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July 2022

COWLES FOUNDATION DISCUSSION PAPER NO. 2321

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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Informational Intermediation, Market Feedback, and Welfare Losses*

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July 28, 2022

Abstract

This paper examines the welfare implications of third-party informational intermediation. A seller sets the price of a product that is sold through an informational intermediary. The intermediary can disclose information about the product to consumers and earns a fixed percentage of sales revenue in each period. The intermediary's market base grows at a rate that increases with past consumer surplus. We characterize the stationary equilibria and the set of subgame perfect equilibrium payoffs. When *market feedback* (i.e., the extent to which past consumer surplus affects future market bases) increases, welfare may decrease in the Pareto sense.

KEYWORDS: Informational intermediary, market base, market feedback, consumer surplus, Pareto-inferior outcomes, stationary-Markov equilibrium, subgame perfect equilibrium.

JEL CLASSIFICATION: C73, D61, D82, D83, L15, M37

*This paper was previously circulated under the titles, "Information Design in Bilateral Trade and Unintended Welfare Loss" and "Market-Minded Informational Intermediary and Unintended Welfare Loss." This version supersedes the previous ones. We thank Pak Hung Au, James Bergin, Ben Brooks, Daniele Condorelli, Hector Chade, Yeon-Koo Che, Yi-Chun Chen, Joyee Deb, Laura Doval, Jack Fanning, Mira Frick, Nima Haghpanagh, Marina Halac, Wei He, Johannes Hörner, Yunzhi Hu, Ryota Iijima, Emir Kamenica, Navin Kartik, Changhwa Lee, Fei Li, Yunan Li, Muxin Li, Elliot Lipnowski, Wooyoung Lim, Ting Liu, Qingming Liu, Fei Long, Stephen Morris, Kota Murayama, Aniko Öry, Jacopo Perego, John Quah, Doron Ravid, Phil Reny, Anne-Katrin Roesler, Hamid Sabourian, Evan Sadler, Tomasz Sadzik, Larry Samuelson, Anna Sanktjohanser, Andy Skrzypacz, Philipp Strack, Roland Strausz, Tomasz Strzalecki, Nancy Stokey, Rui Tang, Mark Whitmeyer, Yu Fu Wong, Qinggong Wu, Nathan Yoder, Andy Zapechelnyuk, Chen Zhao, and Jidong Zhou for their valuable comments. We also thank the participants of the various seminars and conferences at which this paper was presented. All errors are our own.

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1 Introduction

In many markets, products are sold through an intermediary, who facilitates trade and provides product information to consumers before they make purchasing decisions. For example, financial advisors serve as intermediaries through whom issuers sell financial products (e.g., securities) to investors. At the same time, these financial advisors provide information about the financial products to investors. In insurance markets, insurance companies collaborate with insurance brokers who provide information to their customers and persuade them to buy the companies' insurance plans. Similarly, in the emerging online market, intermediaries such as influencers and key opinion leaders (KOL) provide product information to their followers, who then use that information to make purchasing decisions. These *informational intermediaries* often share two common features: (i) they operate independently of the product sellers and collect a certain percentage of sales revenue as a commission through revenue-sharing arrangements;[1] and (ii) they have their own market bases that are affected by past consumer satisfaction. When collaborating with product sellers, intermediaries provide accesses to their market bases that would otherwise be difficult for product sellers to reach by themselves.

As intermediaries' market bases depend on past consumer satisfaction, forward-looking and revenue-maximizing intermediaries and sellers are "consumer-minded" and care about consumer surplus, even if revenue is solely derived from sales, because, after all, a larger market base leads to increased future revenue. Consequently, the level of *market feedback*—the degree to which consumer satisfaction affects the intermediary's future market base—is crucial for understanding the incentives at play, as well as the welfare outcomes. The level of market feedback can be affected by various factors. For example, market feedback may be higher if a product is naturally more visible, leading to increased consumer engagement in interpersonal, word-of-mouth communication after purchasing (see, e.g., Keller, Fay, and Berry (2007)), if there are transparent channels for consumer reviews (e.g., rating systems and recommendation algorithms), or if consumers have multiple alternative sources of information (e.g., competition among intermediaries). Higher levels of market feedback mean that consumer satisfaction is more consequential to the intermediary's market base and therefore intermediaries and sellers are more consumer-minded.

Nonetheless, due to the separation between product sellers and intermediaries, sellers and

[1] It is more common that traditional intermediaries like insurance brokers and financial brokers are paid through commissions and such arrangements in the influencer market are less frequent. Nevertheless, one of the major income sources for online influencers are *affiliate links*. That is, product sellers partner with third party websites such as rStyle or ShopStyle and invite influencers to present their products to consumers while sharing an affiliate link generated by the third party websites. These affiliate links keep track of which influencer brought in the customer, and the influencers are then paid a fixed share of revenue per purchase. For our purpose, this business model can also be thought of as a revenue-sharing arrangement between sellers and influencers.

intermediaries may be "consumer-minded" to different degrees. While both product sellers and intermediaries may benefit from a larger future market base, a particular intermediary's market base is only relevant to a product seller when they collaborate. In contrast, intermediaries' market bases matter to the intermediaries throughout their entire careers. In other words, the nature of informational intermediation implicitly creates a gap in the degree of consumer-mindedness between product sellers and informational intermediaries during collaborations. As intermediaries are always more motivated to manage their own market bases, they are naturally more consumer-minded than product sellers.

In this paper, we study how market feedback affects welfare in the presence of third-party informational intermediation. Our central question is: Does a higher level of market feedback always benefit consumers? Or, more generally, how does market feedback affect welfare outcomes? It may seem natural to conclude that higher market feedback always benefits consumers because it strengthens intermediaries' and sellers' incentives to improve consumer surplus. However, we show that higher market feedback does not always benefit consumers and may in fact *decrease* welfare in the Pareto sense.

These welfare losses stem from the misalignment of incentives between product sellers and informational intermediaries. As market feedback increases, intermediaries become more consumer-minded relative to sellers. To realign incentives, sellers raise prices so that intermediaries get higher commissions for each unit sold and become more willing to sacrifice consumer surplus to generate sales. In equilibrium, consumers do not benefit from higher market feedback as higher prices offset its effect. In the meantime, higher prices lead to fewer sales and lower revenues, resulting in Pareto-inferior outcomes.

Specifically, we consider a dynamic game in discrete time with a seller (he) and an intermediary (she). To capture the aforementioned economic features, we assume that the intermediary has a market base which grows at a rate that increases with past consumer surplus. Coupled with an evolving market base, we model the separation between sellers and intermediaries by assuming that the product seller's discount factor is smaller than that of the intermediary. In other words, the seller values the future market base less because the intermediary's market base is valuable for the seller only when the seller is actively collaborating with the intermediary. For ease of exposition, our baseline model assumes that the seller's discount factor is zero and that the market growth rate is affine in consumer surplus.[2] In each period, a mass of short-lived consumers with unit demands arrive. These consumers have heterogeneous tastes but do not know their values for the product upon arrival. The seller first chooses a price for his product. After observing the price, the intermediary then discloses information about the product to consumers so that they receive signals about their values for the product. Each consumer then decides whether to buy the product based on

[2] Extensions that relax these assumptions can be found in Section 7.

the information provided by the intermediary. The intermediary and the seller then divide the total sales revenue according to a fixed percentage.

The welfare loss result is established by completely characterizing the equilibrium outcomes in our model. We first restrict attention to stationary-Markov equilibrium outcomes (Theorem 1). When the level of market feedback is low, there is a unique stationary-Markov equilibrium outcome wherein consumer surplus is zero and the seller and the intermediary extract all of the surplus. As the level of market feedback increases, the intermediary cares more about consumer surplus. In response, the seller raises the price to suppress the intermediary's desire to improve consumer surplus. In equilibrium, the information provided is such that consumers with values above a (nonzero) cutoff have the same posterior expected value, which is equal to the price chosen by the seller. As a result, consumer surplus remains zero and sales revenue decreases as the price increases, leading to Pareto inferior outcomes.

In addition to the stationary-Markov equilibrium outcomes, we characterize the set of subgame perfect equilibrium payoffs for any fixed discount factor and market feedback level (Theorem 2). A subgame perfect equilibrium outcome that is Pareto-dominated by all other outcomes exists when the level of market feedback is below a certain threshold. Moreover, like the stationary-Markov equilibrium outcomes, this least efficient outcome worsens (in the Pareto sense) as market feedback increases.

Consequently, our results serve as a cautionary tale about improvements in market feedback in third-party-intermediated markets. Increases in the level of market feedback (e.g., due to changes in market structures or technologies) may be detrimental to the entire market, even though these increases bolster the incentives to improve consumer surplus. Meanwhile, policies that seek to improve market feedback (e.g., improving digital recommender systems or encouraging word-of-mouth communication among consumers)—or, in the same regard, policies that aim to incentivize intermediaries to enhance consumer surplus—should be evaluated and implemented carefully. After all, higher market feedback may be undesirable in the presence of third-party informational intermediation.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. We then characterize the stationary-Markov equilibrium outcomes in Section 4 and the subgame perfect equilibrium payoffs in Section 5. Section 6 and Section 7 discuss the policy implications and extensions of the model, respectively. Section 8 then concludes.

2 Related Literature

This paper builds upon the growing literature on information design and pricing, which considers the design of information under various market structures in which pricing decisions are made, including monopoly (e.g., Bergemann, Brooks, and Morris (2015), Roesler and

Szentes (2017), Ravid, Roesler, and Szentes (2022), and Libgober and Mu (2021));[3] oligopoly (e.g., Boleslavsky, Hwang, and Kim (2019), Armstrong and Zhou (2022); and Elliot, Galeotti, Koh, and Li (2021)); auctions (e.g., Bergemann and Pesendorfer (2007), Shi (2012), Chen and Yang (2022), Kim and Koh (2022), Brooks and Du (2021), and Terstiege and Wasser (2022)); consumer search (e.g., Anderson and Renault (2006), Board and Lu (2018), Au and Whitmeyer (forthcoming), and Bergemann, Brooks, and Morris (2021)); and third-party intermediation (e.g., Yang (2022)).

Among the aforementioned articles, our model is the most related to those of Roesler and Szentes (2017), Ravid, Roesler, and Szentes (2022), and Libgober and Mu (2021). Roesler and Szentes (2017) examines the optimal information for consumers in a monopolistic setting in which the monopolist always chooses a price optimally based on the information structure available to consumers. Ravid, Roesler, and Szentes (2022) characterizes the equilibrium outcomes when the monopolist chooses a price and consumers acquire information simultaneously. Libgober and Mu (2021) considers a dynamic pricing problem in which consumers can delay purchases and the seller can commit to a price path, and nature chooses the worst-case consumer information for each period after observing the realized price. In our paper, we consider a dynamic game in which each stage game involves a monopolist choosing a price and an intermediary choosing an information structure for consumers after seeing the price. As such, pricing and information disclosure occur in every period in our model.[4] Moreover, as the informational intermediary in our paper is long-lived, future continuation plays may affect what current information is disclosed by the intermediary, which in turn may affect outcomes.

As the intermediary seeks to enhance and manage her market base, which in turn depends on her past behavior, our paper is also related to the literature on reputation, including the general theory of reputation (e.g., Fundenberg and Levine (1989) and Fundenberg and Levine (1992)), the effect of reputation on firm competition and on inducing efficient levels of effort (e.g., Mailath and Samuelson (2001) and Hörner (2002)), and its effect on expert credibility

[3] See also: Haghpanah and Siegel (forthcomingb), Haghpanah and Siegel (forthcominga), Deb and Roesler (2021), Yang (2021), and Bergemann, Heumann, and Morris (2022) for examples of multi-product counterparts and Doval and Skreta (2022) for an example of a monopolist with limited commitment.

[4] Furthermore, even when focusing on stationary-Markov equilibria, the stage game in our model differs from those in the aforementioned papers. The seller chooses a selling mechanism either after or while the information structure is chosen in Roesler and Szentes (2017) and Ravid, Roesler, and Szentes (2022), respectively, whereas in our paper, the seller chooses the price before the information structure is chosen. In Libgober and Mu (2021), the seller can commit to any selling mechanism and nature always plays against the seller, while in this paper, the seller is restricted to choosing a posted-price mechanism and the information structure is chosen to maximize a linear combination of sales revenue and consumer surplus. In the meantime, information structures are chosen in response to the posted prices both in our model and in that of Libgober and Mu (2021).

and their abilities to communicate information (e.g., Ely and Välimäki (2003), Ottaviani and Sørensen (2006), and Vong (2022b)). A key distinction of our paper is that we abstract from the endogenous formulation of reputation and model the intermediary's market base via an exogenous evolution process in which the growth rate in each period is a function of consumer surplus during the same period.[5] Our assumption that the market growth rate depends only on consumer surplus in the previous period resembles models in which there is only limited records (e.g., Liu (2011) and Liu and Skrzypacz (2014)).

Methodologically, the intermediary's disclosure problem in our model can be regarded as a Bayesian persuasion problem in which only the expected value of the state is payoff relevant (see Gentzkow and Kamenica (2016), Dworczak and Martini (2019), and Kolotilin, Mylovanov, and Zapechelnyuk (forthcoming)).[6] In addition, our model can be regarded as a dynamic game with a long-lived player and a short-lived player, as studied by Fundenberg and Levine (1989) and Fudenberg, Kreps, and Maskin (1990), with the distinction that there is a history-dependent state that scales the stage game payoffs in our model.

In terms of applications, this paper is also related to the literature on certification (e.g., Biglaiser (1993), Lizzeri (1999), Stahl and Strausz (2017), Harbaugh and Rasmusen (2018), Vong (2022a), and Ali, Haghpanah, Lin, and Siegel (2022)) and the recent studies on online influencers. In terms of the literature on certification, our model is closer to that of Lizzeri (1999), who analyzes the optimal disclosure policy of a certifier who can charge the seller a fee, in exchange for providing credible product information to buyers in a market featuring adverse selection. In contrast, our informational intermediary discloses product information to consumers after seeing the seller's price. Moreover, there is no adverse selection problem in our setting without the presence of the intermediary, as sellers do not possess any private information.

In terms of the literature on online influencers. Fainmesser and Galeotti (2021) studies a market in which influencers can be paid to endorse products as sponsored recommendations with an opportunity cost of recommending fewer carefully selected, high-quality products to consumers, which in turn affects their follower base. In our model, the intermediary shares a similar trade-off between long-term market base and short-term revenue. Our model differs, however, by focusing on the information provision services of an intermediary as opposed to product recommendations and endorsements. Mitchell (2021) also examines the economic implications of an influencer's trade-off between advertisement content and good advice and characterizes the optimal dynamic contract for followers. Our paper is complementary in the sense that we abstract from consumers' long-term relationships with the intermediary and focus on pricing and information provision, while Mitchell (2021) abstracts from pric-

[5]Variations in the growth rate can arise from different consumer communication networks, different word-of-mouth behaviors, different rating systems, or different degrees of competition.

[6]See Kamenica (2018) for a comprehensive review of the Bayesian Persuasion literature.

ing and information provision and examines the relationship between the follower and the influencer. Pei and Mayzlin (2022) studies influencers' paid promotional content using a Bayesian persuasion framework. In their model, a product seller can pay an influencer to increase the likelihood of a positive signal in a two-state-two-signal Blackwell experiment. The main difference between their model and ours is that our intermediary can use any Blackwell experiment to inform consumers about their values, and the information cannot be altered by the seller ex-post.

3 Model

3.1 Primitives

Time $t \in \mathbb{N} \cup \{0\}$ is discrete. There is a long-lived informational intermediary with discount $\delta \in (0, 1)$ and a sequence of short-lived sellers.[7] In each period t, the intermediary's market base is denoted by $m_t \geq 0$, which is the mass of (short-lived) consumers arriving in period t. Consumers have different values for the seller's product. Across consumers, values are distributed according to a (commonly known) demand function $\overline{D} : \mathbb{R}_+ \to [0, 1]$, where $\overline{D}(p)$ is the share of consumers with values above p. Assume that $\overline{D}$ is *regular*, in the sense that it is continuously differentiable, has a non-zero derivative on an interval in $\mathbb{R}_+$, and induces a decreasing marginal revenue function.[8] Upon arrival, consumers do not know their values and must learn about the product and determine their values through the information provided by the intermediary.

3.2 Timing and Payoffs

In each period $t \in \mathbb{N} \cup \{0\}$, the timing of events is as follows: (i) a mass m_t of consumers arrive, (ii) the seller chooses a price p_t, (iii) the intermediary observes the chosen price and then provides information about v to consumers (see Section 3.3), (iv) consumers then decide whether to buy the product after receiving information and observing the price, and (v) payoffs are realized. A consumer has payoff $v - p_t$ if his value is v and he buys the product at

[7] As explained in Section 1, the assumption that sellers are short-lived aims to capture the idea that the intermediary operates independently of the sellers and that sellers only have access to the intermediary's market base during the period of collaboration. A collaboration period is typically short term because sellers may collaborate with different intermediaries during different product cycles. Consequently, sellers are less concerned about the intermediary's market base when selling through a particular intermediary. In essence, the sellers being short-lived is only a simplifying assumption. In Section 7, we generalize the model and consider long-lived sellers who discount the future faster than the intermediary. Our main result remains qualitatively similar.

[8] That is, the function $q \mapsto q\overline{D}^{-1}(q)$ is concave. This is equivalent to assuming that $1 - \overline{D}$ is regular in the Myersonian sense.

price p_t; and has zero payoff if he does not buy the product. The seller and the intermediary share the revenue according to a fixed proportion $\alpha \in (0,1)$. The seller's payoff in period t is $1-\alpha$ share of sales revenue in period t and the intermediary's payoff in period t is α share of sales revenue in period t.

The intermediary's market base m_t in period t depends on outcomes in previous periods. m_0 is normalized to 1. In each period $t+1 \in \mathbb{N}$, the growth rate of m_t is an affine function of the average consumer surplus in period t. That is, for all $t \in \mathbb{N} \cup \{0\}$,

$$m_{t+1} = m_t(\gamma + \beta\sigma_t), \tag{1}$$

for some $\beta \geq 0$ and $\gamma \in [0, 1/\delta)$, where σ_t denotes the average consumer surplus in period t. Henceforth, β represents *market feedback* because it determines the extent to which past consumer surplus affects future market bases.

3.3 Information

In each period, consumers receive information about v from the intermediary. As consumers have quasi-linear payoffs, their purchasing decisions are solely based on their posterior expected values given the information. Thus, information can be represented as distributions of posterior expected values, which are known to be mean-preserving contractions of $\overline{D}$ (see Strassen, 1965). Specifically, let $\mathcal{D}$ be the collection of nonincreasing, upper-semicontinuous functions $D : \mathbb{R}_+ \to [0,1]$ such that $D(0) = 1$ and

$$\int_p^\infty D(v)\,\mathrm{d}v \leq \int_p^\infty \overline{D}(v)\,\mathrm{d}v, \tag{2}$$

for all $p \geq 0$, with equality at $p = 0$. Note that $\mathcal{D}$ describes the collection of mean-preserving contractions of $\overline{D}$ and is illustrated in Figure 1, where each $D \in \mathcal{D}$ corresponds to a decreasing and convex function whose graph is located in the highlighted area. A *disclosure policy* is defined as an element D of $\mathcal{D}$.[9]

When information is represented in this way, consumer choices are entirely determined by choices of price and disclosure policy (except for consumers who are indifferent). Indeed, for any price $p \geq 0$ and for any disclosure policy $D \in \mathcal{D}$, since $D(p)$ reflects the share of consumers whose posterior expected values are at least p, the share of consumers who purchase must be between $D(p^+)$ and $D(p)$ (which in turn corresponds to a subgradient of the convex function representing D in Figure 1).

[9] This model of information disclosure is similar to that in Lewis and Sappington (1991) and Johnson and Myatt (2006) and aligns with the modeling approach of Anderson and Renault (2006), Roesler and Szentes (2017), Libgober and Mu (2021), Armstrong and Zhou (2022), and Ravid, Roesler, and Szentes (2022). In this approach, consumers have differentiated values for the same product. The information disclosed can be viewed as information about the characteristics of the product and informs consumers about their true values for the product.

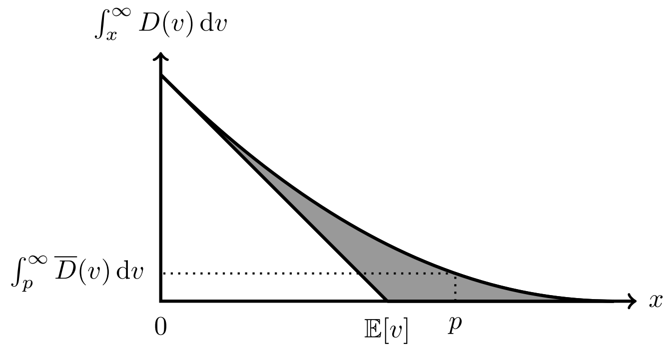


Figure 1: Feasible Disclosure Policies $\mathcal{D}$

3.4 Strategies and Solution Concepts

Based on how information is defined in Section 3.3, the model can be represented by a perfect-information dynamic game with a state variable m_t. There is a long-lived player (the intermediary) with discount δ and a sequence of short-lived players (sellers and "tie-breakers"). In each period $t \in \mathbb{N} \cup \{0\}$, a short-lived seller arrives and chooses a price p_t. The intermediary sees p_t and chooses a disclosure policy $D_t \in \mathcal{D}$. A short-lived tie-breaker then sees p_t and D_t and chooses $q_t \in [D_t(p_t^+), D_t(p_t)]$. The intermediary and the seller get α share and $1-\alpha$ share of the sales revenue $m_t \cdot p_t \cdot q_t$ respectively. The tie-breaker always gets a constant payoff. The state variable in the next period m_{t+1} is given by

$$m_{t+1} = m_t \left(\gamma + \beta \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v \right). \tag{3}$$

In this game, histories consist of all past plays in previous periods.[10] In every period $t \in \mathbb{N} \cup \{0\}$, the seller's strategy maps past histories up to period $t-1$ to a price $p_t \geq 0$; the intermediary's strategy maps past histories up to period $t-1$ and the seller's price p_t in period t to a disclosure policy $D_t \in \mathcal{D}$; and the tie-breaker's strategy maps past histories up to period $t-1$, the seller's price p_t in period t, and the intermediary's disclosure policy D_t in period t, to some $q_t \in [D_t(p_t^+), D_t(p_t)]$.

