COWLES COMMISSION DISCUSSION PAPERS: ECONOMICS NO. 226 A

Abstract: Measurable Utility and the Theory of Assets

(Abbreviated and Revised Version of Economics No. 226)

1. PROBLEM:

To define conditions sufficient to make utility indicators unique up to a linear (and not merely up to any monotone) transformation. We study advisable (rather than actual) behavior, and assume that the individual has "complete information" on the relevant probability distributions.

2. DEFINITIONS:

The individual's consumption of commodity h at time t:

$x_h(t) \equiv x_{h+tH}$; $h = 1, \ldots, H$; $t = 0, 1, \ldots, T$

The individual's "future history" $\equiv X \equiv \{x_1, \ldots, x_{H+TH}\} \equiv$ a random vector with probability distribution

$P(X)$ of the mutually exclusive values X_n, $n = 1, \ldots, N$.

Prospect $P^i \equiv P^i(X) \equiv$ vector $\{p_1^i, \ldots, p_N^i\}$ where $p_n^i \equiv$ probability that $X = X_n$ when $P = P^i$.

Sure prospect $P^n \equiv$ vector $\{\delta_{1n}, \ldots \delta_{Nn}\}$.

Prospect of second order $Q^j \equiv Q^j(P) \equiv \{q_1^j, \ldots\}$ where $q_i^j \equiv$ probability that $P = P^i$. Evidently $Q^j(P) \equiv P^k(X) \equiv \{p_1^k, \ldots, p_N^k\}$ where $p_n^k \; (n = 1, \ldots, N) = \sum_i p_n^i q_i^j$.

Balance sheet $S \equiv \{s_1, \ldots\} \equiv$ set of assets, possibly constrained by $g(S) = 0$.

Environment $E \equiv$ the set of achievable prospects: P is in E if there exists S such that $g(S) = 0$ and $P = f(S)$ (production function).

3. AXIOMS A and B

Axiom A (existence of indifference surfaces): There exists a set of sets $\Omega \equiv (\omega(1), \ldots)$ and a relation called "preferred to" with the following properties:

(A.1) Each element of $\omega^{(r)}$ $(r = 1,\ldots)$ is some prospect P.

(A.2) Each prospect P belongs to one and only one of the sets $\omega^{(1)},\ldots$

(A.3) For any two distinct elements of Ω - say $\omega^{(r)}$, $\omega^{(s)}$ -- either $\omega^{(r)} > \omega^{(s)}$ (read: $\omega^{(r)}$ is preferred to $\omega^{(s)}$) or $\omega^{(s)} > \omega^{(r)}$.

(A.4) If $\omega^{(r)} > \omega^{(s)}$ and $\omega^{(s)} > \omega^{(t)}$ then $\omega^{(r)} > \omega^{(t)}$.

(A.5) If $\omega^{(r)} > \omega^{(s)} > \omega^{(t)}$ then there exist two positive numbers a, b $(a + b = 1)$ and two prospects P^i in $\omega^{(r)}$ and P^k in $\omega^{(t)}$ such that the prospect $(aP^i + bP^k)$ belongs to $\omega^{(s)}$.

Remark. Axiom A was implied by the old theory of assets. The ω's are "indifference sets (surfaces)".

Axiom B (placing of higher-order prospects): If P^i, P^j both belong to $\omega^{(r)}$ then, for any P^k, there exist a set $\omega^{(s)}$ and two positive numbers a, b $(a + b = 1)$ such that the prospects $(aP^i + bP^k)$, $(aP^j + bP^k)$ both belong to $\omega^{(s)}$.

4. THEOREMS I - IV:

Theorem I (indifference surfaces in prospect space are parallel hyper-planes): There exists a real vector $C \equiv (c_1,\ldots,c_N)$, and, for every set $\omega = \omega^{(s)}$, a constant $u^{(s)}$ such that:

(I.1) if P is in $\omega^{(s)}$ then $\sum c_n p_n = u^{(s)}$; and

(I.2) if $\omega^{(r)} > \omega^{(s)} > \omega^{(t)}$ then either $u^{(r)} > u^{(s)} > u^{(t)}$ or $u^{(r)} < u^{(s)} < u^{(t)}$.

Definition: $u^{(s)}$ is an utility indicator for all prospects in $\omega^{(s)}$.

Theorem II (utility indicator is a linear function of probabilities):

(II) $u^{(r)} = \sum_n u^{[n]} p_n^i$ = "average utility of P^i", where $u^{[n]} \equiv$ utility indicator of sure prospect $P^{[n]}$.

Theorem III (utility indicator is unique up to a linear transformation):

(III) If $u^{(1)}, u^{(2)}, \dots$ and $v^{(1)}, v^{(2)}, \dots$ are two sets of real numbers such that if $\omega^{(r)} \succ \omega^{(s)} \succ \omega^{(t)}$ then $u^{(r)} > u^{(s)} > u^{(t)}$ and $v^{(r)} > v^{(s)} > v^{(t)}$, then, for any q, $u^{(q)} = g_0 + g_1 \cdot v^{(q)}$, where g_0, g_1 are two real numbers and $g_1 > 0$.

