuity of $\succ_k$, there exists some $\beta_j > \beta_{i_0}$ such that $\big(f_{\beta_{i_0}}(t), C(f_{\beta_{i_0}}, y_{\beta_{i_0}})\big) \succ_t \big(f_{\beta_j}(t), C(f_{\beta_j}, y_{\beta_j})\big)$. This leads to a contradiction, hence, for all $\omega \in \Omega$, we have $\big(f_{\beta_{i_0}}(\omega), C(f_{\beta_{i_0}}, y_{\beta_{i_0}})\big) \not\succ_\omega \big(f_\lambda(\omega), C(f_\lambda, y_\lambda)\big)$. Pick some $\beta_l > \beta_{i_0}$. Then there exists $s \in \Omega$ such that $\big(f_{\beta_l}(s), C(f_{\beta_l}, y_{\beta_l})\big) \succ_s \big(f_{\beta_{i_0}}(s), C(f_{\beta_{i_0}}, y_{\beta_{i_0}})\big)$. By negative transitivity, we have $\big(f_\lambda(s), C(f_\lambda, y_\lambda)\big) \succ_s \big(f_{\beta_{i_0}}(s), C(f_{\beta_{i_0}}, y_{\beta_{i_0}})\big)$. As the choice of i_0 is arbitrary, we conclude that $(f_\lambda, y_\lambda) \succ_P (f_{\beta_i}, y_{\beta_i})$ for all $i \in I$. As $\{\beta_i : i \in I\}$ is a co-final subset of $\{\beta : \beta < \lambda\}$, by transitivity of $\succ_P$, we have $(f_\lambda, y_\lambda) \succ_P (f_\beta, y_\beta)$ for all $\beta < \lambda$, completing the proof. □

By Claim A.6 and Claim A.7, as Claim A.7 holds for any ordinal α, we have a contradiction. Thus, there exists a consumption-production pair in F such that it is Pareto optimal among F. □

References

[AD54] K. J. Arrow and G. Debreu. "Existence of an equilibrium for a competitive economy". *Econometrica* 22.3 (1954), pp. 265–290.

[AH71] K. Arrow and F. Hahn. *General Competitive Analysis*. Advanced textbooks in economics. Holden-Day, 1971. ISBN: 9780816202751.

[Ber76] T. C. Bergstrom. "How to discard "free disposability"—at no cost". *J. Math. Econom.* 3.2 (1976), pp. 131–134. ISSN: 0304-4068.

[Deb59] G. Debreu. *Theory of value: an axiomatic analysis of economic equilibrium.* Cowles Foundation for Research in Economics at Yale University, Monograph 17. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1959, pp. xii+114.

[Deb62] G. Debreu. "NEW CONCEPTS AND TECHNIQUES FOR EQUILIBRIUM ANALYSIS". *International Economic Review* 3 (1962), p. 257.

[Flo03] M. Florenzano. *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria.* Boston: Kluwer Academic Publishers, 2003.

[Flo99] M. Florig. "On Irreducible Economies". *Annals of economics and statistics* (1999), pp. 183–199.

[Har05] C. Hara. "Existence of equilibria in economies with bads". *Econometrica* 73.2 (2005), pp. 647–658.

[Hil74] W. Hildenbrand. *Core and equilibria of a large economy.* Princeton Studies in Mathematical Economics, No. 5. With an appendix to Chapter 2 by K. Hildenbrand. Princeton University Press, Princeton, N.J., 1974, pp. viii+251.

[HK75] O. D. Hart and H. W. Kuhn. "A proof of the existence of equilibrium without the free disposal assumption". *Journal of Mathematical Economics* 2.3 (1975), pp. 335–343. ISSN: 0304-4068.

[Mak81] V. L. Makarov. "Some results on general assumptions about the existence of economic equilibrium". *J. Math. Econom.* 8.1 (1981), pp. 87–99. ISSN: 0304-4068.

[McK59] L. W. McKenzie. "On the existence of general equilibrium for a competitive market". *Econometrica* 27 (1959), pp. 54–71. ISSN: 0012-9682.

[McK81] L. W. McKenzie. "The classical theorem on existence of competitive equilibrium". *Econometrica* 49.4 (1981), pp. 819–841. ISSN: 0012-9682.

[PS93] H. Polemarchakis and P. Siconolfi. "Competitive equilibria without free disposal or nonsatiation". *Journal of Mathematical Economics* 22.1 (1993), pp. 85–99. ISSN: 0304-4068.

Robert M. Anderson, University of California, Berkeley, Department of Economics

Haosui Duanmu, Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, China

M. Ali Khan, The Johns Hopkins University, Department of Economics