In the spirit of Maskin and Tirole (2001), we say that a strategy profile is *stationary-Markov* if the seller's price does not depend on any past histories, the intermediary's disclosure policy only depends on the seller's price in the same period, and the tie-breaker's strategy only depends on the seller's price and the intermediary's disclosure policy in the same period. A subgame perfect equilibrium is said to be *stationary-Markov* (or simply *stationary* hereafter) if it is a stationary-Markov strategy profile.

[10] See the Online Appendix for a formal definition of histories.

From (3), there may exist equilibria in which the intermediary's payoff is infinite, as the market base is not bounded. In these equilibria, the intermediary is indifferent when choosing among many strategies that lead to diverging continuation payoffs given the seller's strategy. Hence, the seller can be disciplined to charge many different prices, leading to multiple equilibria. Discussions of these equilibria are relagated to the Online Appendix.

Henceforth, we slightly abuse the terminology and focus only on equilibria in which the intermediary's continuation payoffs are finite in every subgame. In Section 4, we characterize the set of stationary equilibrium outcomes. Section 5 further characterizes the payoffs under all subgame perfect equilibria.

We conclude this section by noting that an increase in β has three effects in our model: Higher β leads to (i) a more consumer-minded intermediary; (ii) a higher market growth rate; and (iii) possible changes to the equilibrium strategies. It is noteworthy that effect (i) is always (weakly) beneficial for consumers, while effect (ii) is always (weakly) beneficial for all players. Nonetheless, our main results suggest that a higher β may lead to Pareto worse outcomes despite the positive effects of (i) and (ii). In other words, the main takeaway of our results is that effect (iii) could possibly dominate effects (i) and (ii).

4 Inefficiency of Higher Market Feedback: Stationary

In this section, we characterize all stationary equilibrium outcomes and show that higher market feedback may lead to Pareto-inferior outcomes. Given any strategy profile, we refer to the sequence of normalized sales revenues (i.e., sales revenues per unit of market base), average consumer surpluses, the intermediary's normalized continuation payoffs (i.e., continuation payoffs per unit of market base), prices, and market bases across different periods as an *outcome*, which we denote by $\mathbf{z} := \{r_t, \sigma_t, \omega_t, p_t, m_t\}$.

Definition 1. For any two outcomes $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ and $\mathbf{z}' = \{r'_t, \sigma'_t, \omega'_t, p'_t, m'_t\}$, we say that $\mathbf{z}$ *dominates* $\mathbf{z}'$ if for all t, $m'_t \leq m_t$, $r'_t \leq r_t$, $\sigma'_t \leq \sigma_t$, and $\omega'_t \leq \omega_t$, with at least one inequality being strict.

Note that, if $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ dominates $\mathbf{z}' = \{r'_t, \sigma'_t, \omega'_t, p'_t, m'_t\}$, then for all t, $m'_t r'_t \leq m_t r_t$, $m'_t \sigma'_t \leq m_t \sigma_t$, and $m'_t \omega'_t \leq m_t \omega_t$, with at least one inequality being strict. Thus, $\mathbf{z}'$ must be less efficient than $\mathbf{z}$ in the Pareto sense.

In stationary equilibria, players' strategies do not depend on histories in previous periods, and hence only the market bases can depend on t. As a result, we can write a stationary equilibrium outcome as $\mathbf{z}^{\mathrm{s}} = (r^{\mathrm{s}}, \sigma^{\mathrm{s}}, \omega^{\mathrm{s}}, p^{\mathrm{s}}, \{m^{\mathrm{s}}_t\})$.

In what follows, we characterize all stationary equilibrium outcomes and study how the equilibrium outcomes vary when we adjust the value of market feedback β. Let $\overline{p}$ be the unique solution to $\max_p p\overline{D}(p)$, and let

$$\underline{\beta} := \frac{1-\gamma\delta}{\delta\mathbb{E}[v]} \quad \text{and} \quad \overline{\beta} := \frac{1-\gamma\delta}{\delta\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v}.$$

Proposition 1 (Inefficiency of High Feedback—Stationary)**.** *For any $\beta < \overline{\beta}$, there exists a unique stationary equilibrium outcome $\mathbf{z}^{\mathrm{s}}(\beta)$. Furthermore, $\mathbf{z}^{\mathrm{s}}(\beta)$ dominates $\mathbf{z}^{\mathrm{s}}(\beta')$ for all β, β' such that $\underline{\beta} < \beta < \beta' < \overline{\beta}$.*

According to Proposition 1, higher market feedback does not necessarily benefit consumers, even though it makes the intermediary care more about consumer surplus. In fact, higher market feedback may even lead to Pareto inferior outcomes. Whenever $\beta \in (\underline{\beta}, \overline{\beta})$, an increase in market feedback always leads to a Pareto-worse outcome.

Proposition 1 is established by characterizing all of the stationary equilibrium outcomes. We begin the analysis by noting that the one-shot deviation principle still holds when considering strategy profiles that yield a finite continuation payoff for the intermediary at every history. The reason for this is that stage game payoffs are bounded from below and the intermediary's payoff is additively separable. Lemma 1 summarizes this observation.

Lemma 1 (One-Shot Deviation Principle)**.** *Given any strategies of the seller and the tie-breaker. For any history h^t in any period t, and for any strategy of the intermediary that yields a finite continuation payoff, there is a profitable deviation from the continuation strategy at h^t if and only if there is a profitable one-shot deviation at some history after h^t.*

We now outline the characterization of the stationary equilibrium outcomes. Note that with Lemma 1, stationary equilibria can be characterized by the best responses of both the intermediary and the (short-lived) seller in each period while holding each other's strategy fixed. This leads to the following lemma.

Lemma 2. *A stationary equilibrium is characterized by a tuple $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ where $\omega^{\mathrm{s}}, p^{\mathrm{s}} \in [0,\infty)$ and $\mathbf{D}^{\mathrm{s}} : \mathbb{R}_+ \to \mathcal{D}$ satisfy the following conditions:*

$$\omega^{\mathrm{s}} = \sup_{D\in\mathcal{D}}\left[\alpha p^{\mathrm{s}} D(p^{\mathrm{s}}) + \delta\omega^{\mathrm{s}}\left(\gamma + \beta\int_{p^{\mathrm{s}}}^{\infty} D(v)\,\mathrm{d}v\right)\right], \tag{4}$$

$$p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}}) \geq p\mathbf{D}^{\mathrm{s}}(p|p), \tag{5}$$

for all $p \geq 0$,

$$\alpha p\mathbf{D}^{\mathrm{s}}(p|p) + \delta\omega^{\mathrm{s}}\left(\gamma + \beta\int_{p}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p)\,\mathrm{d}v\right) \geq \alpha p D(p) + \delta\omega^{\mathrm{s}}\left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right), \tag{6}$$

for all $p \geq 0$ and for all $D \in \mathcal{D}$. Furthermore, the outcome of any stationary equilibrium $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ is given by $(r^{\mathrm{s}}, p^{\mathrm{s}}, \sigma^{\mathrm{s}}, \omega^{\mathrm{s}}, \{m_t^{\mathrm{s}}\})$, where $r^{\mathrm{s}} = p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}})$, $\sigma^{\mathrm{s}} = \int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\,\mathrm{d}v$, and $m_t^{\mathrm{s}} = (\gamma + \beta\sigma^{\mathrm{s}})^t$ for all $t \in \mathbb{N}\cup\{0\}$.

One of the implications of this characterization is that in each period, the tie-breaker always breaks ties to maximize sales revenue. We therefore do not explicitly keep track of the tie-breakers' strategies in the analysis of stationary equilibria.

Using Lemma 2, the characterization of stationary equilibria becomes essentially a static problem that consists of the following: (i) solving for the intermediary's per-period best response given a price p chosen by the seller during that period and given the intermediary's continuation value (solving (6) given p and ω^{s}); (ii) solving for the seller's best response given the solution to (i) and given the intermediary's continuation value (solving (5) given ω^{s} and the intermediary's best response derived in (i)); and (iii) finding a consistent continuation payoff (verifying (4) given the solutions to (i) and (ii)). We now introduce two lemmas that characterize the solutions to (i) and (ii) for a fixed continuation value.

For any $p \geq 0$, let $v(p) := \mathbb{E}[v|v \geq p]$ and let $v^{-1}(p) := \inf\{x \geq 0 | v(x) \geq p\}$.[11] Notice that both v and v^{-1} are nondecreasing and $v^{-1}(p) = 0$ for all $p \in [0, \mathbb{E}[v]]$.

Lemma 3. *For any $p, \omega \in [0, \infty)$,*

$$\Delta(p|\omega) := \operatorname*{argmax}_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta\omega \left(\gamma + \beta \int_p^\infty D(v)\, \mathrm{d}v \right) \right]$$

is nonempty. Moreover, for any $D \in \Delta(p|\omega)$, $D(v) = \overline{D}(\xi(p|\omega))$, for all $v \in [\xi(p|\omega), p]$ and

$$\int_{\xi(p|\omega)}^\infty D(v)\, \mathrm{d}v = \int_{\xi(p|\omega)}^\infty \overline{D}(v)\, \mathrm{d}v,$$

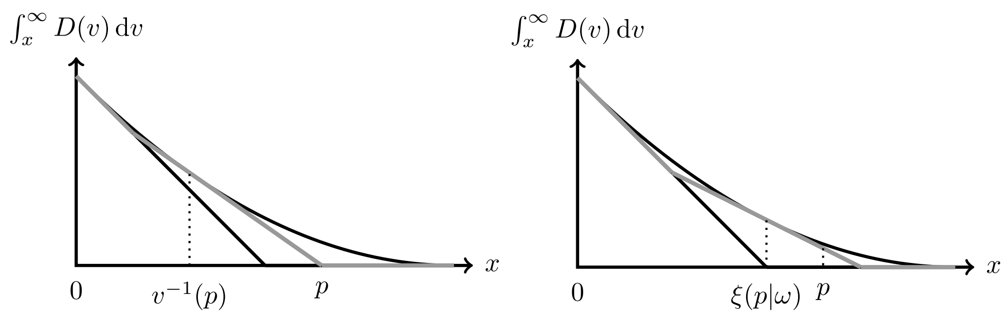
where

$$\xi(p|\omega) := \max\left\{ \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p, v^{-1}(p) \right\}.$$

Lemma 3 provides a characterization of the intermediary's optimal disclosure policy given a price p and a continuation value ω. For any $p, \omega \in [0, \infty)$, the intermediary essentially faces a static problem where she chooses a demand $D \in \mathcal{D}$ to maximize a linear combination of sales revenue and consumer surplus.

To understand the intuition behind this result, consider first the case when $\beta = 0$. In this case, the intermediary seeks to maximize sales revenue, which, from her perspective, is equivalent to maximizing sales volume for each given price. For any price $p \geq 0$, the intermediary can achieve this goal by simply disclosing whether v is above a threshold $v^{-1}(p)$. In doing so, consumers with values above $v^{-1}(p)$ would have a posterior expected value of $\max\{p, \mathbb{E}[v]\}$ and would buy the product (see Figure 2a), whereas consumers with values below $v^{-1}(p)$ would not buy. Note that consumer surplus is zero for all price. Now suppose

[11] As a convention, if p is greater than the upper bound of the support of $\overline{D}$, we define $\mathbb{E}[v|v \geq p]$ as this upper bound. If p is greater than $\max_{x \geq 0} v(x)$, we define $v^{-1}(p)$ as $\max_{x \geq 0} v(x)$.



(a) Optimal cutoff $= v^{-1}(p)$

(b) Optimal Cutoff $= \xi(p|\omega)$

Figure 2: Optimal Cutoff

that $\beta > 0$ and is large enough. In this case, the intermediary will benefit from leaving some surplus to consumers. This means that she may wish to prevent low-value consumers from buying the product at a high price. As a result, the intermediary would increase the threshold to $\xi(p|\omega)$ and encourage fewer consumers to buy (see Figure 2b). As $\beta \to \infty$, the cutoff $\xi(p|\omega)$ converges to the seller's price p, ensuring that every purchasing consumer retains a nonnegative surplus.

For any $\omega \in [0, \infty)$, when anticipating the intermediary's best response, the seller effectively solves a revenue maximization problem in which the demand at price p is given by the sales volume induced by the intermediary's best response to price p. Lemma 4 characterizes the solution to this problem.

Lemma 4. *For any $\omega \in [0, \infty)$ and for any selection $\mathbf{D}$ of $\Delta(\cdot|\omega)$, the maximization problem*

$$\max_{p \geq 0} p\mathbf{D}(p|p)$$

has a unique solution $\tilde{p}$. Furthermore,

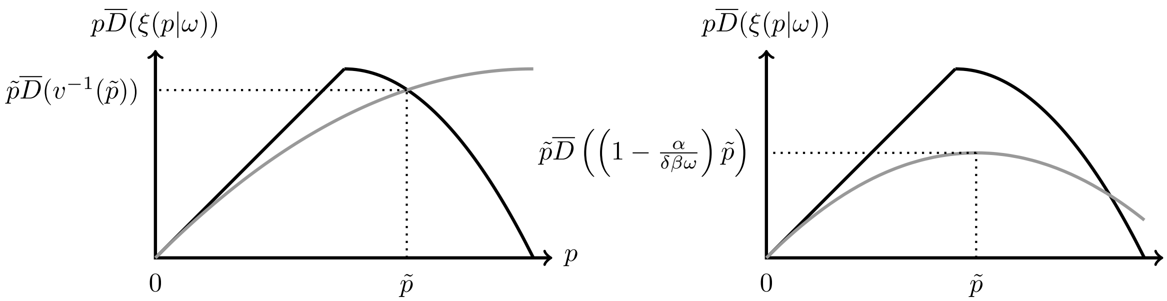
$$v^{-1}(\tilde{p}) \leq \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} \leq \overline{p}, \tag{7}$$

with at least one binding inequality. In particular,

$$\int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\,\mathrm{d}v = 0 \iff \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} = v^{-1}(\tilde{p}).$$

To better understand Lemma 4, notice that by Lemma 3, $p\mathbf{D}(p|p) = p\overline{D}(\xi(p|\omega))$ for any $p, \omega \in [0, \infty)$ and for any selection $\mathbf{D}$ of $\Delta(\cdot|\omega)$. As a result, the seller's revenue maximization problem can be written as

$$\max_{p \geq 0} p\overline{D}(\xi(p|\omega)) = \max_{p \geq 0} \left[\min \left\{ p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} p \right), p\overline{D}(v^{-1}(p)) \right\} \right].$$



(a) First Inequality of (7) Binds (b) Second Inequality of (7) Binds

Figure 3: Optimal Price $\tilde{p}$

If $\delta\beta\omega \le \alpha$, the function above coincides with $p\overline{D}(v^{-1}(p))$, the optimal price is $\mathbb{E}[v]$, the first inequality of (7) binds, and consumer surplus is zero. Meanwhile, if $\delta\beta\omega > \alpha$, there are two possibilities, as depicted by Figure 3. The first possibility is illustrated by Figure 3a, where the gray curve represents (a part of) the first function in the min operator, while the black curve represents the second, and the optimal price is the price at which the graphs of the two functions intersect. In this case, the first inequality of (7) binds and the consumer surplus is zero. Another possibility is illustrated by Figure 3b, where the optimal price is the price at which the first function is maximized. In this case, the second inequality of (7) binds and the consumer surplus is positive.

Even with a fixed ω, Lemma 4 already highlights the main driving force behind Proposition 1. When β is close enough to zero (i.e., when $\delta\beta\omega \le \alpha$), the induced sales revenue is $\mathbb{E}[v]$ and the allocation is efficient. As β increases, at first, the optimal price increases, consumer surplus remains zero, sales revenue decreases, and the size of the market base remains the same as the case of a lower β, leading to a Pareto-inferior outcome. Only when β is large enough will consumer surplus become positive and will the price begin to decrease.

The intuition behind this is reminiscent of the logic of the hold-up problem. As the intermediary discloses information after observing the seller's price and seeks to enhance consumer surplus in addition to sales revenue, the seller—in anticipation of the intermediary's response—charges a higher price to suppress the intermediary's desire to improve consumer surplus, better aligning her interests with those of the seller. This is because when the price is high, the marginal revenue of inducing more sales outweighs the marginal loss in consumer surplus that results from more consumers buying at a price above their values. When prices are higher, fewer consumers end up purchasing and therefore sales revenue decreases. In the meantime, consumers do not benefit as the intermediary's enhanced "consumer-mindedness" is offset by the seller's higher price. These together lead to a Pareto-worse outcome.

Remark 1. On a separate note, Lemma 3 and Lemma 4 describe the solution to a static problem in which the seller chooses a price to maximize revenue and the intermediary chooses a disclosure policy to maximize a weighed sum of the seller's revenue and consumer surplus after observing the seller's price. In this regard, Proposition 1 can be interpreted as an "unintended welfare loss" result—the outcome may become Pareto inferior when the intermediary assigns more weight to consumer surplus. The dynamic model we consider here can be viewed as a micro-foundation of the intermediary's payoff. Other micro-foundations are also possible, including competition among intermediaries to attract customers.

To complete the proof of Proposition 1, we characterize the stationary equilibrium outcomes by finding the appropriate continuation value ω^{s} and equilibrium price p^{s} that satisfies (4), (5) and (6). To describe stationary equilibrium outcomes, it is convenient to define

$$g^{\beta}(p) := \frac{\alpha}{\delta\beta}\left(1 + \frac{p\overline{D}(p)}{\int_p^\infty \overline{D}(v)\,\mathrm{d}v}\right),$$

for all $p \in [0, \overline{p}]$. Note that the function $(\beta, p) \mapsto g^{\beta}(p)$ is continuous, strictly decreasing in β, and strictly increasing in p on $[0, \overline{p}]$. Meanwhile, let

$$p^{\beta} := \inf\left\{p \geq 0 \middle| \delta\left(\gamma + \beta\int_p^\infty \overline{D}(v)\,\mathrm{d}v\right) \geq 1\right\}.$$

By definition, $p^{\beta} \in (0, \overline{p})$ whenever $\beta \in (\underline{\beta}, \overline{\beta})$. Moreover, p^{β} is strictly decreasing in β on $[\underline{\beta}, \overline{\beta}]$. Stationary equilibrium outcomes can then be characterized by Theorem 1 below.[12]

Theorem 1 (Stationary Equilibrium Outcomes)**.** *For any $\beta \in [0, \overline{\beta}]$, the following are equivalent:*

1. $\mathbf{z}^{\mathrm{s}} = (r^{\mathrm{s}}, \sigma^{\mathrm{s}}, \omega^{\mathrm{s}}, p^{\mathrm{s}}, \{m_t^{\mathrm{s}}\})$ *is a stationary equilibrium outcome.*

2. $\omega^{\mathrm{s}} \geq g^{\beta}(\overline{p})$ *if* $\beta = \overline{\beta}$, *while*

$$\omega^{\mathrm{s}} = \begin{cases} \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta}, & \text{if } \beta \in [0, \underline{\beta}] \\ g^{\beta}(p^{\beta}), & \text{if } \beta \in (\underline{\beta}, \overline{\beta}) \end{cases}.$$

Moreover,

$$p^{\mathrm{s}} = \begin{cases} \mathbb{E}[v], & \text{if } \beta \in [0, \underline{\beta}] \\ v(p^{\beta}), & \text{if } \beta \in (\underline{\beta}, \overline{\beta}) \\ \frac{\delta\beta\omega^{\mathrm{s}}}{\delta\beta\omega^{\mathrm{s}}-\alpha}\overline{p}, & \text{if } \beta = \overline{\beta} \end{cases}; \quad r^{\mathrm{s}} = \begin{cases} \mathbb{E}[v], & \text{if } \beta \in [0, \underline{\beta}] \\ \frac{(1-\gamma\delta)}{\alpha}g^{\beta}(p^{\beta}), & \text{if } \beta \in (\underline{\beta}, \overline{\beta}) \\ \frac{\delta\beta\omega^{\mathrm{s}}}{\delta\beta\omega^{\mathrm{s}}-\alpha}\overline{p}\overline{D}(\overline{p}), & \text{if } \beta = \overline{\beta} \end{cases};$$

$$\sigma^{\mathrm{s}} = \begin{cases} 0, & \text{if } \beta \in [0, \overline{\beta}) \\ \int_{\overline{p}}^\infty \overline{D}(v)\,\mathrm{d}v - \frac{\alpha}{\delta\beta\omega^{\mathrm{s}}-\alpha}\overline{p}\overline{D}(\overline{p}), & \text{if } \beta = \overline{\beta} \end{cases};$$

and $m_t^{\mathrm{s}} = (\gamma + \beta\sigma^{\mathrm{s}})^t$, *for all* $t \geq 1$.

[12]Recall that we focus on stationary equilibria in which the intermediary's continuation value is finite. When $\beta > \overline{\beta}$, the intermediary's continuation value diverges in any stationary equilibrium. See the Online Appendix for more details.

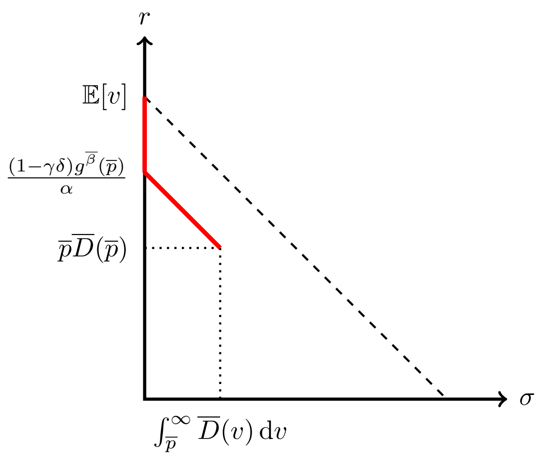


Figure 4: Surplus Divisions under Stationary Equilibria

Proposition 1 then immediately follows from Theorem 1. Figure 4 plots the set of normalized surplus divisions induced by stationary equilibria across all $\beta \in [0, \overline{\beta}]$. For any $\beta \in [0, \underline{\beta}]$, sales revenue equals $\mathbb{E}[v]$ and consumer surplus is zero. For any $\beta \in (\underline{\beta}, \overline{\beta})$, sales revenue equals $(1-\gamma\delta)g^{\beta}(p^{\beta})/\alpha$ and consumer surplus remains zero. In particular, sales revenue decreases as β increases. Finally, when $\beta = \overline{\beta}$, there are multiple stationary equilibrium outcomes and every outcome induces a total surplus of $(1-\gamma\delta)g^{\beta}(\overline{p})/\alpha$. When $\beta \in (\underline{\beta}, \overline{\beta})$, higher market feedback leads to Pareto inferior outcomes. As a benchmark, the dashed line in Figure 4 indicates the efficient frontier where all gains from trade are realized.