Theorem IV (relation between utility differences and probabilities): If $P^i \in \omega^{(r)}$, $P^k \in \omega^{(r)}$, $aP^i + (1-a) P^k \equiv P^j \in \omega^{(s)}$, $0 < a < 1$, then

(IV) $(u^{(s)} - u^{(t)})/(u^{(r)} - u^{(s)}) = a/(1-a)$.

5. CONCLUSIONS:

(a) Comparison with Neumann-Morgenstern. Their verbal presentation uses Theorem IV above as axiom (with $a=1/2$). In their mathematical presentation their axiom (3:c:B) playing the role of our Axiom B uses operations on non-measurable "entities" (our sets $\omega^{(1)}, \dots$ rather than the prospects $P^1, \dots$). In our terms it would read thus: "Consider prospects P^h in $\omega^{(q)}$ and P^k in $\omega^{(t)}$; let a, b, a', b' be positive numbers, $a + b = a' + b' = 1$. Consider $P^i \equiv aP^h + bP^k \in \omega^{(s)}$, $P^j \equiv a'P^i + b'P^k \in \omega^{(r)}$, $P^h_* \in \omega^{(q)}$, $P^k_* \in \omega^{(t)}$. Then $aa' P^h_* + (1-aa') P^k_* \in \omega^{(r)}$."

(b) Maximization of "average utility" (defined in Theorem II): Choose P^i in E such that for every P^j in E

$$\sum_n u[n] p^i_n \geq \sum_n u[n] p^j_n.$$

(c) Why should statisticians minimize average loss?

(d) Applications to economic theory. Remember that $X \equiv \{x_1, x_2, \dots\}$. Theorem III can be rewritten as

$$U(P) = \sum_X u(X) P(X) \equiv \sum_{x_1, x_2, \dots} u(x_1, x_2, \dots) P(x_1, x_2, \dots)$$

where U is the utility of a prospect, $u(X_n)$ is the utility of having X_n with certainty, and P(X) is the joint probability distribution of X. Define the moments $\mu_g = \mathcal{E} x_g$, $\mu_{gk} = \mathcal{E}(x_g - \mu_g)(x_k - \mu_k)$, etc. Expand u(X) about $M \equiv \{\mu_1, \mu_2, \dots\}$ and obtain

(V) $U(P) = u(M) + 1/2 \sum_{g,k} \mu_{gk} \cdot u_{gk}(M) + \ldots$ where $u_{gk} \equiv \partial^2 u / \partial x_g \partial x_k$;

hence

$$\frac{\partial U(P)}{\partial \mu_{gk}} = \frac{1}{2} u_{gk}, \text{ etc; } g,k = 1,\ldots,G$$

For $g = k$, u_{gk} is the "rate of decrease of marginal utility" and the result (for the case of a unique commodity, "income") goes back to Marshall and D. Bernoulli: if the marginal utility of x_g is a decreasing function, a prospect with high variance of x_g is undesirable. "Disutility of gambling" requires thus additional postulates about $u(X)$. It is not implied by the Axioms A,B on "advisable behavior".

(e) For $g \neq k$, u_{gk} is Pareto's measure of complementarity (as distinct from Hicks' measure): a prospect with high correlation between two complementary (competing) goods is desirable (undesirable).

(f) Risk premium $r = r(P)$ has been defined, for the case of a single commodity ($G = 1$), in terms of moments $\mu_1 = \mu_1(P)$, $\mu_{11} = \mu_{11}(P)$ in two different ways:

(1) (Lange): by $U(P) = u(\mu_1 - r)$;

(2) by $r = \mu_{11} \cdot (d\mu_1/d\mu_2)$ for $U(P) =$ constant.

With utility measurable, both definitions permit the evaluation of r in terms of the μ's and the derivatives of u, with the help of (V).

3.6. The Axioms and Their Interpretation *

3.6.1. Our axioms are these:

We consider a system U of entities $u, v, w, \ldots$. In U a relation is given, $u > v$, and for any number α, $(0 < \alpha < 1)$, an operation

$$\alpha u + (1-\alpha)v = w.$$

These concepts satisfy the following axioms:

(3:A) $u > v$ is a complete ordering of U.

This means: Write $u < v$ when $v > u$. Then:

(3:A:a) For any two u, v one and only one of the three following relations holds:

$$u = v, \quad u > v, \quad u < v.$$

(3:A:b) $u > v$, $v > w$ imply $u > w$.

(3:B) Ordering and combining.

(3:B:a) $u < v$ implies that $u < \alpha u + (1-\alpha)v$.

(3:B:b) $u > v$ implies that $u > \alpha u + (1-\alpha)v$.

(3:B:c) $u < w < v$ implies the existence of an α with

$$\alpha u + (1-\alpha)v < w.$$

(3:B:d) $u > w > v$ implies the existence of an α with

$$\alpha u + (1-\alpha)v > w.$$

(3:C) Algebra of combining.

(3:C:a) $$\alpha u + (1-\alpha)v = (1-\alpha)v + \alpha u.$$

(3:C:b) $\alpha(\beta u + (1-\beta)v) + (1-\alpha)v = \gamma u + (1-\gamma)v$ where $\gamma = \alpha\beta$.

* Von Neumann and Morgenstern, Theory of Games and Economic Behavior, Page 26.