5 Equilibrium Payoff Set and the Least Efficient Outcome

In this section, we move beyond stationary equilibria and characterize the payoffs that can be supported by a subgame perfect equilibrium. As the game is infinitely repeated, there are inevitably many subgame perfect equilibrium outcomes. The results in this section outline these outcomes and demonstrate another version of welfare losses resulting from higher market feedback β. We show that whenever $\beta \leq \overline{\beta}$, there exists a unique *Pareto-worst* subgame perfect equilibrium outcome. Moreover, this least efficient outcome may worsen in the Pareto sense as β increases.

To begin with, let

$$r^* := \sup_{p \geq 0} \inf_{D \in \mathcal{D}} \inf_{q \in [D(p^+), D(p)]} p \cdot q \tag{8}$$

denote the *revenue guarantee*. That is, r^* is the sales revenue that the seller can secure in each period, regardless of the strategies of the intermediary and the tie-breaker. By definition, for any $r < r^*$, there exists $p \geq 0$ such that for any $D \in \mathcal{D}$, $pD(p^+) > r$.

Note that for any $p \geq \mathbb{E}[v]$, $\inf_{D \in \mathcal{D}} pD(p^+) = 0$ because the demand $\underline{D} \in \mathcal{D}$ that has a

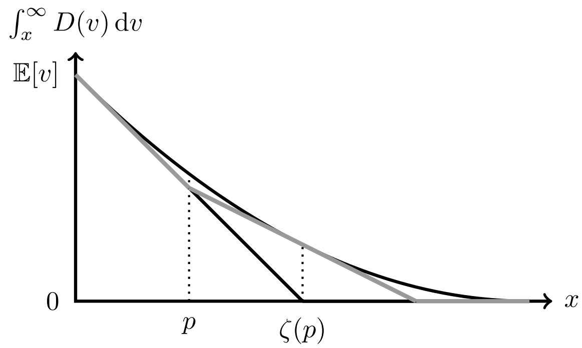


Figure 5: Minimizing Revenue Given p

mass of 1 at $\mathbb{E}[v]$ gives $p\underline{D}(p^+) = 0$. For any $p \in [0, \mathbb{E}[v])$, there exists a unique $\zeta(p) \geq p$ such that $\mathbb{E}[v|v \leq \zeta(p)] = p$. It then follows that $\inf_{D\in\mathcal{D}} pD(p^+) = p\overline{D}(\zeta(p))$ (see Figure 5).[13] Therefore, we must have

$$r^* = \max_{p\in[0,\mathbb{E}[v]]} p\overline{D}(\zeta(p)).$$

Furthermore, as the function $p \mapsto p\overline{D}(v^{-1}(p))$ is quasi-concave, there exists a unique $p^* \geq \mathbb{E}[v]$ such that $r^* = p^*\overline{D}(v^{-1}(p^*))$. In other words, when the seller charges a price p^*, the highest possible sales revenue is exactly r^*. Consequently, in any subgame perfect equilibrium, the seller does not charge any price $p > p^*$ because the best revenue given that price is below the revenue guarantee r^*.[14] With the definitions of r^* and p^*, we define

$$\omega^* := \frac{\alpha p^*\overline{D}(v^{-1}(p^*))}{1-\gamma\delta} = \frac{\alpha r^*}{1-\gamma\delta},$$

and note that ω^* is the present discounted profit of the intermediary if the market growth rate is γ (i.e., consumer surplus is zero) and sales revenue equals the revenue guarantee in every period. Meanwhile, let $\hat{p}$ be the unique price $p \leq \overline{p}$ such that $p\overline{D}(p) = r^*$, and define

$$\widehat{\beta} := \frac{1-\gamma\delta}{\delta\int_{\hat{p}}^\infty \overline{D}(v)\,\mathrm{d}v} \quad \text{and} \quad \beta^* := \frac{1-\gamma\delta}{\delta\int_{p^*}^\infty \overline{D}(v)\,\mathrm{d}v}.$$

Note that $0 < \underline{\beta} < \widehat{\beta} < \overline{\beta} < \beta^*$.

We now characterize the intermediary's payoffs in all subgame perfect equilibria. We first identify a useful lower bound for the intermediary's equilibrium payoff. As sales revenue in every period is at least r^* and the market growth rate is at least γ, the intermediary's payoff must be at least ω^*. If the seller prices at p^* in every period and the intermediary chooses the myopic best response, then the intermediary's payoff would be ω^*. However, it is not always

[13]The function $p \mapsto p\overline{D}(\zeta(p))$ is equivalent to the "pressed" $\overline{D}$ introduced by Libgober and Mu (2021).

[14]p^* plays the same role here as the *min-max strategy* does in canonical repeated games with perfect monitoring, and p^* coincides with the min-max strategy in Fudenberg, Kreps, and Maskin (1990).

incentive compatible for the seller and the intermediary to adopt these strategies. Even when the intermediary anticipates that the seller will always choose p^* and that the normalized continuation value will always be ω^*, it may not be optimal for the intermediary to choose the myopic best response and maximize sales revenue when market feedback is high enough. We therefore construct a tighter lower bound to account for this.

Let

$$h^\beta(\omega) := \delta\left(\gamma + \beta \int_{\left(1-\frac{\alpha}{\delta\beta\omega}\right)^+ p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega,$$

for all $\omega \geq 0$, and let

$$\underline{\omega}^\beta := \inf\{\omega \geq \omega^* | h^\beta(\omega') \leq \omega',\ \forall \omega' \geq \omega\}.$$

Note that $\underline{\omega}^\beta = \omega^*$ whenever $\beta \leq \overline{\beta}$, and that $\underline{\omega}^\beta \to \infty$ as $\beta \uparrow \beta^*$.

Lemma 5. *In any subgame perfect equilibrium, the intermediary's payoff is at least $\underline{\omega}^\beta$.*

For any $\beta \in [0, \beta^*)$, let $\Omega^*(\beta)$ denote the set of the intermediary's payoffs in all subgame perfect equilibrium outcomes.[15]

Theorem 2 (Subgame Perfect Equilibrium Payoffs)**.** *For any $\beta \in [0, \beta^*)$,*

$$\Omega^*(\beta) = [\underline{\boldsymbol{\omega}}^*(\beta), \overline{\boldsymbol{\omega}}^*(\beta)]\backslash\{\infty\},$$

for some $\underline{\omega}^\beta \leq \underline{\boldsymbol{\omega}}^*(\beta) \leq \overline{\boldsymbol{\omega}}^*(\beta) \leq \infty$. Moreover, $\underline{\boldsymbol{\omega}}^*$ is nonincreasing on $[0, \overline{\beta}]$ and $\overline{\boldsymbol{\omega}}^*$ is nondecreasing on $[0, \beta^*)$; while $\underline{\boldsymbol{\omega}}^*(\beta) = \underline{\omega}^\beta$ whenever $\beta \in [\widehat{\beta}, \beta^*)$; and

$$\overline{\boldsymbol{\omega}}^*(\beta) = \begin{cases} \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta}, & \text{if } \beta \in [0, \underline{\beta}] \\ \infty, & \text{if } \beta \in [\widehat{\beta}, \beta^*) \end{cases}.$$

Remark 2. Theorem 2 characterizes the long run player's equilibrium payoff for every fixed discount $\delta \in (0,1)$ rather than only characterizing these payoffs when δ approaches 1 as in Fudenberg, Kreps, and Maskin (1990). Furthermore, the characterization does not explicitly rely on fixed-point arguments and the notion of self-generation as in Abreu, Pearce, and Stacchetti (1986) and Abreu, Pearce, and Stacchetti (1990).[16] Instead, the bounds $\underline{\boldsymbol{\omega}}^*(\beta)$ and $\overline{\boldsymbol{\omega}}^*(\beta)$ are determined by the value of a constrained optimization problem (see Appendix A.7 for exact definitions).

[15]Recall that we focus on subgame perfect equilibria in which the intermediary's continuation payoff at any history is finite. Such equilibria only exists when $\beta < \beta^*$. For characterizations of subgame perfect equilibrium outcomes when $\beta \geq \beta^*$, see the Online Appendix.

[16]The reason is that—in addition to perfect monitoring—the intermediary and the seller's stage game payoffs are linearly dependent, and the feasible payoffs in a stage game are represented as a line segment. As a result, characterizing the intermediary's equilibrium payoff is essentially equivalent to finding two endpoints.

In addition to characterizing the intermediary's equilibrium payoff, it is useful to explore the sales revenue, consumer surplus, and prices that may occur in any period under a subgame perfect equilibrium. Knowing how consumer surplus and sales revenue are structured in a subgame perfect equilibrium helps identify further welfare implications. To this end, for any $\beta \geq 0$, let

$$\underline{r}(\beta) := \begin{cases} \frac{(1-\gamma\delta)\underline{\boldsymbol{\omega}}^*(\beta)}{\alpha}, & \text{if } \beta \leq \widehat{\beta}, \\ r^*, & \text{if } \beta > \widehat{\beta}, \end{cases}$$

and define the set $\mathbf{Z}^*(\beta)$ as follows:

$$\mathbf{Z}^*(\beta) := \left\{ (r,\sigma,p) \in \mathbb{R}^3_+ \;\middle|\; \begin{array}{c} (\mathbb{E}[v]-p)^+ \leq \sigma \leq S\left(\frac{r}{p}\right) - r \\ \underline{r}(\beta) \leq r \leq p \\ \alpha r + \delta(\gamma+\beta\sigma)\overline{\boldsymbol{\omega}}^*(\beta) \geq \sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta \int_p^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right] \end{array} \right\},$$

where $S(q) := \int_0^q \overline{D}^{-1}(x)\,\mathrm{d}x$ is the sum of normalized consumer surplus and normalized sales revenue when the quantity sold is q and when the demand is $\overline{D}$.

Using Theorem 2, subgame perfect equilibrium outcomes can be characterized and are given by Corollary 1.

Corollary 1 (Subgame Perfect Equilibrium Outcomes)**.**

1. *For any $\beta \in [0,\beta^*)$ and for any subgame perfect equilibrium outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$, $(r_t, \sigma_t, p_t) \in \mathbf{Z}^*(\beta)$ for all $t \in \mathbb{N} \cup \{0\}$.*

2. *For any $\beta \in [0,\beta^*)$, for any $T \in \mathbb{N} \cup \{0\}$, and for any $(r,\sigma,p) \in \mathbf{Z}^*(\beta)$, there exists a subgame perfect equilibrium outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ such that $r_T = r$, $\sigma_T = \sigma$, and $p_T = p$.*

An immediate consequence of Corollary 1 is that any outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ in which $r_t = \underline{r}(\beta)$, $\sigma_t = 0$ and $\omega_t = \underline{\boldsymbol{\omega}}^*(\beta)$ for all t is dominated by all other equilibrium outcomes. Such an equilibrium outcome always exists whenever $\beta \leq \overline{\beta}$. In this equilibrium, the seller always charges a price $\underline{p} \in [\mathbb{E}[v], p^*]$ so that $\underline{p}\overline{D}^{-1}(v^{-1}(\underline{p})) = \underline{r}(\beta)$, and the intermediary always chooses her myopic best response, which leads to zero consumer surplus in every period. If the seller deviates and charges any other price, the intermediary chooses a disclosure policy to minimize revenue subject to a constraint that future continuation play is enough to reward this punishment, and the tie-breaker breaks tie against the seller. If the intermediary adopts this punishment strategy, the continuation play gives the intermediary an equilibrium payoff of $\overline{\boldsymbol{\omega}}^*(\beta)$. If the intermediary does not, the continuation play gives her a payoff of $\underline{\boldsymbol{\omega}}^*(\beta)$.[17]

[17]By the definitions of $\underline{\boldsymbol{\omega}}^*(\beta)$ and $\overline{\boldsymbol{\omega}}^*(\beta)$ (which can be found in Appendix A.7), this makes the intermediary's punishment incentive compatible.

As a result, whenever $\beta \leq \overline{\beta}$, there exists a subgame perfect equilibrium outcome that is dominated by every other subgame perfect equilibrium outcome. As $\underline{\boldsymbol{\omega}}^*$ is nonincreasing in β, this dominated outcome becomes worse in the Pareto sense as β increases.

Proposition 2 (Inefficiency of High Feedback—Nonstationary)**.** *For any $\beta \in [0, \overline{\beta}]$, there exists a subgame perfect equilibrium outcome $\mathbf{z}^*(\beta)$ that is dominated by all other subgame perfect equilibrium outcomes. Furthermore, for any γ, δ such that $\gamma\delta \leq 1/2$, there exists $\widehat{\beta}(\gamma, \delta) \in (0, \widehat{\beta})$ such that $\mathbf{z}^*(\beta)$ dominates $\mathbf{z}^*(\beta')$ for any $0 < \beta < \beta' < \widehat{\beta}(\gamma, \delta)$.*

Proposition 2 highlights another version of welfare loss that results from increased market feedback. Unlike Proposition 1, Proposition 2 considers *all* subgame perfect equilibrium outcomes. Despite the multiplicity of subgame perfect equilibrium outcomes, Proposition 2 shows that there is always a Pareto-worst outcome and that this least efficient outcome can get worse as β increases.

It is noteworthy that the least efficient subgame perfect equilibrium outcome $\mathbf{z}^*(\beta)$ introduced in Proposition 2 is not stationary. Moreover, $\mathbf{z}^*(\beta)$ is the outcome of an equilibrium in which the sellers are incentivized to always choose a high price, and the intermediary's best response is to maximize the current sales revenue and leave zero surplus to consumers. To incentivize the sellers to choose this high price, the intermediary punishes a seller's deviation by generating a low sales revenue. For this punishment to be incentive-compatible, the intermediary must a sufficient reward for carrying out the punishment. Higher market feedback β allows the market to grow faster and therefore allows for higher continuation payoff to reward the intermediary. As a result, when β is higher, more severe punishments can be supported and more extreme prices can be incentivized, which in turn leads to worse outcomes.

6 Subscription-Based Model

As demonstrated by Proposition 1 (and Proposition 2), a higher level of market feedback may lead to Pareto inferior outcomes. Proposition 1 is driven by the fact that the intermediary prefers both higher sales revenue *and* higher consumer surplus. Therefore, as the level of market feedback increases, the sellers can raise prices to compel the intermediary to provide information in a way that better aligns with the sellers' interests. This suggests that the welfare losses caused by higher market feedback may not be present if the sellers' and the intermediary's incentives are decoupled. We now explore an alternative business model for the intermediary, the subscription-based model, in which her revenue comes directly from consumers.

In each period $t \in \mathbb{N} \cup \{0\}$, suppose the sequence of events remain the same: the seller chooses a price p_t, the intermediary sees p_t and chooses a disclosure policy $D_t \in \mathcal{D}$, and the tie-breaker sees p_t and D_t and chooses a tie-breaking rule $q_t \in [D_t(p_t^+), D_t(p_t)]$. However, instead of sharing with the intermediary, the seller captures all sales revenue $m_t \cdot p_t \cdot q_t$ in each period. The intermediary, on the other hand, receives $\widetilde{\alpha} \in (0,1)$ share of consumer surplus in each period: Upon entering the market, consumers pay a share $\widetilde{\alpha}$ of their ex-ante surplus to the intermediary in exchange for the product information. The share of consumer surplus captured by the intermediary can be interpreted as subscription fees or revenue from monetizing the market base (e.g., advertising revenue).

The subscription-based model decouples the seller's and the intermediary's interests. Rather than seeking to raise sales revenue while keeping consumers surplus high enough to sustain future revenue, the intermediary's only goal in the subscription-based model is to maximize consumer surplus, as both her stage game payoff and the market base depend only on consumer surplus. As a result, a higher market feedback always leads to more efficient outcomes.

Proposition 3 (Subscription-Based Model)**.** *For any $\beta < \overline{\beta}$, there exists a unique stationary equilibrium outcome $\mathbf{y}^{\mathrm{s}}(\beta)$. Furthermore, for any $0 < \beta < \beta' < \overline{\beta}$, $\mathbf{y}^{\mathrm{s}}(\beta')$ dominates $\mathbf{y}^{\mathrm{s}}(\beta)$.*

Although it is clear from Proposition 1 and Proposition 3 that compared with the revenue-sharing model, the subscription-based model can better translate a higher level of market feedback into more efficient outcomes, the intermediary may not always prefer the subscription based model. While consumers enjoy higher surplus and market bases are larger under the subscription-based model, more surplus is extracted from consumers under the revenue-sharing model. As demonstrated by Proposition 4, it is possible that the intermediary may prefer the revenue-sharing model, even if higher market feedback causes inefficiency.

For any $\beta < \overline{\beta}$, let $\omega^{\mathrm{s}}(\beta)$ denote the intermediary's payoff in the unique stationary equilibrium under the revenue-sharing model, and let $\rho^{\mathrm{s}}(\beta)$ denote the intermediary's payoff in the unique stationary equilibrium under the subscription-based model.

Proposition 4. *There exists $\beta^0 \geq 0$ such that $\omega^{\mathrm{s}}(\beta) > \rho^{\mathrm{s}}(\beta)$ for all $\beta \in [0, \beta^0)$. Moreover, $\beta^0 > 0$ if and only if*

$$\frac{\widetilde{\alpha}}{\alpha} < \frac{\mathbb{E}[v]}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v};$$

and $\beta^0 > \underline{\beta}$ if and only if

$$\frac{\widetilde{\alpha}}{\alpha} + 1 < \frac{\mathbb{E}[v]}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v}.$$

Proposition 4 underlines the possibility that the intermediary may prefer the revenue-sharing model to the subscription-based model, even if β falls in the range $(\underline{\beta}, \overline{\beta})$, in which a higher level of market feedback leads to welfare losses under the revenue-sharing model. Consequently, while the subscription-based model may better translate higher market feedback into efficiency, the intermediary may not voluntarily adopt this model.

7 Extensions

7.1 Long-Lived Seller

In this section, we relax the assumption that the seller is short-lived and consider a case in which both the seller and the intermediary are long-lived, but the seller is less patient than the intermediary.

The seller is long-lived and has discount $\rho \in (0, \delta)$. The sequence of events and strategies of all players remain the same: In each period t, the seller observes all past histories and then chooses a price p_t; the intermediary then sees all past histories and p_t before choosing a disclosure policy $D_t \in \mathcal{D}$; the tie-breaker then sees all past histories, p_t and D_t and chooses a tie breaking rule $q_t \in [D_t(p_t^+), D_t(p_t)]$. Given any strategy profile, the seller's payoff is

$$\pi = (1-\alpha)p_0 q_0 + \sum_{t=1}^{\infty} \rho^t \prod_{s=0}^{t} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\, \mathrm{d}v \right) (1-\alpha) p_t q_t,$$

while the intermediary's payoff is

$$\omega = \alpha p_0 q_0 + \sum_{t=1}^{\infty} \delta^t \prod_{s=0}^{t} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\, \mathrm{d}v \right) \alpha p_t q_t,$$

where $\{p_t, D_t, q_t\}$ are the on-path actions chosen by the seller, the intermediary, and the tie-breaker, respectively. As a result, the seller's and the intermediary's interests are better aligned than in the baseline model but may still differ when $\rho < \delta$. As demonstrated by Proposition 5, our main result still holds qualitatively even if the seller is long-lived.

Proposition 5 (Long-Lived Seller)**.** *There exists a continuously decreasing function $\boldsymbol{\beta}$ with $\boldsymbol{\beta}(0) = \overline{\beta}$ and $\lim_{\rho \uparrow \delta} \boldsymbol{\beta}(\rho) = \underline{\beta}$ such that for any $\rho \in [0, \delta)$ and for any $\beta \in (\underline{\beta}, \boldsymbol{\beta}(\rho))$, there exists a unique stationary equilibrium outcome $\tilde{\mathbf{z}}^{\mathrm{s}}(\beta)$. $\tilde{\mathbf{z}}^{\mathrm{s}}(\beta)$ dominates $\tilde{\mathbf{z}}^{\mathrm{s}}(\beta')$ for any $\beta, \beta' \in (\underline{\beta}, \boldsymbol{\beta}(\rho))$ such that $\beta < \beta'$.*

7.2 Nonlinear Growth Rate

The baseline model assumes that the market growth rate is linear in consumer surplus. In this section, we consider an extension of the model that captures a broad scope of non-linear market growth. In each period $t \geq 1$, the growth rate of the market base m_{t+1}/m_t is a nonlinear function of consumer surplus in period t:

$$\frac{m_{t+1}}{m_t} = f\left(\int_{p_t}^{\infty} D_t(v)\, \mathrm{d}v \right).$$

for some function f, which we refer to as the growth function. In this section, we assume that the function $p \mapsto p\overline{D}(p)$ is strictly concave. We now characterize the set of stationary equilibria in this alternative setting.

Let $\mathcal{F}$ be the collection of twice-differentiable, increasing, and concave functions on $\mathbb{R}_+$ with $f(0) = \gamma$, and let

$$\mathcal{F}_1 := \{f \in \mathcal{F} | f'(0) \in [0, \underline{\beta}]\}.$$

Furthermore, for any $\beta \in (\underline{\beta}, \overline{\beta})$ and for any $\eta \geq 0$, let

$$\mathcal{F}_2(\beta, \eta) := \{f \in \mathcal{F} | f'(0) = \beta,\ \|f''\| \leq \eta\}.$$

Note that for any $\beta \in (\underline{\beta}, \overline{\beta})$, $\mathcal{F}_1$ and $\mathcal{F}_2(\beta, \eta)$ are disjoint sets. As shown in Proposition 6, higher market feedback may still lead to welfare losses even with nonlinear market growth rate.

Proposition 6 (Inefficiency of High Feedback—Nonlinear Growth)**.** *There exists a continuously decreasing function $h : (\underline{\beta}, \overline{\beta}) \to \mathbb{R}_{++}$ such that every $f \in \mathcal{F}_1 \cup [\bigcup_{\beta \in (\underline{\beta}, \overline{\beta})} \mathcal{F}_2(\beta, h(\beta))]$ induces a unique stationary equilibrium outcome $\mathbf{z}^{\mathrm{s}}(f)$. Furthermore, for any β, β' such that $\underline{\beta} < \beta < \beta' < \overline{\beta}$, $\mathbf{z}^{\mathrm{s}}(f_1)$ dominates $\mathbf{z}^{\mathrm{s}}(f_2)$ for all $f_1 \in \mathcal{F}_2(\beta, h(\beta))$ and $f_2 \in \mathcal{F}_2(\beta', h(\beta'))$.*

According to Proposition 6, for any growth function in the set $\bigcup_{\beta \in (\underline{\beta}, \overline{\beta})} \mathcal{F}_2(\beta, h(\beta))$, an increase in the level of market feedback when consumer surplus is zero (i.e., $f'(0)$) leads to Pareto inferior outcomes. Hence, our main finding does not rely on the linearity of the growth function. Rather, the linearity assumption in the baseline model is mainly imposed to simplify the exposition.

7.3 Non-Stationary Revenue-Sharing Rule

Thus far, we have assumed that the seller and the intermediary share sales revenue in each period according to a fixed $\alpha \in (0, 1)$. In reality, however, it is reasonable to expect non-constant sharing rules to be present. After all, in a market with many intermediaries and many sellers, intermediaries with different market bases may have different outside options. In this section, we consider an extension of the model in which the revenue-sharing rule depends on the current market base m, so that the intermediary can obtain $\boldsymbol{\alpha}(m)$ share of sales revenue when the market base is m. When the sharing rule depends on the market base, the stage game is no longer stationary. As a result, Markov strategies may depend on market bases and therefore the intermediary's best response may not have a closed form solution in general. Nonetheless, as shown below, under certain parametrization of the function $\boldsymbol{\alpha}$, our previous analyses can be extended.

Assume that at any history in which the market base is $m \geq 1$, the intermediary can retain $\boldsymbol{\alpha}(m) := m^{-\alpha}$ share of sales revenue, for some $\alpha \in (0, 1)$.[18] With this assumption,

[18]Under this assumption, $\boldsymbol{\alpha}$ is decreasing in m, which means that the intermediary's per-consumer per-sale commission decreases with her market base. The equilibrium prices for intermediaries' services in Fainmesser and Galeotti (2021) exhibit similar features, although under a different model.

under any strategy profile, for any $t \in \mathbb{N} \cup \{0\}$, we note that

$$\widetilde{m}_{t+1} := m_{t+1}\boldsymbol{\alpha}(m_{t+1}) = f(\sigma_t) m_t \boldsymbol{\alpha}(m_t) =: f(\sigma_t)\widetilde{m}_t,$$

where $f(\sigma) := (\gamma + \beta\sigma)^{1-\alpha}$ for some $\gamma \geq 1$ and σ_t denotes the consumer surplus induced by this strategy profile in period t. Note that f is an increasing and concave function and so we may simply replace $\{m_t\}$ with $\{\widetilde{m}_t\}$ and apply the results in Section 7.2.

Thus, even if the revenue sharing rule between the seller and the intermediary is non-stationary, higher market feedback may still lead to Pareto-worse outcomes.

8 Conclusions

In this paper, we show that higher market feedback may lead to Pareto inferior outcomes in a setting featuring informational intermediation. The results are driven by the diverging interests of the seller and the intermediary under a revenue-sharing business model: The intermediary is more consumer-minded than the seller because of the intermediary's concern about her own future market base. We show that for a range of market feedback levels under the unique stationary equilibrium outcome, higher market feedback leads to Pareto inferior outcomes, even though higher market feedback means that the market grows faster and that the intermediary has more incentive to enhance consumer surplus. Additionally, across all subgame perfect equilibria, we show that the Pareto-worst subgame perfect equilibrium outcome worsens as market feedback increases.

Our results serve as a cautionary tale for policy-making, as they highlight the possibility that changes in the level of market feedback may have counter-intuitive policy implications. As a comparison, we show that under an alternative subscription-based business model, higher market feedback always leads to more efficient outcomes.

Several directions for future research emerge from our analyses. For example, while we focus only on the revenue-sharing and information-provision business model, informational intermediaries may operate under other business models, including the pay-sponsorship model. It would be valuable to understand how intermediaries and the market choose among these models and the welfare implications of these choices. Additionally, from a policy point of view, it would also be useful to better understand the broader impact of market feedback under different market structures.

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Appendix

This appendix contains proofs of results in Section 4 and Section 5. Proofs for the extensions are relegated to the Online Appendix.

A.1 Proof of Lemma 1

The "if" part immediately follows from the definition of strategies. For the "only if" part, it suffices to show that there is a profitable deviation only if there is a finite-shot deviation. To this end, consider any period t, any history h^t, and any strategy profile that gives the intermediary a finite continuation payoff. Let $\sigma|_{h^t}$ denote the intermediary's continuation strategy and let

$$\omega_t = \sum_{s=t}^{\infty} \delta^s m_s p_s q_s < \infty$$

denote the intermediary's continuation payoff, where $p_s \geq 0$, $D_s \in \mathcal{D}$ and $q_s \in [D_s(p_s^+), D_s(p_s)]$ are the price charged by the seller, the disclosure policy adopted by the intermediary and the tie-breaking rule chosen by the tie-breaker on path in each period $s \geq t$, respectively, and $\{m_s\}$ are the induced market bases on path. Suppose that, when holding the seller's and the tie-breaker's strategies fixed, there is another continuation strategy $\tilde{\sigma}|_{h^t}$ at h^t that gives the intermediary a continuation payoff

$$\tilde{\omega}_t = \sum_{s=t}^{\infty} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s > \omega_t,$$

where $\tilde{p}_s \geq 0$, $\tilde{D}_s \in \mathcal{D}$, and $\tilde{q}_s \in [\tilde{D}_s(\tilde{p}_s^+), \tilde{D}_s(\tilde{p}_s)]$ are the price charged by the seller, the disclosure policy adopted by the intermediary, and the tie-breaking rule chosen by the tie-breaker in each period $s \geq t$ on the path induced by $\tilde{\sigma}|_{h^t}$, the seller's strategy and the tie-breaker's strategy, respectively, and $\{\tilde{m}_s\}$ are the associated market bases. Furthermore, for any $T > t$, let $\tilde{\omega}_t^T$ be the intermediary's continuation payoff at history h^t when following $\tilde{\sigma}|_{h^t}$ until period $T > t$ and then return to $\sigma|_{h^t}$ from period T onward. Clearly, since stage game payoffs are bounded from below, both $\tilde{\omega}_t$ and $\tilde{\omega}_t^T$ are well-defined. Moreover, for $x \in \{\tilde{\omega}_t, \tilde{\omega}_t^T\}$, either $x = \infty$ or $x < \infty$. Clearly, if $\limsup_{T\to\infty} \tilde{\omega}_t^T = \infty$, then since $\omega_t < \infty$, there exists $\hat{T}$ such that $\tilde{\omega}_t^{\hat{T}} > \omega_t$ and hence deviating to $\tilde{\sigma}|_{h^t}$ for $\hat{T}$ periods then and return to $\sigma|_{h^t}$ is profitable for the intermediary. Thus, it is without loss to assume that $\limsup_{T\to\infty} \tilde{\omega}_t^T < \infty$. In the meantime, if $\tilde{\omega}_t = \infty$, then there exists $\hat{T}$ such that

$$\sum_{s=t}^{\hat{T}} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s > \omega_t,$$

which in turn implies that $\tilde{\omega}_t^{\hat{T}} > \omega_t$ since stage game payoffs are nonnegative. Therefore, it is also without loss to assume that $\tilde{\omega}_t < \infty$. Furthermore, since $\limsup_{T\to\infty} \tilde{\omega}_t^T < \infty$, $\{\tilde{\omega}_t^T\}$ is bounded. Thus, there exists a convergent subsequence $\{\tilde{\omega}_t^{T_n}\}$. We claim that it must be

$$\lim_{n\to\infty} \tilde{\omega}_t^{T_n} \geq \tilde{\omega}_t.$$

Indeed, suppose the contrary, that $\lim_{n\to\infty} \tilde{\omega}_t^{T_n} < \tilde{\omega}_t$. For any $n \in \mathbb{N}$, since $\tilde{\omega}_t^{T_n} < \infty$, it can be written as

$$\tilde{\omega}_t^{T_n} = \sum_{s=t}^{T_n} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s + \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n,$$

where for any $n \in \mathbb{N}$, $\tilde{p}_s^n \geq 0$, $\tilde{D}_s^n \in \mathcal{D}$ and $\tilde{q}_s^n \in [\tilde{D}_s^n(\tilde{p}_s^{n+}), \tilde{D}_s^n(\tilde{p}_s^n)]$ are the price charged by the seller, the disclosure policies adopted by the intermediary, and the tie-breaking rule chosen by the tie-breaker in period $s \geq T_n + 1$ on path, respectively, and $\{\tilde{m}_s\}_{s=T_n+1}^{\infty}$ are the associated market bases. Hence, for any $n \in \mathbb{N}$,

$$\tilde{\omega}_t^{T_n} - \tilde{\omega}_t = \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n - \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s.$$

Therefore, since $\tilde{\omega}_t < \infty$, and since $\sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n \leq \tilde{\omega}_t^{T_n}$ for all $n \in \mathbb{N}$,

$$\begin{aligned}
0 &> \lim_{n\to\infty} [\tilde{\omega}_t^{T_n} - \tilde{\omega}_t] \\
&= \lim_{n\to\infty} \left[\sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n - \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s \right] \\
&= \lim_{n\to\infty} \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n - \lim_{n\to\infty} \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s \tilde{p}_s \tilde{q}_s \\
&= \lim_{n\to\infty} \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n.
\end{aligned}$$

However, since stage game payoffs are nonnegative, $\delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n \geq 0$ for all $n \in \mathbb{N}$ and for all $s \geq T_n + 1$, which implies that

$$\lim_{n\to\infty} \sum_{s=T_n+1}^{\infty} \delta^s \tilde{m}_s^n \tilde{p}_s^n \tilde{q}_s^n \geq 0,$$

a contradiction. Thus, it must be that $\lim_{n\to\infty} \tilde{\omega}_t^{T_n} \geq \tilde{\omega}_t$.

As a result, since $\tilde{\omega}_t > \omega_t$, there exists $n \in \mathbb{N}$ such that $\tilde{\omega}_t^{T_n} > \omega_t$, as desired. ■

A.2 Proof of Lemma 2

Consider any stationary equilibrium. Since both the intermediary's and the seller's strategy do not depend on past histories in any stationary equilibrium, the intermediary's normalized equilibrium continuation value in a given period must be a constant. Therefore, for any t, the intermediary's normalized continuation payoff at the beginning of period t can be written as $\omega^{\mathrm{s}} \in [0, \infty)$. Meanwhile, since the seller's strategy does not depend on history either, the price charged by the seller in period t must be a constant $p^{\mathrm{s}} \in [0, \infty)$ as well. Therefore, since both the intermediary and the seller are best responding in any stationary equilibrium at any history, (4), (5), and (6) must hold.

Conversely, given any tuple $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ that satisfies the conditions required by the lemma, the strategy profile where the seller chooses p^{s} and the intermediary chooses $\mathbf{D}^{\mathrm{s}}(\cdot|p) \in \mathcal{D}$ whenever the seller chooses posted price $p \geq 0$ in the same period is immune to one-shot deviations. Moreover, since $\omega^{\mathrm{s}} < \infty$, Lemma 1 then implies that this strategy profile is indeed a subgame perfect equilibrium. This completes the proof. ■

A.3 Proof of Lemma 3

First, notice that for any $D \in \mathcal{D}$,

$$(\mathbb{E}[v] - p)^+ \leq \int_p^{\infty} D(v)\,\mathrm{d}v \leq \int_p^{\infty} \overline{D}(v)\,\mathrm{d}v$$

for all $p \geq 0$. Moreover, notice that the function $\xi \mapsto \int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)$ is strictly decreasing on $[0,p]$, with a value of 0 at $\xi = v^{-1}(p)$ and a value of $\int_p^\infty \overline{D}(v)\,\mathrm{d}v$ at $\xi = p$, there must exist a unique $\xi(p) \in [v^{-1}(p), p]$ such that $\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi) = \int_p^\infty D(v)\,\mathrm{d}v$.

Now consider any $\widehat{D} \in \mathcal{D}$ such that

$$\int_{\xi(p)}^\infty \widehat{D}(v)\,\mathrm{d}v = \int_{\xi(p)}^\infty \overline{D}(v)\,\mathrm{d}v \tag{A.9}$$

and that

$$\widehat{D}(v) = \overline{D}(\xi(p)), \tag{A.10}$$

for all $v \in (\xi(p), p]$. Such $\widehat{D}$ exists since when $\widehat{D}$ is defined as

$$\widehat{D}(v) := \begin{cases} 1, & \text{if } v \in [0, \mathbb{E}[v|v \leq \xi(p)]] \\ \overline{D}(\xi(p)), & \text{if } v \in (\mathbb{E}[v|v \leq \xi(p)], \mathbb{E}[v|v \geq \xi(p)]] \\ 0, & \text{if } v \in (\mathbb{E}[v|v \geq \xi(p)], \infty) \end{cases},$$

we have $\int_{\xi(p)}^\infty \widehat{D}(v)\,\mathrm{d}v = \int_{\xi(p)}^\infty \overline{D}(v)\,\mathrm{d}v$. As a result, for any $\widehat{D}$ satisfying (A.9) and (A.10), by definition of $\xi(p)$, $\int_p^\infty \widehat{D}(v)\,\mathrm{d}v = \int_p^\infty D(v)\,\mathrm{d}v$.

Moreover, for any $D \in \mathcal{D}$, (A.9) implies that

$$0 \leq \int_{\xi(p)}^\infty (\overline{D}(v) - D(v))\,\mathrm{d}v \leq \int_{\xi(p)}^p (\overline{D}(\xi(p)) - D(v))\,\mathrm{d}v \leq (p - \xi(p))(\overline{D}(\xi(p)) - D(p)),$$

and hence $D(p) \leq \overline{D}(\xi(p))$. Together, we have

$$\alpha p D(p) + \delta\omega\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right) \leq \alpha p \overline{D}(\xi(p)) + \delta\omega\left(\gamma + \beta\left(\int_{\xi(p)}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi(p))\overline{D}(\xi(p))\right)\right).$$

Lastly, notice that for any $\widehat{D}$ satisfying (A.9) and (A.10),

$$\alpha p \overline{D}(\xi(p)) + \delta\omega\left(\gamma + \beta\left(\int_{\xi(p)}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi(p))\overline{D}(\xi(p))\right)\right) = \alpha p \widehat{D}(p) + \delta\omega\left(\gamma + \beta\int_p^\infty \widehat{D}(v)\,\mathrm{d}v\right).$$

As a result, for any D, there exists another $\widehat{D} \in \mathcal{D}$ satisfying (A.9) and (A.10) such that

$$\begin{aligned} \alpha p D(p) + \delta\omega\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right) \leq& \alpha p \widehat{D}(p) + \delta\omega\left(\gamma + \beta\int_p^\infty \widehat{D}(v)\,\mathrm{d}v\right) \\ =& \alpha p \overline{D}(\xi(p)) + \delta\omega\left(\gamma + \beta\left(\int_{\xi(p)}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi(p))\overline{D}(\xi(p))\right)\right). \end{aligned}$$

Therefore, the maximization problem

$$\sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta\omega\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right)\right]$$

can be simplified to

$$\max_{\xi \in [v^{-1}(p), p]} \left[\alpha p \overline{D}(\xi) + \delta\omega\left(\gamma + \beta\left(\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\right)\right], \tag{A.11}$$

which, by continuity of $\overline{D}$, has a solution. This implies that $\Delta(p|\omega)$ is nonempty. Moreover, the first-order Kuhn-Tucker condition of (A.11) implies its solution $\xi(p|\omega)$ is given by

$$\xi(p|\omega) = \max\left\{\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p, v^{-1}(p)\right\}.$$

This in turn implies that any $\widehat{D} \in \mathcal{D}$ satisfying the condition given by the lemma must be in $\Delta(p|\omega)$. This completes the proof. ■

A.4 Proof of Lemma 4

By Lemma 3, for any selection $\mathbf{D}$ of $\Delta(\cdot|\omega)$,

$$\begin{aligned} p\mathbf{D}(p|p) = p\overline{D}(\xi(p|\omega)) &= p\overline{D}\left(\max\left\{\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p, v^{-1}(p)\right\}\right) \\ &= \min\left\{p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right), p\overline{D}(v^{-1}(p))\right\}, \end{aligned}$$

where the last equality follows from the fact that $\overline{D}$ is strictly decreasing. Furthermore, notice that for any p, if $1 \leq \alpha/\delta\beta\omega$, then

$$p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right) = p.$$

Meanwhile, if $1 > \alpha/\delta\beta\omega$, let $\tilde{p} := (1 - \alpha/\delta\beta\omega)p$, then

$$p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right) = \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\tilde{p}\overline{D}(\tilde{p}).$$

Thus, since $\overline{D}$ is regular, $p \mapsto \overline{D}((1 - \alpha/\delta\beta\omega)p)$ is quasi-concave as well. Lastly, by the definition of v^{-1} the function $p \mapsto \overline{D}(v^{-1}(p))$ is also quasi-concave. Together, the function $p \mapsto p\mathbf{D}(p|p)$ is quasi-concave since it is a minimum of two quasi-concave functions. Thus, $\max_{p \geq 0} p\mathbf{D}(p|p)$ has a unique solution.

Moreover, if $1 \leq \alpha/\delta\beta\omega$, then $p\mathbf{D}(p|p) = p\overline{D}(v^{-1}(p))$ and hence $\tilde{p} = \mathbb{E}[v]$, which in turn implies that

$$\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ \tilde{p} = 0 = v^{-1}(\tilde{p}) \leq \overline{p}.$$

Meanwhile, if $1 > \alpha/\delta\beta\omega$, notice that the function $p \mapsto \overline{D}(v^{-1}(p))$ is maximized at $p = \mathbb{E}[v]$ and that $\mathbb{E}[v]\overline{D}(v^{-1}(\mathbb{E}[v])) = \mathbb{E}[v] \geq \mathbb{E}[v]\overline{D}((1 - \alpha/\delta\beta\omega)\mathbb{E}[v])$. As a result, since $p \mapsto p\mathbf{D}(p|p)$ is quasi-concave and hence single-peaked, it must attain its maximum at either price p such that $p\overline{D}(v^{-1}(p)) = p\overline{D}((1-\alpha/\delta\beta\omega)p)$ or the maximizer of $p\overline{D}((1 - \alpha/\delta\beta\omega)p)$, whichever is smaller. Together with the fact that the maximizer of $p\overline{D}((1 - \alpha/\delta\beta\omega)p)$ is given by $\delta\beta\omega\overline{p}/(\delta\beta\omega - \alpha)$, it then follows that either

$$\left(1 - \frac{\alpha}{\delta\beta\omega}\right)\tilde{p} = v^{-1}(\tilde{p}) \text{ and } \tilde{p} \leq \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p}$$

or

$$\tilde{p} = \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p} \leq v(\overline{p}).$$

As a result, it must be that

$$v^{-1}(\tilde{p}) \leq \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} \leq \overline{p},$$

with at least one inequality binding.

Lastly, suppose that

$$\int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\, \mathrm{d}v = 0.$$

Then, by Lemma 3,

$$\int_{\xi(\tilde{p}|\omega)}^{\infty} \overline{D}(v)\, \mathrm{d}v - (\tilde{p} - \xi(\tilde{p}|\omega))\overline{D}(\xi(\tilde{p}|\omega)) = 0,$$

which is equivalent to

$$\mathbb{E}[v|v \geq \xi(\tilde{p}|\omega)] = \tilde{p} \iff \xi(\tilde{p}|\omega) = v^{-1}(\tilde{p}).$$

Moreover, notice that for any $p \in [0, \mathbb{E}[v]]$, $p\overline{D}(v^{-1}(p)) = p \geq p\overline{D}((1 - \alpha/\delta\beta\omega)^{+}p)$, and that $p \mapsto p\overline{D}(v^{-1}(p))$ is uniquely maximized at $p = \mathbb{E}[v]$. Single-peakness of $p \mapsto \mathbf{D}(p|p)$ and $\xi(\tilde{p}|\omega) = v^{-1}(\tilde{p})$ then imply that

$$\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} = v^{-1}(\tilde{p}).$$

Conversely, suppose that

$$\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} = v^{-1}(\tilde{p}).$$

Then $\xi(\tilde{p}|\omega) = v^{-1}(\tilde{p})$ and hence, by Lemma 3,

$$\int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\, \mathrm{d}v = \int_{v^{-1}(\tilde{p})}^{\infty} \overline{D}(v)\, \mathrm{d}v - (\tilde{p} - v^{-1}(\tilde{p}))\overline{D}(v^{-1}(\tilde{p})) = \overline{D}(v^{-1}(\tilde{p}))(\tilde{p} - v(v^{-1}(\tilde{p}))) = 0.$$

This completes the proof. ∎

A.5 Proof of Theorem 1

We first show that any $(r^{\mathrm{s}}, p^{\mathrm{s}}, \sigma^{\mathrm{s}}, \omega^{\mathrm{s}}, \{m_t^{\mathrm{s}}\})$ described in the statement of the theorem is indeed a stationary equilibrium. To this end, we will show that for any such tuple, there exists $\mathbf{D}^{\mathrm{s}} : \mathbb{R}_+ \to \mathcal{D}$ such that $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ satisfies the conditions of Lemma 2. Consider three cases separately.

Case 1: $\beta \in [0, \underline{\beta}]$.
In this case, notice that

$$\delta\beta\omega^{\mathrm{s}} = \delta\beta\frac{\alpha\mathbb{E}[v]}{1 - \gamma\delta} = \frac{\beta\alpha}{\underline{\beta}} \leq \alpha,$$

and therefore $\omega^{\mathrm{s}} \leq \alpha/\delta\beta$. Consider any selection $\mathbf{D}^{\mathrm{s}}$ of $\Delta(\cdot|\omega^{\mathrm{s}})$. Since $p^{\mathrm{s}} = \mathbb{E}[v]$ and thus $v^{-1}(p^{\mathrm{s}}) = 0$, Lemma 4 implies that $p^{\mathrm{s}} \in \operatorname{argmax}_p p\mathbf{D}^{\mathrm{s}}(p|p)$, which establishes (5). Meanwhile, by Lemma 4, it must be that

$$\int_{p^{\mathrm{s}}}^{\infty} \mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\, \mathrm{d}v = 0.$$

Moreover, given $p^{\mathrm{s}} = \mathbb{E}[v]$, Lemma 3 implies that $\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}}) = \overline{D}(0) = 1$. Together,

$$\begin{aligned}\sup_{D\in\mathcal{D}}\left[\alpha p^{\mathrm{s}} D(p^{\mathrm{s}}) + \delta\left(\gamma + \beta\int_{p^{\mathrm{s}}}^{\infty} D(v)\,\mathrm{d}v\right)\omega^{\mathrm{s}}\right] =& \alpha p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}}) + \delta\left(\gamma + \beta\int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\,\mathrm{d}v\right)\omega^{\mathrm{s}}\\ =&\alpha\mathbb{E}[v] + \gamma\delta\omega^{\mathrm{s}}\\ =&\omega^{\mathrm{s}}\\ =&\frac{\alpha p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}})}{1-\delta\left(\gamma+\beta\int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p)\,\mathrm{d}v\right)}\end{aligned}$$

which establishes (4) and (6).

Case 2: $\beta \in (\underline{\beta}, \overline{\beta})$.
In this case, take any selection $\mathbf{D}^{\mathrm{s}}$ of $\Delta(\cdot|\omega^{\mathrm{s}})$. Notice that by definition, $1 > \alpha/\delta\beta\omega^{\mathrm{s}}$, and hence

$$\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)^{+} p^{\mathrm{s}} = \left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right) p^{\mathrm{s}} = v^{-1}(p^{\mathrm{s}}),$$

which in turn implies that, by Lemma 4,

$$\int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\,\mathrm{d}v = 0$$

and therefore,

$$\omega^{\mathrm{s}} = \alpha p^{\mathrm{s}}\overline{D}(\xi(p^{\mathrm{s}}|\omega^{\mathrm{s}})) + \gamma\delta\omega^{\mathrm{s}} = \alpha p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}}) + \delta\left(\gamma+\beta\int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\,\mathrm{d}v\right)\omega^{\mathrm{s}},$$

which establishes (4). Furthermore, since $p^{\beta} < \overline{p}$, $p^{\mathrm{s}} < \delta\beta\omega^{\mathrm{s}}\overline{p}/(\delta\beta\omega^{\mathrm{s}} - \alpha)$ and hence p^{s} is the unique maximizer of $p\overline{D}(\xi(p|\omega^{\mathrm{s}}))$ according to Lemma 4. Thus, by Lemma 3, $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ satisfies (5) and (6).

Case 3: $\beta = \overline{\beta}$.
In this case, consider any selection $\mathbf{D}^{\mathrm{s}}$ of $\Delta(\cdot|\omega^{\mathrm{s}})$. By definition, $1 > \alpha/\delta\beta\omega^{\mathrm{s}}$ and

$$\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right) p^{\mathrm{s}} > v^{-1}(p^{\mathrm{s}}),$$

and thus, by Lemma 3 and Lemma 4,

$$p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}}) = p^{\mathrm{s}}\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)p^{\mathrm{s}}\right) \geq p\overline{D}\left(\max\left\{\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)p, v^{-1}(p)\right\}\right) = p\mathbf{D}^{\mathrm{s}}(p|p)$$

As a result, $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ satisfies (4), (5), and (6) and therefore induces a stationary equilibrium, as desired.

We now show that for any stationary equilibrium, its outcome $(r^{\mathrm{s}}, p^{\mathrm{s}}, \sigma^{\mathrm{s}}, \omega^{\mathrm{s}}, \{m_t^{\mathrm{s}}\})$ must satisfy the conditions given by Theorem 1. By Lemma 2, there exists $(\omega^{\mathrm{s}}, p^{\mathrm{s}}, \mathbf{D}^{\mathrm{s}})$ satisfying (4), (5), and (6) such that $r^{\mathrm{s}} = p^{\mathrm{s}}\mathbf{D}^{\mathrm{s}}(p^{\mathrm{s}}|p^{\mathrm{s}})$, $\sigma^{\mathrm{s}} = \int_{p^{\mathrm{s}}}^{\infty}\mathbf{D}^{\mathrm{s}}(v|p^{\mathrm{s}})\,\mathrm{d}v$ and $m_t^{\mathrm{s}} = (1+\beta\sigma^{\mathrm{s}})^t$. It follows immediately that $r^{\mathrm{s}}, \sigma^{\mathrm{s}}, \{m_t^{\mathrm{s}}\}$ satisfy the condition given by Theorem 1 if ω^{s} and p^{s} satisfy these conditions. Thus, it suffices to show that $\omega^{\mathrm{s}}, p^{\mathrm{s}}$ satisfy these conditions. To this end, notice that By Lemma 4,

$$v^{-1}(p^{\mathrm{s}}) \leq \left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)^{+} p^{\mathrm{s}} \leq \overline{p}, \tag{A.12}$$

with at least one inequality binding. Now consider three cases separately.

Case 1: $\omega^{\mathrm{s}} \leq \alpha/\delta\beta$.
In this case, it immediately follows that

$$\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)^{+} p^{\mathrm{s}} = 0 = v^{-1}(p^{\mathrm{s}})$$

and hence $p^{\mathrm{s}} = \mathbb{E}[v]$, which in turn, by (4), implies that $\omega^{\mathrm{s}} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$. For this to be consistent with $\omega^{\mathrm{s}} \leq \alpha/\delta\beta$, it must be that $\beta \leq \underline{\beta}$.

Case 2: $\omega^{\mathrm{s}} > \alpha/\delta\beta$ and

$$\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right) p^{\mathrm{s}} = v^{-1}(p^{\mathrm{s}}). \tag{A.13}$$

In this case, Lemma 3 implies that

$$\omega^{\mathrm{s}} = \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right)p^{\mathrm{s}}} \overline{D}(v)\,\mathrm{d}v\right)\omega^{\mathrm{s}},$$

and hence, together with (A.12), it must be that $\beta \in [\underline{\beta}, \overline{\beta}]$ and

$$\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right) p^{\mathrm{s}} = p^{\beta}.$$

Meanwhile, since (A.13) is equivalent to

$$\omega^{\mathrm{s}} = g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{s}}}\right) p^{\mathrm{s}}\right),$$

it must be that $\omega^{\mathrm{s}} = g^{\beta}(p^{\beta})$ and hence $p^{\mathrm{s}} = v(p^{\beta})$.

Case 3: $\omega^{\mathrm{s}} > \alpha/\delta\beta$ and

$$p^{\mathrm{s}} = \frac{\delta\beta\omega^{\mathrm{s}}}{\delta\beta\omega^{\mathrm{s}} - \alpha}\overline{p}. \tag{A.14}$$

In this case, Lemma 3 implies that

$$\omega^{\mathrm{s}} = \delta\left(\gamma + \beta\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega^{\mathrm{s}},$$

which means this case can only occur when $\beta = \overline{\beta}$.

Together with observations that $p^{\beta} = 0$ if and only if $\beta \leq \underline{\beta}$, that $p^{\beta} = \overline{p}$ if and only if $\beta = \overline{\beta}$, and that the second inequality of (A.12) is equivalent to $\omega^{\mathrm{s}} \geq g^{\beta}(\overline{p})$, it then follows that $\omega^{\mathrm{s}}, p^{\mathrm{s}}$ must be the same as described in Theorem 1 in all three cases. This completes the proof. ■

A.6 Proof of Lemma 5

Consider any subgame perfect equilibrium. At any history h^t, choosing a myopically optimal demand in every future periods is always a feasible strategy. Suppose that the intermediary deviates to this strategy.

Notice that although the seller's prices may change after this deviation even if the seller's strategy remains unchanged (as the history of the play may be different), the seller must always be best responding in each period as he is short-lived. Therefore, prices in each period must be within $[r^*, p^*]$. As a result, the intermediary's stage game payoff after this deviation must be at least

$$\min_{p\in[r^*,p^*]} \alpha p\overline{D}(v^{-1}(p)) = \alpha p^*\overline{D}(v^{-1}(p^*)) = r^*.$$

Hence, the intermediary's continuation payoff at history h^t in this equilibrium must be at least

$$\frac{\alpha r^*}{1-\gamma\delta} = \omega^*.$$

If $\underline{\omega}^\beta = \omega^*$, then the proof is complete. Thus, it is without loss to assume that $\underline{\omega}^\beta > \omega^*$. Now consider any subgame perfect equilibrium and history h^t. Let $\omega_0 := \omega^*$. For any $n \in \mathbb{N}$, let

$$\omega_n := h^\beta(\omega_{n-1}) = \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{n-1}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{n-1}.$$

We now claim that the intermediary's continuation payoff at any history in any subgame perfect equilibrium must be at least ω_n for all $n \in \mathbb{N}\cup\{0\}$, which will in turn imply that her equilibrium payoff is at least $\underline{\omega}^\beta$.

We prove this claim by induction. Consider any subgame perfect equilibrium, since $\omega_0 = \omega^*$, the intermediary's continuation payoff at any history must be at least ω_0. For any $n \in \mathbb{N}$, suppose that the intermediary's continuation payoff at any history is at least ω_{n-1}. It suffices to show that her continuation payoff at any history is at least ω_n as well. To this end, consider any history h^t at any period t. Suppose that the seller charges p. Then, since the intermediary's continuation payoff starting from the next period is at least ω_{n-1}, regardless of the outcomes in this period, her continuation payoff at (h^t, p) must be at least

$$\mathbf{W}(p|\omega_{n-1}) = \sup_{D\in\mathcal{D}}\left[\alpha pD(p) + \delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)\omega_{n-1}\right].$$

Since price p must be the seller's best response given the intermediary's strategy, it must be that $p \leq p^*$. Moreover, since $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega$ whenever $\alpha \geq \delta\beta\omega$ and $\mathbf{W}(p|\omega)$ is decreasing on $[0, p^*]$ whenever $\alpha < \delta\beta\omega$ (see Lemma A.1 below), it must be that $\mathbf{W}(p|\omega_{n-1}) \geq \mathbf{W}(p^*|\omega_{n-1})$ for all $p \leq p^*$. Therefore, at any history h^t, the seller's continuation payoff must be at least

$$\mathbf{W}(p^*|\omega_{n-1}) = \sup_{D\in\mathcal{D}}\left[\alpha p^*D(p^*) + \delta\left(\gamma+\beta\int_{p^*}^\infty D(v)\,\mathrm{d}v\right)\omega_{n-1}\right].$$

Moreover, by Lemma 3,

$$\begin{aligned}
\mathbf{W}(p^*|\omega_{n-1}) =& \alpha p^*\overline{D}(\xi(p^*|\omega_{n-1})) \\
&+ \delta\left(\gamma+\beta\left(\int_{\xi(p^*|\omega_{n-1})}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi(p^*|\omega_{n-1}))p^*\overline{D}(\xi(p^*|\omega_{n-1}))\right)\right)\omega_{n-1} \\
=& \max\left\{\alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\omega_{n-1}, \delta\left(\gamma+\beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{n-1}}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v\right)\omega_{n-1}\right\} \\
=& \max\{\alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\omega_{n-1}, h^\beta(\omega_{n-1})\} \\
\geq& h^\beta(\omega_{n-1}) \\
=& \omega_n.
\end{aligned}$$

As a result, if $\{\omega_n\}$ does not converge, then since $\underline{\omega}^\beta$ is the unique solution to $\omega = h^\beta(\omega)$, and since $h^\beta(\omega) < \omega$ for all $\omega > \underline{\omega}^\beta$, it must be that $\limsup_{n\to\infty} \omega_n \geq \underline{\omega}^\beta$. Meanwhile, if $\{\omega_n\}$ converges to some $\tilde{\omega} < \infty$, since h is continuous,

$$\tilde{\omega} = \lim_{n\to\infty} \omega_n = \lim_{n\to\infty} h^\beta(\omega_{n-1}) = h^\beta(\lim_{n\to\infty} \omega_{n-1}) = h^\beta(\tilde{\omega}),$$

which in turn implies that $\tilde{\omega} = \underline{\omega}^\beta$. Together, the intermediary's continuation payoff at any history in any subgame perfect equilibrium must be at least $\underline{\omega}^\beta$, as desired. ■

A.7 Proof of Theorem 2

The proof of Theorem 2 involves some additional definitions and lemmas.

Lemma A.1. *Suppose that $\beta < \beta^*$. Then for any $\omega \geq 0$, $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega$ for all $p \geq 0$ whenever $\alpha \geq \delta\beta\omega$. Meanwhile, if $\alpha < \delta\beta\omega$, then $\mathbf{W}(\cdot|\omega)$ is differentiable except at countably many $p \in [0, p^*]$, is decreasing on $[0, p^*]$, and $\mathbf{W}(p^*|\underline{\omega}^\beta) = \underline{\omega}^\beta$.*

Proof. By Lemma 3, for any $p \geq 0$

$$\begin{aligned}\mathbf{W}(p|\omega) &= \max_{\xi\in[v^{-1}(p),p]} \left[\alpha p\overline{D}(\xi) + \delta\left(\gamma + \beta\left(\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\omega\right)\right] \\ &= \left[\alpha p\overline{D}(\xi(p|\omega)) + \delta\left(\gamma + \beta\left(\int_{\xi(p|\omega)}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi(p|\omega))\overline{D}(\xi(p|\omega))\right)\right)\omega\right].\end{aligned}$$

Therefore, $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega$ if $\alpha \geq \delta\beta\omega$.

In the meantime, if $\alpha < \delta\beta\omega$, then $\mathbf{W}(\cdot|\omega)$ is continuous on $[0, p^*]$, and is differentiable at all $p \in [0, p^*]$ except for those such that $(1-\alpha/\delta\beta\omega)^+ p = v^{-1}(p)$, which is at most a countable subset of $[0, p^*]$. Therefore, by lemma 1 of Milgrom and Segal (2002),

$$\frac{\partial}{\partial p}\mathbf{W}(p|\omega) = (\alpha - \delta\beta\omega)\overline{D}(\xi(p|\alpha,\beta,\delta,\omega)) \leq 0,$$

for all $p \in [0, p^*]$ at which $(1-\alpha/\delta\beta\omega)p \neq v^{-1}(p)$ and hence $\mathbf{W}(\cdot|\omega)$ is decreasing on $[0, p^*]$. Now notice that if $\underline{\omega}^\beta = \omega^*$, then

$$\mathbf{W}(p^*|\underline{\omega}^\beta) = \sup_{D\in\mathcal{D}}\left[\alpha p^* D(p^*) + \delta\left(\gamma + \beta\int_{p^*}^\infty D(v)\,\mathrm{d}v\right)\underline{\omega}^\beta\right] \geq \alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega}^\beta = \underline{\omega}^\beta$$

Meanwhile, if $\underline{\omega}^\beta > \omega^*$, then since

$$\alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega}^\beta = (1-\gamma\delta)\omega^* + \gamma\delta\underline{\omega}^\beta < \underline{\omega}^\beta$$

and since $h^\beta(\underline{\omega}^\beta) = \underline{\omega}^\beta$, by Lemma 3, we must have

$$\mathbf{W}(p^*|\underline{\omega}^\beta) = \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}^\beta}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v\right)\underline{\omega}^\beta = h^\beta(\underline{\omega}^\beta) = \underline{\omega}^\beta.$$

This completes the proof. ■

Lemma A.2. *Suppose that $\beta \le \underline{\beta}$. Then in any subgame perfect equilibrium, the intermediary's equilibrium payoff is at most $\alpha\mathbb{E}[v]/(1-\gamma\delta)$.*

Proof. Consider any subgame perfect equilibrium. Let ω denote the intermediary's equilibrium payoff. According to the structure of the game, ω can be written as

$$\omega = \alpha p_0 D_0(p_0) + \sum_{t=1}^{\infty} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\,\mathrm{d}v\right) \alpha p_t D_t(p_t),$$

for some sequence $\{p_t, D_t\}$ with $p_t \in \mathbb{N} \cup \{0\}$ and $D_t \in \mathcal{D}$ for all t. In the meantime, for any $T \in \mathbb{N}$, let

$$\omega_T := \sup_{\{\tilde{D}_t, \tilde{p}_t\}_{t=0}^T} \left[\alpha \tilde{p}_0 \tilde{D}_0(\tilde{p}_0) + \sum_{t=1}^{T} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{\tilde{p}_s}^{\infty} \tilde{D}_s(v)\,\mathrm{d}v\right) \alpha \tilde{p}_t \tilde{D}_t(\tilde{p}_t)\right]$$

For any $T \in \mathbb{N}$, note ω_T can be recursively written as

$$\omega_T = \sup_{\tilde{p} \ge 0, \tilde{D} \in \mathcal{D}} \left[\alpha \tilde{p} \tilde{D}(\tilde{p}) + \delta \left(\gamma + \beta \int_{\tilde{p}}^{\infty} \tilde{D}(v)\,\mathrm{d}v\right) \omega_{T-1}\right]$$

where $\omega_0 := \alpha\mathbb{E}[v]$.

As a result, since $\omega < \infty$, and since the seller's payoff in each period must be at least r^* in any subgame perfect equilibrium , it must be that $\omega \le \limsup_{T\to\infty} \omega_T$.

Now fix any $T \in \mathbb{N}$ and suppose that $\omega_{T-\tau} = \alpha \sum_{s=0}^{\tau} \delta^s \mathbb{E}[v]$ for some $\tau < T$. Then since $\beta \le \underline{\beta}$,

$$\delta\beta\omega_{T-\tau} \le \delta\beta \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta} \le \alpha.$$

Therefore, for any $p \ge 0$, $\xi(p|\omega^*_{T-\tau}) = v^{-1}(p)$. Hence, by Lemma 3,

$$\begin{aligned}\omega_{T-\tau-1} = & \sup_{p \in [r^*, p^*]} [\alpha p \overline{D}(v^{-1}(p)) + \gamma\delta\omega_{T-\tau}] \\ = & \sum_{s=0}^{\tau+1} \gamma^s \delta^s \alpha\mathbb{E}[v].\end{aligned}$$

Therefore, by induction, it must be that $\omega_{T-\tau} = \sum_{s=0}^{\tau} \gamma^s\delta^s\alpha\mathbb{E}[v]$, for all $\tau \in \{0, \dots, T\}$. In particular, $\omega_T = \sum_{s=0}^{T} \delta^s \alpha\mathbb{E}[v]$. Therefore,

$$\omega \le \limsup_{T\to\infty} \omega_T = \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta},$$

as desired. ■

Lemma A.3. *Suppose that $\beta \in (\underline{\beta}, \widehat{\beta})$, and that in any subgame perfect equilibrium , the total revenue in each period is at least $\underline{r} \le \mathbb{E}[v]$. Then in any subgame perfect equilibrium , the intermediary's normalized continuation value at any history is at most*

$$u(\underline{r}) := \frac{\alpha}{\delta\beta} \frac{\underline{r}}{\underline{r} - p^\beta \overline{D}(p^\beta)}. \tag{A.15}$$

Proof. Consider any subgame perfect equilibrium and consider any history. Let $\omega < \infty$ denote the intermediary's normalized continuation payoff at this history. According to the structure of the game, ω can be written as

$$\omega = \alpha p_0 D_0(p_0) \sum_{t=1}^{\infty} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\,\mathrm{d}v \right) \alpha p_t D_t(p_t),$$

for some sequence $\{p_t, D_t\}$ with $p_t \in \mathbb{N} \cup \{0\}$ and $D_t \in \mathcal{D}$ for all t. In the meantime, for any $T \in \mathbb{N}$, let

$$\begin{aligned} \omega_T := \sup_{\{\tilde{D}_t, \tilde{p}_t\}_{t=0}^T} & \left[\alpha \tilde{p}_0 \tilde{D}_0(\tilde{p}_0) + \sum_{t=1}^{T} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{\tilde{p}_s}^{\infty} \tilde{D}_s(v)\,\mathrm{d}v \right) \alpha \tilde{p}_t \tilde{D}_t(p_t) \right] \\ \text{s.t. } & \tilde{p}_t \tilde{D}_t(\tilde{p}_t) \geq \underline{r}, \text{ for all } t. \end{aligned}$$

Moreover, notice that for any $T \in \mathbb{N}$, ω_T can be recursively written as

$$\begin{aligned} \omega_T = \sup_{\tilde{p} \geq 0, \tilde{D} \in \mathcal{D}} & \left[\alpha \tilde{p} \tilde{D}(\tilde{p}) + \delta \left(\gamma + \beta \int_{\tilde{p}}^{\infty} \tilde{D}(v)\,\mathrm{d}v \right) \omega_{T-1} \right] \\ \text{s.t. } & \tilde{p} \tilde{D}(\tilde{p}) \geq \underline{r} \end{aligned}$$

where $\omega_0 := \alpha \mathbb{E}[v]$.

Lastly, since $\omega < \infty$, and since the total revenue in each period must be at least $\underline{r}$ in any subgame perfect equilibrium , it must be that $\omega \leq \limsup_{T \to \infty} \omega_T$.

Since $\beta > \underline{\beta}$, $\alpha \mathbb{E}[v]/(1 - \gamma\delta) > \alpha/\delta\beta$, and hence there exists $\bar{\tau}$ such that $\sum_{s=0}^{\tau} \gamma^s \delta^s \alpha \mathbb{E}[v] > \alpha/\delta\beta$ for all $\tau \geq \bar{\tau}$. Now consider any $T > \bar{\tau}$. For any $\tau \in \{1, \dots, \bar{\tau}\}$, any $p \geq 0$, and for any $\omega_{T-\tau} > \alpha/\delta\beta$, by Lemma 3,

$$\begin{aligned} \omega_{T-\tau+1} = \max_{p \geq 0} & \Bigg[\alpha p \overline{D}(\xi(p|\omega_{T-\tau})) \\ & + \delta \left(\gamma + \beta \left(\int_{\xi(p|\omega_{T-\tau})}^{\infty} \overline{D}(v)\,\mathrm{d}v - (p - \xi(p|\omega_{T-\tau}))\overline{D}(\xi(p|\omega_{T-\tau})) \right) \right) \omega_{T-\tau} \Bigg] \\ \text{s.t. } & p \overline{D}(\xi(p|\omega_{T-\tau})) \geq \underline{r}, \end{aligned}$$

where

$$\begin{aligned} \xi(p|\omega_{T-\tau}) &= \max \left\{ \left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}} \right) p, v^{-1}(p) \right\} \\ &= \begin{cases} \left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}} \right) p, & \text{if } \omega_{T-\tau} \geq g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}} \right) p \right) \\ v^{-1}(p), & \text{if } \omega_{T-\tau} \leq g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}} \right) p \right) \end{cases}, \end{aligned}$$

for all $p \geq 0$, which in turn can be written as

$$\omega_{T-\tau+1} = \max_{p \in [0, \overline{p}]} \psi(p|\omega_{T-\tau}) \quad \text{s.t. } p\overline{D}(\xi(p|\omega_{T-\tau})) \geq \underline{r}.$$

where

$$\psi(p|\tilde{\omega}) := \begin{cases} \delta \left(\gamma + \beta \int_p^{\infty} \overline{D}(v)\,\mathrm{d}v \right) \tilde{\omega}, & \text{if } g^{\beta}(p) \geq \tilde{\omega} \\ \frac{\alpha\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha} p \overline{D}\left(v^{-1}\left(\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha} p \right) \right) + \gamma\delta\tilde{\omega}, & \text{if } g^{\beta}(p) < \tilde{\omega} \end{cases},$$

for all $p \geq 0$ and for all $\tilde{\omega} \geq 0$. Therefore, suppose that for some $\tau \in \{1, \dots, \bar{\tau}\}$, $\omega_{T-\tau} > \alpha/\delta\beta$. Then

$$\omega_{T-\tau+1} \geq \alpha\mathbb{E}[v] + \gamma\delta\omega_{T-\tau} > \frac{\alpha}{\delta\beta}.$$

By induction, it then follows that $\omega_{T-\tau} > \alpha/\delta\beta$ for all $\tau \in \{1, \dots, \bar{\tau}\}$.

Furthermore, since g^β is increasing on $[0, \overline{p}]$, the functions $p \mapsto \delta\left(\gamma + \beta\int_p^\infty \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}$ and $p \mapsto \alpha\delta\beta\tilde{\omega}/(\delta\beta\tilde{\omega} - \alpha)p\overline{D}(v^{-1}(\delta\beta\tilde{\omega}(\delta\beta\tilde{\omega} - \alpha)p))$ cross at most once on $[0, \overline{p}]$ for any $\tilde{\omega} > \alpha/\delta\beta$, and whenever they cross, it must be that $p \mapsto \delta\left(\gamma + \beta\int_p^\infty \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}$ is decreasing at the crossing point. Therefore, if these two functions cross and the crossing point is below p^β, then it must be that $\omega_{T-\tau+1} = \alpha\mathbb{E}[v] + \gamma\delta\omega_{T-\tau} > \omega_{T-\tau}$. Otherwise, it must be that

$$\omega_{T-\tau+1} = \delta\left(\gamma + \beta\int_{p_{T-\tau}}^\infty \overline{D}(v)\,\mathrm{d}v\right)\omega_{T-\tau},$$

where $p_{T-\tau} \in [0, \overline{p}]$ is the unique solution to

$$\frac{\delta\beta\omega_{T-\tau}}{\delta\beta\omega_{T-\tau} - \alpha}p\overline{D}(p) = \underline{r}.$$

Meanwhile, notice that since $\alpha\mathbb{E}[v]/(1-\gamma\delta) > \alpha/\delta\beta$, the crossing point must be above p^β for T large enough and τ small enough.

Finally, for any $\tilde{\omega} > \alpha/\delta\beta$, let $p(\tilde{\omega})$ be the unique solution to

$$\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha}p\overline{D}(p) = \underline{r}.$$

and let

$$\Pi(\tilde{\omega}) := \delta\left(\gamma + \beta\int_{p(\tilde{\omega})}^\infty \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}.$$

Notice that by definition, $\Pi(\tilde{\omega}) > \tilde{\omega}$ if $\tilde{\omega} < u(\underline{r})$; $\Pi(\tilde{\omega}) < \tilde{\omega}$ if $\tilde{\omega} > u(\underline{r})$ and $\Pi(u(\underline{r})) = u(\underline{r})$. Together, it then follows that for any T large enough,

$$\omega_T = \Pi(\omega_{T-1}).$$

Therefore, $\{\omega_T\}$ has a limit and $\lim_{T\to\infty}\omega_T = u(\underline{r})$, and hence

$$\omega \leq \lim_{T\to\infty}\omega_T = u(\underline{r}),$$

as desired. ■

Proof of Theorem 2. We first introduce a family of regimes that describe the intermediary's and the seller's strategies. Then we will use these regimes to construct subgame perfect equilibria to support the desired payoffs,

> **Regime** p-MYOPIC: The seller plays p. The intermediary chooses any $D \in \Delta(p|0)$. The tie-breaker then breaks ties in favor of the seller (i.e. the tie breaker chooses $q = D(p)$).
>
> **Regime** (p,ω)-TRANSITION: The seller plays p, the intermediary chooses any $D \in \Delta(p|\omega)$ and the tie-breaker breaks ties in favor of the seller.
>
> **Regime** D-PUNISH: The intermediary chooses D after seeing the seller's price, and the tie-breaker breaks ties against the seller.

We now characterize the set of equilibrium payoffs by four cases separately.

Case 1: $\beta \leq \underline{\beta}$.

In this case, since

$$\frac{1-\gamma\delta}{\delta\beta} \geq \mathbb{E}[v] > p^*\overline{D}(v^{-1}(p^*)),$$

$\delta\beta\omega^* < \alpha$. Thus,

$$h^\beta(\omega^*) = \delta(\gamma + \beta\mathbb{E}[v])\omega^* \leq \omega^*,$$

which in turn implies, by the definition of $\underline{\omega}^\beta$, $\underline{\omega}^\beta = \omega^* < \alpha/\delta\beta$.

Now let $\overline{\omega} := \alpha\mathbb{E}[v]/(1-\gamma\delta)$. For any $p \geq 0$ and for any $\tilde{\omega} \in [\underline{\omega}^\beta, \overline{\omega}]$, let $\Lambda^\beta(p, \tilde{\omega})$ be the value of the following constrained maximization problem:

$$\sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right)\overline{\omega}\right] \tag{A.16}$$

$$\text{s.t. } \frac{\alpha p D(p)}{1-\gamma\delta} \leq \tilde{\omega}. \tag{A.17}$$

Note that if the constraint does not bind,

$$\Lambda^\beta(p,\tilde{\omega}) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\overline{\omega},$$

while if the constraint binds,

$$\Lambda^\beta(p,\tilde{\omega}) = (1-\gamma\delta)\tilde{\omega} + \delta\left(\gamma + \beta\int_{\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)p}^\infty \overline{D}(v)\,\mathrm{d}v\right)\overline{\omega},$$

and

$$\frac{\partial}{\partial p}\Lambda^\beta(p,\underline{\omega}^\beta) = ((1-\lambda)\alpha - \delta\beta\overline{\omega})\overline{D}\left(\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)p\right) \leq 0,$$

by the envelope theorem, where λ solves

$$\frac{\alpha p\overline{D}\left(\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)p\right)}{1-\gamma\delta} = \tilde{\omega}.$$

Since $\tilde{\omega} \in [\underline{\omega}^\beta, \overline{\omega}]$, there exists $\tilde{p} \in [E(v), p^*]$ such that

$$\frac{\alpha p\overline{D}(v^{-1}(\tilde{p}))}{1-\gamma\delta} = \tilde{\omega}.$$

This implies that the constraint of (A.16) binds if and only if $p \leq \tilde{p}$. Together, $\Lambda^\beta(\cdot,\tilde{\omega})$ is a decreasing function on $[0, p^*]$.

In the meantime, for any $\tilde{\omega} \in [\underline{\omega}^\beta, \overline{\omega}]$, notice that the function $p \mapsto \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$ is quasi-concave and has a maximum at $p = \mathbb{E}[v]$. Moreover, since for any $\lambda \in [1-\delta\beta\overline{\omega}/\alpha, 1]$ and for any $p \in [0, p^*]$,

$$\delta\beta\overline{\omega}\int_{\left(1-\frac{\alpha(1-\lambda)}{\delta\beta\overline{\omega}}\right)p}^\infty \overline{D}(v)\,\mathrm{d}v \geq (1-\lambda)\alpha p\overline{D}(v^{-1}(p)) \iff \overline{\omega} \geq g^\beta\left(\left(1-\frac{\alpha(1-\lambda)}{\delta\beta\overline{\omega}}\right)p\right),$$

the functions $p \mapsto \delta\beta\overline{\omega}\int_{(1-\alpha(1-\lambda)/\delta\beta\overline{\omega})p}^{\infty}\overline{D}(v)\,\mathrm{d}v$ and $p \mapsto \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$ cross at most once.

Together, for any $\tilde{\omega} \in [\underline{\omega}^{\beta}, \overline{\omega}]$,

$$\Lambda^{\beta}(p,\tilde{\omega}) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega},\ \forall p \geq 0$$

if and only if

$$\Lambda^{\beta}(\mathbb{E}[v],\tilde{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\tilde{\omega}.$$

We now define $\underline{\boldsymbol{\omega}}^{*}(\beta)$ as

$$\underline{\boldsymbol{\omega}}^{*}(\beta) := \inf\{\tilde{\omega} \in [\underline{\omega}^{\beta}, \overline{\omega}] | \Lambda^{\beta}(\mathbb{E}[v],\tilde{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\tilde{\omega}\}.$$

Notice that $\underline{\boldsymbol{\omega}}^{*}(\beta)$ is well-defined since $\Lambda^{\beta}(\mathbb{E}[v],\overline{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\overline{\omega}$. Moreover, by definition, since Λ^{β} is increasing in β, $\underline{\boldsymbol{\omega}}^{*}$ is nonincreasing in β on $[0, \underline{\beta}]$.

We now claim that in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^{*}(\beta)/\alpha$. Indeed, suppose the contrary, that there exists a subgame perfect equilibrium in which the lowest total revenue on the equilibrium path, say $\tilde{r}$, is strictly below $\underline{r}$. Then, since the seller must be best responding in the period where the total revenue is $\tilde{r}$, for any $p \geq 0$, there must exist $D \in \mathcal{D}$ such that $pD(p) \leq \tilde{r}$, so that the intermediary would respond by choosing $D \in \mathcal{D}$ if the seller deviates to any price $p \geq 0$. Let $\omega(p, D)$ denote the continuation value at this history. In the meantime, at any such history, since it is always feasible for the intermediary to choose the myopically optimal demand, and since her continuation payoff must be at least α share of the present discounted value of the sum of total revenues onward, which in turn, by hypothesis, is no less than $\tilde{r}$, it must be that

$$\alpha pD(p) + \delta\left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\overline{\omega} \geq \alpha pD(p) + \delta\left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\omega(p,D) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\frac{\alpha\tilde{r}}{1-\gamma\delta},$$

where the first inequality follows from Lemma A.2. Therefore, let $\tilde{\omega} := \alpha\tilde{r}/(1-\gamma\delta)$, it follows that for any $p \geq 0$, there exists D such that

$$\alpha pD(p) + \delta\left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\overline{\omega} \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

and that

$$\frac{\alpha pD(p)}{1-\gamma\delta} \leq \tilde{\omega}$$

and hence

$$\Lambda^{\beta}(p,\tilde{\omega}) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

for all $p \geq 0$, but $\tilde{\omega} < \underline{\boldsymbol{\omega}}^{*}(\beta)$, which contradicts to the definition of $\underline{\boldsymbol{\omega}}^{*}(\beta)$. Thus, in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^{*}(\beta)/\alpha$. This in turn implies that the intermediary's continuation payoff at any history must be at least

$$\frac{\alpha\underline{r}}{1-\gamma\delta} = \underline{\boldsymbol{\omega}}^{*}(\beta).$$

Together with Lemma A.2, by letting $\overline{\boldsymbol{\omega}}^{*}(\beta) := \overline{\omega} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$, it then follows that the intermediary's equilibrium payoff must be at least $\underline{\boldsymbol{\omega}}^{*}(\beta)$ and at most $\overline{\boldsymbol{\omega}}^{*}(\beta)$.

Now consider any $\hat{\omega} \in [\underline{\boldsymbol{\omega}}^*(\beta), \overline{\boldsymbol{\omega}}^*(\beta)]$. By continuity of the function $p \mapsto p\overline{D}(v^{-1}(p))$, there exists $\hat{p}, \underline{p}$ such that $\mathbb{E}[v] \leq \hat{p} \leq \underline{p} \leq p^*$ and that $\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta) = \hat{\omega}$, $\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) = \underline{\boldsymbol{\omega}}^*(\beta)$. Moreover, for any $p \geq 0$, fix any solution to (A.16) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^*(\beta)$ and denote it by D_p. We claim that the following strategy profile constitutes a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$:

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to $p' \neq \hat{p}$, then enter regime $D_{p'}$-PUNISH immediately. If the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, stay in the same regime.

- For any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $\mathbb{E}[v]$-MYOPIC.

- Under regime $\underline{p}$-MYOPIC and regime $\mathbb{E}[v]$-MYOPIC, if the seller deviates to $p' \neq p$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

To see that this is a subgame perfect equilibrium, first notice that the intermediary's continuation payoff induced by the strategy profile is finite in every subgame. Therefore, Lemma 1 implies that it suffices to show there are no incentives to deviate in each regime. Indeed, for any $p \in [\mathbb{E}[v], \underline{p}]$, under regime p-MYOPIC, if all players follow their strategies, the seller's revenue must be $(1-\alpha)p\overline{D}(v^{-1}(p))$. Meanwhile, given that the intermediary follows her strategy, if the seller deviates to $p' \neq p$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution to (A.16) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$(1-\alpha)p'D_{p'}(p') \leq \frac{(1-\alpha)(1-\gamma\delta)}{\alpha}\underline{\boldsymbol{\omega}}^*(\beta) = (1-\alpha)\underline{p}\overline{D}(v^{-1}(\underline{p})) \leq (1-\alpha)p\overline{D}(v^{-1}(p)).$$

Therefore, the seller does not have any incentive to deviate. As for the intermediary, in each period, the present discounted value of playing according to regime $\hat{p}$-MYOPIC is $\hat{\omega} = \alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta)$, while the present discounted value of the best payoff she can obtain by deviating is

$$\sup_{D \in \mathcal{D}} \left[\alpha\hat{p}D(\hat{p}) + \delta\left(\gamma + \beta\int_{\hat{p}}^{\infty} D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right],$$

which, by Lemma 3 and by the fact that $\underline{\boldsymbol{\omega}}^*(\beta) \leq \alpha\mathbb{E}[v]/(1-\gamma\delta) < \alpha/\delta\beta$, is given by

$$\alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \gamma\delta\underline{\boldsymbol{\omega}}^*(\beta) \leq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \frac{\gamma\delta\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \hat{\omega},$$

where the last equality follows from $\underline{\boldsymbol{\omega}}^*(\beta) = \alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) \leq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta)$. Thus, the intermediary does not have an incentive to deviate either.

In the meantime, for any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\overline{\omega}$, whereas if she deviates, her payoff would be at most $\alpha p'\overline{D}(v^{-1}(p')) + \gamma\delta\underline{\boldsymbol{\omega}}^*(\beta)$. Since $D_{p'}$ is a solution to (A.16) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$\alpha p'D_{p'}(p') + \delta\left(\gamma + \beta\int_{p'}^{\infty} D_{p'}(v)\,\mathrm{d}v\right)\overline{\omega} = \Lambda^{\beta}(p', \underline{\boldsymbol{\omega}}^*(\beta)) \geq \alpha p'\overline{D}(v^{-1}(p')) + \gamma\delta\underline{\boldsymbol{\omega}}^*(\beta),$$

where the last inequality follows from the definition of $\underline{\boldsymbol{\omega}}^*(\beta)$. Thus, the intermediary does not have any incentives to deviate under this regime.

Together, the strategy profile described above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is $\hat{\omega}$.

Case 2: $\delta\beta/(1-\gamma\delta) \in (\underline{\beta}, \widehat{\beta})$.

We first claim that $\underline{\omega}^\beta = \omega^*$ in this case as well. To see this, recall that since $\beta \in (\underline{\beta}, \overline{\beta}]$, there exists $p^\beta \in (0, \overline{p}]$ such that

$$\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v = \frac{1-\gamma\delta}{\delta\beta}. \tag{A.18}$$

Let $\omega^\beta := g^\beta(p^\beta)$. Lemma 4 then implies that

$$\frac{\delta\beta\omega^\beta}{\delta\beta\omega^\beta - \alpha} p^\beta \in \operatorname*{argmax}_{p \geq 0} p\mathbf{D}(p|p),$$

for any selection $\mathbf{D}$ of $\Delta(\cdot|\omega^\beta)$. In particular,

$$\frac{\delta\beta\omega^\beta}{\delta\beta\omega^\beta - \alpha} p^\beta \leq p^*. \tag{A.19}$$

Rearranging, we have

$$p^\beta + \frac{1}{\overline{D}(p^\beta)} \int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v \leq p^*,$$

which in turn implies, by (A.18) and (A.19),

$$\begin{aligned} p^* - p^\beta \geq & \frac{1}{\overline{D}(p^\beta)} \int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v \\ = & \frac{1}{\overline{D}(p^\beta)} \frac{1-\gamma\delta}{\delta\beta} \\ \geq & \frac{1-\gamma\delta}{\delta\beta} \overline{D}(v^{-1}(p^*)). \end{aligned}$$

Therefore,

$$p^* - \frac{1-\gamma\delta}{\delta\beta} \overline{D}(v^{-1}(p^*)) \geq p^\beta.$$

Together with the definition of ω^*, we have

$$\int_{\left(1-\frac{\alpha}{\delta\beta\omega^*}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v = \int_{p^* - \frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(p^*))}^\infty \overline{D}(v)\,\mathrm{d}v \leq \int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v = \frac{1-\gamma\delta}{\delta\beta}$$

and hence

$$\delta\left(\gamma + \beta \int_{\left(1-\frac{\alpha}{\delta\beta\omega^*}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v\right) \leq 1.$$

Thus, $h^\beta(\omega^*) \leq \omega^*$, which implies that $\underline{\omega}^\beta = \omega^*$.

Next, for any $\omega \in [\underline{\omega}^\beta, \omega^\beta]$ and for any $p \geq 0$, let $\Lambda^\beta(p, \omega)$ be the value of the constrained maximization problem

$$\begin{aligned} &\sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right) u\left(\frac{1-\gamma\delta}{\alpha}\omega\right)\right] \\ &\text{s.t. } \frac{\alpha p D(p)}{1-\gamma\delta} \leq \omega, \end{aligned} \tag{A.20}$$

where u is defined in (A.15). Let $\underline{\boldsymbol{\omega}}^*(\beta)$ be defined as

$$\underline{\boldsymbol{\omega}}^*(\beta) := \inf\{\omega \in [\underline{\omega}^\beta, \omega^\beta] | \Lambda^\beta(\mathbb{E}[v], \omega) \geq \alpha\mathbb{E}[v] + \gamma\delta\omega\}.$$

By the same arguments as in **Case 1**, $\underline{\boldsymbol{\omega}}^*(\beta) \geq \underline{\omega}^\beta$ is well-defined and $\Lambda^\beta(p, \omega) \geq \alpha\mathbb{E}[v] + \gamma\delta\omega$ if and only if $\omega \in [\underline{\boldsymbol{\omega}}^*(\beta), \omega^\beta]$.

Also, let

$$\overline{\boldsymbol{\omega}}^*(\beta) := u\left(\frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^*(\beta)\right).$$

We now argue that in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^*(\beta)/\alpha$. Indeed, suppose the contrary, that there exists a subgame perfect equilibrium in which the lowest total revenue on the equilibrium path, say $\tilde{r}$, is strictly below $\underline{r}$. Then, since the seller must be best responding in the period where the total revenue is $\tilde{r}$, for any $p \geq 0$, there must exists $D \in \mathcal{D}$ such that $pD(p) \leq \tilde{r}$, so that the intermediary would respond by choosing $D \in \mathcal{D}$ if the seller deviates to any price $p \geq 0$. Let $\omega(p, D)$ denote the continuation value at this history. In the meantime, at any such history, since it is always feasible for the intermediary to choose the myopically optimal demand, and since her continuation payoff must be at least α share of the present discounted value of the sum of total revenues onward, which in turn, by hypothesis, is no less than $\tilde{r}$, it must be that

$$\alpha pD(p)+\delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)u(\tilde{r}) \geq \alpha pD(p)+\delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)\omega(p,D) \geq \alpha p\overline{D}(v^{-1}(p))+\gamma\delta\frac{\alpha\tilde{r}}{1-\gamma\delta},$$

where the first inequality follows from Lemma A.3 since the total revenue in any period is at least $\tilde{r}$. Therefore, let $\tilde{\omega} := \alpha\tilde{r}/(1-\gamma\delta)$, it follows that for any $p \geq 0$, there exists D such that

$$\alpha pD(p) + \delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)u\left(\frac{1-\gamma\delta}{\alpha}\tilde{\omega}\right) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

and that

$$\frac{\alpha pD(p)}{1-\gamma\delta} \leq \tilde{\omega}$$

and hence

$$\Lambda^\beta(p, \tilde{\omega}) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

for all $p \geq 0$, but $\tilde{\omega} < \underline{\boldsymbol{\omega}}^*(\beta)$, which contradicts to the definition of $\underline{\boldsymbol{\omega}}^*(\beta)$. Thus, in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^*(\beta)/\alpha$. This in turn implies that the intermediary's continuation payoff at any history must be at least

$$\frac{\alpha\underline{r}}{1-\gamma\delta} = \underline{\boldsymbol{\omega}}^*(\beta).$$

Furthermore, by Lemma A.3, the intermediary's equilibrium payoff must be below $\overline{\boldsymbol{\omega}}^*(\beta)$. Together, the intermediary's equilibrium payoff must be at least $\underline{\boldsymbol{\omega}}^*(\beta)$ and at most $\overline{\boldsymbol{\omega}}^*(\beta)$.

Consider any $\hat{\omega} \in [\underline{\boldsymbol{\omega}}^*(\beta), \omega^\beta]$. By continuity of the function $p \mapsto p\overline{D}(v^{-1}(p))$, since $\alpha\mathbb{E}[v]/(1-\gamma\delta) \geq \omega^\beta$, there exists $\hat{p}, \underline{p}$ such that $\hat{p} \leq \underline{p} \leq p^*$ and that $\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta) = \hat{\omega}$, $\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) = \underline{\boldsymbol{\omega}}^*(\beta)$.

Notice that since $\hat{\omega} \leq \omega^\beta \leq \alpha\mathbb{E}[v]/(1-\gamma\delta)$,[19] there exists $\tilde{p}^\beta \in [\mathbb{E}[v], \hat{p}]$ such that $\omega^\beta = \alpha\tilde{p}^\beta\overline{D}(v^{-1}(\tilde{p}^\beta))/(1-\gamma\delta)$. Moreover, since $\omega^\beta = g^\beta(p^\beta)$,

$$p^\beta = v^{-1}\left(\frac{\delta\beta\omega^\beta}{\delta\beta\omega^\beta - \alpha}p^\beta\right),$$

and hence

$$v(p^\beta) = \frac{\delta\beta\omega^\beta}{\delta\beta\omega^\delta_\beta - \alpha}p^\beta = \tilde{p}^\beta \leq \hat{p}.$$

Rearranging, we have

$$\hat{p} - p^\beta \geq \frac{1}{\overline{D}(p^\beta)}\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v = \frac{1}{\overline{D}(p^\beta)}\frac{1-\gamma\delta}{\delta\beta} \geq \frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(\hat{p})),$$

and hence

$$\left(1 - \frac{\alpha}{\delta\beta\hat{\omega}}\right)\hat{p} \geq p^\beta,$$

which in turn implies that

$$\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p}) \tag{A.21}$$

For any $p \geq 0$, fix any solution to (A.20) with $\omega = \underline{\boldsymbol{\omega}}^*(\beta)$ and denote it by D_p. We now construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to $p' \neq \hat{p}$, then enter regime $D_{p'}$-PUNISH immediately. If the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, stay in the same regime.

- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha), \overline{\boldsymbol{\omega}}^*(\beta))$-TRANSITION.

- Under regime $\underline{p}$-MYOPIC, if the seller deviates to $p' \neq \underline{p}$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

- Under regime $(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha), \overline{\boldsymbol{\omega}}^*(\beta))$-TRANSITION, if the seller deviates to any $p' \geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

To see that this is a subgame perfect equilibrium, first note that the intermediary's continuation payoff induced by this strategy profile is finite in every subgame. As a result, Lemma 1 implies that it suffices to show there are no incentives to deviate in each regime. Indeed, under regime $(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta) -$

[19] The second inequality follows from the definition of p^β, which implies

$$\begin{aligned}\omega^\beta = g^\beta(p^\beta) =& \frac{\alpha}{\delta\beta}\left(1 + \frac{p^\beta\overline{D}(p^\beta)}{\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v}\right)\\ =& \frac{\alpha}{\delta\beta}\frac{\delta\beta}{1-\gamma\delta}\left(p^\beta\overline{D}(p^\beta) + \int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v\right)\\ \leq& \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta}.\end{aligned}$$

$\alpha), \overline{\boldsymbol{\omega}}^*(\beta))$-TRANSITION. First notice that if all players follow their strategies, by Lemma 3, since $\overline{\boldsymbol{\omega}}^*(\beta) \geq \omega^\beta = g^\beta(p^\beta)$,

$$p^\beta \geq v^{-1}\left(\frac{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha}\right)$$

and hence the intermediary's payoff would be

$$\delta\left(\gamma+\beta\int_{p^\beta}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\overline{\boldsymbol{\omega}}^*(\beta)=\overline{\boldsymbol{\omega}}^*(\beta).$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha)|\overline{\boldsymbol{\omega}}^*(\beta))$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha}p^\beta\overline{D}(p^\beta),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution to (A.20) with $\omega = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^*(\beta)=\frac{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^*(\beta)-\alpha}p^\beta\overline{D}(p^\beta),$$

where the equality follows from the definition of $\overline{\boldsymbol{\omega}}^*(\beta)$. Thus, the seller does not have any incentive to deviate either.

In the meantime, for $p \in \{\hat{p}, \underline{p}\}$, under regime p-myopic, if all players follow their strategies, the seller's payoff would be $(1-\alpha)p\overline{D}(v^{-1}(p))$ and the intermediary's payoff would be $\alpha p\overline{D}(v^{-1}(p))/(1-\gamma\delta)$. Meanwhile, if the seller deviates to any $p' \geq 0$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution to (A.20) with $\omega = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^*(\beta)=\underline{p}\overline{D}(v^{-1}(\underline{p})) \leq p\overline{D}(v^{-1}(p)).$$

Thus, the seller does not have an incentive to deviate. As for the intermediary, if she deviates from this strategy, her continuation value would be at most $\underline{\boldsymbol{\omega}}^*(\beta)$. Therefore, since

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta}=\alpha\hat{p}\overline{D}(v^{-1}(p))+\gamma\delta\hat{\omega} \geq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))+\gamma\delta\underline{\boldsymbol{\omega}}^*(\beta)$$

and

$$\frac{\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))}{1-\gamma\delta}=\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))+\gamma\delta\underline{\boldsymbol{\omega}}^*(\beta),$$

by (A.21), the intermediary's payoff from deviation is at most

$$\sup_{D\in\mathcal{D}}\left[\alpha pD(p)+\delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right]=\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\underline{\boldsymbol{\omega}}^*(\beta)$$

and hence the intermediary does not have an incentive to deviate either.

Lastly, for any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\overline{\boldsymbol{\omega}}^*(\beta)$, whereas if he deviates, her payoff would be at most $\alpha p'\overline{D}(v^{-1}(p'))+\gamma\delta\underline{\boldsymbol{\omega}}^*(\beta)$. Since $D_{p'}$ is a solution to (A.20) with $\omega = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$\alpha p'D_{p'}(p')+\delta\left(\gamma+\beta\int_{p'}^\infty D_{p'}(v)\,\mathrm{d}v\right)\overline{\boldsymbol{\omega}}^*(\beta)=\Lambda^\beta(p',\underline{\boldsymbol{\omega}}^*(\beta)) \geq \alpha p'\overline{D}(v^{-1}(p'))+\gamma\delta\underline{\boldsymbol{\omega}}^*(\beta),$$

where the last inequality follows from the definition of $\underline{\boldsymbol{\omega}}^*(\beta)$. Thus, the intermediary does not have any incentives to deviate under this regime.

Therefore, the strategy profile above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \hat{\omega}.$$

Now consider any $\hat{\omega} \in [\omega^\beta, \overline{\boldsymbol{\varpi}}^*(\beta)]$. We claim that the following strategy profile consitititutes a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha), \hat{\omega})$-TRANSITION. If the seller deviates to any $p' \geq 0$, move to regime p'-PUNISH. Otherwise, stay in the same regime.
- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)-\alpha), \overline{\boldsymbol{\varpi}}^*(\beta))$-TRANSITION.
- Under regime $\underline{p}$-MYOPIC, if the seller deviates to $p' \neq \underline{p}$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.
- Under regime $(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)-\alpha), \overline{\boldsymbol{\varpi}}^*(\beta))$-TRANSITION, if the seller deviates to any $p' \geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

From the same arguments as above, it follows that both the intermediary and the seller do not have incentives to deviate under regime p'-PUNISH, regime $\underline{p}$-MYOPIC, and regime $(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\varpi}}^*(\beta)-\alpha), \overline{\boldsymbol{\varpi}}^*(\beta))$-TRANSITION. Therefore, by Lemma 1, since the intermediary's continuation payoff are finite in all subgames under this strategy profile, it suffices to show that both the intermediary and the seller do not have incentives to deviate under regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha), \hat{\omega})$-TRANSITION and that the intermediary's equilibrium payoff is $\hat{\omega}$. Indeed, since $\hat{\omega} \geq \omega^\beta = g^\beta(p^\beta)$,

$$p^\beta \geq v^{-1}\left(\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\right)$$

and hence by Lemma 3, the intermediary's payoff would be

$$\delta\left(\gamma + \beta\int_{p^\beta}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\hat{\omega} = \hat{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha)|\hat{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution to (A.20) with $\omega = \underline{\boldsymbol{\omega}}^*(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^*(\beta) \leq \frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

where the second inequality follows from $\hat{\omega} \leq \overline{\boldsymbol{\varpi}}^*(\beta)$ and from the definition of $\overline{\boldsymbol{\varpi}}^*(\beta)$. Thus, the seller does not have any incentive to deviate either, as desired.

Case 3: $\beta \in [\widehat{\beta}, \overline{\beta}]$.

In this case, using the same argument as in **Case 2**, it follows that $\underline{\omega}^\beta = \omega^*$ as well. Let $\underline{\boldsymbol{\omega}}^*(\beta) := \underline{\omega}^\beta$ and let $\overline{\boldsymbol{\omega}}^*(\beta) = \infty$. For any $\hat{\omega} \geq \underline{\omega}^\beta$, we will construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$. First, consider any $\hat{\omega} \in [\underline{\omega}^\beta, \omega^\beta]$. Since $\omega^\beta \leq \alpha\mathbb{E}[v]/(1-\gamma\delta)$,[20] there exists $\hat{p} \in [\mathbb{E}[v], p^*]$ such that $\hat{p}\overline{D}(v^{-1}(\hat{p})) = (1-\gamma\delta)\hat{\omega}/\alpha$. Moreover, as shown in **Case 2**, it must be that

$$\left(1 - \frac{\alpha}{\delta\beta\hat{\omega}}\right)\hat{p} \geq p^\beta$$

and therefore

$$\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p}). \tag{A.22}$$

We now construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$. To this end, notice that by definition of p^*, for any p', there exists $D_{p'} \in \mathcal{D}$ such that $p'D_{p'}(p'^+) \leq p^*\overline{D}(v^{-1}(p^*))$. For any $p' \geq 0$, fix any such $D_{p'} \in \mathcal{D}$. In the meantime, take any $\tilde{\omega} > \max\{\alpha\mathbb{E}[v]/\gamma\delta + \underline{\omega}^\beta, \omega^\beta\}$. Now consider the following strategy profile:

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to any $p' \neq \hat{p}$, move to $D_{p'}$-PUNISH. If the intermediary deviates, move to regime p^*-MYOPIC. Otherwise, stay in the same regime.
- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, move to regime p^*-MYOPIC. Otherwise, move to regime $(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION.
- Under regime $(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION, if the seller deviates to $p' \geq 0$, move to regime $D_{p'}$-PUNISH. Otherwise, stay in the same regime.
- Under regime p^*-MYOPIC, if the seller deviates to any $p' \neq p^*$, move to $D_{p'}$-PUNISH. Otherwise, stay in the same regime.

We claim that the strategy profile above constitutes a subgame perfect equilibrium and that the intermediary's payoff is $\hat{\omega}$. To see this, first note since the intermediary's continuation induced by this strategy profile in every subgame is finite, by Lemma 1, it suffices to show that both the intermediary and the seller do not have incentives to deviate under each of the regimes above, given that the other player plays according to this strategy.

Under regime $(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION. First notice that if all players follow their strategies, by Lemma 3, since $\tilde{\omega} \geq \omega^\beta = g^\beta(p^\beta)$,

$$p^\beta \geq v^{-1}\left(\frac{\delta\beta\omega^\beta}{\delta\beta\omega^\beta - \alpha}\right)$$

and hence the intermediary's payoff would be

$$\delta\left(\gamma + \beta\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega} = \tilde{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha)|\tilde{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

[20] See footnote 19.

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq p^*\overline{D}(v^{-1}(p^*))$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega}^\beta \leq p^\beta\overline{D}(p^\beta) \leq \frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

where the first inequality follows from $\beta \geq \widehat{\beta}$, which in turn implies that $p^\beta\overline{D}(p^\beta) \geq p^*\overline{D}(v^{-1}(p^*))$. Thus, the seller does not have any incentive to deviate either.

In the meantime, for $p \in \{\hat{p}, p^*\}$, under regime p-MYOPIC, if all players follow their strategies, the seller's payoff would be $(1-\alpha)p\overline{D}(v^{-1}(p))$ and the intermediary's payoff would be $\alpha p\overline{D}(v^{-1}(p))/(1-\gamma\delta)$. Meanwhile, if the seller deviates to any $p' \geq 0$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq p^*\overline{D}(v^{-1}(p^*))$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega}^\beta = p^*\overline{D}(v^{-1}(p^*)) \leq p\overline{D}(v^{-1}(p)).$$

Thus, the seller does not have an incentive to deviate. As for the intermediary, if she deviates from this strategy, her continuation value would be at most $\underline{\omega}^\beta$. Therefore, since

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \alpha\hat{p}\overline{D}(v^{-1}(p)) + \gamma\delta\hat{\omega} \geq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \gamma\delta\underline{\omega}^\beta$$

and

$$\frac{\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))}{1-\gamma\delta} = \alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega}^\beta,$$

by (A.22), the intermediary's payoff from deviation is at most

$$\begin{aligned}&\sup_{D\in\mathcal{D}}\left[\alpha pD(p) + \delta\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right)\underline{\omega}^\beta\right]\\ =&\alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\underline{\boldsymbol{\omega}}^*(\beta)\end{aligned}$$

and hence the intermediary does not have an incentive to deviate either.

Lastly, for any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\tilde{\omega}$, whereas if she deviates, her payoff would be at most $\alpha p'\overline{D}(v^{-1}(p')) + \gamma\delta\underline{\omega}^\beta$. Since $\tilde{\omega} > \alpha\mathbb{E}[v]/\gamma\delta + \underline{\omega}^\beta$, it must be that

$$\alpha p'D_{p'}(p') + \delta\left(\gamma + \beta\int_{p'}^\infty D_{p'}(v)\,\mathrm{d}v\right)\tilde{\omega} \geq \gamma\delta\tilde{\omega} > \alpha\mathbb{E}[v] + \gamma\delta\underline{\omega}^\beta \geq \alpha p'\overline{D}(v^{-1}(p')) + \gamma\delta\underline{\omega}^\beta.$$

Thus, the intermediary does not have any incentives to deviate under this regime.

Therefore, the strategy profile above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \hat{\omega}.$$

Next, consider any $\hat{\omega} > \omega^\beta$. Again, take any $\tilde{\omega} > \max\{\alpha\mathbb{E}[v]/\gamma\delta, \omega^\beta\}$, we claim that the following strategy profile is a subgame perfect equilibrium and the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha), \hat{\omega})$-TRANSITION. If the seller deviates to any $p' \geq 0$, move to regime p'-PUNISH. Otherwise, stay in the same regime.

- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime p^*-MYOPIC. Otherwise, move to regime $(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha),\tilde{\omega})$-TRANSITION.

- Under regime p^*-MYOPIC, if the seller deviates to $p' \neq p^*$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

- Under regime $(\delta\beta\tilde{\omega}(\alpha,\beta,\delta)p^\beta/(\delta\beta\tilde{\omega}-\alpha),\tilde{\omega})$-TRANSITION, if the seller deviates to any $p' \geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

From the same arguments as above, it follows that both the intermediary and the seller do not have incentives to deviate under regime p'-PUNISH, regime p^*-MYOPIC, and regime $(\delta\beta\tilde{\omega}p^\beta/(\delta\beta\tilde{\omega}-\alpha),\tilde{\omega})$-TRANSITION. Therefore, by Lemma 1, since the intermediary's continuation payoff in any subgame is finite under this strategy profile, it suffices to show that both the intermediary and the seller do not have incentives to deviate under regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha),\hat{\omega})$-TRANSITION and that the intermediary's equilibrium payoff is $\hat{\omega}$. Indeed, since $\hat{\omega} \geq \omega^\beta = g^\beta(p^\beta)$,

$$p^\beta \geq v^{-1}\left(\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\right)$$

and hence by Lemma 3, the intermediary's payoff would be

$$\delta\left(\gamma + \beta\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v\right)\hat{\omega} = \hat{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha)|\hat{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq \underline{\omega}^\beta$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega}^\beta \leq p^\beta\overline{D}(p^\beta)\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\overline{D}(p^\beta),$$

where the second inequality follows from the fact that $\beta \geq \widehat{\beta}$ and the definition of $\widehat{\beta}$. Thus, the seller does not have any incentive to deviate either, as desired.

Case 4: $\beta \in (\overline{\beta}, \beta^*)$.
If $\underline{\omega}^\beta = \omega^*$, then the argument of **Case 3** applies. Therefore, it suffices to consider the case where $\underline{\omega}^\beta > \omega^*$. In this case, it must be that

$$\delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\beta\delta\underline{\omega}^\beta}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v\right) = 1 \iff h^\beta(\underline{\omega}^\beta) = \underline{\omega}^\beta,$$

which also implies that $\underline{\omega}^\beta > \alpha/\delta\beta$. Furthermore, by Lemma A.1, since $\mathbf{W}(p^*|\underline{\omega}^\beta) = \underline{\omega}^\beta$ and since

$$\delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}^\beta}\right)p^*}^\infty \overline{D}(v)\,\mathrm{d}v\right) = 1,$$

we have

$$\underline{\omega}^{\beta} = \mathbf{W}(p^*|\underline{\omega}^{\beta}) \geq \alpha p^* \overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega}^{\beta},$$

and hence

$$\delta\beta\underline{\omega}^{\beta} \int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}^{\beta}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v \geq \alpha p^* \overline{D}(v^{-1}(p^*)),$$

which in turn is equivalent to

$$\left(1 - \frac{\alpha}{\delta\beta\underline{\omega}^{\beta}}\right) p^* \geq v^{-1}(p^*). \tag{A.23}$$

In the meantime, notice that for any $\omega \geq \underline{\omega}^{\beta}$, since $p \mapsto p\overline{D}((1-\alpha/\delta\beta\omega)p)$ is quasi-concave,

$$\overline{p}\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega_{t+1}}\overline{p}\right)\right) > \overline{p}\overline{D}(\overline{p}) \geq p^*\overline{D}(v^{-1}(p^*)),$$

where the last inequality follows from the definition of p^*. Moreover, for any $\omega \geq \underline{\omega}^{\beta}$, by (A.23),

$$p^*\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)p^*\right) < p^*\overline{D}(v^{-1}(p^*)),$$

and hence there exists a unique $\boldsymbol{p}(\omega) \in [\overline{p}, p^*]$ such that

$$\boldsymbol{p}(\omega)\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)\boldsymbol{p}(\omega)\right) = p^*\overline{D}(v^{-1}(p^*)). \tag{A.24}$$

By definition, the function $\boldsymbol{p}$ is continuous, bounded by $\overline{p}$ and p^*, and such that $\omega \mapsto (1-\alpha/\beta\delta\omega)\boldsymbol{p}(\omega)$ is decreasing. Define

$$\overline{\omega} := \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}^{\beta}}\right)\boldsymbol{p}(\underline{\omega}^{\beta})}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\underline{\omega}^{\beta}.$$

It then follows that $\overline{\omega} > \underline{\omega}^{\beta}$.

We now introduce a recursive formula.

Algorithm 1. For any $t \in \mathbb{N}\cup\{0\}$, given any $\omega_t \geq \overline{\omega}$, take ω_{t+1} so that[21]

$$\omega_t = \left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{t+1}}\right)\boldsymbol{p}(\omega_{t+1})}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{t+1}.$$

Then, let $p_t := \boldsymbol{p}(\omega_{t+1})$.

If $\omega_{t+1} \geq \overline{\omega}$, then repeat the procedure by letting $\omega_t = \omega_{t+1}$. Otherwise, stop. $\diamond$

From Algorithm 1, for any initial value $\omega_0 \geq \overline{\omega}$, we may obtain sequences $\{\omega_t\}_{t=0}^{T}$ and $\{p_t\}_{t=0}^{T-1}$ such that

$$\omega_t = \left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{t+1}}\right)p_t}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{t+1}, \tag{A.25}$$

[21] Such ω_{t+1} exists since $\omega_t \geq \overline{\omega}$ and

$$\lim_{\omega\to\infty} \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega}\right)\boldsymbol{p}(\omega)}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega = \infty.$$

that

$$p_t\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\omega_{t+1}}\right)p_t\right)=p^*\overline{D}(v^{-1}(p^*)),\tag{A.26}$$

and that

$$\left(1-\frac{\alpha}{\delta\beta\omega_{t+1}}\right)p_t\geq v^{-1}(p_t).\tag{A.27}$$

Consider any $\hat{\omega}\geq\underline{\omega}^{\beta}$. We will now construct subgame perfect equilibria that give the intermediary equilibrium payoff $\hat{\omega}$. First, suppose that $\hat{\omega}\leq\overline{\omega}$. In this case, take any $T\in\mathbb{N}$ such that [22]

$$\frac{1}{\delta^T}\left(\underline{\omega}^{\beta}-\frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta}\alpha p^*\overline{D}(v^{-1}(p^*))\right)>\max\left\{\overline{\omega},\frac{1}{\delta}\alpha\mathbb{E}[v]+(\gamma+\beta\mathbb{E}[v])\,\underline{\omega}^{\beta}\right\}.$$

Meanwhile, for any $\omega\in[\underline{\omega}^{\beta},\overline{\omega}]$, let

$$\phi(\omega):=\frac{1}{\delta^T}\left(\omega-\frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta}\alpha p^*\overline{D}(v^{-1}(p^*))\right),$$

and let $\{\boldsymbol{\omega}_t(\omega)\}_{t=0}^{T^{\omega}}$ and $\{\boldsymbol{p}_t(\omega)\}_{t=0}^{T^{\omega}-1}$ denote the sequences obtain from Algorithm 1 with $\phi(\omega)$ being the initial value. Meanwhile, for any $p'\geq 0$, by the definition of p^*, there exists $D_{p'}\in\mathcal{D}$ such that $p'D_{p'}(p^{'+})\leq p^*\overline{D}(v^{-1}(p^*))$. Fix any such $D_{p'}\in\mathcal{D}$ for all p'. We now describe the desired subgame perfect equilibrium. For expositional convenience, we index strategies by a state variable ω.

- Set the state as $\hat{\omega}$. Start by playing regime p^*-MYOPIC with state $\hat{\omega}$.
- For any state $\omega\in[\underline{\omega}^{\beta},\overline{\omega}]$, under regime p^*-MYOPIC with state $\omega\in[\underline{\omega}^{\beta},\overline{\omega}]$, if no one has deviated and if this regime has been played for less than T periods, stay in the same regime and the same state; if the seller deviates to any $p'\neq p^*$, move to regime $D_{p'}$-PUNISH immediately while keeping the state unchanged; if the intermediary deviates, reset the count, set the state to $\underline{\omega}^{\beta}$, and stay under the same regime. Otherwise, keep the state unchanged and move to regime $(\boldsymbol{p}_0(\omega),\boldsymbol{\omega}_1(\omega))$-TRANSITION.
- For any state $\omega\in[\underline{\omega}^{\beta},\overline{\omega}]$, under regime $(\boldsymbol{p}_{t-1}(\omega),\boldsymbol{\omega}_t(\omega))$-TRANSITION, if the seller deviates to any $p'\neq p^*$, move to regime $D_{p'}$-PUNISH immediately and set the state to $\underline{\omega}^{\beta}$. Otherwise, move to regime $(\boldsymbol{p}_t(\omega),\boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION while keeping the state unchanged in the next period if $t<T^{\omega}-1$, and move to regime p^*-MYOPIC while setting the state as $\boldsymbol{\omega}_T(\omega)$ if $t=T^{\omega}-1$.
- Under regime $D_{p'}$-PUNISH with any state $\omega\in[\underline{\omega}^{\beta},\overline{\omega}]$, if the intermediary deviates, then move to regime p^*-MYOPIC while setting the state as $\underline{\omega}^{\beta}$. Otherwise, set the state to $\overline{\omega}$ and move to regime $(\boldsymbol{p}_0(\overline{\omega}),\boldsymbol{\omega}_0(\overline{\omega}))$-TRANSITION.

To see that this constitutes a subgame perfect equilibrium, first note that the intermediary's continuation payoff in every subgame is finite under this strategy profile. Thus, by Lemma 1, it suffices to show that there are no incentives for one-shot deviations for each player.

[22]Such T exists because

$$\underline{\omega}^{\beta}>\omega^*=\frac{\alpha p^*\overline{D}(v^{-1}(p^*))}{1-\gamma\delta}.$$

To see this, under regime p^*-MYOPIC with any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$, given that the seller does not deviate, if the intermediary follows the strategy, her present discounted payoff in each period would be at least $\hat{\omega}$.[23] Alternatively, if the intermediary deviates in any period and maintain her continuation strategy, since the continuation play will enter into regime p^*-MYOPIC with state being $\underline{\omega}^\beta$, her continuation value would be $\underline{\omega}^\beta$. Thus, the best present discounted value she can obtain would be

$$\sup_{D \in \mathcal{D}} \left[\alpha p^* D(p^*) + \delta \left(\gamma + \beta \int_{p^*}^{\infty} \overline{D}(v) \, \mathrm{d}v \right) \underline{\omega}^\beta \right] = \underline{\omega}^\beta,$$

where the equality follows from $h^\beta(\underline{\omega}^\beta) = \underline{\omega}^\beta$. Therefore, the intermediary does not have profitable one-shot deviation under regime p^*-MYOPIC with any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$. Meanwhile, if the seller follows his strategy, his payoff would be $(1-\alpha)p^*\overline{D}(v^{-1}(p^*))$, while if he deviate to any $p' \geq 0$, his payoff would be $(1-\alpha)p' D_{p'}(p') \leq (1-\alpha)p^*\overline{D}(v^{-1}(p^*))$.

Moreover, for any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$ and for any $t \in \{0, \dots, T^\omega - 1\}$, under regime-$(\boldsymbol{p}_t(\omega), \boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION, by (A.25), (A.27), and Lemma 3,

$$\boldsymbol{\omega}_t(\omega) = \sup_{D \in \mathcal{D}} \left[\alpha \boldsymbol{p}_t(\omega) D(\boldsymbol{p}_t(\omega)) + \delta \left(\gamma + \beta \int_{\boldsymbol{p}_t(\omega)}^{\infty} \overline{D}(v) \, \mathrm{d}v \right) \boldsymbol{\omega}_{t+1}(\omega) \right]$$

and hence the intermediary would not have any incentive to deviate and her payoff in this subgame would be $\boldsymbol{\omega}_t(\omega)$. Meanwhile, if the seller deviates to any $p' \neq \boldsymbol{p}_t(\omega)$, his payoff would be $(1-\alpha)p' D_{p'}(p') \leq (1-\alpha)p^*\overline{D}(v^{-1}(p^*)) = \boldsymbol{p}_t(\omega)\overline{D}((1-\alpha/\delta\beta\omega)\boldsymbol{p}_t(\omega))$, due to (A.26). Thus, the seller does not have incentives to deviate either.

Lastly, under regime $D_{p'}$-PUNISH with any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$, if the intermediate deviates, her continuation payoff would be $\underline{\omega}^\beta$ and hence the best payoff from deviation is

$$\alpha \mathbb{E}[v] + \delta \left(\gamma + \beta \mathbb{E}[v] \right) \underline{\omega}^\beta.$$

Meanwhile, if she follows the strategy, her continuation payoff would be $\phi(\overline{\omega})$ and hence her payoff would be at least $\delta\phi(\overline{\omega})$. By the definition of T and ϕ,

$$\delta\phi(\overline{\omega}) = \frac{1}{\delta^{T-1}} \left[\overline{\omega} - \frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta} \alpha p^* \overline{D}(v^{-1}(p^*)) \right] > \alpha \mathbb{E}[v] + \delta \left(\gamma + \beta \mathbb{E}[v] \right) \underline{\omega}^\beta.$$

Thus, the intermediary does not have one-shot deviation incentives.

As a result, neither players have profitable one-shot deviations. Moreover, as shown above, since the intermediary' continuation payoff after playing T rounds of p^*-MYOPIC with state $\hat{\omega}$ is exactly $\boldsymbol{\omega}_0(\hat{\omega})$, her on-path payoff under this strategy profile is given by

$$\sum_{t=0}^{T} \gamma^t \delta^t \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta \boldsymbol{\omega}_0(\hat{\omega}) = \frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta} \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta\phi(\hat{\omega}) = \hat{\omega} < \infty.$$

[23] This is because the continuation value at the beginning of the $(\boldsymbol{p}_0(\hat{\omega}), \boldsymbol{\omega}_1(\hat{\omega}))$-TRANSITION regime is $\phi(\hat{\omega})$ and since

$$\hat{\omega} = \frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta} \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta^T \phi(\hat{\omega}) = \sum_{s=0}^{T-1} \gamma^s \delta^s \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta^T \phi(\hat{\omega}).$$

Alternatively, if $\hat{\omega} > \overline{\omega}$, we may construct the same type of strategy as follows: Let $\omega_0 := \hat{\omega}$ and let $\{\omega_t\}_{t=0}^{\bar{T}}$ and $\{p_t\}_{t=0}^{\bar{T}-1}$ denote the sequences obtained from Algorithm 1 with the initial value being ω_0. Consider the following strategy profile:

- Start by playing regime (p_0, ω_1)-TRANSITION with a null state $\emptyset$.
- For any $t \in \{1, \ldots, \bar{T}-1\}$, under regime (p_{t-1}, ω_t)-TRANSITION with any state, if the seller deviates to any p', move to regime $D_{p'}$-PUNISH immediately while setting the state as $\underline{\omega}^\beta$. Otherwise, move to regime (p_t, ω_{t+1})-TRANSITION while keeping the state unchanged if $t < \bar{T}-1$, and move to regime p^*-MYOPIC while setting the state to $\omega_{\bar{T}} \in [\underline{\omega}^\beta, \overline{\omega}]$ if $t = \bar{T}-1$.
- For any state ω, under regime p^*-MYOPIC with state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$, if no one has deviated and if this regime has been played for less than T periods, stay in the same regime and the same state; if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately while keeping the state unchanged; if the intermediary deviates, reset the count, set the state to $\underline{\omega}^\beta$, and stay under the same regime. Otherwise, keep the state unchanged and move to regime $(\boldsymbol{p}_0(\omega), \boldsymbol{\omega}_1(\omega))$-TRANSITION.
- For any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$, under regime $(\boldsymbol{p}_{t-1}(\omega), \boldsymbol{\omega}_t(\omega))$-TRANSITION, if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately and set the state to $\underline{\omega}^\beta$. Otherwise, move to regime $(\boldsymbol{p}_t(\omega), \boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION while keeping the state unchanged in the next period if $t < T^\omega - 1$, and move to regime p^*-MYOPIC while setting the state as $\boldsymbol{\omega}_T(\omega)$ if $t = T^\omega - 1$.
- Under regime p'-PUNISH with any state $\omega \in [\underline{\omega}^\beta, \overline{\omega}]$, if the intermediary deviates, then move to regime p^*-MYOPIC while setting the state as $\underline{\omega}^\beta$. Otherwise, set the state to $\overline{\omega}$ and move to regime $(\boldsymbol{p}_0(\overline{\omega}), \boldsymbol{\omega}_0(\overline{\omega}))$-TRANSITION.

By the same arguments as those for the case $\hat{\omega} \in [\underline{\omega}^\beta, \overline{\omega}]$, there are no profitable one-shot deviations for all players. Thus, by Lemma 1, since the intermediary's payoff following this strategy profile is $\omega_0 = \hat{\omega}$, this is also a subgame perfect equilibrium.

Together, it follows that whenever $\beta < \beta^*$, there exists $\underline{\omega}^\beta \leq \underline{\boldsymbol{\omega}}^*(\beta) \leq \overline{\boldsymbol{\omega}}^*(\beta) \leq \infty$ such that $\Omega^*(\beta) = [\underline{\boldsymbol{\omega}}(\beta), \overline{\boldsymbol{\omega}}(\beta)] \backslash \{\infty\}$. This completes the proof. ■

A.8 Proof of Corollary 1

Consider any $\beta \geq 0$ and any subgame perfect equilibrium . Let $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ be its outcome. Since r^* is the revenue guarantee, $r_t \geq r^*$ for all t. Moreover, by the proof of Theorem 2, $r_t \geq (1-\gamma\delta)\underline{\boldsymbol{\omega}}^*(\beta)/\alpha$ for all t whenever $\beta \leq \overline{\beta}$. Thus, $r_t \geq \underline{r}(\beta)$ for all t. Meanwhile, since $D(0) = 1$ for all $D \in \mathcal{D}$, it must be that $r_t \leq p_t$ for all t.

For any $t \in \mathbb{N} \cup \{0\}$, let $D_t \in \mathcal{D}$ be the disclosure policy chosen by the intermediary on the equilibrium path in period t so that

$$\sigma_t = \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v.$$

Since $D_t \in \mathcal{D}$, $\sigma_t \geq (\mathbb{E}[v] - p_t)^+$. Moreover, since $D_t \in \mathcal{D}$ is nonincreasing, it must be that

$$\int_{p_t}^{\infty} \overline{D}(v)\,\mathrm{d}v \geq \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v \geq \sigma_t - (p - p_t)D_t(p_t),$$

for all $p \geq 0$. As a result,

$$\sigma_t \leq \min_{p\geq 0}\left[\int_p^\infty \overline{D}(v)+(p-p_t)D_t(p_t)\right] = S\left(\frac{r_t}{p_r}\right)-r_t,$$

where the equality follows from the first order condition of the minimization problem, which implies that at the solution $\hat{p}_t$, $\overline{D}(\hat{p}_t)=D_t(p_t)=r_t/p_t$. Together, we have

$$(\mathbb{E}[v]-p_t)^+ \leq \sigma_t \leq S\left(\frac{r_t}{p_r}\right)-r_t,$$

for all $t\in\mathbb{N}\cup\{0\}$.

Lastly, for any $t\in\mathbb{N}\cup\{0\}$, Theorem 2 implies that $\omega_t \leq \overline{\boldsymbol{\omega}}^*(\beta)$. Moreover, subgame perfection implies that

$$\alpha r_t+\delta(\gamma+\beta\sigma_t)\omega_t \geq \sup_{D\in\mathcal{D}}\left[\alpha p_t+\delta\left(\gamma+\beta\int_{p_t}^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right].$$

Together, it must be that

$$\alpha r_t+\delta(\gamma+\beta\sigma_t)\underline{\boldsymbol{\omega}}^*(\beta) \geq \sup_{D\in\mathcal{D}}\left[\alpha p_t+\delta\left(\gamma+\beta\int_{p_t}^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right],$$

as desired.

Conversely, given any $(r,\sigma,p)\in\mathbf{Z}^*(\beta)$ and any $T\in\mathbb{N}$, it suffices to construct a subgame perfect equilibrium with outcome $\mathbf{z}=\{r_t,\sigma_t,\omega_t,p_t,m_t\}$ with $r_0=r$, $\sigma_0=\sigma$, and $p_0=p$, since we may fix this equilibrium play and augment the strategy profile using backward induction for T periods. To this end, we first claim that there exists $D_0\in\mathcal{D}$ such that $pD_0(p)=r$ and $\int_p^\infty D_0(v)\,\mathrm{d}v=\sigma$. Indeed, define D_0 as follows:

$$D_0(v) := \begin{cases} 1, & \text{if } v\in\left[0,\frac{p}{p-r}(\mathbb{E}[v]-r-\sigma)\right] \\ \frac{r}{p}, & \text{if } v\in\left(\frac{p}{p-r}(\mathbb{E}[v]-r-\sigma),p+\frac{p}{r}\sigma\right] \\ 0, & \text{if } v>p+\frac{p}{r}\sigma \end{cases}.$$

Since $\sigma\in[(\mathbb{E}[v]-p)^+,S(r/p)-r]$, it follows that $D_0\in\mathcal{D}$. Moreover, by definition,

$$pD_0(p)=p\cdot\frac{r}{p}=r,$$

and

$$\int_p^\infty D_0(v)\,\mathrm{d}v=\mathbb{E}[v]-\frac{p}{p-r}(\mathbb{E}[v]-r-\sigma)-\frac{r}{p}\left(p-\frac{p}{p-r}(\mathbb{E}[v]-r-\sigma)\right)=\sigma,$$

as desired. Meanwhile, for any $p'\geq 0$ and for any $\omega\geq 0$, consider the following maximization problem:

$$\sup_{D\in\mathcal{D},q\in[D(p'^+),D(p')]}\left[p'q+\delta\left(\gamma+\beta\int_{p'}^\infty D(v)\,\mathrm{d}v\right)\omega\right]$$
$$\text{s.t. } p'q\leq \underline{r}(\beta)$$

and denote that solution by $(D_{p'},q_{p'})$ and the value by $\widetilde{\Lambda}(p',\omega)$. Notice that whenever $\underline{r}(\beta)>r^*$, $q_{p'}=D_{p'}(p')$, while $q_{p'}=D_{p'}(p'^+)$ if $\underline{r}(\beta)=r^*$. Moreover, by the definitions of $\overline{\boldsymbol{\omega}}^*(\beta)$ and $\underline{\boldsymbol{\omega}}^*(\beta)$ in the proof of Theorem 2,

$$\widetilde{\Lambda}(p',\overline{\boldsymbol{\omega}}^*(\beta)) \geq \sup_{D\in\mathcal{D}}\left[\alpha p'D(p')+\delta\left(\gamma+\int_{p'}^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^*(\beta)\right]$$

Lastly, for any $\beta \geq 0$, since

$$\alpha r + \delta(\gamma + \beta\sigma)\overline{\boldsymbol{\omega}}^*(\beta) \geq \sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta \left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right) \underline{\boldsymbol{\omega}}^*(\beta)\right],$$

there exists $\widetilde{\boldsymbol{\omega}}^*(\beta) \leq \overline{\boldsymbol{\omega}}^*(\beta)$ such that $\widetilde{\boldsymbol{\omega}}^*(\beta) < \infty$,

$$\alpha r + \delta(\gamma + \beta\sigma)\widetilde{\boldsymbol{\omega}}^*(\beta) \geq \sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta \left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right) \underline{\boldsymbol{\omega}}^*(\beta)\right], \tag{A.28}$$

and

$$\widetilde{\Lambda}(p', \widetilde{\boldsymbol{\omega}}^*(\beta)) \geq \sup_{D \in \mathcal{D}} \left[\alpha p' D(p') + \delta \left(\gamma + \int_{p'}^{\infty} D(v)\,\mathrm{d}v\right) \underline{\boldsymbol{\omega}}^*(\beta)\right]. \tag{A.29}$$

Now consider the following strategy profile: In period 0, the seller charges price p; the intermediary chooses $D_0 \in \mathcal{D}$ if the seller charges price p, and chooses $D_{p'}$ if the seller charges any other price $p' \neq p$; the tie-breaker chooses $q = D_0(p)$ if the seller charges p and the intermediary chooses D_0, and chooses $q_{p'}$ if the seller charges $p' \neq p$ and the intermediary chooses $D_{p'}$, and always breaks ties in favor of the seller otherwise. Starting from period 1, if the seller charges price p and the intermediary chooses D_0 in period 0, or if the seller charges any $p' \neq p$ and the intermediary chooses $D_{p'}$ in period 0 then they play the subgame perfect equilibrium that gives the intermediary payoff $\widetilde{\boldsymbol{\omega}}^*(\beta)$. Otherwise, they play the subgame perfect equilibrium that gives the intermediary payoff $\underline{\boldsymbol{\omega}}^*(\beta)$.

We claim that this strategy profile is indeed a subgame perfect equilibrium . To see this, notice first that since all players always play a subgame perfect equilibrium from period 1 onward, it suffices to verify that there are no incentives for the seller and the intermediary to deviate from the aforementioned strategies. For the seller, for any $p' \geq 0$,

$$p' q_{p'} \leq \underline{r}(\beta) \leq r = p D_0(p),$$

and hence the seller does not have an incentive to deviate. For the intermediary, given that the seller charges p, and given the continuation play, choosing D_0 gives

$$\alpha r + \delta(\gamma + \beta\sigma)\widetilde{\boldsymbol{\omega}}^*(\beta),$$

whereas the highest payoff the intermediary can obtain from any deviation is

$$\sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta \left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right) \underline{\boldsymbol{\omega}}^*(\beta)\right].$$

By (A.28), the intermediary has no incentive to deviate when the seller charges p. Finally, if the seller chargers any price $p' \neq p$, following the aforementioned strategy and choosing $D_{p'}$ gives the intermediary payoff

$$\alpha p' q_{p'} + \delta \left(\gamma + \beta \int_{p'}^{\infty} D_{p'}(v)\,\mathrm{d}v\right) \widetilde{\boldsymbol{\omega}}^*(\beta) = \Lambda^{\beta}(p', \widetilde{\boldsymbol{\omega}})^*(\beta),$$

whereas the highest payoff she can obtain from any deviation is

$$\sup_{D \in \mathcal{D}} \left[\alpha p D(p) + \delta \left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right) \underline{\boldsymbol{\omega}}^*(\beta)\right].$$

By (A.29), the intermediary has no incentive to deviate when the seller charges any $p' \neq p$ either. Together, this aforementioned strategy profile is indeed a subgame perfect equilibrium .

By construction, this subgame perfect equilibrium induces an outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ with $r_0 = r$, $\sigma_0 = \sigma$, and $p_0 = p$, as desired. This completes the proof. ∎

A.9 Proof of Proposition 2

Consider any $\beta \in [0, \overline{\beta}]$. We first show that there exists a subgame perfect equilibrium that is dominated by any other subgame perfect equilibrium . By Corollary 1, it suffices to find a subgame perfect equilibrium where the intermediary's normalized continuation payoff is $\underline{\boldsymbol{\omega}}^*(\beta)$ and the normalized sales revenue is $\underline{r}(\beta)$, while the consumer surplus is zero in every period. On the equilibrium path of the subgame perfect equilibrium that gives the intermediary payoff $\underline{\boldsymbol{\omega}}^*(\beta)$, which is constructed in the proof of Theorem 2, the seller charges a price $\underline{p}$ such that $\underline{p}\overline{D}(v^{-1}(\underline{p})) = (1-\gamma\delta)\underline{\boldsymbol{\omega}}^*(\beta)/\alpha = \underline{r}(\beta)$; the intermediary chooses the myopic best response when the seller charges $\underline{p}$, which in turn leaves consumers no surplus. As a result, this subgame perfect equilibrium induces an outcome $\mathbf{z}^*(\beta)$ that is dominated by any other subgame perfect equilibrium outcomes .

Moreover, notice that when $\gamma\delta \leq 1/2$, by its definition, $\underline{\boldsymbol{\omega}}^*(\beta) > \underline{\omega}^{\beta} = \omega^*$ if $\beta < \widehat{\beta}$ is small enough. Therefore, since $\underline{\boldsymbol{\omega}}^*$ is noninreasing on $[0, \overline{\beta}]$, for any γ, δ such that $\gamma\delta \leq 1/2$, there exists $\widehat{\beta}(\gamma, \delta) \in (0, \widehat{\beta})$ such that $\underline{\boldsymbol{\omega}}^*(\beta) > \omega^*$ for all $\beta \in (0, \widehat{\beta}(\gamma, \delta))$. As a result, by its definition, $\underline{\boldsymbol{\omega}}^*$ is strictly decreasing on $(0, \widehat{\beta}(\gamma, \delta))$ and hence for any $\beta, \beta' \in (0, \widehat{\beta}(\gamma, \delta))$ with $\beta' > \beta$, $\mathbf{z}^*(\beta)$ dominates $\mathbf{z}^*(\beta')$. This completes the proof. ■